PROPAGATION OF UNIFORM BOUNDEDNESS OF ENTROPY AND INHOMOGENEOUS REGULARITIES FOR VISCOUS AND HEAT CONDUCTIVE GASES WITH FAR FIELD VACUUM IN THREE DIMENSIONS

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ABSTRACT. Due to the highly degeneracy and singularities of the entropy equation, the physical entropy for viscous and heat conductive polytropic gases behave singularly in the presence of vacuum and it is thus a challenge to study its dynamics. It is shown in this paper that the uniform boundedness of the entropy and the inhomogeneous Sobolev regularities of the velocity and temperature can be propagated for viscous and heat conductive gases in $\mathbb{R}^3$, provided that the initial vacuum occurs only at far fields with suitably slow decay of the initial density. Precisely, it is proved that for any strong solution to the Cauchy problem of the heat conductive compressible Navier–Stokes equations, the corresponding entropy keeps uniformly bounded and the L^2 regularities of the velocity and temperature can be propagated, up to the existing time of the solution, as long as the initial density vanishes only at far fields with a rate no more than $O(\frac{1}{|x|^2})$. The main tools are some singularly weighted energy estimates and an elaborate De Giorgi type iteration technique. The De Giorgi type iterations are carried out to different equations in establishing the lower and upper bounds of the entropy.

1. Introduction

In this paper, we consider the following heat conductive compressible Navier–Stokes equations in $\mathbb{R}^3$:

$$\partial_t\rho + \mathrm{div}\,(\rho u) = 0, \tag{1.1}$$

$$\rho(\partial_t u + (u\cdot\nabla)u) - \mu\Delta u - (\mu+\lambda)\nabla\mathrm{div}\,u + \nabla p = 0, \tag{1.2}$$

$$c_v\rho(\partial_t\theta + u\cdot\nabla\theta) + p\mathrm{div}\,u - \kappa\Delta\theta = \mathcal{Q}(\nabla u), \tag{1.3}$$

where the unknowns $\rho\in[0,\infty)$, $u\in\mathbb{R}^3$, $\theta\in[0,\infty)$, and $p\in[0,\infty)$, respectively, represent the density, velocity, temperature, and pressure. Here, c_v is a positive constant, μ and λ are the viscous coefficients, both assumed to be constants and satisfy the physical constraints

$$\mu>0,\quad 2\mu+3\lambda>0,$$

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κ is the heat conductive coefficient, assumed to be a positive constant, and $\mathcal{Q}(\nabla u)$ is a quadratic term of ∇u given as

$$\mathcal{Q}(\nabla u) = \frac{\mu}{2}|\nabla u + (\nabla u)^T|^2 + \lambda(\operatorname{div} u)^2.$$

System (1.1)–(1.3) is complemented with some constitutive equations. The equations of state for ideal gases are given by

$$p = R\rho\theta, \quad e = c_v\theta,$$

for a positive constant R, where e is the specific internal energy. By the Gibbs equation $\theta Ds = De + pD(\frac{1}{\rho})$, where s is the specific entropy, it holds that

$$p = Ae^{\frac{s}{c_v}}\rho^\gamma$$

for some positive constant A, where $\gamma - 1 = \frac{R}{c_v}$. It is clear that $\gamma > 1$. In terms of ρ and θ, the specific entropy s can be expressed as

$$s = c_v\left(\log\frac{R}{A} + \log\theta - (\gamma-1)\log\rho\right), \tag{1.4}$$

satisfying

$$\rho(\partial_t s + u\cdot\nabla s) - \frac{\kappa}{c_v}\Delta s = \kappa(\gamma-1)\operatorname{div}\left(\frac{\nabla\rho}{\rho}\right) + \frac{1}{\theta}\left(\mathcal{Q}(\nabla u) + \kappa\frac{|\nabla\theta|^2}{\theta}\right), \tag{1.5}$$

in the region where both ρ and θ are positive.

The compressible Navier–Stokes equations have been studied extensively with many significant results. One of the important issues in this theory is the vacuum, which, if occurs, means that the density of the fluid vanishes at some points of the domain occupied by the fluid or at far fields. Indeed, the possible presence of vacuum is one of the main difficulties in the theory of global well-posedness of general solutions to the compressible Navier–Stokes equations, and one of the main reasons is that system (1.1)–(1.3) changes its types in the sense that it is a hyperbolic-parabolic coupled system in the non-vacuum region but degenerates to a hyperbolic-elliptic one near the vacuum region. It is even more difficult to analyze the properties of the entropy in the presence of vacuum, as the governing equation (1.5) for entropy is highly degenerate and singular in the vacuum region. Due to this, most of the mathematical studies on the compressible Navier–Stokes equations in the presence of vacuum focus on system (1.1)–(1.3) regardless of the entropy.

The mathematical theory for the one-dimensional compressible Navier–Stokes equations is satisfactory and in particular the global well-posedness has already been known. In the absence of vacuum, global well-posedness of strong solutions was established by Kazhikov–Shelukin [24] and Kazhikov [25], which were later extended in the setting of weak solutions, see, e.g., [2, 23, 54, 55]; see Li–Liang [32] for the large time behavior of solutions with large initial data. In the presence of vacuum, the corresponding global well-posedness of strong solutions were recently established by the first author of this paper in [29, 30], for both heat conductive and non-heat conductive ideal gases without considering the entropy.

One major difference between the one-dimensional and multidimensional cases for the compressible Navier–Stokes equations is the possible formation of vacuum. As shown by Hoff–Smoller [17], in the one-dimensional case, no vacuum can be formed later in finite time from non-vacuum initial data, while such a result remains open for the multidimensional case.

Comparing with the one-dimensional case, mathematical theory for the multidimensional compressible Navier–Stokes equations is far from complete and some fundamental questions remain challenging, which include the global well-posedness of smooth solutions and uniqueness of weak solutions. Global existence of finite energy weak solutions with possible vacuum to the isentropic compressible Navier–Stokes equations was first proved by Lions [36, 37], later improved by Feireisl–Novotný–Petzeltová [12], Jiang–Zhang [22], and more recently Bresch–Jabin [1]. For the full compressible Navier–Stokes equations, global existence of variational weak solutions was proved by Feireisl [14] under some assumptions on the equations of state. For suitably regular initial data, the compressible Navier–Stokes equations admit a unique strong or classic solution, at least in a short time: in the absence of vacuum, this was proved by Nash [43] and Serrin [45] long time ago, and later developed in many works, see, e.g., [21, 38, 47, 48, 50]; in the presence of vacuum, this was proved in [5–7, 46] under some initial compatibility conditions which were removed in [15, 18, 31] recently. However, global well-posedness of solutions with arbitrary large initial data is still open. For the time being, global well-posedness was established only under some additional conditions on the initial data: the case with small perturbed initial data around non-vacuum equilibriums was achieved by Matsumura–Nishida [39–42], and later developed in many works, see, e.g., [3, 4, 8–11, 16, 26, 44, 49]; while the case with initial data of small energy but allowing large oscillations and vacuum was proved by Huang–Li–Xin [20] and Li–Xin [33] for the isentropic case, and later generalized to the full system in [19, 28, 51].

It should be noted that significant differences exist in the mathematical theories for the compressible Navier–Stokes equations between the vacuum and non-vacuum cases and new phenomena may occur depending on the locations and states of vacuum. In the non-vacuum case, one can establish solutions in both the homogeneous and inhomogeneous spaces depending on the properties of the initial data, and the solution spaces guarantee the uniform boundedness of the entropy. This fails in general in the presence of vacuum. Indeed, in the case that the density has compact support, the solution, no matter locally or globally, can be established only in the homogeneous spaces, see, e.g., [5–7, 19, 20, 51], but not in the inhomogeneous spaces, see Li–Wang–Xin [27], and the global solutions may have unbounded entropy. Further more, the blowup results of Xin [52] and Xin–Yan [53] imply that the global solutions established in [19, 28, 51] must have unbounded entropy, if initially there is an isolated mass group surrounded by the vacuum region. However, it is somewhat surprisingly that if the initial density vanishes only at far fields with a rate no more than $O(\frac{1}{|x|^2})$, then, as for the non-vacuum case, the solutions can be established in both the homogeneous and inhomogeneous spaces, and the entropy can be uniformly bounded up to any finite time, at least in the one-dimensional case, see the recent works by the authors [34, 35].

Mathematically, since system (1.1)–(1.3) is already closed, one can establish the corresponding theory for it, regardless of the entropy. However, since the entropy is one of the fundamental states for describing the status of the fluids, it is physically important to analyze its dynamical behavior. Unfortunately, the theories developed previously for system (1.1)–(1.3) do not provide any information about the entropy near the vacuum region.

Technically, due to the lack of the expression of the entropy in the vacuum region and the high singularity and degeneracy of the entropy equation near the vacuum region, in spite of its importance, the mathematical analysis of the entropy for the viscous compressible fluids in the presence of vacuum was rarely carried out before. Recently, we have initiated studies on these issues in the one dimensional case in [34, 35]. It was proved in [34, 35] that the one-dimensional compressible Navier–Stokes equations, with or without heat conducting, can propagate the uniform boundedness of the entropy locally or globally in time, as long as the initial density vanishes only at far fields with a rate no more than $O(\frac{1}{x^2})$. However, the problems in the multi-dimensional case have not been studied.

In this paper, we continue our studies, initiated in [34, 35], on the uniform boundedness of the entropy and well-posedness of strong solutions in inhomogeneous spaces for the multi-dimensional full compressible Navier–Stokes equations in the presence of vacuum. We will focus on the heat conductive flows. Note that for the heat conductive case one only need to deal with the the far field vacuum, as the heat conductivity makes the temperature strictly positive everywhere after the initial time, which implies that the entropy becomes unbounded instantaneously if the interior vacuum occurs initially. It is noted that the problem of the existence of solutions in the inhomogeneous Sobolev spaces, under some conditions on the initial density allowing vacuum at the far fields has been studied in [34] for the non-heat conductive compressible flows in one dimension, but it has not yet been studied either for the heat conductive flows or in multi dimensions.

We will consider the Cauchy problem only in this paper and, thus, complement the system with the following initial condition:

$$(\rho, u, \theta)|_{t=0} = (\rho_0, u_0, \theta_0). \tag{1.6}$$

The following conventions are used throughout this paper. For $1 \leq q \leq \infty$ and positive integer m, $L^q = L^q(\mathbb{R}^3)$ and $W^{m,q} = W^{m,q}(\mathbb{R}^3)$, respectively, are the standard Lebesgue and Sobolev spaces, and $H^m = W^{m,2}$. $D_0^1 = D_0^1(\mathbb{R}^3)$ and $D^{m,q} = D^{m,q}(\mathbb{R}^3)$

are the homogeneous Sobolev spaces defined, respectively, as

$$D_0^1 = \{u \in L^6(\mathbb{R}^3) | \nabla u \in L^2(\mathbb{R}^3)\},$$
$$D^{m,q} = \{u \in L^1_{loc}(\mathbb{R}^3) | \nabla^\alpha u \in L^q(\mathbb{R}^3), 1 \le |\alpha| \le m\}.$$

For $q = 2$, $D^{m,q}$ is simply denoted as D^m. For simplicity, X is used to denote both the space X itself and its N product space X^N. $\|u\|_q$ is the L^q norm of u, and $\|(f_1, f_2, \cdots, f_n)\|_X$ is the sum $\sum_{i=1}^N \|f_i\|_X$ or the equivalent norm $\left(\sum_{i=1}^N \|f_i\|_X^2\right)^{\frac{1}{2}}$. The integral of f over $\mathbb{R}^3$ is abbreviated as $\int f dx$.

Strong solutions considered in this paper are defined as follows.

Definition 1.1. *Given a positive time T and assume that*

$$0 \le \rho_0 \in H^1 \cap W^{1,q}, \quad u_0 \in D_0^1 \cap D^2, \quad 0 \le \theta_0 \in D_0^1 \cap D^2, \tag{1.7}$$

for some $q \in (3, 6]$. A triple (ρ, u, θ) is called a strong solution to system (1.1)–(1.3) in $\mathbb{R}^3 \times (0, T)$, subject to (1.6), if it has the regularities

$$\rho \in C([0, T]; H^1 \cap W^{1,q}), \quad \rho_t \in C([0, T]; L^2 \cap L^q),$$
$$(u, \theta) \in C([0, T]; D_0^1 \cap D^2) \cap L^2(0, T; D^{2,q}),$$
$$(u_t, \theta_t) \in L^2(0, T; D_0^1), \quad \sqrt{\rho} u_t, \sqrt{\rho}\theta_t \in L^\infty(0, T; L^2),$$

satisfies equations (1.1)–(1.3) a.e. $(x, t) \in \mathbb{R}^3 \times (0, T)$, and fulfills the condition (1.6).

Definition 1.2. *A triple (ρ, u, θ) is called a global strong solution to system (1.1)–(1.3), subject to (1.6), if it is a strong solution to the same system in $\mathbb{R}^3 \times (0, T)$, for any finite time T.*

Before stating the main result of this paper, let us recall the following two theorems on the local and global well-posedness of strong solutions to system (1.1)–(1.3), subject to (1.6), which are cited from [7] and [28], respectively.

Theorem 1.1 (Local well-posedness, see [7]). *Let $q \in (3, 6]$ and assume in addition to (1.7) that*

$$-\mu \Delta u_0 - (\mu + \lambda)\nabla \, div u_0 + \nabla p_0 = \sqrt{\rho_0} g_1, \quad -\kappa \Delta \theta_0 - \mathcal{Q}(\nabla u_0) = \sqrt{\rho_0} g_2, \tag{1.8}$$

for given functions $g_1, g_2 \in L^2$, where $p_0 = R\rho_0\theta_0$. Then, there exists a positive time T_* depending only on the initial data, such that system (1.1)–(1.3), subject to (1.6), admits a unique strong solution (ρ, u, θ) in $\mathbb{R}^3 \times (0, T_*)$.

Theorem 1.2 (Global well-posedness, see [28]). *Assume in addition to the conditions (1.7) and (1.8) that $2\mu > \lambda$. Then, there is a positive number ε_0 depending only on R, γ, μ, λ, and κ, such that system (1.1)–(1.3), subject to (1.6), has a unique global strong solution, provided that*

$$\mathscr{N}_0 := \|\rho_0\|_\infty (\|\rho_0\|_3 + \|\rho_0\|_\infty^2 \|\sqrt{\rho_0} u_0\|_2^2)(\|\nabla u_0\|_2^2 + \|\rho_0\|_\infty \|\sqrt{\rho_0} E_0\|_2^2) \le \varepsilon_0.$$

Note that the solutions in Theorem 1.1 and Theorem 1.2 are both in the homogenous Sobolev spaces and, as indicated in [27], the inhomogeneous Sobolev regularities can not be propagated by the compressible Navier–Stokes equations in general, if the initial density is compactly supported. Besides, these solutions may have infinite entropy in the vacuum region, if the initial density contains interior vacuum. Therefore, in order to guarantee the uniform boundedness of the entropy or the inhomogeneous Sobolev regularities, some additional assumptions on the initial density are required, as shown in the following main result of this paper.

Theorem 1.3. *Assume that in addition to the conditions (1.7) and (1.8), the initial density ρ_0 is positive on $\mathbb{R}^3$ and satisfies*

$$|\nabla\rho_0(x)| \leq K_1\rho_0^{\frac{3}{2}}(x), \quad \forall x\in\mathbb{R}^3, \tag{H1}$$

for a positive constants K_1. Denote

$$s_0 = c_v\left(\log\frac{R}{A} + \log\theta_0 - (\gamma-1)\log\rho_0\right), \quad \underline{s}_0 = \inf_{x\in\mathbb{R}^3} s_0(x),$$
$$\overline{s}_0 = \sup_{x\in\mathbb{R}^3} s_0(x), \quad \mathscr{S}_0 = -\mu\Delta u_0 - (\mu+\lambda)\nabla\, div\, u_0 + R\nabla(\rho_0\theta_0).$$

Let (ρ, u, θ) be an arbitrary strong solution to system (1.1)–(1.3) in $\mathbb{R}^3\times(0,T)$, subject to (1.6), and s the corresponding entropy given by (1.4).

Then, the following statements hold:

(i) The additional regularity $u\in L^\infty(0,T;L^2)$ holds, if $u_0\in L^2$.

(ii) The additional regularity $\left(\frac{u}{\sqrt{\rho}},\theta\right)\in L^\infty(0,T;L^2)$ holds, if $\left(\frac{u_0}{\sqrt{\rho_0}},\theta_0\right)\in L^2$.

(iii) Assume in addition that

$$|\Delta\rho_0(x)| \leq K_2\rho_0^2(x), \quad \forall x\in\mathbb{R}^3, \tag{H2}$$

for a positive constant K_2. Then, it holds that

$$\inf_{\mathbb{R}^3\times(0,T)} s(x,t) > -\infty, \qquad \text{as long as } \underline{s}_0 > -\infty.$$

(iv) Assume in addition that (H2) holds, then

$$\sup_{\mathbb{R}^3\times(0,T)} s(x,t) < +\infty,$$

as long as $\overline{s}_0 < +\infty$ and $\rho_0^{\frac{1-\gamma}{2}}u_0, \rho_0^{-\frac{\gamma}{2}}\nabla u_0, \rho_0^{1-\frac{\gamma}{2}}\theta_0, \rho_0^{1-\frac{\gamma}{2}}\nabla\theta_0, \rho_0^{-\frac{\gamma}{2}}\mathscr{S}_0 \in L^2$.

As a corollary of Theorems 1.1–1.3, we have the following:

Corollary 1.1. *Assume that the conditions on the initial data in Theorem 1.3 hold. Then, there is a positive time T depending only on the initial data but independent of K_1 and K_2 in (H1)–(H2), such that system (1.1)–(1.3), subject to (1.6), has a unique strong solution (ρ, u, θ) in $\mathbb{R}^3\times(0,T)$, and that the corresponding entropy given by (1.4) is uniformly bounded. If assume in addition that the conditions in Theorem*

1.2 hold, then the corresponding entropy of the solution in Theorem 1.2 is uniformly bounded up to any finite time.

Remark 1.1 (About the conditions (H1)–(H2))**.** *(i) Assumptions (H1)–(H2) mean essentially that ρ_0 decays at a rate no faster than $O(\frac{1}{|x|^2})$ at far fields. Indeed, for initial density of the form*

$$\rho_0(x) = \frac{K}{\langle x \rangle^\ell}, \quad K \in (0,\infty), \ell \in [0,\infty), \quad \text{where } \langle x \rangle = (1+|x|^2)^{\frac{1}{2}},$$

(H1)–(H2) hold if and only if $\ell \leq 2$. For this reason, (H1)–(H2) will be called slow decay assumptions. Such conditions have already been introduced and employed in [34, 35] to deal with the corresponding problem in one dimension.

(ii) Set

$$\rho_0 = \frac{K}{\langle x \rangle^\ell}, \quad u_0 \in C_c^\infty(\mathbb{R}^3), \quad \theta_0 = \frac{A}{R} e^{\frac{1}{c_v}} \rho_0^\gamma,$$

with

$$\gamma > \frac{5}{4}, \quad \max\left\{\frac{1}{2(\gamma-1)}, \frac{3}{2}\right\} < \ell \leq 2, \quad K \in (0,\infty).$$

Then, (1.7)–(1.8) and (H1)–(H2) hold. Therefore, the set of the initial data satisfying conditions in Theorems 1.1–1.3 is not empty.

Remark 1.2 (Propagation of the inhomogeneous regularities)**.** *Theorem 1.3 implies that the inhomogeneous regularity of the velocity can be propagated by the compressible Navier–Stokes equations in the presence of far field vacuum, as long as the initial density decays sufficiently slowly. If moreover $\frac{u_0}{\sqrt{\rho_0}} \in L^2$, then the propagation of the inhomogeneous regularity of the temperature holds also. Notably, this is in sharp contrast to the case with compactly supported initial density [27], where the inhomogeneous regularity can not be propagated even in a short time.*

Remark 1.3 (About the compatibility condition (1.8))**.** *(i) Assumption (1.8) is not required and condition (1.7) can be relaxed in the proof of (iii) of Theorem 1.3. Besides, some weaker regularities on the strong solutions than those stated in Definition 1.1 are sufficient to justify the arguments. In fact, strong solutions established in [31] can be chosen to guarantee (iii) of Theorem 1.3.*

(ii) The first condition in (1.8) is crucially used in the proof of (iv) of Theorem 1.3, see Proposition 2.5, but the second one in (1.8) is not used. However, since the second condition in (1.8) is assumed in the local well-posedness result, Theorem 1.1, we still put this assumption in Theorem 1.3. It is an interesting question to see if conclusion (iv) of Theorem 1.3 still holds without assuming (1.8).

The main tools of proving Theorem 1.3 are some singularly weighted energy estimates and an elaborate De Giorgi type iteration technique exploited in our previous works [34, 35], where the corresponding problems in one dimension were addressed. Although singular weights chosen in the multidimensional case are exactly the same as those in the one-dimensional case, in deriving the low order singular energy estimates; however, in the multi-dimensional case as considered in this paper, singularly weighted energy estimates for higher order derivatives are also required, which require elaborately chosen singular weights, see Proposition 2.4 and Proposition 2.5. Moreover, in the one-dimensional case, the singularly weighted energy estimates for u_x are carried out in the Lagrangian coordinates via the dynamical equation of the viscous flux G, as this equation can be clearly written out in the Lagrangian coordinates. However, in higher dimensions, estimates of ∇u require the control of both the effective viscous flux G and the vorticity $\nabla \times v$ that are strongly coupled, for which the Lagrangian formulation has no advantages. Thus, we will work directly with the momentum equations in the Eulerian coordinates and derive the singular energy estimates for both ∇u and the material derivative $\dot{u}$, where we used the idea of Hoff to apply the material derivative to the momentum to derive the governing system for $\dot{u}$, see Proposition 2.3 and Proposition 2.5.

The rest of this paper is arranged as follow: in Section 2 and Section 3, we derive important a priori singular energy estimates and carry out the De Giorgi type iterations with singular weights, respectively, which are used to prove Theorem 1.3 in Section 4, and the last section is an appendix on some elementary calculations.

2. SINGULARLY WEIGHTED A PRIORI ESTIMATES

This section is devoted to carrying out some a priori energy estimates with singular weights which will be used in the next section to carry out suitable De Giorgi iterations. Throughout this section, as well as the next one, (ρ, u, θ) is always assumed to be a strong solution to system (1.1)–(1.3) in $\mathbb{R}^3 \times (0, T)$, subject to (1.6), for a given positive time T. The initial density is assumed to satisfy (H1).

To simplify the notations, we set

$$\phi(t) := 1 + \|\sqrt{\rho_0} u\|_\infty^2(t) + \|\nabla u\|_\infty^2(t), \quad \Phi_T := \int_0^T \phi dt. \tag{2.1}$$

2.1. A transport estimate on ρ.

Proposition 2.1. *It holds that*

$$e^{-C_*\Phi_T}\rho_0(x) \le \rho(x,t) \le e^{C_*\Phi_T}\rho_0(x), \quad \forall (x,t) \in \mathbb{R}^3 \times (0,T),$$

for a positive constant C_* depending only on K_1.

Proof. Define $J = \frac{\rho}{\rho_0}$. Then, (1.1) implies that

$$\partial_t J + u \cdot \nabla J + \left(\operatorname{div} u + u \cdot \frac{\nabla \rho_0}{\rho_0} \right) J = 0.$$

Let $X(x,t)$ be the particle path starting from x, that is,

$$\begin{cases} \partial_t X(x,t) = u(X(x,t),t), \\ X(x,0) = x. \end{cases}$$

Then,

$$J(X(x,t),t) = e^{-\int_0^t \left(\operatorname{div} u + u\cdot\frac{\nabla\rho_0}{\rho_0}\right)(X(x,\tau),\tau)d\tau}.$$

Thanks to this and using the fact that the mapping $X(\cdot,t) : \mathbb{R} \to \mathbb{R}$ is bijective for any $t \in (0,T)$, one can easily obtain

$$e^{-\int_0^T \left(\|\operatorname{div} u\|_\infty + \left\|u\cdot\frac{\nabla\rho_0}{\rho_0}\right\|_\infty\right)dt} \le J \le e^{\int_0^T \left(\|\operatorname{div} u\|_\infty + \left\|u\cdot\frac{\nabla\rho_0}{\rho_0}\right\|_\infty\right)dt}, \quad \text{on } \mathbb{R}^3 \times (0,T),$$

which leads to the conclusion by the Cauchy inequality and assumption (H1). $\square$

2.2. Singularly weighted $L^\infty(L^2)$ estimates.

Proposition 2.2. *It holds for any $\alpha > 0$ that*

$$\sup_{0\le t\le T} \|(\rho_0^{\frac{1-\alpha}{2}} u, \rho_0^{1-\frac{\alpha}{2}}\theta)\|_2^2 + \int_0^T \|(\rho_0^{-\frac{\alpha}{2}}\nabla u, \rho_0^{\frac{1-\alpha}{2}}\nabla\theta)\|_2^2 dt \le C\|(\rho_0^{\frac{1-\alpha}{2}} u_0, \rho_0^{1-\frac{\alpha}{2}}\theta_0)\|_2^2$$

for a positive constant C depending only on $\alpha, c_v, \mu, \lambda, \kappa, K_1$, and Φ_T.

Proof. For any fixed $0 < \delta < 1$ set $\rho_{0\delta} = \rho_0 + \delta$. Choose a nonnegative cut off function $\chi \in C_0^\infty(B_2)$ satisfying $\chi \equiv 1$ on B_1 and set $\chi_R(x) = \phi(\frac{x}{R})$ for $R > 0$.

Multiplying (1.2) with $\rho_{0\delta}^{-\alpha} u\chi_R^2$ and integrating over $\mathbb{R}^3$ yield

$$\int [\rho(\partial_t u + (u\cdot\nabla)u) - (\mu\Delta u + (\mu+\lambda)\nabla\operatorname{div} u)]\cdot\rho_{0\delta}^{-\alpha} u\chi_R^2 dx = -\int \nabla p\cdot\rho_{0\delta}^{-\alpha} u\chi_R^2 dx. \quad (2.2)$$

(1.1) implies that

$$\begin{aligned}\int \rho(\partial_t u + (u\cdot\nabla)u)\cdot\rho_{0\delta}^{-\alpha} u\chi_R^2 dx &= \frac{1}{2}\int \rho\left(\partial_t|u|^2 + u\cdot\nabla|u|^2\right)\rho_{0\delta}^{-\alpha}\chi_R^2 dx \\ &= \frac{1}{2}\frac{d}{dt}\int \rho|u|^2\rho_{0\delta}^{-\alpha}\chi_R^2 dx - \frac{1}{2}\int \rho u\cdot\nabla\left(\rho_{0\delta}^{-\alpha}\chi_R^2\right)|u|^2 dx. \quad (2.3)\end{aligned}$$

By (H1), it holds that

$$\begin{aligned}\left|\nabla\left(\rho_{0\delta}^{-\alpha}\chi_R^2\right)\right| &= \left|-\alpha\rho_{0\delta}^{-(\alpha+1)}\nabla\rho_{0\delta}\chi_R^2 + 2\rho_{0\delta}^{-\alpha}\chi_R\nabla\chi_R\right| \\ &\le C\left(\rho_{0\delta}^{-(\alpha+1)}\rho_0^{\frac{3}{2}}\chi_R^2 + (R\delta^\alpha)^{-1}\chi_R 1_{\mathcal{C}_R}\right) \\ &\le C\left(\rho_{0\delta}^{-\alpha}\sqrt{\rho_0}\chi_R^2 + (R\delta^\alpha)^{-1}\chi_R 1_{\mathcal{C}_R}\right), \quad (2.4)\end{aligned}$$

where $1_{\mathcal{C}_R}$ is the characteristic function of the set $\mathcal{C}_R := B_{2R}\setminus B_R$. This and the Sobolev embedding inequality yields that

$$\begin{aligned}\left|\int \rho u\cdot\nabla\left(\rho_{0\delta}^{-\alpha}\chi_R^2\right)|u|^2 dx\right| &\le C\int \rho|u|^3\left(\rho_{0\delta}^{-\alpha}\sqrt{\rho_0}\chi_R^2 + (R\delta^\alpha)^{-1}\chi_R\right)dx \\ &\le C\|\sqrt{\rho_0}u\|_\infty\|\sqrt{\rho}\rho_{0\delta}^{-\frac{\alpha}{2}} u\chi_R\|_2^2 + C(R\delta^\alpha)^{-1}\|u\|_\infty\|\sqrt{\rho}u\|_2^2 \\ &\le C\phi(t)\|\sqrt{\rho}\rho_{0\delta}^{-\frac{\alpha}{2}} u\chi_R\|_2^2 + C(R\delta^\alpha)^{-1}\|u\|_\infty\|\sqrt{\rho}u\|_2^2.\end{aligned}$$

Substituting this into (2.3) yields

$$\begin{aligned}&\int\rho(\partial_tu+(u\cdot\nabla)u)\cdot\rho_{0\delta}^{-\alpha}u\chi_R^2dx\\ \geq\ &\frac12\frac{d}{dt}\|\sqrt{\rho}\rho_{0\delta}^{-\frac{\alpha}{2}}u\chi_R\|_2^2-C\phi(t)\|\sqrt{\rho}\rho_{0\delta}^{-\frac{\alpha}{2}}u\chi_R\|_2^2-C(R\delta^\alpha)^{-1}\|u\|_\infty\|\sqrt{\rho}u\|_2^2.\end{aligned}\tag{2.5}$$

It follows from integrating by parts that

$$-\int\Delta u\cdot\rho_{0\delta}^{-\alpha}u\chi_R^2dx=\int|\nabla u|^2\rho_{0\delta}^{-\alpha}\chi_R^2dx+\int\partial_iu\cdot u\partial_i\left(\rho_{0\delta}^{-\alpha}\chi_R^2\right)dx.\tag{2.6}$$

It follows from (2.4), the Hölder, Sobolev, and Cauchy inequalities that

$$\begin{aligned}&\left|\int\partial_iu\cdot u\partial_i\left(\rho_{0\delta}^{-\alpha}\chi_R^2\right)dx\right|\\ \leq\ &C\int|u||\nabla u|\left(\rho_{0\delta}^{-\alpha}\sqrt{\rho_0}\chi_R^2+(R\delta^\alpha)^{-1}\chi_R1_{\mathcal{C}_R}\right)dx\\ \leq\ &C\|\rho_{0\delta}^{-\frac{\alpha}{2}}\nabla u\chi_R\|_2\|\sqrt{\rho_0}\rho_{0\delta}^{-\frac{\alpha}{2}}u\chi_R\|_2+C(R\delta^\alpha)^{-1}\|\nabla u\|_{L^2(\mathcal{C}_R)}\|u\|_6\|\chi_R\|_3\\ \leq\ &\frac14\|\rho_{0\delta}^{-\frac{\alpha}{2}}\nabla u\chi_R\|_2^2+C\|\sqrt{\rho_0}\rho_{0\delta}^{-\frac{\alpha}{2}}u\chi_R\|_2^2+C\delta^{-\alpha}\|\nabla u\|_{L^2(\mathcal{C}_R)}\|\nabla u\|_2,\end{aligned}$$

where $\|\chi_R\|_3\leq CR$ was used. Substituting this into (2.6) yields

$$\begin{aligned}-\int\Delta u\cdot\rho_{0\delta}^{-\alpha}u\chi_R^2dx\ \geq\ &\frac34\|\rho_{0\delta}^{-\frac{\alpha}{2}}\nabla u\chi_R\|_2^2-C\|\sqrt{\rho_0}\rho_{0\delta}^{-\frac{\alpha}{2}}u\chi_R\|_2^2\\ &-C\delta^{-\alpha}\|\nabla u\|_{L^2(\mathcal{C}_R)}\|\nabla u\|_2.\end{aligned}\tag{2.7}$$

Similarly, one has

$$\begin{aligned}-\int\nabla\mathrm{div}\,u\cdot\rho_{0\delta}^{-\alpha}u\chi_R^2dx\ \geq\ &\frac12\|\rho_{0\delta}^{-\frac{\alpha}{2}}\mathrm{div}\,u\chi_R\|_2^2-C\|\sqrt{\rho_0}\rho_{0\delta}^{-\frac{\alpha}{2}}u\chi_R\|_2^2\\ &-C\delta^{-\alpha}\|\nabla u\|_{L^2(\mathcal{C}_R)}\|\nabla u\|_2.\end{aligned}\tag{2.8}$$

Thanks to (2.4), it follows from integrating by parts and the Cauchy inequality that

$$\begin{aligned}-\int\nabla p\cdot\rho_{0\delta}^{-\alpha}u\chi_R^2dx\ =\ &\int p\Big[u\cdot\nabla\left(\rho_{0\delta}^{-\alpha}\chi_R^2\right)+\mathrm{div}\,u\rho_{0\delta}^{-\alpha}\chi_R^2\Big]dx\\ \leq\ &C\int\rho\theta\left[|\mathrm{div}u|\rho_{0\delta}^{-\alpha}\chi_R^2+|u|\left(\rho_{0\delta}^{-\alpha}\sqrt{\rho_0}\chi_R^2+(R\delta^\alpha)^{-1}\chi_R\right)\right]dx\\ \leq\ &\frac{\mu+\lambda}{2}\|\mathrm{div}u\rho_{0\delta}^{-\frac{\alpha}{2}}\chi_R\|_2^2+C\left(\|\rho\rho_{0\delta}^{-\frac{\alpha}{2}}\theta\chi_R\|_2^2+\|\sqrt{\rho}\rho_{0\delta}^{-\frac{\alpha}{2}}u\chi_R\|_2^2\right)\\ &+C(R\delta^\alpha)^{-1}(\|\sqrt{\rho}u\|_2^2+\|\sqrt{\rho}\theta\|_2^2).\end{aligned}$$

Substituting (2.5), (2.7)–(2.8), and the above inequality into (2.2), and by Proposition 2.1, one gets

$$\frac{d}{dt}\|\sqrt{\rho}\rho_{0\delta}^{-\frac{\alpha}{2}}u\chi_R\|_2^2+1.5\mu\|\rho_{0\delta}^{-\frac{\alpha}{2}}\nabla u\chi_R\|_2^2$$

$$\begin{aligned}&\le C\phi(t)(\|\rho\rho_{0\delta}^{-\frac{\alpha}{2}}\theta\chi_R\|_2^2+\|\sqrt{\rho}\rho_{0\delta}^{-\frac{\alpha}{2}}u\chi_R\|_2^2)+C\delta^{-\alpha}\|\nabla u\|_{L^2(\mathcal{C}_R)}\|\nabla u\|_2\\&\quad +C(R\delta^\alpha)^{-1}(\|\sqrt{\rho}u\|_2^2+\|\sqrt{\rho}\theta\|_2^2+\|u\|_\infty\|\sqrt{\rho}u\|_2^2).\end{aligned}\tag{2.9}$$

Multiplying (1.3) with $\rho_0\rho_{0\delta}^{-\alpha}\theta\chi_R^2$ and integrating over $\mathbb{R}^3$ yield

$$\int[c_v\rho(\partial_t\theta+u\cdot\nabla\theta)-\kappa\Delta\theta]\rho_0\rho_{0\delta}^{-\alpha}\theta\chi_R^2dx=\int(\mathcal{Q}(\nabla u)-p\mathrm{div}\,u)\rho_0\rho_{0\delta}^{-\alpha}\theta\chi_R^2dx.\tag{2.10}$$

By (1.1), one deduces

$$\begin{aligned}\int\rho(\partial_t\theta+u\cdot\nabla\theta)\rho_0\rho_{0\delta}^{-\alpha}\theta\chi_R^2dx&=\frac12\int\rho(\partial_t\theta^2+u\cdot\nabla\theta^2)\rho_0\rho_{0\delta}^{-\alpha}\chi_R^2dx\\&=\frac12\frac{d}{dt}\|\sqrt{\rho\rho_0}\rho_{0\delta}^{-\frac{\alpha}{2}}\theta\chi_R\|_2^2-\frac12\int\rho u\theta^2\nabla(\rho_0\rho_{0\delta}^{-\alpha}\chi_R^2)dx.\end{aligned}\tag{2.11}$$

It follows from direct calculations and (H1) that

$$\begin{aligned}|\nabla(\rho_0\rho_{0\delta}^{-\alpha}\chi_R^2)|&=|\nabla\rho_0\rho_{0\delta}^{-\alpha}\chi_R^2-\alpha\rho_0\rho_{0\delta}^{-(\alpha+1)}\nabla\rho_0\chi_R^2+2\rho_0\rho_{0\delta}^{-\alpha}\chi_R\nabla\chi_R|\\&\le C(\rho_0^{\frac32}\rho_{0\delta}^{-\alpha}\chi_R^2+\rho_0\rho_{0\delta}^{-\alpha}\chi_R|\nabla\chi_R|).\end{aligned}\tag{2.12}$$

Therefore,

$$\begin{aligned}&\left|\int\rho u\theta^2\nabla(\rho_0\rho_{0\delta}^{-\alpha}\chi_R^2)dx\right|\\&\le C\int\rho|u|\theta^2(\rho_0^{\frac32}\rho_{0\delta}^{-\alpha}\chi_R^2+\rho_0\rho_{0\delta}^{-\alpha}\chi_R|\nabla\chi_R|)dx\\&\le C\|\sqrt{\rho_0}u\|_\infty\|\sqrt{\rho\rho_0}\rho_{0\delta}^{-\frac{\alpha}{2}}\theta\chi_R\|_2^2+C(R\delta^\alpha)^{-1}\|\rho_0u\|_\infty\|\sqrt{\rho}\theta\|_2^2\\&\le C\phi(t)\|\sqrt{\rho\rho_0}\rho_{0\delta}^{-\frac{\alpha}{2}}\theta\chi_R\|_2^2+C(R\delta^\alpha)^{-1}\phi(t)\|\sqrt{\rho}\theta\|_2^2.\end{aligned}$$

Substituting the above inequality into (2.11) gives

$$\begin{aligned}\int\rho(\partial_t\theta+u\cdot\nabla\theta)\rho_0\rho_{0\delta}^{-\alpha}\theta\chi_R^2dx&\ge\frac12\frac{d}{dt}\|\sqrt{\rho\rho_0}\rho_{0\delta}^{-\frac{\alpha}{2}}\theta\chi_R\|_2^2-C\phi(t)\|\sqrt{\rho\rho_0}\rho_{0\delta}^{-\frac{\alpha}{2}}\theta\chi_R\|_2^2\\&\quad -C(R\delta^\alpha)^{-1}\phi(t)\|\sqrt{\rho}\theta\|_2^2.\end{aligned}\tag{2.13}$$

Integration by parts yields

$$-\int\Delta\theta\rho_0\rho_{0\delta}^{-\alpha}\theta\chi_R^2dx=\int|\nabla\theta|^2\rho_0\rho_{0\delta}^{-\alpha}\chi_R^2dx+\int\theta\nabla\theta\cdot\nabla(\rho_0\rho_{0\delta}^{-\alpha}\chi_R^2)dx.\tag{2.14}$$

It follows from (2.12) and the Cauchy inequality that

$$\begin{aligned}&\left|\int\theta\nabla\theta\cdot\nabla(\rho_0\rho_{0\delta}^{-\alpha}\chi_R^2)dx\right|\\&\le C\int\theta|\nabla\theta|(\rho_0^{\frac32}\rho_{0\delta}^{-\alpha}\chi_R^2+\rho_0\rho_{0\delta}^{-\alpha}\chi_R|\nabla\chi_R|)dx\\&\le\frac12\|\sqrt{\rho}_0\rho_{0\delta}^{-\frac{\alpha}{2}}\nabla\theta\chi_R\|_2^2+C\|\rho_0\rho_{0\delta}^{-\frac{\alpha}{2}}\theta\chi_R\|_2^2+C\|\sqrt{\rho_0}\rho_{0\delta}^{-\frac{\alpha}{2}}\theta\nabla\chi_R\|_2^2\end{aligned}$$

$$\leq \frac{1}{2}\|\sqrt{\rho}_0\rho_{0\delta}^{-\frac{\alpha}{2}}\nabla\theta\chi_R\|_2^2+C\|\rho_0\rho_{0\delta}^{-\frac{\alpha}{2}}\theta\chi_R\|_2^2+C(R^2\delta^\alpha)^{-1}\|\sqrt{\rho_0}\theta\|_2^2,$$

which and (2.14) lead to

$$-\int\Delta\theta\rho_0\rho_{0\delta}^{-\alpha}\theta\chi_R^2dx\geq\frac{1}{2}\|\sqrt{\rho}_0\rho_{0\delta}^{-\frac{\alpha}{2}}\nabla\theta\chi_R\|_2^2-C\|\rho_0\rho_{0\delta}^{-\frac{\alpha}{2}}\theta\chi_R\|_2^2-C(R^2\delta^\alpha)^{-1}\|\sqrt{\rho_0}\theta\|_2^2. \tag{2.15}$$

Note that the Cauchy inequality yields that

$$\begin{aligned}
&\int\mathcal{Q}(\nabla u)\rho_0\rho_{0\delta}^{-\alpha}\theta\chi_R^2dx\\
\leq\ & C\int|\nabla u|^2\rho_0\rho_{0\delta}^{-\alpha}\theta\chi_R^2dx\leq C\|\nabla u\|_\infty\|\rho_{0\delta}^{-\frac{\alpha}{2}}\nabla u\chi_R\|_2\|\rho_0\rho_{0\delta}^{-\frac{\alpha}{2}}\theta\chi_R\|_2\\
\leq\ & \frac{\mu}{2}\|\rho_{0\delta}^{-\frac{\alpha}{2}}\nabla u\chi_R\|_2^2+C\phi(t)\|\rho_0\rho_{0\delta}^{-\frac{\alpha}{2}}\theta\chi_R\|_2^2
\end{aligned} \tag{2.16}$$

and

$$\begin{aligned}
&-\int p\,\mathrm{div}\,u\rho_0\rho_{0\delta}^{-\alpha}\theta\chi_R^2dx\leq C\int\rho\theta|\mathrm{div}u|\rho_0\rho_{0\delta}^{-\alpha}\theta\chi_R^2dx\\
\leq\ & C\|\nabla u\|_\infty\|\sqrt{\rho\rho_0}\rho_{0\delta}^{-\frac{\alpha}{2}}\theta\chi_R\|_2^2\leq C\phi(t)\|\sqrt{\rho\rho_0}\rho_{0\delta}^{-\frac{\alpha}{2}}\theta\chi_R\|_2^2.
\end{aligned} \tag{2.17}$$

Substituting (2.13) and (2.15)–(2.17) into (2.10) and by Proposition 2.1, one gets that

$$\begin{aligned}
&c_v\frac{d}{dt}\|\sqrt{\rho\rho_0}\rho_{0\delta}^{-\frac{\alpha}{2}}\theta\chi_R\|_2^2+\kappa\|\sqrt{\rho_0}\rho_{0\delta}^{-\frac{\alpha}{2}}\nabla\theta\chi_R\|_2^2\\
\leq\ & \frac{\mu}{2}\|\rho_{0\delta}^{-\frac{\alpha}{2}}\nabla u\chi_R\|_2^2+C\phi(t)\|\sqrt{\rho\rho_0}\rho_{0\delta}^{-\frac{\alpha}{2}}\theta\chi_R\|_2^2\\
&+C[(R\delta^\alpha)^{-1}\phi(t)+(R^2\delta^\alpha)^{-1}]\|\sqrt{\rho_0}\theta\|_2^2.
\end{aligned} \tag{2.18}$$

Summing (2.9) with (2.18), it follows from Proposition 2.1 that

$$\begin{aligned}
&\mu\|\rho_{0\delta}^{-\frac{\alpha}{2}}\nabla u\chi_R\|_2^2+\kappa\|\sqrt{\rho_0}\rho_{0\delta}^{-\frac{\alpha}{2}}\nabla\theta\chi_R\|_2^2\\
&+\frac{d}{dt}(\|\sqrt{\rho}\rho_{0\delta}^{-\frac{\alpha}{2}}u\chi_R\|_2^2+c_v\|\sqrt{\rho\rho_0}\rho_{0\delta}^{-\frac{\alpha}{2}}\theta\chi_R\|_2^2)\\
\leq\ & C\phi(t)(\|\sqrt{\rho}\rho_{0\delta}^{-\frac{\alpha}{2}}u\chi_R\|_2^2+c_v\|\sqrt{\rho\rho_0}\rho_{0\delta}^{-\frac{\alpha}{2}}\theta\chi_R\|_2^2)+C\delta^{-\alpha}\|\nabla u\|_{L^2(\mathcal{C}_R)}\|\nabla u\|_2\\
&+C(R\delta^\alpha)^{-1}(\|\sqrt{\rho}u\|_2^2+\|u\|_\infty\|\sqrt{\rho}u\|_2^2)\\
&+C[(R\delta^\alpha)^{-1}\phi(t)+(R^2\delta^\alpha)^{-1}]\|\sqrt{\rho_0}\theta\|_2^2.
\end{aligned}$$

Applying the the Grönwall inequality to the above and noticing that

$$\int_0^T\|\nabla u\|_{L^2(\mathcal{C}_R)}\|\nabla u\|_2dt\to0,\quad\text{as }R\to\infty,$$
$$(1+\|u\|_\infty)\|\sqrt{\rho}u\|_2^2+\phi(t)\|\sqrt{\rho}\theta\|_2^2)\in L^1((0,T)),$$

guaranteed by the regularities of (ρ, u, θ), one gets by taking $R \to \infty$ and by Proposition 2.1 that

$$(\|\sqrt{\rho_0}\rho_{0\delta}^{-\frac{\alpha}{2}}u\|_2^2 + c_v\|\rho_0\rho_{0\delta}^{-\frac{\alpha}{2}}\theta\|_2^2)(t) + \int_0^t (\|\rho_{0\delta}^{-\frac{\alpha}{2}}\nabla u\|_2^2 + \|\sqrt{\rho_0}\rho_{0\delta}^{-\frac{\alpha}{2}}\nabla\theta\|_2^2)ds$$
$$\leq C\|(\rho_0^{\frac{1-\alpha}{2}}u_0, \rho_0^{1-\frac{\alpha}{2}}\theta_0)\|_2^2.$$

Taking $\delta \downarrow 0$ to the above inequality and by the monotone convergence theorem, the conclusion follows. □

Remark 2.1. *The basic idea of proving Proposition 2.2 is to choose $\rho_0^{-\alpha}u$ and $\rho_0^{1-\alpha}\theta$ as testing functions to (1.2) and (1.3), respectively, in the fashion of integrating by parts formally on the whole space. This argument is justified rigorously by choosing $\rho_{0\delta}^{-\alpha}u\chi_R^2$ and $\rho_0\rho_{0\delta}^{-\alpha}\theta\chi_R^2$ as testing functions to (1.2) and (1.3), respectively, and taking the limits as $R \to \infty$ and $\delta \to 0$, successively, as presented in the proof of Proposition 2.2. For simplicity of presentation, the energy estimates in the remaining part of this paper are carried out in a brief way, that is, we test the relevant equations by some singularly weighted functions directly; however, as already indicated in the proof of Proposition 2.2, due to the assumptions (H1) and (H2), the arguments can be justified rigorously by adopting similar cutoff, approximations, and taking the limits as $R \to \infty$ and $\delta \to 0$.*

2.3. Singularly weighted $L^\infty(H^1)$ estimates.

Proposition 2.3. *It holds that*

$$\sup_{0\leq t\leq T}\|\rho_0^{-\frac{\gamma}{2}}\nabla u\|_2^2 + \int_0^T \|\rho_0^{\frac{1-\gamma}{2}}\partial_t u\|_2^2 dt \leq C\|(\rho_0^{\frac{1-\gamma}{2}}u_0, \rho_0^{1-\frac{\gamma}{2}}\theta_0, \rho_0^{-\frac{\gamma}{2}}\nabla u_0)\|_2^2$$

for a positive constant C depending only on $c_v, \mu, \lambda, \kappa, K_1$, and Φ_T.

Proof. Multiplying (1.2) with $\rho_0^{-\gamma}\partial_t u$ and integrating over $\mathbb{R}^3$ yield

$$-\int(\mu\Delta u + (\mu+\lambda)\nabla\mathrm{div}\, u)\cdot\rho_0^{-\gamma}\partial_t u dx + \|\sqrt{\rho}\rho_0^{-\frac{\gamma}{2}}\partial_t u\|_2^2$$
$$= -\int\rho(u\cdot\nabla)u\cdot\rho_0^{-\gamma}\partial_t u dx - \int\nabla p\rho_0^{-\gamma}\partial_t u dx. \tag{2.19}$$

It follows from integrating by parts, the Cauchy inequality, and Proposition 2.1 that

$$\begin{aligned}-\int\Delta u\rho_0^{-\gamma}\partial_t u dx &= \frac{1}{2}\frac{d}{dt}\|\rho_0^{-\frac{\gamma}{2}}\nabla u\|_2^2 + \int\nabla u : \partial_t u\otimes\nabla\rho_0^{-\gamma}dx\\ &\geq \frac{1}{2}\frac{d}{dt}\|\rho_0^{-\frac{\gamma}{2}}\nabla u\|_2^2 - \frac{1}{8\mu}\|\sqrt{\rho}\rho_0^{-\frac{\gamma}{2}}\partial_t u\|_2^2 - C\|\rho_0^{-\frac{\gamma}{2}}\nabla u\|_2^2,\end{aligned} \tag{2.20}$$

and, similarly,

$$-\int\nabla\mathrm{div} u\rho_0^{-\gamma}\partial_t u dx \geq \frac{1}{2}\frac{d}{dt}\|\rho_0^{-\frac{\gamma}{2}}\mathrm{div}\, u\|_2^2 - \frac{\|\sqrt{\rho}\rho_0^{-\frac{\gamma}{2}}\partial_t u\|_2^2}{8(\mu+\lambda)} - C\|\rho_0^{-\frac{\gamma}{2}}\nabla u\|_2^2. \tag{2.21}$$

The Cauchy inequality and Proposition 2.1 yield

$$-\int \rho(u\cdot\nabla)u\cdot\rho_0^{-\gamma}\partial_t u dx \leq \frac{1}{8}\|\sqrt{\rho}\rho_0^{-\frac{\gamma}{2}}\partial_t u\|_2^2 + C\|\sqrt{\rho_0}u\|_\infty^2\|\rho_0^{-\frac{\gamma}{2}}\nabla u\|_2^2. \tag{2.22}$$

Substituting (2.20)–(2.22) into (2.19) leads to

$$\begin{aligned}&\frac{d}{dt}(\mu\|\rho_0^{-\frac{\gamma}{2}}\nabla u\|_2^2 + (\mu+\lambda)\|\rho_0^{-\frac{\gamma}{2}}\mathrm{div}u\|_2^2) + \frac{5}{4}\|\sqrt{\rho}\rho_0^{-\frac{\gamma}{2}}\partial_t u\|_2^2\\ &\qquad \leq C\phi(t)\|\rho_0^{-\frac{\gamma}{2}}\nabla u\|_2^2 - 2\int \rho_0^{-\gamma}\nabla p\cdot\partial_t u dx.\end{aligned} \tag{2.23}$$

Let G be the effective viscous flux, i.e.,

$$G := (2\mu+\lambda)\mathrm{div}u - p. \tag{2.24}$$

By the Cauchy inequality and Proposition 2.1, it follows

$$\begin{aligned}-\int \rho_0^{-\gamma}\nabla p\cdot\partial_t u dx &= \int p(\nabla\rho_0^{-\gamma}\cdot\partial_t u + \rho_0^{-\gamma}\mathrm{div}\partial_t u)dx\\ &= \frac{d}{dt}\int \rho_0^{-\gamma}\mathrm{div}u p dx + \int (p\nabla\rho_0^{-\gamma}\cdot\partial_t u - \rho_0^{-\gamma}\mathrm{div}u\partial_t p)dx\\ &\leq \frac{d}{dt}\int \rho_0^{-\gamma}\mathrm{div}u p dx - \int \rho_0^{-\gamma}\mathrm{div}u\partial_t p dx\\ &\quad +\eta\|\sqrt{\rho}\rho_0^{-\frac{\gamma}{2}}\partial_t u\|_2^2 + C_\eta\|\rho_0^{1-\frac{\gamma}{2}}\theta\|_2^2\end{aligned} \tag{2.25}$$

for any $\eta > 0$. Note that

$$-\int \rho_0^{-\gamma}\mathrm{div}u\partial_t p dx = -\frac{1}{2(2\mu+\lambda)}\frac{d}{dt}\|\rho_0^{-\frac{\gamma}{2}}p\|_2^2 - \frac{1}{2\mu+\lambda}\int \rho_0^{-\gamma}G\partial_t p dx. \tag{2.26}$$

It follows from (1.1), (1.3), and the equation of state that

$$-\partial_t p = \mathrm{div}(up - \kappa(\gamma-1)\nabla\theta) + (\gamma-1)(\mathrm{div}u p - \mathcal{Q}(\nabla u)).$$

Thanks to this and using Proposition 2.1, one deduces

$$\begin{aligned}-\int \rho_0^{-\gamma}G\partial_t p dx &= \int \rho_0^{-\gamma}G\mathrm{div}(up - \kappa(\gamma-1)\nabla\theta)dx\\ &\quad +(\gamma-1)\int \rho_0^{-\gamma}G(\mathrm{div}u p - \mathcal{Q}(\nabla u))dx\\ &= \int [(\gamma-1)\kappa\nabla\theta - up]\cdot(\rho_0^{-\gamma}\nabla G + \nabla\rho_0^{-\gamma}G)dx\\ &\quad +(\gamma-1)\int \rho_0^{-\gamma}G(\mathrm{div}u p - \mathcal{Q}(\nabla u))dx\\ &\leq C\int (|\nabla\theta| + \rho_0\theta|u|)[\rho_0^{-\gamma}|\nabla G| + \rho_0^{\frac{1}{2}-\gamma}(|\nabla u| + \rho_0\theta)]dx\end{aligned}$$

$$
\begin{aligned}
&+C\int\rho_0^{-\gamma}(|\nabla u|+\rho_0\theta)^2|\nabla u|dx\\
\le\;& \eta\|\rho_0^{-\frac{\gamma+1}{2}}\nabla G\|_2^2+C_\eta\|\rho_0^{\frac{1-\gamma}{2}}\nabla\theta\|_2^2\\
&+C\phi(t)(\|\rho_0^{1-\frac{\gamma}{2}}\theta\|_2^2+\|\rho_0^{-\frac{\gamma}{2}}\nabla u\|_2^2) \qquad (2.27)
\end{aligned}
$$

for any positive η, where $|G|\le C(|\nabla u|+\rho_0\theta)$ has been used.

It follows from (2.23)–(2.27) that

$$
\begin{aligned}
&\frac{d}{dt}\Big(\mu\|\rho_0^{-\frac{\gamma}{2}}\nabla u\|_2^2+(\mu+\lambda)\|\rho_0^{-\frac{\gamma}{2}}\mathrm{div}u\|_2^2+\frac{1}{2\mu+\lambda}\|\rho_0^{-\frac{\gamma}{2}}p\|_2^2\Big)\\
&-2\frac{d}{dt}\int\rho_0^{-\gamma}\mathrm{div}u p dx+\frac{3}{2}\|\sqrt{\rho}\rho_0^{-\frac{\gamma}{2}}\partial_t u\|_2^2\\
\le\;& \eta\|\rho_0^{-\frac{\gamma+1}{2}}\nabla G\|_2^2+C_\eta\|\rho_0^{\frac{1-\gamma}{2}}\nabla\theta\|_2^2+C\phi(t)\|(\rho_0^{1-\frac{\gamma}{2}}\theta,\rho_0^{-\frac{\gamma}{2}}\nabla u)\|_2^2 \qquad (2.28)
\end{aligned}
$$

for any positive η.

It remains to estimate $\|\rho_0^{-\frac{\gamma+1}{2}}\nabla G\|_2^2$. Denote $\dot u=\partial_t u+(u\cdot\nabla)u$. Then (1.2) yields that

$$\Delta G=\mathrm{div}\,(\rho\dot u).$$

Multiplying the above equation by $\rho_0^{-(\gamma+1)}G$ and integrating over $\mathbb{R}^3$ yield

$$-\int\Delta G\rho_0^{-(\gamma+1)}Gdx=-\int\mathrm{div}(\rho\dot u)\rho_0^{-(\gamma+1)}Gdx. \qquad (2.29)$$

Integrating by parts and using (H1) and the Cauchy inequality lead to

$$-\int\Delta G\rho_0^{-(\gamma+1)}Gdx\ge\frac{3}{4}\|\rho_0^{-\frac{\gamma+1}{2}}\nabla G\|_2^2-C\|\rho_0^{-\frac{\gamma}{2}}G\|_2^2. \qquad (2.30)$$

Similarly, one has by Proposition 2.1 that

$$
\begin{aligned}
&-\int\mathrm{div}(\rho\dot u)\rho_0^{-(\gamma+1)}Gdx\\
=\;&\int\rho\dot u\cdot(\rho_0^{-(\gamma+1)}\nabla G+\nabla\rho_0^{-(\gamma+1)}G)dx\\
\le\;& C(\|\sqrt{\rho}\rho_0^{-\frac{\gamma}{2}}\partial_t u\|_2+\|\sqrt{\rho_0}u\|_\infty\|\rho_0^{-\frac{\gamma}{2}}\nabla u\|_2)(\|\rho_0^{-\frac{\gamma+1}{2}}\nabla G\|_2+\|\rho_0^{-\frac{\gamma}{2}}G\|_2)\\
\le\;& C(1+\|\sqrt{\rho_0}u\|_\infty^2)(\|\rho_0^{-\frac{\gamma}{2}}\nabla u\|_2^2+\|\rho_0^{1-\frac{\gamma}{2}}\theta\|_2^2)\\
&+\frac{1}{4}\|\rho_0^{-\frac{\gamma+1}{2}}\nabla G\|_2^2+C\|\sqrt{\rho}\rho_0^{-\frac{\gamma}{2}}\partial_t u\|_2^2.
\end{aligned}
$$

This, together with (2.29) and (2.30), leads to

$$\|\rho_0^{-\frac{\gamma+1}{2}}\nabla G\|_2^2\le C\|\sqrt{\rho}\rho_0^{-\frac{\gamma}{2}}\partial_t u\|_2^2+C\phi(t)(\|\rho_0^{-\frac{\gamma}{2}}\nabla u\|_2^2+\|\rho_0^{1-\frac{\gamma}{2}}\theta\|_2^2). \qquad (2.31)$$

Then, (2.28) and (2.31) yield

$$\frac{d}{dt}\left(\mu\|\rho_0^{-\frac{\gamma}{2}}\nabla u\|_2^2+(\mu+\lambda)\|\rho_0^{-\frac{\gamma}{2}}\mathrm{div}u\|_2^2+\frac{1}{2\mu+\lambda}\|\rho_0^{-\frac{\gamma}{2}}p\|_2^2\right)+\|\sqrt{\rho}\rho_0^{-\frac{\gamma}{2}}\partial_t u\|_2^2$$
$$\leq \quad 2\frac{d}{dt}\int\rho_0^{-\gamma}\mathrm{div}up dx+C\|\rho_0^{\frac{1-\gamma}{2}}\nabla\theta\|_2^2+C\phi(t)(\|\rho_0^{1-\frac{\gamma}{2}}\theta\|_2^2+\|\rho_0^{-\frac{\gamma}{2}}\nabla u\|_2^2).$$

Note that Proposition 2.2 together with Proposition 2.1 yield

$$\sup_{0\leq t\leq T}\left\|\rho_0^{-\frac{\gamma}{2}}p\right\|_2^2\leq C\|(\rho_0^{\frac{1-\gamma}{2}}u_0,\rho_0^{1-\frac{\gamma}{2}}\theta_0)\|_2^2.$$

Thanks to the above two and using the Grönwall inequality, the conclusion follows by applying Propositions 2.1 and 2.2 and the Cauchy inequality. □

Proposition 2.4. *It holds that*

$$\sup_{0\leq t\leq T}\|\rho_0^{1-\frac{\gamma}{2}}\nabla\theta\|_2^2+\int_0^T\|\rho_0^{\frac{3-\gamma}{2}}\partial_t\theta\|_2^2\leq C\|(\rho_0^{\frac{1-\gamma}{2}}u_0,\rho_0^{1-\frac{\gamma}{2}}\theta_0,\rho_0^{-\frac{\gamma}{2}}\nabla u_0,\rho_0^{1-\frac{\gamma}{2}}\nabla\theta_0)\|_2^2$$

for a positive constant C depending only on $c_v,\mu,\lambda,\kappa,K_1,$ and Φ_T.

Proof. Multiplying (1.3) with $\rho_0^{2-\gamma}\partial_t\theta$ and integrating the resultant over $\mathbb{R}^3$, one can get from the Cauchy inequality and Proposition 2.1 that

$$-\kappa\int\Delta\theta\rho_0^{2-\gamma}\partial_t\theta dx+c_v\|\sqrt{\rho}\rho_0^{1-\frac{\gamma}{2}}\partial_t\theta\|_2^2$$
$$=\quad\int(\mathcal{Q}(\nabla u)-p\mathrm{div}u)\rho_0^{2-\gamma}\partial_t\theta dx-c_v\int\rho u\cdot\nabla\theta\rho_0^{2-\gamma}\partial_t\theta dx$$
$$\leq\quad\frac{c_v}{4}\|\sqrt{\rho}\rho_0^{1-\frac{\gamma}{2}}\partial_t\theta\|_2^2+C\phi(t)\|(\rho_0^{\frac{1-\gamma}{2}}\nabla u,\rho_0^{\frac{3-\gamma}{2}}\theta,\rho_0^{1-\frac{\gamma}{2}}\nabla\theta)\|_2^2.$$

Similar to (2.20), one has

$$-\kappa\int\Delta\theta\rho_0^{2-\gamma}\partial_t\theta dx\geq\frac{\kappa}{2}\frac{d}{dt}\|\rho_0^{1-\frac{\gamma}{2}}\nabla\theta\|_2^2-\frac{c_v}{4}\|\sqrt{\rho}\rho_0^{1-\frac{\gamma}{2}}\partial_t\theta\|_2^2-C\|\rho_0^{1-\frac{\gamma}{2}}\nabla\theta\|_2^2.$$

Therefore,

$$\kappa\frac{d}{dt}\|\rho_0^{1-\frac{\gamma}{2}}\nabla\theta\|_2^2+c_v\|\sqrt{\rho}\rho_0^{1-\frac{\gamma}{2}}\partial_t\theta\|_2^2\leq C\phi(t)\|(\rho_0^{\frac{1-\gamma}{2}}\nabla u,\rho_0^{\frac{3-\gamma}{2}}\theta,\rho_0^{1-\frac{\gamma}{2}}\nabla\theta)\|_2^2,$$

from which, by the Grönwall inequality and Propositions 2.1, 2.2, and 2.3, the conclusion follows. □

2.4. **Singularly weighted $L^\infty(L^2)$ estimate for $\dot{u}$.**

Proposition 2.5. *Recall $\dot{u}=\partial_t u+(u\cdot\nabla)u$. Then, it holds that*

$$\sup_{0\leq t\leq T}\|\rho_0^{1-\frac{\gamma}{2}}\dot{u}\|_2^2+\int_0^T\|\rho_0^{\frac{1-\gamma}{2}}\nabla\dot{u}\|_2^2dt$$

$$\leq \ C\|(\rho_0^{\frac{1-\gamma}{2}}u_0, \rho_0^{1-\frac{\gamma}{2}}\theta_0, \rho_0^{-\frac{\gamma}{2}}\nabla u_0, \rho_0^{1-\frac{\gamma}{2}}\nabla\theta_0, \rho_0^{-\frac{\gamma}{2}}\mathscr{S}_0)\|_2^2$$

for a positive constant C depending only on $c_v, \mu, \lambda, \kappa, K_1$, and Φ_T, where

$$\mathscr{S}_0 := \mu\Delta u_0 + (\mu+\lambda)\nabla\, div u_0 - R\nabla(\rho_0\theta_0).$$

Proof. Taking the operator $\partial_t(\,\cdot\,) + \mathrm{div}\,(u\,\cdot\,)$ to (1.2), noticing that

$$\begin{aligned}
\partial_t(\rho\dot{u}_i) + \mathrm{div}\,(u\rho\dot{u}_i) &= \rho(\partial_t\dot{u}_i + u\cdot\nabla\dot{u}_i),\\
\partial_t\partial_i\mathrm{div}\,u + \mathrm{div}\,(u\partial_i\mathrm{div}\,u) &= \partial_i\mathrm{div}\,\dot{u} + \mathrm{div}\,(u\partial_i\mathrm{div}\,u) - \partial_i\mathrm{div}\,((u\cdot\nabla)u), \quad (2.32)\\
\partial_t\Delta u_i + \mathrm{div}\,(u\Delta u_i) &= \Delta\dot{u}_i + \mathrm{div}\,(u\Delta u_i) - \Delta(u\cdot\nabla u_i), \quad (2.33)\\
\partial_t\partial_i p + \mathrm{div}\,(u\partial_i p) &= \partial_i(p_t + \mathrm{div}\,(up)) - \mathrm{div}\,(\partial_i u p)\\
&= R\partial_i(\rho\dot{\theta}) - \mathrm{div}\,(\partial_i u p),
\end{aligned}$$

and applying Lemma 5.1 in the Appendix to the terms $\mathrm{div}\,(u\partial_i\mathrm{div}\,u) - \partial_i\mathrm{div}\,((u\cdot\nabla)u)$ and $\mathrm{div}\,(u\Delta u_i) - \Delta(u\cdot\nabla u_i)$ in (2.32) and (2.33), respectively, one obtains

$$\begin{aligned}
&\rho(\partial_t\dot{u} + (u\cdot\nabla)\dot{u}) - \mu\Delta\dot{u} - (\mu+\lambda)\nabla\mathrm{div}\dot{u}\\
=\ &\mathrm{div}\,((\nabla u)^T p) - R\nabla(\rho\dot{\theta}) + \mu\mathrm{div}\,(\nabla u(\mathrm{div}\,uI - \nabla u - (\nabla u)^T))\\
&+(\mu+\lambda)\mathrm{div}\,[((\mathrm{div}\,u)^2 - \nabla u : (\nabla u)^T)I - \mathrm{div}\,u(\nabla u)^T].
\end{aligned}$$

Multiplying the above with $\rho_0^{1-\gamma}\dot{u}$ and integrating by parts, one gets from the Cauchy inequality, (H1), and Proposition 2.1 that

$$\begin{aligned}
&\int \rho(\partial_t\dot{u} + (u\cdot\nabla)\dot{u})\cdot\rho_0^{1-\gamma}\dot{u}dx - \int[\mu\Delta\dot{u} + (\mu+\lambda)\nabla\mathrm{div}\dot{u}]\cdot\rho_0^{1-\gamma}\dot{u}dx\\
=\ &\int \mathrm{div}\,\Big[(\nabla u)^T p - R\rho\dot{\theta}I + \mu\Big(\nabla u\big(\mathrm{div}\,uI - \nabla u - (\nabla u)^T\big)\Big)\Big]\cdot\rho_0^{1-\gamma}\dot{u}dx\\
&+(\mu+\lambda)\int\mathrm{div}\Big(\big((\mathrm{div}\,u)^2 - \nabla u : (\nabla u)^T\big)I - \mathrm{div}\,u(\nabla u)^T\Big)\cdot\rho_0^{1-\gamma}\dot{u}dx\\
\leq\ & C\int(|\nabla u|^2 + |\nabla u|\rho_0\theta + \rho|\dot{\theta}|)(\rho_0^{1-\gamma}|\nabla\dot{u}| + |\nabla\rho_0^{1-\gamma}||\dot{u}|)dx\\
\leq\ & \eta\|\rho_0^{\frac{1-\gamma}{2}}\nabla\dot{u}\|_2^2 + C\|\rho_0^{\frac{3-\gamma}{2}}\dot{\theta}\|_2^2 + C\phi(t)\|(\rho_0^{\frac{1-\gamma}{2}}\nabla u, \rho_0^{\frac{3-\gamma}{2}}\theta, \rho_0^{1-\frac{\gamma}{2}}\dot{u})\|_2^2. \quad (2.34)
\end{aligned}$$

Using (1.1) and (H1), one deduces

$$\begin{aligned}
&\int\rho(\partial_t\dot{u} + (u\cdot\nabla)\dot{u})\cdot\rho_0^{1-\gamma}\dot{u}dx\\
=\ &\frac{1}{2}\frac{d}{dt}\int\rho\rho_0^{1-\gamma}|\dot{u}|^2dx - \frac{1}{2}\int\rho u\cdot\nabla\rho_0^{1-\gamma}|\dot{u}|^2dx\\
\geq\ &\frac{1}{2}\frac{d}{dt}\|\sqrt{\rho}\rho_0^{\frac{1-\gamma}{2}}\dot{u}\|_2^2 - C\|\sqrt{\rho_0}u\|_\infty\|\sqrt{\rho}\rho_0^{\frac{1-\gamma}{2}}\dot{u}\|_2^2. \quad (2.35)
\end{aligned}$$

Integrating by parts, it follows from (H1) and the Cauchy inequality that

$$\begin{aligned} -\int \Delta\dot{u}\cdot\rho_0^{-\gamma}\dot{u}dx &= \|\rho_0^{-\frac{\gamma}{2}}\nabla\dot{u}\|_2^2+\int\nabla\dot{u}:\nabla\rho_0^{-\gamma}\otimes\dot{u}dx \\ &\geq \frac{3}{4}\|\rho_0^{-\frac{\gamma}{2}}\nabla\dot{u}\|_2^2-C\|\rho_0^{\frac{1-\gamma}{2}}\dot{u}\|_2^2 \end{aligned} \tag{2.36}$$

and, similarly,

$$-\int\nabla\mathrm{div}\dot{u}\cdot\rho_0^{1-\gamma}\dot{u}dx\geq\frac{3}{4}\|\rho_0^{\frac{1-\gamma}{2}}\mathrm{div}\dot{u}\|_2^2-C\|\rho_0^{1-\frac{\gamma}{2}}\dot{u}\|_2^2.$$

Substituting the above three inequalities into (2.34) and using (H1) yield

$$\begin{aligned} &\frac{d}{dt}\|\sqrt{\rho}\rho_0^{\frac{1-\gamma}{2}}\dot{u}\|_2^2+\mu\|\rho_0^{\frac{1-\gamma}{2}}\nabla\dot{u}\|_2^2 \\ \leq\ & C\|\rho_0^{\frac{3-\gamma}{2}}\partial_t\theta\|_2^2+C\phi(t)\|(\rho_0^{\frac{1-\gamma}{2}}\nabla u,\rho_0^{1-\frac{\gamma}{2}}\nabla\theta,\rho_0^{\frac{3-\gamma}{2}}\theta,\rho_0^{1-\frac{\gamma}{2}}\dot{u})\|_2^2, \end{aligned}$$

from which, by the Grönwall inequality and applying Propositions 2.1–2.4, the conclusion follows. □

2.5. A singularly weighted elliptic estimate.

Proposition 2.6. *It holds that*

$$\|\nabla(\rho_0^{-\frac{\gamma}{2}}u)\|_6+\|\rho_0^{-\frac{\gamma}{2}}\nabla u\|_6\leq C\|(\rho_0^{\frac{1-\gamma}{2}}u_0,\rho_0^{1-\frac{\gamma}{2}}\theta_0,\rho_0^{-\frac{\gamma}{2}}\nabla u_0,\rho_0^{1-\frac{\gamma}{2}}\nabla\theta_0,\rho_0^{-\frac{\gamma}{2}}\mathscr{S}_0)\|_2^2$$

for a positive constant C depending only on $c_v,\mu,\lambda,\kappa,K_1$, and Φ_T, where $\mathscr{S}_0$ is defined in Proposition 2.5.

Proof. Note that

$$\begin{aligned} \rho_0^{-\frac{\gamma}{2}}\Delta u &= \Delta(\rho_0^{-\frac{\gamma}{2}}u)-\mathrm{div}(u\otimes\nabla\rho_0^{-\frac{\gamma}{2}})-\nabla u\cdot\nabla\rho_0^{-\frac{\gamma}{2}}, \\ \rho_0^{-\frac{\gamma}{2}}\nabla\mathrm{div}u &= \nabla\mathrm{div}(\rho_0^{-\frac{\gamma}{2}}u)-\nabla(u\cdot\nabla\rho_0^{-\frac{\gamma}{2}})-\mathrm{div}u\nabla\rho_0^{-\frac{\gamma}{2}}, \\ \rho_0^{-\frac{\gamma}{2}}\nabla p &= \nabla(\rho_0^{-\frac{\gamma}{2}}p)-\nabla\rho_0^{-\frac{\gamma}{2}}p. \end{aligned}$$

Therefore, it follows from (1.2) that

$$\begin{aligned} &\mu\Delta(\rho_0^{-\frac{\gamma}{2}}u)+(\mu+\lambda)\nabla\mathrm{div}(\rho_0^{-\frac{\gamma}{2}}u) \\ =\ &\rho_0^{-\frac{\gamma}{2}}\rho\dot{u}+\mathrm{div}\left(\mu u\otimes\nabla\rho_0^{-\frac{\gamma}{2}}+(\mu+\lambda)u\cdot\nabla\rho_0^{-\frac{\gamma}{2}}I+\rho_0^{-\frac{\gamma}{2}}pI\right) \\ &+(\mu\nabla u+(\mu+\lambda)\mathrm{div}uI-pI)\nabla\rho_0^{-\frac{\gamma}{2}}. \end{aligned}$$

It follows from the elliptic estimates, (H1), and Proposition 2.1 that

$$\begin{aligned} \|\nabla(\rho_0^{-\frac{\gamma}{2}}u)\|_6 &\leq C\|\rho_0^{-\frac{\gamma}{2}}\rho\dot{u}+(\mu\nabla u+(\mu+\lambda)\mathrm{div}uI-pI)\nabla\rho_0^{-\frac{\gamma}{2}}\|_2 \\ &\quad+C\left\|\mu u\otimes\nabla\rho_0^{-\frac{\gamma}{2}}+(\mu+\lambda)u\cdot\nabla\rho_0^{-\frac{\gamma}{2}}I+\rho_0^{-\frac{\gamma}{2}}pI\right\|_6 \end{aligned}$$

$$\leq C(\|(\rho_0^{1-\frac{\gamma}{2}}\dot{u}, \rho_0^{\frac{1-\gamma}{2}}\nabla u, \rho_0^{\frac{3-\gamma}{2}}\theta)\|_2 + \|(\rho_0^{\frac{1-\gamma}{2}}u, \rho_0^{1-\frac{\gamma}{2}}\theta)\|_6). \quad (2.37)$$

By the Sobolev inequality and (H1), one can get

$$\|\rho_0^{\frac{1-\gamma}{2}}u\|_6 \leq C(\|\rho_0^{\frac{1-\gamma}{2}}\nabla u\|_2 + \|\nabla\rho_0^{\frac{1-\gamma}{2}}u\|_2) \leq C(\|\rho_0^{\frac{1-\gamma}{2}}\nabla u\|_2 + \|\rho_0^{1-\frac{\gamma}{2}}u\|_2)$$

and, similarly,

$$\|\rho_0^{1-\frac{\gamma}{2}}\theta\|_6 \leq C(\|\rho_0^{1-\frac{\gamma}{2}}\nabla\theta\|_2 + \|\rho_0^{\frac{3-\gamma}{2}}\theta\|_2).$$

Substituting the above two inequalities into (2.37) yields

$$\|\nabla(\rho_0^{-\frac{\gamma}{2}}u)\|_6 \leq C\|(\rho_0^{1-\frac{\gamma}{2}}\dot{u}, \rho_0^{\frac{1-\gamma}{2}}\nabla u, \rho_0^{1-\frac{\gamma}{2}}\nabla\theta, \rho_0^{1-\frac{\gamma}{2}}u, \rho_0^{\frac{3-\gamma}{2}}\theta)\|_2,$$

and further

$$\begin{aligned}\|\rho_0^{-\frac{\gamma}{2}}\nabla u\|_6 &= \|\nabla(\rho_0^{-\frac{\gamma}{2}}u) - \nabla\rho_0^{-\frac{\gamma}{2}}u\|_6 \leq \|\nabla(\rho_0^{-\frac{\gamma}{2}}u)\|_6 + C\|\rho_0^{\frac{1-\gamma}{2}}u\|_6 \\ &\leq C\|(\rho_0^{1-\frac{\gamma}{2}}\dot{u}, \rho_0^{\frac{1-\gamma}{2}}\nabla u, \rho_0^{1-\frac{\gamma}{2}}\nabla\theta, \rho_0^{1-\frac{\gamma}{2}}u, \rho_0^{\frac{3-\gamma}{2}}\theta)\|_2.\end{aligned}$$

Combining the above two inequalities and applying Propositions 2.1–2.5, the conclusion follows. □

3. The De Giorgi iterations

This section is devoted to carrying out suitable De Giorgi iterations, which are preparations for proving the lower and upper bounds of the entropy in the next section. Due to the presence of vacuum at the far fields, which causes the degeneracy of the system, the De Giorgi iterations performed in this section are of singular type, that is, some singular weights are introduced in the testing functions chosen in the iterations. Moreover, the iterations are applied to different equations in establishing the lower and upper bounds of the entropy: in dealing with the lower bound, a De Giorgi iteration is applied to the entropy equation itself, while in dealing with the upper bound, it is applied to the temperature equation. The singularly weighted energy estimates established in the previous section play essential roles in the De Giorgi iteration applied to the temperature equation to deal with the upper bound of the entropy, but not in dealing with the lower bound of the entropy.

Set

$$M_T = \frac{\kappa(\gamma-1)}{c_v}e^{C_*\Phi_T}[(1+|\gamma-2|)K_1^2 + K_2], \quad (3.1)$$

where C_* is the positive constant stated in Proposition 2.1 and Φ_T is given by (2.1).

The following De Giorgi type iteration will be used to get the uniform lower bound of the entropy in the next section.

Proposition 3.1. *Let M_T be given by (3.1) and define*

$$\tilde{s} = \log\theta - (\gamma-1)\log\rho_0 + M_T t, \quad \underline{\tilde{s}}_0 = \frac{\underline{s}_0}{c_v} - \log\frac{R}{A}.$$

Then, the following statements hold:

(i) For any $\ell\leq\underline{\tilde{s}}_0$,

$$\sup_{0\leq t\leq T}\|(\tilde{s}-\ell)_-\|_2^2\,(t)+\int_0^T\left\|\frac{\nabla(\tilde{s}-\ell)_-}{\sqrt{\rho_0}}\right\|_2^2dt\leq C,$$

for a positive constant C *depending only on* $c_v,\mu,\lambda,\kappa,K_1,T,\Phi_T$, *and the initial data.*

(ii) Set

$$\mathcal{Y}_\ell=\mathcal{Y}_\ell(T):=\sup_{0\leq t\leq T}\|(\tilde{s}-\ell)_-\|_2^2+\int_0^T\|\nabla(\tilde{s}-\ell)_-\|_2^2dt,\quad\forall\ell\leq\underline{\tilde{s}}_0,$$

where $f_-:=-\min\{f,0\}$. *Then,*

$$\mathcal{Y}_\ell\leq\frac{C}{(m-\ell)^3}\mathcal{Y}_m^{\frac32},\quad\forall\ell<m\leq\underline{\tilde{s}}_0,$$

for a positive constant C *depending only on* $c_v,\mu,\lambda,\kappa,K_1$, *and* Φ_T.

Proof. It follows from (1.5) and definition of $\tilde{s}$ that

$$\begin{aligned}c_v\rho(\partial_t\tilde{s}+u\cdot\nabla\tilde{s})-\kappa\Delta\tilde{s}\;&=\;c_vM_T\rho+\kappa(\gamma-1)\Delta\log\rho_0-R\rho u\cdot\nabla\log\rho_0\\&\quad-R\rho\operatorname{div}u+\kappa\left|\frac{\nabla\theta}{\theta}\right|^2+\frac{\mathcal{Q}(\nabla u)}{\theta}.\end{aligned}$$

Multiplying the above with $-\frac{(\tilde{s}-\ell)_-}{\rho_0}$ yields

$$\begin{aligned}-c_v&\int\rho(\partial_t\tilde{s}+u\cdot\nabla\tilde{s})\rho_0^{-1}(\tilde{s}-\ell)_-dx+\kappa\int\Delta\tilde{s}\rho_0^{-1}(\tilde{s}-\ell)_-dx\\&\leq-\int[c_vM_T\rho+\kappa(\gamma-1)\Delta\log\rho_0]\rho_0^{-1}(\tilde{s}-\ell)_-dx\\&\quad+R\int\rho u\cdot\nabla\log\rho_0\rho_0^{-1}(\tilde{s}-\ell)_-dx+R\int\rho\operatorname{div}u\rho_0^{-1}(\tilde{s}-\ell)_-dx\\&=:I_1(t)+I_2(t)+I_3(t).\end{aligned}\tag{3.2}$$

In the similar ways as (2.35) and (2.36), one can get

$$\begin{aligned}&-c_v\int\rho(\partial_t\tilde{s}+u\cdot\nabla\tilde{s})\rho_0^{-1}(\tilde{s}-\ell)_-dx\\&=\;c_v\int\rho(\partial_t(\tilde{s}-\ell)_-+u\cdot\nabla(\tilde{s}-\ell)_-)\rho_0^{-1}(\tilde{s}-\ell)_-dx\\&\geq\;\frac{c_v}{2}\frac{d}{dt}\left\|\sqrt{\frac{\rho}{\rho_0}}(\tilde{s}-\ell)_-\right\|_2^2-C\|\sqrt{\rho_0}u\|_\infty\|(\tilde{s}-\ell)_-\|_2^2\end{aligned}\tag{3.3}$$

and

$$\int\Delta\tilde{s}\rho_0^{-1}(\tilde{s}-\ell)_-dx\;=\;-\int\nabla\tilde{s}[\rho_0^{-1}\nabla(\tilde{s}-\ell)_-+\nabla\rho_0^{-1}(\tilde{s}-\ell)_-]dx$$

$$
\begin{aligned}
&= \int \nabla(\tilde{s}-\ell)_-[\rho_0^{-1}\nabla(\tilde{s}-\ell)_- + \nabla\rho_0^{-1}(\tilde{s}-\ell)_-]dx \\
&\geq \frac{3}{4}\left\|\frac{\nabla(\tilde{s}-\ell)_-}{\sqrt{\rho_0}}\right\|_2^2 - C\|(\tilde{s}-\ell)_-\|_2^2. \qquad (3.4)
\end{aligned}
$$

By assumptions (H1)–(H2), Proposition 2.1, and recalling the definition of M_T in (3.1), one deduces that

$$
\begin{aligned}
& c_v M_T \rho + \kappa(\gamma-1)\Delta\log\rho_0 \\
=\ & \rho_0\left(c_v M_T \frac{\rho}{\rho_0} + \kappa(\gamma-1)\frac{\Delta\rho_0}{\rho_0^2} - \kappa(\gamma-1)\frac{|\nabla\rho_0|^2}{\rho_0^3}\right) \\
\geq\ & \rho_0\Big(c_v e^{-C_*\Phi_T} M_T - \kappa(\gamma-1)K_2 - \kappa(\gamma-1)K_1^2\Big) \\
=\ & \varrho_0\kappa(\gamma-1)|\gamma-2|K_1^2 > 0
\end{aligned}
$$

and, thus,

$$
I_1 = -\int [c_v M_T\rho + \kappa(\gamma-1)\Delta\log\rho_0]\rho_0^{-1}(\tilde{s}-\ell)_- dx \leq 0.
$$

Thanks to this, substituting (3.3) and (3.4) into (3.2) yields

$$
\begin{aligned}
& c_v\frac{d}{dt}\left\|\sqrt{\frac{\rho}{\rho_0}}(\tilde{s}-\ell)_-\right\|_2^2 + 1.5\kappa\left\|\frac{\nabla(\tilde{s}-\ell)_-}{\sqrt{\rho_0}}\right\|_2^2 \\
\leq\ & C(1+\|\sqrt{\rho_0}u\|_\infty)\|(\tilde{s}-\ell)_-\|_2^2 + 2(I_2+I_3). \qquad (3.5)
\end{aligned}
$$

(i) By assumption (H1), it follows from Propositions 2.1–2.3 that

$$
I_2 + I_3 \leq C(\|\sqrt{\rho}u\|_2^2 + \|\nabla u\|_2^2 + \|(\tilde{s}-\ell)_-\|_2^2) \leq C(1+\|(\tilde{s}-\ell)_-\|_2^2),
$$

which together with (3.5) yields

$$
c_v\frac{d}{dt}\left\|\sqrt{\frac{\rho}{\rho_0}}(\tilde{s}-\ell)_-\right\|_2^2 + 1.5\kappa\left\|\frac{\nabla(\tilde{s}-\ell)_-}{\sqrt{\rho_0}}\right\|_2^2 \leq C(1+\|\sqrt{\rho_0}u\|_\infty)\|(\tilde{s}-\ell)_-\|_2^2 + C.
$$

Since $\tilde{s}|_{t=0} \geq \underline{s}_0$, it is clear that $(s-\ell)_-|_{t=0} = 0$ for any $\ell \leq \underline{s}_0$. Thanks to this, applying the Grönwall inequality to the above inequality, and by Proposition 2.1, one gets the first conclusion.

(ii) It follows from Proposition 2.1, assumption (H1), and the Cauchy inequality that

$$
\begin{aligned}
2(I_2+I_3) &\leq C\int(|\sqrt{\varrho_0}u| + |\nabla u|)(\tilde{s}-\ell)_- dy \\
&\leq C(\|\sqrt{\rho_0}u\|_\infty^2 + \|\nabla u\|_\infty^2)\|(\tilde{s}-\ell)_-\|_2^2 + C\int_{\{\tilde{s}<\ell\}} 1dx. \qquad (3.6)
\end{aligned}
$$

Substituting (3.6) into (3.5) leads to

$$\begin{aligned} & c_v \frac{d}{dt}\left\|\sqrt{\frac{\rho}{\rho_0}}(\tilde{s}-\ell)_-\right\|_2^2+\kappa\left\|\frac{\nabla(\tilde{s}-\ell)_-}{\sqrt{\rho_0}}\right\|_2^2 \\ \leq\ & C(\|\sqrt{\rho_0}u\|_\infty^2+\|\nabla u\|_\infty^2)\|(\tilde{s}-\ell)_-\|_2^2+C\int_{\{\tilde{s}<\ell\}}1dx. \end{aligned} \tag{3.7}$$

One can check easily that, for any $m>\ell$,

$$1\leq\frac{(\tilde{s}-m)_-}{m-\ell},\qquad \text{on } \{\tilde{s}<\ell\}\subseteq\{\tilde{s}<m\}.$$

Therefore, it follows from the Gagliardo-Nirenberg inequality that

$$\begin{aligned} \int_{\{\tilde{s}<\ell\}}1dx \leq\ & \int_{\{\tilde{s}<\ell\}}\left|\frac{(\tilde{s}-m)_-}{m-\ell}\right|^3dx \\ \leq\ & \int_{\{\tilde{s}<m\}}\left|\frac{(\tilde{s}-m)_-}{m-\ell}\right|^3dx=\frac{\|(\tilde{s}-m)_-\|_3^3}{(m-\ell)^3} \\ \leq\ & \frac{C}{(m-\ell)^3}\|(\tilde{s}-m)_-\|_2^{\frac32}\|\nabla(\tilde{s}-m)_-\|_2^{\frac32}, \end{aligned}$$

which, substituted into (3.7), gives

$$\begin{aligned} & c_v \frac{d}{dt}\left\|\sqrt{\frac{\rho}{\rho_0}}(\tilde{s}-\ell)_-\right\|_2^2+\kappa\left\|\frac{\nabla(\tilde{s}-\ell)_-}{\sqrt{\rho_0}}\right\|_2^2 \\ \leq\ & C(1+\|\sqrt{\rho_0}u\|_\infty^2+\|\nabla u\|_\infty^2)\|(\tilde{s}-\ell)_-\|_2^2 \\ & +\frac{C}{(m-\ell)^3}\|(\tilde{s}-m)_-\|_2^{\frac32}\|\nabla(\tilde{s}-m)_-\|_2^{\frac32}. \end{aligned}$$

Recalling that $\tilde{s}|_{t=0}\geq\underline{\tilde{s}}_0$, applying the Grönwall inequality to the above, and using Proposition 2.1, one obtains the second conclusion. □

To derive a uniform upper bound for the entropy, we need the following De Giorgi type iteration.

Proposition 3.2. *Let M_T be given by (3.1) and define*

$$\overline{S}_0=\frac{A}{R}e^{\frac{\bar{s}_0}{c_v}},\qquad \theta_\ell=\theta-\ell e^{M_Tt}\rho_0^{\gamma-1},\quad \forall\ell\in\mathbb{R},$$

$$\mathcal{Z}_\ell=\mathcal{Z}_\ell(T)=\sup_{0\leq t\leq T}\|\rho_0^{1-\gamma}(\theta_\ell)_+\|_2^2+\int_0^T\|\rho_0^{\frac12-\gamma}\nabla(\theta_\ell)_+\|_2^2dt,\quad \forall\ell\geq\overline{S}_0,$$

where $f_+:=\max\{f,0\}$.

Then, there is a positive constant C depending only on c_v, γ, μ, λ, κ, K_1, K_2, T, Φ_T, and the initial data, such that

$$\mathcal{Z}_\ell\ \leq\ C(\ell^2+1),\quad \forall\ell\geq\overline{S}_0,$$

$$\mathcal{Z}_\ell \leq \frac{C\ell^2}{(\ell-m)^3}\mathcal{Z}_m^{\frac32}, \quad \forall \ell > m \geq \overline{S}_0.$$

Proof. Direct calculations show that

$$\begin{aligned} c_v\rho(\partial_t\theta_\ell + u\cdot\nabla\theta_\ell) - \kappa\Delta\theta_\ell &= -c_v\ell e^{M_Tt}\rho u\cdot\nabla\rho_0^{\gamma-1} - R\rho(\theta_\ell + \ell e^{M_Tt}\rho_0^{\gamma-1})\mathrm{div}\, u \\ &\quad + \ell e^{M_Tt}\left(\kappa\Delta\rho_0^{\gamma-1} - c_vM_T\rho_0^{\gamma-1}\rho\right) + \mathcal{Q}(\nabla u). \end{aligned}$$

Multiplying the above with $\rho_0^{1-2\gamma}(\theta_\ell)_+$ and integrating the resultant over $\mathbb{R}^3$ yield

$$\begin{aligned} & c_v\int\rho(\partial_t\theta_\ell + u\cdot\nabla\theta_\ell)\rho_0^{1-2\gamma}(\theta_\ell)_+dx - \kappa\int\Delta\theta_\ell\rho_0^{1-2\gamma}(\theta_\ell)_+dx \\ = & -c_v\ell e^{M_Tt}\int\rho u\cdot\nabla\rho_0^{\gamma-1}\rho_0^{1-2\gamma}(\theta_\ell)_+dx - R\int\rho\theta_\ell\mathrm{div}\, u\rho_0^{1-2\gamma}(\theta_\ell)_+dx \\ & -R\ell e^{M_Tt}\int\rho\rho_0^{-\gamma}\mathrm{div}\, u(\theta_\ell)_+dx + \ell e^{M_Tt}\int\left(\kappa\Delta\rho_0^{\gamma-1} - c_vM_T\rho_0^{\gamma-1}\rho\right)\rho_0^{1-2\gamma}(\theta_\ell)_+dx \\ & + \int\mathcal{Q}(\nabla u)\rho_0^{1-2\gamma}(\theta_\ell)_+dx =: II_1 + II_2 + II_3 + II_4 + II_5. \end{aligned} \tag{3.8}$$

In the similar ways as in (2.35) and (2.36), one can get

$$\begin{aligned} & c_v\int\rho(\partial_t\theta_\ell + u\cdot\nabla\theta_\ell)\rho_0^{1-2\gamma}(\theta_\ell)_+dx \\ \geq & \frac{c_v}{2}\frac{d}{dt}\|\sqrt{\rho}\rho_0^{\frac12-\gamma}(\theta_\ell)_+\|_2^2 - C\|\sqrt{\rho_0}u\|_\infty\|\sqrt{\rho}\rho_0^{\frac12-\gamma}(\theta_\ell)_+\|_2^2 \end{aligned} \tag{3.9}$$

and

$$-\int\Delta\theta_\ell\rho_0^{1-2\gamma}(\theta_\ell)_+dx \geq \frac34\|\rho_0^{\frac12-\gamma}\nabla(\theta_\ell)_+\|_2^2 - C\|\rho_0^{1-\gamma}(\theta_\ell)_+\|_2^2. \tag{3.10}$$

By (H1)–(H2), (3.1), and Proposition 2.1, one deduces

$$\begin{aligned} \kappa\Delta\rho_0^{\gamma-1} - c_vM_T\rho_0^{\gamma-1}\rho &= \rho_0^\gamma\left(\kappa(\gamma-1)\frac{\Delta\rho_0}{\rho_0^2} + \kappa(\gamma-1)(\gamma-2)\frac{|\nabla\rho_0|^2}{\rho_0^3} - c_vM_T\frac{\rho}{\rho_0}\right) \\ &\leq \rho_0^\gamma\Big(\kappa(\gamma-1)K_2 + \kappa(\gamma-1)|\gamma-2|K_1^2 - c_vM_Te^{-C_*\Phi_T}\Big) \\ &= -\kappa(\gamma-1)\rho_0^\gamma K_1^2 \leq 0 \end{aligned}$$

and, thus,

$$II_4 = \ell e^{M_Tt}\int\left(\kappa\Delta\rho_0^{\gamma-1} - c_vM_T\rho_0^{\gamma-1}\rho\right)\rho_0^{1-2\gamma}(\theta_\ell)_+dx \leq 0. \tag{3.11}$$

Thanks to (3.9)–(3.11), one gets from (3.8) and Proposition 2.1 that

$$\begin{aligned} & c_v\frac{d}{dt}\|\sqrt{\rho}\rho_0^{\frac12-\gamma}(\theta_\ell)_+\|_2^2 + 1.5\kappa\|\rho_0^{\frac12-\gamma}\nabla(\theta_\ell)_+\|_2^2 \\ \leq & C(1+\|\sqrt{\rho_0}u\|_\infty)\|\sqrt{\rho}\rho_0^{\frac12-\gamma}(\theta_\ell)_+\|_2^2 + 2(II_1+II_2+II_3+II_5). \end{aligned} \tag{3.12}$$

(i) By Proposition 2.1 and using (H1), it follows from the Hölder and Cauchy inequalities that

$$\begin{aligned} & II_1+II_2+II_3+II_5 \\ \leq\ & C\ell\|\sqrt{\rho_0}u\|_2\|\rho_0^{1-\gamma}(\theta_\ell)_+\|_2+C\|\nabla u\|_\infty\|\rho_0^{1-\gamma}(\theta_\ell)_+\|_2^2 \\ & +C\ell\|\nabla u\|_2\|\rho_0^{1-\gamma}(\theta_\ell)_+\|_2+C\|\rho_0^{-\frac{\gamma}{2}}\nabla u\|_4^2\|\rho_0^{1-\gamma}(\theta_\ell)_+\|_2 \\ \leq\ & C(\ell^2\|\sqrt{\rho_0}u\|_2^2+\ell^2\|\nabla u\|_2^2+\|\rho_0^{-\frac{\gamma}{2}}\nabla u\|_4^4) \\ & +C(1+\|\nabla u\|_\infty)\|\sqrt{\rho}\rho_0^{\frac12-\gamma}(\theta_\ell)_+\|_2^2. \end{aligned}$$

Due to Propositions 2.2, 2.3, and 2.6, one gets by the Young inequality that

$$\|\sqrt{\rho_0}u\|_2+\|\nabla u\|_2+\|\rho_0^{-\frac{\gamma}{2}}\nabla u\|_4^4\leq C(1+\|\rho_0^{-\frac{\gamma}{2}}\nabla u\|_2^2+\|\rho_0^{-\frac{\gamma}{2}}\nabla u\|_6^6)\leq C.$$

Therefore,

$$II_1+II_2+II_3+II_5\leq C(\ell^2+1)+C(1+\|\nabla u\|_\infty)\|\sqrt{\rho}\rho_0^{\frac12-\gamma}(\theta_\ell)_+\|_2^2,$$

from which and (3.12) one gets

$$\begin{aligned} & c_v\frac{d}{dt}\|\sqrt{\rho}\rho_0^{\frac12-\gamma}(\theta_\ell)_+\|_2^2+1.5\kappa\|\rho_0^{\frac12-\gamma}\nabla(\theta_\ell)_+\|_2^2 \\ \leq\ & C(1+\|\sqrt{\rho_0}u\|_\infty+\|\nabla u\|_\infty)\|\sqrt{\rho}\rho_0^{\frac12-\gamma}(\theta_\ell)_+\|_2^2+C(\ell^2+1). \end{aligned}$$

Applying the Grönwall inequality to the above inequality, by Proposition 2.1, and noticing that $(\theta_\ell)_+|_{t=0}=0$, for any $\ell\geq\overline{S}_0$, the first conclusion follows.

(ii) Using (H1) and Proposition 2.1, one can obtain

$$\begin{aligned} II_1+II_3 \leq\ & C\ell e^{M_Tt}\int(\rho|u|\rho_0^{\frac12-\gamma}(\theta_\ell)_++|\nabla u|\rho_0^{1-\gamma}(\theta_\ell)_+)dx \\ \leq\ & C\ell(\|\sqrt{\rho_0}u\|_\infty+\|\nabla u\|_\infty)\int\rho_0^{1-\gamma}(\theta_\ell)_+dx \\ \leq\ & C\ell^2\int_{\{\theta_\ell>0\}}1dx+C(\|\sqrt{\rho_0}u\|_\infty^2+\|\nabla u\|_\infty^2)\|\rho_0^{1-\gamma}(\theta_\ell)_+\|_2^2, \\ II_2 \leq\ & C\|\nabla u\|_\infty\|\rho_0^{1-\gamma}(\theta_\ell)_+\|_2^2. \end{aligned}$$

As for II_5, by Proposition 2.6, one can deduce from the Hölder and Sobolev inequalities that

$$\begin{aligned} II_5 =\ & \int\mathcal{Q}(\nabla u)\rho_0^{1-2\gamma}(\theta_\ell)_+dx\leq C\int|\rho_0^{-\frac{\gamma}{2}}\nabla u|^2\rho_0^{1-\gamma}(\theta_\ell)_+dx \\ \leq\ & C\|\rho_0^{-\frac{\gamma}{2}}\nabla u\|_6^2\|\rho_0^{1-\gamma}(\theta_\ell)_+\|_6\left(\int_{\{\theta_\ell>0\}}1dx\right)^{\frac12} \\ \leq\ & C\left(\|\rho_0^{1-\gamma}\nabla(\theta_\ell)_+\|_2+\|\rho_0^{\frac32-\gamma}(\theta_\ell)_+\|_2\right)\left(\int_{\{\theta_\ell>0\}}1dx\right)^{\frac12} \end{aligned}$$

$$\leq \frac{\kappa}{8}\|\rho_0^{\frac12-\gamma}\nabla(\theta_\ell)_+\|_2^2+C\|\rho_0^{1-\gamma}(\theta_\ell)_+\|_2^2+C\int_{\{\theta_\ell>0\}}1dx,$$

where the following has been used

$$\begin{aligned}\|\rho_0^{1-\gamma}(\theta_\ell)_+\|_6 &\leq C\|\nabla(\rho_0^{1-\gamma}(\theta_\ell)_+)\|_2\leq \left(\|\rho_0^{1-\gamma}\nabla(\theta_\ell)_+\|_2+\|\nabla\rho_0^{1-\gamma}(\theta_\ell)_+\|_2\right)\\ &\leq C\left(\|\rho_0^{1-\gamma}\nabla(\theta_\ell)_+\|_2+\|\rho_0^{\frac32-\gamma}(\theta_\ell)_+\|_2\right).\end{aligned}$$

Thanks to the estimates on $II_i, i=1,2,\cdots,5$, it follows from (3.12) that

$$\begin{aligned}&c_v\frac{d}{dt}\|\sqrt{\rho}\rho_0^{\frac12-\gamma}(\theta_\ell)_+\|_2^2+\kappa\|\rho_0^{\frac12-\gamma}\nabla(\theta_\ell)_+\|_2^2\\ \leq\ & C(1+\|\sqrt{\rho_0}u\|_\infty^2+\|\nabla u\|_\infty^2)\|\rho_0^{1-\gamma}(\theta_\ell)_+\|_2^2+C\ell^2\int_{\{\theta_\ell>0\}}1dx. \end{aligned}\tag{3.13}$$

Since $M_T>0$, one can check that for any $\ell>m$, it holds that

$$1\leq e^{-M_Tt}\rho_0^{1-\gamma}\frac{\theta_m^+}{\ell-m}\leq\rho_0^{1-\gamma}\frac{\theta_m^+}{\ell-m},\qquad \text{on } \{\theta_\ell>0\}\subseteq\{\theta_m>0\}.$$

Thanks to this, it follows from the Gagliardo-Nirenberg inequality and (H1) that

$$\begin{aligned}\int_{\{\theta_\ell>0\}}1dx &\leq \int_{\{\theta_\ell>0\}}\left|\frac{\varrho_0^{1-\gamma}\theta_m^+}{\ell-m}\right|^3dx\leq\frac{C}{(\ell-m)^3}\|\rho_0^{1-\gamma}\theta_m^+\|_2^{\frac32}\|\nabla(\rho_0^{1-\gamma}\theta_m^+)\|_2^{\frac32}\\ &\leq \frac{C}{(\ell-m)^3}\|\rho_0^{1-\gamma}\theta_m^+\|_2^{\frac32}(\|\rho_0^{1-\gamma}\nabla\theta_m^+\|_2+\|\nabla\rho_0^{1-\gamma}\theta_m^+\|_2)^{\frac32}\\ &\leq \frac{C}{(\ell-m)^3}\|\rho_0^{1-\gamma}\theta_m^+\|_2^{\frac32}(\|\rho_0^{1-\gamma}\nabla\theta_m^+\|_2+\|\rho_0^{1-\gamma}\theta_m^+\|_2)^{\frac32}.\end{aligned}$$

Substituting this into (3.13) gives

$$\begin{aligned}&c_v\frac{d}{dt}\|\sqrt{\rho}\rho_0^{\frac12-\gamma}(\theta_\ell)_+\|_2^2+\kappa\|\rho_0^{\frac12-\gamma}\nabla(\theta_\ell)_+\|_2^2\\ \leq\ &\frac{C\ell^2}{(\ell-m)^3}\|\rho_0^{1-\gamma}\theta_m^+\|_2^{\frac32}(\|\rho_0^{1-\gamma}\nabla\theta_m^+\|_2+\|\rho_0^{1-\gamma}\theta_m^+\|_2)^{\frac32}\\ &+C(1+\|\sqrt{\rho_0}u\|_\infty^2+\|\nabla u\|_\infty^2)\|\rho_0^{1-\gamma}(\theta_\ell)_+\|_2^2.\end{aligned}$$

Note that $(\theta_\ell)_+|_{t=0}=0$, for any $\ell\geq\overline{S}_0$. The second conclusion follows from the above inequality by the Grönwall inequality and Proposition 2.1. □

4. Uniform boundedness of entropy: proof of Theorem 1.3

Let us first recall the following lemma cited from [35].

Lemma 4.1 ([35]). *Let $m_0\in[0,\infty)$ be given and f be a nonnegative non-increasing function on $[m_0,\infty)$ satisfying*

$$f(\ell)\leq\frac{M_0(\ell+1)^\alpha}{(\ell-m)^\beta}f^\sigma(m),\quad\forall\ell>m\geq m_0,$$

for some nonnegative constants M_0,α,β, and σ, with $0\leq\alpha<\beta$ and $\sigma>1$. Then,

$$f(m_0+d)=0,$$

where

$$d=\left[2f^\sigma(m_0)(m_0+M_0+2)^{\frac{2\alpha+2\beta+1}{\sigma-1}+\frac{\beta}{(\sigma-1)^2}+2\alpha+\beta+1}\right]^{\frac{1}{\beta-\alpha}}+2.$$

Then, we can prove Theorem 1.3 as follows.

Proof of Theorem 1.3. (i) and (ii) are direct corollaries of Proposition 2.2, by choosing $\alpha=1$ and $\alpha=2$, respectively.

(iii) Let $\tilde s,\underline{\tilde s}_0$, and $\mathcal{Y}_\ell$ be defined as in Proposition 3.1. By Proposition 3.1, $\mathcal{Y}_\ell$ is finitely valued for $\ell\leq\underline{\tilde s}_0$ and

$$\mathcal{Y}_\ell\leq\frac{C_1}{(m-\ell)^3}\mathcal{Y}_m^{\frac32},\quad\forall\ell<m\leq\underline{\tilde s}_0,\tag{4.1}$$

for a positive constant C_1 depending only on $c_v,\mu,\lambda,\kappa,K_1$, and Φ_T. Define $f(\ell):=\mathcal{Y}_{\underline{\tilde s}_0-\ell}$. Noticing that $\mathcal{Y}_\ell$ is nonnegative and nondecreasing with respect to ℓ, it is clear that $f(\ell)$ is nonnegative and non-increasing with respect to $\ell\in[0,\infty)$. Then, it follows from (4.1) that

$$f(\ell)\leq\frac{C_1}{(m-\ell)^3}f^{\frac32}(m),\quad\forall\ell>m\geq0,$$

from which, by Lemma 4.1, it follows that $f(d_1)=\mathcal{Y}_{\underline{\tilde s}_0-d_1}=0$, for a positive constant d_1 depending only on C_1 and $f(0)=\mathcal{Y}_{\underline{\tilde s}_0}$. With the aid of this and recalling the definition of $\mathcal{Y}_\ell$, one gets $(\tilde s-\underline{\tilde s}_0+d_1)_-=0$ a.e. in $\mathbb{R}^3\times(0,T)$, and, consequently,

$$\inf_{(x,t)\in\mathbb{R}^3\times(0,T)}\tilde s(x,t)\geq\underline{\tilde s}_0-d_1.$$

Therefore, recalling the definition of $\tilde s$ and using Proposition 2.1, one deduces

$$\begin{aligned}s(x,t)&\geq c_v\inf_{(x,t)\in\mathbb{R}^3\times(0,T)}\tilde s+c_v\log\frac{R}{A}+(\gamma-1)\log\left(\inf_{(x,t)\in\mathbb{R}^3\times(0,T)}\frac{\rho_0}{\rho}\right)\\&\geq c_v\left(\underline{\tilde s}_0-d_1+\log\frac{R}{A}\right)-(\gamma-1)C\Phi_T,\quad\forall(x,t)\in\mathbb{R}^3\times(0,T),\end{aligned}$$

for a positive constant C depending only on K_1. This proves the first conclusion.

(iv) Let $\vartheta_\ell,\overline{S}_0$, and $\mathcal{Z}_\ell$ be defined as in Proposition 3.2. It follows from Proposition 3.2 that $\mathcal{Z}_\ell$ is finitely valued for any $\ell\geq\overline{S}_0$ and

$$\mathcal{Z}_\ell\leq\frac{C_2\ell^2}{(\ell-m)^3}\mathcal{Z}_m^{\frac32},\quad\forall\ell>m\geq\overline{S}_0,$$

for a positive constant C_2 depending only on $c_v, \gamma, \mu, \lambda, \kappa, K_1, T, \Phi_T$, and the initial data. Then, by Lemma 4.1, there is a positive constant d_2 depending only on C_2 and $\mathcal{Z}_{\overline{S}_0}$, such that $\mathcal{Z}_{\overline{S}_0+d_2} = 0$. Thanks to this and recalling the definition of $\mathcal{Z}_\ell$, one gets

$$\theta(x,t) \leq (\overline{S}_0 + d_2)\rho_0^{\gamma-1}(x), \quad \forall (x,t) \in \mathbb{R}^3 \times (0,T),$$

from which, since $\theta = \frac{A}{R} e^{\frac{s}{c_v}} \rho^{\gamma-1}$ and by Proposition 2.1, it follows

$$s \leq c_v \log \left[\frac{R}{A}(\overline{S}_0 + d_2) \right] + R \log \frac{\rho_0}{\rho} \leq c_v \log \left[\frac{R}{A}(\overline{S}_0 + d_2) \right] + RC\Phi_T,$$

for a positive constant C depending only on K_1, which yields the second conclusion. Thus, Theorem 1.3 is proved. □

5. Appendix

In this appendix, we prove the following lemma which has already been used in the proof of Proposition 2.5.

Lemma 5.1. *The following two identities hold*

$$\begin{aligned} div\,(\Delta u \otimes u) - \Delta((u\cdot\nabla)u) &= div\,(\nabla u(div\,u I - \nabla u - (\nabla u)^t), \\ div\,(\nabla\, div\, u \otimes u) - \nabla\, div\,((u\cdot\nabla)u) &= div\,[((div\,u)^2 - \nabla u : (\nabla u)^t)I - div\,u(\nabla u)^t]. \end{aligned}$$

Proof. By direct calculations,

$$\begin{aligned} & \operatorname{div}(u\Delta u_i) - \Delta(u\cdot\nabla u_i) \\ = {} & \operatorname{div}(u\partial_k^2 u_i) - \partial_k^2(u\cdot\nabla u_i) \\ = {} & \operatorname{div}(\partial_k(u\partial_k u_i) - \partial_k u \partial_k u_i) - \partial_k(u\cdot\nabla\partial_k u_i) - \partial_k(\partial_k u\cdot\nabla u_i) \\ = {} & \partial_k(u\cdot\nabla\partial_k u_i + \operatorname{div} u\partial_k u_i) - \operatorname{div}(\partial_k u\partial_k u_i) \\ & -\partial_k(u\cdot\nabla\partial_k u_i) - \partial_k(\partial_k u\cdot\nabla u_i) \\ = {} & \partial_k(\operatorname{div} u\partial_k u_i - \partial_k u\cdot\nabla u_i) - \operatorname{div}(\partial_k u\partial_k u_i), \end{aligned}$$

the first conclusion follows. One deduces

$$\begin{aligned} & \operatorname{div}(u\partial_i \operatorname{div} u) - \partial_i \operatorname{div}((u\cdot\nabla)u) \\ = {} & \operatorname{div}\partial_i(u\operatorname{div} u) - \operatorname{div}(\partial_i u\operatorname{div} u) - \partial_i(u\cdot\nabla\operatorname{div} u + \nabla u_k\cdot\partial_k u) \\ = {} & \partial_i(u\cdot\nabla\operatorname{div} u + (\operatorname{div} u)^2) - \operatorname{div}(\partial_i u\operatorname{div} u) \\ & -\partial_i(u\cdot\nabla\operatorname{div} u + \nabla u_k\cdot\partial_k u) \\ = {} & \partial_i((\operatorname{div} u)^2 - \nabla u_k\cdot\partial_k u) - \operatorname{div}(\partial_i u\operatorname{div} u), \end{aligned}$$

the second conclusion follows. □

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