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On the focus order of planar polynomial differential equations

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ABSTRACT

This paper is devoted to finding the highest possible focus order of planar polynomial differential equations. The results consist of two parts: (i) we explicitly construct a class of concrete systems of degree n, where $n+1$ is a prime p or a power of a prime p^k, and show that these systems can have a focus order n^2-n; (ii) we theoretically prove the existence of polynomial systems of degree n having a focus order n^2-1 for any even number n. Corresponding results for odd n and more concrete examples having higher focus orders are given too.

1. Introduction

Consider the following planar polynomial ordinary differential equation:

$$\dot{z} = \mathbf{i}z + P_n(z,\bar{z}), \quad \mathbf{i} = \sqrt{-1}, \ z \in \mathbb{C}, \tag{1}$$

where $P_n(z,\bar{z})$ is a polynomial of degree n consisting of nonlinear terms only. It is well known that such a system always has a center or a focus at the origin, and to obtain criteria to distinguish them is one of the most classical problems in the qualitative theory of ordinary differential equations.

However, to derive such criteria for a given system, is generally theoretically hard and computationally tedious. Indeed, the answer to this problem is available only in a few cases. For example, although Bautin [2] and Sibirskii [11] established well-known "standards" for the calculation of center conditions for quadratic systems and homogeneous cubic systems, respectively, the corresponding

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consideration for the full cubic systems remains very open. In fact, even for the Kukles system, a lop-sided cubic system, the situation has become quite intricate. For recent development on various kinds of polynomial systems for which center conditions have been calculated can be found, say, in [3,4,6, 8–10], and we do not try to summarize them here.

While close attention has been paid to center conditions of families of polynomial systems, extensive interest has also been stimulated in maximal possible focus order of these systems, the twinborn problem of the center conditions. This is because on one hand, the focus order somehow is a symbol of the difficulty in derivation of the criteria and the sophistication of the dynamical properties of the system, on the other hand, it has a close relationship with the questions of the number of limit cycles bifurcating from an equilibrium point of the system, one branch of Hilbert's 16th problem.

A brief survey concerning the subject of focus order of polynomial systems is as follows. According to Bautin [2], the focus order of any quadratic system cannot be more than three. After Bautin, Sibirskii [11] showed that for homogeneous cubic $P_3(z,\bar z)$ in (1) this number is at most five. In [12], eleven limit cycles surrounding an equilibrium point of a cubic system can be bifurcated, hence there are cubic systems which can have a focus with the focus order at least eleven. In [7] explicit examples of homogeneous quartic $P_4(z,\bar z)$ and quintic $P_5(z,\bar z)$ are constructed for which the focus order is at least eighteen in both cases. Investigation on various classes of polynomial systems e.g. the Liénard systems, the Kukles equations, etc. can also be found with great ease in e-resources.

Although one can give a long list of references dealing with different types of polynomial systems, the degrees of these polynomial systems are generally fixed and very low (mainly cubic, quartic, quintic systems). There are very few results available concerning polynomial systems with a general degree n. One of the main reasons is that when facing systems with a general degree, as far as we know, no mature methods have been developed so far. For the related material on this subject, one can refer to, say, [1,3–5,7–10] and the reference therein.

In this paper, we first try to find some concrete systems in an explicit way and study their focus order. To our best knowledge, this is the first paper to construct concrete examples of polynomial systems of arbitrary degree and rigorously prove that their focus order can reach the number given below. We have the following results.

Theorem 1. *For any even number $n \geqslant 4$, if $n+1$ is a prime number p or if it is an integer power p^m of a prime, then the system*

$$\dot z = \mathbf{i}z + z^n - z\bar z^{n-1} + \sigma \bar z^n, \tag{2}$$

where σ is any purely imaginary number, has a focus at the origin with the focus order n^2-n.

In the next theorem, we will present a theoretic result concerning the existence of polynomial systems of degree n, for which the focus order can be slightly higher than that specified in Theorem 1 (for even n). Notice that the nonlinear parts of the systems both in Theorem 1 and in Theorem 2 are homogeneous.

Theorem 2. *For any even number n, $n \geqslant 2$, there exist polynomial systems of the form* (1)*, where $P_n(z,\bar z)$ is a homogeneous polynomial of degree n, which have a fine focus at the origin with the focus order no less than n^2-1.*

For any odd number n, $n \geqslant 3$, there exist polynomial systems of the form (1)*, where $P_n(z,\bar z)$ is a homogeneous polynomial of degree n, which have a fine focus at the origin with the focus order no less than $\frac{1}{2}(n^2-1)$.*

We remark that the construction of the systems in the theorems is based on some close observation and careful analysis of concrete systems such as the quartic and quintic systems. Loosely speaking, these polynomial systems are obtained not by chance at all. To see this point, in what follows, we present yet another class of polynomial systems and show that they have higher focus orders.

Proposition 1. *For $n \in \{4, 6, 8, 10, 12, 14, 16, 18\}$, the system*

$$\dot{z} = \mathbf{i}z - \frac{n}{n-2} z^n + z\bar{z}^{n-1} + \mathbf{i}\tau_n \bar{z}^n, \tag{3}$$

where τ_n is a fixed real number (listed in the proof of the theorem), has a focus at the origin with the focus order of $n^2 + n - 2$.

For $n \in \{3, 5, 7, 9, 11, 13, 15, 17, 19\}$, the following system

$$\dot{z} = \mathbf{i}z + \frac{n}{n-2} z^n + z\bar{z}^{n-1} + (1 + \mathbf{i}\tau)\bar{z}^n, \tag{4}$$

where τ is any transcendental number, has a focus at the origin with the focus order of $\frac{1}{2}(n^2 + n - 2)$.

Remark 1. Notice that τ_n in (3) is a fixed nonzero real number which in fact is obtained as a root of one polynomial; the number τ in (4) in fact can be taken any number except the root of some polynomials, therefore we sufficiently assume that it is any transcendental number. See the proof of the proposition for more details.

In Proposition 1, we assume that $n < 20$ simply because we have checked these cases. However, we have a conjectural feeling that both Theorem 1 and Proposition 1 hold valid for arbitrary n.

2. Preliminary

Since the proof of the theorems is long and technical, in this section we first introduce some definition, notation and symbols in order to make it compact and clear, then we shall collect some lemmas and clarify the main steps of the algorithm.

There are several ways to introduce the order of a fine focus. The reader can find a suitable definition in many standard textbooks of ordinary differential equations. For the coherence of the content, below we give a very brief explanation to bring out the symbols and notation we shall adopt.

Consider the polynomial system (1), where the origin defines a fine focus. The order of the fine focus can be defined to be the multiplicity of $\mathcal{R}e\{z\} = 0$ as a fixed point of the return map, and we shall use this definition in the proof of Theorem 2. Another version of this definition is in the following. There is an analytic function $V(z, \bar{z})$ in a neighborhood of the origin such that $\dot{V}(z, \bar{z})$, the rate of change of V along orbits of the vector field, takes the form $\sum_{k=0}^{\infty} L_k r^{2k+2}$, where r is from the relation $z = re^{\mathbf{i}\theta}$.

The coefficient L_k of the term r^{2k+2} is called the kth Lyapunov constant of system (1) at the origin, and it is a polynomial in the coefficients which arise in $P_n(z, \bar{z})$, and is the focal value. The origin is a center if and only if $L_k = 0$ for all k.

The set of focal values has a finite basis, according to the Hilbert Basis Theorem. The focus order of the system is defined to be the number m such that $L_k = 0$ for all $k < m$ and $L_m \neq 0$. An equivalent way to define the focus order is the number m such that $\dot{r} = g_m r^{2m+1} + \cdots$, where $g_k = 0$ for all $k < m$ but $g_m \neq 0$.

Notice that L_m differs from g_m only by a positive number (see, for example, [5]). The focus order m is an invariant of the system as it has a geometric meaning: The maximum order m of the fine focus implies that at most m limit cycles can bifurcate from the equilibrium point, though this maximum is not always attained.

The calculation of the Lyapunov constants of polynomial systems is a well-trodden ground. For our use, below we shall restrict the algorithm to a specific class of polynomial systems, that is, systems with a homogeneous nonlinearity.

Lemma 1. *(See [7].) If $P_n(z, \bar{z})$ in (1) is a homogeneous polynomial of degree n, then $L_k = 0$ if $\frac{2k}{n-1}$ is not an integer.*

The next lemma is elementary but useful.

Lemma 2. *In $\mathbb{R}$ or $\mathbb{C}$, the set*

$$\left\{\sin^{n+1}\theta, \sin^n\theta\cos\theta, \dots, \sin\theta\cos^n\theta, \cos^{n+1}\theta\right\}$$

and the set

$$\left\{\cos\theta, \cos 3\theta, \dots, \cos(n+1)\theta, \sin\theta, \sin 3\theta, \dots, \sin(n+1)\theta\right\}$$

are equivalent in the sense that each element of one set can be linearly expressed in terms of the elements of the other set. In particular, for each $k \in \{1, \dots, n+2\}$, there exist β_{jk} and γ_{jk}, $j = 0, 1, \dots, n$, such that the following equalities hold:

$$\sum_{j=0}^{n}\left(\beta_{jk}\cos\bigl(2j-(n+1)\bigr)\theta - \gamma_{jk}\sin\bigl(2j-(n+1)\bigr)\theta\right) = \cos^{k-1}\theta\sin^{n+2-k}\theta.$$

Proof. The proof is given only in $\mathbb{C}$. The validity of the lemma in the real case can be obtained by taking the real part of the corresponding relations.

Denote by $\mathcal{V}$ the linear space in $\mathbb{C}$ spanned by

$$\left\{\cos\theta, \cos 3\theta, \dots, \cos(n+1)\theta, \sin\theta, \sin 3\theta, \dots, \sin(n+1)\theta\right\}.$$

Let $\zeta = e^{\mathbf{i}\theta}$. Then

$$\zeta^j\bar{\zeta}^{n+1-j} = \cos\bigl(2j-(n+1)\bigr)\theta + \mathbf{i}\sin\bigl(2j-(n+1)\bigr)\theta \in \mathcal{V},$$

for $j = 0, \dots, n+1$. It follows that

$$\cos^{k-1}\theta\sin^{n+2-k}\theta = \left(\frac{\zeta+\bar{\zeta}}{2}\right)^{k-1}\left(\frac{\zeta-\bar{\zeta}}{2i}\right)^{n+2-k} \in \mathcal{V},$$

and the lemma is proved. □

We point out that all the systems under discussion have a center type linear part and a homogeneous nonlinear part, therefore we shall primarily restrict our preliminaries to this fixed pattern of systems rather than aim at generalizing our discussion.

The following lemma is particularly important in the proof of Theorem 1 and Proposition 1. Since the proof of this lemma is technical and needs more lemmas, in the remaining part of this section we shall primarily deal with the proof of this lemma.

Lemma 3. *Any polynomial differential equation of the following form*

$$\dot{z} = \mathbf{i}z + \mu z^n + \nu z\bar{z}^{n-1} + \sigma\bar{z}^n, \tag{5}$$

where $\mu, \nu \in \mathbb{R}$, $\sigma \in \mathbb{C}$, either has a center at the origin or has a weak focus with the focus order at least $n^2 - n$ when n is even ($n \geqslant 4$) and $\frac{n^2-n}{2}$ when n is odd ($n \geqslant 3$).

Both possibilities mentioned in (3), center or focus, can occur. In fact, it is easy to see that in the following two cases, the system at the origin has a center:

(i) $\nu = -n\mu$;
(ii) $\arg(\sigma) \in \{(\frac{2k_1}{n-1} + k_2)\pi \mid k_1, k_2 \in \mathbb{Z}\}$, or $\sigma = 0$.

The reason why the system is a center is because in these two cases, respectively, it is Hamiltonian and reversible with respect to a linear involution.

2.1. Algorithm of the Lyapunov constants

Denote by $P_n(z)$ its nonlinear part of (5). Namely, $P_n(z) = \mu z^n + \nu z\bar{z}^{n-1} + \sigma\bar{z}^n$. In what follows, we shall mainly consider the case where n is even, for the corresponding study in the case when n is odd can be given in a similar way.

To calculate the Lyapunov constants of the system (5), we introduce a sequence of matrices $\mathcal{M}_1, \mathcal{M}_2, \ldots,$ where $\mathcal{M}_k$ is a matrix of order I_{k+1} by I_k, where, for brevity, we denote

$$I_k = 3 + k(n-1),$$

and

$$\mathcal{M}_k = \big(M_k(i,j)\big)_{I_{k+1}\times I_k}, \quad k = 1, 2, \ldots.$$

The elements $M_k(i,j)$ of $\mathcal{M}_k$ are given in the following way:

(i) if $i - j = 0$, then

$$M_k(i,j) = \begin{cases} \frac{I_k-1}{2j-1-I_k} \cdot \frac{\mu+\nu}{2} + \frac{\nu-\mu}{2}, & \text{if } 2j-1-I_k \neq 0, \\ 0, & \text{if } 2j-1-I_k = 0; \end{cases} \tag{6}$$

(ii) if $i - j = -1$, then

$$M_k(i,j) = \begin{cases} \frac{j-1}{2j-1-I_k} \cdot \sigma, & \text{if } 2j-1-I_k \neq 0, \\ 0, & \text{if } 2j-1-I_k = 0; \end{cases} \tag{7}$$

(iii) if $i - j \in \{n-1, n\}$, then

$$M_k(i,j) = -\overline{M_k(I_{k+1}+1-i, I_k+1-j)}; \tag{8}$$

(iv) in all the other cases,

$$M_k(i,j) = 0. \tag{9}$$

With certain calculation, one sees that all elements of the matrix $\mathcal{M}_k$ except those along the four diagonal lines are zero. To get an impression of such a sequence of matrices, below we illustrate a concrete example. Let $n = 4$. Then $\mathcal{M}_1$ is a matrix of order 9 by 6 and has the following explicit form

$$\begin{pmatrix}
\frac{5}{-5}\varphi-\psi & \frac{1}{-3}\sigma & & & & \\
0 & \frac{5}{-3}\varphi-\psi & \frac{2}{-1}\sigma & & & \\
0 & & \frac{5}{-1}\varphi-\psi & \frac{3}{1}\sigma & & \\
-(\frac{5}{5}\varphi-\psi) & & & \frac{5}{1}\varphi-\psi & \frac{4}{3}\sigma & \\
\frac{5}{5}\bar{\sigma} & -(\frac{5}{3}\varphi-\psi) & & & \frac{5}{3}\varphi-\psi & \frac{5}{5}\sigma \\
 & \frac{4}{3}\bar{\sigma} & -(\frac{5}{1}\varphi-\psi) & & & \frac{5}{5}\varphi-\psi \\
 & & \frac{3}{1}\bar{\sigma} & -(\frac{5}{-1}\varphi-\psi) & & 0 \\
 & & & \frac{2}{-1}\bar{\sigma} & -(\frac{5}{-3}\varphi-\psi) & 0 \\
 & & & & \frac{1}{-3}\bar{\sigma} & -(\frac{5}{-5}\varphi-\psi)
\end{pmatrix},$$

where $\varphi=\frac{\mu+\nu}{2}$, $\psi=\frac{\mu-\nu}{2}$, and all the elements omitted are 0.

Notice that for other values of k the matrix $\mathcal{M}_k$ has the similar pattern, i.e., only those elements in four diagonal lines appear. Notice that in the case that k is even, the number of columns of $\mathcal{M}_k$ is odd and all the elements in the middle column are zero.

Denote by $\mathcal{H}_1$ the following column matrix of order $n+2$

$$\mathcal{H}_1=\big(\sigma,\mu+\nu,0,\ldots,0,-(\mu+\nu),\bar{\sigma}\big)^T \tag{10}$$

and define recursively $\mathcal{H}_s$ as follows

$$\mathcal{H}_{s+1}=\mathbf{i}\cdot\mathcal{M}_s\cdot\mathcal{H}_s,\quad s\geqslant 1. \tag{11}$$

Then $\mathcal{H}_{2s}$ is a column matrix of odd order I_{2s}, namely, $3+2s(n-1)$. Then according to one algorithm (see [7] and the reference therein), the middle element of the column matrix of $\mathcal{H}_{2s}$ precisely gives the Lyapunov constant. More exactly, we have the following

Lemma 4. *(See [7].) The $(n-1)s$th Lyapunov constant $L_{(n-1)s}$ is given by the middle element of the column matrix $\mathcal{H}_{2s}$, namely, $L_{(n-1)s}$ is the $2+s(n-1)$st element of the column matrix $\mathcal{H}_{2s}$.*

By this lemma and (11), to find the middle element of the matrix $\mathcal{H}_{2s}$, we need to calculate the multiplication of some matrices. In order to perform such recursive calculation, we find it more convenient to "imbed" a constant matrix—a matrix where all the elements are constant, into a set of function matrices—matrices where all the elements are functions. Below we describe such a scheme.

Given a matrix $\mathcal{M}=(M(i,j))_{p\times q}$, to each element we attach an exponential function of t to obtain a *functionalized matrix* in the following way:

$$\tilde{\mathcal{M}}=\big(\tilde{M}(i,j)\big)_{p\times q}=\big(M(i,j)\cdot e^{(p-q-2(i-j))t}\big)_{p\times q}.$$

We call $e^{(p-q-2(i-j))t}$ the exponential factor of the element $M(i,j)$.

It is clear that all the elements along each diagonal direction have the same exponential factors. For a column matrix, however, each element has a unique exponential factor. Moreover, when the order of the column matrix is odd, there is a unique element whose exponential factor is identical, and this element precisely locates at the middle of the column matrix. Due to this observation, given a column matrix $\mathcal{H}=(h(1),h(2),\ldots,h(p))^T$, on one hand we have the exponential column matrix

$$\begin{aligned}\tilde{\mathcal{H}}&=\big(\tilde{h}(1),\tilde{h}(2),\ldots,\tilde{h}(p)\big)^T\\&=\big(h(1)e^{(p-1)t},h(2)e^{(p-3)t},\ldots,h(p)e^{-(p-1)t}\big)^T,\end{aligned}$$

on the other hand, we define the following *sum* function

$$\Lambda_{\mathcal{H}}(t) = \tilde{h}(1) + \tilde{h}(2) + \cdots + \tilde{h}(p)$$
$$= h(1)e^{(p-1)t} + h(2)e^{(p-3)t} + \cdots + h(p)e^{-(p-1)t}. \tag{12}$$

Moreover, to a function of the form

$$f(t) = d_0 + \sum_{k\in\mathbb{Z}\setminus\{0\}} d_k e^{kt},$$

we define the integral-like function of f as follows

$$\mathcal{I}(f) = \widetilde{\int} f(t)\,dt = \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{d_k}{k} e^{kt}. \tag{13}$$

One sees that this integral-like function is in fact the indefinite integral of a function without taking into account the constant term. The purpose to introduce the above integral-like function is purely technical, because by doing so we can handle these messy coefficients of the monomials with ease and can obtain a convenient recursive formula (17).

The next lemma can be checked in a straightforward way.

Lemma 5. *If $\mathcal{V}$ and $\mathcal{W}$ are two matrices which make the multiplication $\mathcal{V}\cdot\mathcal{W}$ possible, and $\mathcal{M} = \mathcal{V}\cdot\mathcal{W}$, then $\tilde{\mathcal{M}} = \tilde{\mathcal{V}}\cdot\tilde{\mathcal{W}}$.*

We present an illustrative example to familiarize the notation and definition given above and to introduce some practical symbols for afterward use.

Example 1. Let $\mathcal{H}_1$ be given in (10). Then we have

$$\tilde{\mathcal{H}}_1 = \left(\sigma e^{(n+1)t}, (\mu+\nu)e^{(n-1)t}, 0, \ldots, 0, -(\mu+\nu)e^{-(n-1)t}, \bar{\sigma} e^{-(n+1)t}\right)^T$$
$$= \left(\sigma\zeta_+, (\mu+\nu)\eta_+, 0, \ldots, 0, -(\mu+\nu)\eta_-, \bar{\sigma}\zeta_-\right)^T, \tag{14}$$

where

$$\zeta_\pm = e^{\pm(n+1)t}, \qquad \eta_\pm = e^{\pm(n-1)t}.$$

The *sum* function of $\mathcal{H}_1$ is given as follows

$$\Lambda_{\mathcal{H}_1}(t) = \sigma\zeta_+ + \bar{\sigma}\zeta_- + (\mu+\nu)(\eta_+ - \eta_-). \tag{15}$$

The integral-like function $\mathcal{I}(\Lambda_{\mathcal{H}_1})$ has the following form:

$$\mathcal{I}(\Lambda_{\mathcal{H}_1}) = \frac{1}{n+1}(\sigma\zeta_+ - \bar{\sigma}\zeta_-) + \frac{\mu+\nu}{n-1}(\eta_+ + \eta_-).$$

Notice that an application of Lemma 5 to (11) induces the following relation

$$\tilde{\mathcal{H}}_{s+1} = \mathbf{i} \cdot \tilde{\mathcal{M}}_s \cdot \tilde{\mathcal{H}}_s, \quad s \geqslant 1. \tag{16}$$

With the relation (12), we can rewrite the recursive relation (16) as follows

$$\begin{aligned} \Lambda_{\mathcal{H}_{s+1}}(t) &= \frac{\mathbf{i}}{2}\big(\sigma\zeta_+ - \bar{\sigma}\zeta_- + (\mu - \nu)(\eta_- - \eta_+)\big)\Lambda_{\mathcal{H}_s}(t) \\ &\quad + \mathbf{i}\frac{2 + s(n-1)}{2}\big(\bar{\sigma}\zeta_- - \sigma\zeta_+ + (\mu + \nu)(\eta_- - \eta_+)\big)\mathcal{I}\big(\Lambda_{\mathcal{H}_s}(t)\big). \end{aligned} \tag{17}$$

Now we are ready to present a detailed proof of these theorems.

2.2. Proof of Lemma 3

Given any $s < n$, we know that the Lyapunov constant $L_{(n-1)s}$ is the constant term of the *sum* function $\Lambda_{\mathcal{H}_{2s}}(t)$. Besides, from (15) and (17), one could see that $\Lambda_{\mathcal{H}_k}(t)$ is a homogeneous polynomial of degree k in terms of $\{\eta_+, \eta_-, \sigma\zeta_+, \bar{\sigma}\zeta_-\}$, therefore

$$\begin{aligned} \Lambda_{\mathcal{H}_{2s}}(t) &= \sum \mathtt{Coeff}_{(i_+,i_-,j_+,j_-)} \cdot \eta_+^{i_+}\eta_-^{i_-}(\sigma\zeta_+)^{j_+}(\bar{\sigma}\zeta_-)^{j_-} \\ &= \sum \mathtt{Coeff}_{(i_+,i_-,j_+,j_-)} \cdot \sigma^{j_+}\bar{\sigma}^{j_-}e^{[(n-1)(i_+-i_-)+(n+1)(j_+-j_-)]t}, \end{aligned}$$

where the sum is taken over the set

$$\big\{(i_+, i_-, j_+, j_-) \,\big|\, i_+, i_-, j_+, j_- \geqslant 0,\ i_+ + i_- + j_+ + j_- = 2s\big\}$$

and $\mathtt{Coeff}(i_+, i_-, j_+, j_-)$ is the coefficient of the term $\eta_+^{i_+}\eta_-^{i_-}(\sigma\zeta_+)^{j_+}(\bar{\sigma}\zeta_-)^{j_-}$ which are independent of σ and t but depend on μ and ν.

One sees from the above expression that to obtain the constant term in the summation, the following condition must be met:

$$(n-1)(i_+ - i_-) + (n+1)(j_+ - j_-) = 0.$$

Since n is even, therefore the greatest common divisor of $n+1$ and $n-1$ is 1, namely, $(n+1, n-1) = 1$. Hence, the above equality yields the following relations

$$i_+ - i_- = (n+1)q, \qquad j_+ - j_- = -(n-1)q, \quad q \in \mathbb{Z}. \tag{18}$$

We shall see that q must be 0, since if $q \neq 0$, then

$$2s = i_+ + i_- + j_+ + j_- \geqslant (n+1)|q| + (n-1)|q| = 2n|q|$$

which is not possible for $s < n$. It follows that if and only if $i_+ = i_- = i$, $j_+ = j_- = j$, $i + j = s$, can we have a constant term in $\Lambda_{\mathcal{H}_{2s}}(t)$, which is given as follows

$$L_{(n-1)s} = \sum_{i+j=s,\, i,j\geqslant 0} \mathtt{Coeff}(i, i, j, j) \cdot |\sigma|^{2j}. \tag{19}$$

From the above equality, on one hand, we see that $L_{(n-1)s}$ is a function of $|\sigma|^2$, which means that whether it is 0 or not does not depend on $\arg\sigma$. On the other hand, we know that when $\sigma \in \mathbb{R}$, the original system (5) is reversible with respect to $z \to -\bar{z}$ and consequently the system is a center at the origin. Thus all the Lyapunov constants vanish, i.e., $L_{(n-1)s} = 0$ for all $s \geqslant 1$. It follows that, for any $\sigma \in \mathbb{C}$, we always have $|\sigma|^2 \in \mathbb{R}$, since the above discussion is valid for all $s < n$, we have $L_{(n-1)s} = 0$ for any $s < n$.

Taking $s = n-1$, we proved that system (5) either has a center or has a focus with the first $(n-1)^2$ vanishing focus values. Since by Lemma 1 the next possibly non-vanishing focus order is $(n-1)^2 + (n-1)$, i.e., $n^2 - n$, we proved Lemma 3 when n is even.

If n is odd, we can give a corresponding proof of the lemma, noticing the following two differences with the case that n is even.

(i) Lemma 4 should be changed to the following: *The* $\frac{1}{2}(n-1)$*sth Lyapunov constant* $L_{\frac{n-1}{2}s}$ *is given by the middle element of the column matrix* $\mathcal{H}_s$, *namely,* $L_{\frac{n-1}{2}s}$ *is the* $2 + \frac{1}{2}(n-1)$*sth element of the column matrix* $\mathcal{H}_s$.

(ii) The relation (18) becomes

$$i_+ - i_- = \frac{1}{2}(n+1)q, \qquad j_+ - j_- = -\frac{1}{2}(n-1)q, \quad q \in \mathbb{Z}.$$

We finish the proof of Lemma 3.

3. Proof of Theorem 1

To show the existence of such a system with the desired focus order, we take $\nu = -\mu$ in (5) (we scale them to ± 1 at the end of the proof). Then (15) and (17) take the following forms, respectively,

$$\begin{aligned} \Lambda_{\mathcal{H}_1}(t) &= \sigma\zeta_+ + \bar{\sigma}\zeta_-, \\ \Lambda_{\mathcal{H}_{s+1}}(t) &= \frac{1}{2}\big(\sigma\zeta_+ - \bar{\sigma}\zeta_- + 2\mu(\eta_- - \eta_+)\big)\Lambda_{\mathcal{H}_s}(t) \\ &\quad + \frac{2 + s(n-1)}{2}(\bar{\sigma}\zeta_- - \sigma\zeta_+)\mathcal{I}(\Lambda_{\mathcal{H}_s}). \end{aligned}$$

From the above discussion, we know $L_{(n-1)n}$ consists of three parts:

$$\begin{aligned} L_{(n-1)n} = {} & \sum_{j=0}^{n} \texttt{Coeff}(n-j, n-j, j, j) \cdot |\sigma|^{2j} \\ & + \texttt{Coeff}(n+1, 0, 0, n-1) \cdot \bar{\sigma}^{n-1} \\ & + \texttt{Coeff}(0, n+1, n-1, 0) \cdot \sigma^{n-1}. \end{aligned}$$

With some straight calculation we obtain the following equality:

$$\begin{aligned} \texttt{Coeff}(n+1, 0, 0, n-1) &= -\texttt{Coeff}(0, n+1, n-1, 0) \\ &= \mathbf{i}\frac{\mu^{n+1}}{2^{n-2}} \sum_{1 \leqslant k_1 < \cdots < k_{n-2} \leqslant 2n-1} \prod_{s=1}^{n-2} \frac{1 + k_s(n-1) - ns}{(n-1)(k_s - s) - (n+1)s}. \end{aligned}$$

If $\sigma \in \mathbb{R}$, then the system is reversible (with respect to the imaginary axis) and it has a center at the origin, therefore all the Lyapunov constants are 0. In particular, the sum of the above three parts is 0. Observe that in this case, the sum of the last two parts is 0, hence it implies that the first part is 0 also. Now, if σ is a purely imaginary number, then the first part does not change, namely, it remains to be 0, but in this case the second part is equal to the third part. Below we shall show that the second part, and also the third part, is different from 0. If so, then it follows that the $(n-1)n$th Lyapunov constant is not 0 and we are done. In other words, it suffices for us to show that

$$\Delta = \sum_{1 \leqslant k_1 < \cdots < k_{n-2} \leqslant 2n-1} \prod_{s=1}^{n-2} \frac{1 + k_s(n-1) - ns}{(n-1)(k_s - s) - (n+1)s} \neq 0. \tag{20}$$

Up to now, one sees that Theorem 1 holds true provided the above inequality stands. In other words, if $\Delta \neq 0$, then the system has a focus order of $n(n-1)$. The proof of this inequality, however, perhaps is a number theory problem and, we can only show its validity when $n+1$ is prime p or $n+1$ is an integer power of a prime p^k, and this is the reason why in the theorem we put such an assumption.

Below we only consider the case that $n+1 = p$, where p is a prime. The case that $n+1 = p^k$, where p is a prime and $k \geqslant 2$, can be studied in a similar way.

We denote by

$$\mathbf{K} = (k_1, \ldots, k_{p-3}),$$
$$\mathcal{K} = \{\mathbf{K} \mid 2 \leqslant k_1 < \cdots < k_{p-3} \leqslant 2p-3\}.$$

Here we do not count in the case $k_1 = 1$, this is because if $k_1 = 1$ then

$$\prod_{s=1}^{p-3} \frac{1 + k_s(p-2) - (p-1)s}{(p-2)(k_s - s) - ps} = 0,$$

which does not make contribution to the validity of the (20). It is easy to see that if $k_1 \geqslant 2$, then $s + 1 \leqslant k_s \leqslant s + p$.

Denote by

$$\frac{\mathcal{Q}(\mathbf{K})}{\mathcal{P}(\mathbf{K})} = \prod_{s=1}^{p-3} \frac{1 + k_s(p-2) - (p-1)s}{(p-2)(k_s - s) - ps}, \tag{21}$$

where the fraction is assumed to be irreducible.

Now we can rewrite Δ as follows:

$$\Delta = \sum_{\mathbf{K} \in \mathcal{K}} \frac{\mathcal{Q}(\mathbf{K})}{\mathcal{P}(\mathbf{K})}.$$

To prove that $\Delta \neq 0$, we only need to show the following two points which imply that $\Delta \neq 0$:

(i) $\mathcal{P}(\mathbf{K}) \not\equiv 0 \pmod{p^{p-3}}$, and

$$\mathcal{K}_1 := \big\{\mathbf{K} \bigm| \mathcal{P}(\mathbf{K}) \equiv 0 \ \big(\text{mod } p^{p-4}\big)\big\} \neq \emptyset.$$

(ii) $\sum_{\mathbf{K} \in \mathcal{K}_1} \frac{\mathcal{Q}(\mathbf{K})}{\frac{\mathcal{P}(\mathbf{K})}{p^{p-4}}} \not\equiv 0 \pmod{p}$. If so, then

$$p^{p-4}\sum_{\mathbf{K}\in\mathcal{K}}\frac{\mathcal{Q}(\mathbf{K})}{\mathcal{P}(\mathbf{K})}=\sum_{\mathbf{K}\in\mathcal{K}_1}\frac{\mathcal{Q}(\mathbf{K})}{\frac{\mathcal{P}(\mathbf{K})}{p^{p-4}}}+\sum_{\mathbf{K}\in\mathcal{K}\setminus\mathcal{K}_1}p^{p-4}\frac{\mathcal{Q}(\mathbf{K})}{\mathcal{P}(\mathbf{K})}$$
$$\equiv\sum_{\mathbf{K}\in\mathcal{K}_1}\frac{\mathcal{Q}(\mathbf{K})}{\frac{\mathcal{P}(\mathbf{K})}{p^{p-4}}}\not\equiv 0 \pmod p.$$

Proof of (i). If $(p-2)(k_s-s)-ps\equiv 0 \pmod p$, then $k_s-s\equiv 0 \pmod p$. Notice that $1\leqslant k_s-s\leqslant p$. Therefore we have $k_s-s=p$, i.e., $k_s=s+p$. Thus it follows that

$$\frac{1+k_s(p-2)-(p-1)s}{(p-2)(k_s-s)-ps}=\frac{1-s+p^2-2p}{p(p-2-s)}.$$

Notice that when $s=1$, the right side of the above relation is $\frac{p-2}{p-3}$ and when $s\geqslant 2$, the denominator of the right side can be divided by p but not by p^2, therefore the denominator of the product of these factors can be divided by p^{p-4} but not by p^{p-3}, namely, we have

$$\mathcal{P}(\mathbf{K})\not\equiv 0 \quad \bigl(\text{mod } p^{p-3}\bigr).$$

It is easy to see that $\mathcal{P}(\mathbf{K})\equiv 0 \pmod{p^{p-4}}$ if and only $k_s=s+p$, $s\geqslant 2$, therefore

$$\mathcal{K}_1=\bigl\{(j,p+2,p+3,\dots,2p-3)\bigm| j=2,3,\dots,p+1\bigr\}\neq\emptyset. \qquad \square$$

Proof of (ii). A straightforward calculation yields the following:

$$\sum_{\mathbf{K}\in\mathcal{K}_1}\frac{\mathcal{Q}(\mathbf{K})}{\frac{\mathcal{P}(\mathbf{K})}{p^{p-4}}}=\left(\sum_{j=2}^{p+1}\frac{1+j(p-2)-(p-1)}{(p-2)(j-1)-p}\right)\prod_{s=2}^{p-3}\frac{1-s+p^2-2p}{p-2-s}$$
$$\equiv\Bigl(\underbrace{1+1+\cdots+1}_{p-1}+\frac{p-2}{p-3}\Bigr)\prod_{s=2}^{p-3}\frac{1-s}{-2-s}\equiv\frac{1}{3}\not\equiv 0 \pmod p.$$

We prove the second point and thus the whole theorem. $\square$

4. Proof of Theorem 2

In this part of the paper, we shall prove Theorem 2. We shall only consider the case that n is even, since the case that n is odd can be discussed exactly in the same way.

For any even number n, we consider the following differential equation

$$\dot{z}=\mathbf{i}z-\mathbf{i}zP(z,\delta), \tag{22}$$

where

$$P(z,\delta)=\left(\frac{z+\bar{z}}{2}\right)^{n-1}+\delta\left(\frac{z-\bar{z}}{2\mathbf{i}}\right)^{n-1}, \quad \delta\in\mathbb{R}, \tag{23}$$

where δ is a real number to be determined. Notice that $P(z,\delta)$ is a homogeneous polynomial in z and $\bar{z}$ of degree $n-1$.

It is clear that system (22) has a center at the equilibrium point of the origin. Now we consider its perturbed system

$$\dot{z} = \mathbf{i}z - \mathbf{i}zP(z,\delta) + Q(z,\underline{\varepsilon}), \tag{24}$$

where $Q(z,\underline{\varepsilon})$ is a homogeneous polynomial in z and $\bar{z}$ of degree n having the following explicit expression

$$Q(z,\underline{\varepsilon}) = \sum_{j=0}^{n} \alpha_j(\underline{\varepsilon}) z^j \bar{z}^{n-j}, \tag{25}$$

where $\underline{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_{n+2}) \in \mathbb{R}^{n+2}$, each component ε_k of $\underline{\varepsilon}$, $k = 1, \ldots, n+2$, is assumed to be sufficiently small, and

$$\alpha_j(\underline{\varepsilon}) = \sum_{k=1}^{n+2} (\beta_{jk} + \mathbf{i}\gamma_{jk})\varepsilon_k, \quad \beta_{jk}, \gamma_{jk} \in \mathbb{R}.$$

In the polar coordinates, $z = re^{\mathbf{i}\theta}$, system (24) takes the form

$$\begin{cases} \dot{r} = r^n G_+(\theta,\underline{\varepsilon}), \\ \dot{\theta} = 1 - r^{n-1}\big(P\big(e^{\mathbf{i}\theta},\delta\big) - G_-(\theta,\underline{\varepsilon})\big), \end{cases} \tag{26}$$

where

$$G_+(\theta,\underline{\varepsilon}) = \mathcal{R}e\big(e^{-\mathbf{i}\theta} Q\big(e^{\mathbf{i}\theta},\underline{\varepsilon}\big)\big), \qquad G_-(\theta,\underline{\varepsilon}) = \mathcal{I}m\big(e^{-\mathbf{i}\theta} Q\big(e^{\mathbf{i}\theta},\underline{\varepsilon}\big)\big). \tag{27}$$

Hence, we have the following relation

$$\frac{dr}{d\theta} = \frac{r^n G_+(\theta,\underline{\varepsilon})}{1 - r^{n-1}(P(e^{\mathbf{i}\theta},\delta) - G_-(\theta,\underline{\varepsilon}))}. \tag{28}$$

The solution of this system satisfies the initial condition $r|_{\theta=0} = h$, thus can be expressed as follows

$$r = R(\theta,h,\underline{\varepsilon}) = h + \sum_{j=1}^{\infty} v_{1+j(n-1)}(\theta,\underline{\varepsilon}) h^{1+j(n-1)}, \tag{29}$$

where $v_k(\theta,\underline{\varepsilon})$ are functions of θ.

The linear part with respect to $\underline{\varepsilon}$ in the Poincaré return map $R(2\pi,h,\underline{\varepsilon}) - h$ is

$$\int_0^{2\pi} \frac{h^n G_+(\theta,\underline{\varepsilon})}{1 - h^{n-1}P(e^{\mathbf{i}\theta},\delta)}\, d\theta. \tag{30}$$

From (29) and (30), we can calculate the constant $v_{1+j(n-1)}(2\pi,\underline{\varepsilon})$. Namely,

$$v_{1+j(n-1)}(2\pi,\underline{\varepsilon}) = \int_0^{2\pi} G_+(\theta,\underline{\varepsilon}) P^{j-1}\big(e^{\mathbf{i}\theta},\delta\big)\, d\theta + o(\underline{\varepsilon}), \quad j = 1,2,\ldots. \tag{31}$$

Notice that n is even, therefore by Lemma 1, one can see that

$$v_{1+(2j-1)(n-1)}(2\pi,\underline{\varepsilon}) = 0, \quad j = 1,2,\ldots.$$

Thus we need only to consider

$$v_{1+2j(n-1)}(2\pi,\underline{\varepsilon}) = \int_0^{2\pi} G_+(\theta,\underline{\varepsilon}) P^{2j-1}\big(e^{\mathbf{i}\theta},\delta\big)\,d\theta + o(\underline{\varepsilon}), \quad j = 1,2,\ldots. \tag{32}$$

Now, if we can express $v_{1+2j(n-1)}(2\pi,\underline{\varepsilon})$ in the following form

$$v_{1+2j(n-1)}(2\pi,\underline{\varepsilon}) = \sum_{k=1}^{n+2} a_{jk}(\delta)\varepsilon_k + o(\underline{\varepsilon}), \quad j = 1,2,\ldots, \tag{33}$$

and if there exist $1 \leqslant k_1 < \cdots < k_s \leqslant n+2$ such that

$$\det(a_{jk_l})_{j,l=1,\ldots,s} \neq 0, \tag{34}$$

then, by the reverse function theorem, there exist $\varepsilon_{k_1}, \varepsilon_{k_2}, \ldots, \varepsilon_{k_s}$ such that

$$\begin{aligned} &v_{1+2j(n-1)} = 0, \quad j = 1,\ldots,s-1, \\ &v_{1+2s(n-1)} \neq 0. \end{aligned} \tag{35}$$

Notice that these relations in (35) equivalently say that the focus order of the system is at least $s(n-1)$. Therefore, to prove the theorem, we need only to demonstrate the following two points.

(i) There exists polynomial $Q(z,\underline{\varepsilon})$ having form (25) such that (33) stands.
(ii) The number s in (34) can be taken as high as $n+1$, so that the fine focus can reach the desired order.

Proof of (i). First of all, from (27) it is straightforward to see that

$$\begin{aligned} G_+(\theta,\underline{\varepsilon}) &= \sum_{j=0}^{n}\sum_{k=1}^{n+2} \varepsilon_k\big(\beta_{jk}\cos\big(2j-(n+1)\big)\theta - \gamma_{jk}\sin\big(2j-(n+1)\big)\theta\big) \\ &= \sum_{k=1}^{n+2} \varepsilon_k \sum_{j=0}^{n}\big(\beta_{jk}\cos\big(2j-(n+1)\big)\theta - \gamma_{jk}\sin\big(2j-(n+1)\big)\theta\big). \end{aligned}$$

By Lemma 2, the function $G_+(\theta,\underline{\varepsilon})$ can be rewritten as follows

$$G_+(\theta,\underline{\varepsilon}) = \sum_{k=1}^{n+2} \varepsilon_k \cos^{k-1}\theta \sin^{n+2-k}\theta. \tag{36}$$

Therefore by taking

$$a_{jk}(\delta) = \int_0^{2\pi} \cos^{k-1}\theta \sin^{n+2-k}\theta\, P^{2j-1}\big(e^{\mathbf{i}\theta},\delta\big)\,d\theta,$$

$j,k = 1,\ldots,n+2$, and noticing (32), we in fact obtain the relation (33). □

Proof of (ii). The remaining part of this section is devoted to giving the proof of this point. The proof is technical though elementary.

Consider the following square matrix of order $n+2$

$$\mathcal{A}(\delta) = \big(a_{jk}(\delta)\big)_{(n+1)\times(n+2)} = (A_1, \dots, A_{n+2}),$$

where $A_k = (a_{1k}, \dots, a_{(n+1)k})^T$ and where T means the transpose of a matrix. To prove that $s \geqslant n+1$ is equivalent to showing that the rank of $\mathcal{A}(\delta)$ is $n+1$. To this end, we introduce another square matrix $\mathcal{B}(\delta)$ of order $n+1$ whose elements are linear combinations of that of $\mathcal{A}$'s. Then we prove that this matrix $\mathcal{B}(\delta)$ is of full rank. If so, then we are done. □

We construct the matrix $\mathcal{B}(\delta)$ as follows

$$\mathcal{B}(\delta) = \big(b_{jk}(\delta)\big)_{(n+1)\times(n+1)} = (B_1, \dots, B_{n+1}), \tag{37}$$

where $B_k = (b_{1k}, \dots, b_{(n+1)k})^T$, and where

$$B_k = \begin{cases} (n-1)A_{2k-1}, & k = 1, \dots, m, \\ (2n-2k+1)A_{2k+2-n} + (n-2k)A_{2k-n}, & k = m+1, \dots, n, \\ (n-1)A_{n+1} - \delta((n-1)A_4 - 2A_2), & k = n+1. \end{cases} \tag{38}$$

Noticing the definition of the function $P(z,\delta)$ specified in (23), for simplicity, we introduce the following symbols:

$$\begin{aligned} T &:= P\big(e^{\mathbf{i}\theta}, \delta\big) = \cos^{n-1}\theta + \delta \sin^{n-1}\theta, \\ S_k &:= S_k(\theta) = \cos^k\theta \sin^{n+1-k}\theta. \end{aligned} \tag{39}$$

Then it is straightforward to see that the elements of $\mathcal{B}(\delta)$ have the form

$$b_{jk}(\delta) = \begin{cases} (n-1)\int_0^{2\pi} S_{2k-2}T^{2j-1}\,d\theta, & k = 1, \dots, m, \\ (2n-2k+1)\int_0^{2\pi} S_{2k+1-n}T^{2j-1}\,d\theta & \\ \quad + (n-2k)\int_0^{2\pi} S_{2k-n-1}T^{2j-1}\,d\theta, & k = m+1, \dots, n, \\ \delta((1-n)\int_0^{2\pi} S_3T^{2j-1}d\theta + 2\int_0^{2\pi} S_1T^{2j-1}\,d\theta) & \\ \quad + (n-1)\int_0^{2\pi} S_nT^{2j-1}\,d\theta, & k = n+1. \end{cases} \tag{40}$$

From these expressions, we see that each $b_{jk}(\delta)$ is a polynomial of δ. Moreover, with certain calculation and with the following formula

$$\int_0^{2\pi} \sin^{2m}\theta \cos^{2n}\theta\,d\theta = \frac{(2n-1)!!(2m-1)!!}{(2m+2n)!!} \cdot 2\pi,$$

we can present the explicit form of $b_{jk}(\delta)$ in terms of δ:

$$b_{jk}(\delta) = \begin{cases} \frac{2\pi(2j-1)(n-1)(2n-2k+1)!!}{(2j(n-1)+2)!!} \cdot c_{jk} \cdot \delta + *\delta^3, & k = 1, \dots, m, \\ \frac{2\pi(2j-1)(n-1)(2n-2k+1)!!}{(2j(n-1)+2)!!} \cdot c_{jk} + *\delta^2, & k = m+1, \dots, n, \\ \frac{8\pi}{3} \cdot \frac{(3n-3)!!(2j-1)}{(2j(n-1)+2)!!} \cdot c_{j(n+1)} \cdot \delta^3 + *\delta^5, & k = n+1, \end{cases} \tag{41}$$

where $*$ is a polynomial of δ which does not play any role in further calculation, and where

$$c_{jk} = \begin{cases} (2j(n-1) - 2n + 2k - 1)!!, & k = 1, \dots, n, \\ (j-1) \cdot ((2j-3)(n-1))!!, & k = n+1. \end{cases} \tag{42}$$

Notice that each element of the first m columns of the matrix $\mathcal{B}(\delta)$ has a common factor δ, and that of the last column of the matrix $\mathcal{B}(\delta)$ has a common factor δ^3. Therefore the determinant of $\mathcal{B}(\delta)$ is a polynomial of δ with the lowest degree $m+3$.

Below we shall show that the coefficient of δ^{m+3} given by the following determinant $\mathcal{C}$ is different from 0. If so, then we can choose suitable δ such that $\det \mathcal{B}(\delta) \neq 0$, and this can be done simply by taking δ any transcendental number, noticing that $\det \mathcal{B}(\delta) \neq 0$ is a polynomial of δ.

It is straightforward to see that the coefficient of δ^{m+3} of $\det \mathcal{B}(\delta)$ is $c \cdot \det(\mathcal{C})$, where

$$c = \frac{8\pi}{3}(3n-3)!! \prod_{k=1}^{n} \bigl(2\pi(n-1)(2n-2k-1)!!\bigr) \prod_{j=1}^{n+1} \left(\frac{2j-1}{(2j(n-1)+2)!!} \right) \neq 0$$

and

$$\mathcal{C} = (c_{jk})_{(n+1)\times(n+1)},$$

where c_{jk} is given in (42). Below we shall prove that $\det(\mathcal{C}) \neq 0$.

Notice the special form of the elements in the last column of $\mathcal{C}$. To calculate $\det(\mathcal{C})$, we expand $\mathcal{C}$ by the last column and obtain the following relation:

$$\det(\mathcal{C}) = \sum_{l=1}^{n+1} (-1)^{l+1} c_{l(n+1)} \det \mathcal{C} \begin{bmatrix} 1 & 2 & \cdots & l-1 & l+1 & \cdots & n+1 \\ 1 & 2 & \cdots & l-1 & l & \cdots & n \end{bmatrix},$$

where

$$\mathcal{C} \begin{bmatrix} j_1 & j_2 & \cdots & j_s \\ k_1 & k_2 & \cdots & k_s \end{bmatrix}$$

is the matrix obtained by taking $j_1, \dots, j_s$th rows and $k_1, \dots, k_s$th columns of $\mathcal{C}$.

Noticing the factor $(-1)^{l+1}$ in the expression of $\mathcal{C}$, to prove the non-vanish of $\det(\mathcal{C})$, it suffices to show that

$$c_{l(n+1)} \det \mathcal{C} \begin{bmatrix} 1 & 2 & \cdots & l-1 & l+1 & \cdots & n+1 \\ 1 & 2 & \cdots & l-1 & l & \cdots & n \end{bmatrix}$$

is decreasing with respect to l.

The following lemma gives us an inductive way to calculate the determinant of a matrix of the aforementioned form.

Lemma 6. *For* $1 \leqslant j_1 < \cdots < j_s \leqslant n+1$,

$$\det \mathcal{C} \begin{bmatrix} j_1 & j_2 & \cdots & j_s \\ 1 & 2 & \cdots & s \end{bmatrix} = C(s, j_s) \cdot \det \mathcal{C} \begin{bmatrix} j_1 & j_2 & \cdots & j_{s-1} \\ 1 & 2 & \cdots & s-1 \end{bmatrix},$$

where

$$C(s, j_s) = c_{j_s 1} \cdot (n-1)^{s-1} \prod_{l=1}^{s-1} (2j_s - 2j_l).$$

Proof. For $l = 2, \dots, s$, multiply the $(l-1)$st column of the matrix $\mathcal{C}\left[\begin{smallmatrix} j_1 & j_2 & \cdots & j_s \\ 1 & 2 & \cdots & s \end{smallmatrix}\right]$ by a nonzero number $(1 - 2l + 2n + 2j_s - 2j_s n)$ and then add the multiplication to the lth column of it. Since

$$c_{jl} + (1 - 2l + 2n + 2j_s - 2j_s n) c_{j(l-1)} = (2j_s - 2j)(1-n) c_{j(l-1)},$$

$j = j_1, \dots, j_s$, it is easy to see that under the above elementary transformations, all the elements in the last row of the matrix transformed are 0 except the first one. Moreover, all the elements in the lth row, $l = 1, \dots, s-1$, have a common factor $(2j_s - 2j_l)(1-n)$. Therefore

$$\begin{aligned} \det \mathcal{C}\begin{bmatrix} j_1 & j_2 & \cdots & j_s \\ 1 & 2 & \cdots & s \end{bmatrix} &= (-1)^{s+1} c_{j(s-1)} \cdot (1-n)^{s-1} \prod_{l=1}^{s-1} (2j_s - 2j_l) \det \mathcal{C}\begin{bmatrix} j_1 & j_2 & \cdots & j_{s-1} \\ 1 & 2 & \cdots & s-1 \end{bmatrix} \\ &= c_{j(s-1)} \cdot (n-1)^{s-1} \prod_{l=1}^{s-1} (2j_s - 2j_l) \det \mathcal{C}\begin{bmatrix} j_1 & j_2 & \cdots & j_{s-1} \\ 1 & 2 & \cdots & s-1 \end{bmatrix}. \qquad \square \end{aligned}$$

To prove the non-vanish of $\det(\mathcal{C})$, we apply the lemma repeatedly and obtain the following relation

$$c_{l(n+1)} \cdot \det \mathcal{C}\begin{bmatrix} 1 & 2 & \cdots & l-1 & l+1 & \cdots & n+1 \\ 1 & 2 & \cdots & l-1 & l & \cdots & n \end{bmatrix} = \lambda_0 \cdot D_l,$$

for $l = 2, \dots, n+1$, where

$$\lambda_0 = (2n-2)^{\frac{n(n-1)}{2}} \left(\prod_{s=1}^{n} s! \right) \left(\prod_{s=1}^{n+1} c_{s1} \right),$$

which is independent of l, and where

$$D_l = \frac{c_{l(n+1)}}{c_{l1}(l-1)!(n-l+1)!}, \quad l = 2, \dots, n+1.$$

It remains to prove that $\{D_l\}$ is monotonically decreasing with respect to l. Notice that

$$D_l = \frac{((2l-3)(n-1))!!}{(2l(n-1) - 2n + 1)!!(l-2)!(n-l+1)!}.$$

It follows that

$$\begin{aligned} \frac{D_l}{D_{l+1}} &= \frac{l-1}{n-l+1} \prod_{j=0}^{\frac{n}{2}-2} \frac{2(l+1)(n-1) - 3n + 5 + 2j}{2l(n-1) - 3n + 5 + 2j} \\ &> \frac{l-1}{n-l+1} \left(\frac{2(l+1)(n-1) - 2n + 1}{2l(n-1) - 2n + 1} \right)^{\frac{n}{2}-1} \\ &= \frac{l-1}{n-l+1} \left(1 + \frac{2(n-1)}{2(l-1)(n-1) - 1} \right)^{\frac{n}{2}-1}. \end{aligned}$$

If $\frac{n}{2}-1>2$, i.e. $n\geqslant 6$, then from above we have

$$\frac{D_l}{D_{l+1}} > \frac{l-1}{n-l+1}\left(1+\frac{1}{l-1}\right)^{\frac{n}{2}-1}$$
$$> \frac{l-1}{n-l+1}\left(1+\frac{1}{l-1}\left(\frac{n}{2}-1\right)+\frac{1}{2}\frac{1}{(l-1)^2}\left(\frac{n}{2}-1\right)\left(\frac{n}{2}-2\right)\right).$$

With some straightforward calculation, we know that the right side of the above inequality $\geqslant 1$ is equivalent to the inequality

$$4(l-1)+\frac{(n-2)(n-4)}{4(l-1)}\geqslant n+2,$$

which holds true if $2\sqrt{(n-2)(n-4)}\geqslant n+2$, namely, $n\geqslant 10$.

Apart from this, one can check directly that for $n=4,6,8$, the determinant is different from 0. We prove Theorem 2 for even n.

Finally, if n is odd, one can give a proof of the theorem, noticing the following difference with the case that n is even.

(i) We replace the function P in (23) by $\tilde{P}(z,\bar{z})$ where

$$\tilde{P}(z,\bar{z})=\left(\frac{z+\bar{z}}{2}\right)^{n-1}+\delta\left(\frac{z+\bar{z}}{2}\right)\left(\frac{z-\bar{z}}{2\mathbf{i}}\right)^{n-2},\quad \delta\in\mathbb{R}.$$

(ii) When n is even, one needs only to consider (33) which follows from (32). When n is odd, one has to study $v_{1+j(n-1)}(2\pi,\underline{\varepsilon})$ instead of $v_{1+2j(n-1)}(2\pi,\underline{\varepsilon})$ in (33). In other words, one has to study $v_{1+j(n-1)}(2\pi,\underline{\varepsilon})$ directly from (31).

5. Proof of Proposition 1

For any concrete n, we can demonstrate the validity of Proposition 1 with straightforward calculation by the algorithm mentioned above. In what follows we present the most basic ideas of the proof of the theorem for $n=4,5,6,7$, since for other n, there is no difference in algorithm. In fact, by using some common mathematics softwares, one can check with great ease if the theorem is true or not for other values of n. In Proposition 1 we assume that $n<20$, simply because we have checked all these cases.

Case $n=4$. Consider the system

$$\dot{z}=\mathbf{i}z-2z^4+z\bar{z}^3+\mathbf{i}\tau_4\bar{z}^4.$$

Then according to Theorem 3, we have

$$L_3=L_6=L_9=L_{12}=0.$$

In fact, by applying the algorithm explained in previous sections, we can obtain in a direct way that

$$L_{15}=\frac{7}{320}\tau_4^3\left(20723\tau_4^2-52278\right).$$

Clearly, if we take

$$\tau_4 = \sqrt{\frac{52278}{20723}},$$

then $L_{15} = 0$. Now fixing τ_4, we have

$$L_{18} = -\frac{176320107355600621545 3\sqrt{1083356994}}{1379469510032436458680 0} \neq 0.$$

Therefore the focus order of the system at the origin is 18.

Case $n = 6$. In this case, we consider the system

$$\dot{z} = \mathbf{i}z - \frac{3}{2}z^6 + z\bar{z}^5 + \mathbf{i}\tau_6\bar{z}^6.$$

According to Theorem 3, we have $L_k = 0$, for $k \leqslant 30$. Applying the same algorithm as in the previous case and taking

$$\tau_6 = 6\sqrt{\frac{26750299408255}{958721342366881}},$$

we have $L_{35} = 0$, and $L_{40} = -\frac{a\sqrt{b}}{c}$, where

$$a = 101\ 240\ 219\ 243\ 416\ 368\ 653\ 844\ 889\ 316\ 685\ 004\ 177\ 956\ 784\ 062\ 871$$
$$957\ 297\ 046\ 003\ 663\ 515\ 537\ 5,$$
$$b = 256\ 460\ 829\ 573\ 982\ 160\ 754\ 100\ 026\ 55, \quad \text{and}$$
$$c = 146\ 992\ 054\ 052\ 132\ 819\ 588\ 176\ 942\ 757\ 363\ 561\ 502\ 841\ 620\ 926\ 315$$
$$029\ 177\ 128\ 087\ 497\ 452\ 637\ 695\ 761\ 401\ 3.$$

For other even number n, $n < 20$, we have calculated the number τ_n which occupy pages of digits. In practice, our own computation tells us that it is more convincing and efficient to write a few lines of a program and to see the calculation rather than copy the long list of these digits here.

Cases $n = 5$ *and* 7. Now we present two typical cases where n is odd. Take $n = 5$, and consider the following system

$$\dot{z} = \mathbf{i}z + \frac{5}{3}z^5 + z\bar{z}^4 + (1 + \mathbf{i}\tau)\bar{z}^5.$$

In this case, we have $L_{2k} = 0$, for $k \leqslant 6$, and

$$L_{14} = \frac{14}{729}\left(9\tau^2 - 7\right)\left(\tau^2 - 1\right),$$

which is different from 0 for any transcendental number τ.

If $n = 7$, then we consider the system

$$\dot{z} = \mathbf{i}z + \frac{7}{5}z^7 + z\bar{z}^6 + (1 + \mathbf{i}\tau)\bar{z}^7.$$

In this case, $L_{3k} = 0$, for $k \leqslant 8$, and

$$L_{27} = -\frac{297}{6500000}\left(36055\tau^2 + 11199\right)\left(3\tau^2 - 1\right),$$

which is also different from 0 for any transcendental number τ.

References

[1] J. Bai, Y. Liu, A class of planar degree n (even number) polynomial systems with a fine focus of order $n^2 - n$, Chinese Sci. Bull. 12 (1992) 1063–1065 (in Chinese).

[2] N.N. Bautin, On the number of limit cycles which appear with the variation of coefficients from an equilibrium position of focus or center type, Mat. Sb. 30 (1952) 181–196; English transl. in: Math. USSR Sb. 100 (1954) 397–413.

[3] C. Christopher, C. Li, Limit Cycles of Differential Equations, Birkhäuser Verlag, 2007.

[4] J.P. Françoise, Successive derivatives of a first return map, application to the study of quadratic vector fields, Ergodic Theory Dynam. Systems 16 (1996) 87–96.

[5] F. Gobber, K.D. Willamowskii, Lyapunov approach to multiple Hopf bifurcation, J. Math. Anal. Appl. 71 (1979) 333–350.

[6] J.M. Hill, N. Lloyd, J. Pearson, Centres and limit cycles for an extended Kukles system, Electron. J. Differential Equations 2007 (119) (2007) 1–23.

[7] J. Huang, F. Wang, L. Wang, J. Yang, A quartic system and a quintic system with fine focus of order 18, Bull. Sci. Math. 132 (3) (2008) 205–217.

[8] J. Li, Hilbert's 16th problem and bifurcations of planar polynomial vector fields, Inernat. J. Bifur. Chaos 13 (2003) 47–106.

[9] J. Llibre, Integrability of polynomial differential systems, in: Handbook of Differential Equations (Ordinary Differential Equations, vol. 1), Elsevier, North-Holland, 2004, pp. 437–532.

[10] D. Schlomiuk, Algebraic and geometric aspects of the theory of polynomial vector fields, in: D. Schlomiuk (Ed.), Bifurcations and Periodic Orbits of Vector Fields, in: NATO ASI Ser. C, vol. 408, Kluwer Academic, London, 1993, pp. 429–467.

[11] K.S. Sibirskii, The number of limit cycles in the neighborhood of a singular point, Differential Equations 1 (1965) 36–47.

[12] H. Zoladek, Eleven small limit cycles in a cubic vector field, Nonlinearity 8 (1995) 843–860.