

STABILITY CONDITIONS AND THE A_2 QUIVER

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ABSTRACT. For each integer $n \geq 2$ we describe the space of stability conditions on the derived category of the n -dimensional Ginzburg algebra associated to the A_2 quiver. The form of our results points to a close relationship between these spaces and the Frobenius-Saito structure on the unfolding space of the A_2 singularity.

1. INTRODUCTION

In this paper we study spaces of stability conditions [2] on the sequence of CY_n triangulated categories D_n associated to the A_2 quiver. Our main result is Theorem 1.1 below. There are several striking features. Firstly, we obtain uniform results for all $n \geq 2$: the space of stability conditions quotiented by the action of the spherical twists is independent of n , although the identification maps are highly non-trivial. Secondly, there is a close link between our spaces of stability conditions and the Frobenius-Saito structure on the unfolding space of the A_2 singularity: in fact this structure is precisely what encodes the identifications between our stability spaces for various n . A third interesting feature is that the space of stability conditions on the derived category of the path algebra of the A_2 quiver arises as a kind of limit of the spaces for the categories D_n as $n \rightarrow \infty$.

1.1. Statement of results. For each integer $n \geq 2$ we let $D_n = D_{CY_n}(A_2)$ denote the bounded derived category of the CY_n complex Ginzburg algebra associated to the A_2 quiver. It is a triangulated category of finite type over \mathbb{C} , and is characterised by the following two properties:

- (a) It is CY_n , i.e. for any pair of objects $A, B \in D_n$ there are natural isomorphisms

$$\mathrm{Hom}_{D_n}^\bullet(A, B) \cong \mathrm{Hom}_{D_n}^\bullet(B, A[n])^\vee. \quad (1)$$

- (b) It is generated by two spherical objects S_1, S_2 satisfying

$$\mathrm{Hom}_{D_n}^*(S_1, S_2) = \mathbb{C}[-1]. \quad (2)$$

We denote by D_∞ the bounded derived category of the complex path algebra of the A_2 quiver. It is again a \mathbb{C} -linear triangulated category, and is characterised by the property

that it is generated by two exceptional objects S_1, S_2 , which satisfy (2) and

$$\mathrm{Hom}_{\mathcal{D}_\infty}^*(S_2, S_1) = 0.$$

The notation \mathcal{D}_∞ is convenient: the point being that as n increases, the Serre dual to the extension $S_1 \rightarrow S_2[1]$ occurs in higher and higher degrees, until when $n = \infty$ it doesn't occur at all.

For each $2 \leq n \leq \infty$ we denote by $\mathrm{Stab}(\mathcal{D}_n)$ the space of stability conditions on the category \mathcal{D}_n . We define $\mathrm{Stab}_*(\mathcal{D}_n) \subset \mathrm{Stab}(\mathcal{D}_n)$ to be the connected component containing stability conditions in which the objects S_1 and S_2 are stable of equal phase. Let $\mathrm{Aut}(\mathcal{D}_n)$ denote the group of exact \mathbb{C} -linear autoequivalences of the category \mathcal{D}_n , considered up to isomorphism of functors. We define $\mathrm{Aut}_*(\mathcal{D}_n)$ to be the subquotient consisting of autoequivalences which preserve the connected component $\mathrm{Stab}_*(\mathcal{D}_n)$, modulo those which act trivially on it. When $n < \infty$ the objects S_i are spherical and hence define Seidel-Thomas twist functors $\mathrm{Tw}_{S_i} \in \mathrm{Aut}(\mathcal{D}_n)$. These autoequivalences preserve the connected component $\mathrm{Stab}_*(\mathcal{D}_n)$, and we denote by $\mathrm{Sph}_*(\mathcal{D}_n) \subset \mathrm{Aut}_*(\mathcal{D}_n)$ the subgroup they generate.

The simple complex Lie algebra associated to the A_2 quiver is $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C})$. Its Cartan subalgebra can be described explicitly as

$$\mathfrak{h} = \{(u_1, u_2, u_3) \in \mathbb{C}^3 : \sum_i u_i = 0\}.$$

The complement of the root hyperplanes is

$$\mathfrak{h}^{\mathrm{reg}} = \{(u_1, u_2, u_3) \in \mathfrak{h} : i \neq j \implies u_i \neq u_j\}.$$

There is an obvious action of the Weyl group $W = S_3$ permuting the u_i which is free on $\mathfrak{h}^{\mathrm{reg}}$. The quotient \mathfrak{h}/W is isomorphic to \mathbb{C}^2 , and has natural co-ordinates (a, b) obtained by writing

$$p(x) = (x - u_1)(x - u_2)(x - u_3) = x^3 + ax + b.$$

The image of the root hyperplanes $u_i = u_j$ is the discriminant

$$\Delta = \{(a, b) \in \mathbb{C}^2 : 4a^3 + 27b^2 = 0\}.$$

We can now state the main result of this paper.

Theorem 1.1. (a) For $2 \leq n < \infty$ there is an isomorphism of complex manifolds

$$\mathrm{Stab}_*(\mathcal{D}_n)/\mathrm{Sph}_*(\mathcal{D}_n) \cong \mathfrak{h}^{\mathrm{reg}}/W.$$

Under this isomorphism the central charge map $\text{Stab}_*(D_n) \rightarrow \mathbb{C}^2$ induces the multi-valued map $\mathfrak{h}^{\text{reg}}/W \rightarrow \mathbb{C}^2$ given by

$$\int_{\gamma_i} p(x)^{(n-2)/2} dx$$

for an appropriate basis of paths γ_i connecting the zeroes of the polynomial $p(x)$.

(b) For $n = \infty$ there is an isomorphism of complex manifolds

$$\text{Stab}(D_\infty) \cong \mathfrak{h}/W.$$

Under this isomorphism the central charge map $\text{Stab}(D_\infty) \rightarrow \mathbb{C}^2$ corresponds to the map $\mathfrak{h}/W \rightarrow \mathbb{C}^2$ given by

$$\int_{\delta_i} e^{p(x)} dx$$

for an appropriate basis of paths δ_i which approach ∞ in both directions along rays for which $\text{Re}(x^3) \rightarrow -\infty$.

Theorem 1.1 gives a precise link with the Frobenius-Saito structure on the unfolding space of the A_2 singularity $x^3 = 0$. The corresponding Frobenius manifold is precisely $M = \mathfrak{h}/W$. The maps appearing in part (a) of our result are then the *twisted period maps* of M with parameter $\nu = (n - 2)/2$ (see Equation (5.11) of [6]). The map in part (b) is given by the *deformed flat co-ordinates* of M with parameter $\hbar = 1$ (see [5, Theorem 2.3]).

1.2. Related work. Just as we were finishing this paper, A. Ikeda posted the paper [10] which also proves Theorem 1.1 (a), and indeed generalizes it to the case of the A_k quiver for all $k \geq 1$. The methods we use here are quite different however, and also yield (b), so we feel that this paper is also worth publishing.

As explained above, two of the most interesting features of Theorem 1.1 are the fact that the space $\text{Stab}_*(D_n)/\text{Sph}_*(D_n)$ is independent of $n < \infty$, and that this space embeds in $\text{Stab}(D_\infty)$. At the level of exchange graphs such results were observed for arbitrary acyclic quivers by one of us with A.D. King [14].

The $n < \infty$ case of Theorem 1.1 was first considered by R.P. Thomas in [23]: he obtained the $n = 2$ case and discussed the relationship with Fukaya categories and homological mirror symmetry. The $n = 2$ case was also proved in [3] and generalised to arbitrary ADE Dynkin diagrams. The $n = 3$ case of Theorem 1.1 was proved in [21], and was

extended to all Dynkin quivers of A and D type in [4]. The first statement of part (a), that $\text{Stab}_*(\mathcal{D}) \cong \mathfrak{h}^{\text{reg}}/W$, was proved for all $n < \infty$ in [18].

The case $n = \infty$ of Theorem 1.1 was first considered by King [13] who proved that $\text{Stab}(\mathcal{D}_\infty) \cong \mathbb{C}^2$. This result was obtained by several other researchers since then, and a proof was written down in [18]. The more precise statement of Theorem 1.1 (b) was conjectured by A. Takahashi [22].

Since the first version of this paper was posted, several generalizations and extensions of Theorem 1.1 have appeared. The following seem particularly noteworthy. In [9] a general result relating stability conditions on Fukaya categories of surfaces to spaces of quadratic differentials with exponential singularities is proved. This includes Theorem 1.1 (b) as a very special case.

In [11] a new notion of q -stability conditions is introduced, in terms of which one can make sense of the statement of Theorem 1.1 when the Calabi-Yau dimension $n \geq 2$ is replaced with an arbitrary $s \in \mathbb{C}$ with $\text{Re}(s) \geq 2$. This allows one to see all twisted periods of the Frobenius manifold via central charges.

In [15], an analogue of Theorem 1.1 for the Kronecker quiver is proved. In this case the relevant Frobenius manifold is the quantum cohomology of \mathbb{P}^1 . The case of quivers of type affine A_n was considered in [24]. A general framework for such results, involving Fukaya categories of surfaces, quadratic differentials and Hurwitz spaces is explained in [12].

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2. AUTOEQUIVALENCES AND T-STRUCTURES

In this section we describe the principal components of the exchange graphs of the categories $\mathcal{D}_n = \text{D}_{\text{CY}_n}(A_2)$ and study the action of the group of reachable autoequivalences. We start by recalling some general definitions concerning tilting (see [4, Section 7] for more details).

2.1. Let \mathcal{D} be a triangulated category. We shall be concerned with bounded t -structures on \mathcal{D} . Any such t -structure is determined by its heart $\mathcal{A} \subset \mathcal{D}$, which is a full abelian subcategory. We use the term *heart* to mean the heart of a bounded t -structure. A heart will be called *finite-length* if it is artinian and noetherian as an abelian category.

We say that a pair of hearts $(\mathcal{A}_1, \mathcal{A}_2)$ in \mathcal{D} is a *tilting pair* if the equivalent conditions

$$\mathcal{A}_2 \subset \langle \mathcal{A}_1, \mathcal{A}_1[-1] \rangle, \quad \mathcal{A}_1 \subset \langle \mathcal{A}_2[1], \mathcal{A}_2 \rangle$$

are satisfied. Here the angular brackets signify the extension-closure operation. We also say that \mathcal{A}_1 is a *left tilt* of \mathcal{A}_2 , and that \mathcal{A}_2 is a *right tilt* of \mathcal{A}_1 . Note that $(\mathcal{A}_1, \mathcal{A}_2)$ is a tilting pair precisely if so is $(\mathcal{A}_2[1], \mathcal{A}_1)$.

If $(\mathcal{A}_1, \mathcal{A}_2)$ is a tilting pair in \mathcal{D} , then the subcategories

$$\mathcal{T} = \mathcal{A}_1 \cap \mathcal{A}_2[1], \quad \mathcal{F} = \mathcal{A}_1 \cap \mathcal{A}_2$$

form a torsion pair $(\mathcal{T}, \mathcal{F}) \subset \mathcal{A}_1$. Conversely, if $(\mathcal{T}, \mathcal{F}) \subset \mathcal{A}_1$ is a torsion pair, then the subcategory $\mathcal{A}_2 = \langle \mathcal{F}, \mathcal{T}[-1] \rangle$ is a heart, and the pair $(\mathcal{A}_1, \mathcal{A}_2)$ is a tilting pair.

A special case of the tilting construction will be particularly important. Suppose that \mathcal{A} is a finite-length heart and $S \in \mathcal{A}$ is a simple object. Let $\langle S \rangle \subset \mathcal{A}$ be the full subcategory consisting of objects $E \in \mathcal{A}$ all of whose simple factors are isomorphic to S . Define full subcategories

$${}^\perp S = \{E \in \mathcal{A} : \text{Hom}_{\mathcal{A}}(E, S) = 0\}, \quad S^\perp = \{E \in \mathcal{A} : \text{Hom}_{\mathcal{A}}(S, E) = 0\}.$$

One can either view $\langle S \rangle$ as the torsion-free part of a torsion pair on \mathcal{A} , in which case the torsion part is ${}^\perp S$, or as the torsion part, in which case the torsion-free part is S^\perp . We can then define tilted hearts

$$\mu_S^-(\mathcal{A}) = \langle S[1], {}^\perp S \rangle, \quad \mu_S^+(\mathcal{A}) = \langle S^\perp, S[-1] \rangle,$$

which we refer to as the left and right tilts of the heart \mathcal{A} at the simple S . They fit into tilting pairs $(\mu_S^-(\mathcal{A}), \mathcal{A})$ and $(\mathcal{A}, \mu_S^+(\mathcal{A}))$. Note the relation

$$\mu_{S[1]}^+ \circ \mu_S^-(\mathcal{A}) = \mathcal{A}.$$

The *exchange graph* $\text{EG}(\mathcal{D})$ is the graph with vertices the finite-length hearts in \mathcal{D} and edges corresponding to simple tilts. The group $\text{Aut}(\mathcal{D})$ of triangulated autoequivalences of \mathcal{D} acts on this graph in the obvious way: an auto-equivalence $\Phi \in \text{Aut}(\mathcal{D})$ sends a finite-length heart $\mathcal{A} \subset \mathcal{D}$ to the finite-length heart $\Phi(\mathcal{A}) \subset \mathcal{D}$.

2.2. For each $2 \leq n \leq \infty$ we define the triangulated category \mathcal{D}_n as in the Introduction. It is the bounded derived category of the CY_n Ginzburg algebra associated to the A_2 quiver [7]. This category contains two distinguished objects S_1 and S_2 corresponding to

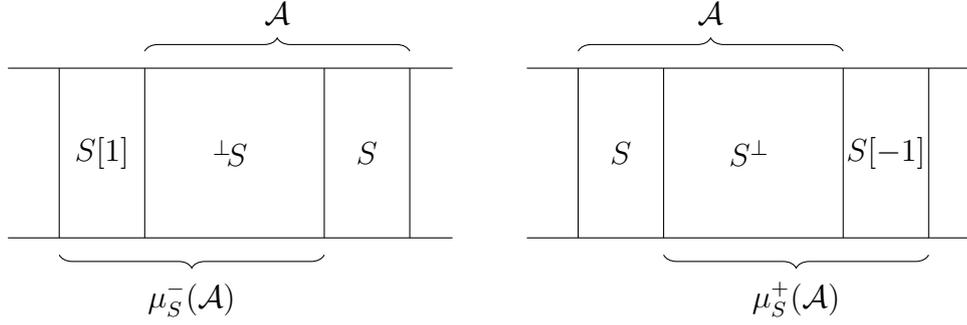


FIGURE 1. Left and right tilts of a heart.

the vertices of the quiver, and a canonical heart

$$\mathcal{A}_n = \langle S_1, S_2 \rangle \subset D_n,$$

which is the extension-closed subcategory generated by these objects. The heart \mathcal{A}_n has finite-length, and hence defines a point of the exchange graph $\text{EG}(D_n)$; we denote by $\text{EG}^\circ(D_n)$ the connected component containing this point. We call $\text{EG}^\circ(D_n)$ the *principal component* of the exchange graph, and refer to the hearts defined by its vertices as *reachable hearts*. We say that a heart $\mathcal{A} \subset D_n$ is *full* if it is equivalent to \mathcal{A}_n as an abelian category.

Remark 2.1. When $n > 2$, the canonical heart \mathcal{A}_n is equivalent to the category $\text{Rep}(A_2)$ of representations of the A_2 quiver; besides the simple objects S_1 and S_2 , it contains one more indecomposable object which we denote by E ; there is a short exact sequence

$$0 \longrightarrow S_2 \longrightarrow E \longrightarrow S_1 \longrightarrow 0. \quad (3)$$

When $n = 2$, the canonical heart \mathcal{A}_2 is equivalent to the category of representations of the preprojective algebra of the A_2 quiver; besides E there is another non-simple indecomposable object fitting into a short exact sequence

$$0 \longrightarrow S_1 \longrightarrow F \longrightarrow S_2 \longrightarrow 0. \quad (4)$$

For this reason, the case $n = 2$ is slightly special, and since our main result Theorem 1.1 is already known (and is considerably easier to prove) in this case [23, 3], in what follows we shall restrict to the case $n \geq 3$.

A triangulated autoequivalence of D_n is called *reachable* if its action on $\text{EG}(D_n)$ preserves the connected component $\text{EG}^\circ(D_n)$. An autoequivalence is called *negligible* if it is reachable and acts by the identity on $\text{EG}^\circ(D_n)$. It is easy to see that an autoequivalence $\Phi \in \text{Aut}(D_n)$ is negligible precisely if $\Phi(S_1) \cong S_1$ and $\Phi(S_2) \cong S_2$. We write $\text{Aut}_*(D_n)$ for

the subquotient of the group $\text{Aut}(\mathcal{D})$ consisting of reachable autoequivalences, modulo negligible autoequivalences. We will show that this agrees with the definition given in the Introduction later: see Section 4.1 and Prop. 4.4.

In the case $n < \infty$ an important role will be played by spherical twist functors [20]. Recall that an object $S \in \mathcal{D}_n$ is called *spherical* if

$$\text{Hom}_{\mathcal{D}_n}^{\bullet}(S, S) = \mathbb{C} \oplus \mathbb{C}[-n].$$

Any such object defines an autoequivalence $\text{Tw}_S \in \text{Aut}(\mathcal{D}_n)$ called a *spherical twist*. This has the property that for each object $E \in \mathcal{D}_n$ there is a triangle

$$\text{Hom}_{\mathcal{D}_n}^{\bullet}(S, E) \otimes S \longrightarrow E \longrightarrow \text{Tw}_S(E),$$

where the first arrow is the evaluation map. In particular, the distinguished objects $S_i \in \mathcal{D}_n$ are spherical, and hence define twist functors $\text{Tw}_{S_i} \in \text{Aut}(\mathcal{D}_n)$.

Lemma 2.2. *Take $2 \leq n < \infty$, and define the following autoequivalences of \mathcal{D}_n :*

$$\Sigma = (\text{Tw}_{S_1} \circ \text{Tw}_{S_2})[n-1], \quad \Upsilon = (\text{Tw}_{S_2} \circ \text{Tw}_{S_1} \circ \text{Tw}_{S_2})[2n-3].$$

Then we have

$$\Sigma(S_1, E, S_2) = (S_2[1], S_1, E), \quad \Upsilon(S_1, S_2) = (S_2, S_1[n-2]).$$

Proof. The defining properties (1) and (2) of the category \mathcal{D}_n , together with the short exact sequence (3), implies that $\text{Tw}_{S_1}(S_2) = E$. In particular E is also spherical. Applying the long exact sequence in cohomology to the short exact sequence (3), and using the fact that the objects S_i are spherical, shows that

$$\text{Hom}_{\mathcal{D}_n}^{\bullet}(S_1, E) = \mathbb{C}[-n], \quad \text{Hom}_{\mathcal{D}_n}^{\bullet}(E, S_2) = \mathbb{C}[-n]. \quad (5)$$

Using the CY_n property we therefore obtain the identities

$$\text{Tw}_{S_1}(S_2) = E, \quad \text{Tw}_E(S_1) = S_2[1], \quad \text{Tw}_{S_2}(E) = S_1.$$

For any spherical object S there is an identity $\text{Tw}_S(S) = S[1-n]$, and for any pair of spherical objects there is a relation

$$\text{Tw}_{S_1} \circ \text{Tw}_{S_2} = \text{Tw}_{\text{Tw}_{S_1}(S_2)} \circ \text{Tw}_{S_1}.$$

Thus we can write $\Sigma = (\text{Tw}_E \circ \text{Tw}_{S_1})[n-1]$. This implies that

$$\Sigma(S_1) = \text{Tw}_E(S_1) = S_2[1], \quad \Sigma(S_2) = \text{Tw}_{S_1}(S_2) = E,$$

and it follows that $\Sigma(E)$ is the unique nontrivial extension of these two objects, namely S_1 . This proves the first claim.

Moving on to the second identity, we use the braid relation

$$\mathrm{Tw}_{S_1} \circ \mathrm{Tw}_{S_2} \circ \mathrm{Tw}_{S_1} = \mathrm{Tw}_{S_2} \circ \mathrm{Tw}_{S_1} \circ \mathrm{Tw}_{S_2}$$

proved by Seidel and Thomas [20]. This implies that

$$\Upsilon(S_1) = \Sigma(S_1[-1]) = S_2, \quad \Upsilon(S_2) = \mathrm{Tw}_{S_2}(E[n-2]) = S_1[n-2],$$

which completes the proof. \square

2.3. The following description of the tilting operation in D_n is the combinatorial underpinning of our main result.

Proposition 2.3. *Take $3 \leq n < \infty$, and consider the hearts obtained by performing simple tilts of the standard heart $\mathcal{A}_n \subset D_n$. We have*

(a) *The left tilt of \mathcal{A}_n at the simple S_2 is another full heart:*

$$\mathcal{A}_n = \langle S_1, S_2 \rangle \rightarrow \langle S_2[1], E \rangle = \Sigma(\mathcal{A}_n).$$

(b) *Repeated left tilts at appropriate shifts of S_1 gives a sequence of hearts*

$$\mathcal{A}_n = \langle S_1, S_2 \rangle \rightarrow \langle S_1[1], S_2 \rangle \rightarrow \langle S_1[2], S_2 \rangle \rightarrow \cdots \rightarrow \langle S_1[n-2], S_2 \rangle = \Upsilon(\mathcal{A}_n).$$

Proof. This can be found in [14, Proposition 5.4], but for the convenience of the reader we give a proof here. Note that since $E \in {}^\perp S_2$ we have $\langle S_2[1], E \rangle = \Sigma(\mathcal{A}_n) \subset \mu_{S_2}^-(\mathcal{A}_n)$. But it is a standard fact, and easily proved, that if one heart is contained in another then they are equal. This gives (a). For the first step in part (b) note that since the only indecomposables in \mathcal{A}_n are S_1, S_2 and E , we have ${}^\perp S_1 = \langle S_2 \rangle$. It follows that $\mu_{S_1}^-(\mathcal{A}_n) = \langle S_1[1], S_2 \rangle$. We have

$$\mathrm{Hom}_{D_n}^1(S_1[k], S_2) = 0 = \mathrm{Hom}_{D_n}^1(S_2, S_1[k]), \quad 0 < k < n-2,$$

so if $n > 3$ the heart $\langle S_1[1], S_2 \rangle$ is semi-simple, with simple objects $S_1[1]$ and S_2 . It follows that its left tilt with respect to the simple $S_1[1]$ is the subcategory $\langle S_1[2], S_2 \rangle$. Repeating the argument we obtain the given sequence of hearts. \square

As explained in the proof of Prop. 2.3, when $n > 3$ each of the intermediate hearts in the sequence in (b) is semi-simple, and in particular non-full. The semi-simplicity implies that

$$\mu_{S_2}^\pm \langle S_1[k], S_2 \rangle = \langle S_1[k], S_2[\pm 1] \rangle,$$

so that tilting these intermediate hearts with respect to S_2 gives hearts of the same kind up to shift.

Lemma 2.4. *Take $3 \leq n < \infty$. Then every reachable heart is of the form*

$$\Phi(\langle S_1[k], S_2 \rangle) \subset D_n,$$

where Φ is a reachable autoequivalence, and $0 \leq k \leq (n-2)/2$.

Proof. What we proved above shows that every time we tilt a heart of the form $\langle S_1[k], S_2 \rangle$ with $0 \leq k < n-2$ we obtain another heart of the same form, up to the action of some autoequivalence, which is necessarily reachable. Since tilting commutes with autoequivalences, in the sense that

$$\mu_{\Phi(S)}^{\pm}(\Phi(\mathcal{A})) = \Phi(\mu_S^{\pm}(\mathcal{A})),$$

it follows that any heart is of the given form, with $0 \leq k < n-2$. To complete the proof, note that if $k_1 + k_2 = n-2$, the autoequivalence Υ exchanges the hearts $\langle S_1[k_i], S_2 \rangle$ up to shift. \square

Lemma 2.5. *Take $3 \leq n < \infty$.*

- (a) *The autoequivalences Σ, Υ and $[1]$ are all reachable.*
- (b) *The spherical twists Tw_{S_i} are reachable.*
- (c) *In the group $\text{Aut}_*(D_n)$ there are relations*

$$\Sigma^3 = [1], \quad \Upsilon^2 = [n-2].$$

Proof. The reachability of Σ and Υ is immediate from Prop. 2.3. Lemma 2.2 shows that $\Sigma^3(\mathcal{A}_n) = \mathcal{A}_n[1] \subset D_n$. Thus the shift functor $[1]$ is also reachable, which gives (a). Part (b) then follows from the relations

$$(\Sigma \circ \text{Tw}_{S_1})[n-2] = \Upsilon = (\text{Tw}_{S_2} \circ \Sigma)[n-2]$$

For part (c) consider the autoequivalence $\Sigma^3[-1]$. Since it fixes the objects S_1, S_2 , it is negligible, and hence defines the identity element in $\text{Aut}_*(D_n)$. The same argument applies to $\Upsilon^2[2-n]$. \square

Lemma 2.6. *For $3 \leq n < \infty$ the action of $\text{Aut}_*(D_n)$ on the set of full reachable hearts is free and transitive.*

Proof. The transitivity follows from Lemma 2.4, since for k in the given range the heart $\langle S_1[k], S_2 \rangle$ is full only for $k = 0$. For the freeness it is therefore enough to consider autoequivalences which preserve the standard heart $\mathcal{A}_n \subset D_n$. Any such autoequivalence Φ must preserve the simple objects S_1, S_2 of \mathcal{A}_n , and it cannot exchange them since it

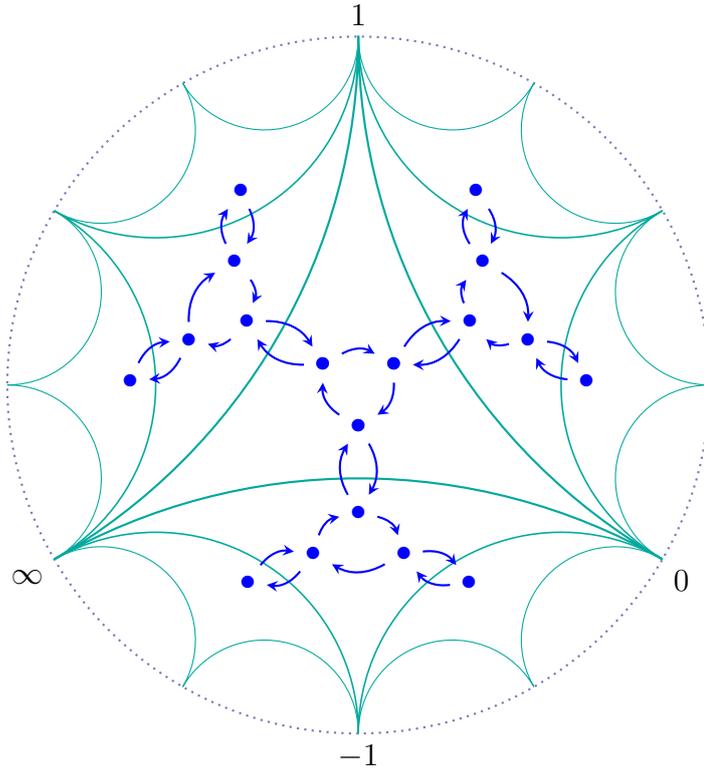


FIGURE 2. The projective exchange graph of D_3 drawn on the hyperbolic disc. The action of $\mathbb{P}\text{Aut}_*(D_3)$ corresponds to the standard action of $\text{PSL}(2, \mathbb{Z})$ on the disc.

preserves the Ext-groups between them, which are asymmetric for $n \geq 3$. Thus $\Phi(S_i) = S_i$ and it follows that Φ defines the identity element of $\text{Aut}_*(D_n)$. \square

Note that when $n = 3$ it follows from Lemma 2.4 that all reachable hearts are full. Thus in this case the principal component $\text{EG}^\circ(D_3)$ is a torsor for the group $\text{Aut}_*(D_3)$.

2.4. We denote by Br_3 the Artin braid group of the A_2 root system; in the notation of the Introduction it is the fundamental group of the quotient $\mathfrak{h}^{\text{reg}}/W$. More concretely, Br_3 is the standard braid group on three strings, and has a presentation

$$\text{Br}_3 = \langle \sigma_1, \sigma_2 : \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle.$$

The centre of Br_3 is generated by the element $\tau = (\sigma_1\sigma_2)^3$.

Proposition 2.7. *Take $3 \leq n < \infty$.*

- (a) *The group $\text{Aut}_*(D_n)$ is generated by Σ, Υ and the shift functor $[1]$.*
- (b) *The group $\text{Aut}_*(D_n)$ is generated by the subgroup $\text{Sph}_*(D_n)$ together with $[1]$.*

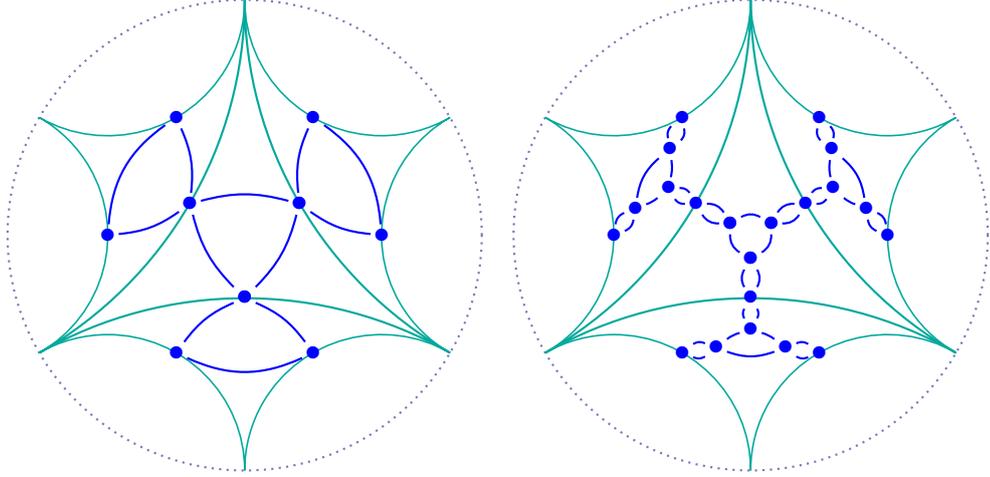


FIGURE 3. Similar pictures of the projective exchange graphs of D_2 and D_4 (orientations omitted). As before, the action of $\mathbb{P}\text{Aut}_*(D_n) \cong \text{PSL}(2, \mathbb{Z})$ corresponds to the standard one.

- (c) There is an isomorphism $\text{Br}_3 \cong \text{Sph}_*(D_n)$ sending the generator σ_i to Tw_{S_i} .
- (d) The isomorphism in (c) sends the central element τ to $[4 - 3n]$.
- (e) The smallest power of $[1]$ contained in $\text{Sph}_*(D_n)$ is $[3n - 4]$. Thus there is a short exact sequence

$$1 \longrightarrow \text{Sph}_*(D_n) \longrightarrow \text{Aut}_*(D_n) \longrightarrow \mu_{3n-4} \longrightarrow 1.$$

Proof. Part (a) follows from the explicit description of tilts given in Prop. 2.3, since any element of $\text{Aut}_*(D_n)$ takes the canonical heart \mathcal{A} to a full reachable heart. Part (b) is then immediate from the definitions of Σ and Υ . It was proved in [20] that there is an injective group homomorphism $\rho: \text{Br}_3 \rightarrow \text{Aut}(D_n)$ sending the generator σ_i to the twist functor Tw_{S_i} . By Lemma 2.5(b) this induces a surjective homomorphism $\bar{\rho}: \text{Br}_3 \rightarrow \text{Sph}_*(D_n)$. It is immediate from Lemma 2.5 and the definition of Υ that $\bar{\rho}(\tau) = [4 - 3n]$, which gives (d). Note that if $g \in \text{Br}_3$ satisfies $\bar{\rho} = 1$, then the autoequivalence $\rho(g)$ is negligible, so commutes with the autoequivalences Tw_{S_i} , which shows that g must be central. But then g must be a power of τ which, by (d), implies that g is the identity. This completes the proof of (c). Part (e) again follows from the fact that τ generates the centre of Br_3 , since any shift $[d]$ lying in the subgroup $\text{Sph}_*(D) \subset \text{Aut}_*(D_n)$ is necessarily central. \square

2.5. It will be useful to introduce the quotient group

$$\mathbb{P}\text{Aut}_*(D_n) = \text{Aut}_*(D_n)/[1].$$

When $3 \leq n < \infty$ the natural action of auto-equivalences on the Grothendieck group $K_0(\mathcal{D}) \cong \mathbb{Z}^{\oplus 2}$ induces a group homomorphism

$$\rho: \mathbb{P} \operatorname{Aut}_*(\mathcal{D}_n) \rightarrow \operatorname{PGL}(K_0(\mathcal{D})). \quad (6)$$

Taking the basis $([S_1], [S_2]) \subset K_0(\mathcal{D})$ we can identify the target of this map with $\operatorname{PGL}(2, \mathbb{Z})$. From the definition of the twist functors we have

$$\rho(\operatorname{Tw}_{S_1}) = \begin{pmatrix} (-1)^{n+1} & 1 \\ 0 & 1 \end{pmatrix}, \quad \rho(\operatorname{Tw}_{S_2}) = \begin{pmatrix} 1 & 0 \\ (-1)^n & (-1)^{n+1} \end{pmatrix}.$$

It is a standard fact that for n odd, the map $\operatorname{Br}_3 \rightarrow \operatorname{PGL}(2, \mathbb{Z})$ sending the generators σ_1, σ_2 to these matrices induces an isomorphism $\operatorname{Br}_3 / \langle \tau \rangle \cong \operatorname{PSL}(2, \mathbb{Z}) < \operatorname{PGL}(2, \mathbb{Z})$. It then follows from Prop. 2.7 that the map (6) is an isomorphism onto its image. Note that

$$\rho(\Sigma) = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \quad \rho(\Upsilon) = \begin{pmatrix} 0 & 1 \\ (-1)^n & 0 \end{pmatrix}. \quad (7)$$

These elements have order 3 and 2 respectively. When considering spaces of stability conditions in Section 4 we will see the actions of these autoequivalences on the dual space $\operatorname{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{C})$, which are given by the transposes of the same matrices.

2.6. The autoequivalence group of the category \mathcal{D}_{∞} is much simpler.

Proposition 2.8. *There is an equality $\operatorname{Aut}_*(\mathcal{D}_{\infty}) = \operatorname{Aut}(\mathcal{D}_{\infty})$. Moreover*

- (a) *The group $\operatorname{Aut}(\mathcal{D}_{\infty}) \cong \mathbb{Z}$ with the Serre functor Σ being a generator.*
- (b) *There is a relation $\Sigma^3 = [1]$.*

Proof. This is easy and well-known. The Auslander-Reiten quiver for \mathcal{D}_{∞} is an infinite sequence

$$\cdots \rightarrow E[-1] \rightarrow S_1[-1] \rightarrow S_2 \rightarrow E \rightarrow S_1 \rightarrow S_2[1] \rightarrow E[1] \rightarrow \cdots$$

and Σ moves along this to the right by one place. □

It follows from this result that $\mathbb{P} \operatorname{Aut}(\mathcal{D}_{\infty}) \cong \mu_3$. Note that our use of the symbol Σ in Prop. 2.8 is reasonably consistent with our earlier use for the category \mathcal{D}_n for $n < \infty$. For example, Prop. 2.3 (a) continues to hold when $n = \infty$. Note also that Prop. 2.3 (b) becomes an infinite chain of tilts of non-full hearts in this case.

3. CONFORMAL MAPS

In this section we describe some explicit conformal maps which will be the analytic ingredients in the proof of our main result. We set $\omega = \exp(2\pi i/3)$. We introduce the Möbius transformation $T(z) = -(z+1)/z$ of order 3 defined by the transpose of the matrix $\rho(\Sigma)$ appearing in (7). Consider the unit circle $C_0 = \{z \in \mathbb{C} : |z| = 1\}$. It is easily checked that (cf. Figure 4)

$$C_{\pm} = T^{\pm 1}(C_0) = \{z \in \mathbb{C} : |z^{\pm 1} + 1| = 1\}, \quad (8)$$

and that $C_- = \{z \in \mathbb{C} : \operatorname{Re}(z) = -\frac{1}{2}\}$. These circles are illustrated in Figure 4, together with the region cut out by the inequalities

$$|z + 1| > 1, \quad |z + 1| > |z|. \quad (9)$$

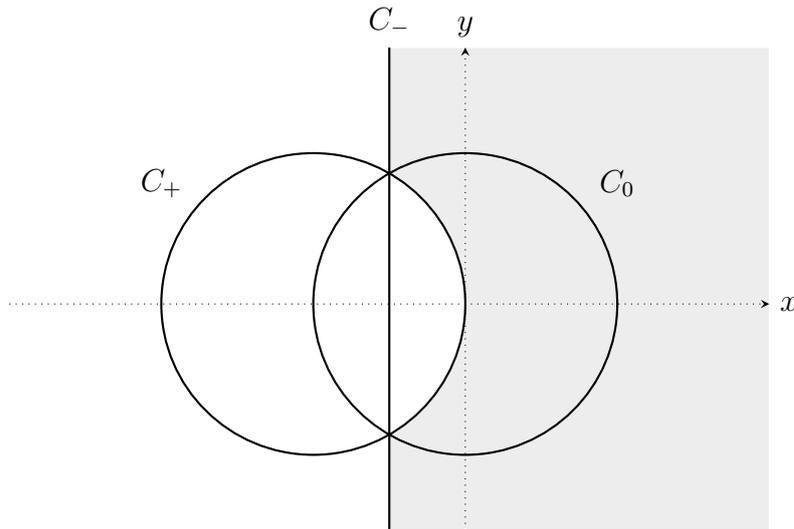


FIGURE 4. The circles C_0 and C_{\pm} .

3.1. Fix an integer $3 \leq n < \infty$ and set $\nu = (n-2)/2$. Consider the domain $R_n \subset \mathbb{C} \subset \mathbb{P}^1$ depicted in Figure 5. It is bounded by the line $\operatorname{Re}(z) = -\nu$ and by the curves ℓ_{\pm} which are the images under the map $z \mapsto (1/\pi i) \log(z)$ of the arcs of the circles C_{\pm} connecting $0^{\pm 1}$ and ω , where $0^{-1} = \infty$. We also consider splitting R_n into two halves R_n^{\pm} by dividing it along the line $\operatorname{Im}(z) = 0$, and we take R_n^- to be the part lying below the real axis. Note that the boundary of the domain R_n^- has three corners, namely $-\nu$, ∞ , $\frac{2}{3}$, and these occur

in anti-clockwise order. Carathéodory's extension of the Riemann mapping theorem, e.g. [1, Theorem 2.8.8], ensures that there is a unique biholomorphism

$$f_n: \mathcal{H} \rightarrow R_n^-$$

which extends homeomorphically over the boundary of the upper half plane $\mathcal{H} \subset \mathbb{P}^1$, and sends $(0, 1, \infty)$ to $(-\nu, \infty, \frac{2}{3})$.

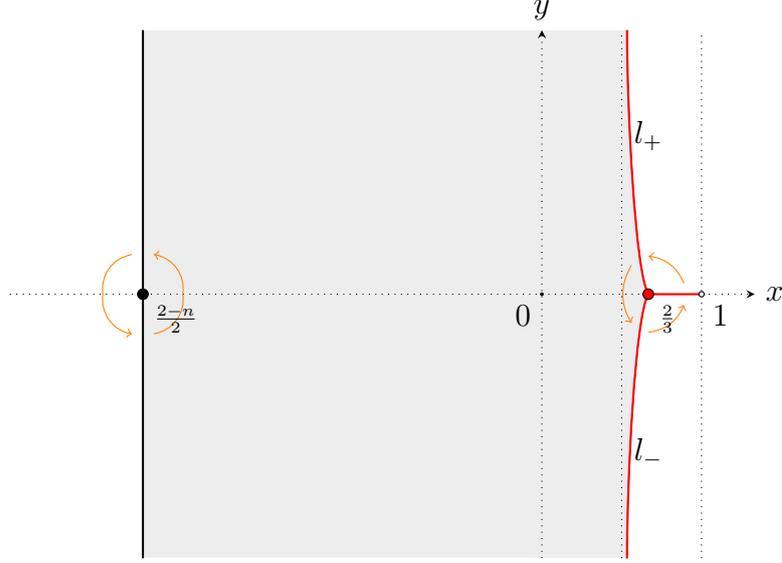


FIGURE 5. The region R_n .

Proposition 3.1. *The function f_n can be written explicitly as*

$$f_n(t) = \frac{1}{\pi i} \log \left(\frac{\phi_n^{(2)}(a, b)}{\phi_n^{(1)}(a, b)} \right),$$

where $t = -27b^2/4a^3$, and

$$\phi_n^{(i)}(a, b) = \int_{\gamma_i} (x^3 + ax + b)^{\frac{n-2}{2}} dx, \quad (10)$$

for appropriately chosen paths γ_i connecting zeroes of the integrand.

Note that the given expression for f_n depends only on the quantity t because rescaling (a, b) with weights $(4, 6)$ rescales both the functions $\phi_n^{(i)}$ with weight $3(n-2) + 2 = 3n-4$, and hence leaves their ratio unchanged. We will split the proof of Proposition 3.1 into two steps. In the first, we show that the function $g_n(t) = \exp(\pi i f_n(t))$ is given by the ratio of

solutions to a hypergeometric equation, and in the second we show that the periods (10) satisfy the same equation.

3.2. The first part of the proof of Proposition 3.1 is a minor extension of the usual proof of the Schwarz triangle theorem, see e.g. [8, Section 4.7], [16, Section V.7]. The basic point is that since the map g_n maps the three connected components of $\mathbb{R} \setminus \{0, 1\}$ to arcs of circles in \mathbb{P}^1 , it is given by a ratio of solutions to a hypergeometric equation, whose coefficients are determined by the angles at which these circles meet. In contrast to the usual setting of the Schwarz triangle theorem, the map g_n is not injective on the upper half-plane, and one of the angles is $> \pi$. Although only small extensions to the usual argument are required to deal with this issue, for the convenience of the reader we will give the argument in full.

The main tool in the proof is the Schwarzian derivative, whose key properties we briefly recall here. Let $f: U \rightarrow \mathbb{C}$ be a holomorphic function on a domain $U \subset \mathbb{C}$. The Schwarzian derivative is defined by the expression

$$\mathcal{S}f(t) = \left(\frac{f''(t)}{f'(t)} \right)' - \frac{1}{2} \left(\frac{f''(t)}{f'(t)} \right)^2.$$

We shall need the following properties:

(S1) Suppose that $f'(t)$ is non-vanishing on a domain $U \subset \mathbb{C}$. Thus $Q(t) = -\mathcal{S}f(t)/2$ defines a holomorphic function $Q: U \rightarrow \mathbb{C}$. Then there exist two holomorphic solutions $y_1(t), y_2(t)$ to the differential equation

$$y''(t) - Q(t)y(t) = 0,$$

such that $f(t) = y_1(t)/y_2(t)$.

(S2) If $R \in \text{PGL}_2(\mathbb{C})$ is a Möbius transformation, then $\mathcal{S}(R \circ f) = \mathcal{S}(f)$.

(S3) If $f: U \rightarrow \mathbb{C}$ and $g: V \rightarrow \mathbb{C}$ are composable holomorphic functions, then

$$\mathcal{S}(f \circ g) = (\mathcal{S}(f) \circ g) \cdot (g')^2 + \mathcal{S}(g).$$

We shall also need the following standard computation.

Lemma 3.2. *Suppose that $k: U \rightarrow \mathbb{C}$ is a holomorphic function defined in a neighbourhood of $t_0 \in U \subset \mathbb{C}$, which satisfies $k(t_0) = 0$ and $k'(t_0) \neq 0$. Take $\alpha \in \mathbb{R}$ and set $q(z) = z^\alpha$, viewed as a multi-valued function on \mathbb{C}^* . Then $\mathcal{S}(q \circ k)$ is a single-valued meromorphic function at $t = t_0$, and satisfies*

$$\mathcal{S}(q \circ k) = \frac{1 - \alpha^2}{2(t - t_0)^2} + O\left(\frac{1}{t - t_0}\right). \quad (11)$$

Proof. Shrinking U if necessary we can assume that the only zero of k on U is at $t = t_0$. Then $q \circ k$ is a multi-valued holomorphic function on $U \setminus \{t_0\}$, but the associated Schwarzian derivative $\mathcal{S}(q \circ k)$ is clearly single-valued, since the various branches of q differ by multiplication by a constant factor. A direct calculation shows that

$$\mathcal{S}q(z) = \frac{1 - \alpha^2}{2z^2}.$$

The result follows by applying property (S3) on subdomains of $U \setminus \{t_0\}$ on which $q \circ z$ is single-valued. \square

3.3. Consider the function $g_n(t) = \exp(\pi i f_n(t))$. By definition this is holomorphic on the upper half-plane $\mathcal{H} \subset \mathbb{C}$, with non-vanishing derivative. Our first objective is to show that the Schwarzian derivative $\mathcal{S}g_n(t)$ extends to a meromorphic function on the Riemann sphere \mathbb{P}^1 , and to understand its leading order behaviour at the points $t = 0, 1, \infty$.

Take a point $t \in \mathbb{R} \setminus \{0, 1\}$ and a neighbourhood $t \in B \subset \mathbb{C} \setminus \{0, 1\}$ invariant under complex conjugation. By construction, the function g_n is holomorphic on $B \cap \mathcal{H}$, with non-vanishing derivative. Moreover, g_n extends continuously over $B \cap \mathbb{R}$, and the extension is injective, and maps $B \cap \mathbb{R}$ onto the arc of a circle $C \subset \mathbb{P}^1$. Let R be a Möbius transformation taking this circle C to the real axis $\mathbb{R} \subset \mathbb{C}$. Then the composite $R \circ g_n$ is real-valued on $B \cap \mathbb{R}$, so by the Schwarz reflection principle extends to a holomorphic function on B . It is easy to see that this function is locally univalent, and hence has non-vanishing derivative. It follows that $\mathcal{S}(R \circ g_n)$ extends to a holomorphic function on B which is real-valued on $B \cap \mathbb{R}$. But by property (S2), this function coincides with $\mathcal{S}g_n$. Thus we conclude that $\mathcal{S}g_n$, which is holomorphic on $\mathcal{H} \subset \mathbb{C}$ by construction, extends to a holomorphic function on a neighbourhood of each point of $\mathbb{R} \setminus \{0, 1\}$, and is real-valued on the real axis. Applying the reflection principle again we find that $\mathcal{S}g_n$ extends to a holomorphic function on $\mathbb{C} \setminus \{0, 1\}$. It thus remains to examine its behaviour at the points $\{0, 1, \infty\}$.

Consider first the point $t = 0$, and again take a small neighbourhood $0 \in B \subset \mathbb{C}$ invariant under complex conjugation. As before, the function g_n is holomorphic on $B \cap \mathcal{H}$, with non-vanishing derivative, and extends continuously to $B \cap \mathbb{R}$. This time the extension maps the two halves of the real axis $B \cap \mathbb{R}_\pm$ into the arcs of two circles C_\pm which meet at an angle $\pi/2$ at the point $g_n(0) = i^{2-n}$. Let R be a Möbius transformation which maps this point to 0, and the circles C_- and C_+ to the circles \mathbb{R} and $\mathbb{R} \cdot i$ respectively. The function $k(t) = R(g_n(t))^2$ is then holomorphic on $B \cap \mathcal{H}$, and continuous and real-valued on $B \cap \mathbb{R}$, so by the reflection principle extends to a holomorphic function $k: B \rightarrow \mathbb{C}$

satisfying $k(0) = 0$. It is easy to see that since g_n is locally univalent on $B \cap \mathcal{H}$, the same is true of k , and hence $k'(0) \neq 0$. Applying Lemma 3.2 to the function $R(g_n(t)) = k(t)^{1/2}$, and using property (S2), we conclude that $\mathcal{S}g_n$ has a double pole at $t = 0$, with leading order behaviour given by the right-hand side of (11), with $t_0 = 0$ and $\alpha = \frac{1}{2}$.

A very similar argument applies at the point $t = \infty$. It is convenient to set $u = t^{-1}$ and consider the behaviour of the map $h_n(u) = g_n(u^{-1})$ near the point $u = 0$. The above argument then applies, with the two circles C_{\pm} now meeting at an angle $\pi/3$ at the point $h_n(0) = \omega$. We consider a Möbius transformation R which maps this point to 0, and the circles C_- and C_+ to the rays of argument 0 and $\pi/3$ respectively. We then consider the function $k(u) = R(h_n(u))^3$. We conclude that $\mathcal{S}h_n$ has a double pole at $u = 0$, with leading order behaviour given by (3.2) with $\alpha = \frac{1}{3}$. Using property (S3) it follows that the leading order behaviour of $\mathcal{S}g_n$ near $t = \infty$ is

$$\mathcal{S}g_n(t) = \frac{4}{9t^2} + O(t^{-3}).$$

In particular, $\mathcal{S}g_n(t)$ has a double zero at $t = \infty$.

Finally, we consider the point $t = 1$. We again take a small neighbourhood $1 \in B \subset \mathbb{C}$ invariant under complex conjugation. A little care is required since the function g_n is not injective on $B \cap \mathcal{H}$. Nonetheless, the two components of $B \cap \mathbb{R} \setminus \{1\}$ are mapped by g_n to the two circles r_- and $\mathbb{R} \cdot i^{2-n}$, which meet at an angle of $\frac{1}{2}(n-1)\pi$ at the point $g_n(1) = \infty$. We can therefore take a Möbius transformation R which maps ∞ to 0, and maps these two arcs of circles to the rays $\mathbb{R} \cdot i^{n-1}$ and \mathbb{R} .

Introduce the functions

$$l(t) = R(g_n(t))^{1/(n-1)}, \quad k(t) = l(t)^2,$$

defined by some fixed branch of $z \mapsto z^{1/(n-1)}$ near 0. Note that if we set $p(z) = z^{n-1}$ then we can write

$$l = (p^{-1} \circ R \circ p) \circ (p^{-1} \circ g_n).$$

Since $R(\infty) = 0$, we have $R(z) = (cz + d)^{-1}$, for some $c, d \in \mathbb{C}$ with $c \neq 0$, and it is then easily checked that $p^{-1} \circ R \circ p$ is holomorphic at ∞ , with non-vanishing derivative. The region R_n^- is closed under the map $z \mapsto z/(n-1)$, so the function $\exp(\pi i f_n(t)/(n-1))$, which is a branch of $p^{-1} \circ g_n(t)$, is holomorphic on $B \cap \mathcal{H}$, and has a continuous extension to $B \cap \mathbb{R}$ sending $t = 1$ to $z = \infty$. Since the region R_n is contained in the strip $|\operatorname{Re}(z)| < \frac{1}{2}(n-1)$,

this function is moreover injective. But note that l cannot be extended to a holomorphic function on B since it maps the two components of $B \cap \mathbb{R} \setminus \{1\}$ to circles meeting at an angle of $\pi/2$.

We have now shown that $k(t) = l(t)^2$ is a holomorphic function on $B \cap \mathcal{H}$, which extends continuously to $B \cap \mathbb{R}$ and satisfies $k(1) = 0$. By construction it is real-valued on $B \cap \mathbb{R}$, so by the reflection principle extends to a holomorphic function on B , and the last remarks of the previous paragraph shows that $k'(1) \neq 0$. Setting $\alpha = (n-1)/2$ we can apply Lemma 3.2 to the function $R(g_n(t)) = k(t)^\alpha$ to conclude that $\mathcal{S}g_n$ has a double pole at $t = 1$, with leading order behaviour given by (11).

3.4. Consider again the function $g_n(t) = \exp(\pi i f_n(t))$ on \mathcal{H} . The property (S1) shows that $g_n(t)$ is given by the ratio $y_2(t)/y_1(t)$ of two solutions to the differential equation

$$y''(t) - Q(t)y(t) = 0, \quad Q(t) = -\frac{1}{2}\mathcal{S}g_n(t). \quad (12)$$

The function $Q(t)$ extends to a meromorphic on \mathbb{P}^1 , with double poles at $t = 0, 1$, no other poles, and a double zero at infinity. It follows that it is uniquely determined by the leading terms in the Laurent expansions at $t = 0, 1, \infty$. Indeed, if two such functions had the same leading terms at these points, then their difference would have a triple zero at infinity, at worst simple poles at $t = 0, 1$, and no other poles, and any such function is zero.

It remains to show that the solutions $y_i(t)$ can be written as period integrals of the form (10). We will show in the next subsection that these period integrals satisfy a hypergeometric equation

$$t(t-1)p''(t) + ((a+b+1)t - c)p'(t) + ab \cdot p(t) = 0, \quad (13)$$

with parameters

$$a = \frac{1}{2}\left(\frac{1}{3} - \nu\right), \quad b = -\frac{1}{2}\left(\frac{1}{3} + \nu\right), \quad c = \frac{1}{2}.$$

A standard calculation, e.g. [16, Section 7], puts this in the Q -form (12), with $-2Q(t)$ having leading order behaviour at the points $t = 0, 1, \infty$ given by expressions of the form (11), suitably interpreted at $t = \infty$. The relevant angles are

$$\alpha_0 = 1 - c = \frac{1}{2}, \quad \alpha_1 = c - a - b = \frac{1}{2} + \nu, \quad \alpha_\infty = b - a = \frac{1}{3}.$$

Since these agree with what we computed above, it follows that the differential equation (12) coincides with the Q -form of the equation (13).

The final thing to check is that we can take the paths γ_i in the integral (10) to be integral cycles rather than complex linear combinations of such cycles. Note that there is

a unique solution to (13) up to scale which vanishes at $t = \infty$. Since $g_n(t) \rightarrow 0$ as $t \rightarrow \infty$ in \mathcal{H} this solution must coincide with y_2 . But this is indeed given by an integral cycle, namely the vanishing cycle. Since the other solution y_1 is obtained by applying monodromy transformations, it follows that it is also integral. Indeed, at $t = 0$ the monodromy has order 2 and preserves the image of the real axis under g_n , namely the unit circle. It follows that it is $z \mapsto \pm z^{-1}$.

3.5. In this section we prove that the twisted periods (10) satisfy the hypergeometric differential equation (13). Let us consider $a \in \mathbb{C}$ to be fixed. As always we set $\nu = (n-2)/2$ with $n \geq 3$. Consider the function

$$f_a(h) = h^{-(\nu+1)} \int e^{h(u^3+au)} du = h^{-(\nu+\frac{4}{3})} \int e^{w^3+h^{2/3}aw} dw,$$

where we set $w = h^{1/3} \cdot u$. Introduce the differential operator

$$D_h = h\partial_h + \nu + 1.$$

Then

$$(D_h + \frac{1}{3})f_a(h) = \frac{2}{3} \cdot h^{-(\nu+\frac{4}{3})} \int h^{\frac{2}{3}} \cdot aw \cdot e^{w^3+h^{2/3}aw} dw.$$

Repeating we obtain

$$(D_h - \frac{1}{3})(D_h + \frac{1}{3})f_a(h) = \frac{4}{9} \cdot h^{-(\nu+\frac{4}{3})} \int h^{\frac{4}{3}} \cdot (aw)^2 \cdot e^{w^3+h^{2/3}aw} dw,$$

and it follows that

$$\left((D_h - \frac{1}{3})(D_h + \frac{1}{3}) + \frac{4a^3}{27} \cdot h^2 \right) f_a(h) = h^{-\nu} \cdot \frac{4a^2}{27} \int (3w^2 + h^{\frac{2}{3}}a) \cdot e^{w^3+h^{2/3}aw} dw = 0.$$

Now consider the inverse Laplace transform

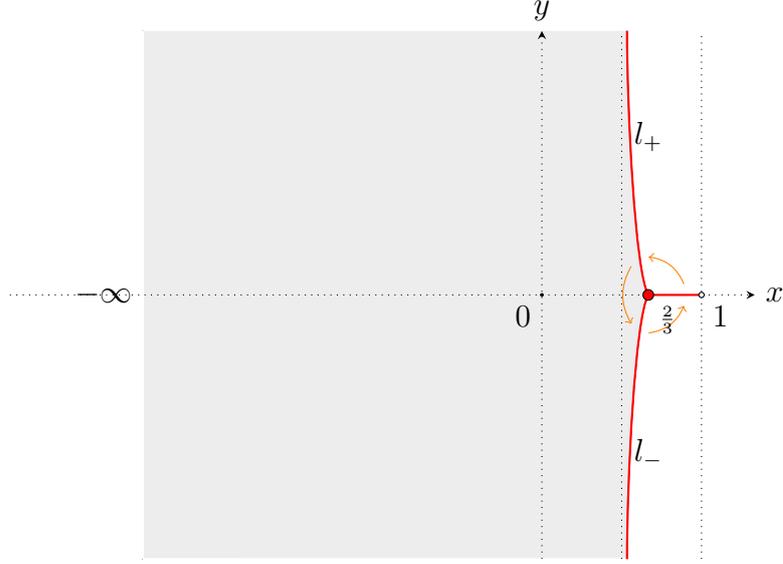
$$p_a(b) = \int e^{bh} f_a(h) dh = \int \int e^{h(u^3+au+b)} \cdot h^{-(\nu+1)} du dh.$$

Exchanging the order of integration and using

$$\int e^{h(s+b)} \cdot h^{-(\nu+1)} dh = \Gamma(-\nu) \cdot (s+b)^\nu,$$

where $\Gamma(x)$ denotes the gamma function, this becomes

$$p_a(b) = \Gamma(-\nu) \cdot \int (x^3 + ax + b)^\nu dx.$$

FIGURE 6. The region R_∞

Under the inverse transform $h\partial_h$ becomes $-b\partial_b - 1$, so the transform of the operator D_h is $-b\partial_b + \nu$. The twisted periods therefore satisfy the differential equation

$$\left((-b\partial_b + \nu + \frac{1}{3})(-b\partial_b + \nu - \frac{1}{3}) + \frac{4a^3}{27} \cdot \partial_b^2 \right) p_a(b) = 0.$$

Setting $t = -27b^2/4a^3$ this becomes

$$\left(t(t-1)\partial_t^2 + \left((1-\nu)t - \frac{1}{2} \right) \partial_t + \frac{1}{4} \left(\nu^2 - \frac{1}{9} \right) \right) p(t) = 0, \quad (14)$$

which coincides with (13).

3.6. For the case $n = \infty$ we instead consider the region R_∞ depicted in Figure 6. It is bounded by the same two curves ℓ_\pm . We again consider the half region R_∞^- consisting of points of R_∞ with negative imaginary part. The boundary has two corners, namely $\frac{2}{3}$ and ∞ . Carathéodory's extension of the Riemann mapping theorem ensures that there is a biholomorphism

$$f_\infty: \mathcal{H} \rightarrow R_\infty^+ \quad (15)$$

which extends continuously over the boundary, and sends the points $(0, \infty)$ to $(\frac{2}{3}, \infty)$. Considering the orientations of the two regions shows that $\mathbb{R}_{<0}$ is mapped to ℓ_- , and $\mathbb{R}_{>0}$ to the open interval of the real axis $(-\infty, 2/3)$. Note that in this case the map $f_\infty(t)$ is not

unique: precomposing it with a dilation of the form $t \mapsto \lambda \cdot t$ with $\lambda \in \mathbb{R}_{>0}$ gives another suitable map.

Proposition 3.3. *One possible choice for the function f_∞ can be written explicitly as*

$$f_\infty(t) = \frac{1}{\pi i} \log \left(\frac{\phi_\infty^{(2)}(a, b)}{\phi_\infty^{(1)}(a, b)} \right),$$

where $t = a^3$, b is arbitrary, and

$$\phi_\infty^{(i)}(a, b) = \int_{\delta_i} \exp(x^3 + ax + b) dx, \quad (16)$$

for certain contours $\delta_i: \mathbb{R} \rightarrow \mathbb{C}$ which satisfy $\operatorname{Re} \delta(x)^3 \rightarrow -\infty$ as $x \rightarrow \pm\infty$.

Note that the function f_∞ only depends on a , because changing b multiplies both functions $\phi_\infty^{(i)}(a, b)$ by an equal factor, and hence leaves their ratio unchanged.

Proof. We shall give a direct proof in this case. Consider the Airy function

$$\operatorname{Ai}(a) = \frac{1}{2\pi i} \int_{\delta} \exp\left(\frac{x^3}{3} - ax\right) dx,$$

where the path of integration $\delta: \mathbb{R} \rightarrow \mathbb{C}$ tends to ∞ along the rays of argument $\pm\pi/3$ as $x \rightarrow \pm\infty$. We shall need two standard properties of this function, see e.g. [8, Section 8.9] or [17, Section 11.8]. Firstly, it is an entire function of $a \in \mathbb{C}$, with zeroes only on the negative real axis. Secondly, there is an asymptotic expansion of $\operatorname{Ai}(a)$ as $a \rightarrow \infty$ which implies that there is a constant $R \in \mathbb{C}$ such that

$$\operatorname{Ai}(a) \cdot \exp\left(\frac{2a^{\frac{3}{2}}}{3}\right) \cdot a^{\frac{1}{4}} \rightarrow R, \quad (17)$$

as $a \rightarrow \infty$ in any closed subsector of $\mathbb{C} \setminus \mathbb{R}_{<0}$. In particular, it follows that $\operatorname{Ai}(a) \rightarrow 0$ as $a \rightarrow \infty$ in the closed sector $|\arg(a)| \leq \pi/3$.

Let us introduce the functions

$$g(a) = \frac{\omega^2 \operatorname{Ai}(-\omega^2 a)}{\omega \operatorname{Ai}(-\omega a)}, \quad h(a) = \frac{1}{\pi i} \log g(a), \quad f(t) = h(t^{1/3}).$$

By what was said above, the function $g(a)$ is meromorphic, with zeroes and poles only on the rays $\mathbb{R}_{>0} \cdot \omega^{\pm 1}$. In particular, it is regular and non-vanishing on a neighbourhood of the closed sector $\Sigma \subset \mathbb{C}$ bounded by the rays of argument 0 and $\pi/3$. Choosing the principal

branch of \log , the function $h(a)$ is then well-defined and holomorphic on this sector, and satisfies $h(0) = 2/3$. To define the function $f(t)$ we choose the branch of the cube root function on \mathcal{H} which lies in Σ . Then $f(t)$ is also well-defined and holomorphic on \mathcal{H} .

We will show that the function $f(t)$ gives a possible choice for the map (15). In view of the obvious relation

$$\omega^i \cdot \text{Ai}(\omega^i a) = \frac{1}{2\pi i} \int_{\omega^{i \cdot \delta}} \exp\left(\frac{x^3}{3} - ax\right) dx, \quad (18)$$

this will be enough to prove the result. Note that the factor of $\frac{1}{3}$ multiplying x^3 can be absorbed by rescaling t by a factor of 3, which is a transformation of the form mentioned just before the statement of the Proposition.

It is an immediate consequence of (18) that

$$\text{Ai}(a) + \omega \text{Ai}(\omega a) + \omega^2 \text{Ai}(\omega^2 a) = 0,$$

and it follows that

$$g(\omega^i a) = T^i(g(a)), \quad (19)$$

where $T(z) = -(z+1)/z$ is the Möbius transformation of order 3 appearing in (8). The definition of the Airy function implies that $\overline{\text{Ai}(a)} = \text{Ai}(\bar{a})$, and hence that

$$\overline{g(a)} = g(\bar{a})^{-1}. \quad (20)$$

It follows that g maps points on the real axis onto the unit circle C_0 . Using (8) and (19) this implies that g maps the ray of argument $\pi/3$, which is part of the circle $\mathbb{R} \cdot \omega^2$, to the circle $C_- = T^{-1}(C_0)$.

Putting all this together we find that the function f is holomorphic on the upper half-plane and extends continuously to the boundary. It satisfies $f(0) = \frac{2}{3}$ and maps the boundary rays $\mathbb{R}_{<0}$ and $\mathbb{R}_{>0}$ to the images under $(1/i\pi)\log(z)$ of the circles C_- and C_0 respectively. The asymptotic property (17) shows that f extends continuously over ∞ .

Let us now compose f with the inverse of the Riemann map (15). The resulting function $k = f \circ f_\infty^{-1}$ defines a holomorphic map $k: \mathcal{H} \rightarrow \mathcal{H}$ which extends continuously over the boundary, and satisfies $k(\mathbb{R}) \subset \mathbb{R}$. By the reflection principle, k extends to a meromorphic function on \mathbb{P}^1 , which has a single pole at ∞ . The asymptotic formula (17) shows that $f(t)$ behaves like a constant multiple of $t^{1/2}$ as $t \rightarrow \infty$. It follows from this that the pole of k is simple, and hence that $k(t) = \lambda \cdot t$ for some $\lambda \in \mathbb{R}_{>0}$. \square

4. STABILITY CONDITIONS

In this section we analyse the space of stability conditions on the categories D_n and give the proof of the main result, Theorem 1.1. We refer to [2] and [4, Section 7] for basic definitions concerning stability conditions. The basic idea is to identify a fundamental domain \mathcal{U}_n for the action of the group $\mathbb{P}\mathrm{Aut}_*(D_n)$ on the space of projective stability conditions $\mathbb{P}\mathrm{Stab}_*(D_n)$, and show that its image under the logarithmic central charge map can be identified with the region R_n of Section 3.

4.1. Take $3 \leq n \leq \infty$ and let $\mathrm{Stab}(D_n)$ denote the space of stability conditions on the triangulated category D_n . In particular, all stability conditions in this space are assumed to satisfy the support property. There is a distinguished connected component $\mathrm{Stab}_*(D_n) \subset \mathrm{Stab}(D_n)$ which contains all stability conditions $\sigma = (Z, \mathcal{P})$ whose heart $\mathcal{P}((0, 1])$ coincides with the canonical heart $\mathcal{A}_n \subset D_n$.

The group of autoequivalences $\mathrm{Aut}(D_n)$ acts on $\mathrm{Stab}(D_n)$ by

$$\Phi \cdot (Z, \mathcal{P}) = (Z', \mathcal{P}'), \quad Z'(E) = Z(\Phi^{-1}(E)), \quad \mathcal{P}'(\phi) = \Phi(\mathcal{P}(\phi)).$$

Recall that an autoequivalence $\Phi \in \mathrm{Aut}(D_n)$ is negligible if it fixes the objects S_1, S_2 . Any such autoequivalence acts trivially on an open subset of $\mathrm{Stab}_*(D_n)$, and hence on the whole connected component $\mathrm{Stab}_*(D_n)$. Conversely, any autoequivalence which acts trivially on $\mathrm{Stab}_*(D_n)$ must be negligible.

There is an action of \mathbb{C} on $\mathrm{Stab}(D_n)$, commuting with the action of $\mathrm{Aut}(D_n)$, given by the rule

$$\lambda \cdot (Z, \mathcal{P}) = (Z', \mathcal{P}'), \quad Z'(E) = \exp(-i\pi\lambda) \cdot Z(E), \quad \mathcal{P}'(\phi) = \mathcal{P}(\phi + \mathrm{Re}(\lambda)).$$

Note that the action of the shift $[k] \in \mathrm{Aut}(D_n)$ coincides with the action of $k \in \mathbb{C}$. This action of \mathbb{C} on $\mathrm{Stab}(D_n)$ is free and proper, and we can consider the quotient complex manifold

$$\mathbb{P}\mathrm{Stab}(D_n) = \mathrm{Stab}(D_n)/\mathbb{C},$$

together with the distinguished connected component $\mathbb{P}\mathrm{Stab}_*(D_n) \subset \mathbb{P}\mathrm{Stab}(D_n)$. The central charge map induces a local isomorphism of complex manifolds

$$\mathbb{P}\mathrm{Stab}_*(D_n) \rightarrow \mathbb{P}\mathrm{Hom}_{\mathbb{Z}}(K_0(D_n), \mathbb{C}) \cong \mathbb{P}^1. \quad (21)$$

We refer to points of the quotient space $\mathbb{P}\mathrm{Stab}_*(D_n)$ as *projective stability conditions*. Note that a projective stability condition determines full subcategories of stable and

semistable objects of D_n , and a well-defined phase difference $\phi(B) - \phi(A) \in \mathbb{R}$ for any two semistable objects $A, B \in D_n$.

4.2. Let us again take $3 \leq n \leq \infty$ and define the following subsets of the space of projective stability conditions $\mathbb{P}\text{Stab}_*(D_n)$.

Definition 4.1. (a) Let $\mathcal{U}_n^+ \subset \mathbb{P}\text{Stab}_*(D_n)$ be the subset of projective stability conditions for which the objects S_1 and S_2 are both stable and

$$0 \leq \phi(S_2) - \phi(S_1) < (n-2)/2. \quad (22)$$

(b) Let $\mathcal{U}_n^- \subset \mathbb{P}\text{Stab}_*(D_n)$ be the subset of projective stability conditions for which the objects S_1, S_2 and E are all stable, and both of the following inequalities hold:

$$\phi(S_1) - \phi(S_2) < \phi(S_2[1]) - \phi(E), \quad \phi(S_1) - \phi(S_2) < \phi(E) - \phi(S_1[-1]). \quad (23)$$

(c) Let $\mathcal{U}_n = \mathcal{U}_n^- \cup \mathcal{U}_n^+ \subset \mathbb{P}\text{Stab}_*(D_n)$.

In Prop. 4.4 below we will show that the subset \mathcal{U}_n is a fundamental domain for the action of $\mathbb{P}\text{Aut}_*(D_n)$. The following result is a first step towards this.

Lemma 4.2. *For any $3 \leq n \leq \infty$ the subset $\mathcal{U}_n \subset \mathbb{P}\text{Stab}_*(D_n)$ is open. Moreover, if a projective stability condition lies in the boundary of \mathcal{U}_n then one of the following two conditions is satisfied:*

(i) *the set of stable objects up to shift is $\{S_1, S_2\}$, and*

$$\phi(S_2) - \phi(S_1) = (n-2)/2; \quad (24)$$

(ii) *the set of stable objects up to shift is $\{S_1, S_2, E\}$, and the inequalities*

$$\phi(S_1) - \phi(S_2) \leq \phi(S_2[1]) - \phi(E), \quad \phi(S_1) - \phi(S_2) \leq \phi(E) - \phi(S_1[-1]) \quad (25)$$

hold, with one or both being an equality.

In particular, when $n = \infty$, only case (ii) can occur.

Proof. It is a general fact [4, Prop. 7.6] that if an object S is stable with respect to some stability condition then the same is true for all stability conditions in an open neighbourhood. Moreover the phase $\phi(S)$ varies continuously in this neighbourhood. It follows immediately that if $\bar{\sigma} \in \mathcal{U}_n$ satisfies $\phi(S_1) \neq \phi(S_2)$, then an open neighbourhood of $\bar{\sigma}$ also lies in \mathcal{U}_n . On the other hand, any projective stability condition for which S_1, S_2 are semistable with $\phi(S_1) = \phi(S_2)$ can be lifted to a stability condition with standard heart \mathcal{A}_n . The object E is then necessarily semistable with $\phi(E) = \phi(S_i)$, and all points in an

open neighbourhood of $\bar{\sigma}$ lie in either \mathcal{U}_n^- or \mathcal{U}_n^+ depending on the sign of $\phi(S_2) - \phi(S_1)$. Thus \mathcal{U}_n is open.

Consider $\bar{\sigma} \in \mathbb{P}\text{Stab}_*(D_n)$ lying in the closure of \mathcal{U}_n^+ . Then the objects S_1 and S_2 are semistable, and the non-strict version of the inequality (22) holds. In particular $n < \infty$. It follows that $\bar{\sigma}$ can be lifted to a stability condition whose heart contains, and hence is equal to, the heart $\mathcal{A}_n(k) = \langle S_1[k], S_2 \rangle$ of Prop. 2.3, for some $0 \leq k \leq (n-2)/2$. This implies that the objects S_1 and S_2 are in fact stable. If $\bar{\sigma}$ lies in the boundary of \mathcal{U}_n^+ , so that $\bar{\sigma} \notin \mathcal{U}_n^+$, it follows that the equality (24) must hold.

Suppose now that $\bar{\sigma} \in \mathbb{P}\text{Stab}_*(D_n)$ lies in the closure of \mathcal{U}_n^- . Then the objects S_1 , S_2 and E are all semistable, and the the inequalities (25) both hold. The triangle associated to the short exact sequence (3) shows that for any lift

$$\phi(S_2) \leq \phi(E) \leq \phi(S_1) \leq \phi(S_2) + 1.$$

We cannot have $\phi(S_1) = \phi(S_2) + 1$ since then either $\phi(E) = \phi(S_1)$, or $\phi(E) = \phi(S_2)$, and in either case one of the equalities (25) fails. Thus $\phi(S_1) - \phi(S_2) < 1$, and hence $\bar{\sigma}$ can be lifted to a stability condition with heart \mathcal{A}_n . If $\bar{\sigma}$ lies in the boundary of \mathcal{U}_n we cannot have $\phi(S_1) = \phi(S_2)$ since as above, such stability conditions lie in \mathcal{U}_n . It follows that S_1 , S_2 and E are all in fact stable, and again, since $\bar{\sigma} \notin \mathcal{U}_n$, at least one of the inequalities (25) must be an equality. \square

4.3. The following result gives the link with the regions considered in Section 3.

Proposition 4.3. *Take $3 \leq n \leq \infty$. Then the function*

$$g_n(\bar{\sigma}) = \frac{1}{\pi i} \log \left(\frac{Z(S_1)}{Z(S_2)} \right)$$

defines a biholomorphic map between the regions \mathcal{U}_n and R_n , where the branch of log is chosen so that $\text{Re } g_n(\sigma) = \phi(S_1) - \phi(S_2)$.

Proof. It follows from the proof of Lemma 4.2 that if $\bar{\sigma} \in \mathcal{U}_n$ then the set of stable objects up to shift is precisely $\{S_1, S_2\}$ or $\{S_1, S_2, E\}$ depending on whether $\bar{\sigma} \in \mathcal{U}_n^+$ or $\bar{\sigma} \in \mathcal{U}_n^-$. Indeed, if $\bar{\sigma} \in \mathcal{U}_n^+$ then it can be lifted to a stability condition whose heart is $\mathcal{A}_n(k) = \langle S_1[k], S_2 \rangle$ for some $0 \leq k \leq (n-2)/2$. If $k > 0$ this heart is semisimple, so the only stable objects are S_1 , S_2 up to shift, and if $k = 0$ the same result holds since $\phi(S_1) \leq \phi(S_2)$. On the other hand, if $\bar{\sigma} \in \mathcal{U}_n^-$ then $0 < \phi(S_1) - \phi(S_2) < 1$, and $\bar{\sigma}$ can

be lifted to a stability condition with heart \mathcal{A}_n , which implies that the stable objects are precisely S_1 , S_2 and E .

Let us consider the two halves \mathcal{U}_n^\pm of the fundamental domain separately. The region \mathcal{U}_n^+ consists of projective stability conditions $\bar{\sigma}$ for which the only stable objects are S_1 and S_2 up to shift. It is easy to see that any such $\bar{\sigma}$ is determined by the value of $z = g_n(\bar{\sigma}) \in \mathbb{C}$, and using the finite-length hearts $\mathcal{A}_n(k) = \langle S_1[k], S_2 \rangle$ of Prop. 2.3 it is also easy to see that all values of $z \in \mathbb{C}$ compatible with the constraint (22) are realised. Thus g_n maps the region \mathcal{U}_n^+ bijectively onto the region $(2-n)/2 < \operatorname{Re}(z) \leq 0$.

We showed above that for all projective stability conditions $\bar{\sigma}$ in the region \mathcal{U}_n^- , the objects S_1, S_2 are stable, and $0 < \phi(S_1) - \phi(S_2) < 1$. It is easy to see that the set of such projective stability conditions is mapped bijectively by g_n onto the strip $0 < \operatorname{Re}(z) < 1$. Since $Z(E) = Z(S_1) + Z(S_2)$, an elementary piece of geometry shows that

$$\phi(E) - \phi(S_2) < \phi(S_1) - \phi(E) \iff |Z(S_2)| < |Z(S_1)|.$$

Recalling from Lemma 2.2 and Prop. 2.8 the action of Σ on objects, and noting that by (7) it induces the Möbius transformation T of Section 3 on the projective space (21), we find that the inequalities (23) can be alternatively expressed as

$$|Z(S_1) + Z(S_2)| > |Z(S_2)|, \quad |Z(S_1) + Z(S_2)| > |Z(S_1)|.$$

In terms of $z = Z(S_2)/Z(S_1)$ this gives the inequalities (9) defining the grey area of Figure 4. Thus the image of \mathcal{U}_n^- is the part of the strip $0 < \operatorname{Re}(z) < 1$ bounded by the images of the circles C_\pm under the map $(1/i\pi) \log(z)$. \square

4.4. The following result will be a key ingredient in our proof of Theorem 1.1.

Proposition 4.4. *Take $3 \leq n \leq \infty$. Then*

- (i) *an autoequivalence $\Phi \in \operatorname{Aut}(\mathbb{D}_n)$ is reachable precisely if its action on $\operatorname{Stab}(\mathbb{D}_n)$ preserves the connected component $\operatorname{Stab}_*(\mathbb{D}_n) \subset \operatorname{Stab}(\mathbb{D}_n)$;*
- (ii) *the open subset $\mathcal{U}_n \subset \mathbb{P}\operatorname{Stab}_*(\mathbb{D}_n)$ is a fundamental domain for the action of $\mathbb{P}\operatorname{Aut}_*(\mathbb{D}_n)$.*

Proof. Suppose that $\bar{\sigma}$ lies in the intersection $\mathcal{U}_n \cap \Phi^{-1}(\mathcal{U}_n)$ for some $\Phi \in \operatorname{Aut}(\mathbb{D}_n)$, so that $\bar{\sigma} \in \mathcal{U}_n$ and also $\Phi(\bar{\sigma}) \in \mathcal{U}_n$. Consider the case that $\bar{\sigma} \in \mathcal{U}_n^+$. Then Φ preserves the set of stable objects $\{S_1, S_2\}$ up to shift. Suppose that Φ exchanges the two objects up to shift. Then given the degrees of the maps between S_1 and S_2 it follows that $n < \infty$. Using

Lemma 2.2 we then obtain

$$\Phi(S_1, S_2) = \Upsilon(S_1, S_2) = (S_2, S_1[n-2])$$

up to shift, and the inequality (22) then gives a contradiction. Thus Φ preserves the objects S_i up to a shift, and it follows easily that $\Phi[d]$ is negligible for some $d \in \mathbb{Z}$, and hence that Φ is reachable and defines the identity element of $\mathbb{P}\text{Aut}_*(D_n)$.

Consider now the case when $\bar{\sigma} \in \mathcal{U}_n^-$. Then Φ preserves the set of stable objects $\{S_1, S_2, E\}$ up to shift. Suppose that Φ defines a non-trivial permutation of these objects up to shift. From Lemma 2.2 and Prop. 2.8 we know that

$$\Sigma(S_1, E, S_2) = (S_2[1], S_1, E), \quad \Sigma^{-1}(S_1, E, S_2) = (E, S_2, S_1[-1]), \quad (26)$$

and from the degrees of the maps between S_1, S_2 and E it follows that $\Phi = \Sigma^{\pm 1}$. Comparing (26) with the form of the inequalities (23) gives a contradiction. Thus again Φ preserves the objects S_i up to a shift, and hence defines the identity element of $\mathbb{P}\text{Aut}_*(D_n)$.

Consider the union

$$W = \bigcup_{\Phi \in \mathbb{P}\text{Aut}_*(D_n)} \Phi(\bar{\mathcal{U}}_n) \subset \mathbb{P}\text{Stab}_*(D_n). \quad (27)$$

By Lemma 4.2, for any projective stability condition $\bar{\sigma}$ lying in the closure of \mathcal{U}_n , the set of stable objects up to shift is contained in $\{S_1, S_2, E\}$, and hence $\bar{\sigma}$ can only lie in the closure of a different region $\Phi(\mathcal{U}_n)$ for finitely many $\Phi \in \mathbb{P}\text{Aut}_*(D_n)$. Thus the union in (27) is locally-finite, and hence W is a closed subset.

We now prove that W is an open subset. Consider $\bar{\sigma}$ lying in the boundary of \mathcal{U}_n , and a small open neighbourhood $\bar{\sigma} \in V \subset \mathbb{P}\text{Stab}_*(D_n)$. According to Lemma 4.2 there are two cases. In case (i) the stable objects are S_1, S_2 up to shift, and if V is small enough these remain stable, and all points of V lie in the closure of either \mathcal{U}_n or $\Upsilon(\mathcal{U}_n)$ depending on which of the positive numbers

$$\phi(S_2) - \phi(S_1), \quad \phi(S_1[n-2]) - \phi(S_2),$$

is the smallest. In case (ii) the stable objects are S_1, S_2 and E . Once again, if V is small enough these objects remain stable, and all points of V lie in the closure of one of the regions $\Sigma^i(\mathcal{U}_n)$ depending on which of the positive numbers

$$\phi(S_1) - \phi(S_2), \quad \phi(S_2[1]) - \phi(E), \quad \phi(E) - \phi(S_1[-1]),$$

is the smallest. This completes the proof that W is open. Since, by Prop. 2.7(a), the autoequivalences Φ and Σ generate $\mathbb{P}\text{Aut}_*(\mathbb{D}_n)$, we also see that W is connected. Thus $W = \mathbb{P}\text{Stab}_*(\mathbb{D}_n)$, and in particular, any reachable autoequivalence preserves the connected component $\text{Stab}_*(\mathbb{D}_n)$.

We have now proved one half of part (i) and part (ii). To complete the proof of (i) take an autoequivalence $\Psi \in \text{Aut}(\mathbb{D}_n)$ preserving the distinguished component $\text{Stab}_*(\mathbb{D}_n)$. Then Ψ maps \mathcal{U}_n into the union (27). Thus we can find a point $\bar{\sigma} \in \mathcal{U}_n$, and a reachable autoequivalence $\Phi \in \text{Aut}_*(\mathbb{D}_n)$ such that $\Psi(\bar{\sigma}) = \Phi(\bar{\sigma})$. Applying the first two paragraphs of this proof to the composite $\Psi \circ \Phi^{-1}$ it follows that Ψ is reachable. \square

Remark 4.5. Suppose a projective stability condition $\bar{\sigma} \in \mathbb{P}\text{Stab}_*(\mathbb{D}_n)$ is fixed by some non-trivial element $\Phi \in \text{Aut}_*(\mathbb{D}_n)$. By Prop. 4.4 it is enough to consider the case when $\bar{\sigma}$ lies in the closure of the fundamental domain \mathcal{U}_n . The proof of Prop. 4.4 then shows that there are two possibilities:

(a) the stable objects are S_1, S_2 up to shift, $\Phi = \Upsilon$, and

$$Z(S_2) = i^{n-2} \cdot Z(S_1), \quad \phi(S_2) - \phi(S_1) = \frac{1}{2}(n-2);$$

(b) the stable objects are S_1, S_2 and E up to shift, $\Phi = \Sigma^{\pm 1}$, and

$$Z(E[1]) = \omega \cdot Z(S_1) = \omega^2 \cdot Z(S_2),$$

$$\phi(S_1) - \phi(S_2) = \phi(S_2[1]) - \phi(E) = \phi(E) - \phi(S_1[-1]) = \frac{2}{3}.$$

These two cases are illustrated in Figure 7. For $n < \infty$ it is easy to see that both possibilities occur, and that the quotient

$$\mathbb{P}\text{Stab}_*(\mathbb{D}_n)/\mathbb{P}\text{Aut}_*(\mathbb{D}_n),$$

therefore has two orbifold points, with stabilizer groups μ_2 and μ_3 respectively. In the case $n = \infty$ only the second case occurs, and we obtain a single μ_3 orbifold point in the quotient.

4.5. We now prove a projectivised version of our main result, Theorem 1.1. Recall from Section 1 that the quotient \mathfrak{h}/W is isomorphic to \mathbb{C}^2 , and has co-ordinates (a, b) obtained by writing

$$p(x) = (x - u_1)(x - u_2)(x - u_3) = x^3 + ax + b.$$

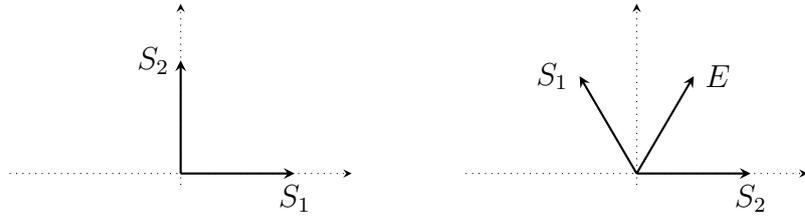
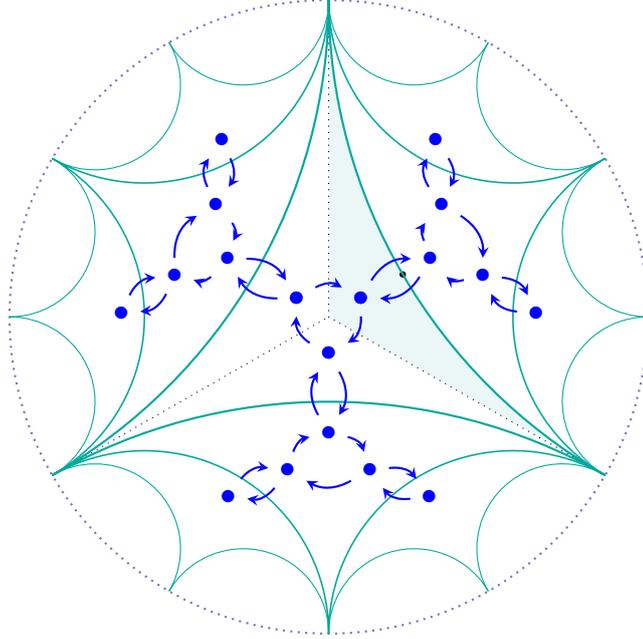

 FIGURE 7. Orbifold points in the boundary of \mathcal{U}_n when n is odd.


FIGURE 8. Exchange graph as the skeleton of space of stability conditions

The image of the root hyperplanes $u_i = u_j$ is the discriminant

$$\Delta = \{(a, b) \in \mathbb{C}^2 : 4a^3 + 27b^2 = 0\}.$$

The space \mathfrak{h} has a \mathbb{C}^* action which rescales the co-ordinates u_i with weight 1. This acts on (a, b) with weights $(2, 3)$. We thus have

$$\mathbb{C}^* \backslash (\mathfrak{h} \setminus \{0\}) / W \cong \mathbb{P}(2, 3).$$

The weighted projective space $\mathbb{P}(2, 3)$ contains two orbifold points which we label by their stabilizer groups: thus $\mu_2 = [1 : 0]$ and $\mu_3 = [0 : 1]$. The image in $\mathbb{P}(2, 3)$ of the discriminant is a single non-orbifold point $\Delta = [-3 : 2]$. The coarse moduli space map is

$$\mathbb{P}(2, 3) \rightarrow \mathbb{P}^1, \quad [a : b] \mapsto [a^3 : b^2].$$

We take the affine co-ordinate $t = -27b^2/4a^3$ on the coarse moduli space. The points (μ_2, Δ, μ_3) then correspond to $t = (0, 1, \infty)$ respectively.

In the case $n = \infty$ we consider the action of μ_3 on \mathbb{C} given by $a \mapsto \omega \cdot a$. The quotient has a single orbifold point at $a = 0$ with stabilizer group μ_3 . We take the co-ordinate $t = a^3$ on the coarse moduli space, which is isomorphic to \mathbb{C} .

Theorem 4.6. (a) *For $3 \leq n < \infty$ the action of $\mathbb{P}\text{Aut}_*(D_n)$ on $\mathbb{P}\text{Stab}_*(D_n)$ is proper and quasi-free and there is an isomorphism of complex orbifolds*

$$\mathbb{P}\text{Stab}_*(D_n)/\mathbb{P}\text{Aut}_*(D_n) \cong \mathbb{P}(2, 3) \setminus \Delta.$$

(b) *The action of $\mathbb{P}\text{Aut}(D_\infty)$ on $\mathbb{P}\text{Stab}(D_\infty)$ is proper and quasi-free and there is an isomorphism of complex orbifolds*

$$\mathbb{P}\text{Stab}(D_\infty)/\mathbb{P}\text{Aut}_*(D_\infty) \cong \mathbb{C}/\mu_3.$$

Proof. Consider part (a) first. We shall actually prove more, namely that there is an isomorphism of complex manifolds

$$h_n: \widetilde{\mathbb{P}(2, 3) \setminus \Delta} \longrightarrow \mathbb{P}\text{Stab}_*(D_n), \quad (28)$$

where the space on the left is the orbifold universal cover, with the following two properties:

- (i) it intertwines the action of the orbifold fundamental group on the left, with the action of $\mathbb{P}\text{Aut}_*(D_n)$ on the right, under the homomorphism

$$\rho: \pi_1(\mathbb{P}(2, 3) \setminus \Delta) \rightarrow \mathbb{P}\text{Aut}_*(D_n), \quad (29)$$

which sends the local generators at the orbifold points μ_2 and μ_3 to the autoequivalences Υ and Σ respectively;

- (ii) it makes the following diagram commute

$$\begin{array}{ccc} \widetilde{\mathbb{P}(2, 3) \setminus \Delta} & \xrightarrow{h_n} & \mathbb{P}\text{Stab}_*(D_n) \\ \downarrow & & \downarrow \\ \mathbb{P}^1 & \xrightarrow{\cong} & \mathbb{P}\text{Hom}_{\mathbb{Z}}(K_0(D_n), \mathbb{C}) \end{array}$$

in which the vertical arrows are local analytic isomorphisms, given on the left by a particular ratio of twisted periods (10), and on the right by the central charge map.

The fact that conditions (i) and (ii) are compatible is the statement that the monodromy representation of the equation (13) agrees with the action of the map $\mathbb{P} \text{Aut}_*(D_n)$ on central charges under the homomorphism ρ . This holds because the Möbius maps specified by the monodromy of the hypergeometric differential equation are precisely those defined by the matrices given in (7)

To construct a suitable map (28), first consider the subset $P_n \subset \mathbb{P}^1$ which is the union of the lower and upper half-planes $\pm \text{Im}(t) > 0$ in the coarse moduli space of $\mathbb{P}(2, 3)$, glued along the segment $(0, \infty) = (\mu_2, \mu_3)$. We can glue the isomorphism $f_n: \mathcal{H} \rightarrow R_n^-$ of Section 3 to its complex conjugate to obtain an isomorphism $f_n: P_n \rightarrow R_n$. Composing this with the inverse of the isomorphism $g_n: \mathcal{U}_n \rightarrow R_n$ of Prop. 4.3 gives a biholomorphic map $h_n = g_n^{-1} \circ f_n: P_n \rightarrow \mathcal{U}_n$. By construction this map satisfies the condition (ii).

Since P_n is simply-connected we can lift it to the universal cover and hence view it as a subset of the space appearing on the left of (28). The subsets P_n and \mathcal{U}_n are then both fundamental domains for the relevant group actions, so there is a unique way to extend h_n uniquely so as to satisfy condition (i) on the dense open subset which is the disjoint union of the translates of the fundamental domains. Since the conditions (i) and (ii) are compatible, the resulting map h_n also satisfies (ii). Using the local homeomorphisms to \mathbb{P}^1 we can then extend h_n over the boundaries of the fundamental domains to obtain the required isomorphism.

The proof of part (b) proceeds along similar lines. The universal cover of the orbifold \mathbb{C}/μ_3 is just \mathbb{C} itself, so in this case we look for an isomorphism

$$h_\infty: \mathbb{C} \rightarrow \mathbb{P} \text{Stab}_*(D_\infty),$$

satisfying the properties

- (i) it intertwines the action of μ_3 on both sides, given on the left by $a \mapsto \omega \cdot a$, and on the right by the autoequivalence Σ ;
- (ii) it makes the following diagram commute

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{h_\infty} & \mathbb{P} \text{Stab}_*(D_\infty) \\ \downarrow & & \downarrow \\ \mathbb{C}/\mu_3 & \xrightarrow{\cong} & \mathbb{P} \text{Hom}_{\mathbb{Z}}(K_0(D_\infty), \mathbb{C}) \end{array}$$

where the local analytic isomorphism on the left is given by a particular ratio of the functions (16).

The fact that these two conditions are compatible follows from the relations (18) and (19), after recalling that the Möbius transformation T is induced by the action of Σ on the space $\mathbb{P}\mathrm{Hom}_{\mathbb{Z}}(K_0(D_\infty), \mathbb{C})$.

To construct the required map h_∞ , we first consider the subset $P_\infty \subset \mathbb{C}$ which is the union of the lower and upper half-planes $\pm \mathrm{Im}(t) > 0$ in the coarse moduli space, glued along the segment $(0, \infty) = (\mu_3, \infty)$. We can glue the isomorphism $f_\infty: \mathcal{H} \rightarrow R_\infty^-$ of Section 3 to its complex conjugate to obtain an isomorphism $f_\infty: P_\infty \rightarrow R_\infty$. Composing with the inverse of the isomorphism $g_\infty: \mathcal{U}_\infty \rightarrow R_\infty$ of Prop. 4.3 gives a biholomorphic map $h_\infty: P_\infty \rightarrow \mathcal{U}_\infty$. We can again lift P_∞ into the universal cover, via the map $a = t^{1/3}$, and so view P_∞ as the sector in the a -plane bounded by the rays of phase $\pm\pi/3$. The resulting map satisfies condition (ii) by construction. We then extend the map h_∞ to a dense open subset of \mathbb{C} using condition (i). The compatibility of the two conditions ensures that this extension also satisfies condition (ii). Finally, we can extend h_∞ over the missing rays using the vertical local isomorphisms in the above commuting diagram. \square

4.6. The final step is to lift Theorem 4.6 to obtain a proof of our main result Theorem 1.1. Consider first the case $3 \leq n < \infty$. We have a diagram of complex manifolds and holomorphic maps

$$\begin{array}{ccc}
 \widetilde{\mathbb{C}^2 \setminus \Delta} & & \mathrm{Stab}_*(D_n) \\
 \downarrow & & \downarrow \\
 \mathbb{P}(\widetilde{2, 3}) \setminus \Delta & \xrightarrow{h_n} & \mathbb{P}\mathrm{Stab}_*(D_n)
 \end{array}$$

The vertical arrows are \mathbb{C} -bundles, and the bottom horizontal arrow h_n is the isomorphism constructed in the proof of Theorem 4.6. We would like to complete the diagram by filling in an upper horizontal isomorphism satisfying the property claimed in Theorem 1.1. Note that by construction the central charge map

$$\mathbb{P}\mathrm{Stab}(D_n) \rightarrow \mathbb{P}^1$$

corresponds under the isomorphism h_n to the map given by ratios of the functions $\phi_n^{(i)}(a, b)$ of Prop. 3.1. These functions are well-defined on the universal cover of $\mathbb{C}^2 \setminus \Delta$, and there

is therefore a unique way to fill in the upper arrow to give an isomorphism

$$h_n: \widetilde{\mathbb{C}^2 \setminus \Delta} \rightarrow \text{Stab}_*(D_n) \quad (30)$$

so that the composition with the central charge map on $\text{Stab}_*(D_n)$ is given by the twisted periods $\phi_n^{(i)}(a, b)$.

The isomorphism (29) lifts to an isomorphism

$$\rho: \pi_1(\mathbb{C}^2 \setminus \Delta) \rightarrow \text{Sph}_*(D_n), \quad (31)$$

by mapping the extra generator corresponding to a loop in the fibre of the \mathbb{C}^* -bundle

$$\pi: \mathbb{C}^2 \setminus \Delta \rightarrow \mathbb{P}(2, 3) \setminus \Delta$$

to the element $[3n - 4]$. This loop is given explicitly by a path of the form

$$(a(\theta), b(\theta)) = (e^{4\pi i \theta} \cdot a, e^{6\pi i \theta} \cdot b), \quad \theta \in [0, 1].$$

By the formula (10) this means that the central charges of the objects S_i vary as

$$Z(S_i)(\theta) = \phi^{(i)}(e^{4\pi i \theta} \cdot a, e^{6\pi i \theta} \cdot b) = e^{(3n-4)\pi i \theta} \cdot \phi^{(i)}(a, b) = e^{(3n-4)\pi i \theta} \cdot Z(S_i),$$

and hence the phases of these objects increase by $3n - 4$. This calculation shows that the map (30) we constructed is equivariant with respect to the group isomorphism (31), and passing to quotients then gives the statement of Theorem 1.1 (a).

In the case $n = \infty$ we have a similar diagram

$$\begin{array}{ccc} \mathbb{C}^2 & & \text{Stab}_*(D_\infty) \\ \downarrow & & \downarrow \\ \mathbb{C} & \xrightarrow{h_\infty} & \mathbb{P} \text{Stab}_*(D_\infty) \end{array}$$

in which the vertical arrows are again \mathbb{C} -bundles. The bundle on the left is just the projection $\mathbb{C}^2 \rightarrow \mathbb{C}$ given by $(a, b) \mapsto a$. By construction, the central charge map

$$\mathbb{P} \text{Stab}_*(D_\infty) \rightarrow \mathbb{P}^1$$

is given by ratios of the functions $\phi_\infty^{(i)}(a, b)$ of Prop. 3.3. These functions lift to \mathbb{C}^2 and there is therefore a unique way to fill in the upper arrow with an isomorphism

$$h_\infty: \mathbb{C}^2 \rightarrow \text{Stab}_*(D_\infty) \quad (32)$$

so that the composition with the central charge map on $\text{Stab}(\mathbb{D}_\infty)$ is given by the oscillating integrals $\phi_\infty^{(i)}(a, b)$. This completes the proof of Theorem 1.1.

Recall that in this case there is an isomorphism of groups

$$\mathbb{Z} \rightarrow \text{Aut}(\mathbb{D}_\infty), \quad 1 \mapsto \Sigma.$$

The map (32) can be made equivariant by letting \mathbb{Z} act on \mathbb{C}^2 via

$$(a, b) \mapsto (e^{2\pi i/3} \cdot a, b + \pi i/3).$$

The element $3 \mapsto \Sigma^3 = [1]$ then fixes a and acts by $b \mapsto b + \pi i$.

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