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Decorated marked surfaces (part B): topological realizations

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Abstract We study categories associated to a decorated marked surface $\mathbf{S}_\triangle$, which is obtained from an unpunctured marked surface $\mathbf{S}$ by adding a set of decorating points. For any triangulation $\mathbf{T}$ of $\mathbf{S}_\triangle$, let $\Gamma_\mathbf{T}$ be the associated Ginzburg dg algebra. We show that there is a bijection between reachable open arcs in $\mathbf{S}_\triangle$ and the reachable rigid indecomposables in the perfect derived category per $\Gamma_\mathbf{T}$. This is the dual of the bijection, between simple closed arcs in $\mathbf{S}_\triangle$ and reachable spherical objects in the 3-Calabi-Yau category $\mathcal{D}_{fd}(\Gamma_\mathbf{T})$, constructed in the prequel (Qiu in Math Ann 365:595–633, 2016). Moreover, we show that Amiot's quotient per $\Gamma_\mathbf{T}/\mathcal{D}_{fd}(\Gamma_\mathbf{T})$ that defines the generalized cluster categories corresponds to the forgetful map $\mathbf{S}_\triangle \to \mathbf{S}$ (forgetting the decorating points) in a suitable sense.

Keywords Calabi-Yau categories · Spherical twists · Quivers with potential · Silting objects · Cluster categories

1 Introduction

1.1 Quivers with potential and generalized cluster categories

Quiver mutation (cf. the survey [6]) was invented by Fomin–Zelevinsky as the combinatorial aspect of cluster algebras. Later, mutation was developed by Derksen–Weyman–Zelevinsky for quivers with potential. The first (additive) categorification of cluster algebras (associated to acyclic quivers) was due to Buan–Marsh–Reineke–Reiten–Todorov, via representations of the corresponding quivers. Amiot [1] introduced the generalized cluster categories via Ginzburg dg algebras for quivers with potential. In her construction, the cluster category $\mathcal{C}(\Gamma)$ is defined by the following short exact sequence of triangulated categories

$$0 \to \mathcal{D}_{fd}(\Gamma) \to \operatorname{per}\Gamma \xrightarrow{\pi} \mathcal{C}(\Gamma) \to 0, \tag{1.1}$$

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where $\Gamma = \Gamma(Q, W)$ is the Ginzburg dg algebra of the quiver with potential (Q, W) and per Γ (resp. $\mathcal{D}_{fd}(\Gamma)$) is the perfect (resp. finite-dimensional) derived category of Γ. Here, $\mathcal{D}_{fd}(\Gamma)$ is a 3-Calabi-Yau (CY) category originally arising from the study of mirror symmetry.

There is an *oriented* exchange graph associated to each of the categories in (1.1), namely:

- the reachable hearts/t-structures in $\mathcal{D}_{fd}(\Gamma)$ as vertices and simple tilting as edges for the exchange graph $\mathrm{EG}^\circ(\mathcal{D}_{fd}(\Gamma))$;
- the reachable silting sets in per Γ as vertices and mutation as edges for the silting exchange graph $\mathrm{SEG}^\circ(\mathrm{per}(\Gamma))$;
- the cluster tilting sets in $\mathcal{C}(\Gamma)$ as vertices and mutation as edges for the cluster exchange graph $\mathrm{CEG}(\mathcal{C}(\Gamma))$.

They play a crucial role in categorifying cluster algebras, understanding quantum dilogarithm identities and computing stability conditions (cf. [9]). By the simple-projective duality, there is a canonical isomorphism between the first two graphs. Moreover, they are coverings of the third (cf. [7]) by the spherical twist group action (cf. (2.4)).

1.2 Triangulations of marked surfaces

A geometric aspect of cluster theory was explored by Fomin–Shapiro–Thurston (FST) [5]. They constructed a quiver $Q_{\mathbf{T}}$ for each (tagged) triangulation $\mathbf{T}$ of a marked surface $\mathbf{S}$ and showed that flipping triangulations corresponds to mutation of quivers. Here, the marked surface $\mathbf{S}$ is a surface with marked points on its boundaries and punctures in its interior. Further, Labardini-Fragoso gave a 'good' rigid potential $W_{\mathbf{T}}$ for each FST quiver $Q_{\mathbf{T}}$ that is compatible with mutation (cf. [8]). Then one can construct the Ginzburg dg algebra $\Gamma_{\mathbf{T}} = \Gamma(Q_{\mathbf{T}}, W_{\mathbf{T}})$ and the associated categories mentioned above.

In this series of papers, we deal the case when $\mathbf{S}$ is unpunctured. In the previous paper [10], we introduce the decorated marked surface $\mathbf{S}_\triangle$, which is obtained from $\mathbf{S}$ by decorating it with a set $\triangle$ of points, as a topological model for these categories. The number of points in $\triangle$ equals the number of triangles in any triangulation of $\mathbf{S}$. Note that when considering the mapping class group of $\mathbf{S}_\triangle$, these decorating points are serving as punctures in topology; however, we reserve the terminology 'punctures' for the FST setting of marked surfaces. This decorating idea already appeared in various mathematical/physical contexts. In the theory of Bridgeland–Smith (cf. [2]), the decorating points are the simple zeroes of some quadratic differentials while the boundaries of $\mathbf{S}$ are the real blow-up of higher order (≥ 3) pole of some quadratic differentials.

A triangulation of $\mathbf{S}_\triangle$ is a maximal collection of simple open arcs that divides $\mathbf{S}_\triangle$ into triangles such that each triangle contains exactly one decorating point. The dual triangulations consists of simple closed arcs, i.e. the simple arcs connecting different decorating points. In the theory of Bridgeland–Smith, these closed arcs correspond to stable objects (w.r.t. some stability conditions) and saddle connections (w.r.t. some quadratic differentials).

1.3 Contents

In sum, we give topological realizations for the followings in this paper (Theorem 3.6, Corollary 3.3, Remark 3.7 and Proposition 3.10):

1° reachable rigid indecomposables in the perfect category as reachable simple open arcs in the decorated marked surface, which generalizes the result for the corresponding cluster category (cf. [4,8,11]);

Table 1 Correspondences

Topological side		Categorical side
Braid twists	$\xrightarrow{\cong}$	Spherical twists
Simple closed arcs in $\mathbf{S}_\triangle$ Dual Tri. with Whitehead moves	$\xrightarrow[\text{up to }[1]]{\text{1-1}}$	Spherical obj. in $\mathcal{D}_{fd}(\Gamma_\mathbf{T})$ Hearts with simple tilting
graph dual $\updownarrow$		$\updownarrow$ sim.-proj. dual
Reachable open arcs in $\mathbf{S}_\triangle$ Triangulations with flips	$\xrightarrow{\text{1-1}}$	Reachable ind. in per $\Gamma_\mathbf{T}$ Silting with mutation
$\mathbf{S}_\triangle$ $\downarrow$ forgetful map $\mathbf{S}$		per $\Gamma_\mathbf{T}$ $\downarrow$ quotient map $\mathcal{C}_\mathbf{S}$
Open arcs in $\mathbf{S}$ Triangulations with flips	$\xrightarrow{\text{1-1}}$	Rigid ind. in $\mathcal{C}_\mathbf{S}$ Cluster tilting with mutation

2° the simple-projective duality for Ginzburg dg algebras as graph duality between dual-triangulations and triangulations;
3° Amiot's triangulated quotient that defines the cluster category as forgetful map from decorated marked surface to the original marked surface;
4° the shift functor for the silting sets in the perfect category as the universal rotation in the marked mapping class group of decorated marked surface, which generalizes the result in [3,4] for the corresponding cluster category.

Table 1 provides a list of all the correspondences in this paper as well as in [10].

2 Settings

2.1 Decorated marked surfaces

We collect notions and notations from [10] about (decorated) marked surfaces. See [10, § 3] for further details (Table 2).

Let $\mathbf{S}$ be an unpunctured *marked surface*, which is determined by its genus g, the number $|\partial\mathbf{S}|$ of boundary components and the integer partition of $|\mathbf{M}|$ into $|\partial\mathbf{S}|$ parts describing the number of marked points on $\partial\mathbf{S}$.

- The *decorated marked surface* $\mathbf{S}_\triangle$ is a marked surface $\mathbf{S}$ together with a fixed set $\triangle$ of $\aleph$ 'decorating' points (in the interior of $\mathbf{S}$). Here, $\aleph$ equals the number of triangles of a triangulation of $\mathbf{S}$, which only depends on $\mathbf{S}$.
- a *closed arc* in $\mathbf{S}_\triangle$ is (the isotopy class of) a curve in $\mathbf{S}_\triangle - \triangle$ that connects different decorating points in $\triangle$. Denote by $\mathrm{CA}(\mathbf{S}_\triangle)$ the set of simple closed arcs.
- An *open arc* in $\mathbf{S}_\triangle$ (or $\mathbf{S}$) is (the isotopy class of) a curve in $\mathbf{S}_\triangle - \triangle$ that connects two marked points in $\mathbf{M}$.

Table 2 List of notations

$\mathbf{S}$ ($\mathbf{S}_\triangle$)	(Decorated) marked surface
$\mathrm{OA}(\mathbf{S})$	Set of open arcs in $\mathbf{S}$
$\mathrm{EG}(\mathbf{S})$	Exchange graph of triangulations in $\mathbf{S}$
$\mathrm{OA}^\circ(\mathbf{S}_\triangle)$	Set of (reachable) open arcs in $\mathbf{S}_\triangle$
$\mathrm{CA}(\mathbf{S}_\triangle)$	Set of (simple) closed arcs in $\mathbf{S}_\triangle$
$\mathrm{EG}^\circ(\mathbf{S}_\triangle)$	(c.c.) Exchange graph of triangulations in $\mathbf{S}_\triangle$
$\mathrm{BT}(\mathbf{S}_\triangle)$	Braid twist group of $\mathbf{S}_\triangle$
$\mathbf{T}_0$	The initial triangulation of $\mathbf{S}_\triangle$
$\mathrm{EG}^\circ_3(\mathbf{T}_0)$	Fundamental domain for $\mathrm{EG}^\circ(\mathbf{S}_\triangle)/\,\mathrm{BT}$
Γ_0	Ginzburg dg algebra associated to $\mathbf{T}_0$
$\mathcal{D}_{fd}(\Gamma_0)$	Finite dimensional derived category of Γ_0
$\mathrm{EG}^\circ(\mathcal{D}_{fd}(\Gamma_0))$	(c.c.) Exchange graph of hearts in $\mathcal{D}_{fd}(\Gamma_0)$
$\mathrm{ST}(\Gamma_0)$	Spherical twist group of $\mathcal{D}_{fd}(\Gamma_0)$
$\mathrm{EG}^\circ_3(\mathcal{H}_0)$	Fundamental domain for $\mathrm{EG}^\circ(\mathcal{D}_{fd}(\Gamma_0))/\,\mathrm{ST}$
$\mathrm{Sph}(\Gamma_0)$	Set of (reachable) spherical object in $\mathcal{D}_{fd}(\Gamma_0)$
$\mathrm{per}\,\Gamma_0$	Perfect derived category of Γ_0
$\mathrm{SEG}^\circ(\mathrm{per}\,\Gamma_0)$	(c.c.) Exchange graph of silting sets in $\mathrm{per}\,\Gamma_0$
$\mathrm{RR}(\mathrm{per}\,\Gamma_0)$	Set of reachable rigid indecomposables in $\mathrm{per}\,\Gamma_0$
$\mathcal{C}(\Gamma_0)$	Cluster category of Γ_0
$\mathrm{CEG}(\Gamma_0)$	Exchange graph of cluster tilting sets in $\mathcal{C}(\Gamma_0)$
$\mathrm{RR}(\mathcal{C}(\Gamma_0))$	Set of rigid indecomposables in $\mathcal{C}(\Gamma_0)$

N.B. c.c. stands for ‘the connected component of’

- A *triangulation* of $\mathbf{S}_\triangle$ is a collection of simple open arcs that divides $\mathbf{S}_\triangle$ into triangles, such that each triangle contains exactly one decorating point. We will fix an initial triangulation $\mathbf{T}_0 = \{\gamma_i\}$ of $\mathbf{S}_\triangle$ with dual triangulation $\mathbf{T}_0^* = \{s_i\}$ (consisting of simple closed arcs). The key feature of triangulations of $\mathbf{S}_\triangle$ is that the flips have directions, cf. Fig. 1 and [10, § 3].
- Denote by $\mathrm{EG}^\circ(\mathbf{S}_\triangle)$ the connected component in the *(oriented) exchange graph* $\mathrm{EG}(\mathbf{S}_\triangle)$ of triangulations in $\mathbf{S}_\triangle$ that contains $\mathbf{T}_0$.
- Let $\mathrm{OA}^\circ(\mathbf{S}_\triangle)$ be the set of (simple) *reachable open arcs* in $\mathbf{S}_\triangle$, which consists of open arcs in triangulations in $\mathrm{EG}^\circ(\mathbf{S}_\triangle)$.
- The *mapping class group* $\mathrm{MCG}(\mathbf{S}_\triangle)$ is the group of isotopy classes of homeomorphisms of $\mathbf{S}_\triangle$, where all homeomorphisms and isotopies are required to fix $\partial\mathbf{S}_\triangle(\supset \mathbf{M})$ pointwise and fix the decorating points set $\triangle$ (but allow to permutate points in it).
- The *braid twist group* $\mathrm{BT}(\mathbf{S}_\triangle)$ is the subgroup of $\mathrm{MCG}(\mathbf{S}_\triangle)$ generated by the *braid twist* B_η (cf. Fig. 2) along the closed arc η in $\mathrm{CA}(\mathbf{S}_\triangle)$.
- The forgetful map $F\colon \mathbf{S}_\triangle \to \mathbf{S}$ induces an isomorphism

$$F_*\colon\ \mathrm{EG}^\circ(\mathbf{S}_\triangle)/\,\mathrm{BT}(\mathbf{S}_\triangle) \cong \mathrm{EG}(\mathbf{S}), \tag{2.1}$$

where $\mathrm{EG}(\mathbf{S})$ is the exchange graph of triangulations of $\mathbf{S}$. Note that $\mathrm{EG}(\mathbf{S})$ is unoriented; however, in (2.1), we regard an unoriented edge on the $\mathrm{EG}(\mathbf{S})$ as an oriented 2-cycle. Denote by $\mathbb{T}_0$ the initial triangulation of $\mathbf{S}$ induced by $\mathbf{T}_0$.

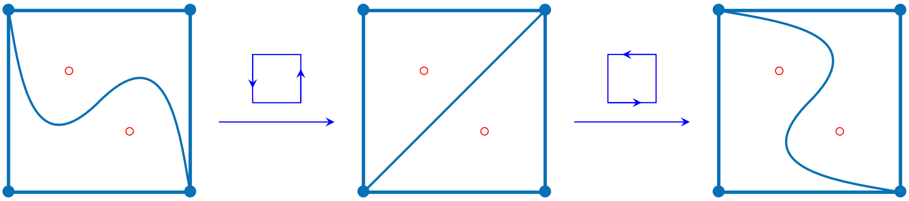


Fig. 1 Forward flips

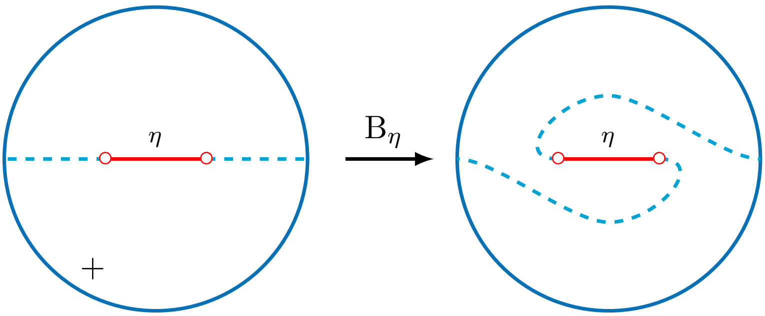


Fig. 2 The braid twist

Note that any triangulation of $\mathbf{S}$ (or $\mathbf{S}_\triangle$) consists of $n = 6g + 3|\partial\mathbf{S}| + |\mathbf{M}| - 6$ open arcs and divides $\mathbf{S}$ into $\aleph = (2n + |\mathbf{M}|)/3$ triangles.

2.2 Amiot's triangulated quotient

We recall the relative categories from [10]. There is a quiver with potential $(Q_\mathbf{T}, W_\mathbf{T})$ associated to each triangulation $\mathbf{T}$ of $\mathbf{S}_\triangle$. Consider the quiver with potential (Q_0, W_0) associated to $\mathbf{T}_0$. Let Γ_0 be the corresponding *Ginzburg dg algebra*; there are three triangulated categories associated to Γ_0.

- the *finite dimensional derived category* $\mathcal{D}_{fd}(\Gamma_0)$;
- the *perfect derived category* $\operatorname{per}\Gamma_0$ (that contains $\mathcal{D}_{fd}(\Gamma_0)$);
- the *cluster category* $\mathcal{C}(\Gamma_0)$, which is the quotient $\operatorname{per}\Gamma_0/\mathcal{D}_{fd}(\Gamma_0)$ that fits into the short exact sequence of triangulated categories

$$0 \to \mathcal{D}_{fd}(\Gamma_0) \to \operatorname{per}\Gamma_0 \xrightarrow{\pi} \mathcal{C}(\Gamma_0) \to 0. \tag{2.2}$$

  Denote the quotient map by π.

We will study various exchange graphs of these categories and compare them with the exchange graph of triangulations.

2.3 The exchange graph of hearts

A *torsion pair* in an abelian category $\mathcal{C}$ is a pair of full subcategories $\langle\mathcal{F}, \mathcal{T}\rangle$ of $\mathcal{C}$, such that $\operatorname{Hom}(\mathcal{T}, \mathcal{F}) = 0$ and furthermore every object $E \in \mathcal{C}$ fits into a short exact sequence $0 \longrightarrow E^{\mathcal{T}} \longrightarrow E \longrightarrow E^{\mathcal{F}} \longrightarrow 0$ for some objects $E^{\mathcal{T}} \in \mathcal{T}$ and $E^{\mathcal{F}} \in \mathcal{F}$. The set of (isomorphism classes of) simple objects in $\mathcal{C}$ will be denoted by $\operatorname{Sim}\mathcal{C}$.

A *t-structure* on a triangulated category $\mathcal{D}$ is a full subcategory $\mathcal{P} \subset \mathcal{D}$ with $\mathcal{P}[1] \subset \mathcal{P}$ such that, if one defines $\mathcal{P}^{\perp} = \{G \in \mathcal{D} : \mathrm{Hom}_{\mathcal{D}}(F, G) = 0, \forall F \in \mathcal{P}\}$, then, for every object $E \in \mathcal{D}$, there is a unique triangle $F \to E \to G \to F[1]$ in $\mathcal{D}$ with $F \in \mathcal{P}$ and $G \in \mathcal{P}^{\perp}$. A t-structure $\mathcal{P}$ is *bounded* if $\mathcal{D} = \bigcup_{i,j \in \mathbb{Z}} \mathcal{P}^{\perp}[i] \cap \mathcal{P}[j]$. The *heart* of a t-structure $\mathcal{P}$ is the full subcategory

$$\mathcal{H} = \mathcal{P}^{\perp}[1] \cap \mathcal{P},$$

which determines $\mathcal{P}$ uniquely.

Recall, e.g. from [7, §3], that we can forward/backward tilt a heart $\mathcal{H}$ to get a new one, with respect to any torsion pair in $\mathcal{H}$ in the sense of Happel–Reiten–Smalø. Further, all forward/backward tilts with respect to torsion pairs in $\mathcal{H}$, correspond one–one to all hearts between $\mathcal{H}$ and $\mathcal{H}[\pm 1]$. Here, the partial order between hearts is defined by: $\mathcal{H}_1 \le \mathcal{H}_2$ if and only if $\mathcal{P}_2 \subset \mathcal{P}_1$ for the corresponding t-structures. Note that $\mathcal{H} \le \mathcal{H}[1]$ for any $\mathcal{H}$. In particular there is a special kind of tilting which is called simple tilting (cf. [7, Definition 3.6]), with respect to a rigid simple of a heart. We denote by $\mathcal{H}_S^{\sharp}$ and $\mathcal{H}_S^{\flat}$, respectively, the simple forward/backward tilts of a heart $\mathcal{H}$, with respect to a simple S.

Definition 2.1 The *exchange graph* of a triangulated category $\mathcal{D}$ is the oriented graph whose vertices are all hearts in $\mathcal{D}$ and whose edges correspond to simple forward tiltings between them.

A triangulated category $\mathcal{D}$ is called 3-*Calabi-Yau* (3-CY) if, for any objects X, X' in $\mathcal{D}$ we have a natural isomorphism

$$\mathfrak{S} : \mathrm{Hom}^{s}_{\mathcal{D}}(X, X') \xrightarrow{\sim} \mathrm{Hom}^{3-s}_{\mathcal{D}}(X', X)^{*}. \tag{2.3}$$

An object S is 3-*spherical* if $\mathrm{Hom}^{\bullet}(S, S) = \mathbf{k} \oplus \mathbf{k}[-3]$ and (2.3) holds functorially for $X = S$ and X' in $\mathcal{D}$. The *twist functor* ϕ *of a spherical object* S is defined by

$$\phi_S(X) = \mathrm{Cone}\left(S \otimes \mathrm{Hom}^{\bullet}(S, X) \to X\right) \tag{2.4}$$

We have the following facts

- $\mathcal{D}_{fd}(\Gamma_0)$ is 3-CY and admits a canonical heart $\mathcal{H}_0$.
- The simples $\{S_i\}$ of $\mathcal{H}_0$ are (3-)spherical objects.

and denote by

- $\mathrm{EG}^{\circ}(\mathcal{D}_{fd}(\Gamma_0))$ the connected component of $\mathrm{EG}(\mathcal{D}_{fd}(\Gamma_0))$ that contains $\mathcal{H}_0$;
- $\mathrm{ST}(\Gamma_0)$ the spherical twist group, which is the subgroup of $\mathrm{Aut}^{\circ}\,\mathcal{D}_{fd}(\Gamma_0)$ generated by ϕ_S for $S \in \mathrm{Sim}\,\mathcal{H}_0$;
- $\mathrm{Sph}(\Gamma_0) = \mathrm{ST}(\Gamma_0) \cdot \mathrm{Sim}\,\mathcal{H}_0$ the set of reachable spherical objects in $\mathcal{D}_{fd}(\Gamma_0)$.

Here,

$$\mathrm{Aut}^{\circ}\,\mathcal{D}_{fd}(\Gamma_0) = \mathrm{Aut}\,\mathcal{D}_{fd}(\Gamma_0)/\sim \tag{2.5}$$

is the auto-equivalence group up to isotopy $\sim$ in the sense that: $\psi \sim \psi'$ in $\mathrm{Aut}\,\mathcal{D}_{fd}(\Gamma_0)$ if $\psi^{-1} \circ \psi'$ acts trivially on $\mathrm{EG}^{\circ}(\mathcal{D}_{fd}(\Gamma_0))$.

Lemma 2.2 $\mathrm{Sph}(\Gamma_0) = \bigcup_{\mathcal{H} \in \mathrm{EG}^{\circ}(\mathcal{D}_{fd}(\Gamma_0))} \mathrm{Sim}\,\mathcal{H}$.

Proof By the tilting formulae in [7, Proposition 5.2 and Remark 7.2], we know that the simples in the tilts of a heart $\mathcal{H}$ is of the form $\phi_S^{\pm 1}(X)$, where S and X are the simples in $\mathcal{H}$. Thus, by induction, we deduce that

$$\operatorname{Sim}\mathcal{H} \subset \operatorname{Sph}(\Gamma_0), \quad \forall \mathcal{H} \in \operatorname{EG}^\circ(\mathcal{D}_{fd}(\Gamma_0)).$$

On the other hand (see [6] and cf. [7, Corollary 8.4]), two backward tiltings on a heart $\mathcal{H}$ with respect to the same simple (up to shift) is equivalent to apply the twist along the simple on $\mathcal{H}$. Thus, we deduce that $\operatorname{ST}(\Gamma_0) \cdot \mathcal{H}_0 \subset \operatorname{EG}^\circ(\mathcal{D}_{fd}(\Gamma_0))$, which implies

$$\operatorname{Sph}(\Gamma_0) = \bigcup_{\mathcal{H}\in\operatorname{ST}(\Gamma_0)\cdot\mathcal{H}_0} \operatorname{Sim}\mathcal{H} \subset \bigcup_{\mathcal{H}\in\operatorname{EG}^\circ(\mathcal{D}_{fd}(\Gamma_0))} \operatorname{Sim}\mathcal{H}$$

that completes the proof. □

2.4 The silting exchange graph

A *silting set* $\mathbf{P}$ in a category $\mathcal{D}$ is an $\operatorname{Ext}^{>0}$-configuration, i.e. a maximal collection of non-isomorphic indecomposables such that $\operatorname{Ext}^i(P, T) = 0$ for any $P, T \in \mathbf{P}$ and integer $i > 0$. The *forward mutation* $\mu_P^\sharp$ at an element $P \in \mathbf{P}$ is another silting set $\mathbf{P}_P^\sharp$, obtained from $\mathbf{P}$ by replacing P with

$$P^\sharp = \operatorname{Cone}\left(P \to \bigoplus_{T\in\mathbf{P}-\{P\}} \operatorname{Irr}(P, T)^* \otimes T\right), \tag{2.6}$$

where $\operatorname{Irr}(X, Y)$ is the space of irreducible maps $X \to Y$, in the additive subcategory $\operatorname{Add}\bigoplus_{T\in\mathbf{P}} T$ of $\mathcal{D}$. The *backward mutation* $\mu_P^\flat$ at an element $P \in \mathbf{P}$ is another silting set $\mathbf{P}_P^\flat$, obtained from $\mathbf{P}$ by replacing P with

$$P^\flat = \operatorname{Cone}\left(\bigoplus_{T\in\mathbf{P}-\{P\}} \operatorname{Irr}(T, P) \otimes T \to P\right)[-1]. \tag{2.7}$$

Definition 2.3 The *silting exchange graph* $\operatorname{SEG}(\mathcal{D})$ of a triangulated category $\mathcal{D}$ is the oriented graph whose vertices are all silting sets in $\mathcal{D}$ and whose edges correspond to forward mutations between them.

Note that Γ_0, considered as a set of its indecomposable summands, is a silting set in $\operatorname{per}\Gamma_0$. Denote by $\operatorname{SEG}^\circ(\operatorname{per}\Gamma_0)$ the principal component of the exchange graph $\operatorname{SEG}(\operatorname{per}\Gamma_0)$, that is the connected component containing Γ_0.

2.5 The cluster exchange graph

A *cluster tilting set* $\mathbf{P}$ in a category $\mathcal{C}$ is an Ext^1-configuration, i.e. a maximal collection of non-isomorphic indecomposables such that $\operatorname{Ext}^1(P, T) = 0$ for any $P, T \in \mathbf{P}$. We will only consider this structure on the cluster categories, which is 2-CY. The *forward mutation* $\mu_P^\sharp$ at an element $P \in \mathbf{P}$ is another cluster tilting set $\mathbf{P}_P^\sharp$, obtained from $\mathbf{P}$ by replacing P with $P^\sharp$ in (2.6). Similarly, we have the *backward mutation* $\mu_P^\flat$ using formula (2.7). In fact, since $\mathcal{C}$ is 2-CY, we have $\mu_P^\sharp\mathbf{P} = \mu_P^\flat\mathbf{P}$ and we will denote the mutation by μ_P.

Definition 2.4 The *cluster exchange graph* $\operatorname{CEG}(\mathcal{C})$ is the (unoriented) graph whose vertices are cluster tilting sets and whose edges correspond to the mutations.

We will write $\mathrm{CEG}(\Gamma_0)$ for $\mathrm{CEG}(\mathcal{C}(\Gamma_0))$.

Remark 2.5 For each cluster tilting set $\mathbf{P}$, denote by $Q_{\mathbf{P}}$ the Gabriel quiver of $\mathrm{End}(\mathrm{add} \bigoplus_{P \in \mathbf{P}} P)$. Then mutation on cluster tilting sets becomes mutation on the corresponding Gabriel quivers (cf. [6]).

3 The topological realizations

First recall the following main result from [10].

Theorem 3.1 [10] *There is a bijection*

$$\widetilde{X}\colon \mathrm{CA}(\mathbf{S}_{\triangle}) \to \mathrm{Sph}(\Gamma_0)/[1] \tag{3.1}$$

that induces a canonical isomorphism

$$\iota\colon \mathrm{BT}(\mathbf{T}_0) \to \mathrm{ST}(\Gamma_0), \tag{3.2}$$

sending the braid twist B_η of a closed arc η to the spherical twist $\phi_{\widetilde{X}(\eta)}$ of the corresponding spherical object $\widetilde{X}_\eta$.

Note that by construction, for any closed arcs s_i in the dual of the initial triangulation $\mathbf{T}_0$, we have $\widetilde{X}(s_i) = S_i[\mathbb{Z}]$, where S_i is the corresponding simple in the canonical heart $\mathcal{H}_0$ (of $\mathcal{D}_{fd}(\Gamma_0)$) and $S_i[\mathbb{Z}]$ its shift orbit.

3.1 The relations between various exchange graphs

We list the known relations between the exchange graphs mentioned above (many results here are due to Keller-Nicolás, see [6] for details).

(a) There is a canonical isomorphism

$$\mathrm{EG}^{\circ}(\mathcal{D}_{fd}(\Gamma_0)) \cong \mathrm{SEG}^{\circ}(\mathrm{per}\,\Gamma_0), \tag{3.3}$$

where the canonical heart $\mathcal{H}_0$ corresponds to Γ_0. Moreover, if a heart $\mathcal{H}$ corresponds to a silting set $\mathbf{P} = \{P_i\}_{i=1}^n$ under (3.3), then its simples can be labeled as $\{X_i\}_{i=1}^n$ such that

$$\mathrm{Hom}^{\bullet}(P_i, X_j) = \delta_{ij}\mathbf{k}. \tag{3.4}$$

(b) The quotient map $\pi\colon \mathrm{per}\,\Gamma_0 \to \mathcal{C}(\Gamma_0)$ induces an isomorphism

$$\pi_*\colon \mathrm{SEG}^{\circ}(\mathrm{per}\,\Gamma_0)/\,\mathrm{ST}(\Gamma_0) \cong \mathrm{CEG}(\Gamma_0). \tag{3.5}$$

Here, we consider a 2-cycle in the quotient graph $\mathrm{SEG}^{\circ}(\mathrm{per}\,\Gamma_0)/\,\mathrm{ST}(\Gamma_0)$ as an unoriented edge, cf. [7, §9]. Denote the image of Γ_0 by $\mathbf{P}_0$.

(c) A fundamental domain for $\mathrm{EG}^{\circ}(\mathcal{D}_{fd}(\Gamma_0))/\,\mathrm{ST}(\Gamma_0)$ is the full subgraph

$$\mathrm{EG}_3^{\circ}(\mathcal{H}_0) := \{\mathcal{H} \in \mathrm{EG}^{\circ}(\mathcal{D}_{fd}(\Gamma_0)) \,\big|\, \mathcal{H}_0 \le \mathcal{H} \le \mathcal{H}_0[1]\}$$

in $\mathrm{EG}^{\circ}(\mathcal{D}_{fd}(\Gamma_0))$ (cf. [7]). In particular, $\mathrm{EG}_3^{\circ}(\mathcal{H}_0) \cong \mathrm{CEG}(\Gamma_0)$ as unoriented graphs. Denote $\mathrm{SEG}_3^{\circ}(\Gamma_0)$ to be the full subgraph of $\mathrm{SEG}^{\circ}(\mathrm{per}\,\Gamma_0)$ that corresponds to $\mathrm{EG}_3^{\circ}(\mathcal{H}_0)$ under the isomorphism in (a). So in particular, $\mathrm{SEG}_3^{\circ}(\Gamma_0) \cong \mathrm{CEG}(\Gamma_0)$.

(d) There is a canonical isomorphism

$$\underline{\wp}\colon\ \mathrm{EG}(\mathbf{S}) \cong \mathrm{CEG}(\Gamma_0) \tag{3.6}$$

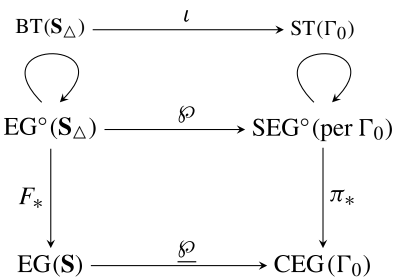
such that $\mathbf{P}_0$ corresponds to the initial triangulation $\mathbb{T}_0$.

Using (3.2), we could extend the isomorphism in (d) above as follows.

Proposition 3.2 *There is a canonical isomorphism (between graphs)*

$$\wp\colon\ \mathrm{EG}^\circ(\mathbf{S}_\triangle) \cong \mathrm{SEG}^\circ(\mathrm{per}\,\Gamma_0) \tag{3.7}$$

sending $\mathbf{T}$ *to* Γ_0 *and fitting into the following commutative diagram:*



(3.8)

where the upper commutativity means $\wp \circ \psi(\mathbf{T}) = \iota(\psi)\,(\wp(\mathbf{T}))$ *for any* $\mathbf{T} \in \mathrm{EG}^\circ(\mathbf{S}_\triangle)$ *and* $\psi \in \mathrm{BT}(\mathbf{S}_\triangle)$.

Proof Combine (b) and (d) above, we have $\mathrm{SEG}^\circ_3(\Gamma_0) \cong \mathrm{EG}(\mathbf{S})$. In particular, $\mathrm{EG}(\mathbf{S})$ inherits the orientation of $\mathrm{SEG}^\circ_3(\Gamma_0)$. Note that $\mathrm{SEG}^\circ_3(\Gamma_0)$ has a unique source $\mathcal{H}_0$ and a unique sink $\mathcal{H}_0[1]$. Lifting $\mathrm{EG}(\mathbf{S})$ to $\mathrm{EG}^\circ(\mathbf{S}_\triangle)$ with respect to such an orientation, such that $\mathbb{T}_0$ (corresponds to $\mathcal{H}_0$) become $\mathbf{T}$, we obtain a fundamental domain $\mathrm{EG}^\circ_3(\mathbf{T}_0)$ in $\mathrm{EG}^\circ(\mathbf{S}_\triangle)$ for $\mathrm{EG}^\circ(\mathbf{S}_\triangle)/\,\mathrm{BT}(\mathbf{S}_\triangle)$, which is isomorphic to $\mathrm{EG}^\circ_3(\mathcal{H}_0)$.

Next, we claim $\mathrm{BT}(\mathbf{S}_\triangle)$ and $\mathrm{ST}(\Gamma_0)$ act freely on $\mathrm{EG}^\circ(\mathbf{S}_\triangle)$ and $\mathrm{SEG}^\circ(\mathrm{per}\,\Gamma_0)$, respectively. If so, by isomorphism ι in (3.2), we can extend $\mathrm{EG}^\circ_3(\mathbf{T}_0) \cong \mathrm{EG}^\circ_3(\mathcal{H}_0)$ to the required isomorphism.

For $\mathrm{BT}(\mathbf{S}_\triangle)$, the freeness follows from the Alexander method. A triangulation of $\mathbf{S}_\triangle$ divides $\mathbf{S}_\triangle$ into once-punctured triangles (disks); hence if $\psi \in \mathrm{BT}(\mathbf{S}_\triangle)$ preserves any triangulation, then $\psi = 1$ in $\mathrm{MCG}(\mathbf{S}_\triangle)$. For $\mathrm{ST}(\Gamma_0)$, the freeness follows by definition (as we have quotient out the part that acts trivially on $\mathrm{EG}^\circ(\mathcal{D}_{fd}(\Gamma_0)) \cong \mathrm{SEG}^\circ(\mathrm{per}\,\Gamma_0)$). □

3.2 Graph duality and simple-projective duality

Let $\mathbf{T}$ be a triangulation in $\mathrm{EG}^\circ(\mathbf{S}_\triangle)$ consisting of open arcs $\{\nu_i\}_{i=1}^n$ and its dual $\mathbf{T}^*$ consists of corresponding closed arcs $\{\eta_i\}_{i=1}^n$. As

$$\pi_* \circ \wp(\mathbf{T}) = \underline{\wp} \circ F_*(\mathbf{T})$$

in (3.8), let $\mathbf{P} = \wp(\mathbf{T})$ consist of indecomposables $\{P_i\}_{i=1}^n$, such that $\pi(P_i) = \underline{\wp} \circ F(\nu)$. Equivalently, P_i is the unique element in the intersection

$$\wp(\mathbf{T}) \cap \left(\pi^{-1} \circ \underline{\wp} \circ F(\nu)\right) \tag{3.9}$$

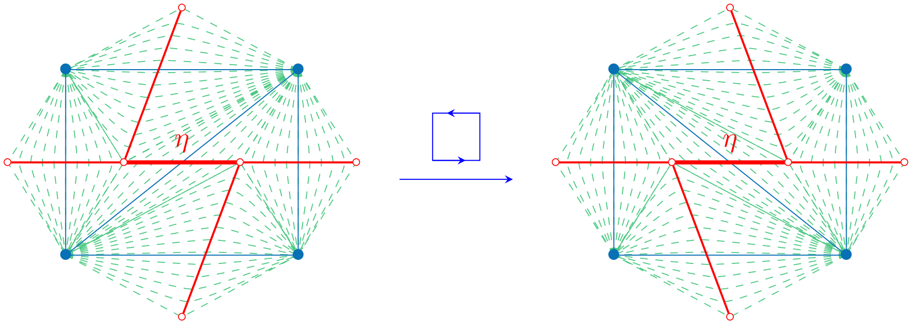


Fig. 3 The Whitehead move for the dual triangulations (*red*) (color figure online)

for $\nu = \nu_i$. Furthermore, under the isomorphism (3.3), let $\mathcal{H}$ with simples $\{X_i\}_{i=1}^n$ be the heart corresponding to $\mathbf{P}$. So we obtain a diagram:

$$\begin{array}{ccc} \mathbf{T} = \{\nu_i\} & \xrightarrow{\wp} & \mathbf{P} = \{P_i\} \\ \text{graph dual} \updownarrow & & \updownarrow \text{proj.-sim. dual} \\ \mathbf{T}^* = \{\eta_i\} & \xrightarrow[\text{up to shift}]{\widetilde{X}} & \operatorname{Sim}\mathcal{H} = \{X_i\}. \end{array} \tag{3.10}$$

We have the following, which says the graph duality (between triangulation and its dual) corresponds to the simple-projective duality.

Corollary 3.3 *Under the notation above, we have* $\widetilde{X}(\eta_i) = X_i[\mathbb{Z}]$ *for* $i = 1, \ldots, n$.

Proof Use induction, on the number of flips linking $\mathbf{T}'$ and $\mathbf{T}$, starting with the case when $\mathbf{T}' = \mathbf{T}$. Then $P_i = \Gamma_i = e_i\Gamma_0, \eta_i = s_i$ and $X_i = S_i$. The claim follows from the construction of $\widetilde{X}$. For the inductive step, assume that the claim holds for a triangulation $\mathbf{T}_1$ and consider the flip $\mathbf{T}_2$ of it with respect to an open arc ν_j. Without loss of generality, assume the flip is forward. Then we have the following:

- Flip of triangulations corresponds Whitehead move of dual triangulations (see Fig. 3). Let $\mathbf{T}_1^* = \{\alpha_i\}$, where $\eta = \alpha_j$ is the dual of ν_j for some j, and $Q_{\mathbf{T}_1}$ be the quiver associated to $\mathbf{T}_1$ whose vertices are indexed by $\{i\}$. Then $\mathbf{T}_2^* = \{\beta_i\}$ are

$$\beta_i = \begin{cases} \mathrm{B}_{\alpha_j}^{-1}(\alpha_i), & \text{if there is an arrow } i \to j \text{ in } Q_{\mathbf{T}_1}, \\ \alpha_i, & \text{otherwise.} \end{cases}$$

- Flip of triangulations corresponds simple tilting of hearts. Let $\mathcal{H}_i$ be the hearts corresponding to $\mathbf{T}_i$ and $\mathbf{T}_1$ has simples $\{X_{\alpha_i}\}$. By the tilting formulae in [7, Proposition 5.2 and Remark 7.2], the simples $\{X_{\beta_i}\}$ of $\mathcal{H}_2$ are

$$X_{\beta_i} = \begin{cases} \phi_{X_{\alpha_j}}^{-1}(X_{\alpha_i}), & \text{if there is an arrow } i \to j \text{ in } Q_{\mathbf{T}_1}, \\ X_{\alpha_j}[1], & \text{if } i = j, \\ X_{\alpha_i}, & \text{otherwise.} \end{cases}$$

By the inductive assumption, we have $\widetilde{X}(\alpha_i) = X_{\alpha_i}[\mathbb{Z}]$. By [10, Corollary 6.4], we have

$$\widetilde{X}(\beta_i) = \widetilde{X}\left(\mathrm{B}_{\alpha_j}^{-1}(\alpha_i)\right) = \iota(\mathrm{B}_{\alpha_j}^{-1})\widetilde{X}(\alpha_i) = \phi_{X_{\alpha_j}}^{-1}(X_{\alpha_i})[\mathbb{Z}] = X_{\beta_i}[\mathbb{Z}],$$

for i, when there is an arrow $i \to j$ in $Q_{\mathbf{T}_1}$. Clearly $\widetilde{X}(\beta_i) = X_{\beta_i}[\mathbb{Z}]$ also holds for other i, which completes the induction. □

3.3 Reachable open arcs and rigid indecomposables

This subsection is devoted to construct a bijection between open arcs and 'silting summands', which induces the isomorphism $\wp$ in Proposition 3.2.

First, we recall the corresponding result for cluster categories. Let $\mathrm{OA}(\mathbf{S})$ be the set of simple open arcs in $\mathbf{S}$ and $\mathrm{RR}(\mathcal{C}(\Gamma_0))$ be the set of rigid indecomposables in $\mathcal{C}(\Gamma_0)$, that is

$$\mathrm{RR}(\mathcal{C}(\Gamma_0)) = \bigcup_{\mathbf{P}\in \mathrm{CEG}(\Gamma_0)} \mathbf{P}.$$

Then there is a bijection ([4] cf. [11])

$$\underline{\rho}\colon\ \mathrm{OA}(\mathbf{S}) \to \mathrm{RR}(\mathcal{C}(\Gamma_0)), \tag{3.11}$$

which induces the isomorphism $\underline{\wp}$ in (3.6), i.e. $\underline{\wp} = \underline{\rho}_*$, in the sense that

$$\underline{\wp}(\mathbb{T}) = \{\underline{\rho}(\underline{\nu}) \big| \underline{\nu} \in \mathbb{T}\}.$$

We proceed to construct the analogue bijection for $\mathbf{S}_\triangle$ and $\operatorname{per}\Gamma_0$.

Recall that $\mathrm{OA}^\circ(\mathbf{S}_\triangle)$ is the set of reachable simple open arcs in $\mathbf{S}_\triangle$, that is

$$\mathrm{OA}^\circ(\mathbf{S}_\triangle) = \bigcup_{\mathbf{T}\in \mathrm{EG}^\circ(\mathbf{S}_\triangle)} \mathbf{T},$$

and $\mathrm{RR}(\operatorname{per}\Gamma_0)$ the set of reachable rigid indecomposables in $\operatorname{per}\Gamma_0$, that is

$$\mathrm{RR}(\operatorname{per}\Gamma_0) = \bigcup_{\mathbf{P}\in \mathrm{SEG}^\circ(\operatorname{per}\Gamma_0)} \mathbf{P}.$$

Note that in the case for $\mathbf{S}$ and $\mathcal{C}(\Gamma_0)$, all simple open arcs/indecomposables are reachable.

Remark 3.4 Note that for RR per Γ_0, we only consider the 'reachable' rigid indecomposables as we don't know if there are more such objects in this category.

Next, we prove a lemma. Let $\mathrm{EG}_3^\circ(\mathbf{T}_0) = \wp^{-1}(\mathrm{SEG}_3^\circ(\Gamma_0))$, which is a full subgraph of $\mathrm{EG}^\circ(\mathbf{S}_\triangle)$ (cf. **(b)** in § 3.1 for the definition of $\mathrm{SEG}_3^\circ(\Gamma_0)$). Then it is a fundamental domain for $\mathrm{EG}^\circ(\mathbf{S}_\triangle)/\mathrm{BT}$ and we have

$$F_*\colon\ \mathrm{EG}_3^\circ(\mathbf{T}_0) \cong \mathrm{EG}(\mathbf{S}). \tag{3.12}$$

Lemma 3.5 *Let $\nu_i \in \mathbf{T}_i$ and $\mathbf{T}_i \in \mathrm{EG}_3^\circ(\mathbf{T}_0)$ for $i = 1, 2$. If $F(\nu_1) = F(\nu_2)$, then $\nu_1 = \nu_2$.*

Proof Let $\underline{\nu} = F(\nu_i)$ and $\mathbb{T}_i = F_*(\mathbf{T}_i)$ that contains $\underline{\nu}$. Consider the surface $\mathbf{S}\backslash\underline{\nu}$, which is obtained from $\mathbf{S}$ by cutting along $\underline{\nu}$ (see [11] for the procedure of cutting). Denote by $\mathrm{EG}_{\underline{\nu}}(\mathbf{S})$ the full subgraph of $\mathrm{EG}(\mathbf{S})$ consisting of triangulations that contain $\underline{\nu}$. We have $\mathrm{EG}_{\underline{\nu}}(\mathbf{S}) \cong \mathrm{EG}(\mathbf{S}\backslash\underline{\nu})$, which is connected. Thus, there is a path $\underline{p}$ in $\mathrm{EG}_{\underline{\nu}}(\mathbf{S})$ connecting $\mathbb{T}_1$ and $\mathbb{T}_2$, which lifts, via F_* above, to a path p in $\mathrm{EG}_3^\circ(\mathbf{T}_0)$ connecting $\mathbf{T}_1$ and $\mathbf{T}_2$. Notice that any triangulation in $\underline{p}$ contains $\underline{\nu}$, or equivalently, $\underline{\nu}$ remains unchanged during these flips.

Thus, by looking at the lifted flips in p, we deduce that ν_1 in $\mathbf{T}_1$ corresponds to ν_2 in $\mathbf{T}_2$, which is unchanged as required. □

Theorem 3.6 *There is a canonical bijection*

$$\rho\colon \ \mathrm{OA}^\circ(\mathbf{S}_\triangle) \to \mathrm{RR}(\mathrm{per}\,\Gamma_0)$$

sending initial arcs $\gamma_i \in \mathbf{T}$ *to* $\Gamma_i = e_i\Gamma_0$ *and fitting into the following commutative diagram:*

$$\begin{array}{ccc} \mathrm{OA}^\circ(\mathbf{S}_\triangle) & \xrightarrow{\quad\rho\quad} & \mathrm{RR}(\mathrm{per}\,\Gamma_0) \\ \Big\downarrow F & & \Big\downarrow \pi \\ \mathrm{OA}(\mathbf{S}) & \xrightarrow{\quad\underline{\rho}\quad} & \mathrm{RR}(\mathcal{C}(\Gamma_0)) \end{array} \quad . \tag{3.13}$$

Further, it induces the isomorphism $\wp$ *in* (3.7)*, i.e.* $\wp = \rho_*$ *in the sense that*

$$\wp(\mathbf{T}) = \{\rho(\nu) \,\big|\, \nu \in \mathbf{T}\}.$$

Proof Consider a pair $(\nu, \mathbf{T})$, where ν is an open arc in a triangulation $\mathbf{T}$ of $\mathbf{S}_\triangle$. Define $\rho(\nu, \mathbf{T})$ to be the element in the silting set $\wp(\mathbf{T})$ whose image under π in $\mathcal{C}(\Gamma_0)$ is $\underline{\rho} \circ F(\nu)$. That is precisely (3.9). Note that for any $\psi \in \mathrm{BT}(\mathbf{S}_\triangle)$, we have

$$\begin{aligned} \rho\left(\psi(\nu), \psi(\mathbf{T})\right) &= (\wp \circ \psi(\mathbf{T})) \cap L\left(\psi(\nu)\right) \\ &= (\wp \circ \psi(\mathbf{T})) \cap L(\nu) \\ &= (\iota(\psi) \circ \wp(\mathbf{T})) \cap L(\nu) \\ &= \iota(\psi)\left(\wp(\mathbf{T}) \cap L(\nu)\right) \\ &= \iota(\psi)\left(\rho(\nu, \mathbf{T})\right), \end{aligned} \tag{3.14}$$

where $L = \pi^{-1} \circ \underline{\rho} \circ F$ satisfying $L \circ \psi = L = \iota(\psi) \circ L$ and the third equality follows from the commutativity of the upper square in (3.8).

To finish the proof, we only need to show that $\rho(\nu, \mathbf{T})$ is independent of the choice of $\mathbf{T}$, or equivalently,

$$\rho(\nu, \mathbf{T}_1) = \rho(\nu, \mathbf{T}_2) \tag{3.15}$$

for any $\mathbf{T}_1$ and $\mathbf{T}_2$ containing ν. If so, $\rho(\nu)\colon = \rho(\nu, \mathbf{T})$ clearly satisfies all the required conditions.

First, consider the case when $\mathbf{T}_1$ and $\mathbf{T}_2$ are related by a flip. In such a case, they only differ by one close arc and so does the corresponding silting set $\wp(\mathbf{T}_1)$. As $\nu \in \mathbf{T}_1 \cap \mathbf{T}_2$, the flip is not with respect to ν, which implies the claim (3.15).

Second, consider the case when $\mathbf{T}_1$ and $\mathbf{T}_2$ are both in the fundamental domain $\mathrm{EG}_3^\circ(\mathbf{T}_0) \cong \mathrm{EG}^\circ(\mathbf{S}_\triangle)/\,\mathrm{BT}$. Recall that we have (3.12). Let $\mathrm{EG}_{\underline{\nu}}(\mathbf{S})$ be the full subgraph of $\mathrm{EG}(\mathbf{S})$, consisting of triangulations that contains $\underline{\nu} = F(\nu)$. By the first case above, it is sufficient to show that $\mathbf{T}_1$ and $\mathbf{T}_2$ are connected by a path in $\mathrm{EG}_3^\circ(\mathbf{T}_0)$ such that any triangulation in this path contains ν. This is equivalent to $\mathbb{T}_1 = F_*(\mathbf{T}_1)$ and $\mathbb{T}_2 = F_*(\mathbf{T}_2)$ are connected in $\mathrm{EG}_{\underline{\nu}}(\mathbf{S})$. Consider the cut surface $\mathbf{S}\backslash\underline{\nu}$ as in Lemma 3.5. We have $\mathrm{EG}_{\underline{\nu}}(\mathbf{S}) \cong \mathrm{EG}(\mathbf{S}\backslash\underline{\nu})$, which is connected. Thus, the claim (3.15) holds in this situation.

Third, consider the case when $\mathbf{T}_2 = \psi(\mathbf{T}_1)$ and $\psi(\nu) = \nu$ for some $\psi \in \mathrm{BT}(\mathbf{S}_\triangle)$. By the former condition, we have $\rho(\nu, \mathbf{T}_2) = \iota(\psi)\left(\rho(\nu, \mathbf{T}_1)\right)$. What is left to prove in this case is that the later condition implies $\iota(\psi)$ preserves $P = \rho(\nu, \mathbf{T}_1)$. Consider the cut surface $\mathbf{S}\backslash\underline{\nu}$ again, which inherits all the decorating points and $(\mathbf{S}\backslash\underline{\nu})_\triangle$ inherits a triangulation $\mathbf{T}_1\backslash\nu$. Since ψ

preserves ν, it is actually in $\mathrm{BT}(\mathbf{S}\backslash\underline{\nu}) = \mathrm{BT}(\mathbf{T}_1\backslash\nu)$ (cf. [10, Proposition 4.3]). As $\mathrm{BT}(\mathbf{T}_1\backslash\nu)$ is generated by B_η, for the closed arc $\eta \in (\mathbf{T}_1\backslash\nu)^*$ dual to some open arc $\gamma \in \mathbf{T}_1\backslash\nu$, we only need to show that $\iota(\mathrm{B}_\eta)$ preserves P. Consider $\rho(\gamma, \mathbf{T}_1)$ in the silting set $\wp(\mathbf{T}_1)$ and the corresponding simple X in the corresponding heart. Then, we have $\mathrm{Hom}^\bullet(P, X) = 0$ by (3.4) and, by Corollary 3.3, $\widetilde{X}(\eta) = X[\mathbb{Z}]$. Using formulae in [10, Corollary 6.4], we have

$$\iota(\mathrm{B}_\eta)(P) = \phi_{\widetilde{X}(\eta)}(P) = \phi_X(P) = P,$$

as required.

Finally, consider the general case. Let $\mathbf{T}_i = \psi_i(\mathbf{T}'_i)$ for some $\psi_i \in \mathrm{BT}(\mathbf{S}_\triangle)$ and $\mathbf{T}'_i$ are in the fundamental domain $\mathrm{EG}^\circ_3(\mathbf{T}_0)$, $i = 1, 2$. Let $\nu'_i = \psi_i(\nu)$ and then $F(\nu'_1) = F(\nu) = F(\nu'_2)$, which implies

$$\nu'_1 = \nu'_2 =: \nu',$$

by Lemma 3.5. Since $\nu' \in \mathbf{T}'_1 \cap \mathbf{T}'_2$, by the second case above, we have

$$\rho(\nu', \mathbf{T}'_1) = \rho(\nu', \mathbf{T}'_2). \tag{3.16}$$

Since $\psi_2^{-1} \circ \psi_1(\nu) = \psi_2^{-1}(\nu') = \nu$, by the third case above, we have

$$\rho(\nu, \mathbf{T}_1) = \rho(\nu, \psi_2^{-1} \circ \psi_1(\mathbf{T}_1)). \tag{3.17}$$

Combining (3.16), (3.17) and formula (3.14), we have

$$\begin{aligned}
\rho(\nu, \mathbf{T}_1) &= \rho(\nu, \psi_2^{-1} \circ \psi_1(\mathbf{T}_1)) \\
&= \iota(\psi_2^{-1})\big(\rho\,(\psi_2(\nu), \psi_1(\mathbf{T}_1))\big) \\
&= \iota(\psi_2^{-1})\big(\rho(\nu', \mathbf{T}'_1)\big) \\
&= \iota(\psi_2^{-1})\big(\rho(\nu', \mathbf{T}'_2)\big) \\
&= \rho\big(\psi_2^{-1}(\nu'), \psi_2^{-1}(\mathbf{T}'_2)\big) \\
&= \rho(\nu, \mathbf{T}_2),
\end{aligned}$$

which finishes the proof. □

Remark 3.7 By Theorem 3.6, the forgetful map $F\colon \mathbf{S}_\triangle \to \mathbf{S}$ is a 'topological realization' of Amiot's quotient map $\pi\colon\ \mathrm{per}\,\Gamma_0 \to \mathcal{C}(\Gamma_0)$:

- the correspondence from open arcs to reachable rigid indecomposables (ρ and $\underline{\rho}$) commutes with them;
- the closed arcs get killed under F, so do the spherical objects under π (Fig. 4).

3.4 Rotations in marked mapping class groups

Definition 3.8 The *marked mapping class group* $\mathrm{MMCG}(\mathbf{S})$ of a marked surface $\mathbf{S}$ is the group of isotopy classes of homeomorphisms of $\mathbf{S}$, where all homeomorphisms and isotopies are required to

- fix the set $\mathbf{M}$ of marked points as a set.

Note that the boundaries are NOT required to be fixed pointwise. Therefore, for each boundary component $C \in \partial\mathbf{S}$ with m marked points, denote by ξ_C the m-th root of the Dehn twist around C, that is, simultaneous (anticlockwise) rotation to the next marked point on C. Then the *universal rotation* $\underline{\xi}$, as an element in $\mathrm{MMCG}(\mathbf{S})$, is

$$\underline{\xi} = \prod_{C \subset \partial\mathbf{S}} \rho_C.$$

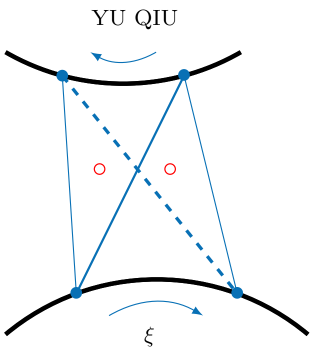


Fig. 4 Flip as universal rotation in a chosen triangulation

Here, the product is over all connected components C of $\partial \mathbf{S}$.

Similar for the definition of marked mapping class group of $\mathbf{S}_\triangle$ (requires fixing the set $\mathbf{M}$ of marked points and the set $\triangle$ of decorating points as sets) and the universal rotation $\xi \in \mathrm{MMCG}(\mathbf{S}_\triangle)$.

Remark 3.9 For the punctured case, the marked mapping class group should be upgraded to the tagged mapping class group, cf. [2,3].

Recall that, under the bijection $\underline{\rho}$ in (3.11), the action $\underline{\xi}$ on $\mathrm{OA}(\mathbf{S})$ corresponds to the shift (i.e. [1], or equivalently, the Auslander-Reiten translation) on $\mathrm{RR}(\mathcal{C}(\Gamma_0))$. In other words, we have (cf. [4] and [3])

$$\underline{\rho}(\underline{\gamma})[1] = \underline{\rho}(\underline{\xi}(\underline{\gamma})), \tag{3.18}$$

for any $\underline{\gamma} \in \mathrm{OA}(\mathbf{S})$.

We prove the analogue result for $\mathbf{S}_\triangle$.

Proposition 3.10 *For any* $\gamma \in \mathrm{OA}^\circ(\mathbf{S}_\triangle)$*, we have*

$$\rho(\gamma)[1] = \rho(\xi(\gamma)). \tag{3.19}$$

Proof This follows exactly the same way as in Case I in the proof of [3, Lemma 3.5] (cf. [3, Fig. 6]), noticing that the (forward/backward) mutation formulae for silting/cluster tilting coincide indeed. □

Note that, under the forgetful map, ξ becomes $\underline{\xi}$; under Amiot's quotient, [1] in $\mathrm{per}\,\Gamma_0$ becomes [1] in $\mathcal{C}(\Gamma_0)$. And they are compatible:

$$\begin{array}{ccc}
\mathbf{S}_\triangle \curvearrowleft \xi & \overset{\rho}{\dashrightarrow} & [1] \curvearrowright \mathrm{per}\,\Gamma_0 \\
\downarrow F & & \downarrow \pi \\
\mathbf{S} \curvearrowleft \underline{\xi} & \overset{\underline{\rho}}{\dashrightarrow} & [1] \curvearrowright \mathcal{C}(\Gamma_0)
\end{array}.$$

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