

Tagged mapping class groups: Auslander–Reiten translation

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Abstract We give a geometric realization, the tagged rotation, of the AR-translation on the generalized cluster category associated to a surface S with marked points and non-empty boundary, which generalizes Brüstle–Zhang’s result for the puncture free case. As an application, we show that the intersection of the shifts in the 3–Calabi–Yau derived category $\mathcal{D}(\Gamma_S)$ associated to the surface and the corresponding Seidel–Thomas braid group of $\mathcal{D}(\Gamma_S)$ is empty, unless S is a polygon with at most one puncture (i.e. of type A or D).

Keywords Mapping class group · Auslander–Reiten translation · Triangulated surface · Cluster theory · Braid group

1 Introduction

Fomin, Shapiro and Thurston studied in [13] the cluster combinatorics of a marked surface, that is, an oriented surface S with a finite set of marked points M on the boundary and a finite set of punctures P inside S . We do not consider closed surfaces in this paper, that is we assume S to have a non-empty boundary. Studying the relation between Teichmüller theory and cluster algebras, it has been shown in [11, 12, 14] that the flip of an arc in a (ideal) triangulation corresponds to a mutation of a cluster variable. However, self-folded triangles do not permit to flip the internal arc, whereas the mutation of a cluster variable is always defined. In order to allow for flips at all arcs, Fomin, Shapiro and Thurston introduce

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in [13] the concept of tagged triangulations, which are (ideal) triangulations equipped with a tagging at each puncture. The thus obtained exchange graph $EG^\times(\mathbf{S})$ of tagged triangulations, encoding flips of tagged triangulations, is isomorphic to the cluster exchange graph of the cluster algebra $\mathcal{A}(\mathbf{S})$ defined by the marked surface \mathbf{S} .

The aim of this paper is to compare geometric properties of the marked surface \mathbf{S} with properties of the categorification $\mathcal{C}(\mathbf{S})$ of the cluster algebra $\mathcal{A}(\mathbf{S})$: Labardini–Fragoso associated in [20] a quiver with potential to the marked surface \mathbf{S} which allows to use Amiot’s construction in [1] to define a 2-Calabi–Yau category $\mathcal{C}(\mathbf{S})$. It is shown in [9] that the cluster-tilting objects in $\mathcal{C}(\mathbf{S})$ correspond bijectively to the clusters of the cluster algebra $\mathcal{A}(\mathbf{S})$. Moreover, the set $\mathbf{A}^\times(\mathbf{S})$ of tagged arcs in \mathbf{S} corresponds bijectively to the set $\mathcal{C}^\times(\mathbf{S})$ of reachable indecomposable objects in $\mathcal{C}(\mathbf{S})$, and we denote by

$$\text{Aut}_0 \mathcal{C}(\mathbf{S}) = \text{Aut } \mathcal{C}^\times(\mathbf{S}) / \text{stab } \mathcal{C}^\times(\mathbf{S})$$

the group of auto-equivalences of $\mathcal{C}(\mathbf{S})$ that preserve the set of reachable indecomposable objects modulo the subgroup of autoequivalences that act as identity on $\mathcal{C}^\times(\mathbf{S})$. We view this group as an analogue of the group of cluster algebra automorphisms studied in [2].

On the geometric side, we consider the group of orientation-preserving diffeomorphisms of \mathbf{S} that preserve (not necessarily pointwise) the set of marked points \mathbf{M} , the set of punctures \mathbf{P} and the boundary. The marked mapping class group $\text{MCG}_\bullet(\mathbf{S})$ of \mathbf{S} is the quotient of this group by the subgroup of diffeomorphisms that are isotopic to the identity. Denote by

$$\text{MCG}_\times(\mathbf{S}) = \text{MCG}_\bullet(\mathbf{S}) \times (\mathbb{Z}_2)^P$$

the tagged mapping class group of \mathbf{S} consisting of elements (φ, δ) , where $\varphi \in \text{MCG}_\bullet(\mathbf{S})$ and δ is a $\{\pm 1\}$ -sign on each point in \mathbf{P} . An element in $\text{MCG}_\times(\mathbf{S})$ acts on the set $\mathbf{A}^\times(\mathbf{S})$ and hence on $\mathcal{C}^\times(\mathbf{S})$, which induces an embedding

$$\text{MCG}_\times(\mathbf{S}) \hookrightarrow \text{Aut}_0 \mathcal{C}(\mathbf{S}) \tag{1.1}$$

The category $\mathcal{C}(\mathbf{S})$ admits a distinguished automorphism, the Auslander–Reiten translation, which in this context coincides with the suspension functor of the triangulated category $\mathcal{C}(\mathbf{S})$, see [17]. The first aim of this article is to show that the Auslander–Reiten translation can be realized by an element in the tagged mapping class group $\text{MCG}_\times(\mathbf{S})$. For each boundary component Y with m marked points, denote by ρ_Y the m -th root of the Dehn twist around Y , that is, simultaneous rotation to the next marked point on Y . Then we define the (universal) tagged rotation ϱ as the permutation on $\mathbf{A}^\times(\mathbf{S})$ induced by the element

$$\varrho = \prod_{Y \subset \partial \mathbf{S}} \rho_Y \cdot \prod_{P \in \mathbf{P}} \delta^P$$

in $\text{MCG}_\times(\mathbf{S})$ where the first product is over all connected components Y of $\partial \mathbf{S}$ and the second product is a simultaneous change of tagging. As one of the main results of this paper we show in Theorem 3.8 that the tagged rotation $\varrho \in \text{MCG}_\times(\mathbf{S})$ on $\mathbf{A}^\times(\mathbf{S})$ becomes the shift [1] on $\mathcal{C}^\times(\mathbf{S})$.

This result is known from [25] for the case of a punctured disc and from [5] for all unpunctured surfaces. In fact, the cluster category of an unpunctured surface is explicitly given by a combinatorial description of string and band objects, and the shift functor [1] is well-known from the theory of string algebras. These methods are not available in the general situation since there is no explicit description of the (indecomposable) objects and morphisms yet. Instead, we show that one can always choose a triangulation such that the operation of the shift functor in the category $\mathcal{C}(\mathbf{S})$ corresponds to the tagged flip of an arc.

The proof is thus an interplay between the triangulated structure of the category $\mathcal{C}(\mathbf{S})$ and the structure of the triangulations of the surface \mathbf{S} . In fact, the result we obtain helps to understand the indecomposable objects and irreducible morphisms of $\mathcal{C}(\mathbf{S})$, since these are encoded in the Auslander–Reiten triangles of the triangulated category $\mathcal{C}(\mathbf{S})$. A further study aiming to generalize results in [5] to the unpunctured case is undertaken in [24].

One motivation to study such a geometric realization of the shift functor comes from physics: to compute the complete spectrum of a BPS particle, one studies maximal green sequences which go from a triangulation to its shift in $\mathcal{C}(\mathbf{S})$, see [7]. Our result allows to determine the tagged triangulation for the endpoint of any maximal green mutation sequence from the tagged triangulation of the starting point, see [6] (cf. [23]). Note that we do not prove the existence of such a (finite) maximal green sequence, we can only provide a guidance in the search for maximal green sequences: if one exists, it needs to end in the triangulation obtained by applying the tagged rotation ϱ .

An invariant induced by the tagged rotation is the *order* of a tagged arc α , that is, the minimal natural number m satisfying $X_\alpha[m] = X_\alpha$, where X_α is the object corresponding to α in $\mathcal{C}(\mathbf{S})$. By convention, the order is infinite if such an m does not exist. We study in the last section of this article the order m of the shift operator [1] of $\mathcal{C}(\mathbf{S})$, and compare the action of the shift functor to the action of the Seidel–Thomas braid group $\text{Br } \mathbf{S}$ which is generated by spherical twists. We show in Theorem 4.4 that the intersection of the shifts and $\text{Br } \mathbf{S}$ is empty, unless \mathbf{S} is a polygon with at most one puncture (i.e. of type A or D).

Having realized one distinguished automorphism of $\mathcal{C}(\mathbf{S})$ geometrically, we further show (Proposition 4.7) that every cluster automorphism is induced from elements in the corresponding tagged mapping class group except for three cases (the marked surfaces corresponding to type D_2 , D_4 and \tilde{D}_4).

2 Preliminaries

2.1 Quiver with potential from surface

Throughout the article, \mathbf{S} denotes a *marked surface* in the sense of [13], that is, a connected Riemann surface with a fixed orientation endowed with a finite set of marked points \mathbf{M} on the boundary $\partial\mathbf{S}$ and a finite set of punctures \mathbf{P} inside \mathbf{S} such that each connected component of the boundary of \mathbf{S} contains at least one marked point. Unless otherwise stated, we will always suppose that $\partial\mathbf{S} \neq \emptyset$.

We recall the following definitions from [13], see there for the precise conditions for everything being well-defined: Curves in \mathbf{S} are considered up to isotopy with respect to the sets of marked points and punctures. Two curves are called *compatible* if they do not intersect in their relative interior. A *simple* curve has no self-intersection except possibly on the endpoints, and an *arc* in \mathbf{S} is a simple curve whose both endpoints are marked points or punctures and which is not isotopic to a boundary component.

An *ideal triangulation* \mathbf{T} of \mathbf{S} is a maximal collection of pairwise compatible arcs in \mathbf{S} . Any triangulation \mathbf{T} of \mathbf{S} consists of

$$n = 6g + 3p + 3b + m - 6 \tag{2.1}$$

arcs (ordinary or tagged), where g is the genus of \mathbf{S} , b is the number of boundary components and m and p denote the number of marked points and punctures. The number n is called the

Fig. 1 A self-folded triangle and the corresponding tagged version

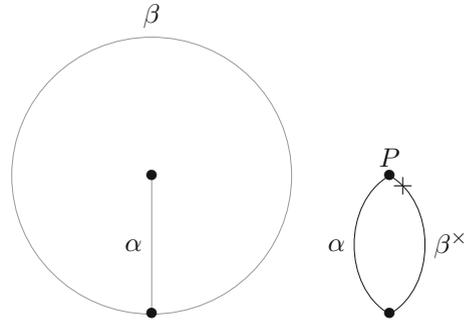
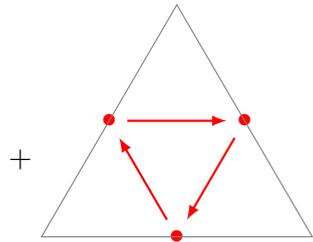


Fig. 2 The quiver associated to a triangle



rank of the surface S . To exclude a few cases where the surface does not admit a triangulation we assume from now on that $n \geq 1$.

The flip of an arc α in a triangulation T is the unique arc $\alpha' \neq \alpha$ which forms a triangulation with the remaining arcs of T . We denote by $EG^\circ(S)$ the exchange graph of ideal triangulations which is formed by the ideal triangulations of S , with an edge between T and T' whenever T' is obtained from T by the flip of an arc. An arc in an ideal triangulation can always be flipped except when it is the arc α connecting to the internal puncture of a self-folded triangle as in the left side of Fig. 1. To overcome this shortcoming, Fomin, Shapiro and Thurston introduced in [13] the concept of tagging which allows to distinguish two ways an arc can end in a puncture (tagged or untagged). They subsequently develop the concepts of tagged triangulations, tagged flips and the exchange graph of tagged triangulations, which is denoted by $EG^\times(S)$. Fomin, Shapiro and Thurston note that $EG^\times(S)$ can be obtained by gluing 2^p copies of $EG^\circ(S)$.

Definition 2.1 We have the following data associated to a tagged triangulation T_\times :

- The signed adjacency matrix $B = B(T_\times)$. The rows and columns of B are naturally labeled by the arcs in T_\times (say from 1 to n). It is skew-symmetric and all its entries b_{ij} are in $\{0, \pm 1, \pm 2\}$, see [13, Definition 4.1] for more details.
- The quiver $Q = Q(T_\times)$ corresponding to $B(T_\times)$ is the quiver whose vertices are the arcs in T_\times and the number of arrows from i to j equals b_{ij} . For instance, the quiver for a triangle is shown in Fig. 2.
- The potential $W = W(T_\times)$ is defined as a sum of certain cycles in the complete path algebra $\widehat{\mathbf{k}Q}$, where \mathbf{k} is an algebraic closed field, see [8]. If $\partial S \neq \emptyset$, this potential is rigid (and thus non-degenerate) by [20].

2.2 Jacobian algebra and Ginzburg algebra

Let Q be a finite quiver and W a potential on Q . The *Jacobian algebra* of the quiver with potential (Q, W) , denoted by $J(Q, W)$, is the quotient of the complete path algebra $\widehat{\mathbf{k}Q}$ by the closure of the ideal generated by $\partial_a W$, where a runs over all arrows of Q . The pair (Q, W) is called *Jacobi-finite* provided $J(Q, W)$ is finite-dimensional. The quiver with potential associated to a triangulation of a surface with non-empty boundary is Jacobi-finite by [8]. The Jacobian algebra can be seen as the zeroth cohomology of the *Ginzburg dg algebra* $\Gamma = \Gamma(Q, W)$ of (Q, W) which is constructed as follows ([16, Section 7.2]):

Let \overline{Q} be the graded quiver with the same vertices as Q and whose arrows are the arrows in Q in degree 0, an arrow $a^* : j \rightarrow i$ with degree -1 for each arrow $a : i \rightarrow j$ in Q and a loop $e^* : i \rightarrow i$ with degree -2 for each vertex e in Q .

The underlying graded algebra of $\Gamma(Q, W)$ is the completion of the graded path algebra $\widehat{\mathbf{k}\overline{Q}}$. The differential d of $\Gamma(Q, W)$ is the unique continuous linear endomorphism homogeneous of degree 1 which satisfies the Leibniz rule and takes the following values on the arrows of \overline{Q} :

$$\begin{cases} da = 0, \quad da^* = \partial_a W, & \forall a \in Q_1, \\ d \sum_{e \in Q_0} e^* = \sum_{a \in Q_1} [a, a^*]. \end{cases}$$

Denote by $\mathcal{D}_{fd}(\Gamma)$ and $\text{per}(\Gamma)$ the *finite dimensional derived category* of $\Gamma(Q, W)$ and the *perfect derived category* of $\Gamma(Q, W)$, respectively (cf. [16, Section 7.3]). In case (Q, W) is Jacobi-finite, Amiot introduced in [1] the *generalized cluster category* $\mathcal{C}(\Gamma)$ as the quotient category $\text{per}(\Gamma)/\mathcal{D}_{fd}(\Gamma)$. In other words, there is an exact sequence of triangulated categories

$$0 \rightarrow \mathcal{D}_{fd}(\Gamma) \rightarrow \text{per}(\Gamma) \rightarrow \mathcal{C}(\Gamma) \rightarrow 0. \tag{2.2}$$

A *cluster tilting set* $\mathcal{P} = \{P_j\}_{j=1}^n$ in $\mathcal{C}(\Gamma)$ is an Ext-configuration, i.e. a maximal collection of non-isomorphic indecomposables such that $\text{Ext}_{\mathcal{C}(\Gamma)}^1(P_i, P_j) = 0$. Note that n is the number of vertices in Q . The *mutation* μ_i at the i -th object acts on a cluster tilting set $\{P_j\}_{j=1}^n$ by replacing P_i by another object P'_i which can be calculated as follows: Define

$$P_i^\sharp = \text{Cone} \left(P_i \rightarrow \bigoplus_{j \neq i} \text{Irr}(P_i, P_j)^* \otimes P_j \right) \tag{2.3}$$

and

$$P_i^\flat = \text{Cone} \left(\bigoplus_{j \neq i} \text{Irr}(P_j, P_i) \otimes P_j \rightarrow P_i \right) [-1] \tag{2.4}$$

where $\text{Irr}(P_i, P_j)$ is the space of irreducible maps $P_i \rightarrow P_j$ in the additive subcategory $\text{add } \mathbf{P}$ of $\mathcal{C}(\Gamma)$, for $\mathbf{P} = \bigoplus_{i=1}^n P_i$. One then checks that the two objects P_i^\sharp and P_i^\flat are isomorphic and defines P'_i to be either of them, see [1].

Obviously, the indecomposable summands of Γ form a cluster tilting set of $\mathcal{C}(\Gamma)$. An arbitrary cluster tilting set of $\mathcal{C}(\Gamma)$ is called *reachable* if it can be obtained from Γ by a finite sequence of mutations. An *indecomposable reachable object* of $\mathcal{C}(\Gamma)$ is an object that is contained in a reachable cluster tilting set.

A key property of mutation is that it is an involution, i.e. we have

$$\mu_i^2(\mathcal{P}) = \mathcal{P} \text{ for all possible } i.$$

If $\Gamma = \Gamma(Q, W)$ is given by the quiver and potential associated to a triangulation \mathbf{T} of \mathbf{S} , then the category $\mathcal{D}_{fd}(\Gamma)$ and the cluster category $\mathcal{C}(\Gamma)$ are (up to derived equivalence) independent of the choice of the triangulation \mathbf{T} : as shown in [13], any two tagged triangulations of a surface with non-empty boundary are connected by a sequence of flips. In [21] it is further shown that the corresponding quivers with potential are also related by a sequence of mutations, and it follows then from [16] that the corresponding categories $\mathcal{D}_{fd}(\Gamma)$ are equivalent as triangulated categories, likewise for $\mathcal{C}(\Gamma)$. We therefore write simply $\mathcal{D}(\mathbf{S}) = \mathcal{D}_{fd}(\Gamma)$ and $\mathcal{C}(\mathbf{S}) = \mathcal{C}(\Gamma)$ in this case. Moreover, we denote by $\text{CEG}_*(\mathbf{S})$ the *exchange graph* of reachable cluster tilting sets of $\mathcal{C}(\mathbf{S})$, that is, the connected unoriented graph whose vertices are the reachable cluster tilting sets and whose edges are the mutations. We refer to [4] for more details on and equivalent definitions of the graph $\text{CEG}_*(\mathbf{S})$. Let $\mathcal{C}^\times(\mathbf{S})$ be the set consisting of objects that appear in some cluster tilting set \mathcal{P} in $\text{CEG}_*(\mathbf{S})$.

2.3 The canonical bijection

By the correspondence between tagged arcs and cluster variables ([13, Theorem 7.11]) and the correspondence between cluster variables and indecomposable reachable objects ([9, Corollary 3.5]), we get a bijection between $\mathbf{A}^\times(\mathbf{S})$ and $\mathcal{C}^\times(\mathbf{S})$. Both correspondences are defined using a fixed triangulation, but are shown to commute with mutations. We therefore obtain the following lemma.

Lemma 2.2 *There is a canonical bijection*

$$\zeta : \mathbf{A}^\times(\mathbf{S}) \rightarrow \mathcal{C}^\times(\mathbf{S}) \tag{2.5}$$

which induces the isomorphism (between graphs)

$$\text{EG}^\times(\mathbf{S}) \cong \text{CEG}_*(\mathbf{S}), \tag{2.6}$$

which sends a tagged triangulation \mathbf{T}_\times to the cluster tilting set consisting of objects $\{\zeta(\alpha) \mid \alpha \in \mathbf{T}_\times\}$ and a tagged flip in $\text{EG}^\times(\mathbf{S})$ to a mutation in $\text{CEG}_(\mathbf{S})$.*

Note that we are using in this lemma the fact that the surface has non-empty boundary, and/or the definition of the exchange graph given by reachable cluster tilting sets: The mutation graph of all cluster tilting sets has two connected components for the case of a once punctured torus without boundary, and thus is not isomorphic to the (connected) graph $\text{EG}^\times(\mathbf{S})$.

3 The tagged rotation

3.1 Tagged mapping class group

Consider the group of orientation-preserving diffeomorphisms of \mathbf{S} that preserve (not necessarily pointwise) the set of marked points \mathbf{M} , the set of punctures \mathbf{P} and the boundary. The quotient of this group by the subgroup of diffeomorphisms that are isotopic to the identity through isotopies that fix the set of marked points pointwise is called the *marked mapping class group* $\text{MCG}_\bullet(\mathbf{S})$ of \mathbf{S} . As in [2], we denote by

$$\text{MCG}_\times(\mathbf{S}) = \text{MCG}_\bullet(\mathbf{S}) \times (\mathbb{Z}_2)^p$$

the *tagged mapping class group* of \mathbf{S} consisting of elements (φ, δ) where $\varphi \in \text{MCG}_\bullet(\mathbf{S})$ and δ is a $\{\pm 1\}$ -sign on each point in \mathbf{P} , endowed with the multiplication defined by

$$(\varphi', \delta') \circ (\varphi, \delta) = (\varphi' \circ \varphi, (\delta' \circ \varphi) \cdot \delta)$$

using the action

$$((\delta' \circ \varphi) \cdot \delta)(P) = \delta'(\varphi(P)) \cdot \delta(P).$$

We can identify $\text{MCG}_\bullet(\mathbf{S})$ with

$$\text{MCG}_\bullet(\mathbf{S}) \times \mathbf{1} \subset \text{MCG}_\times(\mathbf{S}),$$

where $\mathbf{1}(P) = 1$ for any $P \in \mathbf{P}$.

Denote by $\text{Aut } \mathcal{C}(\mathbf{S})$ the group of auto-equivalences of $\mathcal{C}(\mathbf{S})$ and by $\text{stab } \mathcal{C}^\times(\mathbf{S})$ the subgroup of $\text{Aut } \mathcal{C}(\mathbf{S})$ that preserves (pointwise) the set $\mathcal{C}^\times(\mathbf{S})$. Let $\text{Aut}_0 \mathcal{C}(\mathbf{S}) = \text{Aut } \mathcal{C}(\mathbf{S}) / \text{stab } \mathcal{C}^\times(\mathbf{S})$.

Any element in $\text{MCG}_\times(\mathbf{S})$ acts on the set $\mathbf{A}^\times(\mathbf{S})$ and hence on $\mathcal{C}^\times(\mathbf{S})$ via the canonical bijection ζ in Lemma 2.2. The following lemma is an analogue of [2, Theorem 4.11] in terms of the cluster category. Note however that one cannot directly use the result from [2] to define a group homomorphism from $\text{MCG}_\times(\mathbf{S})$ to $\text{Aut}_0 \mathcal{C}(\mathbf{S})$ since the cluster variables correspond only to (reachable) rigid objects, whereas the cluster category can contain more (in fact infinite families of) indecomposable objects.

Lemma 3.1 *There is a canonical injection $\iota_{\mathbf{S}}: \text{MCG}_\times(\mathbf{S}) \hookrightarrow \text{Aut}_0 \mathcal{C}(\mathbf{S})$.*

Proof We first recall how the cluster category $\mathcal{C}(\mathbf{S})$ is defined. Given a triangulation \mathbf{T} , there is an associated quiver with potential $(Q_{\mathbf{T}}, W_{\mathbf{T}})$ with corresponding Ginzburg dg algebra $\Gamma_{\mathbf{T}}$. Then $\mathcal{C}(\mathbf{T})$ is defined to be the quotient category $\text{per } \Gamma_{\mathbf{T}} / \mathcal{D}_{fd}(\Gamma_{\mathbf{T}})$ and there is a corresponding bijection $\zeta_{\mathbf{T}}$ in (2.5). For any two triangulations \mathbf{T} and \mathbf{T}' related by a flip f , the corresponding quivers with potential are right equivalent by [21, Theorem 8.1], and there are two canonical derived equivalences (see e.g. [16, Theorem 7.4])

$$\Phi_{\pm} : \text{per } \Gamma_{\mathbf{T}} \cong \text{per } \Gamma_{\mathbf{T}'}$$

Moreover, the composition $\Phi_{\pm}^{-1} \circ \Phi_{\pm}$ yields a spherical twist (cf. Definition 4.3) and, by [15, Theorem 5.6], the two functors induce the same triangulated equivalence

$$E_f : \mathcal{C}(\mathbf{T}) \cong \mathcal{C}(\mathbf{T}')$$

Further, the bijections $\zeta_{\mathbf{T}}$ and $\zeta_{\mathbf{T}'}$ are compatible under this equivalence, i.e.

$$E_f(\zeta_{\mathbf{T}}(\gamma)) = \zeta_{\mathbf{T}'}(\gamma). \tag{3.1}$$

Given two triangulations \mathbf{T} and \mathbf{T}' and a sequence s of flips that relates them, there is an equivalence E_s between $\mathcal{C}(\mathbf{T})$ and $\mathcal{C}(\mathbf{T}')$ defined as the composition of the equivalences E_f where f runs through the flips in the sequence s . For any two such sequences s and s' transforming \mathbf{T} into \mathbf{T}' , we have $E_{s'}^{-1} \circ E_s = \text{id} \in \text{Aut}_0 \mathcal{C}(\mathbf{T})$ since they act identically on $\mathcal{C}^\times(\mathbf{T})$ by (3.1). Thus we can identify all the cluster categories $\mathcal{C}(\mathbf{T})$ as $\mathcal{C}(\mathbf{S})$, and this identification $E_{\mathbf{T}} : \mathcal{C}(\mathbf{T}) \rightarrow \mathcal{C}(\mathbf{S})$ is unique when studying elements in the quotient group $\text{Aut}_0 \mathcal{C}(\mathbf{S})$.

Now consider an element $\varphi \in \text{MCG}_\times(\mathbf{S})$ which takes a triangulation \mathbf{T} to \mathbf{T}' . So we have the following canonical identifications as above:

$$\begin{array}{ccc}
 \mathcal{C}(\mathbf{T}) & \overset{E_\varphi}{\dashrightarrow} & \mathcal{C}(\mathbf{T}') \\
 \searrow E_{\mathbf{T}} & & \swarrow E_{\mathbf{T}'} \\
 & \mathcal{C}(\mathbf{S}) &
 \end{array} \tag{3.2}$$

As $\mathbf{T}' = \varphi(\mathbf{T})$, the associated quivers with potential are right equivalent (in particular, the associated quivers are isomorphic in a canonical way), which induces an isomorphism $\Gamma_{\mathbf{T}} \rightarrow \Gamma_{\mathbf{T}'}$, see [18, Proposition 3.5] for such type of isomorphisms. Consider the corresponding derived equivalence $i_\varphi: \text{per } \Gamma_{\mathbf{T}} \rightarrow \text{per } \Gamma_{\mathbf{T}'}$ which induces an equivalence $E_\varphi: \mathcal{C}(\mathbf{T}) \cong \mathcal{C}(\mathbf{T}')$. Then E_φ maps the canonical cluster tilting set $\mathbf{M}_{\mathbf{T}}$ in $\mathcal{C}(\mathbf{T})$ to the canonical cluster tilting set $\mathbf{M}_{\mathbf{T}'}$ in $\mathcal{C}(\mathbf{T}')$. Note that $\mathbf{M}_{\mathbf{T}}$ and $\mathbf{M}_{\mathbf{T}'}$ are in fact the images of $\Gamma_{\mathbf{T}}$ and $\Gamma_{\mathbf{T}'}$ under the projection from the perfect derived category to the cluster category. In general these two cluster tilting sets do not map under the canonical identifications $E_{\mathbf{T}}$ and $E_{\mathbf{T}'}$ to the same cluster tilting set in $\mathcal{C}(\mathbf{S})$ in (3.2).

Moreover, since E_φ preserves the canonical cluster tilting sets it also preserves the set of reachable rigid objects. Define $\iota_{\mathbf{S}}(\varphi)$ to be the auto-equivalence

$$E_{\mathbf{T}'} \circ E_\varphi \circ E_{\mathbf{T}}^{-1},$$

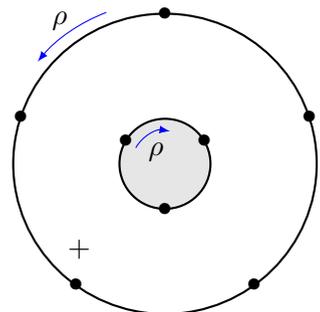
which is well-defined as an element in $\text{Aut}_0 \mathcal{C}(\mathbf{S})$.

The injectivity of $\iota_{\mathbf{S}}$ follows from the well-known Alexander’s method from topology. More precisely, the element in $\text{MCG}_\times(\mathbf{S})$ that preserves a (tagged) triangulation must be the identity. □

The fixed orientation \mathcal{O} of \mathbf{S} induces an orientation \mathcal{O} on each component of $\partial\mathbf{S}$ such that the surface \mathbf{S} lies to the left of every component. This allows to define a rotation ρ on the set of marked points \mathbf{M} in $\partial\mathbf{S}$ which sends each marked point M to the next marked point M' in $\partial\mathbf{S}$ along \mathcal{O} (cf. Fig. 3). We would like to extend this rotation to an element in $\text{MCG}_\bullet(\mathbf{S})$:

Example 3.2 For each component Y in the boundary $\partial\mathbf{S}$ with $m_Y = |\mathbf{M} \cap Y|$ marked points there is a rotation ρ_Y in $\text{MCG}_\bullet(\mathbf{S}) \subset \text{MCG}_\times(\mathbf{S})$ which sends each marked point M in Y to the next marked point in Y . The m_Y -th power $\rho_Y^{m_Y}$ of the rotation ρ_Y coincides with the (positive) Dehn twist D_Y along Y (cf. Fig. 4 and see [10] for a definition of Dehn twists). As an element in the mapping class group $\text{MCG}_\bullet(\mathbf{S})$, ρ_Y acts on the set of arcs of \mathbf{S} and

Fig. 3 The rotation on \mathbf{S}



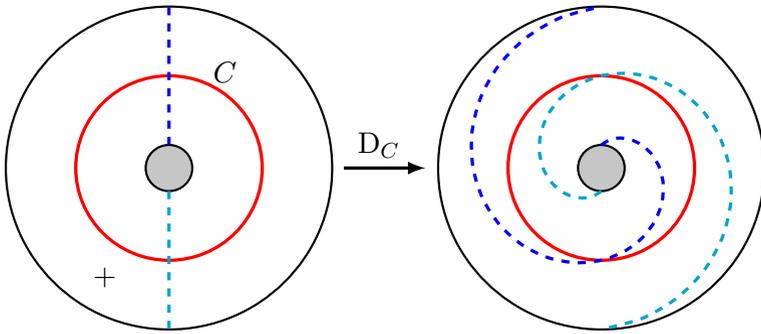


Fig. 4 The Dehn twist

thus induces a permutation on $A^\times(S)$ where the tagging of $\rho_Y(\alpha)$ at any puncture equals the tagging of the arc α at that puncture.

Example 3.3 For each puncture $P \in \mathbf{P}$, there is a tagging switch $\delta^P = (\text{id}, \delta_P)$ in $\text{MCG}_\times(S)$, such that $\delta_P(P') = -1$ if and only if $P' = P$.

Further, δ^P induces a permutation on the set of tagged arcs $A^\times(S)$ that preserves the underlying ordinary arcs but changes the tagging at the puncture P .

Definition 3.4 The (universal) tagged rotation ϱ is the permutation on $A^\times(S)$ induced by the element

$$\varrho = \prod_{Y \subset \partial S} \rho_Y \cdot \prod_{P \in \mathbf{P}} \delta^P \tag{3.3}$$

in $\text{MCG}_\times(S)$ where the first product is over all connected components Y of ∂S . Note that the order in (3.3) does not matter since the ρ_Y commute.

An instance of a tagged rotation is shown in Fig. 5.

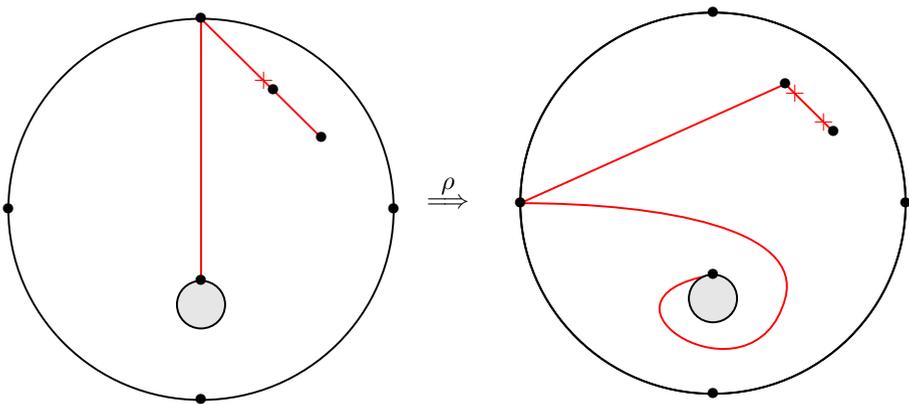


Fig. 5 The tagged rotation on arcs in S

3.2 AR-translation on marked surfaces

This subsection is devoted to show that the tagged rotation corresponds to the shift in the cluster category via the bijection ζ in Lemma 2.2.

Lemma 3.5 *For a tagged arc $\alpha \in \mathbf{A}^\times(\mathbf{S})$ whose end points are distinct and at least one of which is not a puncture, we have $\zeta(\varrho(\alpha)) = \zeta(\alpha)[1]$.*

Proof By formula (2.3), for any cluster tilting set $\mathcal{P} = \{P_j\}_{j=1}^n$, an object P_i in \mathcal{P} becomes $P_i[1]$ in $\mu_i(\mathcal{P})$ if

$$\text{Irr}(P_i, P_j) = 0, \quad \forall j. \tag{3.4}$$

Since the dimensions of the spaces of irreducible morphisms between the objects P_k correspond to the numbers of arrows between vertices in the Gabriel quiver $Q_{\mathcal{P}}$ of $\text{End}(\bigoplus_{k=1}^n P_k)^{\text{op}}$ condition (3.4) is equivalent to P_i being a source in $Q_{\mathcal{P}}$. Thus, via the correspondence ζ in Lemma 2.2, we only need to show that there exists a tagged triangulation \mathbf{T}_\times containing α such that α is a sink in the corresponding associated quiver and $\varrho(\alpha)$ is the tagged arc obtained by the tagged flip of \mathbf{T}_\times at α .

Without loss of generality, suppose that α is not tagged at either end. We consider two cases:

- I. If $\alpha = \widehat{AC}$ is an arc whose endpoints A, C both lie in M , we denote $B = \varrho(A)$ and $D = \varrho(C)$. The arc $\varrho(\alpha)$ is then the arc connecting B and D that can be obtained as a contraction of the union

$$\widehat{BA} \cup \widehat{AC} \cup \widehat{CD},$$

where \widehat{BA} and \widehat{CD} are boundary arc segments, see Fig. 6. We choose a triangulation \mathbf{T}_\times containing the triangles $\triangle ABC$ and $\triangle ACD$ as shown in the left picture of Fig. 6.

- II. If $\alpha = \widehat{AP}$ such that $A \in \mathbf{M}$ and $P \in \mathbf{P}$, we denote $B = \varrho(A)$. Choose a triangulation \mathbf{T}_\times containing α and the arc \widehat{AB} that only encloses the puncture P . Moreover, \mathbf{T}_\times can either contain the ordinary arc \widehat{BP} or the tagged arc \widehat{AP} , as shown in the middle and right pictures of Fig. 6.

In both cases it is straightforward to see that α corresponds to a source in the quiver $Q_{\mathcal{P}}$ and the tagged rotation $\varrho(\alpha)$ of α is indeed the tagged arc obtained by the tagged flip of \mathbf{T}_\times at α , as required. □

We consider now a loop α whose endpoint A is a marked point in M , and such that α is not contractible in the surface \mathbf{S} , that is, α is non-trivial in the homology group even if we

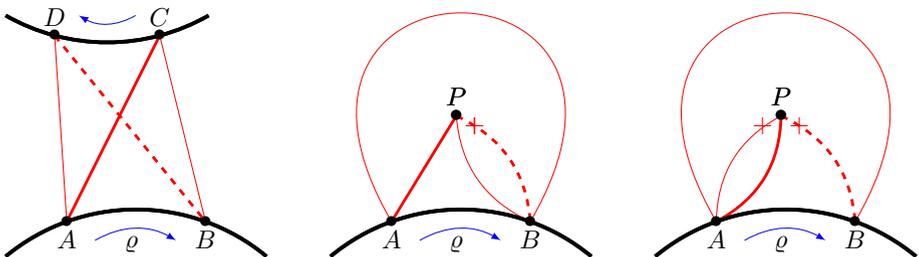


Fig. 6 The flip/tagged rotation of certain types of arcs

glue a disc to each boundary component and ignore all the marked points. This implies that α is a non-separating loop with endpoint in \mathbf{M} . In this case the surface has genus at least one, and we can realize the arc α as the arc labeled by 1 in the upper left picture in Fig. 7. The blue triangle depicts the rest of the surface, in particular, the blue region might contain punctures, boundary components and handle bodies. It is also possible that the arcs labeled 4 and 5 coincide, in which case \mathbf{S} is just a torus with one boundary component and one marked point. The rotation ϱ yields the next point $\varrho(A) = B$ in \mathbf{M} on the same boundary component. We do not insist that A and B are different points, as for in the case of a torus with one boundary component and one marked point the points A and B coincide.

Lemma 3.6 *Let \mathbf{S} be a surface of genus at least one, and let α be a loop with endpoint in \mathbf{M} such that α is not contractible in the surface \mathbf{S} . Then we have $\zeta(\varrho(\alpha)) = \zeta(\alpha)[1]$.*

Proof We choose a triangulation of \mathbf{S} containing the arc α as the arc labeled 1 in Fig. 7, with a presentation of one handle body given by the arcs labeled 2 and 3. The arc $\varrho(\alpha)$ is then the arc labeled with 9. In order to show the statement of the lemma, let X_i denote the rigid object in the cluster category which is obtained under the map ζ from the arc labeled by i . We have a mutation sequence $\mu_{321} = \mu_3 \circ \mu_2 \circ \mu_1$ as shown in Fig. 7, starting from the upper left picture to the lower left one, then the middle one and finally the right one. The mutation sequence of the corresponding quivers is shown in Fig. 8. The corresponding exchange triangles of those three mutations are

$$\begin{aligned} X_1 &\rightarrow X_2 \rightarrow X_7 \rightarrow X_1[1], \\ X_2 &\rightarrow X_3 \oplus X_7 \rightarrow X_8 \rightarrow X_2[1], \\ X_3 &\rightarrow X_8 \rightarrow X_9 \rightarrow X_3[1]. \end{aligned}$$

The first two triangles fit into the following diagram of triangles

$$\begin{array}{ccccccc} & & X_2 & \xlongequal{\quad} & X_2 & & \\ & & \downarrow & & \downarrow & & \\ X_3 & \longrightarrow & X_3 \oplus X_7 & \longrightarrow & X_7 & \longrightarrow & X_3[1] \\ & & \downarrow & & \downarrow & & \parallel \\ \parallel & & & & & & \\ X_3 & \longrightarrow & X_8 & \cdots \longrightarrow & X_1[1] & \cdots \longrightarrow & X_3[1] \\ & & \downarrow & & \downarrow & & \parallel \\ & & X_2[1] & \xlongequal{\quad} & X_2[1] & & \end{array} \tag{3.5}$$

It is straightforward to see that the leftmost and the top square are commutative. For instance, the map from X_2 to X_7 in the top square is the irreducible map corresponding to the arrow $7 \rightarrow 2$ in the second quiver of Fig. 8. By the Octahedral Axiom, the third line (with dotted arrows) in (3.5) is a triangle, which coincides with the third exchange relation. In other words, $X_9 = X_1[1]$, as was to show. \square

Remark 3.7 Presumably, it is possible to use the same method as in Lemma 3.6 for all cases of tagged arcs. However, this becomes complicated when the arc connects two punctures. Instead, we use the tagged mapping class group to give a more conceptual proof.

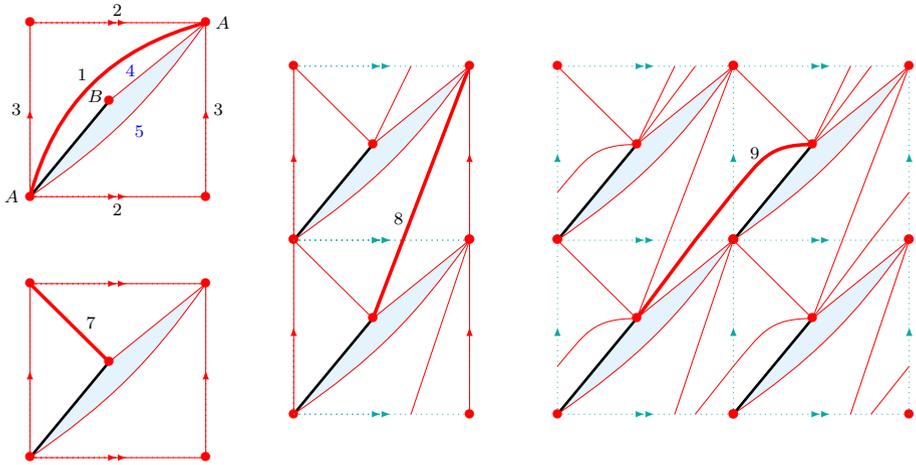


Fig. 7 The mutation sequence μ_{321} on a higher genus surface

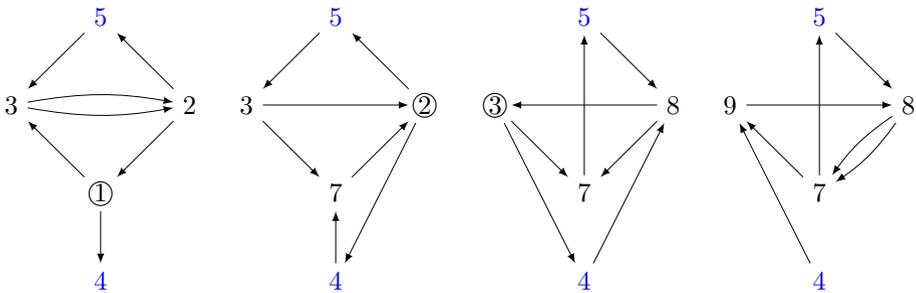


Fig. 8 The mutation sequence μ_{321} of the quiver

Theorem 3.8 *The tagged rotation $\varrho \in \text{MCG}_\times(\mathbb{S})$ on $\mathbf{A}^\times(\mathbb{S})$ becomes the shift [1] on $\mathcal{C}^\times(\mathbb{S})$, i.e. $\iota_{\mathbb{S}}(\varrho) = [1]$.*

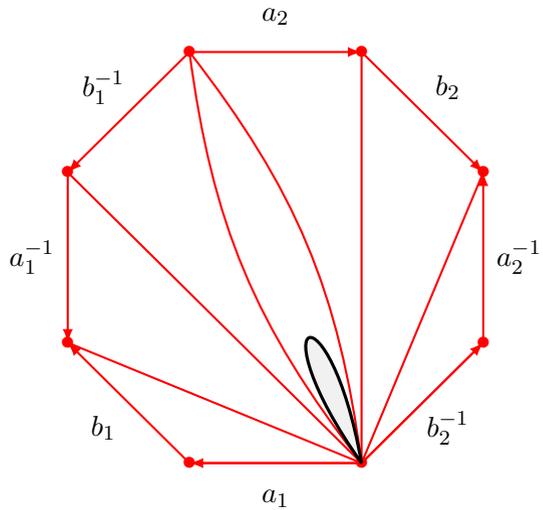
Proof We only need to show that there exists a tagged triangulation \mathbf{T}_\times consisting of the types of arcs considered in Lemmas 3.5 and 3.6. Since the action on $\mathcal{C}^\times(\mathbb{S})$ of an element in $\text{Aut}_0 \mathcal{C}(\mathbb{S})$ is determined by the action on a particular cluster tilting set, the theorem then follows by Lemma 3.1.

Any marked surface falls into at least one of the following cases:

- I. $g > 0$,
- II. $g = 0, b \geq 2$,
- III. $g = 0, b = 1, p \geq 2$,
- IV. $g = 0, b = 1, p = 1, m \geq 2$,
- V. $g = 0, b = 1, p = 0$ and $m \geq 4$.

We use induction on the number $n = 6g + 3p + 3b + m - 6$ defined in Eq. 2.1, starting from one of the following basic cases which represent the minimal possible n in each of the five cases I to V above:

Fig. 9 The basic case **i**: $4g$ -gon presentation of a genus $g = 2$ surface



- i.** $g > 0, b = m = 1$ and $p = 0,$
- ii.** $g = 0, b = m = 2$ and $p = 0,$
- iii.** $g = 0, b = m = 1$ and $p = 2,$
- iv.** $g = 0, b = p = 1$ and $m = 2,$
- v.** $g = 0, b = 1, p = 0$ and $m = 4.$

Any marked surface can be obtained from one of the corresponding basic cases by adding

- (a) a marked point to an existing boundary component,
- (b) a boundary component with one marked point, or
- (c) a puncture.

As start of the induction, we first show that in each of the basic cases **i** to **v** there exists a tagged triangulation consisting of the two types of arcs considered in Lemmas 3.5 and 3.6. For the basic case **i**, Fig. 9 shows for $g = 2$ a tagged triangulation where every arc is a non-contractible loop with endpoint in \mathbf{M} . Thus, all arcs are of the type considered in Lemma 3.6, and an analogous triangulation can be defined for arbitrary $g \geq 1$. For the basic cases **ii**, **iii**, **iv** and **v**, we show in Fig. 10 a triangulation where each arc has two different endpoints, at least one of them belonging to \mathbf{M} . Thus, these triangulations consist of the type of arc considered in Lemma 3.5. Note that in each of these basic cases, there is a triangle with a boundary arc.

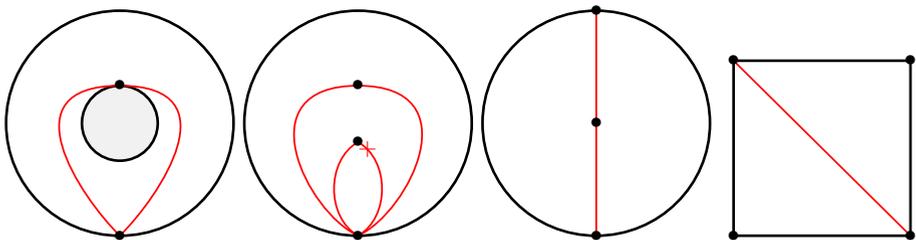


Fig. 10 The basic cases **ii**, **iii**, **iv**, **v**

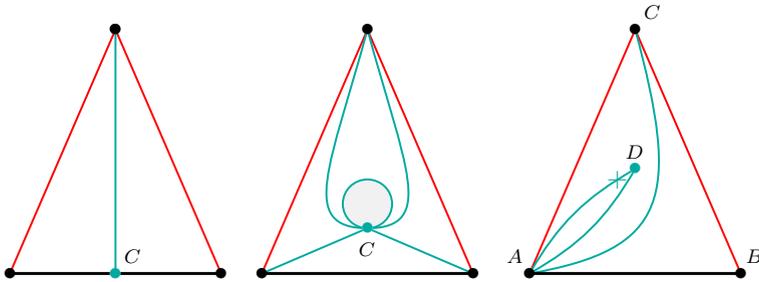


Fig. 11 Three cases of adding (in green) (color figure online)

We now proceed to the inductive step. Figure 11 shows that in the each of the three cases (a), (b) and (c) of adding, we can modify the tagged triangulation \mathbf{T}_\times of the old surface (within the triangle with a boundary arc) to get a tagged triangulation \mathbf{T}'_\times for the new surface, such that \mathbf{T}'_\times only consists of the types of arcs considered in Lemmas 3.5 and 3.6, which completes the proof.

More precisely, in case (a) of adding a marked point C to an existing boundary component, we illustrate in the left picture of Fig. 11 how to add a new arc of the type considered in Lemma 3.5.

In case (b) of adding a boundary component with one marked point C, we show in the middle picture of Fig. 11 how to add the component into a triangle with a boundary edge such that all new arcs are of the type considered in Lemma 3.5.

Finally, in case (c) of adding a puncture D into a triangle ABC with a boundary edge AB, we depict the situation in the right picture of Fig. 11. The added arcs connecting A and the puncture D are of the type considered in Lemma 3.5, as well as the new arc connecting A and C in case these are different points. The case when A and C coincide within a triangle with boundary edge AB occurs in our construction only when one starts with type **i**, thus in this case the new arc is of the type considered in Lemma 3.6. □

4 Applications

4.1 On the Jacobian algebras

As discussed earlier, a tagged triangulation \mathbf{T} of a marked surface \mathbf{S} with non-empty boundary defines a Jacobi-finite quiver Q with potential W . We denote the corresponding Jacobi algebra $J(Q, W)$ by $J(\mathbf{T})$, and $\text{mod } J(\mathbf{T})$ denotes the category of finitely generated $J(\mathbf{T})$ -modules. Moreover, we denote by $\text{add } \zeta(\mathbf{T})$ the additive subcategory of $\mathcal{C}(\mathbf{S})$ generated by the object $\zeta(\mathbf{T})$, that is, all direct sums of summands of $\zeta(\mathbf{T})$.

Lemma 4.1 [1, 17] *There is a canonical equivalence*

$$\theta_{\mathbf{T}}: \mathcal{C}(\mathbf{S}) / \text{add } \zeta(\mathbf{T}) \rightarrow \text{mod } J(\mathbf{T}).$$

By the proof in [17, Section 3.5], the AR-translation on $J(\mathbf{T})$ is induced from the AR-translation τ (or the shift [1]) of the cluster category $\mathcal{C}(\mathbf{S})$. Thus we have the following corollary.

Corollary 4.2 *Under the bijection $\theta_{\mathbf{T}}$ in Lemma 4.1, the tagged rotation ϱ on $\mathbf{A}^\times(\mathbf{S})$ induces the AR-translation τ on the objects in $\text{mod } J(\mathbf{T})$ that are the images of the reachable rigid objects.*

4.2 Shifts and Seidel–Thomas braid group

Let $\Gamma = \Gamma(Q, W)$ be the Ginzburg dg algebra of any quiver with potential. It is known that $\mathcal{D}_{fd}(\Gamma)$ is a Calabi–Yau-3 category, which admits a standard heart \mathcal{H}_Γ generated by simple Γ -modules S_e , for $e \in Q_0$, each of which is a 3-spherical object. Recall that an object S is N -spherical when $\text{Hom}^\bullet(S, S) = \mathbf{k} \oplus \mathbf{k}[-N]$. Moreover, we recall (e.g. [26]) a distinguished family of auto-equivalences of $\mathcal{D}_{fd}(\Gamma)$.

Definition 4.3 The twist functor ϕ of a spherical object S is defined by

$$\phi_S(X) = \text{Cone}(X \rightarrow S \otimes \text{Hom}^\bullet(X, S)^\vee)[-1] \tag{4.1}$$

with inverse

$$\phi_S^{-1}(X) = \text{Cone}(S \otimes \text{Hom}^\bullet(S, X) \rightarrow X) \tag{4.2}$$

Her we denote the graded dual of a graded \mathbf{k} -vector space $V = \bigoplus_{j \in \mathbb{Z}} V_j[j]$ by $V^\vee = \bigoplus_{j \in \mathbb{Z}} V_j^*[-j]$. The Seidel–Thomas braid group, denoted by $\text{Br } \Gamma$, is the subgroup of $\text{Aut } \mathcal{D}_{fd}(\Gamma)$ generated by the twist functors of the simples in $\text{Sim } \mathcal{H}_\Gamma$.

For each simple object S in a reachable heart \mathcal{H} in $\mathcal{D}_{fd}(\Gamma)$ there are two different simple tilts that can be applied to \mathcal{H} , and the resulting tilted hearts (there forward and the backward tilt with respect to S) are related to each other by a spherical twist, see [16]. On the level of the cluster category the two different tilts coincide, thus the cluster exchange graph can be identified with the exchange graph of reachable hearts modulo the braid group. We apply this to surface triangulations: Given a tagged triangulation \mathbf{T} of \mathbf{S} , we recall from Sect. 2 that the category $\mathcal{D}(\mathbf{S}) = \mathcal{D}_{fd}(\Gamma)$ is independent (up to derived equivalence) of the triangulation \mathbf{T} used to define the dg algebra Γ . We accordingly denote the Seidel–Thomas braid group of $\mathcal{D}(\mathbf{S})$ by $\text{Br } \mathbf{S}$ and we also denote by $\text{EG}^\circ(\mathcal{D}(\mathbf{S}))$ the principal component of the exchange graph of hearts in $\mathcal{D}(\mathbf{S})$, which consists of all hearts that can be obtained through iterated (simple) tilts from the standard heart \mathcal{H}_Γ . Let $\text{CEG}(\mathbf{S})$ be the oriented exchange graph of $\mathcal{C}(\mathbf{S})$, which can be obtained from $\text{CEG}_*(\mathbf{S})$ by replacing each unoriented edge with a 2-cycle. By [1] (cf. [19, Theorem 8.6]), there is a map

$$\nu : \text{EG}^\circ(\mathcal{D}(\mathbf{S})) \rightarrow \text{CEG}(\mathbf{S}) \tag{4.3}$$

which induces (cf. [15] for the general case and [19] for the acyclic case) an isomorphism

$$\text{CEG}(\mathbf{S}) \cong \text{EG}^\circ(\mathcal{D}(\mathbf{S}))/\text{Br } \mathbf{S}.$$

More precisely, the map ν is defined as follows. Let \mathcal{H} be a heart in $\text{EG}^\circ(\mathcal{D}(\mathbf{S}))$ which corresponds to a t-structure \mathcal{P} in $\mathcal{D}_{fd}(\Gamma)$. Lift \mathcal{P} to a t-structure in $\text{per}(\Gamma)$ via the inclusion in (2.2). Such a t-structure corresponds to a silting object of $\text{per}(\Gamma)$. The image of this silting object under the functor $\text{per}(\Gamma) \rightarrow \mathcal{C}(\Gamma)$ is defined to be $\nu(\mathcal{H})$. Furthermore, it is not hard to show that ν commutes with the shift functor, i.e.

$$\nu(\mathcal{H}[1]) = \nu(\mathcal{H})[1].$$

Theorem 4.4 *Let \mathbf{S} be a marked surface which is not a polygon or a once-punctured polygon. Then the intersection of $\text{Br } \mathbf{S}$ with the subgroup $\mathbb{Z}[1]$ of shifts in $\text{Aut } \mathcal{D}(\mathbf{S})$ is trivial.*

Proof Suppose that $[m] \in \text{Br } \mathbf{S}$ for some integer m . Since the induced action of the braid group on $\text{CEG}(\mathbf{S})$ is trivial, $[m]$ acts trivially on $\text{CEG}(\mathbf{S})$, or $[m] = \text{id}$ in $\text{Aut}_0 \mathcal{C}(\mathbf{S})$. On the other hand, we claim that there always exists an arc γ such that the orbit $\{\varrho^z(\gamma) \mid z \in \mathbb{Z}\}$ has infinite order: If \mathbf{S} admits two different boundary components, we can choose any arc γ connecting the two components. Likewise, if \mathbf{S} has more than one puncture, for any arc γ connecting one puncture to a marked point on the boundary the set $\{\varrho^z(\gamma) \mid z \in \mathbb{Z}\}$ contains infinitely many non-isotopic curves.

Thus \mathbf{S} can have at most one boundary component and one puncture. Moreover, the genus of \mathbf{S} must be zero since the Dehn twist D of a non-contractible closed curve in a higher genus surface has infinite order. More precisely, for any arc γ whose endpoints coincide on the one marked point on the boundary and such that γ is non-trivial viewed as an element in the fundamental group of the surface obtained from \mathbf{S} by gluing a disc to the boundary component, the orbit of γ under D is infinite. □

Let (Q, W) be a quiver with potential associated to some triangulation of a marked surface \mathbf{S} . Then \mathbf{S} is not a polygon or a once-punctured polygon if and only if (Q, W) is not mutation-equivalent to a quiver of type A or D. The theorem can thus be rephrased by saying the Thomas–Seidel braid group intersects trivially with the shifts in $\text{Aut } \mathcal{D}(\mathbf{S})$ except in the cluster types A or D.

4.3 Center of the braid group

In this subsection, we discuss the centers of the braid groups of type A or D. Let Q be a quiver of type A or D, and denote by Br_Q the *braid group of Q* which is defined by the generating set $\mathbf{b} = \{b_i \mid i \in Q_0\}$ and relations

$$b_i b_j = b_j b_i$$

if there is no arrow between the vertices i and j , and

$$b_i b_j b_i = b_j b_i b_j$$

in case there is an arrow between i and j . Recall that the quasi-center of Br_Q is the subgroup of elements $\Delta(Q)$ in Br_Q satisfying $\Delta(Q) \cdot \mathbf{b} \cdot \Delta(Q)^{-1} = \mathbf{b}$, and that this subgroup is an infinite cyclic group generated by a special element \tilde{z} of Br_Q , called *fundamental element*. The center $Z(\text{Br}_Q)$ of Br_Q is an infinite cyclic group. The element $z_Q = \tilde{z}$ generates $Z(\text{Br}_Q)$ if Q is of type D_n for even n , and $z_Q = \tilde{z}^2$ generates $Z(\text{Br}_Q)$ if Q is of type D_n for odd n or of type A. If we denote by Γ the Ginzburg dg algebra associated to the quiver Q with zero potential $W = 0$, then there is a quotient map

$$\pi : \text{Br}_Q \rightarrow \text{Br } \Gamma$$

since the spherical twists satisfy the braid relation cf. [19, (7.4)].

Example 4.5 [22] Let \mathbf{S} be an $(n + 3)$ -gon as shown in the left picture of Fig. 12. Then the tagged rotation has order $n + 3$. Further, for the shift [1] in $\mathcal{D}(\mathbf{S})$, we have

$$\pi(\tilde{z}^2) = [n + 3].$$

where Q is a quiver of type A_n .

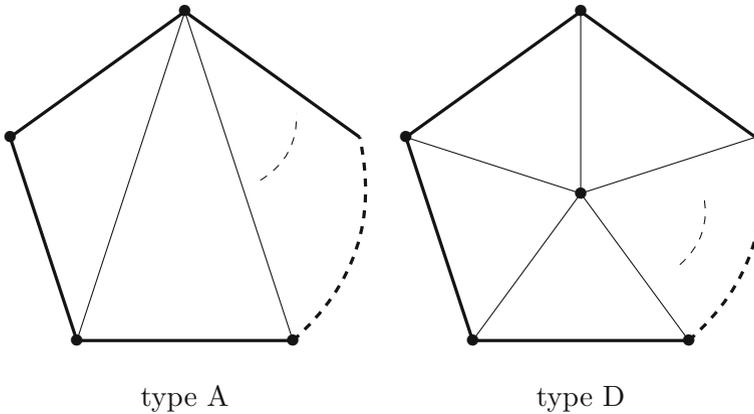


Fig. 12 The triangulations

Example 4.6 [22] Similarly, if \mathbf{S} is an n -gon with a puncture as shown in the right picture of Fig. 12, then the tagged rotation has order n if n is even and order $2n$ if n is odd. Further, for the shift [1] in $\mathcal{D}(\mathbf{S})$, we have

$$\pi(\tilde{z}) = [n],$$

where Q is a quiver of type D_n .

4.4 Isomorphism theorem

In this subsection, we discuss when the canonical injection $\iota_{\mathbf{S}}: \text{MCG}_{\times}(\mathbf{S}) \hookrightarrow \text{Aut}_0 \mathcal{C}(\mathbf{S})$ from Lemma 3.1 is an isomorphism. The result can be obtained by cutting the given marked surface along arcs and using induction similar to the methods in Sect. 3. The details are worked out in [3], from where we derive the main result of this subsection:

Theorem 4.7 *The canonical injection $\iota_{\mathbf{S}}: \text{MCG}_{\times}(\mathbf{S}) \hookrightarrow \text{Aut}_0 \mathcal{C}(\mathbf{S})$ is an isomorphism except when \mathbf{S} is a once-punctured disc with 2 or 4 marked points on the boundary or a twice-punctured disc with 2 marked points on the boundary.*

Proof An element F in $\text{Aut}_0 \mathcal{C}(\mathbf{S})$ acts on the exchange graph $\text{CEG}_{*}(\mathbf{S})$ of clusters $(\mathbf{x}, Q_{\mathbf{x}})$, and it is shown in [2] that F maps a quiver $Q_{\mathbf{x}}$ of a cluster to an isomorphic quiver. But then it follows from Proposition 8.5 in [3] that F is induced by an element in the mapping class group $\text{MCG}_{\times}(\mathbf{S})$ provided the surface is not one of the exceptions listed in the proposition, hence $\iota_{\mathbf{S}}: \text{MCG}_{\times}(\mathbf{S}) \hookrightarrow \text{Aut}_0 \mathcal{C}(\mathbf{S})$ is surjective. \square

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