GLOBAL SMALL SOLUTIONS OF HEAT CONDUCTIVE COMPRESSIBLE NAVIER-STOKES EQUATIONS WITH VACUUM: SMALLNESS ON SCALING INVARIANT QUANTITY

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ABSTRACT. In this paper, we consider the Cauchy problem to the heat conductive compressible Navier-Stokes equations in the presence of vacuum and with vacuum far field. Global well-posedness of strong solutions is established under the assumption, among some other regularity and compatibility conditions, that the scaling invariant quantity $\|\rho_0\|_\infty(\|\rho_0\|_3+\|\rho_0\|_\infty^2\|\sqrt{\rho_0}u_0\|_2^2)(\|\nabla u_0\|_2^2+\|\rho_0\|_\infty\|\sqrt{\rho_0}E_0\|_2^2)$ is sufficiently small, with the smallness depending only on the parameters $R,\gamma,\mu,\lambda,$ and κ in the system. Notably, the smallness assumption is imposed on the above scaling invariant quantity exclusively, and it is independent of any norms of the initial data, which is different from the existing papers, see, e.g., [21, 22, 58]. The total mass can be either finite or infinite. A new equation for the density, more precisely for the cubic of the density, derived from combing the continuity and momentum equations, is employed to get the $L^\infty_t(L^3)$ type estimate of the density.

1. Introduction

In this paper, we consider the following heat conductive compressible Navier-Stokes equations for the ideal gas:

$$\partial_t\rho+\operatorname{div}(\rho u)=0, \tag{1.1}$$

$$\rho(\partial_t u+(u\cdot\nabla)u)-\mu\Delta u-(\mu+\lambda)\nabla\operatorname{div}u+\nabla p=0, \tag{1.2}$$

$$c_v\rho(\partial_t\theta+u\cdot\nabla\theta)+p\operatorname{div}u-\kappa\Delta\theta=\mathcal{Q}(\nabla u), \tag{1.3}$$

in $\mathbb{R}^3\times(0,\infty)$, where the unknowns $\rho\geq 0, u\in\mathbb{R}^3$, and $\theta\geq 0$, respectively, represent the density, velocity, and absolute temperature, $p=R\rho\theta$, with positive constant R, is the pressure, $c_v>0$ is a constant, constants μ and λ are the bulk and shear viscous coefficients, respectively, positive constant κ is the heat conductive coefficient, and

$$\mathcal{Q}(\nabla u)=\frac{\mu}{2}|\nabla u+(\nabla u)^t|^2+\lambda(\operatorname{div}u)^2,$$

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with $(\nabla u)^t$ being the transpose of ∇u. The viscous coefficients μ and λ satisfy the physical constraints

$$\mu > 0, \quad 2\mu + 3\lambda \geq 0.$$

The additional assumption $2\mu > \lambda$ will also be use in this paper.

Due to their fundamental importance in the fluid dynamics, extensive studies have been carried out and many developments have been achieved on the compressible Navier-Stokes equations in the last seventy years. The mathematical studies on the compressible Navier-Stokes equations started with the uniqueness results by Graffi [18] in 1953 for barotropic fluid and by Serrin [51] in 1959 for general fluids, and the local existence result by Nash [49] in 1962 for the Cauchy problem. Since then, comprehensive mathematical theories have been developed for the compressible Navier-Stokes equations.

The mathematical theory for the compressible Navier-Stokes equations in 1D is satisfied and, in particular, the corresponding global well-posedness, for arbitrary large initial data, and the initial density can either be uniformly positive or only nonnegative (that is, it can vanish on some subset of the domain). For the case that the initial density is uniformly positive, the global well-posedness of strong solutions, with large initial data, was first proved in [28], for the isentropic case, and later in [30], for the general case, and the corresponding large time behavior was recently proved in [36], see also [2, 27, 29, 61, 62] for some related results. For the case that the initial density contains vacuum, the corresponding global well-posedness of strong solutions was recently proved by the author and his collaborator, see [35, 38, 39].

Compared with the one dimensional case, the mathematical theory for the multi-dimensional case is far from satisfied and, in particular, some basic problems such as the global existence of strong solutions and uniqueness of weak solution are still unknown. For the case that the initial density is uniformly positive, the local well-posedness was proved long time ago, see [24, 43, 49, 52, 53, 55] and, in particular, the inflow and outflow were allowed in [43]; however, the general global well-posedness is still unknown. Global well-posedness of strong solutions with small initial data was first proved in [44–47], and later further developed in many papers, see, e.g., [3, 7, 11–14, 19, 31, 50, 54]. For the case that the initial density allows vacuum, global existence of weak solutions was first proved in [41, 42], see [1, 15–17, 26] for further developments, but the uniqueness is still an open problem. Local well-posedness of strong solutions was proved in [8–10], and the global well-posedness, with small initial data, was proved in [22], and see [21, 37, 58] for further developments.

The aim of this paper is to establish the global existence of strong solutions to the Cauchy problem of (1.1)–(1.3), under some smallness assumptions on the initial data, in the presence of initial vacuum, and with vacuum far field. The main novelty of this paper is that the smallness assumption is imposed on some quantities that are scaling invariant with respect to the following scaling transformation:

$$(\rho_{0\lambda}(x), u_{0\lambda}(x), \theta_{0\lambda}(x)) = (\rho_0(\lambda x), \lambda u_0(\lambda x), \lambda^2\theta_0(\lambda x)), \quad \forall \lambda \neq 0. \tag{1.4}$$

This scaling transformation on the initial data inheres in the following natural scaling invariant property of system (1.1)–(1.3):

$$\rho_\lambda(x,t)=\rho(\lambda x,\lambda^2t),\quad u_\lambda(x,t)=\lambda u(\lambda x,\lambda^2t),\quad \theta_\lambda(x,t)=\lambda^2\theta(\lambda x,\lambda^2t),$$

that is, if (ρ,u,θ) is a solution, with initial data (ρ_0,u_0,θ_0), then $(\rho_\lambda,u_\lambda,\theta_\lambda)$ is also a solution, for any nonzero λ, but with initial data $(\rho_{0\lambda},u_{0\lambda},\theta_{0\lambda})$.

The reason for us to focus on the smallness assumptions on the scaling invariant quantities, rather than on those not, is the following fact: if assuming that $\mathscr{M}$ is a functional, satisfying

$$\mathscr{M}(\rho_{0\lambda},u_{0\lambda},\theta_{0\lambda})=\lambda^\ell(\rho_0,u_0,\theta_0),\quad \forall\lambda\neq0,\quad \text{for some constant } \ell\neq0,$$

and that the global well-posedness holds, for any initial data (ρ_0,u_0,θ_0), such that $\mathscr{M}(\rho_0,u_0,\theta_0)\leq\varepsilon_0$, for some $\varepsilon_0>0$ depending only on the parameters of the system, then, by suitably choosing the scaling parameter λ, one can show that the system is actually globally well-posed, for arbitrary large initial data; however, this global well-posedness for arbitrary large initial data is far from what we already know.

Before stating the main results, we first clarify some necessary notations being used throughout this paper. For $1\leq q\leq\infty$ and positive integer m, we use $L^q=L^q(\mathbb{R}^3)$ and $W^{1,q}=W^{m,q}(\mathbb{R}^3)$ to denote the standard Lebesgue and Sobolev spaces, respectively, and in the case that $q=2$, we use H^m instead of $W^{m,2}$. For simplicity, we also use notations L^q and H^m to denote the N product spaces $(L^q)^N$ and $(H^m)^N$, respectively. We always use $\|u\|_q$ to denote the L^q norm of u. For shortening the expressions, we sometimes use $\|(f_1,f_2,\cdots,f_n)\|_X$ to denote the sum $\sum_{i=1}^N\|f_i\|_X$ or its equivalent norm $\left(\sum_{i=1}^N\|f_i\|_X^2\right)^{\frac12}$. We denote

$$D^{k,r}=\left\{u\in L^1_{loc}(\mathbb{R}^3)\,\middle|\,\|\nabla^ku\|_r<\infty\right\},\quad D^k=D^{k,2},$$

$$D^1_0=\left\{u\in L^6\,\middle|\,\|\nabla u\|_2<\infty\right\}.$$

For simplicity of notations, we adopt the notation

$$\int fdx=\int_{\mathbb{R}^3}fdx.$$

We are now ready to state the main result of this paper.

Theorem 1.1. *Assume $2\mu>\lambda$ and let $q\in(3,6]$ be a fixed constant. Assume that the initial data (ρ_0,u_0,θ_0) satisfies*

$$\rho_0,\theta_0\geq0,\quad \rho\leq\bar\rho,\quad \rho_0\in H^1\cap W^{1,q},\quad \sqrt{\rho_0}\theta_0\in L^2,\quad (u_0,\theta_0)\in D^1_0\cap D^2,$$
$$-\mu\Delta u_0-(\mu+\lambda)\nabla\,div\,u_0+\nabla p_0=\sqrt{\rho_0}g_1,\quad \kappa\Delta\theta_0+\mathcal{Q}(\nabla u)=\sqrt{\rho_0}g_2,$$

for a positive constant $\bar\rho$ and some $(g_1,g_2)\in L^2$, where $p_0=R\rho_0\theta_0$.

Then, there is a positive number ε_0 depending only on $R, \gamma, \mu, \lambda,$ and κ, such that system (1.1)–(1.3), with initial data (ρ_0, u_0, θ_0), has a unique global solution (ρ, u, θ), satisfying

$$\rho \in C([0,\infty); H^1 \cap W^{1,q}), \quad \rho_t \in C([0,\infty)L^2 \cap L^q),$$
$$(u,\theta) \in C([0,\infty); D_0^1 \cap D^2) \cap L^2_{loc}([0,\infty); D^{2,q}),$$
$$(u_t, \theta_t) \in L^2_{loc}([0,\infty); D_0^1), \quad (\sqrt{\rho} u_t, \sqrt{\rho}\theta_t) \in L^\infty_{loc}([0,\infty); L^2),$$

provided

$$\mathscr{N}_0 := \bar\rho(\|\rho_0\|_3 + \bar\rho^2 \|\sqrt{\rho_0} u_0\|_2^2)(\|\nabla u_0\|_2^2 + \bar\rho \|\sqrt{\rho_0} E_0\|_2^2) \leq \varepsilon_0.$$

Remark 1.1. *(i) One can easily check that the quantity $\mathscr{N}_0$ in Theorem 1.1 is scaling invariant, with respect to the scaling transformation (1.4). Therefore, Theorem 1.1 provides the global well-posedness of system (1.1)–(1.3) under some smallness assumption on a scaling invariant quantity, for the case that the vacuum is allowed. We were not aware of such kind results before for the compressible Navier-Stokes equations, even for the isentropic case.*

(ii) Global well-posedness of strong solutions to the Cauchy problem of system (1.1)–(1.3) in the presence of vacuum has been proved in [21] and [58], with non-vacuum far field and vacuum far field, respectively. The assumptions concerning the smallness in [21] and [58] are imposed as

$$\begin{aligned} C_0 &= \int \left(\frac{\rho_0}{2}|u_0|^2 + R(\rho_0 \log\rho_0 - \rho_0 + 1) + \frac{R}{\gamma-1}\rho_0(\theta_0 - \log\theta_0 + 1) \right) dx \\ &\leq \varepsilon_0 = \varepsilon_0(\|\rho_0\|_\infty, \|\theta_0\|_\infty, \|\nabla u_0\|_2, R, \gamma, \mu, \lambda, \kappa) \end{aligned}$$

and

$$\int \rho_0 dx \leq \varepsilon_0 = \varepsilon_0(\|\rho_0\|_\infty, \|\sqrt{\rho_0}\theta_0\|_2, \|\nabla u_0\|_2, R, \gamma, \mu, \lambda, \kappa),$$

respectively. However, since the explicit dependence of ε_0 on $\|\rho_0\|_\infty, \|\theta_0\|_\infty, \|\sqrt{\rho_0}\theta_0\|_2$, and $\|\nabla u_0\|_2$ are not derived in [21, 58], the scaling invariant quantities, on which the smallness guarantees the global well-posedness, can not be identified there.

(iii) Comparing with the global well-posedness result in [58], our result, Theorem 1.1, allows the initial mass to be infinite. This will be crucial for obtaining the global entropy-bounded solutions in our forthcoming paper [40].

Comparing with the isentropic case considered in [22], the additional difficulty for studying the global well-posedness of the full compressible Navier-Stokes equations is that the following basic energy inequality does not provide any dissipation estimates:

$$\int \rho \left(\frac{|u|^2}{2} + c_v\theta \right) dx = \int \rho_0 \left(\frac{|u_0|^2}{2} + c_v\theta_0 \right) dx.$$

Note that the dissipation estimates of the form $\int_0^T \|\nabla u\|_2^2 \leq C$, which can be guaranteed by the basic energy estimates for the isentropic case, is crucial in the arguments of [22]. To overcome this difficulty, some kinds of dissipative estimates were recovered for the full compressible Navier-Stokes equations in [21] and [58], for the cases that with non-vacuum and vacuum far field, respectively, by using the entropy inequality and the conservation of mass. Notcing that the entropy inequality (a crucial tool in [21]) holds only for the non-vacuum far field case and the finiteness of mass is crucial in [58], and recalling that we consider the case that with vacuum far field and allowing possible infinite mass, the arguments in [21, 58] do not work for our case.

A crucial ingredient of obtaining the dissipative estimates in this paper is the following new equation (see the proof in Proposition 2.4)

$$\frac{2\mu+\lambda}{2}(\partial_t\rho^3+\mathrm{div}\,(u\rho^3))+\rho^3p+\rho^3\Delta^{-1}\mathrm{div}\,(\rho u)_t+\rho^3\Delta^{-1}\mathrm{div}\,\mathrm{div}\,(\rho u\otimes u)=0,$$

which is derived by combining the continuity equation and the momentum equation; note that the temperature equation plays no role in deriving this. This equation is employed to get the $L^\infty(0,T;L^3)$ estimate of ρ. Comparing with the continuity equation, the main advantage of the above is that it enables us to get the time independent $L^\infty(0,T;L^3)$ estimate of ρ, without appealing to the L^∞ of $\mathrm{div}\, u$. In fact, the above equation leads to the following kind of inequality

$$\begin{aligned}&\sup_{0\le t\le T}\|\rho\|_3^3+\int_0^T\int\rho^3p\,dxdt\\ \le\ & C\sup_{0\le t\le T}(\|\rho\|_\infty^{\frac23}\|\sqrt\rho u\|_2^{\frac13}\|\sqrt\rho|u|^2\|_2^{\frac13}\|\rho\|_3^3)\\ &+C\int_0^T\|\rho\|_\infty^2\|\rho\|_3^2\|\nabla u\|_2^2dt+C\|\rho_0\|_3^3,\end{aligned}\tag{1.5}$$

see Proposition 2.4. It seems to us that such kind energy inequality is new even for the isentropic case. A key point of the above estimate is that the time integration term $\int_0^T\|\rho\|_\infty^2\|\rho\|_3^2\|\nabla u\|_2^2dt$ is quadratic in $\|\nabla u\|_2^2$ and, thus, can be expected to be time independent, due to the presence of viscosity in the system. Moreover, the above inequality also provides time independent estimate for $\int_0^T\int\rho^3pdxdt$, which, thought is not used at all in this paper, will be expectably useful for studying the large time behavior of the global solutions. Note that if using the density equation, i.e., (1.1), to perform the same type estimate, then the corresponding inequality reads as

$$\sup_{0\le t\le T}\|\rho\|_3^3\le\|\rho_0\|_3^3+2\int_0^T\int|\mathrm{div}u|\rho^3dxdt,$$

from which the time independent estimate for $\|\rho\|_3$ can not be derived, if without the help of some suitable decaying estimates of the solutions.

The energy inequality (1.5) motivates us to impose a smallness condition on the quantity $\|\rho_0\|_\infty^2\|\sqrt{\rho_0}u_0\|_2\|\sqrt{\rho_0}|u_0|^2\|_2$ (this is one of the terms of $\mathscr{N}_0$ in Theorem 1.1) to get the bound of $\|\rho\|_3$. Inequality (1.5) also suggests us to carry out the estimates on $\|\sqrt\rho u\|_{L^\infty(0,T;L^2)}$, $\|\sqrt\rho E\|_{L^\infty(0,T;L^2)}$, and $\|\rho\|_{L^\infty(0,T;L^\infty)}$, which are performed in Propositions 2.2, 2.3, and 2.6, respectively. Higher order estimates are required in the estimate for $\|\rho\|_{L^\infty(0,T;L^\infty)}$, and they are carried out with the help of $\omega = \nabla \times u$ and $G = (2\mu + \lambda)\mathrm{div}\, u - p$, see Proposition 2.5. Combining Proposition 2.2, 2.3, 2.4, 2.6, and 2.5, by continuity arguments, we are able to get time-independent estimate on a scaling invariant quantity $\mathscr{N}_T$ (its expression is given in Proposition 2.7) as long as it is sufficiently small at the initial time. With this a priori estimate for $\mathscr{N}_T$, one can further get the time-independent a priori estimates of $\|\nabla u\|_{L^\infty(0,T;L^2)}$ and $\|\rho\|_{L^\infty(0,T;L^\infty)}$, based on which, the blow-up criteria apply, and, thus, the global well-posedness follows.

Throughout this paper, we use C to denote a general positive constant which may vary from line to line. $A \lesssim B$ means $A \leq CB$ for some positive constant C.

2. A priori estimates

This section is devoted to deriving some a priori estimates for the solutions to the Cauchy problem of system (1.1)–(1.3). The existence of solution is guaranteed by the following local well-posedness result proved in [10]:

Proposition 2.1. *Under the conditions in Theorem 1.1, there is a positive time T_*, such that system (1.1)–(1.3), with initial data (ρ_0, u_0, θ_0), has a unique solution (ρ, u, θ), on $\mathbb{R}^3 \times (0, T_*)$, satisfying*

$$\begin{gathered}\rho \in C([0,T_*]; H^1 \cap W^{1,q}), \quad \rho_t \in C([0,T_*]L^2 \cap L^q),\\ (u,\theta) \in C([0,T_*]; D_0^1 \cap D^2) \cap L^2(0,T_*; D^{2,q}),\\ (u_t,\theta_t) \in L^2(0,T_*; D_0^1), \quad (\sqrt{\rho}u_t, \sqrt{\rho}\theta_t) \in L^\infty_{loc}(0,T_*; L^2).\end{gathered}$$

In the rest of this section, we always assume that (ρ, u, θ), is a solution to system (1.1)–(1.3), on $\mathbb{R}^3 \times (0, T)$, for some positive time T, satisfying the regularities in Proposition 2.1 with T_* there replaced by T, with initial data (ρ_0, u_0, θ_0).

2.1. **Energy inequalities.**

Proposition 2.2. *The following estimate holds:*

$$\sup_{0\leq t\leq T} \|\sqrt{\rho}u\|_2^2 + \int_0^T \|\nabla u\|_2^2 dt \leq C\|\sqrt{\rho_0}u_0\|_2^2 + C\int_0^T \|\rho\|_3^2 \|\nabla\theta\|_2^2 dt,$$

for a positive constant C depending only on R, γ, μ, λ, and κ.

Proof. Multiplying (1.2) by u, integration the resultant over $\mathbb{R}^3$, and noticing that $\mu + \lambda > 0$, it follows from integration by parts and the Cauchy inequality that

$$\begin{aligned} &\frac{d}{dt}\|\sqrt{\rho}u\|_2^2 + \mu\|\nabla u\|_2^2 + (\mu+\lambda)\|\mathrm{div}\, u\|_2^2 \\ \leq\ & R\|\rho\|_3\|\theta\|_6\|\mathrm{div} u\|_2 \leq C\|\rho\|_3\|\nabla\theta\|_2\|\mathrm{div}\, u\|_2 \\ \leq\ & (\mu+\lambda)\|\mathrm{div}\, u\|_2^2 + C\|\rho\|_3^2\|\nabla\theta\|_2^2, \end{aligned}$$

from which, the conclusion follows by integrating in t. □

Proposition 2.3. *Assume that $2\mu > \lambda$. Then, the following estimate holds:*

$$\begin{aligned}
&\sup_{0\leq t\leq T}\|\sqrt{\rho}E\|_2^2+\int_0^T(\|\nabla\theta\|_2^2+\||u|\nabla u\|_2^2)dt\\
\leq\ & C\|\sqrt{\rho_0}E_0\|_2^2+C\int_0^T\|\rho\|_\infty\|\rho\|_3^{\frac12}\|\sqrt{\rho}\theta\|_2\|(\nabla\theta,|u|\nabla u)\|_2^2dt,
\end{aligned}$$

for a positive constant C depending only on R,γ,μ,λ, and κ, where $E=\frac{|u|^2}{2}+c_v\theta$.

Proof. One can verify

$$\rho(\partial_tE+u\cdot\nabla E)+\mathrm{div}\,(up)-\kappa\Delta\theta=\mathrm{div}\,(\mathcal{S}\cdot u), \tag{2.1}$$

where $\mathcal{S}=\mu(\nabla u+(\nabla u)^t)+\lambda\mathrm{div}\,uI$. Multiplying (2.1) by E, integrating the resultant over $\mathbb{R}^3$, it follows from integration by parts that

$$\begin{aligned}
&\frac12\frac{d}{dt}\|\sqrt{\rho}E\|_2^2+\kappa c_v\|\nabla\theta\|_2^2\\
=\ &\int\left[-\frac{\kappa}{2}\nabla\theta\cdot\nabla|u|^2+(up-\mathcal{S}\cdot u)\cdot\left(c_v\nabla\theta+\frac{\nabla|u|^2}{2}\right)\right]dx\\
\leq\ &\frac{c_v\kappa}{2}\|\nabla\theta\|_2^2+C\||u|\nabla u\|_2^2+C\int\rho^2\theta^2|u|^2dx,
\end{aligned}$$

which yields

$$\frac{d}{dt}\|\sqrt{\rho}E\|_2^2+\kappa c_v\|\nabla\theta\|_2^2\lesssim\||u|\nabla u\|_2^2+\int\rho^2\theta^2|u|^2dx. \tag{2.2}$$

Multiplying (1.2) by $|u|^2u$, integrating the resultant over $\mathbb{R}^3$, it follows from integration by parts that

$$\begin{aligned}
\frac14\frac{d}{dt}\|\sqrt{\rho}|u|^2\|_2^2-\int(\mu\Delta u+(\mu+\lambda)\nabla\mathrm{div}\,u)\cdot|u|^2udx&=-\int p\mathrm{div}\,(|u|^2u)dx\\
&\leq\left(\mu-\frac{\lambda}{2}\right)\int|u|^2|\nabla u|^2dx+C\int\rho^2\theta^2|u|^2dx,
\end{aligned}$$

Some elementary calculations show that

$$-\int(\mu\Delta u+(\mu+\lambda)\nabla\mathrm{div}\,u)\cdot|u|^2udx\geq(2\mu-\lambda)\int|u|^2|\nabla u|^2dx.$$

Combining the above two inequalities leads to

$$\frac{d}{dt}\|\sqrt{\rho}|u|^2\|_2^2+2(2\mu-\lambda)\||u|\nabla u\|_2^3\lesssim\int\rho^2\theta^2|u|^2dx. \tag{2.3}$$

Multiplying (2.3) by a sufficient large number K depending only on R,γ,μ,λ, and κ, and summing the resultant with (2.2), one obtains

$$\frac{d}{dt}(\|\sqrt{\rho}E\|_2^2+K\|\sqrt{\rho}|u|^2\|_2^2)+\kappa c_v\|\nabla\theta\|_2^2+(2\mu-\lambda)K\||u|\nabla u\|_2^2\lesssim\int\rho^2\theta^2|u|^2dx,$$

from which, noticing that the Hölder and Sobolev inequalities yield

$$
\begin{aligned}
\int \rho^2\theta^2|u|^2dx &\leq \|\sqrt{\rho}\theta\|_2\|\theta\|_6\||u|^2\|_6\|\rho\|_{\frac{3}{2}}^{9} \\
&\lesssim \|\sqrt{\rho}\theta\|_2\|\nabla\theta\|_2\|\nabla|u|^2\|_2\|\rho\|_\infty\|\rho\|_3^{\frac{1}{2}}, \qquad (2.4)
\end{aligned}
$$

one obtains

$$
\begin{aligned}
\frac{d}{dt}(\|\sqrt{\rho}E\|_2^2 + K\|\sqrt{\rho}|u|^2\|_2^2) + \kappa c_v\|\nabla\theta\|_2^2 + (2\mu-\lambda)K\||u|\nabla u\|_2^2 \\
\lesssim \|\rho\|_\infty\|\rho\|_3^{\frac{1}{2}}\|\sqrt{\rho}\theta\|_2\|\nabla\theta\|_2\|\nabla|u|^2\|_2.
\end{aligned}
$$

Integrating this in t and using the Cauchy inequality, the conclusion follows. ☐

The following proposition on the $L^\infty(0,T;L^3)$ estimate for ρ is crucial in the proof of this paper. The key point, as mentioned in the introduction, is that the integral in the time integration on the right-hand side of the inequality in the next proposition is quadratic in ∇u, but if using the density equation to derive the $L^\infty(0,T;L^3)$ estimate for ρ, the corresponding term is linear in ∇u.

Proposition 2.4. *The following estimate holds*

$$
\begin{aligned}
\sup_{0\leq t\leq T}\|\rho\|_3^3 + \int_0^T\int \rho^3 p\,dxdt &\leq C\sup_{0\leq t\leq T}(\|\rho\|_\infty^{\frac{2}{3}}\|\sqrt{\rho}u\|_2^{\frac{1}{3}}\|\sqrt{\rho}|u|^2\|_2^{\frac{1}{3}}\|\rho\|_3^3) \\
&\quad + C\int_0^T\|\rho\|_\infty^2\|\rho\|_3^2\|\nabla u\|_2^2dt + C\|\rho_0\|_3^3,
\end{aligned}
$$

for a positive constant C depending only on $R,\gamma,\mu,\lambda,$ and κ.

Proof. Applying the operator $\Delta^{-1}\mathrm{div}$ to (1.2) yields

$$
\Delta^{-1}\mathrm{div}\,(\rho u)_t + \Delta^{-1}\mathrm{div}\,\mathrm{div}\,(\rho u\otimes u) - (2\mu+\lambda)\mathrm{div}\,u + p = 0. \qquad (2.5)
$$

Multiplying the above equation by ρ^3 and noticing that

$$
\partial_t\rho^3 + \mathrm{div}\,(u\rho^3) + 2\mathrm{div}\,u\rho^3 = 0,
$$

one obtains

$$
\frac{2\mu+\lambda}{2}(\partial_t\rho^3 + \mathrm{div}\,(u\rho^3)) + \rho^3 p + \rho^3\Delta^{-1}\mathrm{div}\,(\rho u)_t + \rho^3\Delta^{-1}\mathrm{div}\,\mathrm{div}\,(\rho u\otimes u) = 0. \qquad (2.6)
$$

Integrating the above equation over $\mathbb{R}^3$ yields

$$
\begin{aligned}
\frac{2\mu+\lambda}{2}\frac{d}{dt}\|\rho\|_3^3 + \int\rho^3 p\,dx + \int\rho^3\Delta^{-1}\mathrm{div}\,(\rho u)_t dx \\
= -\int\rho^3\Delta^{-1}\mathrm{div}\,\mathrm{div}\,(\rho u\otimes u)dx. \qquad (2.7)
\end{aligned}
$$

Using (1.1), one deduces

$$\begin{aligned}
&\int \rho^3\Delta^{-1}\mathrm{div}\,(\rho u)_t dx \\
= \; &\frac{d}{dt}\int \rho^3\Delta^{-1}\mathrm{div}\,(\rho u)dx + \int [\mathrm{div}\,(\rho^3 u) + 2\mathrm{div}\,u\rho^3]\Delta^{-1}\mathrm{div}\,(\rho u)dx \\
= \; &\int [2\mathrm{div}\,u\rho^3\Delta^{-1}\mathrm{div}\,(\rho u) - \rho^3 u\cdot\nabla\Delta^{-1}\mathrm{div}\,(\rho u)]dx \\
&+\frac{d}{dt}\int \rho^3\Delta^{-1}\mathrm{div}\,(\rho u)dx.
\end{aligned}$$

Therefore, it follows from (2.7) that

$$\begin{aligned}
&\frac{d}{dt}\int \left(\frac{2\mu+\lambda}{2} + \Delta^{-1}\mathrm{div}\,(\rho u)\right)\rho^3 dx + \int \rho^3 p dx \\
= &\int \left[\rho^3(u\cdot\nabla\Delta^{-1}\mathrm{div}\,(\rho u) - \Delta^{-1}\mathrm{divdiv}\,(\rho u\otimes u)) - 2\mathrm{div}\,u\rho^3\Delta^{-1}\mathrm{div}\,(\rho u)\right]dx. \quad (2.8)
\end{aligned}$$

Noticing that

$$\begin{aligned}
\|\nabla\Delta^{-1}\mathrm{div}\,(\rho u)\|_2 &\lesssim \|\rho u\|_2 \lesssim \|\rho\|_3\|u\|_6, \\
\|\Delta^{-1}\mathrm{div}\,\mathrm{div}\,(\rho u\otimes u)\|_{\frac32} &\lesssim \|\rho|u|^2\|_{\frac32} \lesssim \|\rho\|_3\|u\|_6^2,
\end{aligned}$$

it follows from the Hölder and Sobolev embedding inequality that

$$\begin{aligned}
&\int \rho^3(u\cdot\nabla\Delta^{-1}\mathrm{div}\,(\rho u) - \Delta^{-1}\mathrm{divdiv}\,(\rho u\otimes u))dx \\
&\qquad\lesssim \|\rho\|_9^3\|\rho\|_3\|u\|_6^2 \lesssim \|\rho\|_\infty^2\|\rho\|_3^2\|\nabla u\|_2^2. \quad (2.9)
\end{aligned}$$

By the Sobolev embedding and elliptic estimates

$$\begin{aligned}
\|\Delta^{-1}\mathrm{div}\,(\rho u)\|_6 &\lesssim \|\nabla\Delta^{-1}\mathrm{div}\,(\rho u)\|_2 \lesssim \|\rho u\|_2 \\
&\lesssim \|\rho\|_3\|u\|_6 \lesssim \|\rho\|_3\|\nabla u\|_2,
\end{aligned}$$

and, thus, the Hölder inequality yields

$$\left|\int \mathrm{div}\,u\rho^3\Delta^{-1}\mathrm{div}\,(\rho u)dx\right| \lesssim \|\mathrm{div}u\|_2\|\rho\|_9^3\|\rho\|_3\|\nabla u\|_2 \lesssim \|\rho\|_\infty^2\|\rho\|_3^2\|\nabla u\|_2^2. \quad (2.10)$$

By the Gagliardo-Nirenberg inequality and using the elliptic estimates, it follows

$$\begin{aligned}
\|\Delta^{-1}\mathrm{div}\,(\rho u)\|_\infty &\lesssim \|\Delta^{-1}\mathrm{div}\,(\rho u)\|_6^{\frac13}\|\nabla\Delta^{-1}\mathrm{div}\,(\rho u)\|_4^{\frac23} \\
&\lesssim \|\rho u\|_2^{\frac13}\|\rho u\|_4^{\frac23} \lesssim \|\rho\|_\infty^{\frac23}\|\sqrt{\rho}u\|_2^{\frac13}\|\sqrt{\rho}|u|^2\|_2^{\frac13}. \quad (2.11)
\end{aligned}$$

Integrating (2.8) in t, using (2.9)–(2.11), and by some straightforward calculations, the conclusion follows. □

Proposition 2.5. *Assume*

$$\sup_{0\leq t\leq T}\|\rho\|_\infty\leq 4\bar\rho.$$

Then, there is a positive constant C depending only on R,γ,μ,λ, and κ, such that

$$\begin{aligned}&\sup_{0\leq t\leq T}\|\nabla u\|_2^2+\int_0^T\left\|\left(\sqrt\rho u_t,\frac{\nabla G}{\sqrt{\bar\rho}},\frac{\nabla\omega}{\sqrt{\bar\rho}}\right)\right\|_2^2dt\\ \leq\;& C\|\nabla u_0\|_2^2+C\bar\rho\sup_{0\leq t\leq T}\|\sqrt\rho\theta\|_2^2+C\bar\rho^3\int_0^T\|\nabla u\|_2^4\|(\nabla u,\sqrt{\bar\rho}\sqrt\rho\theta)\|_2^2dt\\ &+C\int_0^T(\bar\rho+\bar\rho^2\|\rho\|_3^{\frac12}\|\sqrt\rho\theta\|_2)\|(\nabla\theta,|u|\nabla u)\|_2^2dt,\end{aligned}$$

where $G=(2\mu+\lambda)\,\mathrm{div}\,u-p$ and $\omega=\nabla\times u$.

Proof. Multiplying (1.2) by u_t, integrating the resultant over $\mathbb{R}^3$, it follows from integration by parts that

$$\begin{aligned}\frac12\frac{d}{dt}(\mu\|\nabla u\|_2^2+(\mu+\lambda)\|\mathrm{div}\,u\|_2^2)-\int p\mathrm{div}\,u_tdx+\|\sqrt\rho u_t\|_2^2&\\ =-\int\rho(u\cdot\nabla)u\cdot u_tdx.&\qquad(2.12)\end{aligned}$$

Noticing that $\mathrm{div}\,u=\frac{G+p}{2\mu+\lambda}$, it follows

$$\begin{aligned}&-\int p\mathrm{div}\,u_tdx=-\frac{d}{dt}\int p\mathrm{div}\,udx+\int p_t\mathrm{div}\,udx\\ =\;&-\frac{d}{dt}\int p\mathrm{div}\,udx+\frac{1}{2(2\mu+\lambda)}\frac{d}{dt}\|p\|_2^2+\frac{1}{2\mu+\lambda}\int p_tGdx.\qquad(2.13)\end{aligned}$$

Noticing that (1.3) implies

$$p_t=(\gamma-1)(\mathcal{Q}(\nabla u)-p\mathrm{div}\,u+\kappa\Delta\theta)-\mathrm{div}\,(up),$$

and, thus, integration by parts gives

$$\int p_tGdx=\int[(\gamma-1)(\mathcal{Q}(\nabla u)-p\mathrm{div}\,u)G+(up-\kappa(\gamma-1)\nabla\theta)\cdot\nabla G]dx.\qquad(2.14)$$

Substituting (2.14) into (2.13), then the resultant into (2.12), and noticing that $\|\nabla u\|_2^2=\|\omega\|_2^2+\|\mathrm{div}\,u\|_2^2$, by some straightforward calculations, one obtains

$$\begin{aligned}&\frac12\frac{d}{dt}\left(\mu\|\omega\|_2^2+\frac{\|G\|_2^2}{2\mu+\lambda}\right)+\|\sqrt\rho u_t\|_2^2\\ =\;&-\int\rho(u\cdot\nabla)u\cdot u_tdx+\frac{1}{2\mu+\lambda}\int(\kappa(\gamma-1)\nabla\theta-up)\cdot\nabla Gdx\\ &-\frac{\gamma-1}{2\mu+\lambda}\int(\mathcal{Q}(\nabla u)-p\mathrm{div}\,u)Gdx.\qquad(2.15)\end{aligned}$$

Use $\Delta u = \nabla \mathrm{div}\, u - \nabla\times\nabla\times u$ to rewrite (1.2) as

$$\rho(u_t + u\cdot\nabla u) = \nabla G - \mu\nabla\times\omega. \tag{2.16}$$

Testing this by ∇G, noticing $\int \nabla G\cdot\nabla\times\omega dx = 0$, and recalling $\|\rho\|_\infty \le 4\bar\rho$ yield

$$\begin{aligned}\|\nabla G\|_2^2 &= \int \rho(u_t + u\cdot\nabla u)\cdot\nabla G dx\\ &\le \int\left(\frac{|\nabla G|^2}{2} + 2\bar\rho\rho|u_t|^2\right)dx + \int \rho u\cdot\nabla u\cdot\nabla G dx,\end{aligned}$$

which gives

$$\frac{\|\nabla G\|_2^2}{16\bar\rho} \le \frac14\|\sqrt{\rho}u_t\|_2^2 + \frac{1}{8\bar\rho}\int \rho(u\cdot\nabla)u\cdot\nabla G dx. \tag{2.17}$$

Similarly

$$\frac{\mu^2\|\nabla \omega\|_2^2}{16\bar\rho} \le \frac14\|\sqrt{\rho}u_t\|_2^2 + \frac{1}{8\bar\rho}\int \rho(u\cdot\nabla)u\cdot\nabla\times\omega dx. \tag{2.18}$$

Thanks to (2.17) and (2.18), one obtains from (2.15) that

$$\begin{aligned}&\frac12\frac{d}{dt}\left(\mu\|\omega\|_2^2 + \frac{\|G\|_2^2}{2\mu+\lambda}\right) + \frac12\|\sqrt{\rho}u_t\|_2^2 + \frac{1}{16\bar\rho}(\|\nabla G\|_2^2 + \mu^2\|\nabla\omega\|_2^2)\\ \le& C\int \rho|u||\nabla u|\left[|u_t| + \frac{1}{\bar\rho}(\nabla G| + |\nabla\omega|)\right]dx + C\int(|\nabla\theta| + \rho\theta|u|)|\nabla G|dx\\ &+ C\int(|\nabla u|^2 + \rho\theta|\nabla u|)|G|dx =: I_1 + I_2 + I_3.\end{aligned} \tag{2.19}$$

The terms I_1, I_2, and I_3 are estimated as follows. For I_1, by the Hölder and Young inequalities, one obtains

$$\begin{aligned}I_1 &\lesssim \sqrt{\bar\rho}\||u|\nabla u\|_2\|\sqrt{\rho}u_t\|_2 + \||u|\nabla u\|_2(\|\nabla G\|_2 + \|\nabla\omega\|_2)\\ &\le \frac16\left[\frac12\|\sqrt{\rho}u_t\|_2^2 + \frac{1}{16\bar\rho}(\|\nabla G\|_2^2 + \mu^2\|\nabla\omega\|_2^2)\right] + C\bar\rho\||u|\nabla u\|_2^2.\end{aligned}$$

Recalling (2.4), it follows from the Hölder and Young inequalities that

$$\begin{aligned}I_2 &\lesssim \|\nabla\theta\|_2\|\nabla G\|_2 + \|\rho\theta u\|_2\|\nabla G\|_2\\ &\lesssim \|\nabla\theta\|_2\|\nabla G\|_2 + \sqrt{\bar\rho}\|\rho\|_3^{\frac14}\|\sqrt{\rho}\theta\|_2^{\frac12}\|\nabla\theta\|_2^{\frac12}\|\nabla|u|^2\|_2^{\frac12}\|\nabla G\|_2\\ &\le \frac{\|\nabla G\|_2^2}{96\bar\rho} + C\left(\bar\rho^2\|\rho\|_3^{\frac12}\|\sqrt{\rho}\theta\|_2 + \bar\rho\right)(\|\nabla\theta\|_2^2 + \||u|\nabla u\|_2^2).\end{aligned}$$

The elliptic estimates and Sobolev embedding inequality yield

$$\begin{aligned}\|\nabla u\|_6 &\lesssim \|\nabla\times u\|_6 + \|\mathrm{div}\, u\|_6 \lesssim \|\omega\|_6 + \|G\|_6 + \|\rho\theta\|_6\\ &\lesssim \|\nabla\omega\|_2 + \|\nabla G\|_2 + \bar\rho\|\nabla\theta\|_2.\end{aligned} \tag{2.20}$$

Using (2.20), by the Hölder, Sobolev, and Young inequalities, one deduces

$$\begin{aligned} I_3 &\lesssim \|\nabla u\|_2\|\nabla u\|_6\|G\|_3+\|\nabla u\|_2\|\rho\theta\|_6\|G\|_3\\ &\lesssim C\|\nabla u\|_2(\|\nabla G\|_2+\|\nabla\omega\|_2+\bar\rho\|\nabla\theta\|_2)\|G\|_2^{\frac12}\|\nabla G\|_2^{\frac12}\\ &\quad+\bar\rho\|\nabla u\|_2\|\nabla\theta\|_2\|G\|_2^{\frac12}\|\nabla G\|_2^{\frac12}\\ &\leq \frac{1}{96\bar\rho}(\|\nabla G\|_2^2+\mu^2\|\nabla\omega\|_2^2)+C\bar\rho^3\|\nabla u\|_2^4\|G\|_2^2+C\bar\rho\|\nabla\theta\|_2^2. \end{aligned}$$

Substituting the estimates for $I_i, i=1,2,3$, into (2.19) yields

$$\begin{aligned} &\frac{d}{dt}\left(\mu\|\omega\|_2^2+\frac{\|G\|_2^2}{2\mu+\lambda}\right)+\frac12\|\sqrt{\rho}u_t\|_2^2+\frac{1}{16\bar\rho}(\|\nabla G\|_2^2+\mu^2\|\nabla\omega\|_2^2)\\ &\quad\lesssim(\bar\rho+\bar\rho^2\|\rho\|_3^{\frac12}\|\sqrt{\rho}\theta\|_2)(\|\nabla\theta\|_2^2+\||u|\nabla u\|_2^2)+\bar\rho^3\|\nabla u\|_2^4\|G\|_2^2, \end{aligned}$$

from which, integrating in t and using

$$\|\nabla u\|_2\lesssim\|\omega\|_2+\|G\|_2+\|\rho\theta\|_2\lesssim\|\omega\|_2+\|G\|_2+\sqrt{\bar\rho}\|\sqrt{\rho}\theta\|_2,$$

the conclusion follows by straightforward calculations. □

Proposition 2.6. *Assume*

$$\sup_{0\leq t\leq T}\|\rho\|_\infty\leq 4\bar\rho.$$

Then, there is a positive constant C depending only on R,γ,μ,λ, and κ, such that

$$\sup_{0\leq t\leq T}\|\rho\|_\infty \leq \|\rho_0\|_\infty e^{C\bar\rho^{\frac23}\sup_{0\leq t\leq T}\|\sqrt{\rho}u\|_2^{\frac13}\|\sqrt{\rho}|u|^2\|_2^{\frac13}+C\bar\rho\int_0^T\|\nabla u\|_2\|(\nabla G,\nabla\omega,\bar\rho\nabla\theta)\|_2dt}.$$

Proof. Denote $\mathcal{O}=\{x\in\mathbb{R}^3|\rho_0(x)=0\}$ and $\Omega=\{x\in\mathbb{R}^3|\rho_0(x)>0\}$. Let $X(x,t)$ be the particle path govern by the velocity field u and starting from x, that is

$$\partial_tX(x,t)=u(X(x,t),t),\quad X(x,0)=x.$$

Then $\rho(X(x,t),t)\equiv0$, for any $x\in\mathcal{O}$, and $\rho(X(x,t),t)>0$, for any $x\in\Omega$. One can verify that $\{X(x,t)|x\in\mathbb{R}^3\}=\mathbb{R}^3$, for any $t\in(0,T)$. Therefore

$$\sup_{x\in\mathbb{R}^3}\rho(x,t)=\sup_{x\in\mathbb{R}^3}\|\rho(X(x,t),t)\|_\infty=\sup_{x\in\Omega}\rho(X(x,t),t). \tag{2.21}$$

Rewrite (2.5) as

$$\begin{aligned} &\partial_t\Delta^{-1}\mathrm{div}\,(\rho u)+u\cdot\nabla\Delta^{-1}\mathrm{div}\,\mathrm{div}\,(\rho u)-(2\mu+\lambda)\mathrm{div}\,u+p\\ =\ &u\cdot\nabla\Delta^{-1}\mathrm{div}\,\mathrm{div}\,(\rho u)-\Delta^{-1}\mathrm{div}\,\mathrm{div}\,(\rho u\otimes u)=[u,\mathcal{R}\otimes\mathcal{R}](\rho u), \end{aligned} \tag{2.22}$$

where $\mathcal{R}$ is the Riesz transform on $\mathbb{R}^3$. Using the fact $\frac{d}{dt}(f(X(x,t),t))=(\partial_tf+u\cdot\nabla f)(X(x,t),t)$, it follows from (1.1) that

$$\frac{d}{dt}(\log\rho(X(x,t),t))=-\mathrm{div}\,u(X(x,t),t),\quad\forall x\in\Omega.$$

Therefore, for any $x\in\Omega$, it follows from (2.22) that

$$\begin{aligned}\frac{d}{dt}\Big((2\mu+\lambda)\log\rho(X(x,t),t)+(\Delta^{-1}\mathrm{div}\,(\rho u))(X(x,t),t)\Big)&\\ +p(X(x,t),t)=\Big([u,\mathcal{R}\otimes\mathcal{R}](\rho u)\Big)(X(x,t),t).&\end{aligned}$$

Due to $p\geq 0$ and (2.21), one can easily derive from the above equality that

$$\|\rho\|_\infty\leq\|\rho_0\|_\infty e^{C\left(\sup_{0\leq t\leq T}\|\Delta^{-1}\mathrm{div}\,(\rho u)\|_\infty+\int_0^T\|[u,\mathcal{R}\otimes\mathcal{R}](\rho u)\|_\infty dt\right)}. \tag{2.23}$$

Using the Gagliardo-Nirenberg inequality and the commutator estimates, one deduces

$$\begin{aligned}\|[u,\mathcal{R}\otimes\mathcal{R}](\rho u)\|_\infty&\lesssim\|[u,\mathcal{R}\otimes\mathcal{R}](\rho u)\|_3^{\frac15}\|\nabla[u,\mathcal{R}\otimes\mathcal{R}](\rho u)\|_4^{\frac45}\\ &\lesssim\|u\|_6^{\frac15}\|\rho u\|_6^{\frac15}\|\nabla u\|_6^{\frac45}\|\rho u\|_{12}^{\frac45}\lesssim\bar\rho\|u\|_6^{\frac15}\|u\|_6^{\frac15}\|\nabla u\|_6^{\frac45}\left(\|u\|_6^{\frac34}\|\nabla u\|_6^{\frac14}\right)^{\frac45}\\ &\lesssim\bar\rho\|\nabla u\|_2\|\nabla u\|_6\lesssim\bar\rho\|\nabla u\|_2(\|\nabla G\|_2+\|\nabla\omega\|_2+\bar\rho\|\nabla\theta\|_2),\end{aligned}$$

where, in the last step, (2.20) has been used. Thanks to this and recalling (2.11), the conclusion follows from (2.23). □

2.2. A priori estimates.

Proposition 2.7. *Assume that $2\mu>\lambda$. Denote*

$$\mathscr{N}_T=\bar\rho\sup_{0\leq t\leq T}(\|\rho\|_3+\bar\rho^2\|\sqrt\rho u\|_2^2)(t)\sup_{0\leq t\leq T}(\|\nabla u\|_2^2+\bar\rho\|\sqrt\rho E\|_2^2)(t).$$

Then, there is a positive constant η_0 depending only on $R,\gamma,\mu,\lambda,$ and κ, such that if

$$\eta\leq\eta_0,\quad\sup_{0\leq t\leq T}\|\rho\|_\infty\leq 4\bar\rho,\quad\text{and}\quad\mathscr{N}_T\leq\sqrt\eta,$$

then the following estimates hold

$$\begin{aligned}\sup_{0\leq t\leq T}\|\sqrt\rho E\|_2^2+\int_0^T\|(\nabla\theta,|u|\nabla u)\|_2^2dt&\leq C\|\sqrt{\rho_0}E_0\|_2^2,\\ \sup_{0\leq t\leq T}\|\rho\|_3+\left(\int_0^T\int\rho^3p\,dxdt\right)^{\frac13}&\leq C(\|\rho_0\|_3+\bar\rho^2\|\sqrt{\rho_0}u_0\|_2^2),\\ \bar\rho^2\left(\sup_{0\leq t\leq T}\|\sqrt\rho u\|_2^2+\int_0^T\|\nabla u\|_2^2dt\right)&\leq C(\|\rho_0\|_3+\bar\rho^2\|\sqrt{\rho_0}u_0\|_2^2),\\ \sup_{0\leq t\leq T}\|\nabla u\|_2^2+\int_0^T\left\|\left(\sqrt\rho u_t,\frac{\nabla G}{\sqrt{\bar\rho}},\frac{\nabla\omega}{\sqrt{\bar\rho}}\right)\right\|_2^2dt&\leq C(\|\nabla u_0\|_2^2+\bar\rho\|\sqrt{\rho_0}E_0\|_2^2),\\ \sup_{0\leq t\leq T}\|\rho\|_\infty&\leq\bar\rho e^{C\mathscr{N}_0^{\frac16}+C\mathscr{N}_0^{\frac12}},\end{aligned}$$

for a positive constant C depending only on $R,\gamma,\mu,\lambda,$ and κ, where

$$\mathscr{N}_0=\bar\rho(\|\rho_0\|_3+\bar\rho^2\|\sqrt{\rho_0}u_0\|_2^2)(\|\nabla u_0\|_2^2+\bar\rho\|\sqrt{\rho_0}E_0\|_2^2).$$

Proof. By assumptions, it follows from Proposition 2.3 that

$$\begin{aligned}&\sup_{0\le t\le T}\|\sqrt{\rho}E\|_2^2+\int_0^T(\|\nabla\theta\|_2^2+\||u|\nabla u\|_2^2)dt\\ \le\ & C\|\sqrt{\rho_0}E_0\|_2^2+C\eta_0^{\frac14}\int_0^T(\|\nabla\theta\|_2^2+\||u|\nabla u\|_2^2)dt,\end{aligned}$$

which by choosing η_0 suitably small implies

$$\sup_{0\le t\le T}\|\sqrt{\rho}E\|_2^2+\int_0^T(\|\nabla\theta\|_2^2+\||u|\nabla u\|_2^2)dt\le C\|\sqrt{\rho_0}E_0\|_2^2. \tag{2.24}$$

Thanks to (2.24), using the assumptions, and applying Proposition 2.2, one obtains

$$\begin{aligned}\sup_{0\le t\le T}\|\sqrt{\rho}u\|_2^2+\int_0^T\|\nabla u\|_2^2dt\ \le\ & C\|\sqrt{\rho_0}u_0\|_2^2+C\|\sqrt{\rho_0}E_0\|_2^2\sup_{0\le t\le T}\|\rho\|_3^2\\ \le\ & C\|\sqrt{\rho_0}u_0\|_2^2+C\sup_{0\le t\le T}\|\sqrt{\rho}E\|_2^2\sup_{0\le t\le T}\|\rho\|_3^2\\ \le\ & C\|\sqrt{\rho_0}u_0\|_2^2+\frac{C\sqrt{\eta_0}}{\bar\rho^2}\sup_{0\le t\le T}\|\rho\|_3.\end{aligned} \tag{2.25}$$

Using the assumptions and (2.25), it follows from Proposition 2.4 and the Young inequality that

$$\begin{aligned}&\sup_{0\le t\le T}\|\rho\|_3^3+\int_0^T\int\rho^3pdxdt\\ \le\ & C\|\rho_0\|_3^3+C\eta_0^{\frac{1}{12}}\sup_{0\le t\le T}\|\rho_0\|_3^3+C\bar\rho^2\left(\|\sqrt{\rho_0}u_0\|_2^2+\frac{\sqrt{\eta_0}}{\bar\rho^2}\sup_{0\le t\le T}\|\rho\|_3\right)\sup_{0\le t\le T}\|\rho\|_3^2\\ \le\ & C\|\rho_0\|_3^3+\left(C\eta_0^{\frac{1}{12}}+\frac14+C\sqrt{\eta_0}\right)\sup_{0\le t\le T}\|\rho\|_3^3+C\bar\rho^6\|\sqrt{\rho_0}u_0\|_2^6,\end{aligned}$$

from which, by choosing η_0 sufficiently small, one obtains

$$\sup_{0\le t\le T}\|\rho\|_3+\left(\int_0^T\int\rho^3pdxdt\right)^{\frac13}\le C(\|\rho_0\|_3+\bar\rho^2\|\sqrt{\rho_0}u_0\|_2^2). \tag{2.26}$$

Combing (2.25) with (2.26) yields

$$\bar\rho^2\left(\sup_{0\le t\le T}\|\sqrt{\rho}u\|_2^2+\int_0^T\|\nabla u\|_2^2dt\right)\le C(\|\rho_0\|_3+\bar\rho^2\|\sqrt{\rho_0}u_0\|_2^2). \tag{2.27}$$

Using (2.24) and (2.27), it follows from Proposition 2.5 that

$$\sup_{0\le t\le T}\|\nabla u\|_2^2+\int_0^T\left\|\left(\sqrt{\rho}u_t,\frac{\nabla G}{\sqrt{\bar\rho}},\frac{\nabla\omega}{\sqrt{\bar\rho}}\right)\right\|_2^2dt$$

$$
\begin{aligned}
&\lesssim \|\nabla u_0\|_2^2+\bar\rho\|\sqrt{\rho_0}E_0\|_2^2+\bar\rho^3\int_0^T\|\nabla u\|_2^2dt\sup_{0\le t\le T}\left(\|\nabla u\|_2^2+\bar\rho\|\sqrt{\rho}\theta\|_2^2\right)\\
&\quad\times\sup_{0\le t\le T}\|\nabla u\|_2^2+\left(\bar\rho+\bar\rho^2\sup_{0\le t\le T}\|\rho\|_3^{\frac12}\|\sqrt{\rho}\theta\|_2\right)\int_0^T\|(\nabla\theta,|u|\nabla u)\|_2^2dt\\
&\lesssim \|\nabla u_0\|_2^2+\bar\rho\|\sqrt{\rho_0}E_0\|_2^2+\bar\rho(\|\rho_0\|_3+\bar\rho^2\|\sqrt{\rho_0}u_0\|_2^2)\\
&\quad\times\sup_{0\le t\le T}\left(\|\nabla u\|_2^2+\bar\rho\|\sqrt{\rho}E\|_2^2\right)\sup_{0\le t\le T}\|\nabla u\|_2^2\\
&\quad+\bar\rho^2\sup_{0\le t\le T}\|\rho\|_3^{\frac12}\|\sqrt{\rho}\theta\|_2\|\sqrt{\rho_0}E_0\|_2^2.
\end{aligned}\tag{2.28}
$$

Recalling the definition of $\mathscr{N}_T$ and the assumption that $\mathscr{N}_T\le\sqrt{\eta_0}$, it is clear that

$$
\begin{aligned}
&\bar\rho(\|\rho_0\|_3+\bar\rho^2\|\sqrt{\rho_0}u_0\|_2^2)\sup_{0\le t\le T}\left(\|\nabla u\|_2^2+\bar\rho\|\sqrt{\rho}E\|_2^2\right)\\
\le\ &\bar\rho\sup_{0\le t\le T}(\|\rho\|_3+\bar\rho^2\|\sqrt{\rho}u\|_2^2)\sup_{0\le t\le T}\left(\|\nabla u\|_2^2+\bar\rho\|\sqrt{\rho}E\|_2^2\right)\le\mathscr{N}_T\le\sqrt{\eta_0}
\end{aligned}
$$

and

$$
\bar\rho\sup_{0\le t\le T}\|\rho\|_3^{\frac12}\|\sqrt{\rho}\theta\|_2\le\left(\bar\rho^2\sup_{0\le t\le T}\|\rho\|_3\sup_{0\le t\le T}\|\sqrt{\rho}E\|_2^2\right)^{\frac12}\le\mathscr{N}_T^{\frac12}\le\eta_0^{\frac14}.
$$

Thanks to the above two estimates, by choosing η_0 sufficiently small, one can easily derive from (2.28) that

$$
\sup_{0\le t\le T}\|\nabla u\|_2^2+\int_0^T\left\|\left(\sqrt{\rho}u_t,\frac{\nabla G}{\sqrt{\bar\rho}},\frac{\nabla\omega}{\sqrt{\bar\rho}}\right)\right\|_2^2dt\le C(\|\nabla u_0\|_2^2+\bar\rho\|\sqrt{\rho_0}E_0\|_2^2).\tag{2.29}
$$

The estimate for $\|\rho\|_\infty$ follows from Proposition 2.6 by using (2.24), (2.27), and (2.29). □

Proposition 2.8. *Assume that $2\mu>\lambda$. Let η_0, $\mathscr{N}_T$, and $\mathscr{N}_0$ be as in Proposition 2.7. Then, the following two hold:*

(i) There is a number $\varepsilon_0\in(0,\eta_0)$ depending only on R,γ,μ,λ, and κ, such that if

$$
\sup_{0\le t\le T}\|\rho\|_\infty\le 4\bar\rho,\quad \mathscr{N}_T\le\sqrt{\varepsilon_0},\quad \textit{and}\quad \mathscr{N}_0\le\varepsilon_0.
$$

then

$$
\sup_{0\le t\le T}\|\rho\|_\infty\le 2\bar\rho\quad\textit{and}\quad\mathscr{N}_T\le\frac{\sqrt{\varepsilon_0}}{2}.
$$

(ii) As a consequence of (i), the following estimates hold

$$
\mathscr{N}_T\le\frac{\sqrt{\varepsilon_0}}{2}\quad\textit{and}\quad\sup_{0\le t\le T}\|\rho\|_\infty\le 2\bar\rho,
$$

as long as $\mathscr{N}_0\le\varepsilon_0$.

Proof. (i) Let $\varepsilon_0 \leq \eta_0$ be sufficiently small. By assumptions, all the conditions in Proposition 2.7 hold, and, thus

$$\begin{aligned}\mathscr{N}_T &\leq C\bar\rho(\|\rho_0\|_3 + \bar\rho^2\|\sqrt{\rho_0}u_0\|_2^2)(\|\nabla u_0\|_2^2 + \bar\rho\|\sqrt{\rho_0}E_0\|_2^2)\\ &= C\mathscr{N}_0 \leq C\varepsilon_0 \leq \frac{\sqrt{\varepsilon_0}}{2},\end{aligned}$$

and

$$\sup_{0\leq t\leq T}\|\rho\|_\infty \leq \bar\rho e^{C\mathscr{N}_0^{\frac16}+C\mathscr{N}_0^{\frac12}} \leq \bar\rho e^{C\varepsilon_0^{\frac16}+C\varepsilon_0^{\frac12}} \leq 2\bar\rho,$$

as long as ε_0 is sufficiently small. The first conclusion follows.

(ii) Define

$$T_\# := \max\left\{\mathcal{T}\in(0,T]\,\middle|\,\mathscr{N}_{\mathcal{T}} \leq \sqrt{\varepsilon_0},\ \sup_{0\leq t\leq \mathcal{T}}\|\rho\|_\infty \leq 4\bar\rho\right\}.$$

Then, by (i), we have

$$\mathscr{N}_{\mathcal{T}} \leq \frac{\sqrt{\varepsilon_0}}{2}, \qquad \sup_{0\leq t\leq\mathcal{T}}\|\rho\|_\infty \leq 2\bar\rho, \quad \forall \mathcal{T}\in(0,T_\#). \tag{2.30}$$

If $T_\# < T$, noticing that $\mathscr{N}_{\mathcal{T}}$ and $\sup_{0\leq t\leq\mathcal{T}}\|\rho\|_\infty$ are continuous on $[0,T]$, there is another time $T_{\#\#}\in(T_\#,T]$, such that

$$\mathscr{N}_{T_{\#\#}} \leq \sqrt{\varepsilon_0} \qquad \text{and} \qquad \sup_{0\leq t\leq T_{\#\#}}\|\rho\|_\infty \leq 4\bar\rho,$$

which contradicts to the definition of $T_\#$. Thus, we have $T_\# = T$, and the conclusion follows from (2.30) and the continuity of $\mathscr{N}_{\mathcal{T}}$ and $\sup_{0\leq t\leq\mathcal{T}}\|\rho\|_\infty$ on $[0,T]$. ☐

The following corollary is a straightforward consequence of Proposition 2.7 and (ii) of Proposition 2.8.

Corollary 2.1. *Assume that $2\mu > \lambda$. Let ε_0 be as in Proposition 2.8 and assume $\mathscr{N}_0 \leq \varepsilon_0$. Then, there is a positive constant C depending only on R, γ, μ, λ, κ, $\bar\rho$, $\|\rho_0\|_3$, $\|\sqrt{\rho_0}u_0\|_2$, $\|\sqrt{\rho_0}E_0\|_2$, and $\|\nabla u_0\|_2$, such that the following estimates hold:*

$$\sup_{0\leq t\leq T}(\|(\sqrt\rho E, \sqrt\rho u, \nabla u)\|_2^2 + \|\rho\|_3 + \|\rho\|_\infty) \leq C,$$

$$\int_0^T\left(\|(\nabla\theta, |u|\nabla u, \sqrt\rho u_t, \nabla G, \nabla\omega)\|_2^2 + \|\nabla u\|_6^2 + \int\rho^3 p\,dx\right)dt \leq C.$$

3. Proof of Theorem 1.1

The following blow-up criteria is cited from Huang–Li [20].

Proposition 3.1. *Let $T^* < \infty$ be the maximal time of existence of a solution (ρ, u, θ) to system (1.1)–(1.3), with initial data (ρ_0, u_0, θ_0). Then,*

$$\lim_{T \to T^*} (\|\rho\|_{L^\infty(0,T;L^\infty)} + \|u\|_{L^s(0,T;L^r)}) = \infty,$$

for any (s, r) such that $\frac{2}{s} + \frac{3}{r} \leq 1$ and $3 < r \leq \infty$.

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. Let ε_0 and $\mathscr{N}_T$ be as in Proposition 2.8 and assume $\mathscr{N}_0 \leq \varepsilon_0$. By Proposition 2.1, there is a unique local strong solution (ρ, u, θ) to system (1.1)–(1.3), with initial data (ρ_0, u_0, θ_0). Extend the local solution (ρ, u, θ) to the maximal time of existence $T_{\max}$. If $T_{\max} = \infty$, then (ρ, u, θ) is a global solution and we are down. Assume that $T_{\max} < \infty$. Then, by the blow up criteria in Proposition 3.1, it holds

$$\lim_{T \to T_{\max}} (\|\rho\|_{L^\infty(0,T;L^\infty)} + \|u\|_{L^4(0,T;L^6)}) = \infty. \tag{3.1}$$

By Corollary 2.1, it follows $\sup_{0 \leq t \leq T}(\|\rho\|_\infty + \|\nabla u\|_2^2) \leq C$ which, by the Sobolev embedding inequality, gives

$$\|\rho\|_{L^\infty(0,T;L^\infty)} + \|u\|_{L^4(0,T;L^6)} \leq C,$$

for any $T \in (0, T_{\max})$, and for a positive constant C independent of T. This implies

$$\lim_{T \to T_{\max}} (\|\rho\|_{L^\infty(0,T;L^\infty)} + \|u\|_{L^4(0,T;L^6)}) \leq C < \infty,$$

contradicting to (3.1). Therefore, we must have $T_{\max} = \infty$, proving Theorem 1.1. □

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