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Ext-quivers of hearts of A -type and the orientations of associahedrons

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ABSTRACT

We classify the Ext-quivers of hearts in the bounded derived category $\mathcal{D}(A_n)$ and the finite-dimensional derived category $\mathcal{D}(\Gamma_N A_n)$ of the Calabi–Yau- N Ginzburg algebra $\Gamma_N A_n$. This provides the classification for Buan–Thomas’ colored quivers for higher clusters of A -type. We also give an explicit combinatorial constructions from a binary tree with $n + 2$ leaves to a torsion pair in $\text{mod } \mathbf{k}\overrightarrow{A}_n$ and a cluster tilting set in the corresponding cluster category, for the straight oriented A -type quiver \overrightarrow{A}_n . As an application, we show that the orientation of the n -dimensional associahedron induced by poset structure of binary trees coincides with the orientation induced by poset structure of torsion pairs in $\text{mod } \mathbf{k}\overrightarrow{A}_n$ (under the correspondence above).

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Summary

Assem and Happel [1] gave a classification of repeatedly tilted algebras of A -type using tilting theory decades ago. In the first part of the paper (Section 1 and Section 2), we generalize their result to classify (Theorem 2.11) the Ext-quivers of hearts of A -type (i.e. in $\mathcal{D}^b(A_n)$), in terms of graded gentle trees. As an application, we describe (Corollary 2.12) the Ext-quivers of hearts in $\mathcal{D}(\Gamma_N A_n)$, the finite-dimensional derived category of the Calabi–Yau- N Ginzburg algebra $\Gamma_N A_n$, which correspond (cf. [10, Theorem 8.6]) to colored quivers for $(N - 1)$ -clusters of A -type, in the sense of Buan–Thomas [3].

In the second part of the paper (Section 3), we give explicit combinatorial constructions (Proposition 3.2 and Proposition 3.3), from a binary tree with $n + 2$ leaves (for parenthesizing a word with $n + 2$ letters) to a torsion pair in $\text{mod } \mathbf{k}\overrightarrow{A}_n$ and a cluster tilting sets in the (normal) cluster category $\mathcal{C}(A_n)$, where \overrightarrow{A}_n is a straight oriented A_n quiver. Thus, we obtain the bijections between these sets. As an application, we show (Theorem 3.5) that under the bijection above, the orientation of the

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n -dimensional associahedron induced by poset structure of binary trees (cf. [11]) coincides with the orientation induced by poset structure of torsion pairs (or hearts, in the sense of King–Qiu [10]).

Note that there are many potential orientations for the n -dimensional associahedron, arising from the representation theory of quivers, cf. [10, Fig. 4 and Theorem 9.6]. These orientations are also of interest in physics (see [4]), as they are related to wall crossing formula, quantum dilogarithm identities and Bridgeland’s stability conditions (cf. [7] and [12]).

1. Preliminaries

1.1. Derived category and cluster category

Let Q be a quiver of A -type with n vertices and \mathbf{k} a fixed algebraic closed field. Let $\mathbf{k}Q$ be the path algebra, $\mathcal{H}_Q = \text{mod } \mathbf{k}Q$ its module category and let $\mathcal{D}(Q) = \mathcal{D}^b(\mathcal{H}_Q)$ be the bounded derived category. Note that $\mathcal{D}(Q)$ is independent of the orientation of Q and we will write A_n for Q sometimes.

Denote by τ the AR-functor (cf. [2, Chapter IV]). Let $\mathcal{C}(A_n)$ be the cluster category of $\mathcal{D}(A_n)$, that is the orbit category of $\mathcal{D}(A_n)$ quotiented by $[-1] \circ \tau$. Denote by π_n be the quotient map

$$\pi_n : \mathcal{D}(A_n) \rightarrow \mathcal{C}(A_n).$$

1.2. Calabi–Yau category

Let $N > 1$ be an integer. Denote by $\Gamma_N Q$ the Calabi–Yau- N Ginzburg (dg) algebra associated to Q , which is constructed as follows (cf. e.g. [10, Section 7]):

- Let Q^N be the graded quiver whose vertex set is Q_0 and whose arrows are: the arrows in Q with degree 0; an arrow $a^* : j \rightarrow i$ with degree $2 - N$ for each arrow $a : i \rightarrow j$ in Q ; a loop $e^* : i \rightarrow i$ with degree $1 - N$ for each vertex e in Q .
- The underlying graded algebra of $\Gamma_N Q$ is the completion of the graded path algebra $\mathbf{k}Q^N$ in the category of graded vector spaces with respect to the ideal generated by the arrow of Q^N .
- The differential of $\Gamma_N Q$ is the unique continuous linear endomorphism homogeneous of degree 1 which satisfies the Leibniz rule and takes the following values on the arrow of Q^N

$$d \sum_{e \in Q_0} e^* = \sum_{a \in Q_1} [a, a^*].$$

Write $\mathcal{D}(\Gamma_N Q)$ for $\mathcal{D}_{fd}(\text{mod } \Gamma_N Q)$, the finite-dimensional derived category of $\Gamma_N Q$ and \mathcal{H}_Γ its canonical heart.

Notice that the derived categories are always triangulated. Again, since $\mathcal{D}(\Gamma_N Q)$ is independent of the orientation of Q , we will write $\Gamma_N A_n$ for $\Gamma_N Q$.

1.3. Hearts of triangulated categories

A torsion pair in an abelian category \mathcal{C} is a pair of full subcategories $\langle \mathcal{F}, \mathcal{T} \rangle$ of \mathcal{C} , such that $\text{Hom}(\mathcal{T}, \mathcal{F}) = 0$ and furthermore every object $E \in \mathcal{C}$ fits into a short exact sequence $0 \rightarrow E^{\mathcal{T}} \rightarrow E \rightarrow E^{\mathcal{F}} \rightarrow 0$ for some objects $E^{\mathcal{T}} \in \mathcal{T}$ and $E^{\mathcal{F}} \in \mathcal{F}$.

A t -structure on a triangulated category \mathcal{D} is a full subcategory $\mathcal{P} \subset \mathcal{D}$ with $\mathcal{P}[1] \subset \mathcal{P}$ such that, if one defines $\mathcal{P}^\perp = \{G \in \mathcal{D} : \text{Hom}_{\mathcal{D}}(F, G) = 0, \forall F \in \mathcal{P}\}$, then, for every object $E \in \mathcal{D}$, there is a unique triangle $F \rightarrow E \rightarrow G \rightarrow F[1]$ in \mathcal{D} with $F \in \mathcal{P}$ and $G \in \mathcal{P}^\perp$. A t -structure \mathcal{P} is bounded if

$$\mathcal{D} = \bigcup_{i, j \in \mathbb{Z}} \mathcal{P}^\perp[i] \cap \mathcal{P}[j].$$

The heart of a t-structure \mathcal{P} is the full subcategory

$$\mathcal{H} = \mathcal{P}^\perp[1] \cap \mathcal{P}$$

and any bounded t-structure is determined by its heart. In this paper, we only consider bounded t-structures and their hearts.

Recall that we can forward/backward tilt a heart \mathcal{H} to get a new one, with respect to any torsion pair in \mathcal{H} in the sense of Happel–Reiten–Smalø ([5], see also [10, Proposition 3.2]). Further, all forward/backward tilts with respect to torsion pairs in \mathcal{H} , correspond one–one to all hearts between \mathcal{H} and $\mathcal{H}[\pm 1]$ (in the sense of King–Qiu [10]). In particular there is a special kind of tilting which is called simple tilting (cf. [12, Definition 3.6]). We denote by $\mathcal{H}_S^{\#}$ and \mathcal{H}_S° , respectively, the simple forward/backward tilts of a heart \mathcal{H} , with respect to a simple S .

We define the exchange graph of a triangulated category \mathcal{D} to be the oriented graph, whose vertices are all hearts in \mathcal{D} and whose edges correspond to the simple forward tilting between them. Denote by $EG(A_n)$ the exchange graph of $\mathcal{D}(A_n)$, and $EG^\circ(\Gamma_N A_n)$ the principal component of the exchange graph of $\mathcal{D}(\Gamma_N A_n)$, that is, the connected component containing \mathcal{H}_Γ .

2. Ext-quivers of A-type

2.1. Graded gentle tree

In [1], there is a complete description of all repeatedly tilted algebras of type A_n , namely:

Definition 2.1. (See [2].) Let A be a quiver algebra with acyclic quiver T_A . The algebra $A \cong \mathbf{k}T_A/\mathcal{I}$ is called gentle if the bound quiver (T_A, \mathcal{I}) has the following properties:

- 1°. Each vertex of T_A is the source and the target of at most two arrows.
- 2°. For each arrow $\alpha \in (T_A)_1$, there is at most one arrow β and one arrow γ such that $\alpha\beta \notin \mathcal{I}$ and $\gamma\alpha \notin \mathcal{I}$.
- 3°. For each arrow $\alpha \in (T_A)_1$, there is at most one arrow ξ and one arrow ζ such that $\alpha\xi \in \mathcal{I}$ and $\zeta\alpha \in \mathcal{I}$.
- 4°. The ideal \mathcal{I} is generated by the paths in 3°.

If T_A is a tree, the gentle algebra $A \cong \mathbf{k}T_A/\mathcal{I}$ is called a gentle tree algebra.

Theorem 2.2. Let A be a quiver algebra with bound quiver (T_A, \mathcal{I}) . Then A is (repeatedly) tilted algebras of type A_n if and only if (T_A, \mathcal{I}) is a gentle tree algebra (cf. [1], also [2]).

Considering the special properties of T_A , we can color the arrows of T_A with two colors, such that any two neighbor arrows α, β has the same color if and only if $\alpha\beta \in \mathcal{I}$ or $\beta\alpha \in \mathcal{I}$. Alternatively, we can also color it with two colors, such that any two neighbor arrows α, β has the different colors if and only if $\alpha\beta \in \mathcal{I}$ or $\beta\alpha \in \mathcal{I}$. By the properties above, either coloring is unique up to swapping colors. Hence we have another way to characterize gentle tree algebra as follows.

Definition 2.3. A gentle tree is a quiver T with a 2-coloring, such that each vertex has at most one arrow of each color incoming or outgoing.

For a colored quiver T , there are two natural ideals

$$\begin{aligned} \mathcal{I}_T^+ &: \text{ generated by all unicolor-paths of length two;} \\ \mathcal{I}_T^- &: \text{ generated by all alternating color paths of length two.} \end{aligned}$$

Proposition 2.4. Let $A = \mathbf{k}T/\mathcal{I}$ be a bound quiver algebra. We have the following equivalent statement:

- A is a gentle tree algebra.
- T is some gentle tree with $\mathcal{I} = \mathcal{I}_T^+$ or $\mathcal{I} = \mathcal{I}_T^-$.

Proof. By the one of two ways of coloring, the relations in the ideal and the coloring of the gentle tree can be determined uniquely by each other. \square

Remark 2.5. In fact, there is an interesting result (but not used in this paper) for a gentle tree T , that $\mathbf{k}T/\mathcal{I}_T^+$ and $\mathbf{k}T/\mathcal{I}_T^-$ are Koszul dual to each other.

We are going to generalize [Theorem 2.2](#) to describe all hearts in $\mathcal{D}(A_n)$.

2.2. Ext-quivers of hearts

Recall that a heart \mathcal{H} is finite, if the set of its simples, denoted by $\text{Sim } \mathcal{H}$, is finite and generates \mathcal{H} by means of extensions,

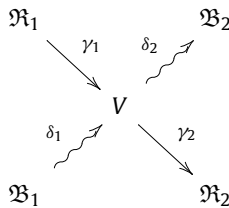
Definition 2.6. Let \mathcal{H} be a finite heart in a triangulated category \mathcal{D} and $\mathbf{S} = \bigoplus_{S \in \text{Sim } \mathcal{H}} S$. The Ext-quiver $\mathcal{Q}(\mathcal{H})$ is the (positively) graded quiver whose vertices are the simples of \mathcal{H} and whose graded edges correspond to a basis of $\text{End}^\bullet(\mathbf{S}, \mathbf{S})$ (where the grading is given by the degree of Hom).

Note that, by [\[10\]](#), \mathcal{H} is finite, rigid and strongly monochromatic for any \mathcal{H} in $\mathcal{D}(A_n)$. By [\[10, Lemma 3.3\]](#), we know that there is at most one arrow between any two vertices in $\mathcal{Q}(\mathcal{H})$.

Definition 2.7. A *graded gentle tree* \mathcal{G} is a gentle tree with a positive grading for each arrow. The associated quiver $\mathcal{Q}(\mathcal{G})$ of \mathcal{G} , is a graded quiver with the same vertex set and an arrow $a : i \rightarrow j$ for each unicolored path $p : i \rightarrow j$ in \mathcal{G} , with the natural grading of p .

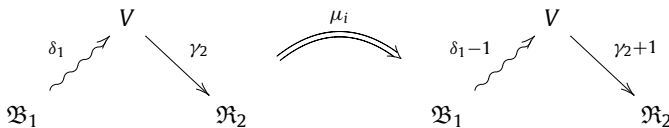
Define a mutation μ on graded gentle tree as follow.

Definition 2.8. For a graded gentle tree \mathcal{G} , let V be a vertex with neighborhood

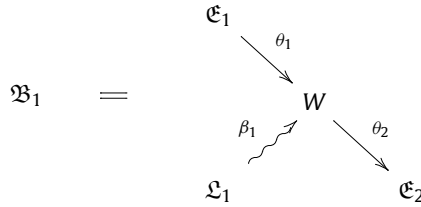


where $\mathcal{B}_i, \mathcal{R}_i$ are the sub trees and γ_i, δ_i are degrees of \mathcal{G} , $i = 1, 2$. The straight line represents one color and the curly line represent the other color. Define the *forward mutation* μ_V at vertex V (on \mathcal{G}) as follows:

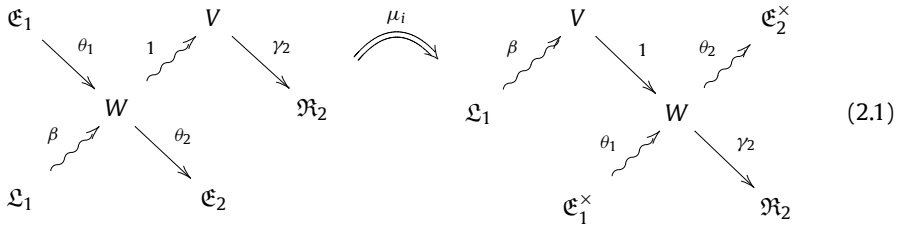
- if $\delta_1 \geq 1$, μ_V on the lower part of the quiver is:



- if $\delta_1 = 1$, denote



and μ_V on the lower part of the quiver is:



where \mathfrak{X}^\times is the operation of swapping colors on a graded gentle trees \mathfrak{X} .

- μ_V on the upper part follows the mirror of the lower part.

Dually, define the backward mutation μ_V^{-1} to be the reverse of μ_V (which follows a similar rule).

Clearly, the set of all graded gentle trees with n vertexes is closed under such mutation. In fact, this set is also connected under (forward/backward) mutation.

Lemma 2.9. Any graded gentle tree with n vertices can be repeatedly mutated from another graded gentle tree with n vertices.

Proof. Use induction, starting from the trivial case when $n = 1$. Suppose that the lemma holds for $n = m$ and consider the case for $n = m + 1$. We only need to show that any graded gentle tree \mathcal{G} with $m + 1$ vertices can be repeatedly mutated from a unicolor graded gentle tree with all degrees equal one. Let V be a sink in \mathcal{G} and the subtree of \mathcal{G} by deleting V is \mathcal{G}' while the connecting arrow from \mathcal{G}' to V has degree d . By backward mutating on V , we can increase d as large as possible without changing \mathcal{G}' . Then the mutation at a vertex other than V on \mathcal{G} restricted to \mathcal{G}' will be the same as mutating at that vertex on \mathcal{G}' . Thus, by the induction assumption, we can mutate \mathcal{G} such that \mathcal{G}' becomes unicolor with all degrees equal zero. Then, repeatedly forward mutating many times on V will turn \mathcal{G} into unicolor with all degrees equal zero. \square

Using Lemma A.4, a direct calculation gives the following proposition.

Proposition 2.10. Let \mathcal{G} be a graded gentle tree and \mathcal{H} be a heart in $\mathcal{D}(A_n)$. If $\mathcal{Q}(\mathcal{G}) = \mathcal{Q}(\mathcal{H})$ with vertex V in \mathcal{G} corresponding to the simple S in \mathcal{H} , then

$$\mathcal{Q}(\mathcal{H}_S^\#) = \mathcal{Q}(\mu_V \mathcal{G}), \quad \mathcal{Q}(\mathcal{H}_S^b) = \mathcal{Q}(\mu_V^{-1} \mathcal{G}). \tag{2.2}$$

Now we can describe all Ext-quiver of hearts of A -type.

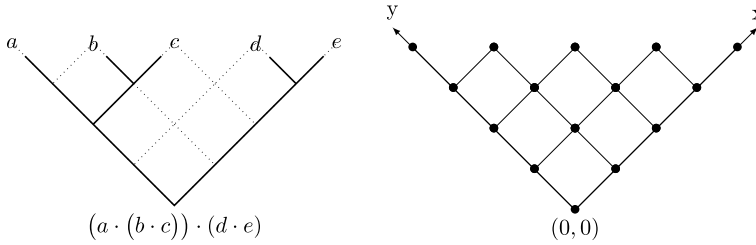


Fig. 1. A parenthesizing of four words on the left and G_4 on the right.

Theorem 2.11. *The Ext-quivers of hearts in $\mathcal{D}(A_n)$ are precisely the associated quivers of graded gentle trees with n vertices.*

Proof. Note that any heart in $\mathcal{D}(A_n)$ can be repeatedly tilted from the standard heart \mathcal{H}_Q , by [9]. Without loss of generality, let Q have straight orientation. Then $\mathcal{Q}(\mathcal{H}_Q)$ certainly is the associated quiver for the graded gentle tree \mathcal{G}_Q with the same orientation and alternating colored arrows. Then, inducting from \mathcal{H}_Q and using (2.2), we deduce that the Ext-quiver of any heart in $\mathcal{D}(A_n)$ is the associated quivers of some graded gentle tree with n vertices. On the other hand, the set of graded gentle trees with n vertices is connected (Lemma 2.9). Then, also by induction, we deduce that the associated quiver of any graded gentle tree with n vertices is the Ext-quiver of some heart, because (2.2) and the fact that we can forward/backward tilt any simples in any heart in $\mathcal{D}(A_n)$ [10, Theorem 5.7]. \square

Recall that we can Calabi–Yau- N (CY- N) double a graded quiver in the sense of [10, Definition 6.2]. Then we have the following corollary.

Corollary 2.12. *The Ext-quivers of hearts in $EG^\circ(\Gamma_N A_n)$ are precisely the CY- N double of the associated quivers of graded gentle trees with n vertices.*

Proof. By [10, Corollary 8.3], any heart \mathcal{H} in $EG^\circ(\Gamma_N A_n)$ is induced from some heart \mathcal{H}' in $\mathcal{D}(A_n)$, while Ext-quiver $\mathcal{Q}(\mathcal{H})$ is the CY- N double of $\mathcal{Q}(\mathcal{H}')$ by [10, Proposition 7.5]. Thus the corollary follows from Theorem 2.11. \square

By [10, Proposition 8.6], the augmented graded quivers of colored quivers for $(N - 1)$ -clusters (cf. [10, Definition 6.1] and [3]) of type A_n are also precisely the CY- N double of the associated quivers of graded gentle trees.

3. Associahedron

3.1. Binary trees

Let BT_m be the set of binary trees with $m + 1$ leaves (and hence with m internal vertices), which can be used to parenthesize a word with $m + 1$ letters (see Fig. 1 and cf. [11]). Let G_m be the full subgraph of the grid \mathbb{Z}^2 induced by

$$G_m = \{(x, y) \mid x \geq 0, y \geq 0, x + y \leq m\} \subset \mathbb{Z}^2.$$

It is well-known that a binary tree with $m + 1$ leaves has a normal form as a subgraph of G_m , such that the leaves are $\{(x, m - x)\}_{x=0}^m$, and we will identify the binary tree with such normal form (see Fig. 1).

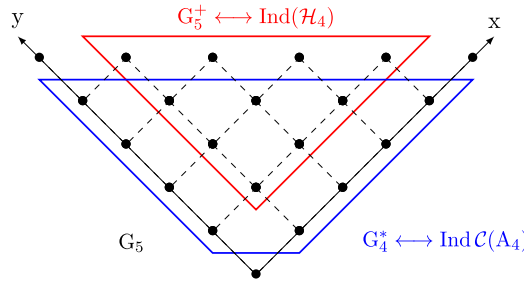


Fig. 2. G_5^+ (red) and G_4^* (blue) sit inside G_5 . (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Example 3.1. Let

$$G_m^+ = G_m \cap \{(x, y) \mid xy > 0\}, \quad G_m^* = G_m - \{(0, 0)\}.$$

Consider the A_n -quiver $\vec{A}_n: n \rightarrow \dots \rightarrow 1$ and let $\mathcal{H}_n = \text{mod } \mathbf{k}\vec{A}_n$ with corresponding simples S_1, \dots, S_n . Then, there are canonical bijections (cf. Fig. 2)

$$\begin{aligned} \xi_n : G_{n+1}^+ &\rightarrow \text{Ind}(\mathcal{H}_n), \\ \zeta_n : G_n^* &\rightarrow \text{Ind } \mathcal{H}_n \cup \text{Proj } \mathcal{H}_n[1] \end{aligned}$$

satisfying $\xi_n(i, j) = \zeta_n(i - 1, j) = M_{i,j}$, where $M_{i,j} \in \text{Ind } \mathcal{H}_n$ is determined by

$$[M_{i,j}] = \sum_i^{n+1-j} [S_k]. \tag{3.1}$$

Let $\zeta_n = \pi_n \circ \zeta_n : G_n^* \rightarrow \text{Ind } \mathcal{C}(A_n)$.

It is known that the following sets (see [6] for more possible sets) can parameterize the vertex set of an n -dimensional associahedron:

- 1°. the set BT_{n+1} of binary trees with $n + 2$ leaves;
- 2°. the set of triangulations of regular $(n + 3)$ -gon;
- 3°. the set $\text{CEG}(A_n)$ of (2-)cluster tilting sets in $\mathcal{C}(A_n)$;
- 4°. the set $\text{TP}(\vec{A}_n)$ of torsion pairs in \mathcal{H}_n (cf. [10] and [5]);
- 5°. the set $\text{EG}(\mathcal{H}_n, \mathcal{H}_n[1])$ of hearts in $\mathcal{D}(A_n)$ between \mathcal{H}_n and $\mathcal{H}_n[1]$ (in the sense of King–Qiu, [10]).

There are natural bijections between these sets (cf. [10,6,8]).

Furthermore, by [10, Section 9], the poset structure of torsion pairs (hearts) gives an orientation O_t of the n -dimensional associahedron, i.e. the orientation of $\text{EG}(\mathcal{H}_n, \mathcal{H}_n[1])$ (considered as a subgraph of $\text{EG}(A_n)$).

On the other hand, there is a natural poset structure on binary trees, inducing by locally flipping a binary tree (as shown in Fig. 3), or equivalently, changing the corresponding parenthesizing of words from $(A \cdot B) \cdot C$ to $A \cdot (B \cdot C)$ (see [11] for details). This poset structure also gives an orientation O_p for the associahedron. We aim to prove $O_t = O_p$ this section.

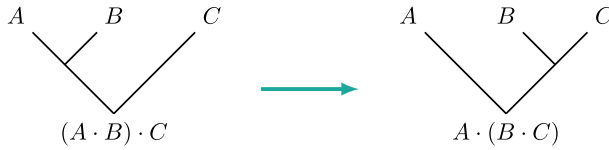


Fig. 3. A local flip of a binary tree (at the word B).

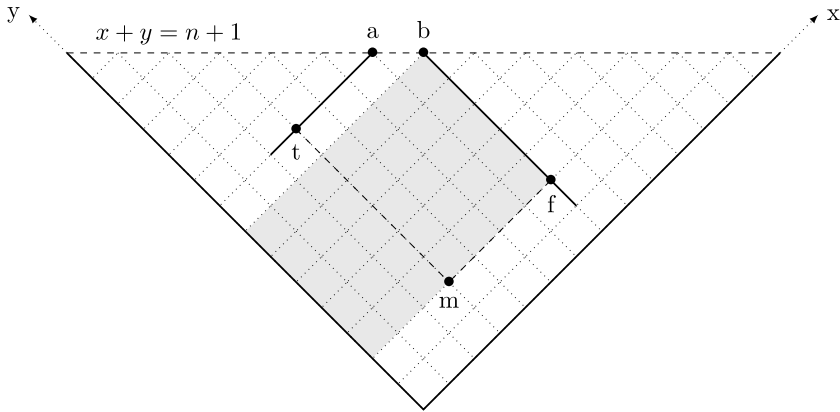


Fig. 4. A short exact sequence in G_{n+1} .

3.2. Combinatorial constructions

First, we give explicit construction of torsion pairs from binary trees. For any $p \in \mathbb{Z}^2$ with coordinate (x_p, y_p) , let $L(p)$ be the edge connecting $(x_p - 1, y_p)$ and p and $R(p)$ be the edge connecting $(x_p, y_p - 1)$ and p . Given a tree \mathbf{b} in BT_{n+1} , define

$$\mathcal{T}(\mathbf{b}) = \langle \xi_n(p) \mid p \in G_{n+1}^+, L(p) \in \mathbf{b} \rangle, \quad \mathcal{F}(\mathbf{b}) = \langle \xi_n(p) \mid p \in G_{n+1}^+, R(p) \in \mathbf{b} \rangle, \quad (3.2)$$

where $\langle \rangle$ means generating by extension.

Proposition 3.2. *There is a bijection $\Theta_n : BT_{n+1} \rightarrow TP(\vec{A}_n)$, sending $\mathbf{b} \in BT_{n+1}$ to $\langle \mathcal{T}(\mathbf{b}), \mathcal{F}(\mathbf{b}) \rangle$.*

Proof. We only need to show that $\Theta_n : \mathbf{b} \mapsto \langle \mathcal{T}(\mathbf{b}), \mathcal{F}(\mathbf{b}) \rangle$ is well-defined (and obviously injective) and hence bijective since both sets have the n th Catalan number many elements.

To do so, we first show that any object $M \in \mathcal{H}_n$ admits a short exact sequence

$$0 \rightarrow T \rightarrow M \rightarrow F \rightarrow 0 \quad (3.3)$$

for some $T \in \mathcal{T}(\mathbf{b})$ and $F \in \mathcal{F}(\mathbf{b})$. Let $m = \xi_n^{-1}(M) \in G_{n+1}^+$. If $m \in \mathbf{b}$ then $M \in \mathcal{T}(\mathbf{b}) \cup \mathcal{F}(\mathbf{b})$ and we have a trivial short exact sequence (3.3). If $m \notin \mathbf{b}$, let t be the vertex in $\mathbf{b} \cap \{(x_m, j) \mid j \geq y_m\}$ with minimal y -coordinate and f be the vertex in $\mathbf{b} \cap \{(i, y_m) \mid i \geq x_m\}$ with minimal x -coordinate; let a and b be the vertices with coordinates $(n + 1 - y_t, y_t)$ and $(x_f, n + 1 - x_f)$, see Fig. 4. By construction and the property of the binary tree, we know that

- edges in the line segments, from m to t and from m to f , are not in \mathbf{b} ;
- edges in the line segments, from a to t and from b to f , are in \mathbf{b} ;
- $(x_a, y_a) + (1, -1) = (x_b, y_b)$, i.e. a, b are neighbors in the line $x + y = n + 1$;
- $L(t)$ and $R(f)$ are in \mathbf{b} .

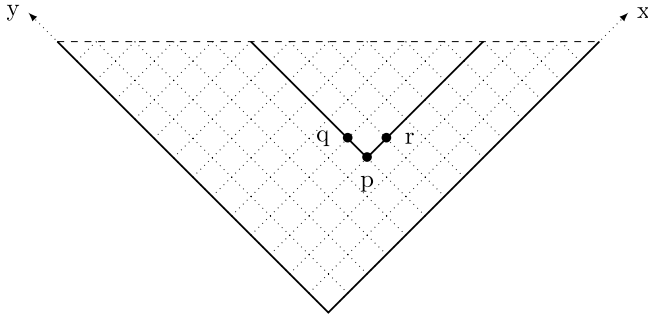


Fig. 5. An interval vertex of a binary tree in G_{n+1} .

Thus $T = \xi_n(t) \in \mathcal{T}(\mathbf{b})$ and $F = \xi_n(f) \in \mathcal{F}(\mathbf{b})$. By (3.1), a direct calculation shows that $[M] = [T] + [F]$, which implies we have (3.3), by Lemma A.1, as required.

To finish, we need to show that $\text{Hom}(\mathcal{T}(\mathbf{b}), \mathcal{F}(\mathbf{b})) = 0$. Let $F = \xi_n(f) \in \mathcal{F}(\mathbf{b})$. As above, edges in the line segments from b to f are in \mathbf{b} . By the property of binary tree, the horizontal edges (i.e. parallel to x -axis) in the shaded area in Fig. 4 are not in \mathbf{b} , which implies, by Lemma A.2, that the modules in \mathcal{H}_n that have nonzero maps to F are not in $\mathcal{T}(\mathbf{b})$, as required. \square

Next, we identify cluster tilting sets from binary trees via ζ_n . For any $\mathbf{b} \in \text{BT}_{n+1}$, let $\text{iv}(\mathbf{b})$ be set of the internal vertices expect $(0, 0)$ so that $\#\text{iv}(\mathbf{b}) = n$. Denote by $\text{Proj } \mathcal{H}$ a complete set of indecomposable projectives of a heart \mathcal{H} . Recall [10, Section 2] that

$$P \in \text{Proj } \mathcal{H} \iff P \in \text{Ind}(\mathcal{P} \cap \tau^{-1}\mathcal{P}^\perp), \tag{3.4}$$

where \mathcal{P} is the t-structure corresponding to \mathcal{H} .

Proposition 3.3. Let $\mathbf{b} \in \text{BT}_{n+1}$ and $\mathcal{H}(\mathbf{b})$ be the heart corresponding to the torsion pair $\Theta_n(\mathbf{b})$ in \mathcal{H}_n . Then we have $\text{Proj } \mathcal{H}(\mathbf{b}) = \zeta_n(\text{iv}(\mathbf{b}))$ and there is a bijection $\zeta_n \circ \text{iv} : \text{BT}_{n+1} \rightarrow \text{CEG}(A_n)$.

Proof. By [10, Corollary 5.12], we know that $\pi_n \text{Proj } \mathcal{H}(\mathbf{b}) \in \text{CEG}(A_n)$ and hence the second claim follows immediately from the first one.

Let $p \in \text{iv}(\mathbf{b})$, which is the intersection of the edges $L(r)$ and $R(q)$, where q, r be the points with coordinates $(x_p, y_p + 1)$ and $(x_p + 1, y_p)$ (see Fig. 5). Note that p is not in the line $x_p + y_p = n$ and thus $q, r \in G_{n+1}$. Let $\mathcal{P}(\mathbf{b})$ be the t-structure corresponding to $\mathcal{H}(\mathbf{b})$. Note that

$$\mathcal{P}(\mathbf{b}) = \mathcal{T}(\mathbf{b}) \cup \bigcup_{j>0} \mathcal{H}_n[j], \quad \mathcal{P}(\mathbf{b})^\perp = \mathcal{F}(\mathbf{b}) \cup \bigcup_{j<0} \mathcal{H}_n[j].$$

If $r \in G_{n+1}^+$, then $P = \zeta_n(p) = \xi_n(r)$ is in $\mathcal{T}(\mathbf{b})$; otherwise, $y_p = 0$ and then $P \in \mathcal{H}_n[1]$. Either way, $P \in \mathcal{P}(\mathbf{b})$. Similarly, if $q \in G_{n+1}^+$, then $\tau P = \xi_n(q)$ is in $\mathcal{F}(\mathbf{b})$; otherwise, $x_p = 0$ and then $\tau P \in \mathcal{H}_n[-1]$. Either way, $\tau P \in \mathcal{P}(\mathbf{b})^\perp$. Therefore $P \in \text{Proj } \mathcal{H}(\mathbf{b})$ by (3.4). Thus $\text{Proj } \mathcal{H}(\mathbf{b})$ contains, and hence equals $\zeta_n(\text{iv}(\mathbf{b}))$ as required, noticing that $\#\text{Proj } \mathcal{H}(\mathbf{b}) = n = \#\text{iv}(\mathbf{b})$.

Example 3.4. Keep the notation in Example 3.1. Then the binary tree in Fig. 1 corresponds to the torsion pair $\mathcal{T} = \langle M_{2,2}, M_{1,2} \rangle$, $\mathcal{F} = \langle M_{1,3}, M_{3,1} \rangle$ and the cluster tilting set $\{M_{1,2}, M_{2,2}, M_{1,1}[1]\}$.

3.3. The orientation

Now we apply the constructions above to show that $O_t = O_p$.

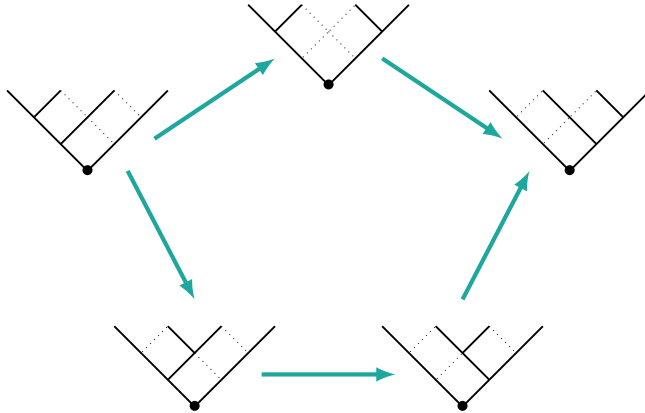


Fig. 6. The orientation of the 2-dimensional associahedron.

Theorem 3.5. Under the bijection Θ_n in Proposition 3.2, the orientations O_t and O_p of the n -dimensional associahedron coincide.

Proof. Consider an edge $e: \mathbf{b}_1 \rightarrow \mathbf{b}_2$ in BT_{n+1} , which corresponds to a local flip as in Fig. 3. Let $\mathcal{H}(\mathbf{b}_i)$ forward tilt of \mathcal{H}_n with respect to $\Theta_n(\mathbf{b}_i)$. We only need to show that $\mathcal{H}(\mathbf{b}_2)$ is a simple forward tilt of $\mathcal{H}(\mathbf{b}_1)$.

By Proposition 3.3, we know that $\text{Proj } \mathcal{H}(\mathbf{b}_i) = \zeta_n(\text{iv}(\mathbf{b}_i))$, for $i = 1, 2$, differ by one object. Denote by $P_i \in \text{Proj } \mathcal{H}(\mathbf{b}_i)$ the different objects. Thus, $\pi_n \text{Proj } \mathcal{H}(\mathbf{b}_i) \in \text{CEG}(A_n)$ are related by one mutation, which implies $\mathcal{H}(\mathbf{b}_i)$ are related by a single simple tilting, by [10, Corollary 5.12], and P_1, P_2 are related by some triangle

$$P_j \rightarrow M \rightarrow P_k \rightarrow P_j[1]$$

in $\mathcal{D}(A_n)$ for some ordering $\{j, k\} = \{1, 2\}$. By Lemma A.3, P_j is a predecessor of P_k . But, from the flip we know that P_1 is the predecessor of P_2 , which implies $j = 1$ and $k = 2$. Thus the forward simple tilting is from $\mathcal{H}(\mathbf{b}_1)$ to $\mathcal{H}(\mathbf{b}_2)$ as required. \square

Example 3.6. Fig. 6 is the orientation of the 2-dimensional associahedron, induced by poset structure of binary trees, which is the oriented pentagon in [10, Fig. 3] and [12, (3.5)], cf. also [7, Fig. 5].

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I would like to thank Alastair King for introducing me to this topic.

Appendix A. Maps and triangles in $\mathcal{D}(A_n)$

In this appendix, we collect several facts about the maps and triangles in $\mathcal{D}(A_n)$. See [2, Chapter IX] for the proofs of the first three lemmas.

Recall there are notions of sectional paths and predecessors in $\mathcal{D}(A_n)$ cf. [12, Section 2.2].

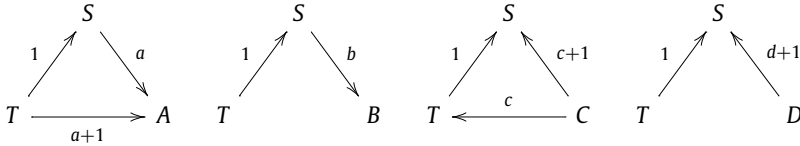
Lemma A.1. Let $M, A, B \in \text{Ind } \mathcal{D}(A_n)$ such that $A \in \text{Ps}^{-1}(M)$ and $B \in \text{Ps}(M) - \text{Ps}(A)$. Then there is a short exact sequence $0 \rightarrow A \rightarrow M \rightarrow B \rightarrow 0$ if and only if $[M] = [A] + [B]$.

Lemma A.2. Let $M, L \in \text{Ind } \mathcal{D}(A_n)$. Then $\text{Hom}(M, L) \neq 0$ if and only if

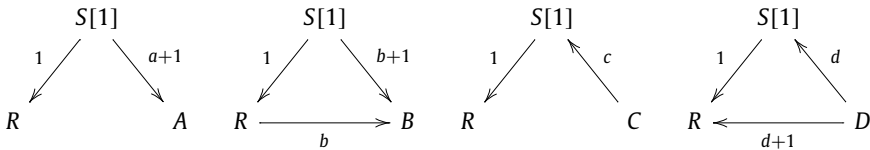
$$L \in [\text{Ps}(M), \text{Ps}^{-1}(\tau(M[1]))], \quad M \in [\text{Ps}(\tau^{-1}(L[-1])), \text{Ps}^{-1}(L)].$$

Lemma A.3. If $\text{Hom}(L, M[1]) \neq 0$ for some M and L in $\text{Ind } \mathcal{D}(A_n)$, then M is a predecessor of L . Any two non-isomorphic indecomposables in $\mathcal{D}(A_n)$ cannot be predecessors of each other.

Lemma A.4. Let \mathcal{H} be a heart in $\mathcal{D}(A_n)$. For any the following sub-quivers



in the Ext-quiver $\mathcal{Q}(\mathcal{H})$ for some $S, T, A, B, C, D \in \text{Sim } \mathcal{H}$ and positive integers a, b, c, d , they become



in the Ext-quiver $\mathcal{Q}(\mathcal{H}_S^\sharp)$, where R is the nontrivial extension of T on top of S .

Proof. We only prove the first case while the other cases are similar. By [10, Theorem 5.7], we know that the simples in \mathcal{H}_S^\sharp corresponding to S, T and A are $S[1], R$ and A . By [12, Lemma 3.3], we have an isomorphism $\text{Hom}^1(T, S) \otimes \text{Hom}^a(S, A) \rightarrow \text{Hom}^{a+1}(T, A)$. Thus, applying $\text{Hom}(-, A)$ to the triangle $S \rightarrow R \rightarrow T \rightarrow S[1]$ gives $\text{Hom}^\bullet(R, A) = 0$. Similarly, a direct calculation of other Hom^\bullet between $S[1], R, A$ shows the new sub-quiver is as required. \square

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