

Finite presentations for spherical/braid twist groups from decorated marked surfaces

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ABSTRACT

We give a finite presentation for the braid twist group of a decorated surface. If the decorated surface arises from a triangulated marked surface without punctures, we obtain a finite presentation for the spherical twist group of the associated 3-Calabi–Yau triangulated category. The motivation/application is that the result will be used to show that the (principal component of) space of stability conditions on the 3-Calabi–Yau category is simply connected in the sequel [King and Qiu, *Invent. Math.*, to appear].

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1. Introduction

Artin’s braid group is a classical object in low-dimensional topology, which links to many areas in mathematics: for example, knot theory, representation theory of algebras, monodromy invariants of algebraic geometry; cf. the survey [4] and the textbook [13]. We are interested in a generalization of braid group, the braid twist groups of decorated surfaces, which fits into the theory of cluster algebras and stability conditions.

1.1. Motivation

In Bridgeland–Smith’s seminal work [5], they established a connection between Teichmüller theory and theory of stability conditions on triangulated categories. More precisely, let \mathbf{S} be a marked surface (that is, an oriented compact surface with or without non-empty boundary, equipped with a collection of marked points in the sense of Fomin–Shapiro–Thurston [8]). There is an associated 3-Calabi–Yau triangulated category $\mathcal{D}(\mathbf{S})$ (see Section 2.3 for the construction). The authors proved that [5, Theorems 1.2 and 1.3] there is an isomorphism of complex orbifolds

$$\text{Stab}^\circ \mathcal{D}(\mathbf{S}) / \text{Aut}^\circ \mathcal{D}(\mathbf{S}) \cong \text{Quad}(\mathbf{S}), \tag{1.1}$$

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where $\text{Quad}(\mathbf{S})$ is the moduli space of quadratic differentials on \mathbf{S} , $\text{Stab}^\circ \mathcal{D}(\mathbf{S})$ the principal component of space of stability conditions on $\mathcal{D}(\mathbf{S})$ and $\text{Aut}^\circ \mathcal{D}(\mathbf{S})$ the group of autoequivalences of $\mathcal{D}(\mathbf{S})$ which preserve this component. Their motivations are coming from string theory in physics, Donaldson–Thomas theory and homological mirror symmetry (cf. [9, 20, 25]).

The main aim of this series of works [18, 20, 23] is to show that $\text{Stab}^\circ \mathcal{D}(\mathbf{S})$ is simply connected. Thus we are interested in the symmetry groups in formula (1.1), in particular, the spherical twist group (see Section 2.3 for the definition) $\text{ST} \mathcal{D}(\mathbf{S}) \subset \text{Aut}^\circ \mathcal{D}(\mathbf{S})$ that sits in the short exact sequence of groups [5, Theorem 9.9]

$$1 \rightarrow \text{ST} \mathcal{D}(\mathbf{S}) \rightarrow \text{Aut}^\circ \mathcal{D}(\mathbf{S}) \rightarrow \text{MCG}(\mathbf{S}) \rightarrow 1,$$

where $\text{MCG}(\mathbf{S})$ is the mapping class group of \mathbf{S} , which consists of the isotopy classes of orientation-preserving homeomorphisms of \mathbf{S} that fix the set of marked points. Such spherical twist groups were first studied by Khovanov, Seidel and Thomas [17, 24] from the two sides of the homological mirror symmetry in the case when \mathbf{S} is a disk.

In the prequel [20], we introduced the decorated marked surface \mathbf{S}_Δ (for \mathbf{S} unpunctured, that is, all marked points are on the boundary of \mathbf{S}), which is obtained from \mathbf{S} by decorating a certain set Δ of points in the interior of \mathbf{S} (see Definition 2.7). Let $\text{MCG}(\mathbf{S}_\Delta)$ be the mapping class group of \mathbf{S}_Δ , that is, the group consisting of the isotopy classes of orientation-preserving homeomorphisms of \mathbf{S}_Δ that fix the boundary of \mathbf{S}_Δ pointwise and preserve the decoration set Δ setwise. We showed that [20, Theorem 1] $\text{ST} \mathcal{D}(\mathbf{S})$ is isomorphic to a subgroup

$$\text{BT}(\mathbf{S}_\Delta) \subset \text{MCG}(\mathbf{S}_\Delta),$$

the braid twist group of \mathbf{S}_Δ (see Definition 2.1 and Section 2.2). In this paper, we give finite presentations for these twist groups, which will play a key role in achieving the aim, that is, proving the simply connectedness of $\text{Stab}^\circ \mathcal{D}(\mathbf{S})$ (in the sequel [18, Theorem 4.16]).

1.2. Braid groups via quivers with potential

The key ingredient in the construction of the 3-Calabi–Yau triangulated category $\mathcal{D}(\mathbf{S})$ associated to a marked surface \mathbf{S} is the so-called quiver with potential in the cluster theory. This notion was introduced by Derksen, Weyman and Zelevinsky [6] in the general case and was developed by Fomin, Shapiro and Thurston [8] and Labardini-Fragoso [19] for the surface case. Given a triangulation \mathbf{T} of a marked surface \mathbf{S} , there is an associated quiver with potential $(Q_{\mathbf{T}}, W_{\mathbf{T}})$ whose vertices are indexed by the arcs in \mathbf{T} , whose arrows are indexed by the oriented angles of triangles of \mathbf{T} and whose potential $W_{\mathbf{T}}$ is the sum of 3-cycles arising from triangles of \mathbf{T} . The category $\mathcal{D}(\mathbf{S})$ is defined to be the finite-dimensional derived category of the Ginzburg dg algebra associated to $(Q_{\mathbf{T}}, W_{\mathbf{T}})$; see Section 2.2 for more details.

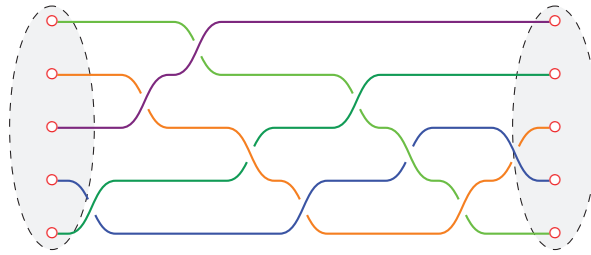
There is a preferable set of generators for the corresponding spherical twist group $\text{ST} \mathcal{D}(\mathbf{S})$, which is indexed by the vertices of the quiver $Q_{\mathbf{T}}$ (see Section 2.3). It is natural to try to find a set of (generating) relations for this set of generators. Then our work fits into a larger program: to introduce (algebraic) braid group $\text{Br}(Q, W)$ associated to a quiver with potential (Q, W) .

For the (classical) braid group on \aleph strands, we have the following well-known Artin presentation [1].

- Generators: $\sigma_i, 1 \leq i \leq \aleph - 1$.
- Relations: (cf. the notation in Section 1.4).
 - $\text{Co}(\sigma_i, \sigma_j)$ if $|i - j| \neq 1$
 - $\text{Br}(\sigma_i, \sigma_j)$ if $|i - j| = 1$.

This naturally generalizes to the Artin group (or generalized braid group) associated to a Dynkin diagram ∇ .

- Generators: σ_i, i vertices of ∇ .

FIGURE 1 (colour online). *Classical braids.*

- Relations:
 - $\text{Co}(\sigma_i, \sigma_j)$ if there is no edge between i and j in ∇
 - $\text{Br}(\sigma_i, \sigma_j)$ if there is exactly one edge between i and j in ∇ .

Recently, Grant and Marsh [11] generalized this to the quivers with potential which are mutation equivalent to a Dynkin quiver; see also [20, Proposition 10.3]. Qiu and Woolf [22] showed that the generalized braid group $\text{Br}(Q, W)$ in this case is indeed isomorphic to the corresponding spherical twist group.

In this paper, we will introduce the braid groups associated to the quivers with potential which comes from marked surfaces and show that they are isomorphic to the corresponding braid/spherical twist groups. Note that the main difficulty lies in showing the predicted relations in [20] are enough (at least in some good cases, see Proposition 5.2). The result is in the same line of the faithfulness of spherical twist actions in [17, 22, 24], which all imply that the corresponding spaces of stability conditions are simply connected (cf. the survey [21]); so does ours in [18].

1.3. Relation with surface braid groups

The (classical) braid group B_{\aleph} on \aleph strands can be realized (topologically) as $\text{MCG}(\mathbf{D}_{\aleph})$, where \mathbf{D}_{\aleph} is a disk \mathbf{D} with \aleph decorating points. One can also realize B_{\aleph} as collections of \aleph strands on \mathbf{D}_{\aleph} (or as the fundamental group of the corresponding configuration space, cf. Figure 1).

This realization leads to the following natural generalization of braid group. Given a decorated surface S_{\aleph} , which is an oriented compact surface S with non-empty boundary and a finite set of \aleph decorating points in its interior, the surface braid group $\text{SBr}(S_{\aleph})$ consists of the collections of strands on S_{\aleph} (cf. [12, Section 2.1]). The surface braid group $\text{SBr}(S_{\aleph})$ can be realized as a subgroup of the mapping class group $\text{MCG}(S_{\aleph})$ (cf. Definition 2.5) and sits in the following short exact sequence of groups

$$1 \rightarrow \text{SBr}(S_{\aleph}) \rightarrow \text{MCG}(S_{\aleph}) \rightarrow \text{MCG}(S) \rightarrow 1$$

(cf. [12, § 2.4, (5)]).

For our purpose, we also want to consider the marking \mathbf{M} on the surface S , which is a set of marked points on the boundary of S . Denote by \mathbf{S} the pair (S, \mathbf{M}) , known as a marked surface, and by \mathbf{S}_{Δ} the associated decorated marked surface as in Section 1.1, where Δ is the decoration set of \aleph points. Our braid twist group $\text{BT}(\mathbf{S}_{\Delta})$ is in fact a subgroup of the surface braid group $\text{SBr}(\mathbf{S}_{\Delta})$. We denote by $\text{FQuad}(\mathbf{S})$ the moduli space of \mathbf{S} -framed quadratic differentials and by $\text{FQuad}(\mathbf{S}_{\Delta})$ the moduli space of \mathbf{S}_{Δ} -framed quadratic differentials. Then we have the following commutative diagram of coverings:

$$\begin{array}{ccc}
 \text{FQuad}(\mathbf{S}_\Delta) & \xrightarrow{\text{SBr}(\mathbf{S}_\Delta)} & \text{FQuad}(\mathbf{S}) \\
 \searrow^{\text{MCG}(\mathbf{S}_\Delta)} & & \swarrow_{\text{MCG}(\mathbf{S})} \\
 & \text{Quad}(\mathbf{S}). &
 \end{array} \tag{1.2}$$

The space $\text{FQuad}(\mathbf{S}_\Delta)$ is not connected in general (unless \mathbf{S} is a disk). For any connected component $\text{FQuad}^\circ(\mathbf{S}_\Delta)$ of $\text{FQuad}(\mathbf{S}_\Delta)$, the isomorphism (1.1) can be upgraded to [18, Theorem 4.14]

$$\kappa^\circ: \text{Stab}^\circ \mathcal{D}(\mathbf{S}) \cong \text{FQuad}^\circ(\mathbf{S}_\Delta), \tag{1.3}$$

and the covering group of the covering

$$\text{FQuad}^\circ(\mathbf{S}_\Delta) \rightarrow \text{FQuad}(\mathbf{S}),$$

is the braid twist group $\text{BT}(\mathbf{S}_\Delta)$, which hence is a subgroup of $\text{SBr}(\mathbf{S}_\Delta)$.

Note that finite presentations of the mapping class group $\text{MCG}(\mathbf{S}_\mathbb{N})$ of a general decorated surface $\mathbf{S}_\mathbb{N}$ have been heavily studied (cf. [7]) and finite presentations of the surface braid group $\text{SBr}(\mathbf{S}_\mathbb{N})$ have been calculated by Bellingeri [2]. Our main strategy is to use the well-studied presentations of $\text{SBr}(\mathbf{S}_\Delta)$ (by many works [2, 3, 12]) to find finite presentations of $\text{BT}(\mathbf{S}_\Delta)$ that fit into our motivation.

1.4. Context and notations

The paper is organized as follows. In Section 2, we review the background of braid/spherical twist groups and surface braid groups. In Section 3, we give an alternative presentation of surface braid group. In Section 4, we find a first finite presentation of the braid twist group of a general decorated surface. In Section 5, we calculate finite presentations for the braid/spherical twist group of a decorated marked surface via quivers with potential. The main results/presentations are

Theorem 4.1: A finite presentation for the braid twist group of a decorating surface.

Theorem 5.9: Finite presentations for the braid twist group of a decorated marked surface via quivers with potential.

Corollary 5.10: Finite presentations for the spherical twist group of the 3-Calabi–Yau triangulated category from a marked surface via quivers with potential.

Theorem 5.11: (Infinite) presentations for braid/spherical twist groups with only conjugation relations.

In mapping class groups, autoequivalence groups or their subgroups, we have the following conventions.

- *Multiplication:* The multiplication ab stands for the composition $a \circ b$, that is, first b then a .
- *Inverse and conjugation:* For simplifying notation in calculations, we will use \underline{s} to denote the inverse s^{-1} of an element s and use a^b to denote the conjugation \underline{bab} of a by b . The easy formula $(a^b)^c = a^{bc}$ will be used frequently.
- *Relation:* We will use the following notations for relations throughout this paper.

Commutation relation	$\text{Co}(a, b)$:	$ab = ba$
Braid relation	$\text{Br}(a, b)$:	$aba = bab$
Skewed commutation relation	$\text{SCo}(x; a, b)$:	$xaxbx = bxa$
Triangle relation	$\text{Tr}(a, b, c)$:	$abca = bcab = cabc$.

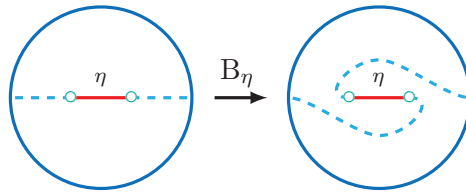


FIGURE 2 (colour online). The braid twist along a simple closed arc η .

The following two subsets of the set \mathbb{N} of natural numbers will be used in the paper:

$$2\mathbb{N}_{\leq g} = \{2h \mid h \in \mathbb{N} \text{ with } 1 \leq h \leq g\}$$

$$2\mathbb{N}_{\leq g} - 1 = \{2h - 1 \mid h \in \mathbb{N} \text{ with } 1 \leq h \leq g\}.$$

2. Preliminaries

2.1. Decorated surfaces and two braid groups

A *decorated surface* S_{\aleph} is a compact connected oriented surface S with non-empty boundary ∂S , endowed with a set $\Delta = \{Z_1, \dots, Z_{\aleph}\}$ of \aleph points in the interior of S . The points Z_i , $1 \leq i \leq \aleph$, are called *decorating points* in S_{\aleph} . Denote by

- g the genus of S , and
- b the number of connected components of ∂S .

The *mapping class group* $\text{MCG}(S_{\aleph})$ of S_{\aleph} is the group of isotopy classes of orientation-preserving homeomorphisms of S , where all homeomorphisms and isotopies are required to: (i) fix ∂S pointwise; (ii) fix the decoration set Δ setwise.

A *closed arc* in S_{\aleph} is (the homotopy class of) a continuous function $\eta : [0, 1] \rightarrow S$ such that

- (1) both $\eta(0)$ and $\eta(1)$ are in Δ with $\eta(0) \neq \eta(1)$, and
- (2) for any $0 < t < 1$, $\eta(t) \notin \Delta$.

A closed arc η is called *simple* if $\eta(t_1) \neq \eta(t_2)$ for any $t_1 \neq t_2$. Denote by $\text{CA}(S_{\aleph})$ the set of simple closed arcs in S_{\aleph} .

DEFINITION 2.1 (Braid twists). For any simple closed arc $\eta \in \text{CA}(S_{\aleph})$, the *braid twist* $B_{\eta} \in \text{MCG}(S_{\aleph})$ along η is shown in Figure 2. The *braid twist group* $\text{BT}(S_{\aleph})$ of a decorated surface S_{\aleph} is the subgroup of $\text{MCG}(S_{\aleph})$ generated by the braid twists.

Note that the braid twist B_{η} does not depend on the orientation of η in the sense that if we define a closed arc η' by $\eta'(t) = \eta(1 - t)$ for $t \in [0, 1]$ then we have $B_{\eta} = B_{\eta'}$. We have the following easy observation for the action of $\text{MCG}(S_{\aleph})$ on $\text{BT}(S_{\aleph})$ by conjugation (cf. [20, equation (3.3)]).

LEMMA 2.2 (Conjugation). For an element Ψ in $\text{MCG}(S_{\aleph})$ and a closed arc η in $\text{CA}(S_{\aleph})$, we have

$$B_{\Psi(\eta)} = (B_{\eta})^{\Psi}. \quad (2.1)$$

This implies that the group $\text{BT}(S_{\aleph})$ is a normal subgroup of $\text{MCG}(S_{\aleph})$.

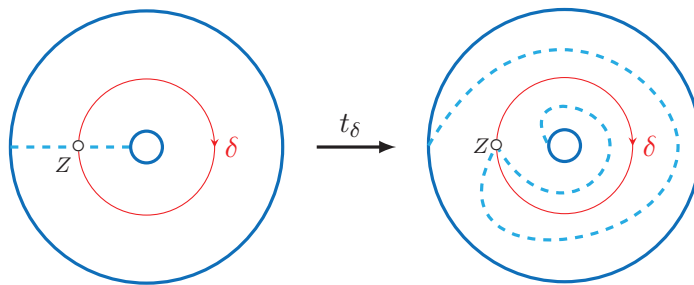


FIGURE 3 (colour online). The twist along a simple L-arc δ .

It is easy to check the following relations on braid twists.

LEMMA 2.3. For any closed arcs a, b, c in $CA(S_{\mathbb{N}})$, the following hold in $BT(S_{\mathbb{N}})$:

- $Co(B_a, B_b)$ if a and b are disjoint;
- $Br(B_a, B_b)$ if a and b are disjoint except sharing a common endpoint;
- $Tr(B_a, B_b, B_c)$ if a, b and c are disjoint except sharing a common endpoint, and they are in clockwise order at that point (see the left picture of Figure 6).

An L-arc δ in $S_{\mathbb{N}}$ is (the homotopy class of) a continuous function $\delta : [0, 1] \rightarrow S$ such that

- (1) $\delta(0) = \delta(1)$ is in Δ ,
- (2) for any $0 < t < 1$, $\delta(t) \notin \Delta$ and
- (3) δ is not homotopic to $\delta(0)$.

An L-arc δ is called *simple* if $\delta(t_1) \neq \delta(t_2)$ for any $t_1 \neq t_2 \in (0, 1)$.

DEFINITION 2.4. For any simple L-arc δ , the twist t_{δ} along δ is an element in $MCG(S_{\mathbb{N}})$, moving the point $Z = \delta(0)$ along δ to $\delta(1)$ as shown in Figure 3.

Note that the twist t_{δ} along an L-arc δ depends on the orientation of δ in the way that $t_{\delta} = \underline{t}_{\delta'}$ where δ' is the L-arc given by $\delta'(t) = \delta(1 - t)$ for $t \in [0, 1]$.

NOTATION 2.5. To simplify the notation, we will denote by η the braid twist B_{η} and denote by δ the twist t_{δ} .

There are several equivalent definitions of surface braid group (the most common one is via configuration space; cf. [12]). We will take the following one as it suits our purpose better.

DEFINITION 2.6 (Surface braid groups). The surface braid group $SBr(S_{\mathbb{N}})$ of a decorated surface $S_{\mathbb{N}}$ is the subgroup of $MCG(S_{\mathbb{N}})$ generated by the twists of simple L-arcs and the braid twists.

It follows directly from the definitions that the braid twist group $BT(S_{\mathbb{N}})$ is the subgroup of $SBr(S_{\mathbb{N}})$ generated by the braid twists. We shall use the following known presentation of $SBr(S_{\mathbb{N}})$ to obtain a presentation of $BT(S_{\mathbb{N}})$ later.

PROPOSITION 2.7 [3]. The group $SBr(S_{\mathbb{N}})$ admits the following presentation.

- Generators: $\sigma_1, \dots, \sigma_{\mathbb{N}-1}, \delta_1, \dots, \delta_{2g+b-1}$ (see Figure 4).
- Relations: for $1 \leq i, j \leq \mathbb{N} - 1$ and $1 \leq r, s \leq 2g + b - 1$,

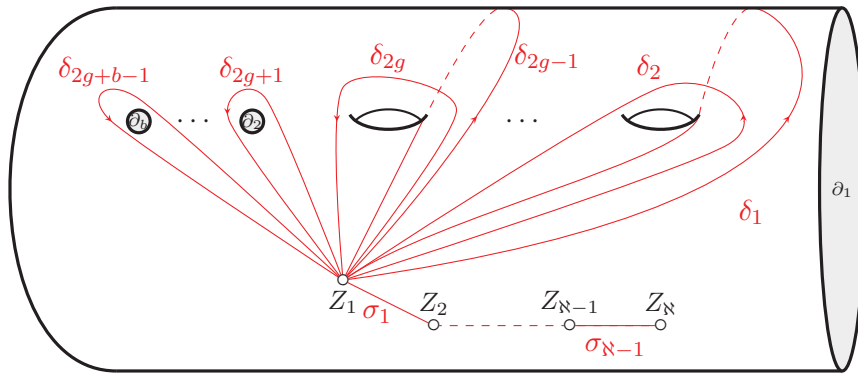


FIGURE 4 (colour online). Generators for $SBr(S_{\aleph})$.

$Co(\sigma_i, \sigma_j)$	if $ i - j > 1$
$Br(\sigma_i, \sigma_j)$	if $ i - j = 1$
$Co(\sigma_i, \delta_r)$	if $i \neq 1$
$Co(\delta_r, \sigma_1 \delta_r \sigma_1)$	
$Co(\delta_r^{\sigma_1}, \delta_s)$	if $s < r$, with $s \notin 2\mathbb{N}_{\leq g} - 1$ or with $s \neq r - 1$
$S Co(\sigma_1; \delta_{s+1}, \delta_s)$	if $s \in 2\mathbb{N}_{\leq g} - 1$.

2.2. Decorated marked surfaces and quivers with potential

A *marked surface* \mathbf{S} without punctures in the sense of [8] is a pair (S, \mathbf{M}) of a compact connected oriented surface S with non-empty boundary and a finite set \mathbf{M} of marked points on the boundary ∂S satisfying that each connected component of ∂S contains at least one marked point.

An (open) *arc* in \mathbf{S} is a curve (up to isotopy) on S whose interior lies in $S - \partial S$, whose endpoints are marked points in \mathbf{M} , and which is neither homotopic to a boundary segment nor to a point. A *triangulation* of \mathbf{S} is a maximal collection of simple arcs in \mathbf{S} which do not cross each other in the interior of \mathbf{S} . Any triangulation of \mathbf{S} consists of $n = 6g + 3b + |\mathbf{M}| - 6$ arcs and divides \mathbf{S} into $\frac{2n+|\mathbf{M}|}{3}$ triangles (cf. [8, Proposition 2.10; 20, equation (2.9)]).

Putting the notion of decorated surface and marked surface together, we have the following.

DEFINITION 2.8 [20, Definition 3.1]. A *decorated marked surface* \mathbf{S}_{Δ} is a marked surface \mathbf{S} with a set Δ of $\aleph = \frac{2n+|\mathbf{M}|}{3}$ decorating points in the interior of \mathbf{S} . A triangulation \mathbf{T} of \mathbf{S}_{Δ} is induced by a triangulation of \mathbf{S} such that each triangle contains exactly one decorating point.

For a decorated marked surface $\mathbf{S}_{\Delta} = (S, \mathbf{M}, \Delta)$, when forgetting the marked points, it becomes a decorated surface S_{\aleph} (where $\aleph = \frac{2n+|\mathbf{M}|}{3}$). In this case, let

$$BT(\mathbf{S}_{\Delta}) := BT(S_{\aleph}) \text{ and } MCG(\mathbf{S}_{\Delta}) := MCG(S_{\aleph}).$$

On the other hand, one can turn a decorated surface S_{\aleph} into a decorated marked surface \mathbf{S}_{Δ} , by adding certain number of marked points, when $\aleph \geq 4g + 3b - 4$.

Let \mathbf{T} be a triangulation of \mathbf{S} . For any arc $\gamma \in \mathbf{T}$, the dual $\gamma^* = \gamma_{\mathbf{T}}^*$ of γ with respect to \mathbf{T} is the unique closed arc in $CA(\mathbf{S}_{\Delta})$ which intersects γ once and does not intersect any other arcs in \mathbf{T} . Let \mathbf{T}^* be the dual of \mathbf{T} , that is, \mathbf{T}^* consists of the duals of the arcs in \mathbf{T} . Let $BT(\mathbf{T})$ be the subgroup of $BT(\mathbf{S}_{\Delta})$ generated by the braid twists $B_{\gamma^*}, \gamma^* \in \mathbf{T}^*$.

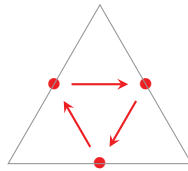


FIGURE 5 (colour online). The (sub)quiver (with potential) associated to a triangle.

LEMMA 2.9 [20, Proposition 4.13; 23, Proposition 2.3]. When specifying a triangulation \mathbf{T} of \mathbf{S}_Δ , the group $\text{BT}(\mathbf{T})$ equals $\text{BT}(\mathbf{S}_\Delta)$.

We recall the notion of quiver with potential from [6]. A quiver is a quadruple $Q = (Q_0, Q_1, s, t)$, where Q_0 is the set of vertices of Q , Q_1 is the set of arrows of Q , and $s, t : Q_1 \rightarrow Q_0$ send an arrow a of Q to its starting vertex $s(a)$ and its ending vertex $t(a)$, respectively. The notation $a : i \rightarrow j$ denotes that a is an arrow of Q with $i = s(a)$ and $j = t(a)$. A path of length d is a sequence $a_1 a_2 \cdots a_d$ with $t(a_i) = s(a_{i+1})$. A path $a_1 a_2 \cdots a_d$ is called a cycle if $t(a_d) = s(a_1)$. A cycle of length d is called a d -cycle. Let \mathbf{k} an algebraically closed field. A potential W is a linear combination of finite cycles in Q up to cyclic permutation. We call the pair (Q, W) a quiver with potential.

Let i be a vertex of Q such that there are no 2-cycles through it. The pre-mutation $\tilde{\mu}_i(Q, W)$ of (Q, W) at i is a new quiver with potential (Q', W') constructed as follows.

- The new quiver Q' is obtained from Q by

Step 1. For any pair of arrows $a : j \rightarrow i$ and $b : i \rightarrow l$, add a new arrow $[ab] : j \rightarrow l$.

Step 2. Reverse each arrow a starting or ending at i , that is, replace a with a new arrow a^* with $s(a^*) = t(a)$ and $t(a^*) = s(a)$.

- The new potential $W' = [W] + \sum b^* a^* [ab]$, where $[W]$ is obtained from W by replacing each composition ab of arrows a and b with ab going through i by $[ab]$, and the sum runs over all composition ab going through i .

We assume that any arrow in a 2-cycle in W' does not occur in any other item in W' . Then the mutation of (Q, W) at the vertex i , denoted by $\mu_i(Q, W)$, is obtained from (Q', W') by removing all 2-cycles from W' and removing all arrows in these 2-cycles from Q' . We remark that in the general case (that is, without the above assumption) the mutation of quiver with potential need to be defined via the notion of right equivalence. However, in the marked surface with punctures case, this assumption always holds (cf. Case 1 in the proof of [19, Theorem 30]), which makes the definition simpler.

Let \mathbf{T} be a triangulation of \mathbf{S}_Δ . There is an associated quiver with potential $(Q_{\mathbf{T}}, W_{\mathbf{T}})$ [8, 19], constructed as follows.

- The vertices of $Q_{\mathbf{T}}$ are (indexed by) the arcs in \mathbf{T} .
- There is an arrow from i to j whenever there is a triangle in \mathbf{T} having i and j as edges with j following i in the clockwise orientation (which is induced by the orientation of \mathbf{S}). For instance, the quiver for a triangle is shown in Figure 5.
- Each triangle in \mathbf{T} yields a unique 3-cycle up to cyclic permutation. The potential $W_{\mathbf{T}}$ is the sum of all such 3-cycles.

DEFINITION 2.10. Let γ be an arc in a triangulation \mathbf{T} of \mathbf{S}_Δ . The arc $\gamma^b = \gamma^b(\mathbf{T})$ is obtained from γ by clockwise moving its endpoints along the quadrilateral in \mathbf{T} whose diagonal is γ (cf. Figure 6), to the next marked points. The backward flip of a triangulation \mathbf{T} of \mathbf{S}_Δ at $\gamma \in \mathbf{T}$

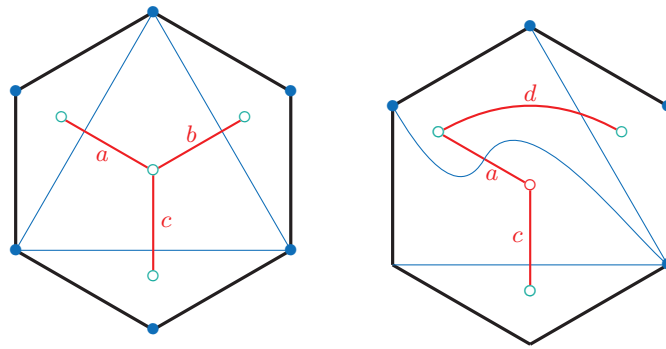


FIGURE 6 (colour online). A backward flip (where $d = a^b = b^a$).

is the triangulation $\mu_\gamma^b(\mathbf{T})$ obtained from \mathbf{T} by replacing the arc γ with γ^b . Similarly, we have the notion of forward flip, which is the inverse of backward flip, that is, $\mathbf{T} = \mu_{\gamma^b}^\#(\mu_\gamma^b(\mathbf{T}))$.

Flip of triangulations is compatible with mutation of quivers with potential in the following sense.

PROPOSITION 2.11 [19, Theorem 30]. *Let γ be an arc in a triangulation \mathbf{T} of \mathbf{S}_Δ . Then the quivers with potential $(Q_{\mu_\gamma^b(\mathbf{T})}, W_{\mu_\gamma^b(\mathbf{T})})$ and $(Q_{\mu_\gamma^\#(\mathbf{T})}, W_{\mu_\gamma^\#(\mathbf{T})})$ coincide with the quiver with potential $\mu_\gamma(Q_{\mathbf{T}}, W_{\mathbf{T}})$.*

2.3. Spherical twists on 3-Calabi–Yau categories

Let (Q, W) be a quiver with potential. For an arrow a of Q and a cycle $a_1 a_2 \cdots a_s$ in Q , define $\partial_a(a_1 a_2 \cdots a_s) = \sum_{a_i=a} a_{i+1} \cdots a_s a_1 \cdots a_{i-1}$. This extends linearly to $\partial_a W$.

DEFINITION 2.12. The complete Ginzburg dg algebra $\Gamma = \Gamma(Q, W)$ is constructed as follows [10]. Let \tilde{Q} be the graded quiver with the same vertices as Q and whose arrows are

- the arrows of Q (with degree 0);
- an arrow $a^* : j \rightarrow i$ of degree -1 for each arrow $a : i \rightarrow j$ of Q ;
- a loop $t_i : i \rightarrow i$ of degree -2 for each vertex i of Q .

The underlying graded algebra of Γ is the completion of the graded path algebra $\mathbf{k}\tilde{Q}$ and the differential d of Γ is linearly determined by the formulas $d(a^*) = \partial_a W$ and $d(\sum_{i: \text{a vertex of } Q} t_i) = \sum_{a: \text{an arrow of } Q} (aa^* - a^*a)$.

Denote by $\mathcal{D}(\Gamma)$ the derived category of Γ . The finite-dimensional derived category $\mathcal{D}_{fd}(\Gamma)$ is the full subcategory of $\mathcal{D}(\Gamma)$ consisting of those dg Γ -modules whose homology is of finite total dimension. The category $\mathcal{D}_{fd}(\Gamma)$ is a 3-Calabi–Yau triangulated category [14] in the sense that for any objects X, Y of $\mathcal{D}_{fd}(\Gamma)$, there is a functorial isomorphism

$$\text{Hom}(X, Y) \cong D \text{Hom}(Y, X[3]),$$

where $D = \text{Hom}_{\mathbf{k}}(-, \mathbf{k})$.

DEFINITION 2.13 (Spherical twists [24]). An object S of $\mathcal{D}_{fd}(\Gamma)$ is called (3-)spherical provided that

$$\text{Hom}_{\mathcal{D}_{fd}(\Gamma)}(S, S[r]) \cong \begin{cases} \mathbf{k} & \text{if } r = 0 \text{ or } 3 \\ 0 & \text{otherwise.} \end{cases}$$

For a spherical object S , there is an induced autoequivalence ϕ_S , called spherical twist, of $\mathcal{D}_{fd}(\Gamma)$ defined by the triangle [24]:

$$\phi_S(X)[-1] \rightarrow S \otimes \text{Hom}^\bullet(S, X) \rightarrow X \rightarrow \phi_S(X)$$

for $X \in \mathcal{D}_{fd}(\Gamma)$.

For any vertex i of Q , the corresponding simple $b\Gamma$ -module S_i is a spherical object in $\mathcal{D}_{fd}(\Gamma)$ (cf. [15, Lemma 2.15]). Denote by $\text{Sim } \mathcal{H}_\Gamma$ the set of simple Γ -modules. The spherical twist group $\text{ST}(\Gamma)$ of $\mathcal{D}_{fd}(\Gamma)$ is defined to be the subgroup of the autoequivalence group $\text{Aut } \mathcal{D}_{fd}(\Gamma)$ generated by ϕ_S , $S \in \text{Sim } \mathcal{H}_\Gamma$. Set $\text{Sph}(\Gamma) = \text{ST}(\Gamma) \cdot \text{Sim } \mathcal{H}_\Gamma$. Then we have that $\text{ST}(\Gamma)$ is also generated by all ϕ_X , $X \in \text{Sph}(\Gamma)$.

For the quiver with potential $(Q_{\mathbf{T}}, W_{\mathbf{T}})$ associated to a triangulation \mathbf{T} of \mathbf{S}_Δ , denote the corresponding Ginzburg dg algebra by $\Gamma_{\mathbf{T}}$.

THEOREM 2.14 [20, Theorem 1]. *There is an isomorphism*

$$\iota: \text{BT}(\mathbf{T}) \rightarrow \text{ST}(\Gamma_{\mathbf{T}}), \tag{2.2}$$

sending the standard generators (that is, the braid twists of closed arcs in \mathbf{T}^) to the standard generators (that is, the spherical twists of the simple $\Gamma_{\mathbf{T}}$ -modules).*

Let \mathbf{T}' be a (forward or backward) flip of \mathbf{T} . Since $(Q_{\mathbf{T}'}, W_{\mathbf{T}'})$ is the mutation of $(Q_{\mathbf{T}}, W_{\mathbf{T}})$ at some vertex (Proposition 2.11), by the main theorem of [15], there is a triangle equivalence between $\mathcal{D}(\Gamma_{\mathbf{T}'})$ and $\mathcal{D}(\Gamma_{\mathbf{T}})$, which restricts to a triangle equivalence $\mathcal{D}_{fd}(\Gamma_{\mathbf{T}'})$ and $\mathcal{D}_{fd}(\Gamma_{\mathbf{T}})$. Then the category $\mathcal{D}_{fd}(\Gamma_{\mathbf{T}})$ is independent of the chosen triangulation \mathbf{T} up to triangle equivalence. Hence one can use $\mathcal{D}(\mathbf{S})$ to denote $\mathcal{D}_{fd}(\Gamma_{\mathbf{T}})$.

3. An alternative presentation of surface braid group

This section devotes to give an alternative (positive) presentation of $\text{SBr}(\mathbf{S}_\aleph)$, which is derived from the presentation in Proposition 2.7 and will be used in the next section. Define ε_r , $1 \leq r \leq 2g + b - 1$, recursively by

$$\varepsilon_r = \begin{cases} \delta_r \varepsilon_{r-1} & \text{if } r \notin 2\mathbb{N}_{\leq g}, \\ \delta_r \varepsilon_{r-2} & \text{if } r \in 2\mathbb{N}_{\leq g}, \end{cases}$$

where for convenience, ε_0 is taken to be the identity. Conversely, we have

$$\delta_r = \begin{cases} \varepsilon_r \varepsilon_{r-1} & \text{if } r \notin 2\mathbb{N}_{\leq g} \\ \varepsilon_r \varepsilon_{r-2} & \text{if } r \in 2\mathbb{N}_{\leq g}. \end{cases}$$

Clearly, σ_i , $1 \leq i \leq \aleph - 1$ and ε_r , $1 \leq r \leq 2g + b - 1$, form new generators for $\text{SBr}(\mathbf{S}_\aleph)$, which are illustrated in Figure 7, where, to make it reader friendly, we use the following notation:

$$\xi_{-r} = \varepsilon_{2r-1}, \xi_r = \varepsilon_{2r}, \zeta_l = \varepsilon_{2g+l}, \quad \text{for } 1 \leq r \leq g \text{ and } 1 \leq l \leq b - 1.$$

PROPOSITION 3.1. *The group $\text{SBr}(\mathbf{S}_\aleph)$ admits the following presentation.*

- *Generators:* σ_i , $1 \leq i \leq \aleph - 1$, ε_r , $1 \leq r \leq 2g + b - 1$.

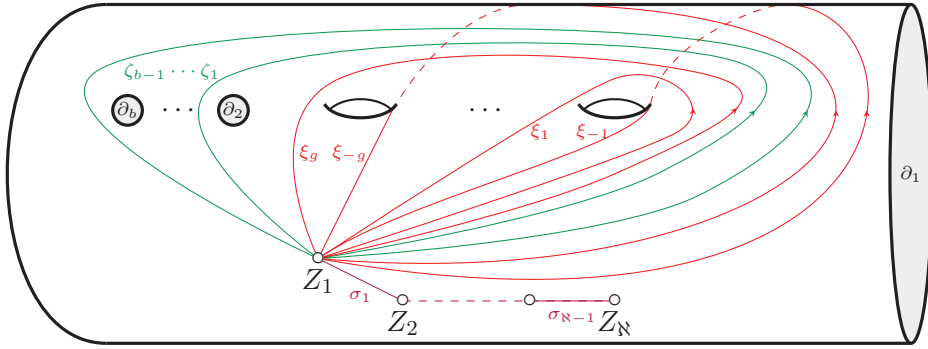


FIGURE 7 (colour online). Alternative generators for $SBr(S_{\aleph})$.

- Relations: for $1 \leq i, j \leq \aleph - 2$ and $1 \leq r, s \leq 2g + b - 1$,

$$\text{Co}(\sigma_i, \sigma_j) \quad \text{if } |i - j| > 1 \tag{3.1}$$

$$\text{Br}(\sigma_i, \sigma_j) \quad \text{if } |i - j| = 1 \tag{3.2}$$

$$\text{Co}(\sigma_i, \varepsilon_r) \quad \text{if } i \neq 1 \tag{3.3}$$

$$\text{Co}(\varepsilon_r, \sigma_1 \varepsilon_r \sigma_1) \tag{3.4}$$

$$\text{Co}(\varepsilon_s, \sigma_1 \varepsilon_r \sigma_1) \quad \text{if } s < r \text{ with } s \notin 2\mathbb{N}_{\leq g} - 1 \tag{3.5}$$

$$\text{SCo}(\sigma_1; \varepsilon_r, \varepsilon_s) \quad \text{if } s < r \text{ with } s \in 2\mathbb{N}_{\leq g} - 1. \tag{3.6}$$

Proof. We need to prove that the relations in Proposition 2.7 are equivalent to those in Proposition 3.1. The relations common to both presentations are

$$\begin{aligned} \text{Co}(\sigma_i, \sigma_j) & \quad \text{if } |i - j| > 1 \\ \text{Br}(\sigma_i, \sigma_j) & \quad \text{if } |i - j| = 1. \end{aligned}$$

By the construction of ε_r , it is easy to see the equivalence between the relations $\text{Co}(\sigma_i, \delta_r)$ and $\text{Co}(\sigma_i, \varepsilon_r)$ for any $i \neq 1$.

Now we prove that the relations in Proposition 2.7 imply the other relations in Proposition 3.1. First, we show a useful relation

$$\text{Co}(\delta_r^{\sigma_1}, \varepsilon_s) \quad \text{if } s < r \text{ with } s \notin 2\mathbb{N}_{\leq g} - 1 \text{ or with } s \neq r - 1,$$

which will be used for many times. Indeed, by construction, we have $\varepsilon_s = \delta_s \delta_{r_1} \cdots \delta_{r_m}$ with $s > r_1 > r_2 > \cdots > r_m$. So each $r_i < r - 1$. Then we have the following relations from Proposition 2.7: $\text{Co}(\delta_r^{\sigma_1}, \delta_s)$ and $\text{Co}(\delta_r^{\sigma_1}, \delta_{r_i})$, $1 \leq i \leq m$, which imply the required relation.

To show $\text{Co}(\varepsilon_r, \sigma_1 \varepsilon_r \sigma_1)$, use induction on r , starting with the trivial case $r = 0$. Assume $\text{Co}(\varepsilon_r, \sigma_1 \varepsilon_r \sigma_1)$ holds for any $r < t$ with some $t > 0$. Consider the case $r = t$. By construction, $\varepsilon_t = \delta_t \varepsilon_{t-x}$, where $x = 1$ if $t \notin 2\mathbb{N}_{\leq g}$ or $x = 2$ if $t \in 2\mathbb{N}_{\leq g}$. So $t - x \notin 2\mathbb{N}_{\leq g} - 1$, which implies that we have the useful relation $\text{Co}(\delta_t^{\sigma_1}, \delta_{t-x})$. Hence we have

$$\begin{aligned} \varepsilon_t \sigma_1 \varepsilon_t \sigma_1 &= \delta_t \varepsilon_{t-x} \sigma_1 \delta_t \varepsilon_{t-x} \sigma_1 \\ &= \delta_t \sigma_1 \delta_t \sigma_1 \varepsilon_{t-x} \sigma_1 \varepsilon_{t-x} \sigma_1 \\ &= \delta_t \sigma_1 \delta_t \varepsilon_{t-x} \sigma_1 \varepsilon_{t-x} \\ &= \sigma_1 \delta_t \sigma_1 \delta_t \varepsilon_{t-x} \sigma_1 \varepsilon_{t-x} \\ &= \sigma_1 \delta_t \varepsilon_{t-x} \sigma_1 \delta_t \varepsilon_{t-x} \\ &= \sigma_1 \varepsilon_t \sigma_1 \varepsilon_t, \end{aligned}$$

where the first and the last equalities are due to $\varepsilon_t = \delta_t \varepsilon_{t-x}$, the second and the fifth equalities use the relation $\text{Co}(\delta_t^{\sigma_1}, \varepsilon_{t-x})$, the third equality uses the inductive assumption $\text{Co}(\varepsilon_{t-x}, \sigma_1 \varepsilon_{t-x} \sigma_1)$ and the fourth equality uses the relation $\text{Co}(\delta_t, \sigma_1 \delta_t \sigma_1)$ from Proposition 2.7.

To show $\text{Co}(\varepsilon_s, \sigma_1 \varepsilon_r \sigma_1)$ for $s < r$ with $s \notin 2\mathbb{N}_{\leq g} - 1$, fix s and use induction on r , starting with the extreme case $r = s$ which was proved above. Assume $\text{Co}(\varepsilon_s, \sigma_1 \varepsilon_r \sigma_1)$ holds for any $r < t$ with some $t > s$. Consider the case $r = t$. By construction, $\varepsilon_t = \delta_t \varepsilon_{t-x}$, where $x = 1$ if $t \notin 2\mathbb{N}_{\leq g}$ or $x = 2$ otherwise. We claim $t - x \geq s$. Indeed, if $t = s + 1$ then $t \notin 2\mathbb{N}_{\leq g}$. So $x = 1$ and $t - x = s$; if $t \geq s + 2$ then $t - x \geq t - 2 \geq s$. Then we have

$$\begin{aligned} \varepsilon_s \sigma_1 \varepsilon_t \sigma_1 &= \varepsilon_s \sigma_1 \delta_t \varepsilon_{t-x} \sigma_1 \\ &= \sigma_1 \delta_t \sigma_1 \varepsilon_s \sigma_1 \varepsilon_{t-x} \sigma_1 \\ &= \sigma_1 \delta_t \varepsilon_{t-x} \sigma_1 \varepsilon_s \\ &= \sigma_1 \varepsilon_t \sigma_1 \varepsilon_s, \end{aligned}$$

where the first and the last equalities are due to $\varepsilon_t = \delta_t \varepsilon_{t-x}$, the second equality uses the useful relation $\text{Co}(\delta_t^{\sigma_1}, \varepsilon_s)$ and the third equality uses the inductive assumption $\text{Co}(\varepsilon_s, \sigma_1 \varepsilon_{t-x} \sigma_1)$ by $t - x \geq s$.

To show $\text{SCo}(\sigma_1; \varepsilon_r, \varepsilon_s)$ for $s < r$ with $s \in 2\mathbb{N}_{\leq g} - 1$, fix s and use induction on r . Write $s = 2i - 1$ for some $1 \leq i \leq g$. We first use induction on i to prove the case $r = s + 1$, starting with the case $i = 1$, where by construction, $\varepsilon_s = \delta_1$ and $\varepsilon_r = \delta_2$, so $\text{SCo}(\sigma_1; \varepsilon_r, \varepsilon_s)$ becomes $\text{SCo}(\sigma_1; \delta_2, \delta_1)$ from Proposition 2.7. Assume $\text{SCo}(\sigma_1; \varepsilon_{2i}, \varepsilon_{2i-1})$ holds for any $i < j$ with some $1 < j \leq g$. Consider the case $i = j$. By construction, $\varepsilon_{2j-1} = \delta_{2j-1} \varepsilon_{2j-2}$ and $\varepsilon_{2j} = \delta_{2j} \varepsilon_{2j-2}$. Then we have

$$\begin{aligned} \varepsilon_{2j-1} \sigma_1 \varepsilon_{2j} &= \delta_{2j-1} \varepsilon_{2j-2} \sigma_1 \delta_{2j} \varepsilon_{2j-2} \\ &= \delta_{2j-1} \sigma_1 \delta_{2j} \sigma_1 \varepsilon_{2j-2} \sigma_1 \varepsilon_{2j-2} \\ &= \sigma_1 \delta_{2j} \sigma_1 \delta_{2j-1} \varepsilon_{2j-2} \sigma_1 \varepsilon_{2j-2} \\ &= \sigma_1 \delta_{2j} \sigma_1 \delta_{2j-1} \sigma_1 \varepsilon_{2j-2} \sigma_1 \varepsilon_{2j-2} \sigma_1 \\ &= \sigma_1 \delta_{2j} \varepsilon_{2j-2} \sigma_1 \delta_{2j-1} \varepsilon_{2j-2} \sigma_1 \\ &= \sigma_1 \varepsilon_{2j} \sigma_1 \varepsilon_{2j-1} \sigma_1, \end{aligned}$$

where the first and the last equalities are due to both $\varepsilon_{2j-1} = \delta_{2j-1} \varepsilon_{2j-2}$ and $\varepsilon_{2j} = \delta_{2j} \varepsilon_{2j-2}$, the second equality uses the useful relation $\text{Co}(\delta_{2j}^{\sigma_1}, \varepsilon_{2j-2})$, the third equality uses the relation $\text{SCo}(\sigma_1; \delta_{2j}, \delta_{2j-1})$ from Proposition 2.7, the fourth equality uses the relation $\text{Co}(\varepsilon_{2j-2}, \sigma_1 \varepsilon_{2j-2} \sigma_1)$ proved above and the fifth equality uses the useful relation $\text{Co}(\delta_{2j-1}^{\sigma_1}, \varepsilon_{2j-2})$. Thus, the proof for the case $r = s + 1$ is complete. Assume now $\text{SCo}(\sigma_1; \varepsilon_r, \varepsilon_s)$ holds for any $r < t$ with some $t > s + 1$. Consider the case $r = t$. By construction, $\varepsilon_t = \delta_t \varepsilon_{t-x}$, where $x = 1$ if $t \notin 2\mathbb{N}_{\leq g}$ or $x = 2$ otherwise. If $t = s + 2$ then $t \notin 2\mathbb{N}_{\leq g}$, which implies $x = 1$ and $t - x > s$; if $t > s + 2$ then $t - x \geq t - 2 > s$. Hence we always have $t - x > s$. Then we have

$$\begin{aligned} \varepsilon_s \sigma_1 \varepsilon_t &= \varepsilon_s \sigma_1 \delta_t \varepsilon_{t-x} \\ &= \sigma_1 \delta_t \sigma_1 \varepsilon_s \sigma_1 \varepsilon_{t-x} \\ &= \sigma_1 \delta_t \varepsilon_{t-x} \sigma_1 \varepsilon_s \sigma_1 \\ &= \sigma_1 \varepsilon_t \sigma_1 \varepsilon_s \sigma_1, \end{aligned}$$

where the first and the last equalities are due to $\varepsilon_t = \delta_t \varepsilon_{t-x}$, the second equality uses the useful relation $\text{Co}(\delta_t^{\sigma_1}, \varepsilon_s)$ by $t > s + 1$ and the third equality uses the inductive assumption $\text{SCo}(\sigma_1; \varepsilon_{t-x}, \varepsilon_s)$ by $t - x > s$.

Conversely, we prove that the relations in Proposition 3.1 imply the other relations in Proposition 2.7. To show $\text{Co}(\delta_r, \sigma_1 \delta_r \sigma_1)$, by construction, $\delta_r = \varepsilon_r \varepsilon_{r-x}$, where $x = 1$ if $r \notin 2\mathbb{N}_{\leq g}$ or $x = 2$ if $r \in 2\mathbb{N}_{\leq g}$. In any case, we always have $r - x \notin 2\mathbb{N}_{\leq g} - 1$, which implies $\text{Co}(\varepsilon_{r-x}, \sigma_1 \varepsilon_r \sigma_1)$ from (3.5). From relation (3.4), we have $\text{Co}(\varepsilon_{r-x}, \sigma_1 \varepsilon_{r-x} \sigma_1)$ and

$\text{Co}(\varepsilon_r, \sigma_1 \varepsilon_r \sigma_1)$, which imply $\underline{\varepsilon_{r-x} \sigma_1 \varepsilon_{r-x} \sigma_1} = \sigma_1 \underline{\varepsilon_{r-x} \sigma_1 \varepsilon_{r-x}}$ and $\varepsilon_r \sigma_1 \varepsilon_r \sigma_1 = \sigma_1 \varepsilon_r \sigma_1 \varepsilon_r$, respectively. Combining these two equalities, we have

$$\varepsilon_r \sigma_1 \varepsilon_r \sigma_1 \underline{\varepsilon_{r-x} \sigma_1 \varepsilon_{r-x} \sigma_1} = \sigma_1 \varepsilon_r \sigma_1 \varepsilon_r \sigma_1 \underline{\varepsilon_{r-x} \sigma_1 \varepsilon_{r-x}}.$$

Now using $\text{Co}(\varepsilon_{r-x}, \sigma_1 \varepsilon_r \sigma_1)$, we have $\underline{\varepsilon_r \varepsilon_{r-x} \sigma_1 \varepsilon_r \varepsilon_{r-x} \sigma_1} = \sigma_1 \underline{\varepsilon_r \varepsilon_{r-x} \sigma_1 \varepsilon_r \varepsilon_{r-x}}$. Then due to $\delta_r = \underline{\varepsilon_r \varepsilon_{r-x}}$, we have $\delta_r \sigma_1 \delta_r \sigma_1 = \sigma_1 \delta_r \sigma_1 \delta_r$, which is $\text{Co}(\delta_r, \sigma_1 \delta_r \sigma_1)$.

We also need to prove the useful relation

$$\text{Co}(\delta_r^{\underline{\sigma_1}}, \varepsilon_s) \quad \text{if } s < r \text{ with } s \notin 2\mathbb{N}_{\leq g} - 1 \text{ or with } s \neq r - 1.$$

By construction, $\delta_r = \underline{\varepsilon_r \varepsilon_{r-x}}$, where $x = 1$ if $r \notin 2\mathbb{N}_{\leq g}$ or $x = 2$ if $r \in 2\mathbb{N}_{\leq g}$. There are two cases depending on whether s is in $2\mathbb{N}_{\leq g} - 1$. If $s \notin 2\mathbb{N}_{\leq g} - 1$, it is easy to see that $r - x \geq s$. Then we have

$$\begin{aligned} \sigma_1 \delta_r \sigma_1 \underline{\varepsilon_s \sigma_1} &= \sigma_1 \varepsilon_r \varepsilon_{r-x} \sigma_1 \underline{\varepsilon_s \sigma_1} \\ &= \sigma_1 \varepsilon_r \sigma_1 \underline{\varepsilon_s \sigma_1 \varepsilon_{r-x}} \\ &= \varepsilon_s \sigma_1 \varepsilon_r \underline{\varepsilon_{r-x}} \\ &= \varepsilon_s \sigma_1 \delta_r, \end{aligned}$$

where the first and the last equalities are due to $\delta_r = \underline{\varepsilon_r \varepsilon_{r-x}}$, the second equality uses $\text{Co}(\varepsilon_s, \sigma_1 \varepsilon_{r-x} \sigma_1)$ from (3.4) if $r - x = s$ or from (3.5) if $r - x > s$, and the third equality uses $\text{Co}(\varepsilon_s, \sigma_1 \varepsilon_r \sigma_1)$ from (3.5). If $s \in 2\mathbb{N}_{\leq g} - 1$, then $s \neq r - 1$. In this case, it is easy to see that $r - x > s$. So we have both $\text{SCo}(\sigma_1; \varepsilon_{r-x}, \varepsilon_s)$ and $\text{SCo}(\sigma_1; \varepsilon_r, \varepsilon_s)$ from (3.6). Then we have

$$\begin{aligned} \sigma_1 \delta_r \sigma_1 \underline{\varepsilon_s \sigma_1} &= \sigma_1 \varepsilon_r \varepsilon_{r-x} \sigma_1 \underline{\varepsilon_s \sigma_1} \\ &= \sigma_1 \varepsilon_r \sigma_1 \underline{\varepsilon_s \sigma_1 \varepsilon_{r-x}} \\ &= \varepsilon_s \sigma_1 \varepsilon_r \underline{\varepsilon_{r-x}} \\ &= \varepsilon_s \sigma_1 \delta_r, \end{aligned}$$

where the first and the last equalities are due to $\delta_r = \underline{\varepsilon_r \varepsilon_{r-x}}$, the second and the third equalities use $\text{SCo}(\sigma_1; \varepsilon_{r-x}, \varepsilon_s)$ and $\text{SCo}(\sigma_1; \varepsilon_r, \varepsilon_s)$, respectively.

To show $\text{Co}(\delta_r^{\underline{\sigma_1}}, \delta_s)$ for $s < r$, with $s \notin 2\mathbb{N}_{\leq g} - 1$ or with $s \neq r - 1$, by construction, we have $\delta_s = \underline{\varepsilon_s \varepsilon_{s-x}}$, where $x = 1$ or 2 . It follows that $s - x < r - 1$. Hence we have $\text{Co}(\delta_r^{\underline{\sigma_1}}, \varepsilon_s)$ and $\text{Co}(\delta_r^{\underline{\sigma_1}}, \varepsilon_{s-x})$, which imply $\text{Co}(\delta_r^{\underline{\sigma_1}}, \delta_s)$.

To show $\text{SCo}(\sigma_1; \delta_{s+1}, \delta_s)$ for $s \in 2\mathbb{N}_{\leq g} - 1$, by construction, we have $\delta_s = \underline{\varepsilon_s \varepsilon_{s-1}}$ and $\delta_{s+1} = \underline{\varepsilon_{s+1} \varepsilon_{s-1}}$. Hence we have

$$\begin{aligned} \sigma_1 \delta_{s+1} \sigma_1 \delta_s \sigma_1 &= \sigma_1 \varepsilon_{s+1} \underline{\varepsilon_{s-1} \sigma_1 \varepsilon_s \varepsilon_{s-1} \sigma_1} \\ &= \sigma_1 \varepsilon_{s+1} \sigma_1 \underline{\varepsilon_s \sigma_1 \varepsilon_{s-1} \sigma_1 \varepsilon_{s-1} \sigma_1} \\ &= \varepsilon_s \sigma_1 \varepsilon_{s+1} \underline{\varepsilon_{s-1} \sigma_1 \varepsilon_{s-1} \sigma_1} \\ &= \varepsilon_s \sigma_1 \varepsilon_{s+1} \sigma_1 \underline{\varepsilon_{s-1} \sigma_1 \varepsilon_{s-1}} \\ &= \varepsilon_s \underline{\varepsilon_{s-1} \sigma_1 \varepsilon_{s+1} \varepsilon_{s-1}} \\ &= \delta_s \sigma_1 \delta_{s+1}, \end{aligned}$$

where the first and the last equalities are due to both $\delta_s = \underline{\varepsilon_s \varepsilon_{s-1}}$ and $\delta_{s+1} = \underline{\varepsilon_{s+1} \varepsilon_{s-1}}$, the second equality uses $\text{Co}(\varepsilon_{s-1}, \sigma_1 \varepsilon_s \sigma_1)$ from (3.5), the third equality uses $\text{SCo}(\sigma_1; \varepsilon_{s+1}, \varepsilon_s)$ from (3.6), the fourth equality uses $\text{Co}(\varepsilon_{s-1}, \sigma_1 \varepsilon_{s-1} \sigma_1)$ from (3.4) and the fifth equality uses $\text{Co}(\varepsilon_{s-1}, \sigma_1 \varepsilon_{s+1} \sigma_1)$ from (3.5). \square

4. A finite presentation for braid twist group

Recall that the braid twist group $\text{BT}(\mathbb{S}_{\aleph})$ of a decorated surface \mathbb{S}_{\aleph} is the subgroup of the surface braid group $\text{SBr}(\mathbb{S}_{\aleph})$ generated by the braid twists along closed arcs in $\text{CA}(\mathbb{S}_{\aleph})$. In the presentation of $\text{SBr}(\mathbb{S}_{\aleph})$ in Proposition 3.1, the generators σ_i , $1 \leq i \leq \aleph - 1$, are braid twists,

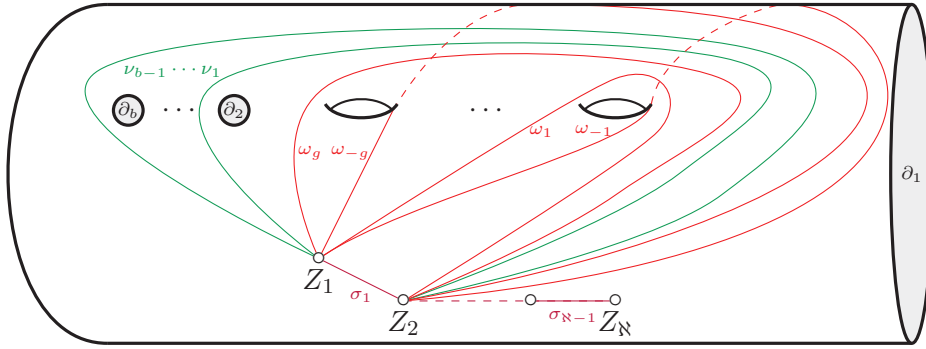


FIGURE 8 (colour online). Generators for $\text{BT}(\mathbb{S}_{\aleph})$.

while the generators ε_r , $1 \leq r \leq 2g + b - 1$, are not. So we introduce the following elements in $\text{BT}(\mathbb{S}_{\aleph})$:

$$\tau_r = \sigma_1^{\varepsilon_r}, \quad 1 \leq r \leq 2g + b - 1 \tag{4.1}$$

which, by Lemma 2.2, are the braid twists along the closed arcs $t_{\varepsilon_r}(\sigma_1)$, $1 \leq r \leq 2g + b - 1$, respectively. So as in Notation 2.5 we denote $t_{\varepsilon_r}(\sigma_1)$ also by τ_r . We illustrate these elements in Figure 8, where similarly as in Figure 7, we use the notation

$$\omega_{-j} = \tau_{2j-1}, \omega_j = \tau_{2j}, \nu_k = \tau_{2g+k}, \text{ for } 1 \leq j \leq g \text{ and } 1 \leq k \leq b - 1.$$

The main result in this section is the following presentation of $\text{BT}(\mathbb{S}_{\aleph})$.

THEOREM 4.1. *Suppose that either $\aleph \geq 5$, or $\aleph = 4$ and $2g + b - 1 \leq 2$. The braid twist group $\text{BT}(\mathbb{S}_{\aleph})$ has the following finite presentation.*

- Generators: σ_i , $1 \leq i \leq \aleph - 1$, τ_r , $1 \leq r \leq 2g + b - 1$.
- Relations: for $1 \leq i, j \leq \aleph - 1$ and $1 \leq r, s \leq 2g + b - 1$,

$$\text{Co}(\sigma_i, \sigma_j) \quad \text{if } |i - j| > 1 \tag{4.2}$$

$$\text{Br}(\sigma_i, \sigma_j) \quad \text{if } |i - j| = 1 \tag{4.3}$$

$$\text{Co}(\tau_r, \sigma_i) \quad \text{if } i > 2 \tag{4.4}$$

$$\text{Br}(\tau_r, x) \tag{4.5}$$

$$\text{Br}(\tau_r, y) \tag{4.6}$$

$$\text{Co}(\tau_r^y, \tau_s^x) \quad \text{if } s < r \text{ and } s \notin 2\mathbb{N}_{\leq g} - 1 \tag{4.7}$$

$$\text{Co}(\tau_r^y, \tau_s^x) \quad \text{if } s < r \text{ and } s \in 2\mathbb{N}_{\leq g} - 1, \tag{4.8}$$

where

$$x := \sigma_1^{\sigma_2} = \sigma_2^{\sigma_1}, \tag{4.9}$$

$$y := \sigma_3^{\sigma_2} = \sigma_2^{\sigma_3}. \tag{4.10}$$

Apply this result to the case of decorated marked surface, we have the following.

COROLLARY 4.2. *Let \mathbf{S}_{Δ} be a decorated marked surface with $|\Delta| \geq 4$. Then $\text{BT}(\mathbf{S}_{\Delta})$ has the above presentation.*

Proof. We only need to show that $2g + b - 1 \leq 2$ when $|\Delta| = 4$. For a decorated marked surface \mathbf{S}_Δ , if $|\Delta| = 4$ then $4g + 2b + |M| - 4 = 4$. Note that $|M| \geq b > 0$. So either $g = 1$ and $b = 1$, or $g = 0$ and $b \leq 2$. In each case, $2g + b - 1 \leq 2$ holds. \square

The remaining of the section devotes to prove Theorem 4.1. First we show σ_i and τ_r form a set of generators for $\text{BT}(\mathbf{S}_\aleph)$.

LEMMA 4.3. *The braid twist group $\text{BT}(\mathbf{S}_\aleph)$ is generated by σ_i , $1 \leq i \leq \aleph - 1$ and τ_r , $1 \leq r \leq 2g + b - 1$.*

Proof. Let N be the subgroup of $\text{BT}(\mathbf{S}_\aleph)$ generated by σ_i , $1 \leq i \leq \aleph - 1$, and τ_r , $1 \leq r \leq 2g + b - 1$. To complete the proof, we only need to show that the braid twist η is in N for any $\eta \in \text{CA}(\mathbf{S}_\aleph)$. Without loss of generality, suppose that Z_1 is an endpoint of η . As in Figure 8, we use the notation

$$\omega_{-j} = \tau_{2j-1}, \omega_j = \tau_{2j}, \nu_k = \tau_{2g+k}, \text{ for } 1 \leq j \leq g \text{ and } 1 \leq k \leq b - 1.$$

Add a marked point on each boundary component of \mathbf{S}_\aleph and take the set \mathbf{T} of (blue/cyan) open arcs in the upper picture in Figure 9 on \mathbf{S}_\aleph that divide \mathbf{S}_\aleph into two polygons: one (denoted by \mathbf{L}) contains the decorating point Z_1 and the other (denoted by \mathbf{R}) contains other decorating points.

Then the closed arcs (cf. the upper picture in Figure 9)

$$\{\sigma_1, \omega_{\pm j}, \nu_k \mid 1 \leq j \leq g, 1 \leq k \leq b - 1\}$$

form a ‘dual’ set $\overrightarrow{\mathbf{T}}$ of \mathbf{T} in the sense that each of which intersect exactly one of the open arcs in \mathbf{T} . Note that \overrightarrow{A}_g consists of b open arcs (where b is the number of boundary components of \mathbf{S}_\aleph), which are dual to ν_k , $1 \leq k \leq b - 1$, and ω_g , respectively (cf. the lower picture in Figure 9).

Use induction on $\text{Int}(\eta, \mathbf{T})$. The initial case is when $\text{Int}(\eta, \mathbf{T}) = 1$. Without loss of generality, suppose that the intersection is on \overrightarrow{A}_1 . Then the braid twist η and ω_1 (the dual of \overrightarrow{A}_1) only differ by an element of

$$\text{MCG}(\mathbf{R}_{\aleph-1}) = \langle \sigma_i \mid 2 \leq i \leq \aleph - 1 \rangle (\cong B_{\aleph-1}),$$

where $\mathbf{R}_{\aleph-1}$ is the right polygon \mathbf{R} with decorating points Z_i , $2 \leq i \leq \aleph$. Therefore, $\eta \in N$. Now consider the case when $\text{Int}(\eta, \mathbf{T}) = m \geq 2$. Let Z_q be the endpoint of η other than Z_1 . The intersections divide η into $m + 1$ segments L_0, \dots, L_m in order (so $L_0 \subset \mathbf{L}$ and $L_m \subset \mathbf{R}$). Note that except L_0 and L_m , L_t has both endpoints on arcs in \mathbf{T} . Choose a decorating point $Z \in \mathbf{R}$ that is not Z_q and connect a line l from Z in $\mathbf{R} - L_m$ (which is still a disk) to one of L_t , $1 \leq t \leq m - 1$ so that it does not intersect η except at the endpoint $Y \in L_t$ (cf. Figure 10). Then we can decompose η into α and β , where α is isotopy to $l \cup Y Z_q$ and β is isotopy to $l \cup Y Z_1$ such that

- $\alpha, \beta \in \text{CA}(\mathbf{S}_\aleph)$ whose intersection numbers with \mathbf{T} are less than m .
- As braid twists, $\eta = \beta^\alpha$.

By inductive assumption, we have $\alpha, \beta \in N$ and hence $\eta \in N$ as required. \square

By Lemmas 2.2 and 2.3, it is easy to see that all relations in Theorem 4.1 hold. Indeed, note that σ_i and σ_j are disjoint if $|i - j| > 1$ or are disjoint except sharing a common endpoint if $|i - j| = 1$, and note that τ_r and σ_i are disjoint if $i > 2$ (see Figure 8), so we have relations (4.2)–(4.4). By Lemma 2.2, the elements $x = \sigma_1^{\sigma_2}$ and $y = \sigma_3^{\sigma_2}$ are the braid twists along the closed arcs $B_{\sigma_2}(\sigma_1)$ and $B_{\sigma_2}(\sigma_3)$, respectively. As in Notation 2.5, we denote $\sigma_2(\sigma_1)$ and $\sigma_2(\sigma_3)$ also by x and y , respectively, see Figure 11. Then x (respectively, y) and τ_r are disjoint except sharing a common endpoint. So by Lemma 2.3, we have relations (4.5) and (4.6). For $s < r$ and

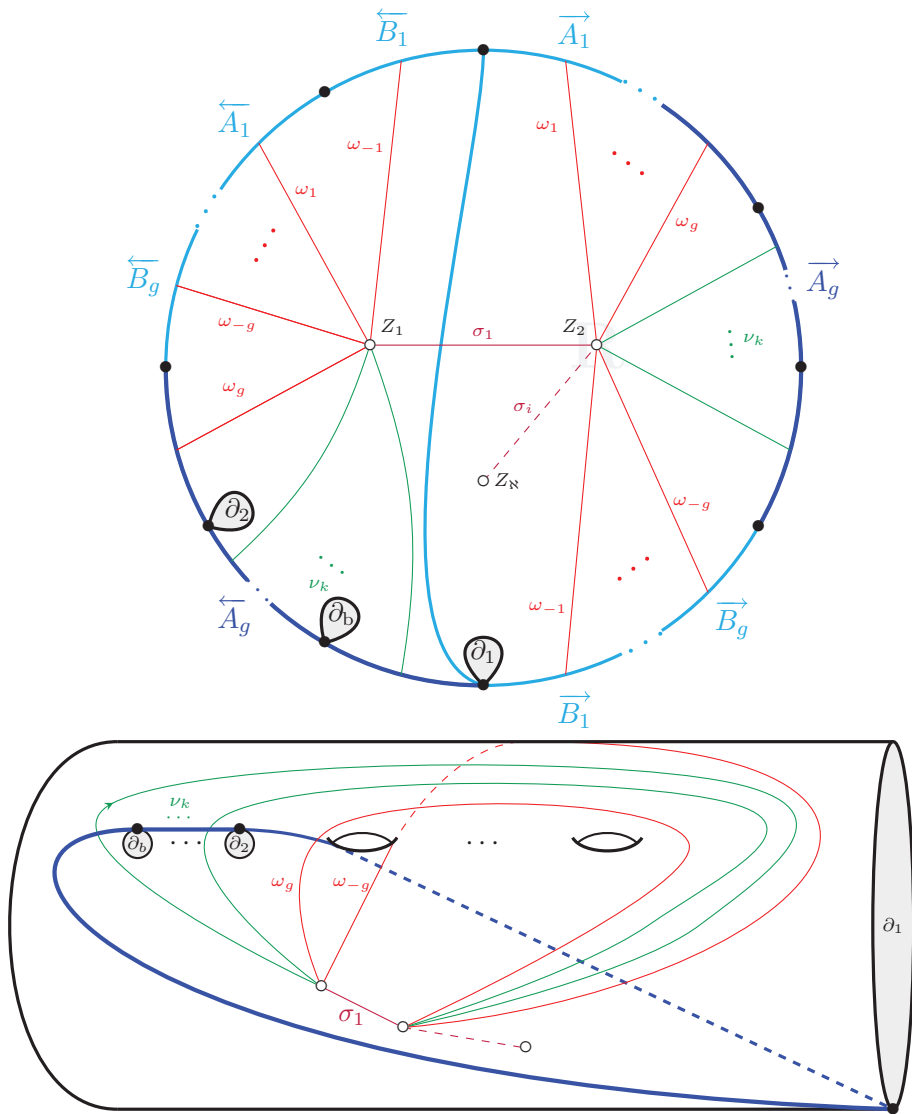


FIGURE 9 (colour online). Dividing S_N into two polygons.

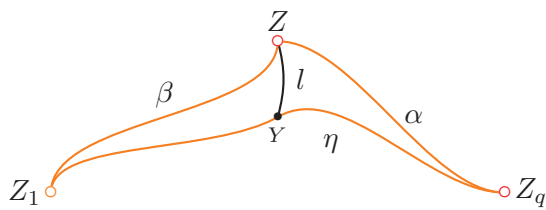


FIGURE 10 (colour online). Decomposing η .

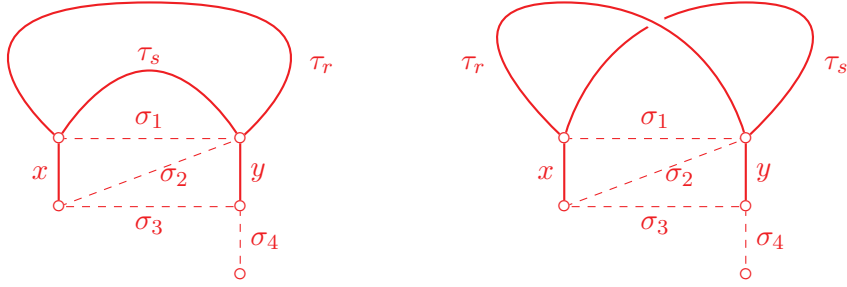


FIGURE 11 (colour online). $s < r$ and $s \notin 2\mathbb{N}_{\leq g} - 1$ on the left, while $s \in 2\mathbb{N}_{\leq g} - 1$ on the right.

$s \notin 2\mathbb{N}_{\leq g} - 1$, the elements τ_r^y and τ_s^x are the braid twists along the closed arcs $\underline{B}_y(\tau_r)$ and $\underline{B}_x(\tau_s)$, respectively, which are disjoint (see the left picture in Figure 11), so we have relation (4.7). Similarly, for $s < r$ and $s \notin 2\mathbb{N}_{\leq g} - 1$, the elements τ_r^y and τ_s^x are the braid twists along the closed arcs $B_y(\tau_r)$ and $\underline{B}_x(\tau_s)$, respectively, which are disjoint (see the right picture in Figure 11), so we have relation (4.8).

To prove that these relations are sufficient, we need to show that a group \tilde{N} with the following presentation.

- Generators: $\tilde{\sigma}_i, 1 \leq i \leq \aleph - 1, \tilde{\tau}_r, 1 \leq r \leq 2g + b - 1$.
- Relations: for $1 \leq i, j \leq \aleph - 1$ and $1 \leq r, s \leq 2g + b - 1$,

$$\text{Co}(\tilde{\sigma}_i, \tilde{\sigma}_j) \quad \text{if } |i - j| > 1 \tag{4.11}$$

$$\text{Br}(\tilde{\sigma}_i, \tilde{\sigma}_j) \quad \text{if } |i - j| = 1 \tag{4.12}$$

$$\text{Co}(\tilde{\tau}_r, \tilde{\sigma}_i) \quad \text{if } i > 2 \tag{4.13}$$

$$\text{Br}(\tilde{\tau}_r, \tilde{x}) \tag{4.14}$$

$$\text{Br}(\tilde{\tau}_r, \tilde{y}) \tag{4.15}$$

$$\text{Co}(\tilde{\tau}_r^y, \tilde{\tau}_s^x) \quad \text{if } s < r \text{ and } s \notin 2\mathbb{N}_{\leq g} - 1 \tag{4.16}$$

$$\text{Co}(\tilde{\tau}_r^y, \tilde{\tau}_s^x) \quad \text{if } s < r \text{ and } s \in 2\mathbb{N}_{\leq g} - 1, \tag{4.17}$$

where

$$\tilde{x} := \tilde{\sigma}_1^{\tilde{\sigma}_2} = \tilde{\sigma}_2^{\tilde{\sigma}_1} \tag{4.18}$$

and

$$\tilde{y} := \tilde{\sigma}_3^{\tilde{\sigma}_2} = \tilde{\sigma}_2^{\tilde{\sigma}_3} \tag{4.19}$$

is isomorphic to $\text{BT}(\mathbb{S}_\aleph)$ by sending $\tilde{\sigma}_i$ to σ_i and sending $\tilde{\tau}_r$ to τ_r .

The following relations in \tilde{N} derived from the above relations will be useful later.

LEMMA 4.4. *The following hold in \tilde{N} .*

$$\text{Co}(\tilde{x}, \tilde{y}) \tag{4.20}$$

$$\text{Br}(\tilde{x}, \tilde{\sigma}_3) \tag{4.21}$$

$$\text{Br}(\tilde{y}, \tilde{\sigma}_3) \tag{4.22}$$

$$\text{Br}(\tilde{\tau}_r, \tilde{\tau}_s^{\tilde{x}}) \quad \text{if } s < r \tag{4.23}$$

$$\text{Br}(\tilde{\tau}_r^{\tilde{y}}, \tilde{\tau}_s) \quad \text{if } s < r \text{ and } s \notin 2\mathbb{N}_{\leq g} - 1 \tag{4.24}$$

$$\text{Br}(\tilde{\tau}_r^{\tilde{y}}, \tilde{\tau}_s) \quad \text{if } s < r \text{ and } s \in 2\mathbb{N}_{\leq g} - 1. \tag{4.25}$$

Proof. Relations (4.20), (4.21) and (4.22) follow directly from (4.18), (4.19), (4.11) and (4.12).

To show (4.23), by (4.15), we have $\text{Br}(\tilde{y}, \tilde{\tau}_s)$. Conjugated by \tilde{x} , we get $\text{Br}(\tilde{y}^{\tilde{x}}, \tilde{\tau}_s^{\tilde{x}})$. Since by (4.20), $\tilde{y}^{\tilde{x}} = \tilde{y}$, we deduce that $\text{Br}(\tilde{y}, \tilde{\tau}_s^{\tilde{x}})$ holds. It follows from $\text{Br}(\tilde{\tau}_r, \tilde{y})$ (4.15) that both $\text{Br}(\tilde{\tau}_r^{\tilde{y}}, \tilde{\tau}_s^{\tilde{x}})$ and $\text{Br}(\tilde{\tau}_r^{\tilde{y}}, \tilde{\tau}_s^{\tilde{x}})$ hold. For the case $s \notin 2\mathbb{N}_{\leq g} - 1$, conjugating $\text{Br}(\tilde{\tau}_r^{\tilde{y}}, \tilde{\tau}_s^{\tilde{x}})$ by $\tilde{\tau}_r^{\tilde{y}}$ and using $\text{Co}(\tilde{\tau}_r^{\tilde{y}}, \tilde{\tau}_s^{\tilde{x}})$ (4.16), we get $\text{Br}(\tilde{\tau}_r, \tilde{\tau}_s^{\tilde{x}})$; for the case $s \in 2\mathbb{N}_{\leq g} - 1$, conjugating $\text{Br}(\tilde{\tau}_r^{\tilde{y}}, \tilde{\tau}_s^{\tilde{x}})$ by $\tilde{\tau}_r^{\tilde{y}}$ and using $\text{Co}(\tilde{\tau}_r^{\tilde{y}}, \tilde{\tau}_s^{\tilde{x}})$ from (4.17), we also get $\text{Br}(\tilde{\tau}_r, \tilde{\tau}_s^{\tilde{x}})$.

Relations (4.24) and (4.25) can be proved similarly. □

To compare \tilde{N} with $\text{BT}(\mathbb{S}_{\aleph})$, we shall consider the following group action. Let H be the subgroup of $\text{SBr}(\mathbb{S}_{\aleph})$ generated by ε_r , $1 \leq r \leq 2g + b - 1$ (which are the generators in the presentation for $\text{SBr}(\mathbb{S}_{\aleph})$ in Proposition 3.1, that are not in $\text{BT}(\mathbb{S}_{\aleph})$). The group H can be regarded as the fundamental group $\pi_1(\mathbb{S}, Z_1)$ of the underlying surface \mathbb{S} at the base point Z_1 . Thus H is freely generated by ε_r , $1 \leq r \leq 2g + b - 1$. Since $\text{BT}(\mathbb{S}_{\aleph})$ is a normal subgroup of $\text{SBr}(\mathbb{S}_{\aleph})$ (by Lemma 2.2), there is an action of H on $\text{BT}(\mathbb{S}_{\aleph})$ by conjugation.

LEMMA 4.5. *The action of H on $\text{BT}(\mathbb{S}_{\aleph})$ by conjugation is given explicitly as follows.*

$$\sigma_i^{\varepsilon_t} = \begin{cases} \tau_t & \text{if } i = 1 \\ \sigma_i & \text{if } i \neq 1 \end{cases}$$

$$\tau_r^{\varepsilon_t} = \begin{cases} x^{\sigma_2 \tau_t} & \text{if } r = t \\ \tau_r^{x \tau_t \sigma_2 \tau_t} & \text{if } r < t \text{ and } r \notin 2\mathbb{N}_{\leq g} - 1 \\ \tau_r^{x \tau_t \sigma_2 \tau_t} & \text{if } r < t \text{ and } r \in 2\mathbb{N}_{\leq g} - 1 \\ \tau_r^{x \tau_t \sigma_2 \tau_t} & \text{if } r > t \text{ and } t \notin 2\mathbb{N}_{\leq g} - 1 \\ \tau_r^{x \tau_t \sigma_2 \tau_t} & \text{if } r > t \text{ and } t \in 2\mathbb{N}_{\leq g} - 1, \end{cases}$$

where $1 \leq i \leq \aleph - 1$ and $1 \leq r, t \leq 2g + b - 1$.

Proof. The first equality follows from (4.1) and (3.3). To show the second equality, for $r = t$, we have

$$\tau_t^{\varepsilon_t} \stackrel{(4.1)}{=} \sigma_1^{\varepsilon_t \varepsilon_t} \stackrel{(3.4)}{=} \sigma_1^{\varepsilon_t \sigma_1 \varepsilon_t} \stackrel{(4.1)}{=} \sigma_1^{\tau_t} \stackrel{(4.9)}{=} x^{\sigma_2 \tau_t}$$

(where each equality is labeled by the relation which is used there); if $r < t$ and $r \notin 2\mathbb{N}_{\leq g} - 1$, we have

$$\tau_r^{x \tau_t \sigma_2 \tau_t} \stackrel{(4.9)}{=} \tau_r^{\sigma_2 \sigma_1 \sigma_2 \tau_t \sigma_2 \tau_t} \stackrel{(4.1)}{=} \sigma_1^{\varepsilon_r \sigma_2 \sigma_1 \sigma_2 \varepsilon_t \sigma_1 \varepsilon_t \sigma_2 \varepsilon_t \sigma_1 \varepsilon_t} \stackrel{(3.3)}{=} \sigma_1^{\sigma_2 \varepsilon_r \sigma_1 \varepsilon_t \sigma_2 \sigma_1 \sigma_2 \sigma_1 \varepsilon_t}$$

$$\stackrel{(3.2)}{=} \sigma_2^{\sigma_1 \varepsilon_r \sigma_1 \varepsilon_t \sigma_1 \sigma_2 \varepsilon_t} \stackrel{(3.5)}{=} \sigma_2^{\varepsilon_t \sigma_1 \varepsilon_r \sigma_2 \varepsilon_t} \stackrel{(3.3)}{=} \sigma_2^{\sigma_1 \sigma_2 \varepsilon_r \varepsilon_t} \stackrel{(3.2)}{=} \sigma_1^{\varepsilon_r \varepsilon_t} \stackrel{(4.1)}{=} \tau_r^{\varepsilon_t};$$

the other three cases can be proved similarly. □

On the other hand, we are going to construct an action of H on \tilde{N} . Denote by Λ the set $\{\tilde{\sigma}_i, 1 \leq i \leq \aleph - 1; \tilde{\tau}_r, 1 \leq r \leq 2g + b - 1\}$ of generators of \tilde{N} . To any generator ε_t of H , we associate a map

$$\rho(\varepsilon_t) : \Lambda \rightarrow \tilde{N} \\ \lambda \mapsto \rho(\varepsilon_t) \cdot \lambda$$

where (comparing with the formulas in Lemma 4.5).

$$\rho(\varepsilon_t) \cdot \tilde{\sigma}_i = \begin{cases} \tilde{\tau}_t & \text{if } i = 1 \\ \tilde{\sigma}_i & \text{if } i \neq 1 \end{cases} \quad (4.26)$$

$$\rho(\varepsilon_t) \cdot \tilde{\tau}_r = \begin{cases} \tilde{x}^{\tilde{\sigma}_2 \tilde{\tau}_t} & \text{if } r = t \\ \frac{\tilde{x} \tilde{\tau}_t \tilde{\sigma}_2 \tilde{\tau}_t}{\tilde{\tau}_r} & \text{if } r < t \text{ and } r \notin 2\mathbb{N}_{\leq g} - 1 \\ \frac{\tilde{x} \tilde{\tau}_t \tilde{\sigma}_2 \tilde{\tau}_t}{\tilde{\tau}_r} & \text{if } r < t \text{ and } r \in 2\mathbb{N}_{\leq g} - 1 \\ \frac{\tilde{x} \tilde{\tau}_t \tilde{\sigma}_2 \tilde{\tau}_t}{\tilde{\tau}_r} & \text{if } r > t \text{ and } t \notin 2\mathbb{N}_{\leq g} - 1 \\ \frac{\tilde{x} \tilde{\tau}_t \tilde{\sigma}_2 \tilde{\tau}_t}{\tilde{\tau}_r} & \text{if } r > t \text{ and } t \in 2\mathbb{N}_{\leq g} - 1 \end{cases} \quad (4.27)$$

To the inverse $\underline{\varepsilon}_t$, we also associate a map

$$\rho(\underline{\varepsilon}_t) : \Lambda \rightarrow \tilde{N} \\ \lambda \mapsto \rho(\underline{\varepsilon}_t) \cdot \lambda,$$

where

$$\rho(\underline{\varepsilon}_t) \cdot \tilde{\sigma}_i = \begin{cases} \tilde{\tau}_t^{\tilde{\sigma}_1} & \text{if } i = 1 \\ \tilde{\sigma}_i & \text{if } i \neq 1 \end{cases} \quad (4.28)$$

$$\rho(\underline{\varepsilon}_t) \cdot \tilde{\tau}_r = \begin{cases} \tilde{\sigma}_1 & \text{if } r = t \\ \frac{\tilde{\sigma}_2 \tilde{\tau}_t \tilde{\sigma}_1 \tilde{\sigma}_2}{\tilde{\tau}_r} & \text{if } r < t \text{ and } r \notin 2\mathbb{N}_{\leq g} - 1 \\ \frac{\tilde{\sigma}_2 \tilde{\tau}_t \tilde{\sigma}_1 \tilde{\sigma}_2}{\tilde{\tau}_r} & \text{if } r < t \text{ and } r \in 2\mathbb{N}_{\leq g} - 1 \\ \frac{\tilde{\sigma}_2 \tilde{\tau}_t \tilde{\sigma}_1 \tilde{\sigma}_2}{\tilde{\tau}_r} & \text{if } r > t \text{ and } t \notin 2\mathbb{N}_{\leq g} - 1 \\ \frac{\tilde{\sigma}_2 \tilde{\tau}_t \tilde{\sigma}_1 \tilde{\sigma}_2}{\tilde{\tau}_r} & \text{if } r > t \text{ and } t \in 2\mathbb{N}_{\leq g} - 1. \end{cases} \quad (4.29)$$

Since H is freely generated by $\varepsilon_1, \dots, \varepsilon_{2g+b-1}$, we have that ρ induces an action of H on \tilde{N} if and only if each $\rho(\varepsilon_r)$ induces an automorphism of \tilde{N} . In this case, one can identify \tilde{N} with $\text{BT}(\mathbb{S}_{\aleph})$ as follows.

LEMMA 4.6. *If both $\rho(\varepsilon_t)$ and $\rho(\underline{\varepsilon}_t)$ induce endomorphisms of the group \tilde{N} for any $1 \leq t \leq 2g + b - 1$, then there is a group isomorphism from \tilde{N} to $\text{BT}(\mathbb{S}_{\aleph})$, sending $\tilde{\sigma}_i$ to σ_i and sending $\tilde{\tau}_r$ to τ_r .*

Proof. We use the same notation $\rho(\varepsilon_t)$ and $\rho(\underline{\varepsilon}_t)$ to denote their induced endomorphisms of \tilde{N} , respectively. We claim that $\rho(\varepsilon_t)$ is an automorphism of \tilde{N} . Since it is easy to see that the elements $\rho(\varepsilon_t) \cdot \tilde{\sigma}_i, 1 \leq i \leq \aleph - 1$, and $\rho(\varepsilon_t) \cdot \tilde{\tau}_r, 1 \leq r \leq 2g + b - 1$, generate \tilde{N} , the map $\rho(\varepsilon_t)$ is surjective. To show it is injective, we only need to show that $\rho(\underline{\varepsilon}_t) \circ \rho(\varepsilon_t)$ is the identity on the set Λ of generators of \tilde{N} . It is straightforward to see $(\rho(\underline{\varepsilon}_t) \circ \rho(\varepsilon_t)) \cdot \tilde{\sigma}_i = \tilde{\sigma}_i$ for any $1 \leq i \leq \aleph - 1$; to prove $(\rho(\underline{\varepsilon}_t) \circ \rho(\varepsilon_t)) \cdot \tilde{\tau}_r = \tilde{\tau}_r$, for $r = t$, we have $\rho(\varepsilon_t) \cdot \tilde{\tau}_r = \tilde{x}^{\tilde{\sigma}_2 \tilde{\tau}_t}$ (4.27), which, by $\tilde{x} = \tilde{\sigma}_1^{\tilde{\sigma}_2}$ (4.18), equals $\tilde{\sigma}_1^{\tilde{\tau}_t}$, so by $\rho(\underline{\varepsilon}_t) \cdot \tilde{\sigma}_1 = \tilde{\tau}_t^{\tilde{\sigma}_1}$ (4.28) and $\rho(\varepsilon_t) \cdot \tilde{\tau}_t = \tilde{\sigma}_1$

(4.29), we have $\rho(\varepsilon_t) \cdot (\rho(\varepsilon_t) \cdot \tilde{\tau}_r) = \rho(\varepsilon_t) \cdot (\tilde{\sigma}_1^{\tilde{\tau}_t}) = (\tilde{\tau}_t^{\tilde{\sigma}_1})^{\tilde{\sigma}_1} = \tilde{\tau}_t$; for $r < t$ and $r \notin 2\mathbb{N}_{\leq g} - 1$, we have $\rho(\varepsilon_t) \cdot \tilde{\tau}_r = \tilde{\tau}_r^{\tilde{\sigma}_1 \tilde{\sigma}_2 \tilde{\tau}_t}$ (4.27), so using (4.28) and (4.29), we have $\rho(\varepsilon_t) \cdot (\rho(\varepsilon_t) \cdot \tilde{\tau}_r) = (\tilde{\tau}_r^{\tilde{\sigma}_2 \tilde{\tau}_t \tilde{\sigma}_1 \tilde{\sigma}_2})^{\tilde{\sigma}_1 \tilde{\sigma}_2 \tilde{\sigma}_1 \tilde{\sigma}_2 \tilde{\sigma}_1 \tilde{\tau}_t \tilde{\sigma}_1 \tilde{\sigma}_2 \tilde{\sigma}_1 \tilde{\sigma}_2 \tilde{\sigma}_1} = \tilde{\tau}_r^{\tilde{\sigma}_2 \tilde{\sigma}_1 \tilde{\sigma}_2 \tilde{\sigma}_1 \tilde{\sigma}_2 \tilde{\sigma}_1} = \tilde{\tau}_r$ where the last equality uses the braid relation $\text{Br}(\tilde{\sigma}_1, \tilde{\sigma}_2)$ (4.12); the other cases can be proved similarly. Hence $\rho(\varepsilon_t) \circ \rho(\varepsilon_t)$ is the identity on Λ . Thus, we complete the proof of the claim that $\rho(\varepsilon_t)$ is an isomorphism.

Since H is generated freely by ε_t , it follows that ρ induces an action of H on \tilde{N} . Let $\tilde{G} = \tilde{N} \rtimes_{\rho} H$ be the outer semidirect product of \tilde{N} and H with respect to ρ . Then \tilde{G} has a presentation whose generators are the union of the generators of \tilde{N} and the generators of H , and whose relations are the union of the relations of \tilde{N} and the relations $\lambda^{\varepsilon_t} = \rho(\varepsilon_t) \cdot \lambda$ for all $\lambda \in \Lambda$.

Note that we have proved the relations in Theorem 4.1 and gave the explicit formulas for the action of H on $\text{Br}(\mathbb{S}_{\aleph})$ in Lemma 4.5. Comparing these with the presentation of \tilde{N} and the action of H on \tilde{N} , it follows that there is a homomorphism Φ from \tilde{G} to $\text{SBr}(\mathbb{S}_{\aleph})$ sending $\tilde{\sigma}_i$ to σ , sending $\tilde{\tau}_r$ to τ_r and sending ε_r to ε_r . By the relations in Proposition 3.1, $\ker \Phi$ is generated by $\tilde{\sigma}_i \tilde{\sigma}_j \tilde{\sigma}_i \tilde{\sigma}_j$ for $|i - j| > 1$, $\tilde{\sigma}_i \tilde{\sigma}_j \tilde{\sigma}_i \tilde{\sigma}_j \tilde{\sigma}_i \tilde{\sigma}_j$ for $|i - j| = 1$, $\tilde{\sigma}_i \varepsilon_r \tilde{\sigma}_i \varepsilon_r$ for $i \neq 1$, $\varepsilon_r \tilde{\sigma}_1 \varepsilon_r \tilde{\sigma}_1 \varepsilon_r \tilde{\sigma}_1 \varepsilon_r \tilde{\sigma}_1$ for all r , $\varepsilon_s \tilde{\sigma}_1 \varepsilon_r \tilde{\sigma}_1 \varepsilon_s \tilde{\sigma}_1 \varepsilon_r \tilde{\sigma}_1$ for $s < r$ with $s \notin 2\mathbb{N}_{\leq g} - 1$, and $\varepsilon_s \tilde{\sigma}_1 \varepsilon_r \tilde{\sigma}_1 \varepsilon_s \tilde{\sigma}_1 \varepsilon_r \tilde{\sigma}_1$ for $s < r$ with $s \in 2\mathbb{N}_{\leq g} - 1$. We calculate these elements in the following. By (4.11) and (4.12), $\tilde{\sigma}_i \tilde{\sigma}_j \tilde{\sigma}_i \tilde{\sigma}_j$ (for $|i - j| > 1$) and $\tilde{\sigma}_i \tilde{\sigma}_j \tilde{\sigma}_i \tilde{\sigma}_j \tilde{\sigma}_i \tilde{\sigma}_j$ (for $|i - j| = 1$) are the identity. For $i \neq 1$, we have

$$\tilde{\sigma}_i \varepsilon_r \tilde{\sigma}_i \varepsilon_r = \tilde{\sigma}_i (\rho(\varepsilon_r) \cdot \tilde{\sigma}_i) = \tilde{\sigma}_i \tilde{\sigma}_i = 1.$$

For all r , we have

$$\varepsilon_r \tilde{\sigma}_1 \varepsilon_r \tilde{\sigma}_1 \varepsilon_r \tilde{\sigma}_1 \varepsilon_r \tilde{\sigma}_1 = (\rho(\varepsilon_r) \cdot \tilde{\sigma}_1) \tilde{\sigma}_1 (\rho(\varepsilon_r) \cdot \tilde{\sigma}_1) \tilde{\sigma}_1 = \tilde{\tau}_t^{\tilde{\sigma}_1} \tilde{\sigma}_1 \tilde{\tau}_t \tilde{\sigma}_1 = 1.$$

For $s < r$ with $s \notin 2\mathbb{N}_{\leq g} - 1$, we have

$$\varepsilon_s \tilde{\sigma}_1 \varepsilon_r \tilde{\sigma}_1 \varepsilon_s \tilde{\sigma}_1 \varepsilon_r \tilde{\sigma}_1 = (\rho(\varepsilon_s) \cdot \tilde{\sigma}_1) (\rho(\varepsilon_s \varepsilon_r) \cdot \tilde{\sigma}_1) (\rho(\varepsilon_s \varepsilon_r \varepsilon_s) \cdot \tilde{\sigma}_1) (\rho(\varepsilon_s \varepsilon_r \varepsilon_s \varepsilon_r) \cdot \tilde{\sigma}_1) \varepsilon_s \varepsilon_r \varepsilon_s \varepsilon_r.$$

For $s < r$ with $s \in 2\mathbb{N}_{\leq g} - 1$, we have

$$\varepsilon_s \tilde{\sigma}_1 \varepsilon_r \tilde{\sigma}_1 \varepsilon_s \tilde{\sigma}_1 \varepsilon_r \tilde{\sigma}_1 = (\rho(\varepsilon_s) \cdot \tilde{\sigma}_1) (\rho(\varepsilon_s \varepsilon_r) \cdot \tilde{\sigma}_1) (\rho(\varepsilon_s \varepsilon_r \varepsilon_s) \cdot \tilde{\sigma}_1) (\rho(\varepsilon_s \varepsilon_r \varepsilon_s \varepsilon_r) \cdot \tilde{\sigma}_1) \varepsilon_s \varepsilon_r \varepsilon_s \varepsilon_r.$$

For $1 \leq s < r \leq 2g + b - 1$, we set

$$n_{s,t} = \begin{cases} (\rho(\varepsilon_s) \cdot \tilde{\sigma}_1) (\rho(\varepsilon_s \varepsilon_r) \cdot \tilde{\sigma}_1) (\rho(\varepsilon_s \varepsilon_r \varepsilon_s) \cdot \tilde{\sigma}_1) (\rho(\varepsilon_s \varepsilon_r \varepsilon_s \varepsilon_r) \cdot \tilde{\sigma}_1) & \text{if } s \notin 2\mathbb{N}_{\leq g} - 1 \\ (\rho(\varepsilon_s) \cdot \tilde{\sigma}_1) (\rho(\varepsilon_s \varepsilon_r) \cdot \tilde{\sigma}_1) (\rho(\varepsilon_s \varepsilon_r \varepsilon_s) \cdot \tilde{\sigma}_1) (\rho(\varepsilon_s \varepsilon_r \varepsilon_s \varepsilon_r) \cdot \tilde{\sigma}_1) & \text{if } s \in 2\mathbb{N}_{\leq g} - 1. \end{cases}$$

Then we have that $\ker \Phi$ is generated by $n_{s,t} \varepsilon_s \varepsilon_r \varepsilon_s \varepsilon_r$ for $1 \leq s < r \leq 2g + b - 1$. Note that $n_{s,t}$ is in \tilde{N} and $\varepsilon_s \varepsilon_r \varepsilon_s \varepsilon_r$, $1 \leq s < r \leq 2g + b - 1$, freely generate a subgroup of H . It follows that $\ker \Phi \cap \tilde{N} = \emptyset$. Hence the homomorphism Φ restricting on \tilde{N} is injective. Therefore we get the required isomorphism. \square

REMARK 4.7. By the above proof, the surface braid group $\text{SBr}(\mathbb{S}_{\aleph})$ is not the (inner) semidirect product of $\text{BT}(\mathbb{S}_{\aleph})$ and H , except that \mathbb{S}_{\aleph} is a decorating disk or a decorating annulus. For a decorating disk, H is trivial; for a decorating annulus, this is the case in [16].

By Lemma 4.6, to complete the proof of Theorem 4.1, we only need to show that $\rho(\varepsilon_t)$ and $\rho(\varepsilon_t)$ induce endomorphisms of \tilde{N} for every $1 \leq t \leq 2g + b - 1$, that is, to show that the images of generating relations (4.11)–(4.17) for \tilde{N} under the action of $\rho(\varepsilon_t)$ and $\rho(\varepsilon_t)$, respectively, still hold in \tilde{N} . Explicitly, we need to show that the following hold: for $1 \leq i, j \leq \aleph - 1$ and $1 \leq r, s \leq 2g + b - 1$,

$$\mathrm{Co}(\rho(\varepsilon_t) \cdot \tilde{\sigma}_i, \rho(\varepsilon_t) \cdot \tilde{\sigma}_j) \quad \text{if } |i - j| > 1 \quad (4.30)$$

$$\mathrm{Br}(\rho(\varepsilon_t) \cdot \tilde{\sigma}_i, \rho(\varepsilon_t) \cdot \tilde{\sigma}_j) \quad \text{if } |i - j| = 1 \quad (4.31)$$

$$\mathrm{Co}(\rho(\varepsilon_t) \cdot \tilde{\tau}_r, \rho(\varepsilon_t) \cdot \tilde{\sigma}_i) \quad \text{if } i > 2 \quad (4.32)$$

$$\mathrm{Br}(\rho(\varepsilon_t) \cdot \tilde{\tau}_r, \rho(\varepsilon_t) \cdot \tilde{x}) \quad (4.33)$$

$$\mathrm{Br}(\rho(\varepsilon_t) \cdot \tilde{\tau}_r, \rho(\varepsilon_t) \cdot \tilde{y}) \quad (4.34)$$

$$\mathrm{Co}((\rho(\varepsilon_t) \cdot \tilde{\tau}_r)^{(\rho(\varepsilon_t) \cdot \tilde{y})}, (\rho(\varepsilon_t) \cdot \tilde{\tau}_s)^{(\rho(\varepsilon_t) \cdot \tilde{x})}) \quad \text{if } s < r \text{ and } s \notin 2\mathbb{N}_{\leq g} - 1 \quad (4.35)$$

$$\mathrm{Co}((\rho(\varepsilon_t) \cdot \tilde{\tau}_r)^{(\rho(\varepsilon_t) \cdot \tilde{y})}, (\rho(\varepsilon_t) \cdot \tilde{\tau}_s)^{(\rho(\varepsilon_t) \cdot \tilde{x})}) \quad \text{if } s < r \text{ and } s \in 2\mathbb{N}_{\leq g} - 1 \quad (4.36)$$

and

$$\mathrm{Co}(\rho(\underline{\varepsilon}_t) \cdot \tilde{\sigma}_i, \rho(\underline{\varepsilon}_t) \cdot \tilde{\sigma}_j) \quad \text{if } |i - j| > 1 \quad (4.37)$$

$$\mathrm{Br}(\rho(\underline{\varepsilon}_t) \cdot \tilde{\sigma}_i, \rho(\underline{\varepsilon}_t) \cdot \tilde{\sigma}_j) \quad \text{if } |i - j| = 1 \quad (4.38)$$

$$\mathrm{Co}(\rho(\underline{\varepsilon}_t) \cdot \tilde{\tau}_r, \rho(\underline{\varepsilon}_t) \cdot \tilde{\sigma}_i) \quad \text{if } i > 2 \quad (4.39)$$

$$\mathrm{Br}(\rho(\underline{\varepsilon}_t) \cdot \tilde{\tau}_r, \rho(\underline{\varepsilon}_t) \cdot \tilde{x}) \quad (4.40)$$

$$\mathrm{Br}(\rho(\underline{\varepsilon}_t) \cdot \tilde{\tau}_r, \rho(\underline{\varepsilon}_t) \cdot \tilde{y}) \quad (4.41)$$

$$\mathrm{Co}((\rho(\underline{\varepsilon}_t) \cdot \tilde{\tau}_r)^{(\rho(\underline{\varepsilon}_t) \cdot \tilde{y})}, (\rho(\underline{\varepsilon}_t) \cdot \tilde{\tau}_s)^{(\rho(\underline{\varepsilon}_t) \cdot \tilde{x})}) \quad \text{if } s < r \text{ and } s \notin 2\mathbb{N}_{\leq g} - 1 \quad (4.42)$$

$$\mathrm{Co}((\rho(\underline{\varepsilon}_t) \cdot \tilde{\tau}_r)^{(\rho(\underline{\varepsilon}_t) \cdot \tilde{y})}, (\rho(\underline{\varepsilon}_t) \cdot \tilde{\tau}_s)^{(\rho(\underline{\varepsilon}_t) \cdot \tilde{x})}) \quad \text{if } s < r \text{ and } s \in 2\mathbb{N}_{\leq g} - 1, \quad (4.43)$$

where

$$\rho(\varepsilon_t) \cdot \tilde{x} := (\rho(\varepsilon_t) \cdot \tilde{\sigma}_2)^{(\rho(\varepsilon_t) \cdot \tilde{\sigma}_1)} = \tilde{\sigma}_2^{\tilde{\tau}_t}, \quad (4.44)$$

$$\rho(\varepsilon_t) \cdot \tilde{y} := (\rho(\varepsilon_t) \cdot \tilde{\sigma}_2)^{(\rho(\varepsilon_t) \cdot \tilde{\sigma}_3)} = \tilde{\sigma}_2^{\tilde{\sigma}_3} = \tilde{y}, \quad (4.45)$$

$$\rho(\underline{\varepsilon}_t) \cdot \tilde{x} := (\rho(\underline{\varepsilon}_t) \cdot \tilde{\sigma}_2)^{(\rho(\underline{\varepsilon}_t) \cdot \tilde{\sigma}_1)} = \tilde{\sigma}_2^{\left(\frac{\tilde{\sigma}_1}{\tilde{\tau}_t}\right)} = \tilde{\tau}_t^{\tilde{\sigma}_1 \tilde{\sigma}_2}, \quad (4.46)$$

$$\rho(\underline{\varepsilon}_t) \cdot \tilde{y} := (\rho(\underline{\varepsilon}_t) \cdot \tilde{\sigma}_2)^{(\rho(\underline{\varepsilon}_t) \cdot \tilde{\sigma}_3)} = \tilde{\sigma}_2^{\tilde{\sigma}_3} = \tilde{y}. \quad (4.47)$$

We show these relations in Appendix A. Thus we finish the proof.

5. Finite presentations for spherical twist groups

In this section, we introduce the braid group associated to a quiver with potential arising from a marked surface and show that mutations of quivers with potential induce isomorphisms of the associated braid groups. Then we show that such braid group is isomorphic to the braid twist group of the corresponding decorated marked surface. Thus, we get finite presentations of braid/spherical twist groups via quivers with potential.

5.1. Braid groups associated to quivers with potential

Let (Q, W) be a quiver with potential arising from a triangulated marked surface, that is, $(Q, W) = (Q_{\mathbf{T}}, W_{\mathbf{T}})$ for a triangulation \mathbf{T} of a decorated marked surface \mathbf{S}_{Δ} . By the construction (see Section 2.2), the quiver with potential (Q, W) has the following properties.

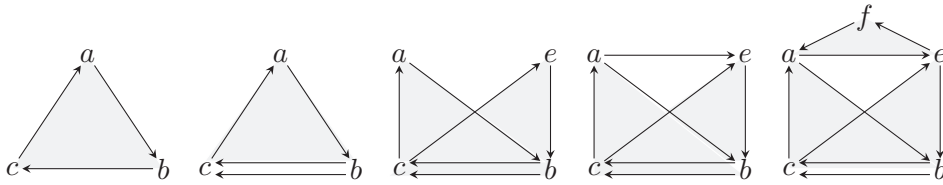


FIGURE 12. Cases of full subquiver of (Q, W) .

- For each vertex of Q , there are at most two arrows starting at it and at most two arrows ending at it. In particular, there are at most two arrows between two vertices.
- There are no 2-cycles in Q . So if there are two arrows between two vertices, then they have the same directions, which are called a double arrow.
- Any two 3-cycles in W do not share an arrow.
- If there is a 3-cycle between vertices a, b, c , then there is at most one double arrow between them and there is exactly one 3-cycle between them contributing a term in W , that is, the full subquiver with potential between these vertices is the first one or the second one in Figure 12, where the gray triangle means that its three edges form a 3-cycle in W .

DEFINITION 5.1. Let (Q, W) be a quiver with potential arising from a triangulated marked surface. We define the associated braid group $\text{Br}(Q, W)$ by the following presentation.

- Generators: vertices of Q .
- Relations (see Section 1.4 for the notation):
 - 1°. $\text{Co}(a, b)$, if there is no arrow between a and b .
 - 2°. $\text{Br}(a, b)$, if there is exactly one arrow between a and b .
 - 3°. $\text{Co}(a^b, c)$, if there is one 3-cycle between a, b and c which contributes a term in W and there are no double arrows between them, see the first full subquiver with potential in Figure 12.
 - 4°. $\text{Br}(a^b, c)$, if there is a 3-cycle between a, b and c which contributes a term in W and there is a double arrow between b and c , see the second full subquiver with potential in Figure 12.
 - 5°. $\text{Co}(c^{ae}, b)$, if there is one 3-cycle between a, b and c and one 3-cycle between e, b and c , which contribute terms in W , and there are no arrows between a and e , see the third full subquiver with potential in Figure 12.
 - 6°. $\text{Br}(c^{ae}, b)$ and $\text{Br}(c^{ea}, b)$, if there is one 3-cycle between a, b and c and one 3-cycle between e, b and c , which contribute terms in W , and there is an arrow between a and e , see the fourth full subquiver with potential in Figure 12.
 - 7°. $\text{Co}(e, f^{abc})$, if in the previous case, additionally there is a 3-cycle between a, e and f which contributes a term in W , see the last full subquiver with potential in Figure 12.

The above presentation will become much simpler if there are no double arrows in the quiver Q . Note that excluding the case that \mathbf{S} is an annulus with one marked point on each of its boundary components or a torus with only one marked point, there always exists a triangulation \mathbf{T} such that there are no double arrows in the associated quiver $Q_{\mathbf{T}}$ (see [20, Lemma 3.12]).

PROPOSITION 5.2. Let (Q, W) be a quiver with potential arising from a triangulated marked surface. If there are no double arrows in Q , then the associated braid group $\text{Br}(Q, W)$ has the following presentation.

- Generators: vertices of Q .

- *Relations:*

- 1°. $\text{Co}(a, b)$ if there is no arrow between a and b .
- 2°. $\text{Br}(a, b)$ if there is exactly one arrow between a and b .
- 3°'. $\text{Tr}(a, b, c)$ if there is a 3-cycle between a , b and c , which contributes a term in W .

Proof. Since there are no double arrows in Q , relations 4°–7° in Definition 5.1 does not occur. So we only need to show that relation 3° is equivalent to 3°' under relations 1° and 2°. This equivalence follows from the following easy equivalences:

$$abca = bcab \iff \underline{b}abc = caba \iff a^b c = cb^a \iff \text{Co}(a^b, c),$$

where the last equivalence follows from $b^a = a^b$ by $\text{Br}(a, b)$. \square

REMARK 5.3. Sometime we prefer this triangle relation $\text{Tr}(a, b, c)$, since it can be generalized to the cyclic relation, which may be used for the quivers with potential which contains cycles of lengths at least 4 in the potential (cf. [11; 20, Definition 10.1]).

The relations in Definition 5.1 are not minimal. See the following two simple cases, where less relations are enough.

LEMMA 5.4. *Let (Q, W) be the first quiver with potential in Figure 12. Then $\text{Br}(Q, W)$ admits the following presentation.*

- *Generators:* a, b, c .
- *Relations:*

$$\text{Br}(a, b), \text{Br}(a, c), \text{Co}(a^b, c),$$

where $\text{Co}(a^b, c)$ can be replaced by $\text{Co}(c^a, b)$.

Proof. By the braid relation $\text{Br}(a, b)$, we have $a^b = b^a$. Then $\text{Co}(a^b, c)$ is equivalent to $\text{Co}(b^a, c)$. The latter is equivalent to $\text{Co}(b, c^a)$ via conjugation by a . Thus, $\text{Co}(a^b, c)$ can be replaced by $\text{Co}(c^a, b)$.

We need to show the relations $\text{Br}(b, c)$ and $\text{Co}(a, b^c)$. Starting from $\text{Br}(a, c)$, conjugated by \underline{a}^b and using the commutation relation $\text{Co}(a^b, c)$, we have $\text{Br}(a(\underline{a}^b), c)$. Then using the braid relation $\text{Br}(a, b)$, we get $\text{Br}(b, c)$. To show the other one, starting from $\text{Co}(a^b, c)$, conjugated by \underline{b} and using the braid relation $\text{Br}(b, c)$, we are done. \square

LEMMA 5.5. *Let (Q, W) be the third quiver with potential in Figure 12. Then $\text{Br}(Q, W)$ admits the following presentation.*

- *Generators:* a, b, c, e .
- *Relations:*

$$\text{Br}(a, b), \text{Br}(a, c), \text{Br}(e, b), \text{Br}(e, c), \text{Co}(a, e), \text{Co}(c^{ae}, b).$$

Proof. We shall prove that the relations $\text{Br}(a^b, c)$ and $\text{Br}(e^b, c)$ hold. Indeed, starting from $\text{Br}(e, b)$, conjugated by c^{ae} and using the commutation relation $\text{Co}(c^{ae}, b)$, we get $\text{Br}(e(c^{ae}), b)$. By the commutation relation $\text{Co}(a, e)$, we have $\text{Br}(e^{cea}, b)$, which implies $\text{Br}(c^a, b)$ using the braid relation $\text{Br}(e, c)$. Conjugated by \underline{a} and using the braid relation $\text{Br}(a, b)$, we deduce $\text{Br}(c, a^b)$ as required. The relation $\text{Br}(c, e^b)$ can be showed similarly. \square

Another case for which less relations are enough is the last case in Figure 12; see Lemma B.5, whose proof is a little complicated.

5.2. Mutations and isomorphisms

In this subsection, we show that the presentations in Definition 5.1 are compatible with mutation of quivers with potential. Recall from Section 2.2 the notion of mutation of quivers with potential.

PROPOSITION 5.6. *Let (Q, W) be a quiver with potential arising from a triangulated marked surface and $(Q', W') = \mu_\gamma(Q, W)$ the mutation of (Q, W) at a vertex γ . Denote by α' the vertex of Q' corresponding to a vertex α of Q . There are mutually inverse canonical isomorphisms of groups*

$$\begin{aligned} \theta_\gamma^b &: \text{Br}(Q, W) \xrightarrow{\cong} \text{Br}(Q', W') \\ \theta_{\gamma'}^\sharp &: \text{Br}(Q', W') \xrightarrow{\cong} \text{Br}(Q, W) \end{aligned}$$

satisfying

$$\theta_\gamma^b(\alpha) = \begin{cases} (\alpha')^{\gamma'} & \text{if there are arrows from } \gamma \text{ to } \alpha \text{ in } Q, \\ \alpha' & \text{otherwise,} \end{cases} \tag{5.1}$$

$$\theta_{\gamma'}^\sharp(\alpha') = \begin{cases} (\alpha)^\gamma & \text{if there are arrows from } \alpha' \text{ to } \gamma' \text{ in } Q', \\ \alpha & \text{otherwise.} \end{cases} \tag{5.2}$$

Proof. Formulas (5.1) and (5.2) define two homomorphisms θ_γ^\sharp and $\theta_{\gamma'}^b$, between the groups freely generated by the vertices of Q and Q' , respectively. Moreover, $\theta_{\gamma'}^\sharp$ is the composition of $\theta_{\gamma'}^b$ with the conjugation by γ , and θ_γ^b and $\theta_{\gamma'}^\sharp$ are mutually inverse. So we only need to show $\theta = \theta_\gamma^b$ preserves the relations in the presentation of $\text{Br}(Q, W)$ in Definition 5.1.

First, we consider several special (local) cases, where we show that the images of the relations in $\text{Br}(Q', W')$ under θ^{-1} are equivalent to the relations in $\text{Br}(Q, W)$. In each case, the left quiver with potential is (a full subquiver of) (Q, W) and the right quiver with potential is (the corresponding full subquiver of) (Q', W') , but whose vertices are indexed by their images under θ^{-1} . The relations common to both sides are listed on the top and an equivalence may use common relations and equivalences above it. We only give proofs in Appendix B for non-easily checked equivalences.

Case (I): Consider the mutation in (5.3). Lemma 5.4 ensures that the relations on the left side are enough.

$$\begin{array}{ccc} \text{Br}(a, b) & \iff & \text{Br}(a, c) \\ \text{Co}(a^b, c) & \iff & \text{Co}(c, d). \end{array} \tag{5.3}$$

Case (II.1/2): Consider the mutations in (5.4) and (5.5), respectively.

$$\begin{array}{ccc} \text{Br}(a, b) & \iff & \text{Br}(a, d) \\ \text{Br}(a^b, c) & \iff & \text{Br}(d, c) \end{array} \tag{5.4}$$

$$\begin{array}{ccc}
 \text{Br}(a^b, c) & \iff & \text{Br}(a, l) \\
 \text{Br}(a, c) & \iff & \text{Br}(a^l, b).
 \end{array} \tag{5.5}$$

Case (III.1/2): Consider the mutations in (5.6) and (5.7), respectively.

For the mutation in (5.6), Lemmas 5.5 and 5.4 ensure the relations on the left side and on the right side are enough, respectively.

$$\begin{array}{ccc}
 \text{Co}(a, e) & & \\
 \text{Br}(a, c) & & \\
 \text{Br}(e, c) & \iff & \text{Br}(e, d) \\
 \text{Br}(e, b) & \iff & \text{Br}(a, d) \\
 \text{Br}(a, b) & \iff & \text{Co}(e^d, c). \\
 \text{Co}(c^{ae}, b) & \iff &
 \end{array} \tag{5.6}$$

For the mutation in (5.7), we show the last three equivalences in Lemma B.1.

$$\begin{array}{ccc}
 \text{Br}(a, b) & & \\
 \text{Co}(a, e) & & \\
 \text{Br}(b, e) & & \\
 \text{Br}(a^b, c) & \iff & \text{Br}(a, l) \\
 \text{Br}(e^b, c) & \iff & \text{Br}(e, l) \\
 \text{Br}(a, c) & \iff & \text{Br}(a^l, b) \\
 \text{Br}(e, c) & \iff & \text{Br}(e^l, b) \\
 \text{Co}(c^{ae}, b) & \iff & \text{Co}(b^{ae}, l).
 \end{array} \tag{5.7}$$

Case (IV.1/2): Consider the mutations in (5.8) and (5.9), respectively.

For the mutation in (5.8), we show the last three equivalences in Lemma B.2.

$$\begin{array}{ccc}
 \text{Br}(a, e) & & \\
 \text{Br}(a, c) & & \\
 \text{Br}(e, c) & & \\
 \text{Br}(e, b) & \iff & \text{Br}(e, h) \\
 \text{Br}(a, b) & \iff & \text{Br}(e^a, h) \\
 \text{Br}(e^b, c) & \iff & \text{Br}(h, c) \\
 \text{Br}(a^b, c) & \iff & \text{Br}(h^{ec}, a) \\
 \text{Br}(c^{ae}, b) & \iff & \text{Br}(c^a, h) \\
 \text{Br}(c^{ea}, b) & \iff & \text{Br}(h^{ce}, a).
 \end{array} \tag{5.8}$$

For the mutation in (5.9), we show the last two equivalences in Lemma B.3.

$$\begin{array}{ccc}
 \text{Br}(a, b) & & \\
 \text{Br}(a, e) & & \\
 \text{Br}(e, b) & & \\
 \text{Br}(a^b, c) & \iff & \text{Br}(a, l) \\
 \text{Br}(a, c) & \iff & \text{Br}(a^l, b) \\
 \text{Br}(e^b, c) & \iff & \text{Br}(e, l) \\
 \text{Br}(e, c) & \iff & \text{Br}(e^l, b) \\
 \text{Br}(c^{ae}, b) & \iff & \text{Br}(b^{ae}, l) \\
 \text{Br}(c^{ea}, b) & \iff & \text{Br}(b^{ea}, l).
 \end{array} \tag{5.9}$$

Case (V.1~4): Consider the mutations in (5.10), (5.11) (5.12) and (5.13), respectively.

For the mutation in (5.10), Lemma B.5, together with Lemma 5.4, ensures that the relations on both sides are enough. We show the last equivalence in Lemma B.4.

		$\begin{aligned} & \text{Br}(f, e) \\ & \text{Br}(e, b) \\ & \text{Br}(e, c) \\ & \text{Co}(f, b) \\ & \text{Co}(f, c) \end{aligned}$	$\begin{aligned} & \text{Br}(f, g) \\ & \text{Br}(g, b) \\ & \text{Br}(g, c) \\ & \text{Co}(g, e) \\ & \text{Co}(c^{ge}, b). \end{aligned}$
	$\begin{aligned} & \text{Br}(f, a) \\ & \text{Br}(a, b) \\ & \text{Br}(a, c) \\ & \text{Co}(f^a, e) \\ & \text{Co}(e, f^{abc}) \end{aligned}$	$\begin{aligned} & \iff \\ & \iff \\ & \iff \\ & \iff \\ & \iff \end{aligned}$	(5.10)

For the mutation in (5.11), the equivalences are from (5.9).

		(5.11)
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For the mutation in (5.12), we show the last equivalence in Lemma B.6.

		$\begin{aligned} & \text{Br}(a, e) \\ & \text{Br}(a, c) \\ & \text{Br}(e, c) \end{aligned}$	$\begin{aligned} & \text{Br}(e, h) \\ & \text{Br}(e^a, h) \\ & \text{Br}(h, c) \\ & \text{Br}(h^{ec}, a) \\ & \text{Br}(e, k) \\ & \text{Co}(h, k) \\ & \text{Co}(e^c, k) \\ & \text{Co}(k, a) \\ & \text{Co}(c, k^{eah}). \end{aligned}$
	$\begin{aligned} & \text{Br}(e, b) \\ & \text{Br}(a, b) \\ & \text{Br}(e^b, c) \\ & \text{Br}(a^b, c) \\ & \text{Br}(e, f) \\ & \text{Co}(b, f) \\ & \text{Co}(c, f) \\ & \text{Co}(e^f, a) \\ & \text{Co}(e, f^{abc}) \end{aligned}$	$\begin{aligned} & \iff \\ & \iff \\ & \iff \\ & \iff \\ & \iff \\ & \iff \\ & \iff \\ & \iff \\ & \iff \end{aligned}$	(5.12)

For the mutation in (5.13), this is a combination of (5.6) for (a, b, c, e) and (5.4) for (f, b, e) .

		(5.13)
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Next, we consider general (global) cases. We claim that for any relation R in $\text{Br}(Q, W)$, $\theta(R)$ holds in $\text{Br}(Q', W')$. This will complete the proof. There are the following cases depending on the type of R in Definition 5.1. Note that the numbers of arrows between two fixed vertices in (Q, W) and (Q', W') differ at most 1.

1°. $R = \text{Co}(c, d)$ is of type 1°. If there is exactly one arrow between c', d' , then locally the mutation is the composition of the inverse of μ_a in (5.3) with the conjugation by a and hence

$\theta(R)$ holds. If there is no arrow between c', d' , then we have $\text{Co}(c', d')$ and there are the following cases.

- Both $\theta(t) = t'$ for $t = c, d$ and then $\text{Co}(\theta(c), \theta(d))$ holds.
- Both $\theta(t) = (t')^{\gamma'}$ for $t = c, d$ and then $\text{Co}(\theta(c), \theta(d))$ holds.
- $\theta(c) = c'$ but $\theta(d) = (d')^{\gamma'}$. So there are arrows from γ to d but no arrows from γ to c in Q . As there no arrows from c' to d' in Q' , there is no arrow from c to γ in Q . So we have $\text{Co}(c, \gamma)$ and then $\text{Co}(c', d')$ implies $\text{Co}(c', (d')^{\gamma'})$, which is $\text{Co}(\theta(c), \theta(d))$.

2°. $R = \text{Br}(b, c)$ is of type 2°. Then there is exactly one arrow between b and c in Q . If the number of arrows between b' and c' is zero or two in Q' , then there is exactly one path between b and c of length 2 and through γ . Hence the mutation is locally μ_a in (5.3) or the composition of the inverse of μ_a in (5.4) (via $\{b, c\}$ equaling $\{d, c\}$ there) with the conjugation by a . So $\theta(R)$ holds. If there is exactly one arrow between b' and c' in Q' , then we have $\text{Br}(b', c')$ and there are the following cases.

- Both $\theta(t) = t'$ for $t = b, c$ and then $\text{Co}(\theta(c), \theta(d))$ holds.
- Both $\theta(t) = (t')^{\gamma'}$ for $t = b, c$ and then $\text{Co}(\theta(c), \theta(d))$ holds.
- $\theta(c) = c'$ but $\theta(b) = (b')^{\gamma'}$. So there are arrows from γ to b but no arrows from γ to c in Q . If there are no arrows from c to γ in Q' , similarly as above, we deduce that $\theta(R)$ holds. If there are arrows from c to γ , then the arrow between b and c is from b to c and exactly one of the following occurs:
 - there are two arrows from γ to b ;
 - there are two arrows from c to γ .

Then the mutation is locally μ_a in (5.5) (via (b, c) equaling (c, a) there) or the composition of the inverse of μ_a in (5.5) (via (b, c) equaling (a, l) there) with the conjugation by b . Hence $\theta(R)$ holds.

3°. $R = \text{Co}(a^b, c)$ is of type 3°. Then the full subquiver between a, b, c in Q is a triangle L which contributes a term in the potential W . If γ is one of a, b, c , without loss of generality, assuming $\gamma = a$, then the mutation locally is μ_a in (5.3) and so $\theta(R)$ holds. If γ is different from any of a, b, c , then there are the following cases.

- There are no arrows from any of a, b, c to γ , or there are no arrows from γ to any of a, b, c . Then the full subquiver with potential between a', b', c' is the same as a, b, c . It is straightforward to check that $\theta(R)$ holds.
- There is an arrow from γ to a and an arrow from b or c to γ . Note that if there is an arrow from b to γ , then there is a 3-cycle between a, b, γ . So the arrow from a to b has to contribute two terms in W , a contradiction. Hence the arrow to γ is from c . Then the mutation is the composition of the inverse of μ_a in (5.6) (via (a, b, c, γ) equaling (c, e, d, a) there) with the conjugation by γ . Hence $\theta(R)$ holds.

4°. R is of type 4° – 7°, then locally, the full subquiver of R in Q , denoted by Q_R , is a subquiver of the left quiver \tilde{Q} in (5.8), (5.10) or (5.13).

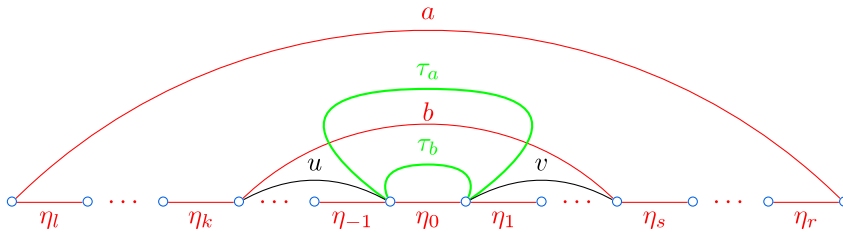
If γ is one of the vertices in \tilde{Q} , then the possible mutations, up to reversing all arrows, are the mutations in Cases (III)–(V) or their inverses composited by some conjugations. Hence $\theta(R)$ holds in this case.

If γ is not one of the vertices in \tilde{Q} , then the full subquiver between the corresponding vertices in Q' is the same as Q_R . There are two cases. First, there is no arrow between γ and a vertex in Q_R except for at most one vertex. As $\text{Co}(t', \gamma') \Leftrightarrow (t')^{\gamma'} = t'$, we have either $\theta(t) = t'$ or $\theta(t) = (t')^{\gamma'}$ for all $t \in Q_R$, which implies that $\theta(R)$ holds. Second, γ has arrows from/to at least two vertices in Q_R . Then Q_R is isomorphic to one of the quivers in (5.7) or (5.9) and γ must have arrows both to or both from vertices a, e . This forces that there is no arrow between a and e , that is, Q_R is a quiver in (5.6), say the left one, and $R = \text{Co}(c^{ae}, b)$. Then either $\theta(t) = t'$ or $\theta(t) = (t')^{\gamma'}$ for both $t = a, e$. Moreover, $\theta(t) = t'$ for $t = b, c$. With $\text{Co}(\gamma', b')$ and $\text{Co}(\gamma', c')$, it is straightforward to check $\theta(R)$ holds. \square

5.3. The main result

We shall state and prove our main result in the paper: finite presentations of braid/spherical twist groups via quivers with potential. Recall (Lemma 2.9) that for a decorated marked surface \mathbf{S}_Δ , its braid twist group $\text{BT}(\mathbf{S}_\Delta) = \text{BT}(\mathbf{T})$ admits a set \mathbf{T}^* of generators for any triangulation \mathbf{T} of \mathbf{S}_Δ , where \mathbf{T}^* is the set of duals of arcs in \mathbf{T} (see Section 2.2). We proceed to prove that the braid twist group $\text{BT}(\mathbf{S}_\Delta)$ is isomorphic to the braid group $\text{Br}(Q_{\mathbf{T}}, W_{\mathbf{T}})$, and thus has the presentation in Definition 5.1, that is, a finite presentation via quiver with potential. We shall need the following two lemmas.

LEMMA 5.7. Suppose that we have a group generated by a, b and $\eta_i, l \leq i \leq r$, where l, r are integers with $l < -1$ and $r > 1$, subject to the relations given by the following configuration of arcs



where

- $\text{Co}(\beta, \gamma)$ holds if the arcs (labeled by) β and γ are disjoint;
- $\text{Br}(\beta, \gamma)$ holds if the arcs β and γ are disjoint except sharing an endpoint;
- $\text{Tr}(b, \eta_{k+1}, \eta_k)$ and $\text{Tr}(b, \eta_s, \eta_{s-1})$ hold for some fixed integers k and s satisfying $l \leq k < -1$ and $1 < s \leq r$.

Let $\tau_a = a^{(\eta_l \cdots \eta_{-1})(\eta_r \cdots \eta_1)}$, $\tau_b = b^{(\eta_{k+1} \cdots \eta_{-1})(\eta_{s-1} \cdots \eta_1)}$ (cf. the green arcs in the figure above). Then we have the relation

$$\text{Co}(\tau_a^{\eta_1}, \tau_b^{\eta_{-1}}). \tag{5.14}$$

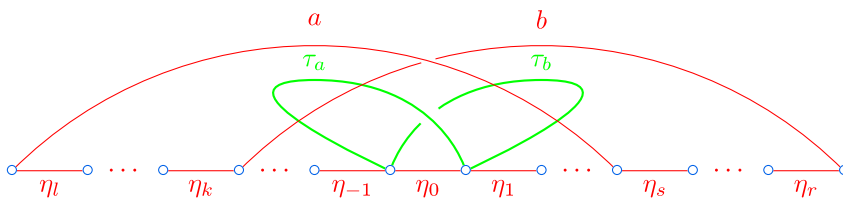
Proof. Let

$$u = \eta_{k+1}^{\eta_{k+2} \cdots \eta_{-1}}, \quad v = \eta_1^{\eta_2 \cdots \eta_{r-1}}$$

which would correspond to the violet arcs in the configuration above. It is straightforward to see that $\text{Co}(a, u), \text{Co}(a, v)$ holds. Conjugating (5.14) with $\eta_2 \cdots \eta_r \eta_{-1} \cdots \eta_l$, we obtain $\text{Co}(a, b^{uv})$, which holds as a commutes with all of b, u, v . \square

Similarly, we have the following.

LEMMA 5.8. Suppose that we have a group generated by a, b and $\eta_i, l \leq i \leq r$ where l, r are integers with $l < -1$ and $r > 1$, subject to the relations given by the following configuration of arcs:



where

- $\text{Co}(\beta, \gamma)$ holds if the arcs (labeled by) β and γ are disjoint (note that a and b are disjoint here);
- $\text{Br}(\beta, \gamma)$ holds if the arcs β and γ are disjoint except sharing an endpoint;
- $\text{Tr}(b, \eta_{k+1}, \eta_k), \text{Tr}(a, \eta_s, \eta_{s-1})$ hold for some fixed integers k and s satisfying $l \leq k < -1$ and $1 < s \leq r$.

Let $\tau_b = b^{(\eta_{k+1} \cdots \eta_{-1})(\underline{\eta_s \cdots \eta_1})}$, $\tau_a = a^{(\eta_{l \cdots \eta_{-1}})(\underline{\eta_{s-1} \cdots \eta_1})}$ and then we have the relation

$$\text{Co}(\tau_a^{\eta_1}, \tau_b^{\eta_{-1}}). \tag{5.15}$$

Now we have the following main result.

THEOREM 5.9 (Presentations for braid twist groups). *Let \mathbf{T} be a triangulation of \mathbf{S}_Δ and $(Q_{\mathbf{T}}, W_{\mathbf{T}})$ the associated quiver with potential. Then there is a canonical isomorphism*

$$\kappa_{\mathbf{T}}: \text{Br}(Q_{\mathbf{T}}, W_{\mathbf{T}}) \rightarrow \text{BT}(\mathbf{S}_\Delta) \tag{5.16}$$

sending the generators γ to the braid twists $\gamma^* \in \mathbf{T}^*$. Thus, $\text{BT}(\mathbf{S}_\Delta)$ admits the finite presentation in Definition 5.1 via $(Q_{\mathbf{T}}, W_{\mathbf{T}})$. Moreover, it is compatible with the canonical isomorphisms in Proposition 5.6, in the sense that the diagram

$$\begin{array}{ccc}
 \text{Br}(Q_{\mathbf{T}}, W_{\mathbf{T}}) & & \\
 \downarrow \theta_\gamma^? & \searrow \kappa_{\mathbf{T}} & \\
 & & \text{BT}(\mathbf{S}_\Delta) \\
 & \nearrow \kappa_{\mathbf{T}'} & \\
 \text{Br}(Q_{\mathbf{T}'}, W_{\mathbf{T}'}) & &
 \end{array} \tag{5.17}$$

commutes, where \mathbf{T}' is the backward and forward flip of \mathbf{T} at γ , for $? = \flat$ and \sharp , respectively.

Proof. First, choose the triangulation \mathbf{T}_0 of \mathbf{S}_Δ as shown in Figure 13. Assume that $\aleph \geq 4$. Then there are no double arrows in the quiver $Q_{\mathbf{T}_0}$. So the braid group $\text{Br}(Q_{\mathbf{T}_0}, W_{\mathbf{T}_0})$ admits the presentation in Proposition 5.2. Hence, by Lemma 2.3, the group homomorphism $\kappa_{\mathbf{T}_0}$ is well defined. By Lemma 2.9, $\kappa_{\mathbf{T}_0}$ is surjective. To show that it is injective, we only need to show that

$$\{\kappa_{\mathbf{T}_0}(R) \mid R \text{ a relation in } \text{Br}(Q_{\mathbf{T}_0}, W_{\mathbf{T}_0})\}$$

generates all the relations of $\text{BT}(\mathbf{S}_\Delta)$. By Corollary 4.2, we just need to consider the relations in Theorem 4.1; cf. Figure 8. It is easy to check the first five relations, while the last two follows from Lemmas 5.7 and 5.8, respectively, taking $\eta_0 = \kappa_{\mathbf{T}_0}(\sigma_1)$, $\eta_{-1} = \kappa_{\mathbf{T}_0}(x)$, $\eta_1 = \kappa_{\mathbf{T}_0}(y)$, $\tau_a = \tau_r$ and $\tau_b = \tau_t$ as in Figure 14. Note that the two cases are determined by the relative position of τ_r and τ_s . For the case $\aleph \leq 3$, we have that either $g = 1, b = 1, |\mathbf{M}| = 1$, or $g = 0, b \leq 2$. A direct calculation can give the bijection $\kappa_{\mathbf{T}_0}$.

Next, using commutativity in diagram (5.17) as definition, we obtain an isomorphism $\kappa_{\mathbf{T}_1}$ for any (forward/backward) flip \mathbf{T}_1 of \mathbf{T}_0 . It is straightforward to check that this isomorphism satisfies the condition that sending $\gamma \in \mathbf{T}_1$ to the corresponding braid twist $\eta = \gamma^* \in \mathbf{T}_1^*$, comparing (5.1), (5.2) and Figure 6. Hence, any mutate sequence p from \mathbf{T}_0 to \mathbf{T} induces an isomorphism $\kappa_{\mathbf{T}}$ satisfying the condition above. This implies that $\kappa_{\mathbf{T}}$ does not depend on p . Hence, the theorem holds for any triangulation in the connected component of the exchange graph $\text{EG}(\mathbf{S}_\Delta)$ (of triangulations with flips; cf. [20, Definition 3.4]) containing \mathbf{T}_0 .

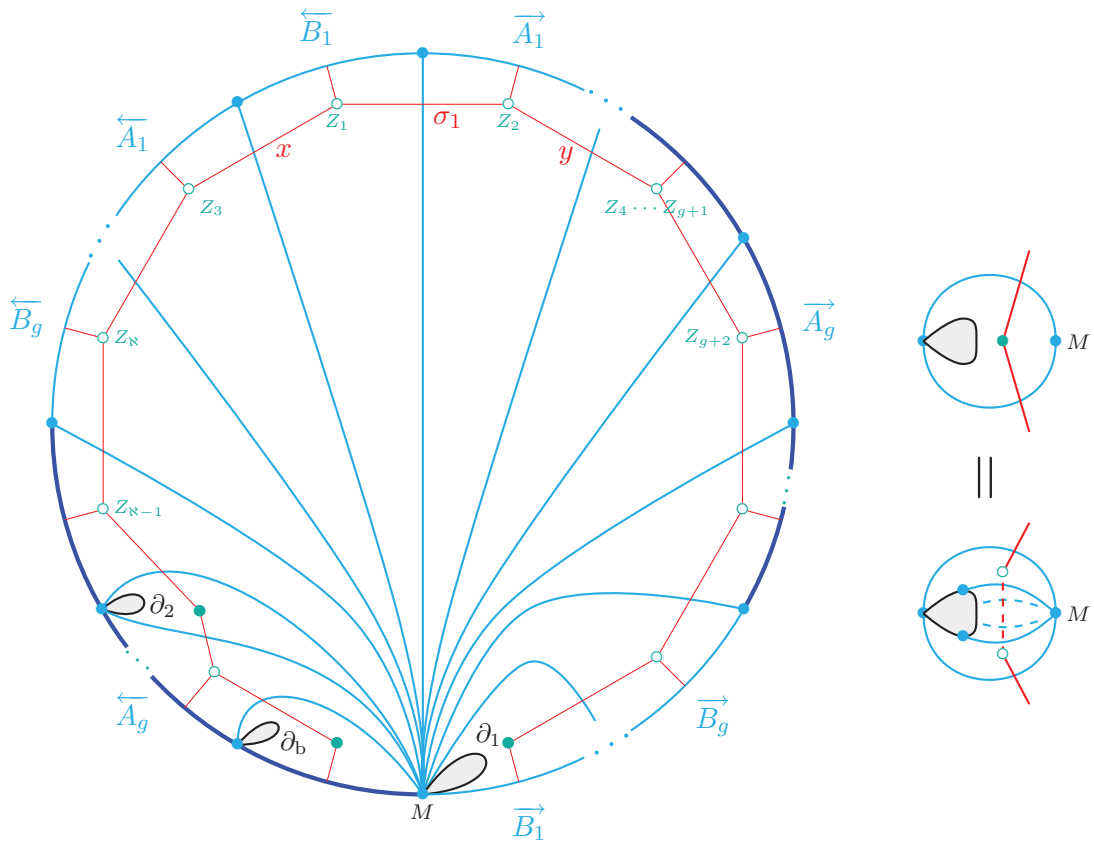


FIGURE 13 (colour online). A triangulation of S_{Δ} .

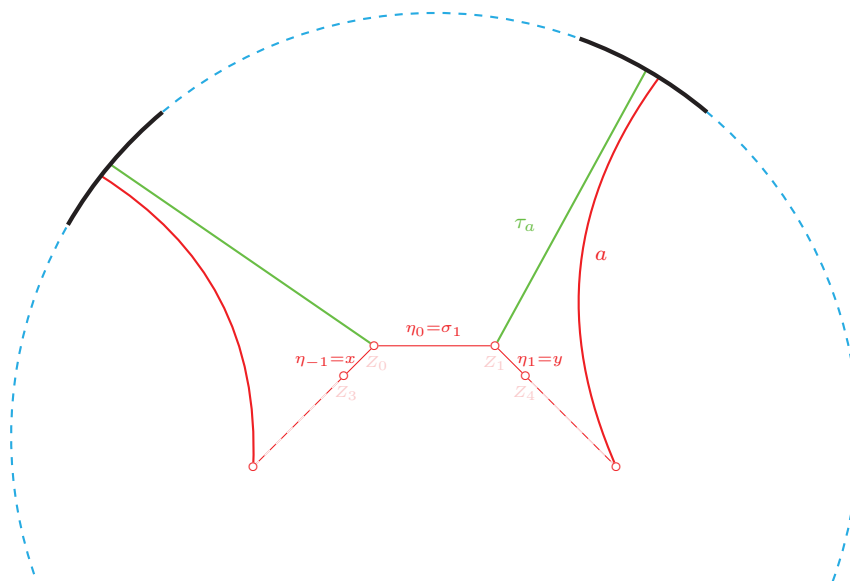


FIGURE 14 (colour online). $\tau_a = \tau_r$ (similarly for $\tau_b = \tau_s$).

By [20, Remark 3.10], all components of the exchange graph $\text{EG}(\mathbf{S}_\Delta)$ are identical, hence the theorem holds in general. \square

By Theorem 2.14, we obtain the following corollary, which provides finite presentations for spherical twist groups.

COROLLARY 5.10 (Presentations for spherical twist groups). *Let (Q, W) be a quiver with potential arising from a triangulated marked surface and Γ the associated Ginzburg dg algebra. Then the spherical twist group $\text{ST}(\Gamma)$ admits the finite presentation given in Definition 5.1 via the following isomorphism of groups:*

$$\begin{aligned} \text{ST}(\Gamma) &\xrightarrow{\cong} \text{Br}(Q, W), \\ \phi_{S_\gamma} &\mapsto \gamma \end{aligned} \quad (5.18)$$

where S_γ is the simple $\Gamma(Q, W)$ -module corresponds to a vertex γ of Q .

We also have the following infinite presentations for these twist groups.

THEOREM 5.11 (Infinite presentations). *The braid twist group $\text{BT}(\mathbf{S}_\Delta)$ admits the following presentation.*

- Generators: $B_\eta, \eta \in \text{CA}(\mathbf{S}_\Delta)$.
- Relations: all conjugation relations, that is,

$$B_{B_\alpha(\beta)} = B_\beta^{B_\alpha}, \quad \forall \alpha, \beta \in \text{CA}(\mathbf{S}_\Delta).$$

Let (Q, W) be a quiver with potential arising from a triangulated marked surface and Γ the associated Ginzburg dg algebra. Then the spherical twist group $\text{ST}(\Gamma)$ admits the following presentation.

- Generators: $\phi_X, X \in \text{Sph}(\Gamma)$.
- Relations: all conjugation relations,

$$\phi_{\phi_X(Y)} = \phi_Y^{\phi_X}, \quad \forall X, Y \in \text{Sph}(\Gamma).$$

Proof. The presentation for $\text{BT}(\mathbf{S}_\Delta)$ follows from the following three facts.

(1) The relations in Definition 5.1 are either commutation or braid between braid twists of conjugations of simple closed arcs, which are still closed arcs.

(2) Commutation/braid relations of two simple closed arcs are given by conjugation relations. More precisely, if two closed arcs α and β are disjoint, then $B_\alpha(\beta) = \beta$, which implies the following implications:

$$B_{B_\alpha(\beta)} = B_\beta^{B_\alpha} \implies B_\beta = B_\beta^{B_\alpha} \implies \text{Co}(B_\alpha, B_\beta);$$

and if two closed arcs α and β are disjoint except share a common endpoint, then $B_\alpha(\beta) = B_\beta(\alpha)$, which implies the following implications:

$$\left. \begin{aligned} B_{B_\alpha(\beta)} &= B_\beta^{B_\alpha} \\ B_{B_\beta(\alpha)} &= B_\alpha^{B_\beta} \end{aligned} \right\} \implies B_\beta^{B_\alpha} = B_\alpha^{B_\beta} \implies \text{Br}(B_\alpha, B_\beta).$$

(3) Any simple closed arcs can be obtained from the generators in Definition 5.1 by a finite sequence of conjugations of arcs; cf. [20, Proposition 4.3; 23, Proposition 2.3].

The presentation for $\text{ST}(\Gamma)$ follows from the isomorphism (2.2) and [23, Proposition 4.6] (cf. also [20, Corollary 6.5]) that the action of a spherical twist is compatible with the action of the corresponding braid twist. \square

Appendix A. Proof of relations (4.30)–(4.43) in Section 4

Set

$$v_{t,r} = \begin{cases} \tilde{x} & \text{if } r = t \\ \frac{\tilde{\tau}_r^{\tilde{x}\tilde{\tau}_t}}{\tilde{\tau}_r} & \text{if } r < t \text{ and } r \notin 2\mathbb{N}_{\leq g} - 1 \\ \frac{\tilde{\tau}_r^{\tilde{x}\tilde{\tau}_t}}{\tilde{\tau}_r} & \text{if } r < t \text{ and } r \in 2\mathbb{N}_{\leq g} - 1 \\ \frac{\tilde{\tau}_r^{\tilde{x}\tilde{\tau}_t}}{\tilde{\tau}_r} & \text{if } r > t \text{ and } t \notin 2\mathbb{N}_{\leq g} - 1 \\ \frac{\tilde{\tau}_r^{\tilde{x}\tilde{\tau}_t}}{\tilde{\tau}_r} & \text{if } r > t \text{ and } t \in 2\mathbb{N}_{\leq g} - 1 \end{cases} \tag{A.1}$$

and

$$v'_{t,r} = \begin{cases} \tilde{y} & \text{if } r = t \\ \frac{\tilde{\tau}_r^{\tilde{y}\tilde{\tau}_t}}{\tilde{\tau}_r} & \text{if } r < t \text{ and } r \notin 2\mathbb{N}_{\leq g} - 1 \\ \frac{\tilde{\tau}_r^{\tilde{y}\tilde{\tau}_t}}{\tilde{\tau}_r} & \text{if } r < t \text{ and } r \in 2\mathbb{N}_{\leq g} - 1 \\ \frac{\tilde{\tau}_r^{\tilde{y}\tilde{\tau}_t}}{\tilde{\tau}_r} & \text{if } r > t \text{ and } t \notin 2\mathbb{N}_{\leq g} - 1 \\ \frac{\tilde{\tau}_r^{\tilde{y}\tilde{\tau}_t}}{\tilde{\tau}_r} & \text{if } r > t \text{ and } t \in 2\mathbb{N}_{\leq g} - 1. \end{cases} \tag{A.2}$$

Using these notation, the equalities (4.27) and (4.29) become

$$\rho(\varepsilon_t) \cdot \tilde{\tau}_r = v_{t,r} \tilde{\sigma}_2 \tilde{\tau}_t \tag{A.3}$$

and

$$\rho(\varepsilon_t) \cdot \tilde{\tau}_r = v'_{t,r} \tilde{\sigma}_3 \tilde{\sigma}_1 \tilde{\sigma}_2, \tag{A.4}$$

respectively. Here, checking the equality (A.3) is straightforward. To show the equality (A.4), for $r = t$, we have

$$\rho(\varepsilon_t) \cdot \tilde{\tau}_t = \tilde{y} \tilde{\sigma}_3 \tilde{\sigma}_1 \tilde{\sigma}_2 \stackrel{(4.19)}{=} \tilde{\sigma}_2 \tilde{\sigma}_1 \tilde{\sigma}_2 \stackrel{(4.12)}{=} \tilde{\sigma}_1;$$

for $r < t$ and $r \notin 2\mathbb{N}_{\leq g} - 1$, we have

$$\rho(\varepsilon_t) \cdot \tilde{\tau}_r = \frac{\tilde{\tau}_r^{\tilde{y}\tilde{\tau}_t} \tilde{\sigma}_3 \tilde{\sigma}_1 \tilde{\sigma}_2}{\tilde{\tau}_r} \stackrel{(4.13)}{=} \frac{\tilde{y} \tilde{\sigma}_3 \tilde{\tau}_t \tilde{\sigma}_1 \tilde{\sigma}_2}{\tilde{\tau}_r} \stackrel{(4.19)}{=} \frac{\tilde{\sigma}_3 \tilde{\sigma}_2 \tilde{\tau}_t \tilde{\sigma}_1 \tilde{\sigma}_2}{\tilde{\tau}_r} \stackrel{(4.13)}{=} \frac{\tilde{\sigma}_2 \tilde{\tau}_t \tilde{\sigma}_1 \tilde{\sigma}_2}{\tilde{\tau}_r};$$

the other cases can be proved similarly.

The following two (classes of) relations will be useful.

LEMMA A.1. *The following hold:*

$$\text{Co}(v_{t,r}, \tilde{y}) \tag{A.5}$$

$$\text{Co}(v'_{t,r}, \tilde{x}) \tag{A.6}$$

for any $1 \leq r, t \leq 2g + b - 1$.

Proof. We only prove (A.5); the proof of (A.6) is similar. For $r = t$, (A.5) is $\text{Co}(\tilde{x}, \tilde{y})$ (4.20). For $r < t$ and $r \notin 2\mathbb{N}_{\leq g} - 1$, by (4.16), we have the commutative relation $\text{Co}(\tilde{\tau}_r^{\tilde{x}}, \tilde{\tau}_t^{\tilde{y}})$. Then we have the following implications:

$$\text{Co}(\tilde{\tau}_r^{\tilde{x}}, \tilde{\tau}_t^{\tilde{y}}) \implies \text{Co}(\tilde{\tau}_r^{\tilde{x}}, \tilde{y} \tilde{\tau}_t) \implies \text{Co}(\tilde{\tau}_r^{\tilde{x}\tilde{\tau}_t}, \tilde{y}) \implies \text{Co}(v_{t,r}, \tilde{y}),$$

where the first implication uses the braid relation $\text{Br}(\tilde{\tau}_t, \tilde{y})$ by (4.15), the second implication is taking the conjugation by $\tilde{\tau}_t$, the last implication uses $v_{t,r} = \tilde{\tau}_r^{\tilde{x}\tilde{\tau}_t}$ (A.1). The case $r < t$ and $r \in 2\mathbb{N}_{\leq g} - 1$ can be proved similarly.

For $r > t$ and $t \notin 2\mathbb{N}_{\leq g} - 1$, we have the commutation relation $\text{Co}(\tilde{\tau}_r^{\tilde{y}}, \tilde{\tau}_t^{\tilde{x}})$ by (4.16). Then we have the following implications:

$$\text{Co}(\tilde{\tau}_r^{\tilde{y}}, \tilde{\tau}_t^{\tilde{x}}) \implies \text{Co}(\tilde{\tau}_r^{\tilde{x}}, \tilde{\tau}_t^{\tilde{y}}) \implies \text{Co}(\tilde{\tau}_r^{\tilde{x}}, \tilde{y}^{\tilde{\tau}_t}) \implies \text{Co}(\tilde{\tau}_r^{\tilde{x}\tilde{\tau}_t}, \tilde{y}) \implies \text{Co}(v_{t,r}, \tilde{y}),$$

where the first implication uses the commutation relation $\text{Co}(\tilde{x}, \tilde{y})$ (4.20), the second implication uses the braid relation $\text{Br}(\tilde{\tau}_t, \tilde{y})$ by (4.15), the third implication is taking the conjugated by $\tilde{\tau}_t$, and the last equality uses $v_{t,r} = \tilde{\tau}_r^{\tilde{x}\tilde{\tau}_t}$ (A.1). The case $r > t$ and $t \in 2\mathbb{N}_{\leq g} - 1$ can be proved similarly. \square

LEMMA A.2. Relations (4.30), (4.31), (4.32), (4.37), (4.38) and (4.39) hold.

Proof. Using (4.26), (4.27), (4.11), (4.12), (4.13), (4.14) and (4.15), it is easy to check relations (4.30), (4.31), (4.32) for $i > 3$, (4.37), (4.38), and (4.39) for $i > 3$. Relation (4.32) for $i = 3$, conjugated by $\tilde{\tau}_t\tilde{\sigma}_2$, becomes (A.5). Relation (4.39) for $i = 3$, conjugated by $\tilde{\sigma}_2\tilde{\sigma}_1\tilde{\sigma}_3$, becomes (A.6). \square

LEMMA A.3. Relations (4.33), (4.34), (4.35) and (4.36) hold if the following relations hold, respectively:

$$\text{Br}(v_{t,r}, \tilde{\sigma}_3) \tag{A.7}$$

$$\text{Br}(v_{t,r}, \tilde{\tau}_t) \tag{A.8}$$

$$\text{Co}(v_{t,r}^{\tilde{\tau}_t}, v_{t,s}^{\tilde{\sigma}_3}) \quad \text{if } s < r \text{ and } s \notin 2\mathbb{N}_{\leq g} - 1 \tag{A.9}$$

$$\text{Co}(v_{t,r}^{\tilde{\tau}_t}, v_{t,s}^{\tilde{\sigma}_3}) \quad \text{if } s < r \text{ and } s \in 2\mathbb{N}_{\leq g} - 1. \tag{A.10}$$

Relations (4.40), (4.41), (4.42) and (4.43) hold if the following relations hold, respectively:

$$\text{Br}(v'_{t,r}, \tilde{\tau}_t) \tag{A.11}$$

$$\text{Br}(v'_{t,r}, \tilde{\sigma}_3) \tag{A.12}$$

$$\text{Co}(v'_{t,r}^{\tilde{\sigma}_3}, v'_{t,s}^{\tilde{\tau}_t}) \quad \text{if } s < r \text{ and } s \notin 2\mathbb{N}_{\leq g} - 1 \tag{A.13}$$

$$\text{Co}(v'_{t,r}^{\tilde{\sigma}_3}, v'_{t,s}^{\tilde{\tau}_t}) \quad \text{if } s < r \text{ and } s \in 2\mathbb{N}_{\leq g} - 1. \tag{A.14}$$

Proof. We only prove the first assertion, since the second can be proved similarly. Conjugated by $\tilde{\tau}_t\tilde{\sigma}_2\tilde{y}$, we have the following easy calculations:

$$(\rho(\varepsilon_t).\tilde{x})^{\tilde{\tau}_t\tilde{\sigma}_2\tilde{y}} = \tilde{\sigma}_2^{\tilde{\tau}_t\tilde{\tau}_t\tilde{\sigma}_2\tilde{y}} = \tilde{\sigma}_2^{\tilde{y}} = \tilde{\sigma}_2^{(\tilde{\sigma}_2^{\tilde{\sigma}_3})} = \tilde{\sigma}_3,$$

where the first equality uses $\rho(\varepsilon_t).\tilde{x} = \tilde{\sigma}_2^{\tilde{\tau}_t}$ by (4.44), the third equality uses $\tilde{y} = \tilde{\sigma}_2^{\tilde{\sigma}_3}$ (4.19), and the last equality uses the braid relation $\text{Br}(\tilde{\sigma}_2, \tilde{\sigma}_3)$ by (4.12);

$$(\rho(\varepsilon_t).\tilde{y})^{\tilde{\tau}_t\tilde{\sigma}_2\tilde{y}} = \tilde{y}^{\tilde{\tau}_t\tilde{\sigma}_2\tilde{y}} = \tilde{\tau}_t^{\tilde{y}\tilde{\sigma}_2\tilde{y}} = \tilde{\tau}_t^{\tilde{\sigma}_2\tilde{\sigma}_3\tilde{\sigma}_2\tilde{\sigma}_2\tilde{\sigma}_3\tilde{\sigma}_2} = \tilde{\tau}_t^{\tilde{\sigma}_3}\tilde{\tau}_t,$$

where the first equality uses $\rho(\varepsilon_t).\tilde{y} = \tilde{y}$ by (4.45), the second equality uses the braid relation $\text{Br}(\tilde{\tau}_t, \tilde{y})$ by (4.15), the third equality uses $\tilde{y} = \tilde{\sigma}_2^{\tilde{\sigma}_3}$ (4.19), the fourth equality uses the braid relation $\text{Br}(\tilde{\sigma}_2, \tilde{\sigma}_3)$ by (4.12), and the last equality uses the commutation relation $\text{Co}(\tilde{\tau}_t, \tilde{\sigma}_3)$ by (4.13);

$$(\rho(\varepsilon_t).\tilde{\sigma}_3)^{\tilde{\tau}_t\tilde{\sigma}_2\tilde{y}} = \tilde{\sigma}_3^{\tilde{\tau}_t\tilde{\sigma}_2\tilde{y}} = \tilde{\sigma}_3^{\tilde{\sigma}_2\tilde{y}}\tilde{y},$$

where the first equality uses $\rho(\varepsilon_t).\tilde{\sigma}_3 = \tilde{\sigma}_3$ (4.26), the second equality uses the commutation relation $\text{Co}(\tilde{\tau}_t, \tilde{\sigma}_3)$ (4.13), and the last equality uses $\tilde{y} = \tilde{\sigma}_3^{\tilde{\sigma}_2}$ (4.19);

$$(\rho(\varepsilon_t).\tilde{\tau}_r)^{\tilde{\tau}_t\tilde{\sigma}_2\tilde{y}} = v_{t,r}^{\tilde{\sigma}_2\tilde{\tau}_t\tilde{\tau}_t\tilde{\sigma}_2\tilde{y}} = v_{t,r}^{\tilde{y}} = v_{t,r},$$

where the first equality uses $\rho(\varepsilon_t).\tilde{\tau}_r = v_{t,r}^{\tilde{\sigma}_2\tilde{\tau}_t}$ (A.3), and the last equality uses the commutation relation $\text{Co}(v_{t,r}, \tilde{y})$.

Then relations (4.33), (4.34), (4.35) and (4.36), after conjugated by $\tilde{\tau}_t\tilde{\sigma}_2\tilde{y}$, become (A.7), (A.8), (A.9) and (A.10), respectively. \square

LEMMA A.4. Relations (A.7), (A.8), (A.11) and (A.12) hold.

Proof. We only show relations (A.7) and (A.8); the other two relations can be shown similarly. To show (A.7), note that by the braid relation $\text{Br}(\tilde{\tau}_r, \tilde{x})$ (4.14), the equality (A.1) becomes

$$v_{t,r} = \begin{cases} \tilde{x} & \text{if } r = t \\ \tilde{x}^{\tilde{\tau}_r\tilde{\tau}_t} & \text{if } r < t \text{ and } r \notin 2\mathbb{N}_{\leq g} - 1 \\ \tilde{x}^{\tilde{\tau}_r\tilde{\tau}_t} & \text{if } r < t \text{ and } r \in 2\mathbb{N}_{\leq g} - 1 \\ \tilde{x}^{\tilde{\tau}_r\tilde{\tau}_t} & \text{if } r > t \text{ and } t \notin 2\mathbb{N}_{\leq g} - 1 \\ \tilde{x}^{\tilde{\tau}_r\tilde{\tau}_t} & \text{if } r > t \text{ and } t \in 2\mathbb{N}_{\leq g} - 1. \end{cases}$$

So $v_{t,r} = \tilde{x}^A$, where A satisfies the commutation relation $\text{Co}(A, \tilde{\sigma}_3)$ because of the commutation relations $\text{Co}(\tilde{\tau}_t, \tilde{\sigma}_3)$ and $\text{Co}(\tilde{\tau}_r, \tilde{\sigma}_3)$ by (4.13). Then we have

$$v_{t,r}^{\tilde{\sigma}_3} = \tilde{x}^{A\tilde{\sigma}_3} = \tilde{x}^{\tilde{\sigma}_3A} = \tilde{\sigma}_3^{\tilde{x}A} = \tilde{\sigma}_3^{A\tilde{x}A} = \tilde{\sigma}_3^{v_{t,r}}$$

which implies (A.7), where the second and the fourth equalities uses the commutation relation $\text{Co}(A, \tilde{\sigma}_3)$, and the third equality uses the braid relation $\text{Br}(\tilde{x}, \tilde{\sigma}_3)$ (4.21).

To show (A.8), for $t = r$, (A.8) is $\text{Br}(\tilde{x}, \tilde{\tau}_r)$ (4.14) by $v_{t,r} = \tilde{x}$ (A.1). For $r < t$ and $r \notin 2\mathbb{N}_{\leq g} - 1$, by (4.23), we have $\text{Br}(\tilde{\tau}_r^{\tilde{x}}, \tilde{\tau}_t)$. Then we have the following easy implications:

$$\text{Br}(\tilde{\tau}_r^{\tilde{x}}, \tilde{\tau}_t) \implies \tilde{\tau}_r^{\tilde{x}} = \tilde{\tau}_t^{(\tilde{\tau}_r^{\tilde{x}})\tilde{\tau}_t} \implies \left(\tilde{\tau}_r^{\tilde{x}\tilde{\tau}_t}\right)^{\tilde{\tau}_t} = \tilde{\tau}_t^{(\tilde{\tau}_r^{\tilde{x}\tilde{\tau}_t})} \implies \text{Br}(\tilde{\tau}_r^{\tilde{x}\tilde{\tau}_t}, \tilde{\tau}_t)$$

which implies (A.8) because $v_{t,r} = \tilde{\tau}_r^{\tilde{x}\tilde{\tau}_t}$ in this case by (A.1). The other cases can be proved similarly. \square

LEMMA A.5. Relations (A.9), (A.10), (A.13) and (A.14) hold if $t = r$ or $t = s$.

Proof. We only show relation (A.9); the other relations can be shown similarly.

To show (A.9), if $t = r$, we have $v_{t,r} = \tilde{x}$ and $v_{t,s} = \tilde{\tau}_s^{\tilde{x}\tilde{\tau}_t}$. By (4.13), we have the commutation relations $\text{Co}(\tilde{\sigma}_3, \tilde{\tau}_s)$ and $\text{Co}(\tilde{\sigma}_3, \tilde{\tau}_t)$. So we have the following implications:

$$\text{Co}(\tilde{\sigma}_3, \tilde{\tau}_s) \implies \text{Co}(\tilde{\sigma}_3^{\tilde{x}}, \tilde{\tau}_s^{\tilde{x}}) \implies \text{Co}(\tilde{x}^{\tilde{\sigma}_3}, \tilde{\tau}_s^{\tilde{x}}) \implies \text{Co}(\tilde{x}^{\tilde{\tau}_t}, \tilde{\tau}_s^{\tilde{x}\tilde{\sigma}_3\tilde{\tau}_t}) \implies \text{Co}\left(\tilde{x}^{\tilde{\tau}_t}, \left(\tilde{\tau}_s^{\tilde{x}\tilde{\tau}_t}\right)^{\tilde{\sigma}_3}\right)$$

which is (A.9), where the first implication is taking the conjugation by \tilde{x} , the second implication uses the braid relation $\text{Br}(\tilde{x}, \tilde{\sigma}_3)$ (4.21), the third implication is taking the conjugation by $\tilde{\sigma}_3\tilde{\tau}_t$, the fourth implication uses the commutation relation $\text{Co}(\tilde{\sigma}_3, \tilde{\tau}_t)$. The case $t = s$ can be proved similarly. \square

Note that in the case when $2g + b - 1 \leq 2$, the assumption in Lemma A.5 holds automatically. It follows that all relations (4.30)–(4.43) hold in this case. Hence by Lemma 4.6, we have the following result.

LEMMA A.6. *If $2g + b - 1 \leq 2$, then the braid twist group $BT(S_{\aleph})$ has the presentation in Theorem 4.1.*

We denote by $S_{\aleph}^{g,b}$ a decorated surface S_{\aleph} with genus g and with b boundary components. So both $S_{\aleph}^{0,3}$ and $S_{\aleph}^{1,1}$ satisfy $2g + b - 1 = 2$, that is, they have generators $\sigma_i, 1 \leq i \leq \aleph - 1, \tau_1$ and τ_2 , where $\tau_1 \notin 2\mathbb{N}_{\leq g} - 1$ for $S_{\aleph}^{0,3}$ while $\tau_1 \in 2\mathbb{N}_{\leq g} - 1$ for $S_{\aleph}^{1,1}$. They can hence give a model for the relations between at most two τ_r 's in the following sense.

LEMMA A.7. *For any $1 \leq s < r \leq 2g + b - 1$, there is a group homomorphism:*

$$\begin{aligned} \psi_{\{s,r\}} : BT(S_{\aleph}^{0,3}) &\rightarrow \tilde{N} & \text{if } s \notin 2\mathbb{N}_{\leq g} - 1 \\ \psi_{\{s,r\}} : BT(S_{\aleph}^{1,1}) &\rightarrow \tilde{N} & \text{if } s \in 2\mathbb{N}_{\leq g} - 1 \end{aligned}$$

sending σ_i to $\tilde{\sigma}_i$, sending τ_1 to $\tilde{\tau}_s$ and sending τ_2 to $\tilde{\tau}_r$.

Proof. This follows directly from Lemma A.6 and the presentation of \tilde{N} . □

Now we can prove the relations in Lemma A.5 for the general case.

LEMMA A.8. *Relations (A.9), (A.10), (A.13) and (A.14) hold.*

Proof. By Lemma A.6, we may assume $\aleph \geq 5$. We only show relation (A.9); the other relations can be showed similarly. Due to Lemma A.5, the case when $t = r$ or $t = s$ has been proved. Since the assumption for (A.9) is $s < r$ and $s \notin 2\mathbb{N}_{\leq g} - 1$, we still need to consider the following cases: (1) $t < s < r$ and $t \notin 2\mathbb{N}_{\leq g} - 1$, (2) $s < t < r$ and $t \notin 2\mathbb{N}_{\leq g} - 1$, (3) $s < r < t$ and $r \notin 2\mathbb{N}_{\leq g} - 1$, (4) $t < s < r$ and $t \in 2\mathbb{N}_{\leq g} - 1$, (5) $s < t < r$ with $t \in 2\mathbb{N}_{\leq g} - 1$, (6) $s < r < t$ with $r \in 2\mathbb{N}_{\leq g} - 1$. We only prove the case (1), since the other cases can be proved similarly.

For case (1), we have $t < s < r, s \notin 2\mathbb{N}_{\leq g} - 1$ and $t \notin 2\mathbb{N}_{\leq g} - 1$. In the surface $S_{\aleph}^{0,3}$, using Lemmas 2.2 and 2.3, one can easily deduce the relations $\tau_2^{x\tau_1} = (\tau_1^{y\sigma_4})^{(\tau_2^{xy\sigma_4})}, \text{Co}(\tau_2^{x\sigma_3}, \tau_1^{xy\sigma_4})$ and $\text{Co}(\tau_2^{x\sigma_3}, \tau_1^{y\sigma_4})$. By Lemma A.7, these three relations can be transferred to the following relations in \tilde{N} : $\tilde{\tau}_s^{\tilde{x}\tilde{\tau}_t} = (\tilde{\tau}_t^{\tilde{y}\tilde{\sigma}_4})^{(\tilde{\tau}_s^{\tilde{x}\tilde{y}\tilde{\sigma}_4})}, \text{Co}(\tilde{\tau}_r^{\tilde{x}\tilde{\sigma}_3}, \tilde{\tau}_s^{\tilde{x}\tilde{y}\tilde{\sigma}_4})$ and $\text{Co}(\tilde{\tau}_r^{\tilde{x}\tilde{\sigma}_3}, \tilde{\tau}_t^{\tilde{y}\tilde{\sigma}_4})$. So we have the relation $\text{Co}(\tilde{\tau}_r^{\tilde{x}\tilde{\sigma}_3}, \tilde{\tau}_s^{\tilde{x}\tilde{\tau}_t})$. Conjugated by $\tilde{\sigma}_3$, we get $\text{Co}(\tilde{\tau}_r^{\tilde{x}}, \tilde{\tau}_s^{\tilde{x}\tilde{\tau}_t\tilde{\sigma}_3})$, which implies $\text{Co}(v_{t,r}^{\tilde{\tau}_t}, v_{t,s}^{\tilde{\sigma}_3})$ (A.9) because in this case $v_{t,r} = \tilde{\tau}_r^{\tilde{x}\tilde{\tau}_t}$ and $v_{t,s} = \tilde{\tau}_s^{\tilde{x}\tilde{\tau}_t}$ by (A.1). □

Appendix B. Calculations for the proof of Proposition 5.6

In the following proofs, the relations and conjugations used for an equivalence, implication or equality will be labeled there.

LEMMA B.1. *If $l = c^b$, $\text{Co}(a, e), \text{Br}(a, b)$ and $\text{Br}(b, e)$ hold, then*

$$\text{Co}(c^{ae}, b) \iff \text{Co}(b^{ae}, l).$$

Proof. We have

$$b^{ea} \xlongequal{\text{Co}(a,e)} b^{ae} \xlongequal{\text{Br}(a,b)} a^{be} \xlongequal{\text{Co}(a,e)} a^{ebe} \xlongequal{\text{Br}(b,e)} a^{beb} \xlongequal{\text{Br}(a,b)} b^{aeb}$$

and hence

$$\text{Co}(c^{ae}, b) \xleftrightarrow{\gamma^{ea}} \text{Co}(c, b^{ea}) \xleftrightarrow{\text{above}} \text{Co}(c, b^{aeb}) \xleftrightarrow{?^b} \text{Co}(c^b, b^{ae}) \xleftrightarrow{l=c^b} \text{Co}(l, b^{ae}). \quad \square$$

LEMMA B.2. If $h = b^e, \text{Br}(a, b), \text{Br}(b, e), \text{Br}(a, c), \text{Br}(a, e), \text{Br}(e, c), \text{Br}(e^b, c)$ hold, then

$$\text{Br}(a^b, c) \iff \text{Br}(h^{ec}, a), \quad \text{Br}(c^{ae}, b) \iff \text{Br}(c^a, h), \quad \text{Br}(c^{ea}, b) \iff \text{Br}(h^{ce}, a).$$

Proof. For the first one, we have

$$\text{Br}(h^{ec}, a) \xleftrightarrow{h=b^e} \text{Br}(b^c, a) \xleftrightarrow[\text{Br}(a,c)]{?e} \text{Br}(b, c^a) \xleftrightarrow[\text{Br}(a,b)]{?a} \text{Br}(a^b, c).$$

For the second one, we have $\text{Br}(c^a, h) \xleftrightarrow[\text{?e}]{h=b^e} \text{Br}(c^{ae}, b)$. For the third one, we have $\text{Br}(e^b, c) \xleftrightarrow[\text{?e}]{\text{Br}(b,e)} \text{Br}(b, c^e)$ and hence we have

$$\text{Br}(h^{ce}, a) \xleftrightarrow{h=b^e} \text{Br}(b^{(c^e)}, a) \xleftrightarrow[\text{?b}]{\text{Br}(b,c^e)} \text{Br}(c^e, a^b) \xleftrightarrow[\text{?a}]{\text{Br}(a,b)} \text{Br}(c^{ea}, b). \quad \square$$

LEMMA B.3. In (5.9), the last two equivalences follow from the relations above them.

Proof. We only show the first one, since the second one is similar.

$$\text{Br}(b^{ae}, l) \xleftrightarrow{?e} \text{Br}(b^a, l^e) \xleftrightarrow[\text{Br}(a,b)]{\text{Br}(e,l)} \text{Br}(a^b, e^l) \xleftrightarrow[\text{?b}]{l=c^b} \text{Br}(a, e^{bc}) \xleftrightarrow[\text{Br}(a,c)]{\text{Br}(e,b)} \text{Br}(c^a, b^e) \xleftrightarrow{?e} \text{Br}(c^{ae}, b). \quad \square$$

LEMMA B.4. If $g = a^f, \text{Br}(e, c), \text{Br}(g, b), \text{Co}(g, e)$ and $\text{Br}(f, a)$ hold, then we have

$$\text{Co}(e, f^{abc}) \iff \text{Co}(c^{ge}, b).$$

Proof. Note that $g = a^f$ and $\text{Br}(f, a)$ imply $g = f^a$. Then we have

$$\text{Co}(e, f^{abc}) \xleftrightarrow{g=f^a} \text{Co}(e, g^{bc}) \xleftrightarrow[\text{Br}(e,c)]{?e} \text{Co}(c^e, g^b) \xleftrightarrow[\text{?g}]{\text{Br}(g,b)} \text{Co}(c^{eg}, b) \xleftrightarrow{\text{Co}(g,e)} \text{Co}(c^{ge}, b). \quad \square$$

LEMMA B.5. Let (Q, W) be the last quiver with potential in Figure 12. Then $\text{Br}(Q, W)$ admits the following presentation.

- Generators: a, b, c, e, f .
- Relations: the relations in Lemma 5.4 for the 3-cycle between $\{f, a, e\}$ and

$$\text{Br}(a, b), \text{Br}(a, c), \text{Br}(e, b), \text{Br}(e, c), \text{Co}(f, b), \text{Co}(f, c), \text{Co}(e, f^{abc}).$$

Proof. We need to prove that the relations $\text{Br}(a^b, c), \text{Br}(e^b, c), \text{Br}(c^{ae}, b)$ and $\text{Br}(c^{ea}, b)$ hold.

To show the first two relations, set $g = a^f \xrightarrow{\text{Br}(f,a)} f^a$. We have $\text{Co}(f^a, e) \xrightarrow{g=f^a} \text{Co}(g, e)$ and $\text{Br}(a, b) \xrightarrow{\text{Co}(f,b)} \text{Br}(a, b^f) \xrightarrow[\text{g=a}^f]{?f} \text{Br}(g, b)$. So by Lemma B.4, $\text{Co}(c^{ge}, b)$ holds. Then by Lemma 5.5, we have $\text{Br}(g^b, c)$ and $\text{Br}(e^b, c)$. Hence we have

$$\text{Br}(g^b, c) \xrightarrow{g=a^f} \text{Br}(a^{fb}, c) \xrightarrow[\text{Co}(f,c)]{\text{Co}(f,b)} \text{Br}(a^b, c).$$

To show the third relation, we have

$$\text{Br}(a, c) \xrightarrow{\text{Co}(f,c)} \text{Br}(a, c^f) \xrightarrow[\text{g=a}^f]{?f} \text{Br}(g, c).$$

So we have

$$\text{Co}(c^{ge}, b) \xrightarrow[\text{Br}(b,e)]{?e} \text{Co}(c^g, e^b) \xrightarrow[\text{?c}]{\text{Br}(g,c)} \text{Co}(g, e^{bc}) \xrightarrow{g=f^a} \text{Co}(f^a, e^{bc})$$

and then

$$\begin{aligned} \mathrm{Br}(f, e) &\xrightarrow[\mathrm{Co}(f, b)]{\mathrm{Co}(f, c)} \mathrm{Br}(f^{cb}, e) \xrightarrow{?bc} \mathrm{Br}(f, e^{bc}) \xrightarrow[\mathrm{Co}(f^a, e^{bc})]{\mathrm{Co}(f^a, e^{bc})} \mathrm{Br}(f^{(f^a)}, e^{bc}) \xrightarrow{\mathrm{Br}(f, a)} \mathrm{Br}(a, e^{bc}) \\ &\xrightarrow[\mathrm{Br}(a, c)]{?e} \mathrm{Br}(c^a, e^b) \xrightarrow[\mathrm{Br}(c^a, e^b)]{\mathrm{Br}(e, b)} \mathrm{Br}(c^{ae}, b). \end{aligned}$$

The fourth relation can be proved similarly. \square

LEMMA B.6. *If $f = e^k, b = e^h, \mathrm{Br}(e, c), \mathrm{Br}(e, h), \mathrm{Co}(k, a), \mathrm{Br}(e, k), \mathrm{Br}(a, c)$, and $\mathrm{Br}(e, a)$ hold, then we have $\mathrm{Co}(e, f^{abc}) \iff \mathrm{Co}(c, k^{eah})$.*

Proof. First, we have

$$\mathrm{Co}(k, a) \xrightarrow{?e} \mathrm{Co}(k^e, a^e) \xrightarrow[\mathrm{Br}(e, a)]{\mathrm{Br}(e, k)} \mathrm{Co}(e^k, e^a) \xrightarrow{?a} \mathrm{Co}(e^{ka}, e) \xrightarrow{f=e^k} \mathrm{Co}(f^a, e).$$

So we have

$$\mathrm{Co}(e, f^{abc}) \xleftarrow[\mathrm{Br}(e, c)]{?e} \mathrm{Co}(c^e, f^{ab}) \xleftarrow{\mathrm{Co}(f^a, e)} \mathrm{Co}(c^e, f^{aeb}) \xleftarrow[\mathrm{Br}(e, k)]{b=e^h} \mathrm{Co}(c, e^{kaee^he}) \xleftarrow[\mathrm{Br}(e, k)]{\mathrm{Br}(e, h)} \mathrm{Co}(c, k^{eah}).$$

\square

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