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A silting theorem ☆

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ABSTRACT

We give a generalization of the classical tilting theorem of Brenner and Butler. We show that for a 2-term silting complex $\mathbf{P}$ in the bounded homotopy category $K^b(\text{proj}\,A)$ of finitely generated projective modules of a finite dimensional algebra A, the algebra $B = \text{End}_{K^b(\text{proj}\,A)}(\mathbf{P})$ admits a 2-term silting complex $\mathbf{Q}$ with the following properties: (i) The endomorphism algebra of $\mathbf{Q}$ in $K^b(\text{proj}\,B)$ is a factor algebra of A, and (ii) there are induced torsion pairs in $\text{mod}\,A$ and $\text{mod}\,B$, such that we obtain natural equivalences induced by Hom- and Ext-functors. Moreover, we show how the Auslander–Reiten theory of $\text{mod}\,B$ can be described in terms of the Auslander–Reiten theory of $\text{mod}\,A$.

0. Introduction

The fundamental idea of tilting theory is to relate the module categories of two algebras by the use of tilting functors. Such functors were introduced by Brenner and Butler, in [8], who were generalizing the ideas in [9] and [6].

In the seminal paper [16], Happel and Ringel introduced the concepts of *tilting modules* and *tilted algebras*. A tilted algebra is the endomorphism ring of a tilting module over a hereditary finite dimensional algebra. Happel [15] and Cline, Parshall, Scott [12] proved that tilting modules induce derived equivalences, and inspired by this Rickard [26] introduced the concept of *tilting complexes*, as a necessary ingredient in developing Morita theory for derived categories.

Over the last 35 years these ideas and concepts have become an essential tool in many branches of mathematics, including algebraic geometry, finite group theory, algebraic group theory and algebraic topology, see [4]. More recently, the development of cluster tilting theory, see [21,25], has spurred further interest in the topic and the relation to cluster algebras [14].

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Let us briefly recall the main ideas from [8] and [16]. Let $\mathbf{k}$ be a field, let A be a finite dimensional algebra over $\mathbf{k}$, and T a tilting module in $\operatorname{mod} A$, the category of (finite dimensional) right A-modules. That is: T is a module with projective dimension at most 1 ($\operatorname{pd} T \leq 1$), with $\operatorname{Ext}^1_A(T,T) = 0$ and such that $|T| = |A|$, where $|X|$ denotes the number of indecomposable direct summands in X, up to isomorphism. Let $B = \operatorname{End}_A(T)$. Then $D(T)_B$ is a cotilting module over B and $A \cong \operatorname{End}_B(D(T)_B)$, where D is the $\mathbf{k}$-dual of finite dimensional $\mathbf{k}$-vector spaces. Cotilting modules are defined by replacing $\operatorname{pd} T \leq 1$ with $\operatorname{id} T \leq 1$ in the definition of tilting modules, where $\operatorname{id} T$ is the injective dimension of T. Moreover, let $\mathcal{T} = \operatorname{Fac} T$ be the full subcategory of $\operatorname{mod} A$ whose objects are generated by T, and let $\mathcal{F}$ be the full subcategory of $\operatorname{mod} A$ with objects X such that $\operatorname{Hom}_A(\mathcal{T}, X) = 0$. Then $(\mathcal{T}, \mathcal{F})$ is a torsion pair in $\operatorname{mod} A$. There is also a torsion pair $(\mathcal{X}, \mathcal{Y})$ in $\operatorname{mod} B$, induced by the cotilting module $D(T)_B$, and Hom- and Ext-functors induce inverse equivalences of $\mathcal{T}$ with $\mathcal{Y}$ and of $\mathcal{F}$ with $\mathcal{X}$.

We generalize these results to the following setting. We consider a 2-term silting complex $\mathbf{P}$ in the bounded homotopy category $K^b(\operatorname{proj} A)$ of finitely generated projective A-modules. This is just a map between projective A-modules, considered as a complex, with the property that $\operatorname{Hom}_{K^b(\operatorname{proj} A)}(\mathbf{P}, \mathbf{P}[1]) = 0$, and such that $\mathbf{P}$ generates $K^b(\operatorname{proj} A)$. Let $B = \operatorname{End}_{K^b(\operatorname{proj} A)}(\mathbf{P})$. It then turns out that $\operatorname{mod} A$ and $\operatorname{mod} B$ can be compared in a way very similar to the setting with tilting modules.

It is known that if $\mathbf{P}$ is a 2-term silting complex in $K^b(\operatorname{proj} A)$ then $H^0(\mathbf{P})$ is a tilting $(A/\operatorname{ann} H^0(\mathbf{P}))$-module and $H^{-1}(\nu\mathbf{P})$ is a cotilting $(A/\operatorname{ann} H^{-1}(\nu\mathbf{P}))$-module, where ν is the Nakayama functor. In particular, the Brenner–Butler tilting theorem and its dual apply in this setting. However, for a general 2-term silting complex $\mathbf{P}$, both $\operatorname{End}_A(H^0(\mathbf{P}))$ and $\operatorname{End}_A(H^{-1}(\nu\mathbf{P}))$ are factor algebras of $\operatorname{End}_{K^b(\operatorname{proj} A)}(\mathbf{P})$, so they are much smaller than $\operatorname{End}_{K^b(\operatorname{proj} A)}(\mathbf{P})$. Hence, in the general case, the Brenner–Butler tilting theorem does not give the expected result.

The concept of silting complexes originated from Keller and Vossieck [22]. In [18], the relation between 2-term silting complexes and torsion pairs in module categories was first considered. They were mainly dealing with abelian categories with arbitrary coproducts, but we adapt many of their results to our setting.

More recently, there have been several papers, starting with [3], often focusing on various (combinatorial) properties on the set of silting complexes. Silting complexes correspond to bounded t-structures having a heart which is a length category, i.e. there are finitely many simples, and all objects have finite length [24].

The set of 2-term silting complexes has a natural structure of an ordered exchange graph, and as beautifully summarized in [10], this gives links (expressed as isomorphisms of exchange graphs, see the figure in their introduction) to a plenitude of other structures which have recently been studied. Among these are support τ-tilting modules [1] in the module category, and certain bounded t-structures in the bounded derived category, see [10, Corollary 4.3]. Starting with a quiver Q, with no loops or oriented 2-cycles, there is a corresponding cluster algebra A_Q, [14], and then we obtain also a correspondence with the clusters in A_Q, see [1]. Given Q as above, and a potential, there is a correspondence with certain bounded t-structures in the finite-dimensional derived category of the corresponding Ginzburg dg algebra [10,23].

In this paper and the forthcoming paper [11], we consider the endomorphism algebras of 2-term silting complexes, which so far have been less studied. These algebras are isomorphic to the 0-th cohomology of the corresponding differential graded endomorphism algebras.

The paper is organized as follows. In the first section, we review some background and notation, and state the main results. In Section 2, we consider links between silting theory, t-structures and torsion pairs. In Section 3, we prove further properties of 2-term silting complexes, and the main result is proved in Section 4. In Section 5 we apply the main result to obtain some information about the AR-theory of the endomorphism ring of a 2-term silting complex, inspired by similar results in classical tilting theory, see [5].

1. Background and main result

Let A be a finite dimensional $\mathbf{k}$-algebra, and $\operatorname{mod} A$ the category of finitely generated right A-modules. Let $D^b(A)$ be the bounded derived category, with shift functor [1]. Whenever we consider subcategories of $\operatorname{mod} A$ or $D^b(A)$, they are assumed to be full and closed under isomorphism. For an object M in an additive category, let $\operatorname{add} M$ denote the additive closure, i.e. the full subcategory generated by all direct summands of direct sums of copies of M.

Recall that a *torsion pair* in $\operatorname{mod} A$, is a pair $(\mathcal{X}, \mathcal{Y})$ of subcategories of $\operatorname{mod} A$, with the properties that

- $\operatorname{Hom}_A(\mathcal{X}, Y) = 0$ if and only if Y is in $\mathcal{Y}$, and
- $\operatorname{Hom}_A(X, \mathcal{Y}) = 0$ if and only if X is in $\mathcal{X}$.

If M is an object in $\operatorname{mod} A$, then there is an exact sequence,

$$0 \to tM \to M \to M/tM \to 0$$

called the *canonical sequence* of M, and with tM in $\mathcal{X}$ and with M/tM in $\mathcal{Y}$. Let $\operatorname{proj} A$ denote the full subcategory of $\operatorname{mod} A$ generated by the projective modules. We consider 2-term complexes $\mathbf{P}$ in $K^b(\operatorname{proj} A)$. These are complexes $\mathbf{P} = \{P^i\}$ with $P^i = 0$ for $i \neq -1, 0$. Such a complex is called *presilting* if $\operatorname{Hom}_{K^b(\operatorname{proj} A)}(\mathbf{P}, \mathbf{P}[1]) = 0$ and *silting* if in addition $\operatorname{thick} \mathbf{P} = K^b(\operatorname{proj} A)$. Here, for an object $\mathbf{X}$ in $K^b(\operatorname{proj} A)$, we denote by $\operatorname{thick} \mathbf{X}$ the smallest triangulated subcategory closed under direct summands containing $\mathbf{X}$. A 2-term silting complex $\mathbf{P}$ is *tilting*, if in addition $\operatorname{Hom}_{K^b(\operatorname{proj} A)}(\mathbf{P}, \mathbf{P}[-1]) = 0$.

Let $\mathbf{P}$ be a 2-term silting complex, and consider the full subcategories of $\operatorname{mod} A$ given by

$\mathcal{T}(\mathbf{P}) = \{X \in \operatorname{mod} A \mid \operatorname{Hom}_{D^b(A)}(\mathbf{P}, X[1]) = 0\}$, and
$\mathcal{F}(\mathbf{P}) = \{Y \in \operatorname{mod} A \mid \operatorname{Hom}_{D^b(A)}(\mathbf{P}, Y) = 0\}$.

Note that if $\mathbf{P}$ is a projective presentation of a tilting module T, then $\mathbf{P}$ is quasi-isomorphic to its 0-th cohomology $T = H^0(\mathbf{P})$. Hence $\mathcal{T}(\mathbf{P}) = \ker \operatorname{Ext}^1_A(T, -)$ and $\mathcal{F}(\mathbf{P}) = \ker \operatorname{Hom}_A(T, -)$ and these are the classes considered in classical tilting theory.

Our main theorem is a generalization of the Brenner–Butler tilting theorem to 2-term silting complexes. Note that (a) is from [18], (b) is from [28], while (c) and (d) can be easily deduced from [10].

Theorem 1.1. *Let $\mathbf{P}$ be a 2-term silting complex in $K^b(\operatorname{proj} A)$, and let $B = \operatorname{End}_{D^b(A)}(\mathbf{P})$.*

(a) *The pair $(\mathcal{T}(\mathbf{P}), \mathcal{F}(\mathbf{P}))$ is a torsion pair in $\operatorname{mod} A$.*
(b) *There is a triangle*

$$A \to \mathbf{P}' \xrightarrow{f} \mathbf{P}'' \to A[1]$$

with $\mathbf{P}'$, $\mathbf{P}''$ in $\operatorname{add} \mathbf{P}$.
Consider the 2-term complex $\mathbf{Q}$ in $K^b(\operatorname{proj} B)$ induced by the map

$$\operatorname{Hom}_{D^b(A)}(\mathbf{P}, f)\colon \operatorname{Hom}_{D^b(A)}(\mathbf{P}, \mathbf{P}') \to \operatorname{Hom}_{D^b(A)}(\mathbf{P}, \mathbf{P}'').$$

(c) *$\mathbf{Q}$ is a 2-term silting complex in $K^b(\operatorname{proj} B)$.*
(d) *There is an algebra epimorphism $\Phi_{\mathbf{P}}\colon A \to \overline{A} = \operatorname{End}_{D^b(B)}(\mathbf{Q})$.*
(e) *$\Phi_{\mathbf{P}}$ is an isomorphism if and only if $\mathbf{P}$ is tilting.*
Let $\Phi_*\colon \operatorname{mod} \overline{A} \hookrightarrow \operatorname{mod} A$ be the induced inclusion functor.

(f) *The restriction of the functors* $\mathrm{Hom}_{D^b(A)}(\mathbf{P}, -)$ *and* $\Phi_* \mathrm{Hom}_{D^b(B)}(\mathbf{Q}, -[1])$ *to* $\mathcal{T}(\mathbf{P})$ *and* $\mathcal{F}(\mathbf{Q})$ *is a pair of inverse equivalences.*
(g) *The restriction of the functors* $\mathrm{Hom}_{D^b(A)}(\mathbf{P}, -[1])$ *and* $\Phi_* \mathrm{Hom}_{D^b(B)}(\mathbf{Q}, -)$ *to* $\mathcal{F}(\mathbf{P})$ *and* $\mathcal{T}(\mathbf{Q})$ *is a pair of inverse equivalences.*

We give a brief explanation on how (c) and (d) can be deduced from [10]. Let $\widetilde{B}$ be the differential graded endomorphsim algebra of $\mathbf{P}$. Then $B \cong H^0(\widetilde{B})$. So the canonical epimorphism $\widetilde{B} \to B$ induces an exact functor $\phi : \mathrm{per}\,\widetilde{B} \to \mathrm{per}\,B$. Note that $\mathrm{per}\,\widetilde{B}$ is equivalent to $K^b(\mathrm{proj}\,A)$ and $\mathrm{per}\,B = K^b(\mathrm{proj}\,B)$. Then by [10, Proposition A.3], the functor ϕ induces a bijection between a certain set of silting complexes in $K^b(\mathrm{proj}\,A)$ and the set of 2-term silting complexes in $K^b(\mathrm{proj}\,B)$. We remark that the complex $\mathbf{Q}$ in (c), which we will construct in Section 3, corresponds (up to isomorphism) to A (as a stalk complex in $K^b(\mathrm{proj}\,A)$) under this bijection. The epimorphism in (d) is then obtained by [10, Proposition A.5].

We will use the following notation. For any subcategory $\mathcal{T}$ of $\mathrm{mod}\,A$, an A-module X in $\mathcal{T}$ is called Ext-*projective* in $\mathcal{T}$ if $\mathrm{Ext}^1_A(X, Y) = 0$ for all Y in $\mathcal{T}$; dually, X in $\mathcal{T}$ is called Ext-*injective* in $\mathcal{T}$ if $\mathrm{Ext}^1_A(Y, X) = 0$ for all Y in $\mathcal{T}$. Furthermore, we let ν denote the Nakayama functor $\nu = D\,\mathrm{Hom}_A(-, A)$, which is an equivalence from $\mathrm{proj}\,A$ to the full subcategory $\mathrm{inj}\,A$ of $\mathrm{mod}\,A$ generated by the injective modules. Then ν induces an equivalence

$$\nu : K^b(\mathrm{proj}\,A) \to K^b(\mathrm{inj}\,A).$$

It is well known that there is an isomorphism

$$\mathrm{Hom}_{D^b(A)}(\mathbf{X}, \nu\mathbf{Y}) \cong D\,\mathrm{Hom}_{D^b(A)}(\mathbf{Y}, \mathbf{X})$$

for any $\mathbf{X}, \mathbf{Y} \in K^b(\mathrm{proj}\,A)$ (see e.g. [15, Chapter 1, Section 4.6]). Note that the derived Nakayama functor ν in general is not a Serre functor on the bounded derived category $D^b(A)$, as the algebra A is not assumed to have finite global dimension.

2. 2-term silting complexes, *t*-structures and torsion pairs

In this section we recall the notion of a t-structure [7] in a triangulated category, and the interplay between t-structures, torsion pairs and 2-term silting complexes.

A pair $(\mathcal{X}, \mathcal{Y})$ of subcategories of $D^b(A)$ is called a t-*structure* if and only if the following conditions hold:

(1): $\mathcal{X}[1] \subset \mathcal{X}$ and $\mathcal{Y}[-1] \subset \mathcal{Y}$;
(2): $\mathrm{Hom}_{D^b(A)}(\mathbf{X}, \mathbf{Y}[-1]) = 0$ for any $\mathbf{X} \in \mathcal{X}$ and $\mathbf{Y} \in \mathcal{Y}$;
(3): for any $\mathbf{C} \in D^b(A)$, there is a triangle

$$\mathbf{X} \to \mathbf{C} \to \mathbf{Y}[-1] \to \mathbf{X}[1]$$

with $\mathbf{X} \in \mathcal{X}$ and $\mathbf{Y} \in \mathcal{Y}$.

Silting complexes give rise to t-structures in a natural way. For an integer m, consider the pair of subcategories

$$D^{\leq m}(\mathbf{P}) = \{\mathbf{X} \in D^b(A) \mid \mathrm{Hom}_{D^b(A)}(\mathbf{P}, \mathbf{X}[i]) = 0, \text{ for } i > m\}$$

and

$$D^{\geq m}(\mathbf{P}) = \{\mathbf{X} \in D^b(A) \mid \mathrm{Hom}_{D^b(A)}(\mathbf{P}, \mathbf{X}[i]) = 0, \text{ for } i < m\}$$

in the derived category $D^b(A)$.

Observe that $\mathcal{T}(\mathbf{P}) = D^{\leq 0}(\mathbf{P}) \cap \mathrm{mod}\, A$ and $\mathcal{F}(\mathbf{P}) = D^{\geq 1}(\mathbf{P}) \cap \mathrm{mod}\, A$. We have the following result. Here, (b) is from [18] and (a) is from [24]. Note also that a version of (a) was proved in [18], in the setting of abelian categories with arbitrary coproducts.

Theorem 2.1. *Let* $\mathbf{P}$ *be a 2-term silting complex in* $K^b(\mathrm{proj}\, A)$*.*

(a) *The pair* $(D^{\leq 0}(\mathbf{P}), D^{\geq 0}(\mathbf{P}))$ *is a t-structure in* $D^b(A)$*.*
(b) *The pair* $(\mathcal{T}(\mathbf{P}), \mathcal{F}(\mathbf{P}))$ *is a torsion pair in* $\mathrm{mod}\, A$*.*

The following lemma will be useful for later.

Lemma 2.2. *For any* $\mathbf{X} \in D^b(A)$ *and* $i \in \mathbb{Z}$*, there is a short exact sequence,*

$$0 \to \mathrm{Hom}_{D^b(A)}(\mathbf{P}, H^{i-1}(\mathbf{X})[1]) \to \mathrm{Hom}_{D^b(A)}(\mathbf{P}, \mathbf{X}[i]) \to \mathrm{Hom}_{D^b(A)}(\mathbf{P}, H^i(\mathbf{X})) \to 0.$$

Proof. See [18, Lemma 2.5], the proof given there works also in our case. □

Let $\mathcal{C}(\mathbf{P}) = D^{\leq 0}(\mathbf{P}) \cap D^{\geq 0}(\mathbf{P})$ be the heart of the t-structure $(D^{\leq 0}(\mathbf{P}), D^{\geq 0}(\mathbf{P}))$. The following summarizes the main features of $\mathcal{C}(\mathbf{P})$.

Theorem 2.3. *Let* $\mathbf{P}$ *be a 2-term silting complex in* $K^b(\mathrm{proj}\, A)$*.*

(a) $\mathcal{C}(\mathbf{P})$ *is an abelian category and the short exact sequences in* $\mathcal{C}(\mathbf{P})$ *are precisely the triangles in* $D^b(A)$ *all of whose vertices are objects in* $\mathcal{C}(\mathbf{P})$*.*
(b) $(\mathcal{F}(\mathbf{P})[1], \mathcal{T}(\mathbf{P}))$ *is a torsion pair in* $\mathcal{C}(\mathbf{P})$*.*
(c) *For a complex* $\mathbf{X}$ *in* $D^b(A)$*, we have that* $\mathbf{X}$ *is in* $\mathcal{C}(\mathbf{P})$ *if and only if* $H^0(\mathbf{X})$ *is in* $\mathcal{T}(\mathbf{P})$*,* $H^{-1}(\mathbf{X})$ *is in* $\mathcal{F}(\mathbf{P})$ *and* $H^i(\mathbf{X}) = 0$ *for* $i \neq -1, 0$*.*
(d) $\mathrm{Hom}_{D^b(A)}(\mathbf{P}, -)\colon \mathcal{C}(\mathbf{P}) \to \mathrm{mod}\, B$ *is an equivalence of (abelian) categories.*

Proof. Note that (a) is a classical result of [7]. Proofs of (b), (c) and (d) can be found in [18] (although there they proved these in the setting of abelian categories with arbitrary coproducts, but their proofs also work in our case, using Theorem 2.1 (a)). We now explain how (b) and (c) can also be seen to follow from [17, Proposition I.2.1 and Corollary I.22], which says that for any torsion pair $(\mathcal{T}, \mathcal{F})$ in $\mathrm{mod}\, A$, we have that the two subcategories

$$\{\mathbf{X} \in D^b(A) \mid H^i(\mathbf{X}) = 0 \text{ for } i > 0 \text{ and } H^0(\mathbf{X}) \in \mathcal{T}\}$$

and

$$\{\mathbf{X} \in D^b(A) \mid H^i(\mathbf{X}) = 0 \text{ for } i < -1 \text{ and } H^{-1}(\mathbf{X}) \in \mathcal{F}\}$$

form a t-structure, and that $(\mathcal{F}[1], \mathcal{T})$ is a torsion pair in the heart of this t-structure.

Note first that by Lemma 2.2 we have that

$$D^{\leq 0}(\mathbf{P}) = \{\mathbf{X} \in D^b(A) \mid \mathrm{Hom}_{D^b(A)}(\mathbf{P}, H^i(\mathbf{X})) = 0 \text{ for } i > 0 \text{ and } \mathrm{Hom}_{D^b(A)}(\mathbf{P}, H^j(\mathbf{X})[1]) = 0, \text{ for } j \geq 0\}.$$

Since for any module M, we have that $\operatorname{Hom}_{D^b(A)}(\mathbf{P}, M) = 0 = \operatorname{Hom}_{D^b(A)}(\mathbf{P}, M[1])$ only if $M = 0$, it follows that

$$D^{\leq 0}(\mathbf{P}) = \{\mathbf{X} \in D^b(A) \mid H^i(\mathbf{X}) = 0 \text{ for } i > 0 \text{ and } \operatorname{Hom}_{D^b(A)}(\mathbf{P}, H^0(\mathbf{X})[1]) = 0\}$$
$$= \{\mathbf{X} \in D^b(A) \mid H^i(\mathbf{X}) = 0 \text{ for } i > 0 \text{ and } H^0(\mathbf{X}) \in \mathcal{T}(\mathbf{P})\}.$$

Similarly, we have that

$$D^{\geq 0}(\mathbf{P}) = \{\mathbf{X} \in D^b(A) \mid H^i(\mathbf{X}) = 0 \text{ for } i < -1 \text{ and } H^{-1}(\mathbf{X}) \in \mathcal{F}(\mathbf{P})\}.$$

Hence (b) and (c) follows.

We also refer to [20, Proposition 3.13] for a different proof of (d). □

As before, for a module M in $\operatorname{mod} A$, we let $\operatorname{Fac} M$ denote the full subcategory whose objects are generated by M, and dually we let $\operatorname{Sub} M$ denote the full subcategory whose objects are cogenerated by M. We then have the following, which is also due to [18].

Proposition 2.4. *Let* $\mathbf{P}$ *be a 2-term silting complex in* $K^b(\operatorname{proj} A)$*. Then, we have*

$$(\mathcal{T}(\mathbf{P}), \mathcal{F}(\mathbf{P})) = (\operatorname{Fac} H^0(\mathbf{P}), \operatorname{Sub} H^{-1}(\nu(\mathbf{P}))).$$

Note that $H^0(\mathbf{P})$ is the support τ-tilting module corresponding to $\mathbf{P}$ by [1, Theorem 3.2], so Proposition 2.4 shows that $(\mathcal{T}(\mathbf{P}), \mathcal{F}(\mathbf{P}))$ is precisely the functorially finite torsion pair associated with $H^0(\mathbf{P})$ via [1, Theorem 2.7].

Consider now the subcategories $\mathcal{X}(\mathbf{P}) = \operatorname{Hom}_{D^b(A)}(\mathbf{P}, \mathcal{F}(\mathbf{P})[1])$ and $\mathcal{Y}(\mathbf{P}) = \operatorname{Hom}_{D^b(A)}(\mathbf{P}, \mathcal{T}(\mathbf{P}))$ of $\operatorname{mod} B$. We have the following direct consequences of Theorem 2.3.

Corollary 2.5. *Let* $\mathbf{P}$ *be a 2-term silting complex in* $K^b(\operatorname{proj} A)$*, then* $(\mathcal{X}(\mathbf{P}), \mathcal{Y}(\mathbf{P}))$ *is a torsion pair in* $\operatorname{mod} B$ *and there are equivalences*

$$\operatorname{Hom}_{D^b(A)}(\mathbf{P}, -)\colon \mathcal{T}(\mathbf{P}) \to \mathcal{Y}(\mathbf{P}),$$

and

$$\operatorname{Hom}_{D^b(A)}(\mathbf{P}, -[1])\colon \mathcal{F}(\mathbf{P}) \to \mathcal{X}(\mathbf{P}).$$

The equivalences send short exact sequences with terms in $\mathcal{T}(\mathbf{P})$ *(resp.* $\mathcal{F}(\mathbf{P})$*) to short exact sequences in* $\operatorname{mod} B$*.*

Proof. This follows from Theorem 2.3 (a) and (d), using that $\mathcal{T}(\mathbf{P}) \cup \mathcal{F}(\mathbf{P})[1] \subset \mathcal{C}(\mathbf{P})$. □

In Section 4 we will provide natural quasi-inverses of these functors.

Corollary 2.6. *Let* $M \in \mathcal{T}(\mathbf{P})$ *and* $N \in \mathcal{F}(\mathbf{P})$*, for a 2-term silting complex* $\mathbf{P}$*. Then we have the following functorial isomorphisms*

$$\operatorname{Hom}_B(\operatorname{Hom}_{D^b(A)}(\mathbf{P}, M), \operatorname{Hom}_{D^b(A)}(\mathbf{P}, N[1])) \cong \operatorname{Hom}_{D^b(A)}(M, N[1]) \cong \operatorname{Ext}^1_A(M, N)$$

and

$$\operatorname{Ext}^1_B(\operatorname{Hom}_{D^b(A)}(\mathbf{P}, M), \operatorname{Hom}_{D^b(A)}(\mathbf{P}, N[1])) \cong \operatorname{Hom}_{D^b(A)}(M, N[2]) \cong \operatorname{Ext}^2_A(M, N).$$

Proof. Note that by Theorem 2.3 (c) both M and $N[1]$ are in $\mathcal{C}(\mathbf{P})$. Then the first isomorphism follows from Theorem 2.3 (d), while the second follows from (a) and (d). □

The following easy observation will be useful later.

Lemma 2.7. *For any A-module X, we have a functorial isomorphism*

$$\mathrm{Hom}_{D^b(A)}(\mathbf{P}, X) \cong \mathrm{Hom}_A(H^0(\mathbf{P}), X)$$

and a monomorphism

$$\mathrm{Hom}_{D^b(A)}(H^0(\mathbf{P}), X[1]) \to \mathrm{Hom}_{D^b(A)}(\mathbf{P}, X[1]).$$

Proof. Note that for any 2-term complex $\mathbf{Y}$ in $D^b(A)$, there is a triangle

$$H^{-1}(\mathbf{Y})[1] \to \mathbf{Y} \to H^0(\mathbf{Y}) \to H^{-1}(\mathbf{Y})[2].$$

Now applying $\mathrm{Hom}_{D^b(A)}(-, X)$ to the triangle

$$H^{-1}(\mathbf{P})[1] \to \mathbf{P} \to H^0(\mathbf{P}) \to H^{-1}(\mathbf{P})[2]$$

and using that there is no non-zero negative extensions between modules, we get the required isomorphism and monomorphism. □

We next describe some useful properties for the torsion pair corresponding to a 2-term silting complex. A consequence of this is that both $\mathcal{T}(\mathbf{P})$ and $\mathcal{F}(\mathbf{P})$ are exact categories with enough projectives and injectives. Note that since Proposition 2.4 implies that $(\mathcal{T}(\mathbf{P}), \mathcal{F}(\mathbf{P}))$ is precisely the torsion pair associated with the support τ-tilting A-module $H^0(\mathbf{P})$, statement (1) also follows from the proof of [1, Theorem 2.7].

Proposition 2.8. *Let $\mathbf{P}$ be a 2-term silting complex and $(\mathcal{T}(\mathbf{P}), \mathcal{F}(\mathbf{P}))$ be the torsion pair induced by $\mathbf{P}$. Then*

(1): *for any $X \in \mathrm{mod}\, A$, we have that $X \in \mathrm{add}\, H^0(\mathbf{P})$ if and only if X is* Ext*-projective in $\mathcal{T}(\mathbf{P})$;*

(2): *for any $X \in \mathcal{T}(\mathbf{P})$, there is a short exact sequence*

$$0 \to L \to T_0 \to X \to 0$$

with $T_0 \in \mathrm{add}\, H^0(\mathbf{P})$ and $L \in \mathcal{T}(\mathbf{P})$;

(3): *for any $X \in \mathrm{mod}\, A$, we have that $X \in \mathrm{add}\, t\nu A$ if and only if X is* Ext*-injective in $\mathcal{T}(\mathbf{P})$;*

(4): *for any $X \in \mathcal{T}(\mathbf{P})$, there is a short exact sequence*

$$0 \to X \to T_0 \to L \to 0$$

with $T_0 \in \mathrm{add}\, t\nu A$ and $L \in \mathcal{T}(\mathbf{P})$;

(5): *for any $X \in \mathrm{mod}\, A$, we have that $X \in \mathrm{add}\, H^{-1}(\nu\mathbf{P})$ if and only if X is* Ext*-injective in $\mathcal{F}(\mathbf{P})$;*

(6): *for any $X \in \mathcal{F}(\mathbf{P})$, there is a short exact sequence*

$$0 \to X \to F_0 \to L \to 0$$

with $F_0 \in \mathrm{add}\, H^{-1}(\nu\mathbf{P})$ and $L \in \mathcal{F}(\mathbf{P})$;

(7): *for any $X \in \mathrm{mod}\, A$, we have that $X \in \mathrm{add}\, A/tA$ if and only if X is* Ext*-projective in $\mathcal{F}(\mathbf{P})$;*

(8): *for any $X \in \mathcal{F}(\mathbf{P})$, there is a short exact sequence*

$$0 \to L \to F_0 \to X \to 0$$

with $F_0 \in \operatorname{add} A/tA$ and $L \in \mathcal{F}(\mathbf{P})$.

Proof. We only prove (1)–(4). The proofs of (5)–(8) are similar.

By the monomorphism in Lemma 2.7, we have that $\operatorname{add} H^0(\mathbf{P})$ is Ext-projective.

Assume M is Ext-projective in $\mathcal{T}(\mathbf{P}) = \operatorname{Fac} H^0(\mathbf{P})$. Then there is an exact sequence

$$0 \to L \to T_0 \xrightarrow{\alpha} M \to 0 \tag{$\sharp$}$$

where $T_0 \xrightarrow{\alpha} M$ is a right $\operatorname{add} H^0(\mathbf{P})$-approximation. Since $\operatorname{Hom}_A(H^0(\mathbf{P}), \alpha)$ is an epimorphism, we have that $\operatorname{Hom}_{D^b(A)}(\mathbf{P}, \alpha)$ is also an epimorphism by Lemma 2.7. Applying $\operatorname{Hom}_{D^b(A)}(\mathbf{P}, -)$ to ($\sharp$), we have an exact sequence

$$\operatorname{Hom}_{D^b(A)}(\mathbf{P}, T_0) \xrightarrow{\operatorname{Hom}_{D^b(A)}(\mathbf{P},\alpha)} \operatorname{Hom}_{D^b(A)}(\mathbf{P}, M) \to \operatorname{Hom}_{D^b(A)}(\mathbf{P}, L[1]) \to 0.$$

Then $\operatorname{Hom}_{D^b(A)}(\mathbf{P}, L[1]) = 0$ which implies that L is in $\mathcal{T}(\mathbf{P})$. Then, by assumption, the sequence ($\sharp$) splits, and hence M is in $\operatorname{add} H^0(\mathbf{P})$. This proves (1). Replacing M with an arbitrary object X in $\mathcal{T}(\mathbf{P})$, we also obtain (2).

For (3) cf. [27] or [5, Proposition VI.1.11].

We now prove (4). For any $X \in \mathcal{T}(\mathbf{P})$, we have an injective envelope $\alpha: X \to I_0$ with $I_0 \in \operatorname{add} \nu A$. Considering the canonical exact sequence of I_0 in $(\mathcal{T}(\mathbf{P}), \mathcal{F}(\mathbf{P}))$:

$$\begin{array}{ccccccccc} & & & \overset{\alpha'}{\swarrow} & X & & & & \\ & & & & \downarrow{\scriptstyle\alpha} & & & & \\ 0 & \longrightarrow & tI_0 & \xrightarrow[\beta]{} & I_0 & \xrightarrow[\gamma]{} & I_0/tI_0 & \longrightarrow & 0 \end{array},$$

we have that $\gamma\alpha = 0$ by $X \in \mathcal{T}(\mathbf{P})$ and $I_0/tI_0 \in \mathcal{F}(\mathbf{P})$. So there is a morphism $\alpha': X \to tI_0$ such that $\alpha = \beta\alpha'$. Note that α' is injective since α is injective. Let $F_0 = tI_0 \in \operatorname{add} t\nu A$ and L be the cokernel of α'. Then L is in $\mathcal{T}(\mathbf{P})$, since $\mathcal{T}(\mathbf{P})$ is closed under taking factor modules. □

3. 2-term silting complexes

The first lemma is the analog, for 2-term silting complexes, of the Bongartz completion of classical tilting modules. It can be deduced from [1, Theorem 2.10] and was proven in [2,13,19,28]. We state the proof in [2] here for self-containedness.

Lemma 3.1. *Let $\mathbf{P}$ be a 2-term presilting complex in $K^b(\operatorname{proj} A)$. Then there exists a triangle*

$$A \to \mathbf{E} \to \mathbf{P}'' \to A[1]$$

with $\mathbf{E}$ being a 2-term complex in $K^b(\operatorname{proj} A)$ such that $\mathbf{P} \oplus \mathbf{E}$ is a 2-term silting complex.

Proof. Let $\mathbf{P}'' \to A[1]$ be a right $\operatorname{add} \mathbf{P}$-approximation of $A[1]$. Extend it to a triangle

$$A \to \mathbf{E} \to \mathbf{P}'' \to A[1]. \tag{$*$}$$

By applying the functors $\mathrm{Hom}_{D^b(A)}(\mathbf{P}, -)$ and $\mathrm{Hom}_{D^b(A)}(-, \mathbf{P})$ to the triangle $(*)$, we have that $\mathrm{Hom}_{D^b(A)}(\mathbf{P}, \mathbf{E}[i]) = 0$ for $i > 0$ and $\mathrm{Hom}_{D^b(A)}(\mathbf{E}, \mathbf{P}[i]) = 0$ for $i > 0$. Applying $\mathrm{Hom}_{D^b(A)}(-, \mathbf{E})$ yields $\mathrm{Hom}_{D^b(A)}(\mathbf{E}, \mathbf{E}[i]) = 0$ for $i > 0$. Hence $\mathbf{P} \oplus \mathbf{E}$ is a 2-term presilting complex in $K^b(\mathrm{proj}\, A)$. The triangle $(*)$ shows that $A \in \mathrm{thick}(\mathbf{P} \oplus \mathbf{E})$ and so $\mathrm{thick}(\mathbf{P} \oplus \mathbf{E}) = K^b(\mathrm{proj}\, A)$ which implies that $\mathbf{P} \oplus \mathbf{E}$ is a silting complex. □

Remark 3.2. Note that if the right $\mathrm{add}\,\mathbf{P}$-approximation in the triangle $(*)$ is minimal, then $\mathbf{E}$ does not contain any direct summands whose 0-th cohomology is zero since $\mathrm{Hom}_{D^b(A)}(A, A[1]) = 0$. Therefore one can deduce [1, Theorem 2.10] from this proof.

We obtain the following characterization of silting complexes.

Corollary 3.3. *Let* $\mathbf{P}$ *be a 2-term presilting complex in* $K^b(\mathrm{proj}\, A)$*. Then the following are equivalent:*

(1): $\mathbf{P}$ *is a silting complex in* $K^b(\mathrm{proj}\, A)$*;*
(2): $|\mathbf{P}| = |A|$*;*
(3): *there is a triangle* $\Delta_{\mathbf{P}}$

$$A \xrightarrow{e} \mathbf{P}' \xrightarrow{f} \mathbf{P}'' \xrightarrow{g} A[1]$$

with $\mathbf{P}', \mathbf{P}'' \in \mathrm{add}\,\mathbf{P}$.

Proof. The equivalence between (1) and (2) is exactly [1, Proposition 3.3], cf. also [3,13,19]. The equivalence between (1) and (3) is an immediate corollary of Lemma 3.1, cf. also [28, Theorem 3.5, Proposition 3.9]. □

Remark 3.4. The map f in the triangle $\Delta_{\mathbf{P}}$ in Corollary 3.3 defines a 2-term complex in $K^b(\mathrm{add}\,\mathbf{P})$, the bounded homotopy category of the additive category $\mathrm{add}\,\mathbf{P}$. Since $\mathrm{Hom}_{D^b(A)}(\mathbf{P}, \mathbf{P}[1]) = 0$, it follows that e is a left $\mathrm{add}\,\mathbf{P}$-approximation, and that g is a right $\mathrm{add}\,\mathbf{P}$-approximation. Moreover, the map e is left minimal if and only if g is right minimal. Hence the resulting 2-term complex is unique up to homotopy equivalence of complexes. Applying $\mathrm{Hom}_{D^b(A)}(\mathbf{P}, -)$ to it, we obtain a unique 2-term complex $\mathbf{Q}$ in $K^b(\mathrm{proj}\, B)$

The following lemmas will be useful later.

Lemma 3.5. *There is a functorial isomorphism*

$$\mathrm{Hom}_{D^b(A)}(\mathbf{P}_0, \mathbf{X}) \cong \mathrm{Hom}_B(\mathrm{Hom}_{D^b(A)}(\mathbf{P}, \mathbf{P}_0), \mathrm{Hom}_{D^b(A)}(\mathbf{P}, \mathbf{X}))$$

sending f *to* $\mathrm{Hom}_{D^b(A)}(\mathbf{P}, f)$*, for any* $\mathbf{P}_0 \in \mathrm{add}\,\mathbf{P}$ *and* $\mathbf{X} \in D^b(A)$.

Proof. This follows from the additivity of the functors and from the fact that the defined map is an isomorphism when $\mathbf{P}_0 = \mathbf{P}$. □

Lemma 3.6. *For each* A*-module* X*, there are isomorphisms*

$$\mathrm{Hom}_{D^b(A)}(\mathbf{P}, X) \cong \mathrm{Hom}_{D^b(A)}(\mathbf{P}, tX)$$

and

$$\mathrm{Hom}_{D^b(A)}(\mathbf{P}, X[1]) \cong \mathrm{Hom}_{D^b(A)}(\mathbf{P}, X/tX[1])$$

as B-modules, where $0 \to tX \to X \to X/tX \to 0$ is the canonical sequence of X with respect to the torsion pair $(\mathcal{T}(\mathbf{P}), \mathcal{F}(\mathbf{P}))$. In particular, $\mathrm{Hom}_{D^b(A)}(\mathbf{P}, X)$ is in $\mathcal{Y}(\mathbf{P})$ and $\mathrm{Hom}_{D^b(A)}(\mathbf{P}, X[1])$ is in $\mathcal{X}(\mathbf{P})$ for any X in $\mathrm{mod}\, A$.

Proof. Applying $\mathrm{Hom}_{D^b(A)}(\mathbf{P}, -)$ to the canonical exact sequence of X in the torsion pair $(\mathcal{T}(\mathbf{P}), \mathcal{F}(\mathbf{P}))$, we have a long exact sequence

$$\begin{array}{ccccccccc} 0 & \to & \mathrm{Hom}_{D^b(A)}(\mathbf{P}, tX) & \to & \mathrm{Hom}_{D^b(A)}(\mathbf{P}, X) & \to & \mathrm{Hom}_{D^b(A)}(\mathbf{P}, X/tX) & & \\ & \to & \mathrm{Hom}_{D^b(A)}(\mathbf{P}, tX[1]) & \to & \mathrm{Hom}_{D^b(A)}(\mathbf{P}, X[1]) & \to & \mathrm{Hom}_{D^b(A)}(\mathbf{P}, X/tX[1]) & \to & 0. \end{array}$$

Note that $\mathrm{Hom}_{D^b(A)}(\mathbf{P}, X/tX) = 0$ by $X/tX \in \mathcal{F}(\mathbf{P})$ and $\mathrm{Hom}_{D^b(A)}(\mathbf{P}, tX[1]) = 0$ by $tX \in \mathcal{T}(\mathbf{P})$. Thus we get the desired isomorphisms. □

Lemma 3.7. *For any 2-term complex $\mathbf{X} : X^{-1} \xrightarrow{x} X^0$ in $K^b(\mathrm{proj}\, A)$, if $H^0(\mathbf{X}) \cong 0 \cong H^{-1}(\nu\mathbf{X})$, then $\mathbf{X} \cong 0$.*

Proof. On the one hand, $H^0(\mathbf{X}) \cong 0$ implies that x is an epimorphism, so x is a retraction. On the other hand, $H^{-1}(\nu\mathbf{X}) \cong 0$ implies that νx is a monomorphism, so νx is a section. Since ν is an equivalence from $\mathrm{proj}\, A$ to $\mathrm{inj}\, A$, we have that x is an isomorphism. Hence $\mathbf{X} \cong 0$. □

Recall from Remark 3.4 that $\mathbf{P}$ determines a unique (up to isomorphism) 2-term complex $\mathbf{Q}$ in $K^b(\mathrm{proj}\, B)$ given by

$$\mathrm{Hom}_{D^b(A)}(\mathbf{P}, \mathbf{P}') \xrightarrow{\mathrm{Hom}_{D^b(A)}(\mathbf{P}, f)} \mathrm{Hom}_{D^b(A)}(\mathbf{P}, \mathbf{P}''),$$

where f is the map from the triangle $\Delta_{\mathbf{P}}$.

Proposition 3.8. *Let $\mathbf{P}$ be a 2-term silting complex. Then the complex $\mathbf{Q}$ defined above is a 2-term silting complex in $K^b(\mathrm{proj}\, B)$. Moreover, $\mathcal{T}(\mathbf{Q}) = \mathcal{X}(\mathbf{P})$ and $\mathcal{F}(\mathbf{Q}) = \mathcal{Y}(\mathbf{P})$.*

Proof. Let $P_1, \cdots, P_n$ be a complete collection of indecomposable, pairwise non-isomorphic projective A-modules. Since the map e from the triangle $\Delta_{\mathbf{P}}$ is a left $\mathrm{add}\,\mathbf{P}$-approximation, there are triangles

$$P_i \xrightarrow{e_i} \mathbf{P}'_i \xrightarrow{f_i} \mathbf{P}''_i \xrightarrow{g_i} P_i[1]$$

such that the direct sum of these triangles is a direct summand of $\Delta_{\mathbf{P}}$. Let $\mathbf{Q}_i$ be the 2-term complex in $K^b(\mathrm{proj}\, B)$ given by

$$\mathrm{Hom}_{D^b(A)}(\mathbf{P}, \mathbf{P}'_i) \xrightarrow{\mathrm{Hom}_{D^b(A)}(\mathbf{P}, f_i)} \mathrm{Hom}_{D^b(A)}(\mathbf{P}, \mathbf{P}''_i),$$

for each $1 \le i \le n$. Then $\oplus_{i=1}^n \mathbf{Q}_i$ is isomorphic to a direct summand of $\mathbf{Q}$. We claim that $\mathbf{Q}_1, \cdots, \mathbf{Q}_n$ are nonzero and each two of them have no common direct summands. Indeed, by Lemma 3.6, for each $1 \le i \le n$,

$$H^0(\mathbf{Q}_i) = \mathrm{coker}\, \mathrm{Hom}_{D^b(A)}(\mathbf{P}, f_i) \cong \mathrm{Hom}_{D^b(A)}(\mathbf{P}, P_i[1]) \cong \mathrm{Hom}_{D^b(A)}(\mathbf{P}, P_i/tP_i[1])$$

and

$$H^{-1}(\nu\mathbf{Q}_i) = \ker \nu\, \mathrm{Hom}_{D^b(A)}(\mathbf{P}, f_i) \overset{(*)}{\cong} \ker \mathrm{Hom}_{D^b(A)}(\mathbf{P}, \nu f_i) \cong \mathrm{Hom}_{D^b(A)}(\mathbf{P}, \nu P_i) \cong \mathrm{Hom}_{D^b(A)}(\mathbf{P}, t\nu P_i)$$

where $(*)$ holds because $\mathrm{Hom}_{D^b(A)}(\mathbf{P}, \nu\mathbf{P}) \cong D\,\mathrm{Hom}_{D^b(A)}(\mathbf{P}, \mathbf{P})$ is an injective generator of $\mathrm{mod}\, B$. If $\mathbf{Q}_i \cong 0$ for some i, then both $H^0(\mathbf{Q}_i)$ and $H^{-1}(\nu\mathbf{Q}_i)$ are isomorphic to zero. Note that $P_i/tP_i \in \mathcal{F}(\mathbf{P})$ and

$t\nu P_i \in \mathcal{T}(\mathbf{P})$. Then by Corollary 2.5, we have $P_i/tP_i \cong 0 \cong t\nu P_i$, where the first isomorphism implies that $P_i \in \operatorname{add}\mathbf{P}$, and the second implies that $P_i[1] \in \operatorname{add}\mathbf{P}$. This is a contradiction. Hence $\mathbf{Q}_i \ncong 0$. Note that P_i is a projective cover of P_i/tP_i (if $P_i/tP_i \neq 0$) and νP_i is an injective envelope of $t\nu P_i$ (if $t\nu P_i \neq 0$). So by Corollary 2.5, for any $i \neq j$, $H^0(\mathbf{Q}_i)$ and $H^0(\mathbf{Q}_j)$ have no common direct summands, and $H^{-1}(\nu\mathbf{Q}_i)$ and $H^{-1}(\nu\mathbf{Q}_j)$ have no common direct summands. If $\mathbf{Q}_i$ and $\mathbf{Q}_j$ have a common direct summand $\mathbf{X}$, then $H^0(\mathbf{X}) \cong 0 \cong H^{-1}(\nu\mathbf{X})$. By Lemma 3.7, $\mathbf{X} \cong 0$. We finish the proof of the claim. Therefore, $|\mathbf{Q}| \geq |A|$.

To prove that $\mathbf{Q}$ is silting, it is by Corollary 3.3, sufficient to prove that $\mathbf{Q}$ is presilting. Let α be a morphism in $\operatorname{Hom}_{K^b(\operatorname{proj} B)}(\mathbf{Q}, \mathbf{Q}[1])$, then it has the following form

$$\begin{array}{ccccc} & & \operatorname{Hom}_{D^b(A)}(\mathbf{P}, \mathbf{P}') & \xrightarrow{\operatorname{Hom}_{D^b(A)}(\mathbf{P}, f)} & \operatorname{Hom}_{D^b(A)}(\mathbf{P}, \mathbf{P}'') \\ & & \downarrow{\scriptstyle \alpha} & & \\ \operatorname{Hom}_{D^b(A)}(\mathbf{P}, \mathbf{P}') & \xrightarrow{\operatorname{Hom}_{D^b(A)}(\mathbf{P}, f)} & \operatorname{Hom}_{D^b(A)}(\mathbf{P}, \mathbf{P}'') & & \end{array}$$

By Lemma 3.5, there is a morphism $h: \mathbf{P}' \to \mathbf{P}''$ such that $\alpha = \operatorname{Hom}_{D^b(A)}(\mathbf{P}, h)$. Since $\operatorname{Hom}_{D^b(A)}(A, A[1]) = 0$, there are morphisms h_1, h_2 such that the following commutative diagram is a morphism of triangles:

$$\begin{array}{ccccccc} A & \xrightarrow{e} & \mathbf{P}' & \xrightarrow{f} & \mathbf{P}'' & \xrightarrow{g} & A[1] \\ {\scriptstyle h_1}\downarrow & & {\scriptstyle h}\downarrow & & {\scriptstyle h_2}\downarrow & & \downarrow{\scriptstyle h_1[1]} \\ \mathbf{P}' & \xrightarrow{f} & \mathbf{P}'' & \xrightarrow{g} & A[1] & \xrightarrow{-e[1]} & \mathbf{P}'[1] \end{array}$$

By Remark 3.4, the morphism g is a right $\operatorname{add}\mathbf{P}$-approximation of $A[1]$. So there is a morphism h_3 such that $h_2 = gh_3$. Then $g(h - h_3 f) = gh - gh_3 f = gh - h_2 f = 0$. Hence there is a morphism h_4 such that $h - h_3 f = f h_4$.

$$\begin{array}{ccccccc} A & \xrightarrow{e} & \mathbf{P}' & \xrightarrow{f} & \mathbf{P}'' & \xrightarrow{g} & A[1] \\ {\scriptstyle h_1}\downarrow & {\scriptstyle h_4}\swarrow & {\scriptstyle h}\downarrow & {\scriptstyle h_3}\swarrow & {\scriptstyle h_2}\downarrow & & \downarrow{\scriptstyle h_1[1]} \\ \mathbf{P}' & \xrightarrow{f} & \mathbf{P}'' & \xrightarrow{g} & A[1] & \xrightarrow{-e[1]} & \mathbf{P}'[1] \end{array}$$

Applying $\operatorname{Hom}_{D^b(A)}(\mathbf{P}, -)$ to $h - h_3 f = f h_4$ yields

$$\alpha = \operatorname{Hom}_{D^b(A)}(\mathbf{P}, h_3) \operatorname{Hom}_{D^b(A)}(\mathbf{P}, f) + \operatorname{Hom}_{D^b(A)}(\mathbf{P}, f) \operatorname{Hom}_{D^b(A)}(\mathbf{P}, h_4)$$

which implies that α, regarded as a map in $\operatorname{Hom}_{D^b(A)}(\mathbf{Q}, \mathbf{Q}[1])$ is null-homotopic. Thus, we have completed the proof that $\mathbf{Q}$ is a 2-term silting complex.

Finally we prove that $\mathcal{T}(\mathbf{Q}) = \mathcal{X}(\mathbf{P})$. The proof of $\mathcal{F}(\mathbf{Q}) = \mathcal{Y}(\mathbf{P})$ is similar. Since we have proven that $H^0(\mathbf{Q}) \cong \operatorname{Hom}_{D^b(A)}(\mathbf{P}, A/tA[1])$, by Proposition 2.4, it is therefore sufficient to prove that $\operatorname{Fac}\operatorname{Hom}_{D^b(A)}(\mathbf{P}, A/tA[1]) = \mathcal{X}(\mathbf{P})$.

Let X be in $\mathcal{X}(\mathbf{P})$. There is then an object X' in $\mathcal{F}(\mathbf{P})$, such that $X = \operatorname{Hom}_{D^b(A)}(P, X'[1])$. By Proposition 2.8 (8), there is a short exact sequence

$$0 \to L \to F_0 \to X' \to 0$$

in $\mathcal{F}(\mathbf{P})$, with F_0 in $\operatorname{add} A/tA$. Then there is an induced triangle $L \to F_0 \to X' \to L[1]$ in $D^b(A)$. Apply now $\operatorname{Hom}_{D^b(A)}(\mathbf{P}, -[1])$ to this triangle, to obtain a short exact sequence in $\operatorname{mod} B$ showing that X is in

$\operatorname{Fac}\operatorname{Hom}_{D^b(A)}(\mathbf{P}, A/tA[1])$, so we have $\mathcal{X}(\mathbf{P}) \subset \operatorname{Fac}\operatorname{Hom}_{D^b(A)}(\mathbf{P}, A/tA)$. On the other hand, $A/tA \in \mathcal{F}(\mathbf{P})$ implies $\operatorname{Hom}_{D^b(A)}(\mathbf{P}, A/tA[1]) \in \mathcal{X}(\mathbf{P})$, hence $\operatorname{Fac}\operatorname{Hom}_{D^b(A)}(\mathbf{P}, A/tA[1]) \subset \mathcal{X}(\mathbf{P})$, since $\mathcal{X}(\mathbf{P})$ is closed under factor objects. This concludes the proof. $\square$

Corollary 3.9. *The induced torsion pair $(\mathcal{X}(\mathbf{P}), \mathcal{Y}(\mathbf{P}))$ by $\mathbf{P}$ in $\operatorname{mod} B$ is functorially finite.*

Proof. This follows from Proposition 3.8, Proposition 2.4 and the main result of [27]. $\square$

4. A silting theorem

If $\mathbf{P}$ is a projective presentation of a tilting A-module T, then $\nu\mathbf{Q}[-1]$ is isomorphic to the cotilting B-module $D(T)_B = D\operatorname{Hom}_A(T, A)$, and moreover, the endomorphism algebra of this cotilting module is canonically isomorphic to A.

It is easy to check that this does not hold in our setting, that is: in general it does not hold that $\operatorname{End}_{D^b(B)}(\mathbf{Q})$ is isomorphic to A, where $\mathbf{Q}$ is the 2-term silting complex in $K^b(\operatorname{proj} A)$, considered in the previous section. However, we prove that $\operatorname{End}_{D^b(B)}(\mathbf{Q})$ is isomorphic to a factor algebra of A and this factor algebra is isomorphic to A if and only if $\mathbf{P}$ is a tilting complex. This will then be used to provide mutual equivalences of torsion pairs, as we have in classical tilting theory.

Consider now, as in Remark 3.4, the map $\mathbf{P}' \xrightarrow{f} \mathbf{P}''$, coming from the triangle $\Delta_{\mathbf{P}}$ in Corollary 3.3, as an object $\hat{\mathbf{Q}}$ in $K^b(\operatorname{add}\mathbf{P})$, by letting $\hat{\mathbf{Q}}^i = 0$ for all $i \neq -1, 0$. Recall that the functor $\operatorname{Hom}_{D^b(A)}(\mathbf{P}, -)$ gives an equivalence between additive categories $\operatorname{add}\mathbf{P}$ and $\operatorname{proj} B$, hence it induces an equivalence of triangulated categories $K^b(\operatorname{add}\mathbf{P})$ and $K^b(\operatorname{proj} B)$. So it induces an algebra isomorphism $\operatorname{End}_{K^b(\operatorname{add}\mathbf{P})}(\hat{\mathbf{Q}}) \to \operatorname{End}_{D^b(B)}(\mathbf{Q})$.

We will define an algebra-homomorphism $\operatorname{End}_A(A) \to \operatorname{End}_{K^b(\operatorname{add}\mathbf{P})}(\hat{\mathbf{Q}})$. For this, represent the object $\mathbf{P}'$ by $P_\Delta^{-1} \xrightarrow{p'} P_\Delta^0$, and represent the object $\mathbf{P}''$ as the mapping cone of $A \to \mathbf{P}'$, that is $P_\Delta^{-1} \oplus A \xrightarrow{(-p'\ e)} P_\Delta^0$.

Now, let $a \in \operatorname{End}_A(A)$ and consider the solid diagram, whose rows are triangles,

$$\begin{array}{ccccccc}
\mathbf{P}''[-1] & \xrightarrow{-g[-1]} & A & \xrightarrow{e} & \mathbf{P}' & \xrightarrow{f} & \mathbf{P}'' \\
 & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \\
\mathbf{P}''[-1] & \xrightarrow{-g[-1]} & A & \xrightarrow{e} & \mathbf{P}' & \xrightarrow{f} & \mathbf{P}''
\end{array}$$

Since $\operatorname{Hom}_{D^b(A)}(\mathbf{P}''[-1], \mathbf{P}') = 0$, there is a map $b \colon \mathbf{P}' \to \mathbf{P}'$ such that $be = ea$. Choose first such a map $b = (b_1, b_2)$. Then, in particular, the following diagram commutes in $\operatorname{proj} A$:

$$\begin{array}{ccc}
P_\Delta^{-1} & \xrightarrow{p'} & P_\Delta^0 \\
\downarrow{\scriptstyle b_1} & & \downarrow{\scriptstyle b_2} \\
P_\Delta^{-1} & \xrightarrow{p'} & P_\Delta^0
\end{array}$$

Now since the diagram

$$\begin{array}{ccc}
A & \xrightarrow{e} & \mathbf{P}' \\
\downarrow{\scriptstyle a} & & \downarrow{\scriptstyle (b_1, b_2)} \\
A & \xrightarrow{e} & \mathbf{P}'
\end{array}$$

commutes in $K^b(\mathrm{proj}\, A)$, the chain map

$$\begin{array}{ccc} 0 & \longrightarrow & A \\ \downarrow & & \downarrow{\scriptstyle ea-b_2e} \\ P_\Delta^{-1} & \xrightarrow{p'} & P_\Delta^0 \end{array}$$

is null-homotopic. Then there is a map $A \xrightarrow{t} P_\Delta^{-1}$, such that $p't = ea - b_2e$.

Next, consider the endomorphism c of

$$P_\Delta^{-1} \oplus A \xrightarrow{(\,-p'\ e\,)} P_\Delta^0$$

given as follows

$$\begin{array}{ccc} P_\Delta^{-1} \oplus A & \xrightarrow{(\,-p'\ e\,)} & P_\Delta^0 \\ \downarrow{\scriptstyle \left(\begin{smallmatrix} b_1 & t \\ 0 & a \end{smallmatrix}\right)} & & \downarrow{\scriptstyle b_2} \\ P_\Delta^{-1} \oplus A & \xrightarrow{(\,-p'\ e\,)} & P_\Delta^0 \end{array}$$

It is straightforward to check, that the map c is a chain map and that we obtain a morphism of triangles

$$\begin{array}{ccccccc} A & \xrightarrow{e} & \mathbf{P}' & \xrightarrow{f} & \mathbf{P}'' & \xrightarrow{g} & A[1] \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle a[1]} \\ A & \xrightarrow{e} & \mathbf{P}' & \xrightarrow{f} & \mathbf{P}'' & \xrightarrow{g} & A[1] \end{array}$$

where f and g now denote the maps

$$\begin{array}{ccc} P_\Delta^{-1} & \xrightarrow{p'} & P_\Delta^0 \\ \downarrow{\scriptstyle \left(\begin{smallmatrix} -1 \\ 0 \end{smallmatrix}\right)} & & \downarrow{\scriptstyle 1} \\ P_\Delta^{-1} \oplus A & \xrightarrow{(\,-p'\ e\,)} & P_\Delta^0 \end{array} \qquad \begin{array}{ccc} P_\Delta^{-1} \oplus A & \xrightarrow{(\,-p'\ e\,)} & P_\Delta^0 \\ \downarrow{\scriptstyle (\,0\ 1\,)} & & \downarrow{\scriptstyle 0} \\ A & \xrightarrow{0} & 0 \end{array}$$

Proposition 4.1. *The map* $\Phi_{\mathbf{P}}\colon \mathrm{End}_A(A) \to \mathrm{End}_{K^b}(\hat{\mathbf{Q}})$ *given by* $a \mapsto (b, c)$, *where* $b = (b_1, b_2)$ *and* $c = (\left(\begin{smallmatrix} b_1 & t \\ 0 & a \end{smallmatrix}\right), b_2)$ *are chain maps, is a well-defined and surjective algebra morphism with kernel given by*

$$\{v\alpha u \mid u \in \mathrm{Hom}_{D^b(A)}(A, \mathbf{P}_I), \alpha \in \mathrm{Hom}_{D^b(A)}(\mathbf{P}_I, \mathbf{P}_{II}[-1]) \text{ and } v \in \mathrm{Hom}_{D^b(A)}(\mathbf{P}_{II}[-1], A) \text{ with } \mathbf{P}_I, \mathbf{P}_{IIT} \in \mathrm{add}\,\mathbf{P}\}.$$

Moreover, we have $\ker \Phi_{\mathbf{P}} = 0$ *if and only if* $\mathrm{Hom}_{D^b(A)}(\mathbf{P}, \mathbf{P}[-1]) = 0$.

Proof. In order to show that $\Phi_{\mathbf{P}}$ is well-defined, for any map $a \in \mathrm{End}_A(A)$, take two maps (b^χ, c^χ), $\chi = 1, 2$, in $\mathrm{End}_{K^b(\mathrm{add}\,\mathbf{P})}(\hat{\mathbf{Q}})$ of the form

$$\begin{array}{ccc} \mathbf{P}' & \xrightarrow{\left(\left(\begin{smallmatrix}-1\\0\end{smallmatrix}\right),1\right)} & \mathbf{P}'' \\ \Big\downarrow{\scriptstyle (b_1^\chi,b_2^\chi)} & & \Big\downarrow{\scriptstyle \left(\left(\begin{smallmatrix}b_1^\chi & t^\chi\\0 & a\end{smallmatrix}\right),b_2^\chi\right)} \\ \mathbf{P}' & \xrightarrow{\left(\left(\begin{smallmatrix}-1\\0\end{smallmatrix}\right),1\right)} & \mathbf{P}'' \end{array} .$$

We need to prove that the two maps (b^χ, c^χ) are homotopic. This is equivalent to showing that their difference

$$\begin{array}{ccc} \mathbf{P}' & \xrightarrow{\left(\left(\begin{smallmatrix}-1\\0\end{smallmatrix}\right),1\right)} & \mathbf{P}'' \\ \Big\downarrow{\scriptstyle (b_1^0,b_2^0)} & & \Big\downarrow{\scriptstyle \left(\left(\begin{smallmatrix}b_1^0 & t^0\\0 & 0\end{smallmatrix}\right),b_2^0\right)} \\ \mathbf{P}' & \xrightarrow{\left(\left(\begin{smallmatrix}-1\\0\end{smallmatrix}\right),1\right)} & \mathbf{P}'' \end{array} \qquad (\star)$$

is null-homotopic in $K^b(\operatorname{add}\mathbf{P})$, where $b_1^0 = b_1^1 - b_1^2$, $b_2^0 = b_2^1 - b_2^2$ and $t^0 = t^1 - t^2$.

Consider the map $\mathbf{P}'' \stackrel{\mu}{\to} \mathbf{P}'$ defined as follows:

$$\begin{array}{ccc} P_\triangle^{-1} \oplus A & \xrightarrow{(\,-p'\ \ e\,)} & P_\triangle^0 \\ \Big\downarrow{\scriptstyle (\,-b_1^0\ \ -t^0\,)} & & \Big\downarrow{\scriptstyle b_2^0} \\ P_\triangle^{-1} & \xrightarrow{p'} & P_\triangle^0 \end{array}$$

Then it is easily verified that $\mu f = (b_1^0, b_2^0)$, and that $f\mu = \left(\left(\begin{smallmatrix}b_1^0 & t^0\\0 & 0\end{smallmatrix}\right), b_2^0\right)$ in $\operatorname{add}\mathbf{P}$. Hence, the map $(\star)$ is null-homotopic and therefore $\Phi_\mathbf{P}$ is well-defined. It is easy to check that it is an algebra homomorphism.

We next show that $\Phi_\mathbf{P}$ is surjective. Consider an arbitrary map (b, c) in $\operatorname{End}_{K^b(\operatorname{add}\mathbf{P})}(\hat{\mathbf{Q}})$ represented by

$$\begin{array}{ccc} \mathbf{P}' & \xrightarrow{\left(\left(\begin{smallmatrix}-1\\0\end{smallmatrix}\right),1\right)} & \mathbf{P}'' \\ \Big\downarrow{\scriptstyle (b_1,b_2)} & & \Big\downarrow{\scriptstyle \left(\left(\begin{smallmatrix}c_1 & c_2\\c_3 & c_4\end{smallmatrix}\right),c_0\right)} \\ \mathbf{P}' & \xrightarrow{\left(\left(\begin{smallmatrix}-1\\0\end{smallmatrix}\right),1\right)} & \mathbf{P}'' \end{array}$$

It is sufficient to show that such map is equivalent to a map of the form

$$\begin{array}{ccc} \mathbf{P}' & \xrightarrow{\left(\left(\begin{smallmatrix}-1\\0\end{smallmatrix}\right),1\right)} & \mathbf{P}'' \\ \Big\downarrow{\scriptstyle (b_1,b_2)} & & \Big\downarrow{\scriptstyle \left(\left(\begin{smallmatrix}b_1 & u\\0 & a\end{smallmatrix}\right),b_2\right)} \\ \mathbf{P}' & \xrightarrow{\left(\left(\begin{smallmatrix}-1\\0\end{smallmatrix}\right),1\right)} & \mathbf{P}'' \end{array}$$

for some value of a, and for a u satisfying $p'u = ea - b_2e$. Here, the condition $p'u = ea - b_2e$, together with (b_1, b_2) being a chain map, ensures that $\left(\left(\begin{smallmatrix}b_1 & u\\0 & a\end{smallmatrix}\right), b_2\right)$ is a chain map.

Since $cf = fb$, we have that the following maps

$$
\begin{array}{ccc}
P^{-1} & \xrightarrow{p'} & P^0 \\
{\scriptstyle \binom{-c_1}{-c_3}}\downarrow & & \downarrow{\scriptstyle c_0} \\
P^{-1}\oplus A & \xrightarrow{(\,-p'\ e\,)} & P^0
\end{array}
\qquad\qquad
\begin{array}{ccc}
P^{-1} & \xrightarrow{p'} & P^0 \\
{\scriptstyle \binom{-b_1}{0}}\downarrow & & \downarrow{\scriptstyle b_2} \\
P^{-1}\oplus A & \xrightarrow{(\,-p'\ e\,)} & P^0
\end{array}
$$

are homotopic in $K^b(\operatorname{proj} A)$. Hence, there exists $\binom{x}{y}: P^0 \to P^{-1} \oplus A$, such that $\binom{x}{y} p' = \binom{c_1 - b_1}{c_3}$ and $b_2 - c_0 = (\,-p'\ e\,)\binom{x}{y} = -p'x + ey$.

It is now straightforward to verify that the map $c = \left(\left(\begin{smallmatrix} c_1 & c_2 \\ c_3 & c_4 \end{smallmatrix}\right), c_0\right)$ is homotopic to the map $\left(\left(\begin{smallmatrix} b_1 & c_2+xe \\ 0 & c_4+ye \end{smallmatrix}\right), b_2\right)$, and that $u := c_2 + xe$ satisfies $p'u = ea - b_2 e$ where $a = c_4 + ye$. This proves the claim, and hence $\Phi_{\mathbf{P}}$ is surjective.

Assume now a is in the kernel in $\Phi_{\mathbf{P}}$, so that (b, c) is homotopic to zero. That is, there exists a chain map $d : \mathbf{P}'' \to \mathbf{P}'$ of the following form

$$
\begin{array}{ccc}
P_\Delta^{-1}\oplus A & \xrightarrow{(\,-p'\ e\,)} & P_\Delta^0 \\
{\scriptstyle (\,d_1\ d_2\,)}\downarrow & & \downarrow{\scriptstyle w} \\
P_\Delta^{-1} & \xrightarrow{p'} & P_\Delta^0
\end{array}
$$

such that $b = df$ and $c = fd$ in $\operatorname{add}\mathbf{P}$. So (b_1, b_2) is homotopic to $(-d_1, w) = ((d_1, d_2), w)\left(\left(\begin{smallmatrix} -1 \\ 0 \end{smallmatrix}\right), 1\right)$ and $\left(\left(\begin{smallmatrix} b_1 & t \\ 0 & a \end{smallmatrix}\right), b_2\right)$ is homotopic to $\left(\left(\begin{smallmatrix} -d_1 & -d_2 \\ 0 & 0 \end{smallmatrix}\right), w\right) = \left(\left(\begin{smallmatrix} -1 \\ 0 \end{smallmatrix}\right), 1\right)((d_1, d_2), w)$.

There is then a map $\delta : P_\Delta^0 \to P_\Delta^{-1}$ such that $p'\delta = b_2 - w$ and $\delta p' = b_1 + d_1$, and a map $\binom{\epsilon}{\theta} : P_\Delta^0 \to P_\Delta^{-1} \oplus A$ such that

$$
\begin{pmatrix} \epsilon \\ \theta \end{pmatrix} (\,-p' \quad e\,) = \begin{pmatrix} -\epsilon p' & \epsilon e \\ -\theta p' & \theta e \end{pmatrix} = \begin{pmatrix} b_1 + d_1 & t + d_2 \\ 0 & a \end{pmatrix}
$$

and such that $(\,-p'\ e\,)\binom{\epsilon}{\theta} = -p'\epsilon + e\theta = b_2 - w$. Combining these equations we obtain

$$
p'(\delta + \epsilon) = e\theta \qquad\qquad (\delta + \epsilon)p' = 0.
$$

Note that in particular we have $\theta p' = 0$ and $\theta e = a$. In this way we obtain that the map $a : A \to A$ factors as follows

$$
\begin{array}{ccccc}
 & & A & & \\
 & & \downarrow{\scriptstyle e} & & \\
P_\Delta^{-1} & \xrightarrow{p'} & P_\Delta^0 & & \\
 & & \downarrow{\scriptstyle \binom{\delta+\epsilon}{\theta}} & & \\
 & & P_\Delta^{-1}\oplus A & \xrightarrow{(\,-p'\ e\,)} & P_\Delta^0 \\
 & & \downarrow{\scriptstyle (\,0\ 1\,)} & & \\
 & & A & &
\end{array}
$$

So we have proved that

$$\ker \Phi_{\mathbf{P}} \subseteq I = \{v\alpha u \mid u \in \mathrm{Hom}_{D^b(A)}(A, \mathbf{P}_I), \alpha \in \mathrm{Hom}_{D^b(A)}(\mathbf{P}_I, \mathbf{P}_{II}[-1]) \text{ and } v \in \mathrm{Hom}_{D^b(A)}(\mathbf{P}_{II}[-1], A) \text{ with } \mathbf{P}_I, \mathbf{P}_{II} \in \operatorname{add} \mathbf{P}\}.$$

Next, we prove that $I \subseteq \ker \Phi_{\mathbf{P}}$. Let a be an element in I. Since the map $e: A \to \mathbf{P}'$ is a left $\operatorname{add}\mathbf{P}$-approximation, and the map $g: \mathbf{P}'' \to A[1]$ is a right $\operatorname{add}\mathbf{P}$-approximation, we have that $a = g[-1]ue$ for some map $u: \mathbf{P}' \to \mathbf{P}''[-1]$.

Now, assume u is represented by $P^0_\Delta \xrightarrow{\binom{u_1}{u_2}} P^{-1}_\Delta \oplus A$, so we have $\binom{u_1 p'}{u_2 p'} = \binom{0}{0}$ and $p'u_1 = eu_2$ and $a = u_2 e$.

Consider the map in $\mathrm{End}_{K^b(\operatorname{add}\mathbf{P})}(\hat{\mathbf{Q}})$ given by

$$\begin{array}{ccc} \mathbf{P}' & \xrightarrow{\quad f \quad} & \mathbf{P}'' \\ \Big\downarrow{\scriptstyle (0, eu_2)} & & \Big\downarrow{\scriptstyle \left(\begin{pmatrix} 0 & 0 \\ 0 & u_2 e \end{pmatrix}, eu_2\right)} \\ \mathbf{P}' & \xrightarrow{\quad f \quad} & \mathbf{P}'' \end{array} .$$

Since $a = u_2 e$, this map must be homotopic to $\Phi_{\mathbf{P}}(a)$. The map $(0, eu_2)$ is nullhomotopic in $K^b(\operatorname{proj} A)$, since $u_1 p' = 0$ and $eu_2 = p'u_1$. Moreover, the map $\left(\begin{pmatrix} 0 & 0 \\ 0 & u_2 e \end{pmatrix}, eu_2\right)$ is also nullhomotopic in $K^b(\operatorname{proj} A)$, since

$$\begin{pmatrix} 0 \\ u_2 \end{pmatrix} \begin{pmatrix} -p' & e \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -u_2 p' & u_2 e \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & u_2 e \end{pmatrix}$$

and $\begin{pmatrix} -p' & e \end{pmatrix} \begin{pmatrix} 0 \\ u_2 \end{pmatrix} = eu_2$. Hence a is in $\ker \Phi_{\mathbf{P}}$.

We are now left with proving that $\ker \Phi_{\mathbf{P}} = 0$ if and only if $\mathrm{Hom}_{D^b(A)}(\mathbf{P}, \mathbf{P}[-1]) = 0$. By the first part, we have that $\mathrm{Hom}_{D^b(A)}(\mathbf{P}, \mathbf{P}[-1]) = 0$ implies that $\ker \Phi_{\mathbf{P}} = 0$. Assume $\mathrm{Hom}_{D^b(A)}(\mathbf{P}, \mathbf{P}[-1]) \neq 0$. Then $\mathrm{Hom}_{D^b(A)}(\mathbf{P}, \mathbf{P}[-1])$ contains a non-zero element η, which is a chain map:

$$\begin{array}{ccccc} P^{-1} & \xrightarrow{p'} & P^0 & & \\ & & \Big\downarrow{\scriptstyle \eta} & & \\ & & P^{-1} & \xrightarrow{p'} & P^0 \end{array}$$

So there are P_i, P_j, indecomposable direct summands of P^0, P^{-1} respectively, such that the component of η from P_i to P_j is not zero. This induces a non-zero morphism a_η in $\mathrm{Hom}_A(A, A)$ which factors through η. Then a_η is in $\ker \Phi_{\mathbf{P}}$. This concludes the proof. $\square$

Remark 4.2. For any map $a \in \mathrm{End}_A(A)$, its image (b, c) under the epimorphism $\Phi_{\mathbf{P}}$ makes the diagram

$$\begin{array}{ccccccc} A & \xrightarrow{\quad e \quad} & \mathbf{P}' & \xrightarrow{\quad f \quad} & \mathbf{P}'' & \xrightarrow{\quad g \quad} & A[1] \\ \Big\downarrow{\scriptstyle a} & & \Big\downarrow{\scriptstyle b} & & \Big\downarrow{\scriptstyle c} & & \Big\downarrow{\scriptstyle a[1]} \\ A & \xrightarrow{\quad e \quad} & \mathbf{P}' & \xrightarrow{\quad f \quad} & \mathbf{P}'' & \xrightarrow{\quad g \quad} & A[1] \end{array}$$

commute in $K^b(\operatorname{proj} A)$. However, the converse is in general not true. For example, let $A = \mathbf{k}Q/I$, where Q is the following quiver

$$1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2$$

and I is generated by $\alpha\beta\alpha$ and $\beta\alpha\beta$. Let $\mathbf{P} = \mathbf{P}_1 \oplus \mathbf{P}_2$ with

$$\mathbf{P}_1: \ 0 \to P_1$$

and

$$\mathbf{P}_2: \ P_2 \to P_1.$$

Then $\mathbf{P}$ is a 2-term silting complex in $K^b(\operatorname{proj} A)$ and we have that $\mathbf{P}' \cong \mathbf{P}_1 \oplus \mathbf{P}_1$ and $\mathbf{P}'' \cong \mathbf{P}_2$. We take a and b to be zero maps and take c to be the following chain map

$$\begin{array}{ccc} P_2 & \longrightarrow & P_1 \\ {\scriptstyle \beta\alpha}\downarrow & & \downarrow{\scriptstyle 0} \\ P_2 & \longrightarrow & P_1 \end{array}$$

It is easily verified that the maps a, b, c make the diagram commutes. But the pair (b, c) regarded as a chain map from $\hat{\mathbf{Q}}$ to itself is not null-homotopic in $K^b(\operatorname{add}\mathbf{P})$. This means that (b, c) is not $\Phi_{\mathbf{P}}(a)$.

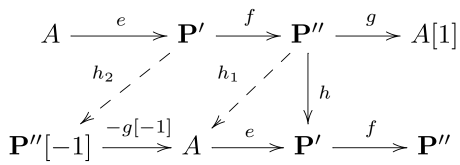
The following corollary shows that in the tilting case, our result covers the classical result.

Corollary 4.3. *Under the same notation as before, $\mathbf{P}$ is a tilting complex if and only if $\Phi_{\mathbf{P}}$ is an isomorphism. In this case, $\mathbf{Q}$ is also tilting.*

Proof. Clearly $\mathbf{P}$ is a tilting complex if and only if $\operatorname{Hom}_{D^b(A)}(\mathbf{P}, \mathbf{P}[-1]) = 0$. Hence, the equivalence follows directly from the last part of Proposition 4.1. Assume now $\operatorname{Hom}_{D^b(A)}(\mathbf{P}, \mathbf{P}[-1]) = 0$. It suffices to prove that then also $\operatorname{Hom}_{D^b(B)}(\mathbf{Q}, \mathbf{Q}[-1]) = 0$. Note that each morphism α in $\operatorname{Hom}_{D^b(B)}(\mathbf{Q}, \mathbf{Q}[-1])$ has the following form:

$$\begin{array}{cccc} \operatorname{Hom}_{D^b(A)}(\mathbf{P}, \mathbf{P}') & \xrightarrow{\operatorname{Hom}_{D^b(A)}(\mathbf{P}, f)} & \operatorname{Hom}_{D^b(A)}(\mathbf{P}, \mathbf{P}'') & \\ & & \downarrow{\scriptstyle \alpha} & \\ & & \operatorname{Hom}_{D^b(A)}(\mathbf{P}, \mathbf{P}') & \xrightarrow{\operatorname{Hom}_{D^b(A)}(\mathbf{P}, f)} \operatorname{Hom}_{D^b(A)}(\mathbf{P}, \mathbf{P}'') \end{array}$$

with $\alpha \operatorname{Hom}_{D^b(A)}(\mathbf{P}, f) = 0 = \operatorname{Hom}_{D^b(A)}(\mathbf{P}, f)\alpha$. By Lemma 3.5, there exist $h: \mathbf{P}'' \to \mathbf{P}'$ with $\alpha = \operatorname{Hom}_{D^b(A)}(\mathbf{P}, h)$ and $hf = 0 = fh$. Hence there exists the following morphism h_1:



such that $h = eh_1$. Due to $eh_1 f = hf = 0$, there exists h_2 such that $-g[-1]h_2 = h_1 f$. But $h_2 \in \operatorname{Hom}_{D^b(A)}(\mathbf{P}', \mathbf{P}''[-1]) = 0$, so h_1 factors through g and then it is zero since $\operatorname{Hom}_{D^b(A)}(A[1], A) = 0$. Therefore, $h = 0$ which implies that $\alpha = 0$. Thus, $\mathbf{Q}$ is tilting. $\square$

By now we have proved parts (d) and (e) of Theorem 1.1, we next finish the proofs of (f) and (g). Adopting earlier notation, we let $\mathcal{X}(\mathbf{Q}) = \mathrm{Hom}_{D^b(B)}(\mathbf{Q}, \mathcal{F}(\mathbf{Q})[1])$ and $\mathcal{Y}(\mathbf{Q}) = \mathrm{Hom}_{D^b(B)}(\mathbf{Q}, \mathcal{T}(\mathbf{Q}))$. Now, by Corollary 2.5, we have that $\mathrm{Hom}_{D^b(b)}(\mathbf{Q}, -)$ induces equivalences $\mathcal{T}(\mathbf{Q}) \to \mathcal{Y}(\mathbf{Q})$ and $\mathcal{F}(\mathbf{Q})[1] \to \mathcal{X}(\mathbf{P})$.

Theorem 4.4. *Let $\Phi_* \colon \mathrm{mod}\,\mathrm{End}_{D^b(B)}(\mathbf{Q}) \hookrightarrow \mathrm{mod}\,A$ be the inclusion functor induced by $\Phi_{\mathbf{P}}$. Then $\Phi_*(\mathcal{X}(\mathbf{Q})) = \mathcal{T}(\mathbf{P})$ and $\Phi_*(\mathcal{Y}(\mathbf{Q})) = \mathcal{F}(\mathbf{P})$.*

Proof. We prove that $\Phi_*(\mathcal{Y}(\mathbf{Q})) = \mathcal{F}(\mathbf{P})$. The proof of $\Phi_*(\mathcal{X}(\mathbf{Q})) = \mathcal{T}(\mathbf{P})$ is similar. By Proposition 3.8, we have that $\mathcal{T}(\mathbf{Q}) = \mathcal{X}(\mathbf{P})$, so we obtain that

$$\mathcal{Y}(\mathbf{Q}) = \mathrm{Hom}_{D^b(B)}(\mathbf{Q}, \mathcal{T}(\mathbf{Q})) = \mathrm{Hom}_{D^b(B)}(\mathbf{Q}, \mathcal{X}(\mathbf{P})) = \mathrm{Hom}_{D^b(B)}(\mathbf{Q}, \mathrm{Hom}_{D^b(A)}(\mathbf{P}, \mathcal{F}(\mathbf{P})[1])).$$

Then to complete the proof, we only need to prove that for any $Y \in \mathcal{F}(\mathbf{P})$, there is an isomorphism of A-modules $Y \cong \mathrm{Hom}_{D^b(B)}(\mathbf{Q}, \mathrm{Hom}_{D^b(A)}(\mathbf{P}, Y[1]))$. Note first that $\mathrm{Hom}_{D^b(B)}(\mathbf{Q}, \mathrm{Hom}_{D^b(A)}(\mathbf{P}, Y[1]))$ is the kernel of the map

$$\mathrm{Hom}_B(\mathrm{Hom}_{D^b(A)}(\mathbf{P}, f), \mathrm{Hom}_{D^b(A)}(\mathbf{P}, Y[1])) \colon \mathrm{Hom}_B\,(\mathrm{Hom}_{D^b(A)}(\mathbf{P}, \mathbf{P}''), \mathrm{Hom}_{D^b(A)}(\mathbf{P}, Y[1])) \to \mathrm{Hom}_B(\mathrm{Hom}_{D^b(A)}(\mathbf{P}, \mathbf{P}'), \mathrm{Hom}_{D^b(A)}(\mathbf{P}, Y[1])).$$

By Lemma 3.5, this is isomorphic to the kernel of

$$\mathrm{Hom}_{D^b(A)}(f, Y[1]) \colon \mathrm{Hom}_{D^b(A)}(\mathbf{P}'', Y[1]) \to \mathrm{Hom}_{D^b(A)}(\mathbf{P}', Y[1]).$$

Applying $\mathrm{Hom}_{D^b(A)}(-, Y[1])$ to the triangle $\Delta_{\mathbf{P}}$, and using that $\mathrm{Hom}_{D^b(A)}(\mathbf{P}', Y) = 0$, since Y is in $\mathcal{F}(\mathbf{P})$, we obtain that $\ker \mathrm{Hom}_{D^b(A)}(f, Y[1]) \cong \mathrm{Hom}_A(A, Y)$. Hence there is an isomorphism

$$\varphi : \ \mathrm{Hom}_A(A, Y) \cong \mathrm{Hom}_{D^b(B)}(\mathbf{Q}, \mathrm{Hom}_{D^b(A)}(\mathbf{P}, Y[1]))$$

as vector spaces and for any map $v \in \mathrm{Hom}_A(A, Y)$, the corresponding map $\varphi(v)$ is the following chain map

$$\begin{array}{ccc}
\mathrm{Hom}_{D^b(A)}(\mathbf{P}, \mathbf{P}') & \xrightarrow{\mathrm{Hom}_{D^b(A)}(\mathbf{P}, f)} & \mathrm{Hom}_{D^b(A)}(\mathbf{P}, \mathbf{P}'') \\
\downarrow & & \downarrow {\scriptstyle \mathrm{Hom}_{D^b(A)}(\mathbf{P}, v[1]g)} \\
0 & \longrightarrow & \mathrm{Hom}_{D^b(A)}(\mathbf{P}, Y[1])
\end{array}$$

Moreover, for any map $a \in \mathrm{End}_A(A)$, by the commutative diagram

$$\begin{array}{ccccccc}
A & \xrightarrow{e} & \mathbf{P}' & \xrightarrow{f} & \mathbf{P}'' & \xrightarrow{g} & A[1] \\
\downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle a[1]} \\
A & \xrightarrow{e} & \mathbf{P}' & \xrightarrow{f} & \mathbf{P}'' & \xrightarrow{g} & A[1]
\end{array}$$

with $\Phi_{\mathbf{P}}(a) = (b, c)$, we have that $\mathrm{Hom}_{D^b(A)}(\mathbf{P}, (va)[1]g) = \mathrm{Hom}_{D^b(A)}(\mathbf{P}, v[1]gc)$. So $\varphi(va) = \Phi_{\mathbf{P}}(a)\varphi(v)$ which implies that the isomorphism φ is an A-module map. Thus, since $Y \cong \mathrm{Hom}_A(A, Y)$, we get the desired isomorphism. $\square$

5. Auslander–Reiten theory

As an application of Theorem 1.1, we show how the AR-theory of $B = \mathrm{End}_{D^b(A)}(\mathbf{P})$ can be understood in terms of the AR-theory of A. In the case where A is hereditary, we obtain particularly strong results. These will turn out to be essential for studying the so-called *silted* algebras, that is: algebras obtained as $\mathrm{End}_{D^b(A)}(\mathbf{P})$, for a 2-term silting complex $\mathbf{P}$ over a hereditary algebra A. Such algebras are investigated and characterized in [11].

5.1. Connecting sequences

In this section we describe almost split sequences in $\mathrm{mod}\, B$. Similarly as in classical tilting theory, we call an almost split sequence in $\mathrm{mod}\, B$ whose left term lies in $\mathcal{Y}(\mathbf{P})$ and whose right term lies in $\mathcal{X}(\mathbf{P})$ a *connecting sequence*. We denote the AR-translation in a module category by τ.

Lemma 5.1. *If $0 \to Y \to E \to X \to 0$ is a connecting sequence, then there exists an indecomposable projective A-module P_i such that $Y \cong \mathrm{Hom}_{D^b(A)}(\mathbf{P}, \nu P_i)$.*

Proof. Since $Y \in \mathcal{Y}(\mathbf{P})$ and $X = \tau^{-1}Y \in \mathcal{X}(\mathbf{P})$, by [27, Lemma 0.1], Y is Ext-injective in $\mathcal{Y}(\mathbf{P})$. Then by Proposition 2.8 (3), there is an indecomposable A-module $Y' \in \mathrm{add}\, t\nu A$ such that $Y \cong \mathrm{Hom}_{D^b(A)}(\mathbf{P}, Y')$. Note that for each indecomposable projective A-module P_i, if $t\nu P_i \ncong 0$, then it is indecomposable since νP_i is its injective envelope. So $Y \cong \mathrm{Hom}_{D^b(A)}(\mathbf{P}, t\nu P_i)$ for some indecomposable projective A-module P_i. Hence $Y \cong \mathrm{Hom}_{D^b(A)}(\mathbf{P}, \nu P_i)$ by Lemma 3.6. □

Note that $\mathrm{Hom}_{D^b(A)}(\mathbf{P}, \nu P_i) = 0$ if and only if $\nu P_i \in \mathcal{F}(\mathbf{P})$ if and only if $\nu P_i \in \mathrm{add}\, H^{-1}(\nu \mathbf{P})$ if and only if $P_i[1] \in \mathrm{add}\, \mathbf{P}$. The following lemma is a generalization of the connecting lemma in tilting theory.

Lemma 5.2. *Let P_i be an indecomposable projective A-module with $P_i[1] \notin \mathrm{add}\, \mathbf{P}$. Then*

$$\tau^{-1} \mathrm{Hom}_{D^b(A)}(\mathbf{P}, \nu P_i) \cong \mathrm{Hom}_{D^b(A)}(\mathbf{P}, P_i[1]).$$

In particular, $\mathrm{Hom}_{D^b(A)}(\mathbf{P}, \nu P_i)$ *is an injective B-module if and only if $P_i \in \mathrm{add}\, \mathbf{P}$.*

Proof. This follows from the fact that $\mathrm{Hom}_{D^b(A)}(\mathbf{P}, P_i[1]) \cong H^0(\mathbf{Q}_i)$ and $\mathrm{Hom}_{D^b(A)}(\mathbf{P}, \nu P_i) \cong H^{-1}(\nu \mathbf{Q}_i)$ for a 2-term complex $\mathbf{Q}_i$ in $K^b(\mathrm{proj}\, B)$, which was proved in the proof of Proposition 3.8. □

Hence, we have shown that the connecting sequences are of the form

$$0 \to \mathrm{Hom}_{D^b(A)}(\mathbf{P}, \nu P_i) \to E \to \mathrm{Hom}_{D^b(A)}(\mathbf{P}, P_i[1]) \to 0.$$

It remains to describe the middle term E.

Corollary 5.3. *Let P_i be an indecomposable projective A-module with $P_i \notin \mathrm{add}\, \mathbf{P}$ and $P_i[1] \notin \mathrm{add}\, \mathbf{P}$ and E be the middle term of the almost split sequence starting at $\mathrm{Hom}_{D^b(A)}(\mathbf{P}, \nu P_i)$. Then the canonical sequence of E in the torsion pair $(\mathcal{X}(\mathbf{P}), \mathcal{Y}(\mathbf{P}))$ is*

$$0 \to \mathrm{Hom}_{D^b(A)}(\mathbf{P}, \mathrm{rad}\, P_i[1]) \to E \to \mathrm{Hom}_{D^b(A)}(\mathbf{P}, \nu P_i/S_i) \to 0$$

where $\mathrm{rad}\, P_i$ denotes the radical of P_i and S_i is the simple module $P_i/\mathrm{rad}\, P_i$.

Proof. Since $(\mathcal{T}(\mathbf{P}), \mathcal{F}(\mathbf{P}))$ is a torsion pair, S_i is either in $\mathcal{T}(\mathbf{P})$ or in $\mathcal{F}(\mathbf{P})$. We refer to the proof of [5, Corollary VI.4.10] where the first part (i.e. the case $S_i \in \mathcal{T}(\mathbf{P})$) works in our case by a small suitable modification. However, the second part does not work in our case, instead, one need to use the dual proof of the first part. So for the convenience of readers, we give a proof for the case $S_i \in \mathcal{F}(\mathbf{P})$. Applying $\mathrm{Hom}_{D^b(A)}(\mathbf{P}, -)$ to the triangle

$$\mathrm{rad}\, P_i \to P_i \to S_i \to \mathrm{rad}\, P_i[1]$$

yields a short exact sequence

$$0 \to \mathrm{Hom}_{D^b(A)}(\mathbf{P}, \mathrm{rad}\, P_i[1]) \to \mathrm{Hom}_{D^b(A)}(\mathbf{P}, P_i[1]) \xrightarrow{\delta} \mathrm{Hom}_{D^b(A)}(\mathbf{P}, S_i[1]) \to 0.$$

Consider the short exact sequences

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
 & & & & t\nu P_i & & & & \\
 & & & & \downarrow & & & & \\
0 & \longrightarrow & S_i & \xrightarrow{\alpha} & \nu P_i & \xrightarrow{\beta} & \nu P_i/S_i & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle \gamma} & & & & \\
 & & & & \nu P_i/t\nu P_i & & & & \\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & &
\end{array}$$

Since $P_i[1] \notin \mathrm{add}\, \mathbf{P}$, we have that $\gamma: \nu P_i \to \nu P_i/t\nu P_i$ is not an isomorphism. Then the composition $\gamma\alpha = 0$. So γ factors through β. Because $\mathrm{Hom}_{D^b(A)}(\mathbf{P}, \gamma[1])$ is an isomorphism by Lemma 3.6, the map $\mathrm{Hom}_{D^b(A)}(\mathbf{P}, \beta[1])$ is a monomorphism. Hence we have a short exact sequence

$$0 \to \mathrm{Hom}_{D^b(A)}(\mathbf{P}, \nu P_i) \xrightarrow{\theta} \mathrm{Hom}_{D^b(A)}(\mathbf{P}, \nu P_i/S_i) \to \mathrm{Hom}_{D^b(A)}(\mathbf{P}, S_i[1]) \to 0$$

where $\theta = \mathrm{Hom}_{D^b(A)}(\mathbf{P}, \beta)$. Since $\mathrm{Hom}_{D^b(A)}(\mathbf{P}, S_i[1]) \in \mathcal{X}(\mathbf{P})$ and $\mathrm{Hom}_{D^b(A)}(\mathbf{P}, \nu P_i/S_i) \in \mathcal{Y}(\mathbf{P})$ by Lemma 3.6 and $\mathrm{Hom}_{D^b(A)}(\mathbf{P}, S_i[1]) \neq 0$ by $S_i \in \mathcal{F}(\mathbf{P})$, the sequence is not split. In particular, θ is not a section, so there is a commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Hom}_{D^b(A)}(\mathbf{P}, \nu P_i) & \longrightarrow & E & \longrightarrow & \mathrm{Hom}_{D^b(A)}(\mathbf{P}, P_i[1]) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow{\scriptstyle h} & & \\
0 & \longrightarrow & \mathrm{Hom}_{D^b(A)}(\mathbf{P}, \nu P_i) & \xrightarrow{\theta} & \mathrm{Hom}_{D^b(A)}(\mathbf{P}, \nu P_i/S_i) & \longrightarrow & \mathrm{Hom}_{D^b(A)}(\mathbf{P}, S_i[1]) & \longrightarrow & 0
\end{array}$$

where the upper sequence is the AR-sequence starting at $\mathrm{Hom}_{D^b(A)}(\mathbf{P}, \nu P_i)$. Note that $h \neq 0$, since otherwise the upper sequence would be split exact.

Since $\mathrm{Hom}_{D^b(A)}(\mathbf{P}, P_i[1]) \cong \mathrm{Hom}_{D^b(A)}(\mathbf{P}, P_i/tP_i[1])$ by Lemma 3.6 and $\mathrm{Hom}_{D^b(A)}(\mathbf{P}, -[1])$ is an equivalence from $\mathcal{F}(\mathbf{P})$ to $\mathcal{X}(\mathbf{P})$ by Corollary 2.5, we have that

$$\mathrm{Hom}_B(\mathrm{Hom}_{D^b(A)}(\mathbf{P}, P_i[1]), \mathrm{Hom}_{D^b(A)}(\mathbf{P}, S_i[1])) \cong \mathrm{Hom}_A(P_i/tP_i, S_i).$$

Using that S_i is in $\mathcal{F}(\mathbf{P})$, by assumption, we have $\mathrm{Hom}_A(P_i/tP_i, S_i) \cong \mathrm{Hom}_A(P_i, S_i)$, which is a one dimensional space. Therefore, since $h \neq 0$, it equals $k\delta$, for an element $k \in \mathbf{k}$. Hence, $\ker h \cong \mathrm{Hom}_{D^b(A)}(\mathbf{P}, \mathrm{rad}\, P_i[1])$. Using the snake lemma, we obtain the following commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathrm{Hom}_{D^b(A)}(\mathbf{P}, \mathrm{rad}\, P_i[1]) & = & \mathrm{Hom}_{D^b(A)}(\mathbf{P}, \mathrm{rad}\, P_i[1]) & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & \mathrm{Hom}_{D^b(A)}(\mathbf{P}, \nu P_i) & \to & E & \to & \mathrm{Hom}_{D^b(A)}(\mathbf{P}, P_i[1]) & \to & 0 \\
 & & \| & & \downarrow & & \downarrow{\scriptstyle h} & & \\
0 & \to & \mathrm{Hom}_{D^b(A)}(\mathbf{P}, \nu P_i) & \xrightarrow{g} & \mathrm{Hom}_{D^b(A)}(\mathbf{P}, \nu P_i/S_i) & \to & \mathrm{Hom}_{D^b(A)}(\mathbf{P}, S_i[1]) & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

where the middle column gives the required short exact sequence. □

5.2. Separating and splitting silting complexes

Recall that a torsion pair $(\mathcal{X}, \mathcal{Y})$ in $\mathrm{mod}\, A$ is called *split* (or sometimes splitting) if each indecomposable A-module lies either in $\mathcal{X}$ or in $\mathcal{Y}$, see [5]. In other words, $(\mathcal{X}, \mathcal{Y})$ is split if and only if $\mathrm{Ext}^1_A(Y, X) = 0$ for all $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$.

Definition 5.4. Let A be a finite dimensional algebra, let $\mathbf{P}$ be a 2-term silting complex in $K^b(\mathrm{proj}\, A)$ and $B = \mathrm{End}_{D^b(A)}(\mathbf{P})$. Then

(1): $\mathbf{P}$ is called *separating* if the induced torsion pair $(\mathcal{T}(\mathbf{P}), \mathcal{F}(\mathbf{P}))$ in $\mathrm{mod}\, A$ is split, and
(2): $\mathbf{P}$ is called *splitting* if the induced torsion pair $(\mathcal{X}(\mathbf{P}), \mathcal{Y}(\mathbf{P}))$ in $\mathrm{mod}\, B$ is split.

Lemma 5.5. *A 2-term silting complex* $\mathbf{P}$ *is splitting if and only if* $\mathrm{Ext}^2_A(\mathcal{T}(\mathbf{P}), \mathcal{F}(\mathbf{P})) = 0$.

Proof. This follows from the second isomorphism in Corollary 2.6. □

Note that in particular Lemma 5.5 implies that if A is hereditary, then all 2-term silting complexes are splitting. In a forthcoming paper, [11], we study endomorphism rings of 2-term silting complexes over hereditary algebras. We now state a result which is of particular importance for describing the AR-theory of silted algebras.

Proposition 5.6. *If a silting complex* $\mathbf{P}$ *is splitting, then any almost split sequence in* $\mathrm{mod}\, B$ *lies entirely in either* $\mathcal{X}(\mathbf{P})$ *or* $\mathcal{Y}(\mathbf{P})$*, or else it is of the form*

$$0 \to \mathrm{Hom}_{D^b(A)}(\mathbf{P}, \nu P_i) \to \mathrm{Hom}_{D^b(A)}(\mathbf{P}, \mathrm{rad}\, P_i[1]) \oplus \mathrm{Hom}_{D^b(A)}(\mathbf{P}, \nu P_i \,/\, S_i) \to \mathrm{Hom}_{D^b(A)}(\mathbf{P}, P_i[1]) \to 0,$$

where P_i is an indecomposable projective A-module with $P_i \notin \operatorname{add}\mathbf{P}$ and $P_i[1] \notin \operatorname{add}\mathbf{P}$. Moreover, almost split sequences in $\mathcal{T}(\mathbf{P})$ and $\mathcal{F}(\mathbf{P})$ are by $\operatorname{Hom}_{D^b(A)}(\mathbf{P},-)$ and $\operatorname{Hom}_{D^b(A)}(\mathbf{P},-[1])$ mapped to almost split sequences in $\mathcal{Y}(\mathbf{P})$ and $\mathcal{X}(\mathbf{P})$, respectively.

Proof. The first statement follows from Lemma 5.2 and Corollary 5.3.

For the second statement, we only prove the statement for $\mathcal{T}(\mathbf{P})$, since the proof for $\mathcal{F}(\mathbf{P})$ is similar. Let

$$0 \to X_1 \xrightarrow{\alpha} X_2 \xrightarrow{\beta} X_3 \to 0$$

be an almost split sequence in $\mathcal{T}(\mathbf{P})$. Then by Corollary 2.5, we have a short exact sequence in $\mathcal{Y}(\mathbf{P})$:

$$0 \to \operatorname{Hom}_{D^b(A)}(\mathbf{P}, X_1) \xrightarrow{\operatorname{Hom}_{D^b(A)}(\mathbf{P},\alpha)} \operatorname{Hom}_{D^b(A)}(\mathbf{P}, X_2) \xrightarrow{\operatorname{Hom}_{D^b(A)}(\mathbf{P},\beta)} \operatorname{Hom}_{D^b(A)}(\mathbf{P}, X_3) \to 0$$

where $\operatorname{Hom}_{D^b(A)}(\mathbf{P}, X_1)$ and $\operatorname{Hom}_{D^b(A)}(\mathbf{P}, X_3)$ are indecomposable. Let Y be an indecomposable B-module, then $Y \in \mathcal{X}(\mathbf{P})$ or $Y \in \mathcal{Y}(\mathbf{P})$. To complete the proof, by e.g. [5, Theorem IV.1.13], it is sufficient to prove the following claim: each homomorphism from Y to $\operatorname{Hom}_{D^b(A)}(\mathbf{P}, X_3)$ which is not a split epimorphism factors through $\operatorname{Hom}_{D^b(A)}(\mathbf{P}, \beta)$. If $Y \in \mathcal{Y}(\mathbf{P})$, then this claim follows from that $\operatorname{Hom}_{D^b(A)}(\mathbf{P}, -)$ is an equivalence from $\mathcal{T}(\mathbf{P})$ to $\mathcal{Y}(\mathbf{P})$. Now we assume that $Y \in \mathcal{X}(\mathbf{P})$. Then $\operatorname{Hom}_B(Y, \operatorname{Hom}_{D^b(A)}(\mathbf{P}, X_3)) = 0$, so there is nothing left to prove. □

Proposition 5.7. *Each separating 2-term silting complex* $\mathbf{P}$ *is a tilting complex.*

Proof. By Corollary 4.3, it is sufficient to prove that $\Phi_{\mathbf{P}}$ is an isomorphism. This is equivalent to prove that the induced functor $\Phi_*\colon \operatorname{mod}\operatorname{End}_{D^b(B)}(\mathbf{Q}) \hookrightarrow \operatorname{mod} A$ is an equivalence. Since Φ_* is always fully faithful, we only need to prove that Φ_* is dense. Since $\mathbf{P}$ is separating, each indecomposable A-module M is either in $\mathcal{T}(\mathbf{P})$ or in $\mathcal{F}(\mathbf{P})$. Then by Theorem 4.4, there is an $N \in \operatorname{mod}\operatorname{End}_{D^b(B)}(\mathbf{Q})$ such that $\Phi_*(N) = M$. Thus, we complete the proof. □

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