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Endomorphism Algebras of 2-term Silting Complexes

Aslak Bakke Buan[1] · Yu Zhou[1]

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Abstract We study possible values of the global dimension of endomorphism algebras of 2-term silting complexes. We show that for any algebra A whose global dimension gl. dim $A \leq 2$ and any 2-term silting complex $\mathbf{P}$ in the bounded derived category $D^b(A)$ of A, the global dimension of $\mathrm{End}_{D^b(A)}(\mathbf{P})$ is at most 7. We also show that for each $n > 2$, there is an algebra A with gl. dim $A = n$ such that $D^b(A)$ admits a 2-term silting complex $\mathbf{P}$ with gl. dim $\mathrm{End}_{D^b(A)}(\mathbf{P})$ infinite.

Keywords 2-term silting complexes · Derived categories · Global dimensions

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1 Introduction

Let A be a finite dimensional algebra over a field k. Let T be a (classical) tilting module in the category mod A of finite dimensional right A-modules; that is the projective dimension pd T is at most 1, we have $\mathrm{Ext}^1_A(T, T) = 0$ and there is an exact sequence $0 \to A \to T_1 \to T_2 \to 0$ with T_1, T_2 in add T, the additive closure of T. Let $B = \mathrm{End}_A(T)$. Then, it is a well-known fact (see for example [8, III, Section 3.4] for a more general statement) that gl. dim $B \leq$ gl. dim $A + 1$, where gl. dim A denotes the global dimension of A.

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✉ Yu Zhou
yu.zhou@math.ntnu.no

Aslak Bakke Buan
aslakb@math.ntnu.no

1 Department of Mathematical Sciences, Norwegian University of Science and Technology, 7491 Trondheim, Norway

In this paper we investigate to which extent this generalizes to the following setting. We now consider a 2-term silting complex $\mathbf{P}$ in the bounded homotopy category of finitely generated projective A-modules, $K^b(\mathrm{proj}A)$. This is just a map between projective A-modules, considered as a complex, with the property that $\mathrm{Hom}_{K^b(\mathrm{proj}A)}(\mathbf{P}, \mathbf{P}[1]) = 0$ where [1] denotes the shift functor, and such that $\mathbf{P}$ generates $K^b(\mathrm{proj}A)$ as a triangulated category. Note that $K^b(\mathrm{proj}A)$ can be considered to be a full triangulated subcategory of the derived category $D^b(A)$.

The concept of silting complexes originated from [11], and has more recently been studied by many authors, often motivated by combinatorial aspects related to mutations, as in [2]. Moreover, the case of 2-term silting is of particular interest, see e.g. [1, 4] and [12].

In the setting of 2-term silting, we have the following theorem:

Theorem 1.1 *Let $B = End_{D^b(A)}(\mathbf{P})$, for a 2-term silting complex $\mathbf{P}$ in $K^b(projA)$. Then the following hold.*

(a) If gl. dim $A = 1$, then gl. dim $B \leq 3$.
(b) If gl. dim $A = 2$, then gl. dim $B \leq 7$.

Moreover, for each $n > 2$, there is an algebra A, with gl. dim $A = n$, such that $K^b(projA)$ admits a 2-term silting complex $\mathbf{P}$ with gl. dim $End_{D^b(A)}(\mathbf{P}) = \infty$.

Note that the projective presentation of a tilting A-module T as defined above, gives rise to a 2-term silting complex $\mathbf{P}_T$ in $K^b(\mathrm{proj}A)$, and that we have an isomorphism of algebras $\mathrm{End}_A(T) \cong \mathrm{End}_{D^b(A)}(\mathbf{P}_T)$.

The situation in part (a) was studied in [6]. In this case B is a called a *silted algebra*, and it was proved that silted algebras are so-called shod algebras [7], in particular this implies that gl. dim $B \leq 3$, by [9].

The main body of this paper is a proof of (b), an example that the global dimension of B actually can be 7 in this case, and a class of examples that justifies the last statement of Theorem 1.1.

We also prove that with a stronger assumption on $\mathbf{P}$, we actually get that gl. dim B is bounded by gl. dim A. More precisely, we show the following.

Theorem 1.2 *With the above notation, and assuming in addition that pd $H^0(\mathbf{P}) \leq 1$, we have gl. dim $B \leq 2$(gl. dim $A) + 2$.*

In the first section, we recall some notation and background concerning 2-term silting complexes and their endomorphism algebras. In the second section, we prove some preliminary general results. Then, in Sections 3 and 4, we prove respectively Theorem 1.2 and Theorem 1.1, while in the last section, we give some examples.

2 Background and Notation

Let A be a finite dimensional algebra with gl. dim $A = d$. Then $K^b(\mathrm{proj}A) = D^b(A) := \mathcal{D}$. Let $\mathbf{P}$ be a 2-term silting complex in $\mathcal{D}$ and let $B = \mathrm{End}_{\mathcal{D}}(\mathbf{P})$. We recall some classical notation (see e.g. [3]) and some results from [5], which will be used freely in the remaining of the paper.

Recall that a pair of subcategories $(\mathcal{X}, \mathcal{Y})$ of mod A, is called a *torsion pair*, if the following hold:

- $\mathrm{Hom}_A(\mathcal{X}, Y) = 0$ if and only if Y is in $\mathcal{Y}$, and
- $\mathrm{Hom}_A(X, \mathcal{Y}) = 0$ if and only if X is in $\mathcal{X}$.

For a given torsion pair $(\mathcal{X}, \mathcal{Y})$ and an object M in mod A, there is a (unique) exact sequence

$$0 \to tM \to M \to M/tM \to 0$$

with tM in $\mathcal{X}$ and M/tM in $\mathcal{Y}$. This is called the *canonical sequence* of M. Furthermore, for an A-module X we let add X denote the additive closure of X in mod A, and we let Fac$X$ denote the full subcategory of all quotients of modules in add X. The first notion is also used for a complex X in $\mathcal{D}$.

For a 2-term silting complex $\mathbf{P}$, consider the full subcategories of mod A given by

- $\mathcal{T}(\mathbf{P}) = \{X \in \text{mod } A \mid \mathrm{Hom}_{\mathcal{D}}(\mathbf{P}, X[1]) = 0$, and
- $\mathcal{F}(\mathbf{P}) = \{Y \in \text{mod } A \mid \mathrm{Hom}_{\mathcal{D}}(\mathbf{P}, Y) = 0$.

Furthermore, let $B = \mathrm{End}_{\mathcal{D}}(\mathbf{P})$. The following summarizes results from [5] which will be essential later in this paper.

Proposition 2.1 *Let* $\mathbf{P}$ *be a 2-term silting complex in* $K^b(\mathrm{proj}A)$*. Then the following hold.*

(a) *The pair* $(\mathcal{T}(\mathbf{P}), \mathcal{F}(\mathbf{P}))$ *is a torsion pair in mod A.*
(b) $\mathcal{T}(\mathbf{P}) = \mathrm{Fac}H^0(\mathbf{P})$.
(c) *The category* $\mathcal{C}(\mathbf{P}) = \{\mathbf{X} \in \mathcal{D} \mid \mathrm{Hom}(\mathbf{P}, \mathbf{X}[i]) = 0 \text{ for } i \neq 0\}$ *is an abelian category with short exact sequences coinciding with the triangles in* $\mathcal{D}$ *whose vertices are in* $\mathcal{C}(\mathbf{P})$.
(d) *Let* $\mathbf{X}$ *be in* $\mathcal{D}$*. Then we have that* $\mathbf{X}$ *is in* $\mathcal{C}(\mathbf{P})$ *if and only if* $H^0(\mathbf{X})$ *is in* $\mathcal{T}(\mathbf{P})$, $H^{-1}(\mathbf{X})$ *is in* $\mathcal{F}(\mathbf{P})$ *and* $H^i(\mathbf{X}) = 0$ *for* $i \neq -1, 0$.
(e) $\mathrm{Hom}_{\mathcal{D}}(\mathbf{P}, -)\colon \mathcal{C}(\mathbf{P}) \to \mathrm{mod}B$ *is an equivalence of (abelian) categories.*

For full subcategories $\mathcal{X}$ and $\mathcal{Y}$ of $\mathcal{D}$, we let $\mathcal{X} * \mathcal{Y}$ denote the full subcategory of $\mathcal{D}$ with objects Z appearing in a triangle

$$X \to Z \to Y \to X[1]$$

with X in $\mathcal{X}$ and Y in $\mathcal{Y}$. It follows from the octahedral axiom that we have $(\mathcal{X} * \mathcal{Y}) * \mathcal{Z} = \mathcal{X} * (\mathcal{Y} * \mathcal{Z})$, for three full subcategories $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$. The subcategory $\mathcal{X}$ is called *extension closed* if $\mathcal{X} * \mathcal{X} = \mathcal{X}$. We will need the following fact, which follows from [10, Propositions 2.1 and 2.4].

Lemma 2.2 *Let* $\mathcal{X}_i$ *be subcategories of* $\mathcal{D}$*, with* $\mathrm{Hom}_{\mathcal{D}}(\mathcal{X}_i, \mathcal{X}_j) = 0 = \mathrm{Hom}_{\mathcal{D}}(\mathcal{X}_i, \mathcal{X}_j[1])$ *for* $i < j$*. Then* $\mathcal{X}_1 * \mathcal{X}_2 * \cdots * \mathcal{X}_n$ *is closed under extensions and direct summands.*

3 Preliminaries

Now, fix a 2-term silting complex $\mathbf{P}$ in $K^b(\mathrm{proj}A)$, and let $\mathcal{P} = \mathrm{add}\,\mathbf{P}$. In this section we include some general observations on projective objects and projective dimensions in $\mathcal{C}(\mathbf{P})$.

For each $\mathbf{P}_0$ in $\mathcal{P}$, given by $P_0^{-1} \xrightarrow{p_0} P_0^0$, consider the canonical exact sequence of $H^{-1}(\mathbf{P}_0)$ relative to the torsion pair $(\mathcal{T}(\mathbf{P}), \mathcal{F}(\mathbf{P}))$:

$$0 \to tH^{-1}(\mathbf{P}_0) \to H^{-1}(\mathbf{P}_0) \to H^{-1}(\mathbf{P}_0)/tH^{-1}(\mathbf{P}_0) \to 0.$$

So $tH^{-1}(\mathbf{P}_0)$ is a submodule of P_0^{-1} and we denote by $\pi : P_0^{-1} \to P_0^{-1}/tH^{-1}(\mathbf{P}_0)$ the canonical epimorphism. Let $\widetilde{\mathbf{P}}_0$ be the complex $P_0^{-1}/tH^{-1}(\mathbf{P}_0) \xrightarrow{\widetilde{p}_0} P_0^0$, where $\widetilde{p}_0$ is the unique homomorphism such that the diagram

$$\begin{array}{ccccc} & & P_0^{-1}/tH^{-1}(\mathbf{P}_0) & & \\ & \stackrel{\pi}{\nearrow} & & \stackrel{\widetilde{p}_0}{\searrow} & \\ P_0^{-1} & & \xrightarrow{\quad p_0 \quad} & & P_0^0 \end{array}$$

commutes.

Let $\mathcal{P}_C = \mathcal{P} \cap \mathcal{C}(\mathbf{P})$.

Lemma 3.1 *Let $\mathbf{P}_0$ be in $\mathcal{P}$. Then $\mathbf{P}_0$ is in $\mathcal{P}_C$ if and only if $\mathbf{P}_0 \cong \widetilde{\mathbf{P}}_0$.*

Proof We have by definition that $\mathbf{P}_0 \cong \widetilde{\mathbf{P}}_0$ if and only if $tH^{-1}(\mathbf{P}_0) = 0$ if and only if $H^{-1}(\mathbf{P}_0)$ is in $\mathcal{F}(\mathbf{P})$ if and only if $\mathrm{Hom}(\mathbf{P}, H^{-1}(\mathbf{P}_0)) = 0$. It is straightforward to check that $\mathrm{Hom}(\mathbf{P}, H^{-1}(\mathbf{P}_0)) = 0$ if and only if $\mathrm{Hom}(\mathbf{P}, \mathbf{P}_0[-1]) = 0$. Moreover, we have that $\mathrm{Hom}(\mathbf{P}, \mathbf{P}_0[-1]) = 0$ if and only if $\mathbf{P}_0$ is in $\mathcal{C}(\mathbf{P})$, and the statement follows from this. □

Lemma 3.2 *With notation as above, the following hold.*

(a) *There is a triangle in $\mathcal{D}$:*

$$tH^{-1}(\mathbf{P})[1] \to \mathbf{P} \to \widetilde{\mathbf{P}} \to tH^{-1}(\mathbf{P})[2].$$

(b) *The object $\widetilde{\mathbf{P}}$ is a projective generator for $\mathcal{C}(\mathbf{P})$.*

Proof The triangle in (a) exists by the construction of $\widetilde{\mathbf{P}}$.

Note that $H^0(\widetilde{\mathbf{P}}) = H^0(\mathbf{P})$ is in $\mathcal{T}(\mathbf{P})$ and $H^{-1}(\widetilde{\mathbf{P}}) = H^{-1}(\mathbf{P})/tH^{-1}(\mathbf{P})$ is in $\mathcal{F}(\mathbf{P})$. Then by Proposition 2.1 (d), we have $\widetilde{\mathbf{P}} \in \mathcal{C}(\mathbf{P})$. Applying the functor $\mathrm{Hom}_{\mathcal{D}}(\mathbf{P}, -)$ to this triangle yields an isomorphism

$$\mathrm{Hom}_{\mathcal{D}}(\mathbf{P}, \mathbf{P}) \cong \mathrm{Hom}_{\mathcal{D}}(\mathbf{P}, \widetilde{\mathbf{P}})$$

as B-modules. Now (b) follows from Proposition 2.1 (e). □

For any integer i, we let $\mathcal{D}^{\leq i}(\mathbf{P}) = \{\mathbf{X} \in \mathcal{D} \mid \mathrm{Hom}_{\mathcal{D}}(\mathbf{P}, \mathbf{X}[j]) = 0 \text{ for } j > i\}$, and we let $\mathcal{D}^{\geq i}(\mathbf{P}) = \{\mathbf{X} \in \mathcal{D} \mid \mathrm{Hom}_{\mathcal{D}}(\mathbf{P}, \mathbf{X}[j]) = 0 \text{ for } j < i\}$.

Lemma 3.3 *With notation as above, we have: $\mathcal{C}(\mathbf{P}) \subset \mathcal{P} * \mathcal{P}[1] * \cdots * \mathcal{P}[d+1]$.*

Proof By [2, Proposition 2.23], we have

$$\mathcal{C}(\mathbf{P}) \subset \mathcal{D}^{\leq 0}(\mathbf{P}) \subset \mathcal{P} * \mathcal{P}[1] * \cdots * \mathcal{P}[l-1] * \mathcal{P}[l]$$

for some $l > 0$. For any $\mathbf{M}$ in $\mathcal{C}(\mathbf{P})$, by Proposition 2.1 (d), we have $H^i(\mathbf{M}) = 0$ for $i \neq -1, 0$. So there is a complex $\mathbf{X}$ of projective A-modules, which is equivalent to $\mathbf{M}$, and such that $H^i(\mathbf{X}) = 0$ for $i > 0$ or $i < -d-1$. So

$$\mathrm{Hom}_{\mathcal{D}}(\mathbf{M}, \mathbf{P}[i]) \cong \mathrm{Hom}_{\mathcal{D}}(\mathbf{X}, \mathbf{P}[i]) = 0, i \geq d+2,$$

which implies that $\mathbf{M}$ is in $\mathcal{P} * \mathcal{P}[1] * \cdots * \mathcal{P}[d+1]$. □

Lemma 3.4 *For a complex* $\mathbf{X}$ *in* $\mathcal{C}(\mathbf{P}) \cap (\mathcal{P}_C * \mathcal{P}_C[1] * \cdots \mathcal{P}_C[m])$ *for some* $m \geq 0$*, we have* $pd\, Hom_{\mathcal{D}}(\mathbf{P}, \mathbf{X})_B \leq m$.

Proof Let $\mathbf{X}_0 = \mathbf{X}$. There are triangles

$$\mathbf{X}_{i+1} \to \mathbf{O}_i \xrightarrow{g_i} \mathbf{X}_i \to \mathbf{X}_{i+1}[1], \quad 0 \leq i \leq m-1$$

where $\mathbf{O}_i$ is in $\mathcal{P}_C$ and $\mathbf{X}_i$ is in $\mathcal{P}_C * \mathcal{P}_C[1] * \cdots * \mathcal{P}_C[m-i]$. Since $\mathrm{Hom}_{\mathcal{D}}(\mathbf{P}, \mathbf{P}[i]) = 0$ for all $i > 0$, we have that g_i is a right $\mathcal{P}$-approximation of $\mathbf{X}_i$. By Lemma 3.1 and Lemma 3.2 (b), each $\mathbf{O}_i$ is projective in $\mathcal{C}(\mathbf{P})$. Assume that $\mathbf{X}_i$ is in $\mathcal{C}(\mathbf{P})$ for some $0 \leq i \leq m-1$. Then, since g_i is a right $\mathcal{P}-$approximation and $\mathbf{O}_i$ is projective in $\mathcal{C}(\mathbf{P})$, we have that g_i is an epimorphism in $\mathcal{C}(\mathbf{P})$. So $\mathbf{X}_{i+1}$ is the kernel of g_i, by Proposition 2.1 (c). Note that $\mathbf{X}_0 \in \mathcal{C}(\mathbf{P})$. Then by induction on i, we have that $\mathbf{X}_i \in \mathcal{C}(\mathbf{P})$ for all $0 \leq i \leq m$ and

$$\mathrm{pd}\,\mathrm{Hom}_{\mathcal{D}}(\mathbf{P}, \mathbf{X}_i)_B \leq \mathrm{pd}\,\mathrm{Hom}_{\mathcal{D}}(\mathbf{P}, \mathbf{X}_{i+1})_B + 1.$$

Therefore $\mathrm{pd}\,\mathrm{Hom}_{\mathcal{D}}(\mathbf{P}, \mathbf{X})_B \leq \mathrm{pd}\,\mathrm{Hom}_{\mathcal{D}}(\mathbf{P}, \mathbf{X}_m)_B + m = m$ since $\mathbf{X}_m \in \mathcal{P}_C$ is projective in $\mathcal{C}(\mathbf{P})$. □

We end this section by considering the following special case. Recall from [13], that a 2-term silting complex $\mathbf{P}$ in $K^b(\mathrm{proj}A)$ is a *tilting complex* if $\mathrm{Hom}_{\mathcal{D}}(\mathbf{P}, \mathbf{P}[-1]) = 0$.

Proposition 3.5 *If the 2-term silting complex* $\mathbf{P}$ *is a tilting complex, then gl. dimEnd*$_{\mathcal{D}}(\mathbf{P}) \leq$ *gl. dim* $A + 1$.

Proof If $\mathbf{P}$ is tilting, then $\mathbf{P}$ is in $\mathcal{C}(\mathbf{P})$. So we infer that $\mathcal{P} = \mathcal{P}_C$. It follows from Lemma 3.3 and Lemma 3.4 that gl. $\mathrm{dimEnd}_{\mathcal{D}}(\mathbf{P}) \leq$ gl. dim $A + 1$. □

Note that the classical situation (as in [8, III, section 3.4]) where $\mathbf{P}$ is the projective resolution of a classical tilting module, is covered by this result.

4 The Partial Tilting Case

Throughout this section, we assume that $\mathrm{pd}\, H^0(\mathbf{P})_A \leq 1$, that is: $H^0(\mathbf{P})$ is a partial tilting A-module. Then we have that $Q = H^{-1}(\mathbf{P})$ is projective as an A-module, and $\mathbf{P} \cong H^0(\mathbf{P}) \oplus Q[1]$. Consider the canonical exact sequence of Q relative to the torsion pair $(\mathcal{T}(\mathbf{P}), \mathcal{F}(\mathbf{P}))$:

$$0 \to tQ \to Q \to Q/tQ \to 0.$$

As before, we let $d =$ gl. dim A. We first prove two technical lemmas.

Lemma 4.1 *With the above notation, we have*

$$tQ \in add\, H^0(\mathbf{P}) * add\, H^0(\mathbf{P})[1] * \cdots * add\, H^0(\mathbf{P})[d-1].$$

Proof We first note that $tQ \in \mathcal{T}(\mathbf{P})$, so by definition $\mathrm{Hom}_{D^b(A)}(\mathbf{P}, tQ[i]) = 0$ for $i \neq 0$. In particular, we have $\mathrm{Hom}_{D^b(A)}(Q[1], tQ[i]) = 0$ for $i \neq 0$. For $i = 0$, since both Q and tQ are A-modules, we also have that $\mathrm{Hom}_{D^b(A)}(Q[1], tQ) = 0$. It follows from $T(\mathbf{P}) \in \mathcal{C}(\mathbf{P})$ that by Proposition 2.1, we have $tQ \in \mathcal{P} * \mathcal{P}[1] * \cdots * \mathcal{P}[d+1]$. Therefore, using $\mathbf{P} \cong H^0(\mathbf{P}) \oplus Q[1]$, we get that tQ is in add $H^0(\mathbf{P}) *$ add $H^0(\mathbf{P})[1] * \cdots *$ add $H^0(\mathbf{P})[d+1]$. By the canonical sequence of Q, we have $\mathrm{pd}\,(tQ)_A \leq \mathrm{pd}\,(Q/tQ)_A - 1 \leq d-1$. Hence, it follows that $\mathrm{Hom}(tQ, \mathbf{P}[d]) = 0 = \mathrm{Hom}(tQ, \mathbf{P}[d+1])$. The claim of the lemma follows. □

Lemma 4.2 *With the above notation, we have* $\mathcal{C}(\mathbf{P}) \subset \mathcal{P} * \mathcal{P}[1] * \cdots * \mathcal{P}[d] *$ *add* $H^0(\mathbf{P})[d+1]$.

Proof Using that $\mathbf{P} \cong H^0(\mathbf{P}) \oplus Q[1]$ in combination with Lemma 3.3, we only need to prove that $\mathrm{Hom}_{\mathcal{D}}(\mathbf{X}, Q[d+2]) = 0$ for $\mathbf{X} \in \mathcal{C}(\mathbf{P})$. This follows from $\mathrm{pd}\ H^i(\mathbf{X})_A \leq d$ for $i = -1, 0$ and $H^i(\mathbf{X}) = 0$ for $i \neq -1, 0$. □

We can now prove the main result of this section.

Theorem 4.3 *If* $\mathrm{pd}\,(H^0(\mathbf{P}))_A \leq 1$, *then gl. dim* $B \leq 2$*gl. dim* $A + 2$.

Proof Let $\mathbf{X}$ be an object in $\mathcal{C}(\mathbf{P})$ with

$$\mathbf{X} \in \mathcal{P} * \cdots * \mathcal{P}[i] * \text{ add } H^0(\mathbf{P})[i+1] * \cdots * \text{ add } H^0(\mathbf{P})[d+1]$$

for some $0 \leq i \leq d$. Then there is a triangle

$$\mathbf{X}_1 \to \mathbf{E} \stackrel{g_{\mathbf{X}}}{\to} \mathbf{X} \to \mathbf{X}_1[1]$$

where $g_{\mathbf{X}}$ is a right $\mathcal{P}-$approximation of $\mathbf{X}$ and $\mathbf{X}_1$ is in

$$\mathcal{P} * \cdots * \mathcal{P}[i-1] * \text{ add } H^0(\mathbf{P})[i] * \cdots * \text{ add } H^0(\mathbf{P})[d].$$

Then $\mathrm{Hom}_{\mathcal{D}}(\mathbf{P}, g_{\mathbf{X}})$ is an epimorphism and $\mathrm{Hom}_{\mathcal{D}}(\mathbf{P}, \mathbf{E})$ is projective in mod$B$.

Recall that $Q = H^{-1}(\mathbf{P})$. Then, by Lemma 3.2 there is a triangle

$$F[1] \to \mathbf{E} \to \widetilde{\mathbf{E}} \to F[2]$$

where F is in add $tQ \subset \mathcal{T}(\mathbf{P}) \subset \mathcal{C}(\mathbf{P})$ and $\widetilde{\mathbf{E}}$ is projective in $\mathcal{C}(\mathbf{P})$. So $\mathrm{Hom}_{\mathcal{D}}(F[1], \mathbf{X}) = 0$ since $\mathbf{X} \in \mathcal{C}(\mathbf{P})$. It follows that the map $g_{\mathbf{X}}$ factors through the map $\mathbf{E} \to \widetilde{\mathbf{E}}$. Then, by the octahedral axiom, we have the following commutative diagram of triangles:

$$\begin{array}{ccccccc}
 & & \mathbf{X}[-1] & = & \mathbf{X}[-1] & & \\
 & & \downarrow & & \downarrow & & \\
F[1] & \longrightarrow & \mathbf{X}_1 & \longrightarrow & \mathbf{X}' & \longrightarrow & F[2] \\
\| & & \downarrow & & \downarrow & & \| \\
F[1] & \longrightarrow & \mathbf{E} & \longrightarrow & \widetilde{\mathbf{E}} & \longrightarrow & F[2] \\
 & & \downarrow{\scriptstyle g_{\mathbf{X}}} & & \downarrow{\scriptstyle \widetilde{g}_{\mathbf{X}}} & & \\
 & & \mathbf{X} & = & \mathbf{X} & &
\end{array}$$

Then we have that

$$\begin{aligned}\mathbf{X}' &\in \operatorname{add}\mathbf{X}_1 * \operatorname{add} F[2]\\ &\subset \big(\mathcal{P} * \cdots * \mathcal{P}[i-1] * \operatorname{add} H^0(\mathbf{P})[i] * \cdots * \operatorname{add} H^0(\mathbf{P})[d]\big)\\ &\quad * \big(\operatorname{add} H^0(\mathbf{P}) * \cdots * \operatorname{add} H^0(\mathbf{P})[d-1]\big)[2]\\ &= \big(\mathcal{P} * \cdots * \mathcal{P}[i-1] * \operatorname{add} H^0(\mathbf{P})[i] * \cdots * \operatorname{add} H^0(\mathbf{P})[d]\big) * \operatorname{add} H^0(\mathbf{P})[d+1]\end{aligned}$$

where the inclusion is due to Lemma 4.1, and the equality follows from

$$\mathcal{P} * \cdots * \mathcal{P}[i-1] * \operatorname{add} H^0(\mathbf{P})[i] * \cdots * \operatorname{add} H^0(\mathbf{P})[d]$$

being closed under extensions by Lemma 2.2. Applying $\operatorname{Hom}_{\mathcal{D}}(\mathbf{P}, -)$ to the above diagram, we obtain a commutative diagram

$$\begin{array}{ccccc}\operatorname{Hom}_{\mathcal{D}}(\mathbf{P},\mathbf{E}) & & \xrightarrow{\ \cong\ } & & \operatorname{Hom}_{\mathcal{D}}(\mathbf{P},\widetilde{\mathbf{E}})\\ & {\scriptstyle \operatorname{Hom}_{\mathcal{D}}(\mathbf{P},g_{\mathbf{X}})}\searrow & & \swarrow{\scriptstyle \operatorname{Hom}_{\mathcal{D}}(\mathbf{P},\widetilde{g}_{\mathbf{X}})} & \\ & & \operatorname{Hom}_{\mathcal{D}}(\mathbf{P},\mathbf{X}) & & \end{array}$$

Using that the map $\operatorname{Hom}_{\mathcal{D}}(\mathbf{P}, g_{\mathbf{X}})$ is an epimorphism in $\operatorname{mod} B$, it follows that the map $\widetilde{g}_{\mathbf{X}}$ is an epimorphism in $\mathcal{C}(\mathbf{P})$. Then $\mathbf{X}'$ is the kernel of $\widetilde{g}_{\mathbf{X}}$ in $\mathcal{C}(\mathbf{P})$. Hence

$$\operatorname{pd}\operatorname{Hom}_{\mathcal{D}}(\mathbf{P},\mathbf{X})_B \leq \operatorname{pd}\operatorname{Hom}_{\mathcal{D}}(\mathbf{P},\mathbf{X}')_B + 1.$$

Using induction on i and Lemma 4.2, we have that for $\mathbf{X} \in \mathcal{C}(\mathbf{P})$, there is $\mathbf{X}'$ such that

$$\mathbf{X}' \in \mathcal{C}(\mathbf{P}) \cap \Big(\operatorname{add} H^0(\mathbf{P}) * \operatorname{add} H^0(\mathbf{P})[1] * \cdots * \operatorname{add} H^0(\mathbf{P})[d+1]\Big) \tag{1}$$

and $\operatorname{pd}\operatorname{Hom}_{\mathcal{D}}(\mathbf{P},\mathbf{X})_B \leq \operatorname{pd}\operatorname{Hom}_{\mathcal{D}}(\mathbf{P},\mathbf{X}')_B + d + 1$. By Lemma 3.4 and Eq. 1, we have $\operatorname{pd}\operatorname{Hom}_{\mathcal{D}}(\mathbf{P},\mathbf{X}')_B \leq d+1$. It then follows that $\operatorname{pd}\operatorname{Hom}_{\mathcal{D}}(\mathbf{P},\mathbf{X})_B \leq 2d+2$, and hence $\operatorname{gl.dim} B \leq 2d+2$. □

5 The Case of Global Dimension 2

In this section, we consider the case when $\operatorname{gl.dim} A \leq 2$. Our aim is to prove part (b) of Theorem 1.1, stating that in this case we have that the global dimension is at most 7 for the endomorphism algebra of any 2-term silting complex.

We prepare by showing four technical lemmas. Let $\mathcal{P}_{\mathcal{C}}^{[0,1]} = (\mathcal{P} * \mathcal{P}[1]) \cap \mathcal{C}(\mathbf{P})$.

Lemma 5.1 *If $\mathbf{X}$ is in $\mathcal{P}_{\mathcal{C}}^{[0,1]}$, then $\operatorname{pd}\operatorname{Hom}_{\mathcal{D}}(\mathbf{P},\mathbf{X})_B \leq 1$.*

Proof Since $\mathbf{X}$ is in $\mathcal{P}*\mathcal{P}[1]$, there is a triangle $\mathbf{O}_1 \to \mathbf{O}_0 \to \mathbf{X} \to \mathbf{O}_1[1]$ with $\mathbf{O}_0, \mathbf{O}_1 \in \mathcal{P}$. Applying the functor $\operatorname{Hom}_{\mathcal{D}}(\mathbf{P}, -)$ to this triangle, we get a long exact sequence

$$\operatorname{Hom}_{\mathcal{D}}(\mathbf{P},\mathbf{X}[-1]) \to \operatorname{Hom}_{\mathcal{D}}(\mathbf{P},\mathbf{O}_1) \to \operatorname{Hom}_{\mathcal{D}}(\mathbf{P},\mathbf{O}_0) \to \operatorname{Hom}_{\mathcal{D}}(\mathbf{P},\mathbf{X}) \to \operatorname{Hom}_{\mathcal{D}}(\mathbf{P},\mathbf{O}_1[1])$$

where the first term is zero since $\mathbf{X}$ is in $\mathcal{C}(\mathbf{P})$, and the last term is zero since $\operatorname{Hom}_{\mathcal{D}}(\mathbf{P},\mathbf{P}[1]) = 0$. Therefore, $\operatorname{pd}\operatorname{Hom}_{\mathcal{D}}(\mathbf{P},\mathbf{X})_B \leq 1$. □

Lemma 5.2 *If $\mathbf{X}$ is in $\mathcal{C}(\mathbf{P}) \cap \Big(\mathcal{P}_{\mathcal{C}}^{[0,1]} * \mathcal{P}_{\mathcal{C}}^{[0,1]}[1] * \mathcal{P}_{\mathcal{C}}^{[0,1]}[2]\Big)$, then $\operatorname{pd}\operatorname{Hom}_{\mathcal{D}}(\mathbf{P},\mathbf{X})_B \leq 3$.*

Proof By $\mathbf{X} \in \mathcal{P}_{\mathcal{C}}^{[0,1]} * \mathcal{P}_{\mathcal{C}}^{[0,1]}[1] * \mathcal{P}_{\mathcal{C}}^{[0,1]}[2]$, there are triangles

$$\mathbf{L} \to \mathbf{D}_1 \to \mathbf{X} \to \mathbf{L}[1] \tag{2}$$

and

$$\mathbf{D}_3 \to \mathbf{D}_2 \to \mathbf{L} \to \mathbf{D}_3[1] \tag{3}$$

with $\mathbf{D}_1, \mathbf{D}_2, \mathbf{D}_3 \in \mathcal{P}_{\mathcal{C}}^{[0,1]}$ and $\mathbf{L} \in \mathcal{P}_{\mathcal{C}}^{[0,1]} * \mathcal{P}_{\mathcal{C}}^{[0,1]}[1] \subset \mathcal{P} * \mathcal{P}[1] * \mathcal{P}[2]$. Applying $\mathrm{Hom}_{\mathcal{D}}(\mathbf{P}, -)$ to triangle (2), we obtain a long exact sequence

$$\begin{aligned}\mathrm{Hom}_{\mathcal{D}}(\mathbf{P}, \mathbf{X}[-2]) &\to \mathrm{Hom}_{\mathcal{D}}(\mathbf{P}, \mathbf{L}[-1]) \to \mathrm{Hom}_{\mathcal{D}}(\mathbf{P}, \mathbf{D}_1[-1]) \to \mathrm{Hom}_{\mathcal{D}}(\mathbf{P}, \mathbf{X}[-1]) \\ &\to \mathrm{Hom}_{\mathcal{D}}(\mathbf{P}, \mathbf{L}) \to \mathrm{Hom}_{\mathcal{D}}(\mathbf{P}, \mathbf{D}_1) \to \mathrm{Hom}_{\mathcal{D}}(\mathbf{P}, \mathbf{X}) \to \mathrm{Hom}_{\mathcal{D}}(\mathbf{P}, \mathbf{L}[1]).\end{aligned}$$

We have $\mathrm{Hom}_{\mathcal{D}}(\mathbf{P}, \mathbf{X}[i]) = 0$ for $i = -1$ or $i = -2$, since $\mathbf{X}$ is in $\mathcal{C}(\mathbf{P})$. Furthermore, we have $\mathrm{Hom}_{\mathcal{D}}(\mathbf{P}, \mathbf{D}_1[-1]) = 0$, by $\mathbf{D}_1 \in \mathcal{C}(\mathbf{P})$ and $\mathrm{Hom}_{\mathcal{D}}(\mathbf{P}, \mathbf{L}[1]) = 0$ by $\mathbf{L} \in \mathcal{P} * \mathcal{P}[1] * \mathcal{P}[2]$. From this it follows that we have a short exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{D}}(\mathbf{P}, \mathbf{L}) \to \mathrm{Hom}_{\mathcal{D}}(\mathbf{P}, \mathbf{D}_1) \to \mathrm{Hom}_{\mathcal{D}}(\mathbf{P}, \mathbf{X}) \to 0$$

and that $\mathrm{Hom}_{\mathcal{D}}(\mathbf{P}, \mathbf{L}[-1]) = 0$. Using this short exact sequence, it follows that

$$\mathrm{pd}\,\mathrm{Hom}_{\mathcal{D}}(\mathbf{P}, \mathbf{X})_B \leq \max\{\mathrm{pd}\,\mathrm{Hom}_{\mathcal{D}}(\mathbf{P}, \mathbf{D}_1)_B, \mathrm{pd}\,\mathrm{Hom}_{\mathcal{D}}(\mathbf{P}, \mathbf{L})_B + 1\}. \tag{4}$$

Applying $\mathrm{Hom}_{\mathcal{D}}(\mathbf{P}, -)$ to the triangle (3), we obtain an exact sequence

$$0 = \mathrm{Hom}_{\mathcal{D}}(\mathbf{P}, \mathbf{L}[-1]) \to \mathrm{Hom}_{\mathcal{D}}(\mathbf{P}, \mathbf{D}_3) \to \mathrm{Hom}_{\mathcal{D}}(\mathbf{P}, \mathbf{D}_2) \to \mathrm{Hom}_{\mathcal{D}}(\mathbf{P}, \mathbf{L}) \to \mathrm{Hom}_{\mathcal{D}}(\mathbf{P}, \mathbf{D}_3[1])$$

where the last term is zero due to $\mathbf{D}_3 \in \mathcal{P}_{\mathcal{C}}^{[0,1]}$. As above, we obtain that

$$\mathrm{pd}\,\mathrm{Hom}_{\mathcal{D}}(\mathbf{P}, \mathbf{L})_B \leq \max\{\mathrm{pd}\,\mathrm{Hom}_{\mathcal{D}}(\mathbf{P}, \mathbf{D}_2)_B, \mathrm{pd}\,\mathrm{Hom}_{\mathcal{D}}(\mathbf{P}, \mathbf{D}_3)_B + 1\}. \tag{5}$$

Now, combining the inequalities (4) and (5) with Lemma 5.1, we obtain $\mathrm{pd}\,\mathrm{Hom}_{\mathcal{D}}(\mathbf{P}, \mathbf{X})_B \leq 3$. □

Lemma 5.3 *If* $\mathbf{N}$ *is in* $\mathcal{D}^{\geq -1}(\mathbf{P}) \cap (\mathcal{P} * \mathcal{P}[1] * \mathcal{P}[2])$, *then there is an object* $\widetilde{\mathbf{N}} \in \mathcal{C}(\mathbf{P})$ *such that* $\mathrm{Hom}_{\mathcal{D}}(\mathbf{P}, \mathbf{N}) \cong \mathrm{Hom}_{\mathcal{D}}(\mathbf{P}, \widetilde{\mathbf{N}})$ *as* $B-$*modules and* $\widetilde{\mathbf{N}} \in \mathrm{add}\,\mathbf{N} * \mathcal{P}_{\mathcal{C}}^{[0,1]}[2]$.

Proof Since $(\mathcal{D}^{\leq 0}(\mathbf{P}), \mathcal{D}^{\geq 0}(\mathbf{P}))$ is a t-structure (see [12, Lemma 5.10]), there is a triangle

$$\mathbf{M} \to \mathbf{N} \to \widetilde{\mathbf{N}} \to \mathbf{M}[1] \tag{6}$$

with $\mathbf{M} \in \mathcal{D}^{\leq 0}(\mathbf{P})[1]$ and $\widetilde{\mathbf{N}} \in \mathcal{D}^{\geq 0}(\mathbf{P})$. Then $\mathbf{M} \in \mathcal{P}[1] * \cdots * \mathcal{P}[l]$ for some l by [2, Proposition 2.23]. Applying the functor $\mathrm{Hom}_{\mathcal{D}}(\mathbf{P}, -)$ to the triangle (6), we have a long exact sequence

$$\cdots \to \mathrm{Hom}_{\mathcal{D}}(\mathbf{P}, \mathbf{M}[i]) \to \mathrm{Hom}_{\mathcal{D}}(\mathbf{P}, \mathbf{N}[i]) \to \mathrm{Hom}_{\mathcal{D}}(\mathbf{P}, \widetilde{\mathbf{N}}[i]) \to \mathrm{Hom}_{\mathcal{D}}(\mathbf{P}, \mathbf{M}[i+1]) \to \cdots$$

Since $\mathrm{Hom}_{\mathcal{D}}(\mathbf{P}, \mathbf{M}[i]) = 0$ for $i \geq 0$ by $\mathbf{M} \in \mathcal{P}[1] * \cdots * \mathcal{P}[l]$ and $\mathrm{Hom}_{\mathcal{D}}(\mathbf{P}, \widetilde{\mathbf{N}}[i]) = 0$ for $i < 0$ by $\widetilde{\mathbf{N}} \in \mathcal{D}^{\geq 0}(\mathbf{P})$, and also $\mathrm{Hom}_{\mathcal{D}}(\mathbf{P}, \mathbf{N}[i]) = 0$ for $i \neq -1, 0$ by the assumption $\mathbf{N} \in \mathcal{D}^{\geq -1}(\mathbf{P}) \cap (\mathcal{P} * \mathcal{P}[1] * \mathcal{P}[2])$, we have that

$$\mathrm{Hom}_{\mathcal{D}}(\mathbf{P}, \mathbf{N}) \cong \mathrm{Hom}_{\mathcal{D}}(\mathbf{P}, \widetilde{\mathbf{N}})$$

as $B-$modules,

$$\mathrm{Hom}_{\mathcal{D}}(\mathbf{P}, \widetilde{\mathbf{N}}[i]) = 0, \ \mathrm{for} i > 0,$$

and

$$\mathrm{Hom}_{\mathcal{D}}(\mathbf{P}, \mathbf{M}[i]) = 0, \text{ for} i < -1.$$

Thus, we obtain $\widetilde{\mathbf{N}} \in \mathcal{D}^{\leq 0}(\mathbf{P}) \cap \mathcal{D}^{\geq 0}(\mathbf{P}) = \mathcal{C}(\mathbf{P})$ and $\mathbf{M} \in \mathcal{D}^{\geq 0}(\mathbf{P})[1] \cap \mathcal{D}^{\leq 0}(\mathbf{P})[1] = \mathcal{C}(\mathbf{P})[1]$. Then by Lemma 3.3, we have $\widetilde{\mathbf{N}} \in \mathcal{P} * \mathcal{P}[1] * \mathcal{P}[2] * \mathcal{P}[3]$. Applying the functor $\mathrm{Hom}_{\mathcal{D}}(-, \mathbf{P})$ to the triangle (6), we obtain a long exact sequence

$$\cdots \to \mathrm{Hom}_{\mathcal{D}}(\widetilde{\mathbf{N}}, \mathbf{P}[i]) \to \mathrm{Hom}_{\mathcal{D}}(\mathbf{N}, \mathbf{P}[i]) \to \mathrm{Hom}_{\mathcal{D}}(\mathbf{M}, \mathbf{P}[i]) \to \mathrm{Hom}_{\mathcal{D}}(\widetilde{\mathbf{N}}, \mathbf{P}[i+1]) \to \cdots$$

We have $\mathrm{Hom}_{\mathcal{D}}(\mathbf{N}, \mathbf{P}[i]) = 0$ for $i > 2$ by $\mathbf{N} \in \mathcal{P} * \mathcal{P}[1] * \mathcal{P}[2]$, and we have $\mathrm{Hom}_{\mathcal{D}}(\widetilde{\mathbf{N}}, \mathbf{P}[i]) = 0$ for $i > 3$ by $\widetilde{\mathbf{N}} \in \mathcal{P} * \mathcal{P}[1] * \mathcal{P}[2] * \mathcal{P}[3]$. From this it follows that $\mathrm{Hom}_{\mathcal{D}}(\mathbf{M}, \mathbf{P}[i]) = 0$ for $i > 2$, and hence $\mathbf{M} \in (\mathcal{P}[1] * \mathcal{P}[2]) \cap \mathcal{C}(\mathbf{P})[1] = \mathcal{P}_{\mathcal{C}}^{[0,1]}[1]$. Therefore we have that

$$\widetilde{\mathbf{N}} \in \operatorname{add} \mathbf{N} * \operatorname{add} \mathbf{M}[1] \subset \operatorname{add} \mathbf{N} * \mathcal{P}_{\mathcal{C}}^{[0,1]}[2].$$

□

Lemma 5.4 *Let* $\mathbf{X} \in \mathcal{C}(\mathbf{P}) \cap (\mathcal{P} * \mathcal{P}[1] * \cdots * \mathcal{P}[t] * \mathcal{H}[t+1])$ *for some* t *with* $0 \leq t \leq 3$, *where* $\mathcal{H} \subset \mathcal{P} * \mathcal{P}[1] * \cdots * \mathcal{P}[2-t]$ *(and* $\mathcal{H} = 0$ *for* $t = 3$*). Then for each* r *with* $0 \leq r \leq \min\{t+1, 3\}$, *there is an object* $\widetilde{\mathbf{X}}_r \in \mathcal{C}(\mathbf{P})$ *such that*

$$pd\, Hom_{\mathcal{D}}(\mathbf{P}, \mathbf{X})_B \leq pd\, Hom_{\mathcal{D}}(\mathbf{P}, \widetilde{\mathbf{X}}_r)_B + r$$

and

$$\widetilde{\mathbf{X}}_r \in \mathcal{P} * \mathcal{P}[1] * \cdots * \mathcal{P}[t-r] * \mathcal{H}[t+1-r] * \mathcal{P}_{\mathcal{C}}^{[0,1]}[3-r] * \cdots * \mathcal{P}_{\mathcal{C}}^{[0,1]}[2]$$

where $\mathcal{P} * \mathcal{P}[1] * \cdots * \mathcal{P}[t-r]$ *is taken to be 0 when* $r = t+1$ *and* $\mathcal{P}_{\mathcal{C}}^{[0,1]}[3-r] * \cdots * \mathcal{P}_{\mathcal{C}}^{[0,1]}[2]$ *is taken to be 0 when* $r = 0$.

Proof Let $\widetilde{\mathbf{X}}_0 = \mathbf{X}$. Then $\widetilde{\mathbf{X}}_0$ satisfies the conditions in the lemma. Assume that $\widetilde{\mathbf{X}}_{r-1}$ satisfying the conditions. By

$$\widetilde{\mathbf{X}}_{r-1} \in \mathcal{P} * \mathcal{P}[1] * \cdots * \mathcal{P}[t-(r-1)] * \mathcal{H}[t+1-(r-1)] * \mathcal{P}_{\mathcal{C}}^{[0,1]}[3-(r-1)] * \cdots * \mathcal{P}_{\mathcal{C}}^{[0,1]}[2],$$

there is a triangle

$$\mathbf{X}_r \to \mathbf{P}_0 \to \widetilde{\mathbf{X}}_{r-1} \to \mathbf{X}_r[1]$$

with $\mathbf{P}_0 \in \mathcal{P}$, $\mathbf{X}_r \in \mathcal{P} * \cdots * \mathcal{P}[t-r] * \mathcal{H}[t+1-r] * \mathcal{P}_{\mathcal{C}}^{[0,1]}[3-r] * \cdots * \mathcal{P}_{\mathcal{C}}^{[0,1]}[1] \subset \mathcal{P} * \mathcal{P}[1] * \mathcal{P}[2]$. The inclusion follows from Lemma 2.2. Applying $\mathrm{Hom}_{\mathcal{D}}(\mathbf{P}, -)$ to this triangle, we have a long exact sequence

$$\cdots \to \mathrm{Hom}_{\mathcal{D}}(\mathbf{P}, \mathbf{X}_r[i]) \to \mathrm{Hom}_{\mathcal{D}}(\mathbf{P}, \mathbf{P}_0[i]) \to \mathrm{Hom}_{\mathcal{D}}(\mathbf{P}, \widetilde{\mathbf{X}}_{r-1}[i]) \to \mathrm{Hom}_{\mathcal{D}}(\mathbf{P}, \mathbf{X}_r[i+1]) \to \cdots$$

Since $\mathrm{Hom}_{\mathcal{D}}(\mathbf{P}, \widetilde{\mathbf{X}}_{r-1}[i]) = 0$ for $i \neq 0$ by $\widetilde{\mathbf{X}}_{r-1} \in \mathcal{C}(\mathbf{P})$, $\mathrm{Hom}_{\mathcal{D}}(\mathbf{P}, \mathbf{X}_r[1]) = 0$ by $\mathbf{X}_r \in \mathcal{P} * \mathcal{P}[1] * \mathcal{P}[2]$, and also $\mathrm{Hom}_{\mathcal{D}}(\mathbf{P}, \mathbf{P}_0[i]) = 0$ for $i < -1$ by $\mathbf{P}$ being 2-term, we have a short exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{D}}(\mathbf{P}, \mathbf{X}_r) \to \mathrm{Hom}_{\mathcal{D}}(\mathbf{P}, \mathbf{P}_0) \to \mathrm{Hom}_{\mathcal{D}}(\mathbf{P}, \widetilde{\mathbf{X}}_{r-1}) \to 0$$

and

$$\mathrm{Hom}_{\mathcal{D}}(\mathbf{P}, \mathbf{X}_r[i]) = 0 \text{ for } i < -1.$$

Then $\mathrm{pd}\,\mathrm{Hom}_{\mathcal{D}}(\mathbf{P}, \widetilde{\mathbf{X}}_{r-1})_B \leq \mathrm{pd}\,\mathrm{Hom}_{\mathcal{D}}(\mathbf{P}, \mathbf{X}_r)_B + 1$ and by Lemma 5.3, there is an object $\widetilde{\mathbf{X}}_r \in \mathcal{C}(\mathbf{P})$ such that $\mathrm{Hom}_{\mathcal{D}}(\mathbf{P}, \widetilde{\mathbf{X}}_r)_B \cong \mathrm{Hom}_{\mathcal{D}}(\mathbf{P}, \mathbf{X}_r)_B$ and

$$\widetilde{\mathbf{X}}_r \in \mathrm{add}\,\mathbf{X}_r * \mathcal{P}_{\mathcal{C}}^{[0,1]}[2] \subset \mathcal{P} * \mathcal{P}[1] * \cdots * \mathcal{P}[t-r] * \mathcal{H}[t+1-r] * \mathcal{P}_{\mathcal{C}}^{[0,1]}[3-r] * \cdots * \mathcal{P}_{\mathcal{C}}^{[0,1]}[1] * \mathcal{P}_{\mathcal{C}}^{[0,1]}[2].$$

□

Now we prove the main result in this section.

Theorem 5.5 *If gl. dim $A \leq 2$, then gl. dim $End_{\mathcal{D}}(\mathbf{P}) \leq 7$ for any 2-term silting complex $\mathbf{P}$ in $K^b(projA)$.*

Proof Let $\mathbf{X} \in \mathcal{C}(\mathbf{P})$. Then by Lemma 3.3, we have that $\mathbf{X} \in \mathcal{P} * \mathcal{P}[1] * \mathcal{P}[2] * \mathcal{P}[3]$. By Lemma 5.4, (taking $t = 3$, $r = 2$ and hence $\mathcal{H} = 0$), there is an $\widetilde{\mathbf{X}} \in \mathcal{C}(\mathbf{P})$ such that $\widetilde{\mathbf{X}} \in \mathcal{P} * \mathcal{P}[1] * \mathcal{P}_{\mathcal{C}}^{[0,1]}[1] * \mathcal{P}_{\mathcal{C}}^{[0,1]}[2]$, and

$$\mathrm{pd}\,\mathrm{Hom}_{\mathcal{D}}(\mathbf{P}, \mathbf{X})_B \leq \mathrm{pd}\,\mathrm{Hom}_{\mathcal{D}}(\mathbf{P}, \widetilde{\mathbf{X}})_B + 2. \tag{7}$$

Then there is a triangle

$$\mathbf{Z} \to \mathbf{Y} \to \widetilde{\mathbf{X}} \to \mathbf{Z}[1]$$

with $\mathbf{Y} \in \mathcal{P} * \mathcal{P}[1]$ and $\mathbf{Z} \in \mathcal{P}_{\mathcal{C}}^{[0,1]} * \mathcal{P}_{\mathcal{C}}^{[0,1]}[1]$. Applying the functor $\mathrm{Hom}_{\mathcal{D}}(\mathbf{P}, -)$ to this triangle, we have a long exact sequence

$$\cdots \to \mathrm{Hom}_{\mathcal{D}}(\mathbf{P}, \mathbf{Z}[i]) \to \mathrm{Hom}_{\mathcal{D}}(\mathbf{P}, \mathbf{Y}[i]) \to \mathrm{Hom}_{\mathcal{D}}(\mathbf{P}, \widetilde{\mathbf{X}}[i]) \to \mathrm{Hom}_{\mathcal{D}}(\mathbf{P}, \mathbf{Z}[i+1]) \to \cdots$$

Since $\mathrm{Hom}_{\mathcal{D}}(\mathbf{P}, \widetilde{\mathbf{X}}[i]) = 0$ for $i \neq 0$ by $\widetilde{\mathbf{X}} \in \mathcal{C}(\mathbf{P})$, and $\mathrm{Hom}_{\mathcal{D}}(\mathbf{P}, \mathbf{Z}[i]) = 0$ for $i \neq -1, 0$ by $\mathbf{Z} \in \mathcal{C}(\mathbf{P}) * \mathcal{C}(\mathbf{P})[1]$, we have a short exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{D}}(\mathbf{P}, \mathbf{Z}) \to \mathrm{Hom}_{\mathcal{D}}(\mathbf{P}, \mathbf{Y}) \to \mathrm{Hom}_{\mathcal{D}}(\mathbf{P}, \widetilde{\mathbf{X}}) \to 0,$$

and

$$\mathrm{Hom}_{\mathcal{D}}(\mathbf{P}, \mathbf{Y}[i]) = 0 \text{ for } i < -1.$$

Then we have that

$$\mathrm{pd}\,\mathrm{Hom}_{\mathcal{D}}(\mathbf{P}, \widetilde{\mathbf{X}})_B \leq \max\{\mathrm{pd}\,\mathrm{Hom}_{\mathcal{D}}(\mathbf{P}, \mathbf{Y})_B, \mathrm{pd}\,\mathrm{Hom}_{\mathcal{D}}(\mathbf{P}, \mathbf{Z})_B + 1\} \tag{8}$$

and $\mathbf{Y}, \mathbf{Z} \in \mathcal{D}^{\geq -1}(\mathbf{P})$. By Lemma 5.3, there are objects $\widetilde{\mathbf{Y}}, \widetilde{\mathbf{Z}} \in \mathcal{C}(\mathbf{P})$ such that:

$$\widetilde{\mathbf{Y}} \in \mathcal{P} * \mathcal{P}[1] * \mathcal{P}_{\mathcal{C}}^{[0,1]}[2] \qquad \widetilde{\mathbf{Z}} \in \mathcal{P}_{\mathcal{C}}^{[0,1]} * \mathcal{P}_{\mathcal{C}}^{[0,1]}[1] * \mathcal{P}_{\mathcal{C}}^{[0,1]}[2]$$

$$\mathrm{Hom}_{\mathcal{D}}(\mathbf{P}, \widetilde{\mathbf{Y}}_B) \cong \mathrm{Hom}_{\mathcal{D}}(\mathbf{P}, \mathbf{Y})_B \qquad \mathrm{Hom}_{\mathcal{D}}(\mathbf{P}, \widetilde{\mathbf{Z}})_B \cong \mathrm{Hom}_{\mathcal{D}}(\mathbf{P}, \mathbf{Z})_B$$

By Lemma 5.4, (taking $t = 1$, $r = 2$ and $\mathcal{H} = \mathcal{P}_{\mathcal{C}}^{[0,1]}[2]$), there is an object $\widetilde{\mathbf{Y}}' \in \mathcal{C}(\mathbf{P})$ such that $\widetilde{\mathbf{Y}}' \in \mathcal{P}_{\mathcal{C}}^{[0,1]} * \mathcal{P}_{\mathcal{C}}^{[0,1]}[1] * \mathcal{P}_{\mathcal{C}}^{[0,1]}[2]$ and

$$\mathrm{pd}\,\mathrm{Hom}_{\mathcal{D}}(\mathbf{P}, \widetilde{\mathbf{Y}})_B \leq \mathrm{pd}\,\mathrm{Hom}_{\mathcal{D}}(\mathbf{P}, \widetilde{\mathbf{Y}}')_B + 2. \tag{9}$$

By Lemma 5.2, we have $\operatorname{pd}\operatorname{Hom}_{\mathcal{D}}(\mathbf{P}, \widetilde{\mathbf{Z}})_B \le 3$ and $\operatorname{pd}\operatorname{Hom}_{\mathcal{D}}(\mathbf{P}, \widetilde{\mathbf{Y}}')_B \le 3$. Hence, combining (7), (8) and (9), we obtain

$$\begin{aligned}
\operatorname{pd}\operatorname{Hom}_{\mathcal{D}}(\mathbf{P}, \mathbf{X})_B &\le \operatorname{pd}\operatorname{Hom}_{\mathcal{D}}(\mathbf{P}, \widetilde{\mathbf{X}})_B + 2 \\
&\le \max\{\operatorname{pd}\operatorname{Hom}_{\mathcal{D}}(\mathbf{P}, \mathbf{Y})_B, \operatorname{pd}\operatorname{Hom}_{\mathcal{D}}(\mathbf{P}, \mathbf{Z})_B + 1\} + 2 \\
&= \max\{\operatorname{pd}\operatorname{Hom}_{\mathcal{D}}(\mathbf{P}, \widetilde{\mathbf{Y}})_B, \operatorname{pd}\operatorname{Hom}_{\mathcal{D}}(\mathbf{P}, \widetilde{\mathbf{Z}})_B + 1\} + 2 \\
&\le \max\{\operatorname{pd}\operatorname{Hom}_{\mathcal{D}}(\mathbf{P}, \widetilde{\mathbf{Y}}')_B + 2, \operatorname{pd}\operatorname{Hom}_{\mathcal{D}}(\mathbf{P}, \widetilde{\mathbf{Z}})_B + 1\} + 2 \\
&\le 7.
\end{aligned}$$

□

6 Examples

6.1 First example

We first give an example to show that the bound in Theorem 4.3 is possible. Let $n \ge 2$ and $A = \mathbf{k}Q/I$ where Q is the following quiver

$$\begin{array}{ccccccccccccc}
2n+3 & \xrightarrow{a_n} & \cdots & \xrightarrow{a_3} & 7 & \xrightarrow{a_2} & 5 & \xrightarrow{a_1} & 3 & & & & \\
 & & & & & & & & & \overset{c_2}{\searrow} & & \overset{c_1}{\searrow} & \\
 & & & & & & & & & & 2 & & 1 \\
 & & & & & & & & & \underset{d_2}{\swarrow} & & \underset{d_1}{\swarrow} & \\
2n+4 & \xleftarrow{b_n} & \cdots & \xleftarrow{b_3} & 8 & \xleftarrow{b_2} & 6 & \xleftarrow{b_1} & 4 & & & &
\end{array}$$

and the ideal I is generated by $c_1d_1 - c_2d_2$, $a_{i+1}a_i$ and b_ib_{i+1}, $1 \le i \le n-1$. Then $\operatorname{gl.dim} A = n$. Let $\mathbf{P}$ be the direct sum of the following complexes in $K^b(\operatorname{proj} A)$:

$$\begin{array}{cccc}
0 & \longrightarrow & \bigoplus_{1\le i\le n+2,\, i\ne 2} P_{2i} & , \\
P_4 & \longrightarrow & P_2 & , \\
P_1 & \longrightarrow & P_3 & , \\
\bigoplus_{1\le i\le n+2,\, i\ne 2} P_{2i-1} & \longrightarrow & 0 & .
\end{array}$$

It is easily verified that $\mathbf{P}$ is a 2-term silting complex. The quiver of the endomorphism ring $\operatorname{End}_{\mathcal{D}}(\mathbf{P})$ is the Dynkin quiver of type A_{2n+4}:

$$\begin{array}{ccccccccccc}
2n+3 & \xrightarrow{a_n} & \cdots & \xrightarrow{a_3} & 7 & \xrightarrow{a_2} & 5 & \xrightarrow{a_1} & 3 & \xrightarrow{a_0} & 1 \\
 & & & & & & & & & & \downarrow{\scriptstyle c} \\
2n+4 & \xleftarrow{b_n} & \cdots & \xleftarrow{b_3} & 8 & \xleftarrow{b_2} & 6 & \xleftarrow{b_1} & 4 & \xleftarrow{b_0} & 2
\end{array}$$

with the relations $a_{i+1}a_i = 0$, $b_ib_{i+1} = 0$, $1 \le i \le n-1$ and $a_0cb_0 = 0$. Hence the global dimension of $\operatorname{End}_{\mathcal{D}}(\mathbf{P})$ is $2n+2$.

6.2 Second example

The next example shows that 7 is a possible value for the global dimension of the endomorphism algebra of a 2-term silting complex over an algebra of global dimension two. Let $A = \mathbf{k}Q/I$ with Q the following quiver

$$1 \xrightarrow{a_1} 2 \xrightarrow{a_2} 3 \xrightarrow{a_3} 4 \underset{a_5}{\overset{a_4}{\rightrightarrows}} 5 \xrightarrow{a_6} 6 \xrightarrow{a_7} 7 \xrightarrow{a_8} 8$$

and I the ideal generated by a_1a_2, $a_3a_4a_6$ and a_7a_8. Then A has global dimension two, and the complex $\mathbf{P}$ given by the direct sum of the complexes

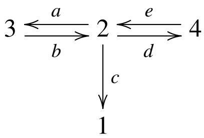
$$\begin{array}{ccc} 0 & \longrightarrow & \bigoplus_{i=5,7,8} P_i \ , \\ P_6 & \longrightarrow & P_5 \ , \\ P_4 & \longrightarrow & P_3 \ , \\ \bigoplus_{i=1,2,4} P_i & \longrightarrow & 0 \ . \end{array}$$

is a 2-term silting complex. It is easily verified, that the quiver of $\operatorname{End}_{\mathcal{D}}(\mathbf{P})$ is a linearly oriented Dynkin quiver of type A_8 and the ideal of relation equals the square of the Jacobson radical. Hence the global dimension of $\operatorname{End}_{\mathcal{D}}(\mathbf{P})$ is 7.

6.3 Third example

The last example shows that there is no bound on the global dimension of the endomorphism algebra of a 2-silting object over an algebra with global dimension $d \geq 3$. This example then completes the proof of Theorem 1.1.

Let first $A = \mathbf{k}Q/I$ where the quiver Q is



and $I = \langle ba, bd, abc, de \rangle$. The indecomposable projective $A-$modules are

$$P_1 = {\scriptstyle 1}, \quad P_2 = {\scriptstyle \begin{smallmatrix} & 2 & \\ 1 & 3 & 4 \\ & 2 & \end{smallmatrix}}, \quad P_3 = {\scriptstyle \begin{smallmatrix} 3 \\ 2 \\ 1 \end{smallmatrix}}, \quad P_4 = {\scriptstyle \begin{smallmatrix} & 4 & \\ & 2 & \\ 1 & 3 & 4 \\ & 2 & \end{smallmatrix}}.$$

The integers here denote the corresponding simples, and the notation indicates the radical filtration. The global dimension of A is 3. Let $\mathbf{P}$ be the direct sum of

$$\mathbf{P}_i = \cdots \to 0 \to P_i \to 0 \to 0 \to \cdots, \quad i = 1, 3, 4,$$

(concentrated in degree -1) and

$$\mathbf{P}_2 = \cdots \to 0 \to P_1 \oplus P_3 \oplus P_4 \xrightarrow{p} P_2 \to 0 \to \cdots$$

where p is a projective presentation of S_2. Then it is easily verified that $\mathbf{P}$ is a 2-term silting complex.

By Proposition 2.1(b) we have that $\mathcal{T}(\mathbf{P}) = \operatorname{Fac} H^0(\mathbf{P})$, and hence $\mathcal{T}(\mathbf{P}) = \operatorname{add} S_2$. We will show the projective dimension of S_2 in $\mathcal{C}(\mathbf{P})$ is infinite, by proving that its third syzygy equals S_2. This implies that a minimal projective resolution of S_2 is periodic and hence infinite.

Using the notation in Section 2, we have that $\widetilde{\mathbf{P}}_1 = \mathbf{P}_1$ and $\widetilde{\mathbf{P}}_3 = \mathbf{P}_3$. Moreover $\widetilde{\mathbf{P}}_4 = (P_4/S_2)[1]$, and $\widetilde{\mathbf{P}}_2$ is given by the complex

$$\cdots \to 0 \to P_1 \oplus P_3 \oplus (P_4/S_2) \xrightarrow{\widetilde{p}} P_2 \to 0 \to \cdots$$

Consider now the triangle

$$\operatorname{cone}(\pi)[-1] \to \widetilde{\mathbf{P}}_2 \xrightarrow{\pi} S_2 \to \operatorname{cone}(\pi)$$

where π is

$$\begin{array}{ccccccccc} \cdots \to & 0 & \to & P_1 \oplus P_3 \oplus (P_4/S_2) & \xrightarrow{\widetilde{p}} & P_2 & \to & 0 & \to \cdots \\ & \downarrow & & \downarrow & & \pi^0 \downarrow & & \downarrow & \\ \cdots \to & 0 & \to & 0 & \to & S_2 & \to & 0 & \to \cdots \end{array}$$

with π^0 being a projective cover of S_2 in mod A.

Then $H^0(\mathrm{cone}(\pi)[-1]) = 0$, $H^{-1}(\mathrm{cone}(\pi)[-1]) \cong H^{-1}(\widetilde{\mathbf{P}}_2) \in \mathcal{F}(\mathbf{P})$ and $H^i(\mathrm{cone}(\pi)[-1]) = 0$ for $i \neq -1, 0$. So $\mathrm{cone}(\pi)[-1]$ is in $\mathcal{C}(\mathbf{P})$, using Proposition 2.1 (d). Hence π is a projective cover of S_2 in $\mathcal{C}(\mathbf{P})$ and $\mathrm{cone}(\pi)[-1]$ is its kernel in $\mathcal{C}(\mathbf{P})$.

Note that $\mathrm{cone}(\pi)[-1] \cong H^{-1}(\mathrm{cone}(\pi)[-1])[1] \cong H^{-1}(\widetilde{\mathbf{P}}_2) \cong P_1[1] \oplus M[1]$, where $M = \begin{smallmatrix} & 2 & \\ 1 & 3 & 4 \end{smallmatrix} \in \mathcal{F}(\mathbf{P})$. Consider the triangle

$$\mathrm{cone}(\pi_1)[-1] \to \widetilde{\mathbf{P}}_1 \oplus \widetilde{\mathbf{P}}_3 \oplus \widetilde{\mathbf{P}}_4 \xrightarrow{\pi_1} M[1] \to \mathrm{cone}(\pi_1)$$

where π_1 is

$$\begin{array}{ccccccccc} \cdots \to & 0 & \to & P_1 \oplus P_3 \oplus (P_4/S_2) & \to & 0 & \to & 0 & \to \cdots \\ & \downarrow & & \pi_1^{-1} \downarrow & & \downarrow & & \downarrow & \\ \cdots \to & 0 & \to & M & \to & 0 & \to & 0 & \to \cdots \end{array}$$

with π_1^{-1} being the unique (up to a scalar) right minimal homomorphism from $P_1 \oplus P_3 \oplus (P_4/S_2)$ to M. Then $H^0(\mathrm{cone}(\pi_1)[-1]) \cong S_2 \in \mathcal{T}(\mathbf{P})$, $H^{-1}(\mathrm{cone}(\pi_1)[-1]) \cong \begin{smallmatrix} 2 \\ 1 \end{smallmatrix} \oplus M \in \mathcal{F}(\mathbf{P})$ (since $\mathrm{Hom}_A(S_2, \begin{smallmatrix} 2 \\ 1 \end{smallmatrix}) = 0$) and $H^i(\mathrm{cone}(\pi_1)[-1]) = 0$ for $i \neq -1, 0$. So $\mathrm{cone}(\pi_1)[-1] \in \mathcal{C}(\mathbf{P})$, hence π_1 is a projective cover of $M[1]$ in $\mathcal{C}(\mathbf{P})$ and $\mathrm{cone}(\pi_1)[-1]$ is its kernel.

Now consider the triangle

$$\mathrm{cone}(\pi_2)[-1] \to \widetilde{\mathbf{P}}_2 \xrightarrow{\pi_2} \mathrm{cone}(\pi_1)[-1] \to \mathrm{cone}(\pi_2)$$

where π_2 is

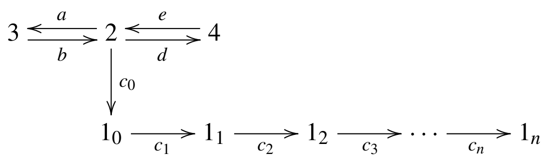
$$\begin{array}{ccccccccc} \cdots \to & 0 & \to & P_1 \oplus P_3 \oplus (P_4/S_2) & \xrightarrow{p} & P_2 & \to & 0 & \to \cdots \\ & \downarrow & & \pi_2^{-1} \downarrow & & \pi_2^0 \downarrow & & \downarrow & \\ \cdots \to & 0 & \to & P_1 \oplus P_3 \oplus (P_4/S_2) & \xrightarrow{\pi_1^{-1}} & M & \to & 0 & \to \cdots \end{array}$$

with π_2^{-1} being the identity map and π_2^0 being a projective cover of M in mod A. Then we have $H^0(\mathrm{cone}(\pi_2)[-1]) \cong S_2 \in \mathcal{T}(\mathbf{P})$ and $H^i(\mathrm{cone}(\pi_1)[-1]) = 0$ for $i \neq 0$. Hence $\mathrm{cone}(\pi_2)[-1] \cong S_2 \in \mathcal{C}(\mathbf{P})$ and we have a short exact sequence in $\mathcal{C}(\mathbf{P})$:

$$0 \to S_2 \to \widetilde{\mathbf{P}}_2 \xrightarrow{\pi_2} (\mathrm{cone}(\pi_1)[-1] \to 0.$$

Thus, the projective resolution of S_2 in $\mathcal{C}(\mathbf{P})$ is periodic and hence the projective dimension is infinite. Therefore, also the global dimension of B is infinite, by Proposition 2.1 (e).

Now, for any $n \geq 0$ consider the quiver Q_n given by



with relations $I_n = \langle ba, bd, abc_0, de, c_0c_1, c_1c_2, \ldots, c_{n-1}c_n\rangle$. Consider the algebra $A(n) = kQ_n/I_n$. It is easy to check that $A(n)$ has global dimension $n+3$. Let $\mathbf{P}'$ be the direct sum of

$$\mathbf{P}'_{1_i} = \cdots \to 0 \to P_{1_i} \to 0 \to 0 \to \cdots, \quad 0 \le i \le n,$$
$$\mathbf{P}'_3 = \cdots \to 0 \to P_3 \to 0 \to 0 \to \cdots,$$
$$\mathbf{P}'_4 = \cdots \to 0 \to P_4 \to 0 \to 0 \to \cdots,$$

(concentrated in degree -1) and

$$\mathbf{P}'_2 = \cdots \to 0 \to P_{1_0} \oplus P_3 \oplus P_4 \xrightarrow{p} P_2 \to 0 \to \cdots.$$

where P_x is the projective module corresponding to a vertex x of Q_n, and p is a projective cover of the simple S_2. A similar discussion as above shows that $\mathbf{P}'$ is a silting complex with $\mathrm{End}_{D^b(A(n))}(\mathbf{P}')$ having infinite global dimension.

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