STOCHASTIC PERSISTENCY OF NEMATIC ALIGNMENT STATE FOR THE JUSTH-KRISHNAPRASAD MODEL WITH ADDITIVE WHITE NOISES

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Abstract. We present a stochastic Justh-Krishnaprasad flocking model describing interactions among individuals in a planar domain with their positions and heading angles. The deterministic counterpart of the proposed model describes the formation of nematic alignment in an ensemble of planar particles moving with a unit speed. When the noise is turned off, we show that the nematic alignment state, in which all heading angles are either same or the opposite, is nonlinearly stable using a Lyapunov functional approach. We employed a diameter-like functional via the rearrangement of heading angles in the 2π-interval. In contrast, under the additive noise, a continuous angle configuration will be deviated asymptotically from the nematic state. Nevertheless, in any finite-time interval, we will see that some part of angle configuration will stay close to the nematic state with a positive probability, where we call this phenomenon as stochastic persistency. We provide a quantitative estimate on the probability for stochastic persistency and compare several numerical examples with analytical results.

Alignment, emergence, Justh-Krishnaprasad model, nematic alignment, stochastic noises. AMS Subject Classification: 70K20, 34D05.

1. Introduction

The emergence of coherent collective motions in natural and man-made complex systems is ubiquitous, e.g., synchronization of biological cells[1, 34], flocking of migratory birds[7, 37, 39], swarming phenomenon[14, 17, 31, 38], consensus of opinions[35] and swarming dynamics of active complex systems [2, 3, 4, 5] etc. In recent years, modeling of such coherent collective motions has received intensive attention in control theory community[24, 29, 33, 36], due to the increasing needs of decentralized control algorithm motivated by biological systems. For the modeling of such collective motions, several phenomenological models were proposed in literature.

In this paper, we are interested in the modeling of nematic alignment which is often observed in a group of Myxobacteria[12, 13] in which bacteria moves either in the same direction or in the opposite direction and form a traveling band asymptotically. In aforementioned literature, it has been studied based on the corresponding hydrodynamic models, which consists of deterministic dynamics and stochastic Poisson process. Unfortunately, it is not easy to analyze temporal evolutions in such hybrid agent-based models. Thus, we propose a simple planar solvable particle model exhibiting formation of nematic alignment. Our model corresponds to the stochastic counterpart of the particle model[19], which generalizes the particle model proposed by Justh and Krishnaprasad[25, 26]. Compared to the Justh-Krishnaprasad (J-K) model, we modified the communication weight and coupling function in order to represent nematic alignment.

Our primary goal is to observe a persistence of the nematic alignment under the effect of additive noise. To fix the idea, let $x_t^j \in \mathbb{R}^2$ and $\theta_t^j \in \mathbb{R}$ be the position and heading-angle for the j-th particle, respectively, which follows the system of stochastic ordinary differential equations (SDEs):

$$\begin{cases} dx_t^j = (\cos\theta_t^j, \sin\theta_t^j)\ dt, \quad t>0,\ j=1,\cdots,N, \\ d\theta_t^j = \dfrac{\kappa}{N}\displaystyle\sum_{k=1}^{N}\psi(\|x_t^k - x_t^j\|)\sin 2(\theta_t^k-\theta_t^j)dt + \sqrt{2\sigma}dB_t^j, \\ (x_0^j,\theta_0^j) = (x_{in}^j, \theta_{in}^j), \end{cases} \tag{1}$$

where κ is a nonnegative coupling strength, $\psi(\cdot)$ is the communication weight and B_t^j are independent and identically distributed Brownian motions. We also assume that the interaction is global but decreases along with the distance: $\psi(\cdot)$ is an analytic function and there exist positive constants ψ_m and ψ_M such that

$$\begin{aligned} &0<\psi_m \le \psi(r) \le \psi_M, \quad \forall\ r\ge 0, \quad [\psi]_{\mathrm{Lip}} < \infty, \\ &(\psi(r_1)-\psi(r_2))(r_1-r_2) \le 0, \quad r_1, r_2 \ge 0. \end{aligned} \tag{2}$$

Since the right-hand side of (1) is Lipschitz continuous and uniformly bounded in state variables, the standard existence theory for SDEs implies a global well-posedness[32].

Note that if the communication weight is constant, $\psi \equiv 1$, then the dynamics of θ_t^j in (1) corresponds to the Kuramoto model[27, 28] with respect to $2\theta_t^j$. In the absence of noise, the generalized J-K model was derived and analyzed by the first author and his collaborators[19] from the Cucker-Smale flocking model[8, 11, 22, 23, 30] with unit speed constraint.

The main results are two-fold. First, we provide a nonlinear stability analysis for the nematic alignment (see Definition 2.1) in the corresponding deterministic system:

$$\begin{cases} \dfrac{dx_t^j}{dt} = (\cos\theta_t^j, \sin\theta_t^j), \quad t>0,\ j=1,\cdots,N, \\ \dfrac{d\theta_t^j}{dt} = \dfrac{\kappa}{N}\displaystyle\sum_{k=1}^{N}\psi(\|x_t^k - x_t^j\|)\sin 2(\theta_t^k-\theta_t^j). \end{cases} \tag{3}$$

Note that the states with

$$2(\theta_t^i - \theta_t^j) \equiv 0, \quad \mod 2\pi \tag{4}$$

are clearly equilibria for θ_t^j in (3). Our first result in Theorem 3.1 shows that nematic alignment states (which is characterized by (4)) are nonlinearly stable. For this, we adopt a Lyapunov functional approach[19], where we extended the previous result to the interactions with doubled angles, $2(\theta_t^k - \theta_t^j)$. More precisely, as long as initial angle configurations are in the range of $\pi/4$ from a pair of antipodal points $\{\theta_0, \theta_0+\pi\}$, the diameter-like Lyapunov functional of the heading angles $\mathcal{L}(\Theta(t))$ decays to zero exponentially.

Second, we present the persistency of nematic state in the presence of white noise. Due to the effect of additive noise, the heading angles will diffuse out from the nematic alignment state. Thus, it is not reasonable to discuss asymptotic dynamics, instead, we estimate the escaping time of heading angles from a given area. Thus, our second result deals with the following probability: For a given T and ε, we estimate the probability:

$$\mathbb{P}\Big\{ \max_{0\le t\le T} \mathcal{L}(\Theta(t)) < \varepsilon \Big\},$$

which is roughly in the order of $(1 - T\exp(-\varepsilon^2))$ (see Theorem 5.1 for detail). We follow the sample-path tracking method[6, 18] to estimate the maximum of the Lyapunov function. Since the underlying dynamics (1) contains a nonlinear kernel $\psi(\cdot)$, we construct several reference processes and use technical arguments involved with several stopping times (see Section 4.2). We leave its detailed discussion in Section 4.1.

The rest of the paper is organized as follows. In Section 2, we briefly study the dynamics of order parameters, and introduce the concept of nematic alignments for the deterministic J-K model (3). We also provide some basic elementary lemmas for the stochastic model (1) to be used for stochastic estimates later. In Section 3, we present the first result, a nonlinear stability result for the nematic alignment state in the deterministic model (3). Here, we will define the diameter-like functional of heading angles $\mathcal{L}(\Theta(t))$ and prove that it decays exponentially. In Section 4, for the preparation of stochastic result, we introduce a concept 'relaxed collision' and provide several estimations on probabilities. In Section 5, we provide the second result, the stochastic persistence on stability, which is a lower bound estimate for the probability that $\mathcal{L}(\Theta)$ stays small for a finite-time interval. In Section 6, we provide several numerical simulations for the deterministic ($\sigma = 0$) and the stochastic case ($\sigma > 0$), and compare them with the analytic results. Finally, Section 7 is devoted to a brief summary of the main results and some discussion on remaining issues for a future work.

Notation: Throughout the paper, we represent $(X(t), \Theta(t))$ and (X_t, Θ_t) as a family of positions and heading angles for deterministic and stochastic J-K models, respectively:

$$X = (x^1, \cdots, x^N) \in \mathbb{R}^{2N}, \quad \Theta = (\theta^1, \cdots, \theta^N) \in \mathbb{R}^N,$$
$$X_t = (x_t^1, \cdots, x_t^N) \in \mathbb{R}^{2N}, \quad \Theta_t = (\theta_t^1, \cdots, \theta_t^N) \in \mathbb{R}^N.$$

For the heading angles, we represent them on both unit circle $\mathbb{S}^1 \simeq \mathbb{R}/(2\pi\mathbb{Z})$ and its covering space $\mathbb{R}$ with natural quotient maps p and $e^{i(\cdot)}$:

$$p : \mathbb{R} \to \mathbb{R}/(2\pi\mathbb{Z}), \quad e^{i(\cdot)} : \mathbb{R} \to \mathbb{S}^1.$$

2. Preliminaries

In this section, we briefly discuss basic properties of the deterministic and stochastic J-K models exhibiting nematic alignment asymptotically, and present a priori estimates for a later use.

2.1. **The deterministic J-K model.** Let (X, Θ) be the phase vector for positions and heading angles of J-K particles without noise. Then, its dynamics is governed by the following ODE model:

$$\begin{aligned} &\dot{x}^j = (\cos\theta^j, \sin\theta^j), \quad t > 0, \quad j = 1, \cdots, N, \\ &\dot{\theta}^j = \frac{\kappa}{N} \sum_{k=1}^N \psi(\|x^k - x^j\|) \sin 2(\theta^k - \theta^j). \end{aligned} \tag{1}$$

In this model, we will analyze symmetries and order parameters which can explain dynamical properties.

2.1.1. *elementary estimates.* First, we consider a transformation $(X, \Theta) \to (\tilde{X}, \tilde{\Theta})$: for any $s \in \mathbb{R}$, we set

$$\tilde{x}^j := \begin{bmatrix} \cos s & -\sin s \\ \sin s & \cos s \end{bmatrix} x^j, \quad \tilde{\theta}^j := \theta^j + s, \quad j = 1, \cdots, N. \tag{2}$$

This implies that system (1) is invariant under the rotational transformation (2). Moreover, as in the Kuramoto model[27], we can see that the total sum of heading angles is conserved along the dynamics (1).

Proposition 2.1. *Let (X, Θ) be a smooth solution to system (1) with the initial data (X_{in}, Θ_{in}). Then, the total sum of heading angles is conserved along the dynamics (1):*

$$\sum_{j=1}^N \theta^j(t) = \sum_{j=1}^N \theta^j_{in}, \quad t \geq 0.$$

Proof. We sum $(1)_2$ over all j and use the index exchange trick $j \Longleftrightarrow k$:

$$\begin{aligned} \frac{d}{dt} \sum_{j=1}^N \theta^j &= \frac{\kappa}{N} \sum_{j,k=1}^N \psi(\|x^k - x^j\|) \sin 2(\theta^k - \theta^j) \\ &= -\frac{\kappa}{N} \sum_{j,k=1}^N \psi(\|x^j - x^k\|) \sin 2(\theta^j - \theta^k) = 0 \end{aligned}$$

to get the desired estimate. □

Next, we present elementary calculus type lemma to be used in Section 3 and Section 4.

Lemma 2.1. [9] *Let $\Theta = (\theta^1, \cdots, \theta^N)$ be a heading angle configuration in $\mathbb{R}^N$ such that*

$$\max_{i,j} |\theta^i - \theta^j| \leq 2\pi.$$

If there exists a constant $\delta \geq 0$ and indices $m, l, M \in \{1, \cdots, N\}$ satisfying

$$\theta^m - \delta \leq \theta^l \leq \theta^M + \delta,$$

then we have

$$\sin(\theta^M - \theta^m) - \sin(\theta^M - \theta^l) - \sin(\theta^l - \theta^m) \leq 2\delta. \tag{3}$$

Proof. For simplicity, we set

$$\theta^{ij} := \theta^i - \theta^j, \quad 1 \le i, j \le N.$$

Then we can transform our target function

$$\sin(\theta^M - \theta^m) - \sin(\theta^M - \theta^l) - \sin(\theta^l - \theta^m)$$

to

$$\begin{aligned}\sin\theta^{Mm} &- \sin\theta^{Ml} - \sin\theta^{lm} \\ &= 2\sin\frac{\theta^{Mm}}{2}\left(\cos\frac{\theta^{Mm}}{2} - \cos\frac{\theta^{Ml}-\theta^{lm}}{2}\right) = -4\sin\frac{\theta^{Mm}}{2}\sin\frac{\theta^{Ml}}{2}\sin\frac{\theta^{lm}}{2}.\end{aligned}$$

Now, we consider the following trichotomy:

$$(1)\ \theta^l \le \theta^m (\le \theta^l + \delta) \quad (2)\ \theta^M \le \theta^l (\le \theta^M + \delta) \quad (3)\ \theta^m < \theta^l < \theta^M.$$

For the first two cases, we have

$$\text{either } |\theta^{lm}| \le \delta \quad \text{or} \quad |\theta^{Ml}| \le \delta.$$

This yields

$$\sin\theta^{Mm} - \sin\theta^{Ml} - \sin\theta^{lm} = -4\sin\frac{\theta^{Mm}}{2}\sin\frac{\theta^{Ml}}{2}\sin\frac{\theta^{lm}}{2} \le 4\cdot 1\cdot 1\cdot \sin\frac{\delta}{2} \le 2\delta.$$

For the third case, since $\frac{\theta^{Mm}}{2}$, $\frac{\theta^{Ml}}{2}$ and $\frac{\theta^{lm}}{2}$ are all contained in $[0, \pi]$ and therefore their sine values are all nonnegative. Thus, we can immediately see that (3) holds. □

Remark 2.1. *In the previous paper* [9]*, the estimate* (3) *was obtained for* $\delta = 0$*. However, for later analysis dealing with the first relaxed collision time* $\tau^0(\delta)$ *in Section 4.2, we will use* (3) *with* $\delta > 0$*.*

2.1.2. *Order parameters.* Before we introduce the order parameters of phase configuration, we first define the notion of asymptotic alignment and nematic alignment.

Definitino 2.1. *Let* $\Theta = (\theta^1, \cdots, \theta^N) \in \mathbb{R}^N$ *be an angle configuration whose dynamics is governed by* (1)*.*

(1) We call the configuration Θ *exhibits asymptotic alignment if*

$$\lim_{t\to\infty}[\theta_i(t) - \theta_j(t)] \equiv 0 \mod 2\pi, \quad \forall 1 \le i, j \le N.$$

(2) We call the configuration Θ *exhibits asymptotic nematic alignment if*

$$\lim_{t\to\infty}[\theta_i(t) - \theta_j(t)] \equiv 0 \mod \pi, \quad \forall 1 \le i, j \le N.$$

It is easy to see that the (asymptotic) alignment of Θ implies (asymptotic) nematic alignment.

Now, for $\Theta = (\theta^1, \cdots, \theta^N)$ and $n \in \mathbb{N}$, we introduce the n-th order parameters (R_n, ϕ_n) by the following implicit relation:

$$(4) \qquad R_n(\Theta)e^{\mathrm{i}n\phi_n(\Theta)} := \frac{1}{N}\sum_{k=1}^N e^{\mathrm{i}n\theta^k}, \quad R_n(\Theta) > 0 \quad \text{and} \quad \phi_n(\Theta) \in \mathbb{R}/(2\pi\mathbb{Z}).$$

By using the order parameters defined above, we can recharacterize the asymptotic alignment and nematic alignment as follows.

Proposition 2.2. *Let $\Theta = (\theta^1, \cdots, \theta^N)$ be an angle configuration whose dynamics is governed by* (1)*.*

(1) *The configuration Θ exhibits alignment asymptotically if and only if*

$$\lim_{t\to\infty} R_1(\Theta(t)) = 1.$$

(2) *The configuration Θ exhibits nematic alignment asymptotically if and only if*

$$\lim_{t\to\infty} R_2(\Theta(t)) = 1.$$

Proof. First, we observe that the order parameter $R_n(\Theta)$ can be represented by the average of N cosine terms:

$$R_n(\Theta) = \frac{1}{N}\sum_{k=1}^N e^{in(\theta^k - \phi_n(\Theta))} = \mathcal{R}e\left[\frac{1}{N}\sum_{k=1}^N e^{in(\theta^k-\phi_n(\Theta))}\right] = \frac{1}{N}\sum_{k=1}^N \cos n(\theta^k - \phi_n(\Theta)).$$

Then, since $\cos(\cdot)$ is even and $\sin(\cdot)$ is odd, we have

$$\begin{aligned} |R_n|^2 &= \left(\frac{1}{N}\sum_{j=1}^N \cos n(\phi_n(\Theta) - \theta^j)\right)\left(\frac{1}{N}\sum_{k=1}^N \cos n(\theta^k - \phi_n(\Theta))\right) \\ &= \left(\frac{1}{N}\sum_{j=1}^N e^{in(\phi_n(\Theta)-\theta^j)}\right)\left(\frac{1}{N}\sum_{k=1}^N e^{in(\theta^k-\phi_n(\Theta))}\right), \end{aligned} \tag{5}$$

and therefore

$$|R_n(\Theta)|^2 = \frac{1}{N^2}\sum_{j,k=1}^N e^{in(\theta^k-\theta^j)} = \mathcal{R}e\left[\frac{1}{N^2}\sum_{j,k=1}^N e^{in(\theta^k-\theta^j)}\right] = \frac{1}{N^2}\sum_{j,k=1}^N \cos n(\theta^k - \theta^j).$$

Hence, the convergence of $R_n(\Theta)$ to 1 is equivalent to the convergence of $\cos n(\theta^k - \theta^j)$ to 1 for all j and k, which concludes the desired result. □

2.1.3. *The diameter of heading angles.* Though the order parameter explains a lot on the bifurcation phenomena such as stability, the diameter function is more useful to show the emergence of collective behavior. In a series of papers [10, 15, 16, 20] for the synchronization estimates to the Kuramoto model, the diameter of heading angles

$$D(\Theta) = \max_{1\le i,j\le N} |\theta^i - \theta^j| \tag{6}$$

and that of the derivatives $D(\dot\Theta) := \max_{i,j} |\dot\theta^i - \dot\theta^j|$ play the key roles as Lyapunov functionals when the heading angles are confined in a half circle. Indeed, at least intuitively, $D(\Theta)$ should refer to the minimal length of the connected arc containing the set $\mathcal{P}(\Theta) := \{e^{\mathrm{i}\theta^1}, \cdots, e^{\mathrm{i}\theta^N}\} \subset \mathbb{S}$ for any given angle configuration Θ, while definition (6) does not always give this 'intuitive' diameter for general angle configuration Θ. The simplest example of this disagreement is when $\Theta = (\theta^1, \theta^2) = (0, 2\pi)$. Although definition (6) may not give the intuitive diameter in $\mathbb{S}$ for every generic phases, (6) has been mainly used as definition of phase diameter in [10, 15, 16, 20], when the configuration $\mathcal{P}(\Theta)$ is confined in a half circle. This is because the oscillator phases $\{\theta^i\}_{i=1}^N$ in Kuramoto model has a $2\pi-$periodicity, and therefore the 2π translation of θ^i does not affect the dynamics. Thus, if $\mathcal{P}(\Theta)$ is in a half circle, we can apply $2\pi-$translations to 'lifted' phases $\theta^i \in \mathbb{R}$ several times if necessary so that the intuitive diameter coincides with (6).

However, to describe the nematic alignment state, this modification of angles by $2\pi-$translation is not allowed anymore. For the nematic alignment state, we may consider $\mathcal{L}(\Theta)$ as the half of (intuitive) phase diameter $D(2\Theta)$ for the heading angle configuration Θ:

$$\mathcal{L}(\Theta) = 0 \quad \Longleftrightarrow \quad D(2\Theta) = 0 \quad \Longleftrightarrow \quad \Theta \ : \ \text{nematic alignment state.}$$

The problem arises when we apply $2\pi-$translation to doubled angles $\{2\theta^i\}_{i=1}^N$ odd times. If so, the corresponding θ must be translated by π odd times and reverse the heading angle of the individual. Therefore, this modification may cause a miscalculation of relative distances $\|x^k - x^j\|$ and affect the evaluation of whole state variables $\{(x^i, \theta^i)\}_{i=1}^N$.

In order to define $\mathcal{L}(\Theta)$ properly, i.e., evaluate intuitive diameter $D(2\Theta)$ unambiguously, we first consider the connected components in the excluded set $\mathbb{S}-\mathcal{P}(2\Theta)$. Since each connected component c of $\mathbb{S} - \mathcal{P}(2\Theta)$ has a connected complement $\mathbb{S} - c$ containing $\mathcal{P}(2\Theta)$, the maximum length among them coincides with the value $2\pi - D(2\Theta)$. Thus, we define a set of intervals $\Gamma(2\Theta)$ and Lyapunov functional $\mathcal{L}(\Theta)$ for nematic alignment states as below:

$$\begin{aligned} &\Gamma(2\Theta) := \text{set of all connected components in } \mathbb{S} - \mathcal{P}(2\Theta), \\ &\mathcal{L}(\Theta) := \frac{1}{2}\Big(2\pi - \max_{\gamma \in \Gamma(2\Theta)} |\gamma|\Big), \end{aligned}$$

where $|\gamma|$ is the length of the interval γ. Therefore, $\mathcal{L}(\Theta)$ denotes the minimal length of arc $\mathcal{A} \subset \mathbb{R}/(2\pi\mathbb{Z})$ which satisfies

$$\mathcal{P}(\Theta) \subset \mathcal{A} \cup (\mathcal{A} + \pi).$$

2.2. **Elementary stochastic estimates.** In this subsection, we recall two elementary lemmas in relation with the stochastic process. First, we list some basic properties of Brownian motion to be used in Section 4 whose proof can be found in [6, 32].

Lemma 2.2. *Let B_t be the standard one-dimensional Brownian motion. Then, the following assertions hold:*

(1) (Andre's reflection Principle[32]): for any time $T > 0$ and positive number a,

$$\mathbb{P}\Big\{\sup_{0 \le t \le T} B_t \ge a\Big\} = 2\mathbb{P}\{B_T \ge a\} \le \sqrt{\frac{2T}{\pi a^2}} e^{-\frac{a^2}{2T}}.$$

(2) (Bounded Ornstein-Uhlenbeck (O-U) process[6]): there exist $c_0, r_0 > 0$ such that if

$$r\left(\frac{\sqrt{\nu}h}{\sigma}, \nu T\right) := \frac{\sigma}{\sqrt{\nu}h} + \nu T e^{-c_0 \nu \frac{h^2}{\sigma^2}} \le r_0,$$

we have

$$\mathbb{P}\left\{\sup_{0 \le t \le T} \sqrt{2}\int_0^t e^{-\nu(t-s)} dB_s \ge \frac{h}{\sigma}\right\} \le A e^{-\frac{\nu h^2}{2\sigma^2}},$$

where $A = A\left(\frac{h}{\sigma}, \nu, T\right)$ has an following order with respect to $\frac{h}{\sigma}, \nu, T$:

$$A = \sqrt{\frac{2\nu^3 T^2 h^2}{\pi\sigma^2}} \left[1 + \mathcal{O}\left(r\left(\frac{\sqrt{\nu}h}{\sigma}, \nu T\right) + \frac{1}{\nu} + \frac{1}{\nu T}\log\left(1 + \frac{\sqrt{\nu}h}{\sigma}\right)\right)\right].$$

Next, we define a martingale process Y_t as a solution of the following SDE: for each $i = 1, \cdots, N$,

$$dY_t = \sqrt{2\sigma} D_t dB_t, \quad t > 0, \qquad Y_0 = 0, \quad \text{where } |D_t| \le 1. \tag{7}$$

In the following lemma, we provide the worst bound estimate from Doob's exponential martingale inequality.

Lemma 2.3. [32] *For a constant $h > 0$, the process Y_t defined by* (7) *satisfies*

$$\mathbb{P}\Big\{\sup_{0\le t\le T} Y_t > h\Big\} \le \exp\Big(-\frac{h^2}{4\sigma T}\Big). \tag{8}$$

Remark 2.2. *Note that the estimate* (8) *has the same order as for Brownian motion (see (1) in Lemma 2.2).*

3. The deterministic Justh-Krishnaprasad flow

In this section, we study a nonlinear stability of the nematic alignment state to the deterministic J-K model. To verify an asymptotic stability, we begin with an initial datum close to a nematic alignment configuration (bi-polar configuration in angle), and show that the particles will stay in their own groups and will align with either ϕ_2 or $\phi_2 + \pi$ asymptotically. We denote $\psi_{\min}$ by the minimum of $\psi(\|x^i - x^j\|)$ for all $x^i, x^j \in \mathbb{R}^2$.

Theorem 3.1. *Let (x^j, θ^j) be a smooth solution of* (1) *with initial configuration $\{(x^j_{in}, \theta^j_{in})\}_{j=1}^N$ satisfying*

$$\mathcal{L}(\Theta_{in}) < \frac{\pi}{2}.$$

Then, the Lyapunov functional $\mathcal{L}(\Theta)$ shows the following decay:

$$\tan\mathcal{L}(\Theta(t)) \le e^{-2\kappa\psi_m t}\tan\mathcal{L}(\Theta_{in}), \quad t \ge 0.$$

For the proof of Theorem 3.1, as in the Kuramoto model, we first show that $\mathcal{L}(\Theta)$ is non-increasing along the flow (1), and then we will prove that it decays to zero exponentially fast, as long as it is smaller than $\pi/2$. In the sequel, the assumption on the analyticity of the communication weight ψ will guarantee the piecewise analyticity of $\mathcal{L}(\Theta)$ in t.

Lemma 3.1. (Contractivity of the Lyapunov functional) *Let (X, Θ) be a smooth solution of* (1) *with an initial configuration (X_{in}, Θ_{in}) satisfying*

$$0 < \mathcal{L}(\Theta_{in}) < \frac{\pi}{2}.$$

Then, the Lyapunov functional $\mathcal{L}(\Theta)$ is non-increasing in t:

$$\mathcal{L}(\Theta(t)) \le \mathcal{L}(\Theta_{in}), \quad \forall\, t \ge 0.$$

Proof. We split the proof into several steps.

• (Step 1): First, from the construction of $\mathcal{L}(\Theta_{in})$, we may choose a well-ordered phase configuration $(\alpha_{in}^1, \cdots, \alpha_{in}^N)$ which represent $D(2\Theta_{in})$ in the sense that

$$2\mathcal{L}(\Theta_{in}) = \max_{j,k} |\alpha_{in}^k - \alpha_{in}^j|, \quad e^{2\mathrm{i}\theta_{in}^j} = e^{\mathrm{i}\alpha_{in}^j} \quad \text{for all } \ j. \tag{1}$$

More explicitly, the angles $\{\alpha_{in}^j\}_{j=1}^N$ above can be defined by following way:

For a maximal open interval $\gamma_0 \in \Gamma(2\Theta_{in})$, we choose one lifted interval (a_0, b_0) of γ_0 satisfying the following constraints:

$$\gamma_0 = \{e^{is} : s \in (a_0, b_0)\}, \quad b_0 - a_0 = |\gamma_0| < 2\pi, \quad 0 \le b_0 < 2\pi.$$

Then, for each j, we denote α_{in}^j as a unique element of $\{2\theta_{in}^j + 2n\pi : n \in \mathbb{Z}\}$ contained in $(a_0, a_0 + 2\pi]$:

$$\{\alpha_{in}^j\} = \{2\theta_{in}^j + 2n\pi : n \in \mathbb{Z}\} \cap (a_0, a_0 + 2\pi].$$

Since $\gamma_0 \in \Gamma(2\Theta_{in})$ and $e^{i\alpha_{in}^j} = e^{2i\theta_{in}^j}$, these α_{in}^js are indeed contained in

$$[b_0, a_0 + 2\pi] = (a_0, a_0 + 2\pi] - (a_0, b_0),$$

and

$$(a_0 + 2\pi) - b_0 = 2\pi - |\gamma_0| = 2\mathcal{L}(\Theta_{in}).$$

One the other hand, the maximality of γ_0 implies that $e^{ia_0} = e^{i(a_0+2\pi)}$ and e^{ib_0} are contained in $\mathcal{P}(2\Theta_{in})$, and therefore

$$\min_{1\le j\le N} \alpha_{in}^j = b_0, \quad \max_{1\le j\le N} \alpha_{in}^j = a_0 + 2\pi,$$

so that the configuration $\{\alpha_{in}^j\}_{j=1}^N$ satisfies (1). Then, we find N integers $n_1, \cdots, n_N$ satisfying

$$\alpha_{in}^j = 2\theta_{in}^j + 2n_j\pi, \quad j = 1, \cdots, N,$$

and define $\alpha^j(t)$ by

$$\alpha^j(t) := 2\theta^j(t) + 2n_j\pi$$

so that $e^{i\alpha^j(t)} = e^{2i\theta^j(t)}$ for all $1 \le j \le N$ and $t \ge 0$.

Now, from the analyticity of the communication weight $\psi(\cdot)$, the solution $\{(x^j, \theta^j)\}_{j=1}^N$ of system (1) is analytic. Since each α^j differs to $2\theta^j$ up to constant, α^j is also analytic. Therefore, any zero set of $\alpha^i - \alpha^j$ has no limit point unless $2\theta^i \equiv 2\theta^j$ along the whole time. This implies that α^M and α^m are continuous and piecewise smooth in t.

• (Step 2): Next, we can choose indices m, M for each time t, where $\alpha^m(t)$ and $\alpha^M(t)$ are the minimum and maximum values in $\mathbb{R}$. Then, from the initial data,

$$\alpha_{in}^M - \alpha_{in}^m = 2\mathcal{L}(\Theta_{in}).$$

From the continuity of $\alpha^j(t)$, there exists a time $T_0 > 0$ satisfying

$$\alpha^M(t) - \alpha^m(t) < \pi, \quad 0 \le t < T_0.$$

Then, it follows from (1) that

$$\begin{aligned} \frac{d\alpha^M}{dt} &= \frac{2\kappa}{N}\sum_{k=1}^N \psi(\|x^k - x^M\|)\sin(\alpha^k - \alpha^M) \le 0, \\ \frac{d\alpha^m}{dt} &= \frac{2\kappa}{N}\sum_{k=1}^N \psi(\|x^k - x^m\|)\sin(\alpha^k - \alpha^m) \ge 0, \end{aligned} \tag{2}$$

for all but countably many t where α^M and α^m are both differentiable at t. Therefore, since $\alpha^M(t) - \alpha^m(t)$ is absolutely continuous and its derivative is nonpositive, it is non-increasing for all time $t \ge 0$.

- (Step 3): Since $\alpha^M(t) - \alpha^m(t) < \pi$ for all $t \ge 0$, the arc

$$\left\{ e^{is} : s \in \left(\alpha^M(t) - 2\pi, \alpha^m(t)\right) \right\}$$

is an element of $\Gamma(2\Theta(t))$ with the length larger than π. Therefore, this interval has the largest arclength among $\Gamma(2\Theta(t))$ and $\alpha^M(t) - \alpha^m(t)$ is still the same as the diameter $D(2\Theta(t))$, i.e.,

$$2\mathcal{L}(\Theta(t)) = 2\pi - (\alpha^m(t) - (\alpha^M(t) - 2\pi)) = (\alpha^M(t) - \alpha^m(t)),$$

and hence $\mathcal{L}(\Theta(t))$ is non-increasing in t. □

Remark 3.1. *1. In the proof of Lemma 3.1, we use the condition $\mathcal{L}(\Theta_{in}) > 0$ to guarantee the existence of α^M and α^m for a.e. t. On the other hand, if the J-K particles are initially nematically aligned, that is, $\mathcal{L}(\Theta_{in}) = 0$, the heading angles of those all particles are unchanged for whole time t. Therefore, we exclude this trivial case from our analysis.*

2. Since $\alpha^m_{in} \le \alpha^m(t) \le \alpha^j(t) \le \alpha^M(t) \le \alpha^M_{in}$, $\alpha^j(t)$ is contained in the fixed interval $[\alpha^m_{in}, \alpha^M_{in}]$ for all $t \ge 0$ and $1 \le j \le N$. Therefore, if we fix two closed intervals

$$\gamma_A := \left\{ e^{is} : s \in \left[\frac{\alpha^m_{in}}{2}, \frac{\alpha^M_{in}}{2}\right]\right\}, \quad \gamma_B := \left\{ e^{is} : s \in \left[\frac{\alpha^m_{in}}{2} + \pi, \frac{\alpha^M_{in}}{2} + \pi\right]\right\},$$

and two index sets

$$A := \left\{ j : e^{i\theta^j_{in}} \in \gamma_A \right\}, \quad B := \left\{ j : e^{i\theta^j_{in}} \in \gamma_B \right\},$$

we have

$$e^{i\theta^j(t)} \in \gamma_A \quad \textit{for} \ \ j \in A, \quad \textit{and} \quad e^{i\theta^j(t)} \in \gamma_B \quad \textit{for} \ \ j \in B.$$

for all time t. That is, there exist two fixed closed connected arcs with arclength $\mathcal{L}(\Theta_{in})$ at the opposite so that each heading angles stay exactly one of them all the time.

Proof. **Proof of Theorem 3.1** We combine the two inequalities in (2) and apply Lemma 2.2 for $\delta = 0$ to get

(3)
$$\begin{aligned}
\frac{d}{dt}\mathcal{L}(\Theta) &= \frac{1}{2}\frac{d}{dt}(\alpha^M - \alpha^m) \\
&= \frac{\kappa}{N}\sum_{k=1}^{N}\Big[\psi(\|x_k - x_M\|)\sin(\alpha_k - \alpha_M) - \psi(\|x_k - x_m\|)\sin(\alpha_k - \alpha_m)\Big] \\
&\le \frac{\kappa}{N}\sum_{k=1}^{N}\psi_m\Big[\sin(\alpha_k - \alpha_M) - \sin(\alpha_k - \alpha_m)\Big] \\
&\le -\kappa\psi_m \sin(\alpha^M - \alpha^m) = -\kappa\psi_m \sin 2\mathcal{L}(\Theta).
\end{aligned}$$

Then, we use

$$\int \csc 2x \ \mathrm{d}x = \frac{1}{2}\log(\tan x) + C$$

to integrate (3):

$$\frac{d}{dt}\left(\log\left(\tan \mathcal{L}(\Theta)\right)\right) + 2\kappa\psi_m \le 0.$$

This yields our desired estimate:

$$\tan \mathcal{L}(\Theta(t)) \le \tan \mathcal{L}(\Theta_{in})e^{-2\kappa\psi_m t}.$$

□

Remark 3.2. *Consider the generalized J-K particles in the whole space $\mathbb{R}^2$ again:*

(4)
$$\begin{aligned}
&\frac{dx^j}{dt} = (\cos\theta^j, \sin\theta^j), \quad t > 0, \quad x^j \in \mathbb{R}^2, \quad j = 1, \cdots, N, \\
&\frac{d\theta^j}{dt} = \frac{\kappa}{N}\sum_{k=1}^{N}\psi(\|x^k - x^j\|)\sin 2(\theta^k - \theta^j), \quad \theta^j \in \mathbb{R}.
\end{aligned}$$

From these equations, one can show that $D(X(t))$ is differentiable for all but countably many (and not accumulating) time t, and its derivative is less than or equal to 2:

(5)
$$\left|\frac{dD(X)}{dt}\right| \le 2.$$

Then, we can combine (3) *and* (5) *to obtain the sharper differential inequality when ψ does not have a positive lower bound:*

$$\frac{d\mathcal{L}(\Theta)}{dt} \le -\kappa\psi(D(X_{in}) + 2t)\sin 2\mathcal{L}(\Theta).$$

This implies

$$\tan \mathcal{L}(\Theta(t)) \le \tan \mathcal{L}(\Theta_{in})e^{-\kappa\int_{D(X_{in})}^{D(X_{in})+2t}\psi(s)ds}.$$

In particular, whenever ψ is not integrable, i.e.,

$$\int_0^\infty \psi(s)ds = \infty,$$

$\mathcal{L}(\Theta)$ vanishes asymptotically and nematic alignment occurs.

4. The stochastic Justh-Krishnaprasad flow

In this section, we present preparatory lemmas for the stochastic persistency estimation in Section 5. We introduce a relaxed version of collision and provide some probability estimates of stopping times related to this relaxed collision event.

Due to additive white noise, the continuous sample trajectory will depart from the nematic alignment state eventually even if they are initially close to it. Thus, from the view point of dynamical systems theory, the nematic alignment state is unstable in the sense that there always exists a continuous sample path depart from the small neighborhood of nematic state asymptotically. In the following two sections, we are interested in the following stochastic persistency question:

> "For a given initial configuration close to nematic alignment state and finite-time window $[0, T]$, what will be the probability that the stochastic flow stays in an ε-neighborhood of the nematic alignment state?" To fix the idea, note that $\mathcal{L}(\Theta_t) = 0$ denotes the nematic alignment state. So for a given T and ε, the probability
>
> $$\mathbb{P}\Big\{ \sup_{0\le t\le T} \mathcal{L}(\Theta_t) \ge \varepsilon \Big\} \tag{1}$$
>
> will measure how much the configuration deviates from the nematic alignment state.

Since the stochastic J-K flow is nonlinear, we might not be able to have an exact probability (1) depending of T and ε. Therefore, our primary goal of the following two sections is to provide a non-trivial lower bound for such probability. To do this, we will employ several stopping times in the sequel analysis. Below, we introduce a relaxed first collision time and its related probabilistic estimate.

4.1. **Relaxed first collision-time.**

Consider a general stochastic Justh-Krishnaprasad model:

$$\begin{aligned} dx_t^j &= (\cos\theta_t^j, \sin\theta_t^j)\ dt, \quad t>0,\ j=1,\cdots,N,\\ d\theta_t^j &= \frac{\kappa}{N}\sum_{k=1}^N \psi(\|x_t^k - x_t^j\|)\sin 2(\theta_t^k-\theta_t^j)dt + \sqrt{2\sigma}dB_t^j. \end{aligned}$$

In the deterministic flow (1), once the communication weight function ψ is analytic, then the collisions between two heading angles occur only finitely in a finite-time interval. Thus, the Lyapunov function $\mathcal{L}(\Theta_t)$ can be described in terms of maximum and minimum phases $\alpha^M(t)$ and $\alpha^m(t)$ which are piecewise analytic.

However, in the stochastic flow, the zero set of phase differences $\alpha^i - \alpha^j$ can be infinite in a finite time interval due to the lack of regularity. Thus, we need to modify the concept of the maximum and minimum phases. We adopt the technique of "*relaxed first-collision time*" [18] and estimate the dynamics of $\mathcal{L}(\Theta_t)$ until the collision time.

For this system of stochastic differential equations, we repeat the phase representation $(\alpha_t^1, \cdots, \alpha_t^N)$ from the proof of Lemma 3.1 with respect to $D(2\Theta_t)$:

$$d\alpha_t^j = \frac{2\kappa}{N}\sum_{k=1}^N \psi(\|x_t^k - x_t^j\|)\sin(\alpha_t^k - \alpha_t^j)dt + 2\sqrt{2\sigma}dB_t^j, \tag{2}$$

whose initial data satisfy

$$2\mathcal{L}(\Theta_{in}) = \max_{j,k} |\alpha_{in}^k - \alpha_{in}^j|.$$

As we have seen in Lemma 3.1, the above equation also holds for $t \geq 0$ whenever

$$\max_{j,k} |\alpha_{in}^k - \alpha_{in}^j| < \pi.$$

For a given $\delta > 0$ and initial configuration $(X_{in}, \Theta_{in}) \in \mathbb{R}^{2N} \times \mathbb{R}^N$, we choose one pair of indices (M_0, m_0) and relaxed first collision-time $\tau^0(\delta) := \tau^0(\delta; X_{in}, \Theta_{in})$ of the time-dependent interval $[\alpha_t^{m_0} - \delta, \alpha_t^{M_0} + \delta]$ as follows:

$$\begin{aligned} &\alpha_{in}^{m_0} := \min_{1\leq j\leq N} \alpha_{in}^j, \quad \alpha_{in}^{M_0} := \max_{1\leq j\leq N} \alpha_{in}^j, \quad \text{and} \\ &\tau^0(\delta; X_{in}, \Theta_{in}) := \inf\Big\{t > 0 : \alpha_t^i \notin [\alpha_t^{m_0} - \delta, \alpha_t^{M_0} + \delta] \quad \text{for some } i\Big\}. \end{aligned} \tag{3}$$

Note that the indices M_0, m_0 are determined only by the initial data and will not be changed for all time t. From definition of $\tau^0(\delta)$, $\alpha_t^{m_0}$ and $\alpha_t^{M_0}$ may not be the minimum or maximum of $\{\alpha_t^1, \cdots, \alpha_t^N\}$ for some $t < \tau^0(\delta)$. On the other hand, for $t < \tau^0(\delta)$, we have the following estimates on the diameter of α_t^j:

$$\max_{j,k} |\alpha_t^k - \alpha_t^j| \leq \alpha_t^{M_0} - \alpha_t^{m_0} + 2\delta.$$

Therefore, we can use the equations (2) to bound $\mathcal{L}(\Theta)$, which will be presented in Section 5. Before this, we need to estimate the collision time $\tau^0(\delta)$.

4.2. **Estimate on the relaxed first collision-time.** In this subsection, we present a quantitative estimate on $\tau^0(\delta)$. Even in the deterministic case, $\tau^0(\delta)$ is not infinity in general. Hence, we estimate the probability that $\tau^0(\delta)$ is less than a small constant value. For this, we set

$$C_\delta := (1 + \sin\delta)\psi_M - \cos\delta\psi_m \quad \text{and} \quad T_\delta := \frac{\delta}{4\kappa C_\delta}. \tag{4}$$

Note that for $0 < \delta \ll 1$,

$$\lim_{\delta\to 0+} C_\delta = \psi_M - \psi_m > 0, \quad \lim_{\delta\to 0+} \frac{T_\delta}{\delta} = \frac{1}{4\kappa(\psi_M - \psi_m)}.$$

Let (X_t, Θ_t) be a stochastic J-K flow issued from the initial state (X_{in}, Θ_{in}). Below, we provide an estimation on the probability:

$$\mathbb{P}\Big\{\tau^0(\delta; X_{in}, \Theta_{in}) < T_\delta\Big\}.$$

Note that the defining relation (3) implies

$$\begin{aligned} &\tau^0(\delta; X_{in}, \Theta_{in}) < T_\delta \\ &\quad\Longleftrightarrow \quad \exists\, j \in \{1, \cdots, N\},\ t < T_\delta \quad \text{such that} \\ &\quad\quad \text{either } \alpha_t^j < \alpha_t^{m_0} - \delta \quad \text{or} \quad \alpha_t^j > \alpha_t^{M_0} + \delta. \end{aligned} \tag{5}$$

Hence, we introduce the collision time of a pair of angles: for $\delta \geq 0$ and a configuration (X_{in}, Θ_{in}),

$$\tau^{ij}(\delta; X_{in}, \Theta_{in}) := \inf\Big\{t > 0 : \alpha_t^i + \delta < \alpha_t^j, (X_0, \Theta_0) = (X_{in}, \Theta_{in})\Big\}. \tag{6}$$

Then, the probability for the event (5) can be expressed in terms of (6):

$$\begin{aligned}
&\mathbb{P}\{\tau^0(\delta; X_{in}, \Theta_{in}) < T_\delta\} \\
&\quad = \mathbb{P}\{\exists\ j \text{ s.t. } \inf_{t<T_\delta}(\alpha_t^j - \alpha_t^{m_0}) < -\delta \text{ or } \inf_{t<T_\delta}(\alpha_t^{M_0} - \alpha_t^j) < -\delta\} \qquad (7)\\
&\quad \le \mathbb{P}\{\exists\ j : \tau^{jm_0}(\delta; X_{in}, \Theta_{in}) < T_\delta\} + \mathbb{P}\{\exists\ j : \tau^{M_0 j}(\delta; X_{in}, \Theta_{in}) < T_\delta\}.
\end{aligned}$$

In the following lemma, we estimate the above probabilities quantitatively.

Lemma 4.1. *For any $\delta \in (0, \frac{\pi}{2})$, the following estimates hold.*

$$\begin{aligned}
&(i)\ \mathbb{P}\{\tau^{jm_0}(\delta; X_{in}, \Theta_{in}) < T_\delta\} \le 4\sqrt{\frac{32\sigma}{\pi\kappa\delta C_\delta}}e^{-\frac{\kappa\delta C_\delta}{32\sigma}}.\\
&(ii)\ \mathbb{P}\{\tau^{M_0 j}(\delta; X_{in}, \Theta_{in}) < T_\delta\} \le 4\sqrt{\frac{32\sigma}{\pi\kappa\delta C_\delta}}e^{-\frac{\kappa\delta C_\delta}{32\sigma}}.
\end{aligned}$$

Proof. Since the derivation of the estimate (ii) will be similar to that of (i), we only consider the estimate (i) below. The proof looks complicated and technical, but basically we want to bound α_t^j by drifted Brownian motions, whose probabilities are discussed in Lemma 2.2. We split the proof into three steps.

• Step A: For each j, from the sample space Ω, consider a continuous sample path $\alpha_t^{jm_0}(\omega) := \alpha_t^j(\omega) - \alpha_t^{m_0}(\omega)$ for $\omega \in \Omega$, where

$$\omega \in \left\{\inf_{t<T_\delta}(\alpha_t^j - \alpha_t^{m_0}) < -\delta\right\} = \{\tau^{jm_0}(\delta; X_{in}, \Theta_{in}) < T_\delta\}.$$

Since each α^j has a continuous path, we know

$$\tau^{jm_0}(0; X_{in}, \Theta_{in}) < \tau^{jm_0}(\delta; X_{in}, \Theta_{in}).$$

Thus, we may change the initial time into $\tau^{jm_0}(0)$ in the following way: For given initial data (X_{in}, Θ_{in}) and $\delta > 0$,

$$\begin{aligned}
\left\{\omega : \inf_{t<T_\delta}\alpha_t^{jm_0}(\omega) < -\delta\right\} &= \left\{\omega : \inf_{\tau^{jm_0}(0)<t<T_\delta}\alpha_t^{jm_0}(\omega) < -\delta\right\}\\
&\subset \left\{\omega : \inf_{\tau^{jm_0}(0)<t<T_\delta+\tau^{jm_0}(0)}\alpha_t^{jm_0}(\omega) < -\delta\right\}\\
&= \left\{\omega : \inf_{0<t<T_\delta}\left(\beta_t^j(\omega) - \beta_t^{m_0}(\omega)\right) < -\delta\right\},
\end{aligned}$$

where $\beta_t^i(\omega) := \alpha^i_{t+\tau^{jm_0}(0)}(\omega)$ for all $i = 1, \cdots, N$ and $t \ge 0$.

Since we do not have enough information on β_0^i, we do the worst-case analysis. Let Λ^{ij} be the set of all initial configuration $(\tilde{X}_{in}, \tilde{\Theta}_{in})$ satisfying $\tilde{\alpha}_{in}^i = \tilde{\alpha}_{in}^j$. Then, the collision-time $\tilde{\tau}^{ij} = \tilde{\tau}^{ij}(\delta, X_{in}, \Theta_{in})$ for $(X_{in}, \Theta_{in}) \in \Lambda^{ij}$ can be defined by

$$\tilde{\tau}^{ij}(\delta; X_{in}, \Theta_{in}) := \inf\left\{t > 0 : |\alpha_t^i - \alpha_t^j| > \delta,\ (X_0, \Theta_0) = (X_{in}, \Theta_{in})\right\}.$$

Since $\beta_0^{jm_0} = 0$ and the stochastic process β_t satisfies the same SDE (2) as α_t, we have

$$
\begin{aligned}
&\mathbb{P}\{\tau^{jm_0}(\delta; X_{in}, \Theta_{in}) < T_\delta\} \\
&\quad = \mathbb{P}\left\{\omega : \inf_{t<T_\delta} \alpha_t^{jm_0}(\omega) < -\delta\right\} \\
&\quad \le \mathbb{P}\left\{\omega : \inf_{t<T_\delta} \beta_t^{jm_0}(\omega) < -\delta\right\} \\
&\quad \le \sup_{(\tilde X_{in}, \tilde\Theta_{in}) \in \Lambda^{jm_0}} \mathbb{P}\{\tilde\tau^{jm_0}(\delta; \tilde X_{in}, \tilde\Theta_{in}) < T_\delta\}.
\end{aligned} \tag{8}
$$

• Step B: Consider the initial configuration $(\tilde X_{in}, \tilde\Theta_{in}) \in \Lambda^{jm_0}$, and for simplicity, we set

$$
\tilde\psi_t^{ij} := \psi(\|\tilde x_t^i - \tilde x_t^j\|) \quad \text{and} \quad \tilde\alpha_t^{ij} := \tilde\alpha_t^i - \tilde\alpha_t^j, \quad i, j = 1, \cdots, N.
$$

We claim: for $t < \tilde\tau^{jm_0}(\delta; \tilde X_{in}, \tilde\Theta_{in}) \wedge T_\delta$,

$$
\begin{aligned}
&\tilde\alpha_t^j - \tilde\alpha_t^{m_0} \ge -\frac{\delta}{2} + 2\sqrt{2\sigma}(B_t^j - B_t^{m_0}), \\
&\tilde\alpha_t^j - \tilde\alpha_t^{m_0} \le \frac{\delta}{2} + 2\sqrt{2\sigma}(B_t^j - B_t^{m_0}).
\end{aligned} \tag{9}
$$

Proof of (9): Note that for $t < \tilde\tau^{jm_0}(\delta; \tilde X_{in}, \tilde\Theta_{in})$, we use the trignometry identity:

$$
\sin \alpha_t^{km_0} = \sin \tilde\alpha_t^{kj} \cos \tilde\alpha_t^{jm_0} + \cos \tilde\alpha_t^{kj} \sin \tilde\alpha_t^{jm_0}
$$

to get

$$
\begin{aligned}
d\tilde\alpha_t^{jm_0} &= \frac{2\kappa}{N} \sum_{k=1}^N \Big(\tilde\psi_t^{kj} \sin \tilde\alpha_t^{kj} - \tilde\psi_t^{km_0} \sin \tilde\alpha_t^{km_0}\Big) dt + 2\sqrt{2\sigma} d(B_t^j - B_t^{m_0}) \\
&= \frac{2\kappa}{N} \sum_{k=1}^N \underbrace{\Big((\tilde\psi_t^{kj} - \tilde\psi_t^{km_0} \cos \tilde\alpha_t^{jm_0}) \sin \tilde\alpha_t^{kj} - \tilde\psi_t^{km_0} \sin \tilde\alpha_t^{jm_0} \cos \tilde\alpha_t^{kj}\Big)}_{=: \mathcal{J}_k^{jm_0}} dt \\
&\quad + 2\sqrt{2\sigma} d(B_t^j - B_t^{m_0}).
\end{aligned} \tag{10}
$$

⋄ Case A (Derivation of $(9)_1$): To derive a worst lower bound for $\mathcal{J}$, we use

$$
|\tilde\alpha_t^j - \tilde\alpha_t^{m_0}| \le \delta, \quad \text{for } t < \tilde\tau^{jm_0}(\delta; \tilde X_{in}, \tilde\Theta_{in})
$$

and

$$
\left|\tilde\psi_t^{kj} - \tilde\psi_t^{km_0} \cos \tilde\alpha_t^{jm_0}\right| \le \psi_M - \psi_m \cos \delta, \quad \left|\tilde\psi_t^{km_0} \sin \tilde\alpha_t^{jm_0} \cos \tilde\alpha_t^{kj}\right| \le \psi_M \sin \delta,
$$

to see

$$
\begin{aligned}
\mathcal{J}_k^{jm_0} &= (\tilde\psi_t^{kj} - \tilde\psi_t^{km_0} \cos \tilde\alpha_t^{jm_0}) \sin \tilde\alpha_t^{kj} - \tilde\psi_t^{km_0} \sin \tilde\alpha_t^{jm_0} \cos \tilde\alpha_t^{kj} \\
&\ge -\Big(\psi_M(1 + \sin \delta) - \psi_m \cos \delta\Big) = -C_\delta.
\end{aligned} \tag{11}
$$

We combine (10) and (11) to get

$$
d\tilde\alpha_t^{jm_0} \ge -2\kappa C_\delta dt + 2\sqrt{2\sigma} d(B_t^j - B_t^{m_0}), \quad t < \tilde\tau^{jm_0}(\delta; \tilde X_{in}, \tilde\Theta_{in}). \tag{12}
$$

Next, we integrate the above inequality (12) using the relations:

$$\tilde{\alpha}_0^j - \tilde{\alpha}_0^{m_0} = 0 \quad \text{and} \quad 2\kappa C_\delta t \le 2\kappa C_\delta T_\delta = \frac{\delta}{2}, \quad \text{for } t < \tilde{\tau}^{jm_0}(\delta; \tilde{X}_{in}, \tilde{\Theta}_{in}) \wedge T_\delta$$

to get

$$\tilde{\alpha}_t^j - \tilde{\alpha}_t^{m_0} \ge -\frac{\delta}{2} + 2\sqrt{2\sigma}(B_t^j - B_t^{m_0}), \quad t < \tilde{\tau}^{jm_0}(\delta; \tilde{X}_{in}, \tilde{\Theta}_{in}) \wedge T_\delta. \tag{13}$$

⋄ Case B (Estimate of $(9)_2$): Similar to Case A, we have

$$\tilde{\alpha}_t^j - \tilde{\alpha}_t^{m_0} \le \frac{\delta}{2} + 2\sqrt{2\sigma}(B_t^j - B_t^{m_0}), \quad t < \tilde{\tau}^{jm_0}(\delta; \tilde{X}_{in}, \tilde{\Theta}_{in}) \wedge T_\delta. \tag{14}$$

Finally, (13) and (14) yield the desired estimate (9).

• Step C: Next, we show the following inclusion relation:

$$\{\tilde{\tau}^{jm_0}(\delta; \tilde{X}_{in}, \tilde{\Theta}_{in}) < T_\delta\} \subset \left\{ \sup_{t \le T_\delta} \left| 2\sqrt{2\sigma}(B_t^j - B_t^{m_0}) \right| \ge \frac{\delta}{2} \right\}. \tag{15}$$

Suppose that there exist a sample point ω satisfying

$$\tilde{\tau}^{jm_0}(\delta; \tilde{X}_{in}, \tilde{\Theta}_{in})(\omega) < T_\delta,$$

and

$$\sup_{t \le T_\delta} \left| 2\sqrt{2\sigma}(B_t^j - B_t^{m_0}) \right| (\omega) < \frac{\delta}{2}.$$

Then, we have

$$\begin{aligned}
&\sup_{t \le \tilde{\tau}^{jm_0}} \left| \tilde{\alpha}_t^j - \tilde{\alpha}_t^{m_0} \right| \\
&\quad \le \sup_{t \le \tilde{\tau}^{jm_0}} \left| \tilde{\alpha}_t^j - \tilde{\alpha}_t^{m_0} - 2\sqrt{2\sigma}(B_t^j - B_t^{m_0}) \right| + \sup_{t \le \tilde{\tau}^{jm_0}} \left| 2\sqrt{2\sigma}(B_t^j - B_t^{m_0}) \right| \\
&\quad \le \sup_{t \le \tilde{\tau}^{jm_0} \wedge T_\delta} \left| \tilde{\alpha}_t^j - \tilde{\alpha}_t^{m_0} - 2\sqrt{2\sigma}(B_t^j - B_t^{m_0}) \right| + \sup_{t \le T_\delta} \left| 2\sqrt{2\sigma}(B_t^j - B_t^{m_0}) \right| \\
&\quad < \frac{\delta}{2} + \frac{\delta}{2} = \delta.
\end{aligned}$$

Since this gives a contradiction to definition of $\tilde{\tau}^{jm_0}$, the inclusion relation (15) holds.

• Final step: Let $(\tilde{X}_{in}, \tilde{\Theta}_{in})$ be any initial configuration in Λ^{jm_0}. Then, we use the relation (15), Andre's reflection principle (see (i) in Lemma 2.3) and the relation

$$B_t^j - B_t^{m_0} = \sqrt{2}\tilde{B}_t \quad \text{for some standard Brownian motion } \tilde{B}_t$$

to obtain

$$\begin{aligned}
&\mathbb{P}\{\tilde{\tau}^{jm_0}(\delta; \tilde{X}_{in}, \tilde{\Theta}_{in}) < T_\delta\} \le \mathbb{P}\left\{ \sup_{t \le T_\delta} \left| 2\sqrt{2\sigma}(B_t^j - B_t^{m_0}) \right| \ge \frac{\delta}{2} \right\} \\
&\quad = 2\mathbb{P}\left\{ \sup_{t \le T_\delta} 2\sqrt{2\sigma}(B_t^j - B_t^{m_0}) \ge \frac{\delta}{2} \right\} = 2\mathbb{P}\left\{ \sup_{t \le T_\delta} \tilde{B}_t \ge \frac{\delta}{8\sqrt{\sigma}} \right\} \\
&\quad \le 16\sqrt{\frac{2\sigma T_\delta}{\pi\delta^2}} e^{-\frac{\delta^2}{128\sigma T_\delta}} = 4\sqrt{\frac{32\sigma}{\pi\kappa\delta C_\delta}} e^{-\frac{\kappa\delta C_\delta}{32\sigma}}.
\end{aligned} \tag{16}$$

Finally, the desired estimate follows from (7),(8) and (16). □

As a direct application of Lemma 4.1, we obtain the probability estimate on the event $\{\tau^0(\delta; X_{in}, \Theta_{in}) < T_\delta\}$ as follows.

Proposition 4.1. *Suppose that for a positive number $\varepsilon > 0$, the system parameters κ, σ and control parameter δ satisfy*

$$\kappa > 0, \quad \sigma > 0, \quad \delta \in \left(0, \frac{\pi}{2}\right), \quad \sqrt{\frac{32\sigma}{\kappa\delta C_\delta}} < \varepsilon, \tag{17}$$

and let (X_t, Θ_t) be a stochastic J-K flow with the initial state (X_{in}, Θ_{in}). Then, for any small $\delta > 0$, we have

$$\mathbb{P}\{\tau^0(\delta; X_{in}, \Theta_{in}) < T_\delta\} \le \frac{8N}{\sqrt{\pi}} \varepsilon e^{-\varepsilon^{-2}}. \tag{18}$$

Proof. We combine (8)-(16) to deduce

$$\begin{aligned}
&\mathbb{P}\{\tau^0(\delta; X_{in}, \Theta_{in}) < T_\delta\} \\
&\le \sum_j \mathbb{P}\{\tilde\tau^{jm_0}(\delta; X_{in}, \Theta_{in}) < T_\delta\} + \sum_j \mathbb{P}\{\tilde\tau^{M_0 j}(\delta; X_{in}, \Theta_{in}) < T_\delta\} \\
&= \sum_j \left[\sup_{(\tilde X_{in}, \tilde\Theta_{in}) \in \Lambda^{jm_0}} \mathbb{P}\{\tilde\tau^{jm_0}(\delta; \tilde X_{in}, \tilde\Theta_{in}) < T_\delta\} + \sup_{(\tilde X_{in}, \tilde\Theta_{in}) \in \Lambda^{Mj}} \mathbb{P}\{\tilde\tau^{Mj}(\delta; \tilde X_{in}, \tilde\Theta_{in}) < T_\delta\} \right] \\
&\le 8N \sqrt{\frac{32\sigma}{\pi\kappa\delta C_\delta}} e^{-\frac{\kappa\delta C_\delta}{32\sigma}}.
\end{aligned}$$

Now, we use the relation (17) to get the desired estimate. □

Remark 4.1. *(1) For the constant $\psi(r) = \psi_\infty$, we can improve the estimate (18) as follows.*

$$\begin{aligned}
d\alpha_t^{jm_0} &= \frac{2\kappa\psi_\infty}{N} \sum_{k=1}^N \left(\sin\alpha_t^{kj} - \sin\alpha_t^{km_0}\right) dt + 2\sqrt{2\sigma} d(B_t^j - B_t^{m_0}) \\
&\ge -2\kappa\psi_\infty \delta dt + 2\sqrt{2\sigma} d(B_t^j - B_t^{m_0}), \quad t < \tilde\tau^{jm_0}(\delta; \tilde X_{in}, \tilde\Theta_{in}).
\end{aligned}$$

Therefore, for any $\delta > 0$, we have

$$\begin{aligned}
&\mathbb{P}\left\{\tilde\tau^{jm_0}(\delta; \tilde X_{in}, \tilde\Theta_{in}) < \frac{1}{4\kappa\psi_\infty}\right\} \le \mathbb{P}\left\{\sup_{t \le 1/(4\kappa\psi_\infty)} \left|2\sqrt{2\sigma}(B_t^j - B_t^{m_0})\right| > \frac{\delta}{2}\right\} \\
&\quad = 2\mathbb{P}\left\{\sup_{t \le 1/(4\kappa\psi_\infty)} 2\sqrt{2\sigma}(B_t^j - B_t^{m_0}) > \frac{\delta}{2}\right\} \le \sqrt{\frac{32\sigma}{\pi\kappa\psi_\infty\delta^2}} e^{-\frac{\kappa\psi_\infty\delta^2}{32\sigma}}, \\
&\mathbb{P}\{\tau^0(\delta; X_0, \Theta_0) < T_\delta\} \le 2N \sqrt{\frac{32\sigma}{\pi\kappa\psi_\infty\delta^2}} e^{-\frac{\kappa\psi_\infty\delta^2}{32\sigma}}.
\end{aligned}$$

Note that this coincides with the result in [18] *for the identical Kuramoto model ($D(\Omega) = 0$) with additive noise.*

(2) The $N-$dependency of the result (18) comes from definition of τ^0 and the additive noise in (1). Since $\tau^0 = \tau^0(\delta)$ is defined as the first collision time of α_t^j to either $\alpha_t^{m_0} - \delta$ or $\alpha_t^{M_0} + \delta$ for at least one j, the probability of the event

$$\{\tau^0 < T_\delta\}$$

can only be controlled by the $N-$times of probability for the collision event for each α_t^j, unless we can relate the event of collsions for different α^js. However, as the additive noises $\{dB_t^j\}_{j=1}^N$ are assumed to be independent to each other, we are not able to acheive this.

5. Stochastic persistency of nematic alignment state

In this section, we conclude the stochastic persistency estimate, which provides us a nontrivial lower bound for the probability in which the system stays near the nematic alignment state. Since the proof for our second main result will be very lengthy, we first briefly discuss our main result and a strategy to prove it, and then provide its detailed proof in the second part.

5.1. **Main result and strategy.** Consider the following system of stochastic differential equations:

$$
\begin{aligned}
&dx_t^j = (\cos\theta_t^j, \sin\theta_t^j)\ dt, \quad t>0,\ j=1,\cdots,N,\\
&d\theta_t^j = \frac{\kappa}{N}\sum_{k=1}^N \psi(\|x_t^k - x_t^j\|)\sin 2(\theta_t^k-\theta_t^j)dt + \sqrt{2\sigma}dB_t^j,
\end{aligned}
\tag{1}
$$

where σ is a positive noise strength and we assume that the communication weight function ψ is Lipschitz continuous and strictly positive such that there exist positive constants ψ_m and ψ_M such that

$$
0<\psi_m \le \psi(\|x-y\|) \le \psi_M, \quad \forall\ x,y\in\mathbb{R}^2. \tag{2}
$$

Our goal for this stochastic differential equation (1) is the derivation of a similar result to Theorem 3.1. Although we cannot expect the asymptotic nematic alignment of the stochastic flow (X_t,Θ_t) as in Section 3, we analyze the first exit time of a process $\mathcal{L}(\Theta_t)$ from finite interval $[0,D_\infty]$, where D_∞ is a given positive constant smaller than $\frac{\pi}{2}$. In fact, we need some time-dependent barrier function $L(s)$ which gives a sharper bound on $\mathcal{L}(\Theta_t)$, i.e., $L(s)\le D_\infty$ for all $s\ge 0$. Then, under some sufficient framework, we estimate the probability of the event

$$
\Big\{\mathcal{L}(\Theta_s) \text{ exceeds } L(s) \text{ at least once for } s\in[0,t]\Big\},
$$

which naturally induces the boundedness (by D_∞) and decreasing behavior of $\mathcal{L}(\Theta_s)$.

Before we present the stochastic persistency estimate for (1) - (2), we introduce several notation including the barrier function $L(s)$: for positive constants δ and $D_\infty<\frac{\pi}{2}$, we set

$$
\begin{aligned}
&R_\infty := \frac{\sin 2D_\infty}{2D_\infty}, \quad C_\delta := (1+\sin\delta)\psi_M - \cos\delta\psi_m, \quad T_\delta := \frac{\delta}{4\kappa C_\delta},\\
&L(s) := (\mathcal{L}(\Theta_0)+\delta)e^{-2\kappa\psi_m R_\infty s} + \frac{\delta\psi_M}{\psi_m R_\infty}(1-e^{-2\kappa\psi_m R_\infty s}) + \delta\\
&\qquad\quad + 2\delta\sum_{r=1}^\infty e^{-2\kappa\psi_m R_\infty(s-rT_\delta)}\chi_{(rT_\delta,\ \infty)},\\
&P_\infty := A\left(\frac{\delta}{\sqrt{2\sigma}}, 2\kappa\psi_m R_\infty, T_\delta\right)e^{-\frac{\kappa\psi_m R_\infty\delta^2}{2\sigma}} + 4N\sqrt{\frac{32\sigma}{\pi\delta\kappa C_\delta}}e^{-\frac{\delta\kappa C_\delta}{32\sigma}},
\end{aligned}
\tag{3}
$$

where $r = r\left(\frac{\sqrt{\nu}h}{\sigma}, \nu T\right)$ and $A = A\left(\frac{h}{\sigma}, \nu, T\right)$ are the functions defined in Lemma 2.2:

$$r\left(\frac{\sqrt{\nu}h}{\sigma}, \nu T\right) = \frac{\sigma}{\sqrt{\nu}h} + \nu T e^{-c_0\nu\frac{h^2}{\sigma^2}} \le r_0,$$

$$A\left(\frac{h}{\sigma}, \nu, T\right) = \sqrt{\frac{2\nu^3 T^2 h^2}{\pi\sigma^2}}\left[1 + \mathcal{O}\left(r\left(\frac{\sqrt{\nu}h}{\sigma}, \nu T\right) + \frac{1}{\nu} + \frac{1}{\nu T}\log\left(1 + \frac{\sqrt{\nu}h}{\sigma}\right)\right)\right].$$

Now, we are ready to state our second main result on the emergent dynamics for (1) - (2).

Theorem 5.1. *Suppose that the initial data Θ_0 and system parameters κ, σ, δ satisfy the following relations:*

$$\begin{aligned} &(i)\ \max\left\{\mathcal{L}(\Theta_0) + 2\delta,\ \frac{\delta\psi_M}{\psi_m R_\infty} + \delta + \frac{2\delta}{1 - e^{-\frac{\psi_m R_\infty \delta}{2C_\delta}}}\right\} < D_\infty < \frac{\pi}{2}, \\ &(ii)\ r\left(\delta\sqrt{\frac{\kappa\psi_m R_\infty}{\sigma}}, 2\kappa\psi_m R_\infty T_\delta\right) \le r_0, \\ &(iii)\ P_\infty < 1, \end{aligned} \tag{4}$$

and let Θ_t be a solution to (1) - (2) *with the initial data Θ_{in} (see Lemma 2.2 for definition of r). Then, for any positive integer $\ell \ge 1$,*

$$\mathbb{P}\left\{\exists s < \ell T_\delta : \mathcal{L}(\Theta_s) > L(s)\right\} \le 1 - (1 - P_\infty)^\ell. \tag{5}$$

Proof. We use iterative methods using the time step T_δ, which is given by the collision time $\tau^0(\delta)$. In order to use induction, we split the proof into several steps. Here, we briefly sketch three main steps:

- Step A: As we explained in Section 4.1, we first define phase process $\alpha_t = (\alpha_t^1, \cdots, \alpha_t^N)$. At $t = 0$, we choose the maximal and minimal values among $\{\alpha_t^1, \cdots, \alpha_t^N\}$, $\alpha_0^{M_0}$ and $\alpha_0^{m_0}$, and fix the indices M_0 and m_0. Then, for $t \in [0, \tau^0(\delta))$, we have
$$\mathcal{L}(\Theta_t) \le \frac{1}{2}(\alpha_t^{M_0} - \alpha_t^{m_0}) + \delta,$$
under the assumption of
$$\max_{j,k} |\alpha_t^k - \alpha_t^j| < 2D_\infty.$$
We estimate the process $\alpha_t^{M_0} - \alpha_t^{m_0}$, which is expected to decrease in a high probability from the deterministic model.
- Step B: Provided that $\tau^0(\delta) \ge T_\delta$, we reindex maximal and minimal indices of α at the time $t = T_\delta$ defined in (3) and do Step A again: We choose the maximum and minimum indices M_1 and m_1 at time T_δ, and estimate the process $\mathcal{L}(\Theta_t)$ until $t = 2T_\delta$ by using α^{M_1} and α^{m_1}. By iterating this procedure, we may estimate the probability on the bounds of $\mathcal{L}(\Theta_{(\ell+1)T_\delta})$ under proper assumptions on $\mathcal{L}(\Theta_{\ell T_\delta})$.

- Step C: After estimating $\mathcal{L}(\Theta_t)$ inductively, we will estimate the probability for the whole interval $[0, t]$. We split this probability into three parts. The first part is from the fluctuation of α_t^j from noise, which can be estimated by Lemma 2.2 (ii). Secondly, the estimation of $\{\tau(\delta) > T_\delta\}$ in Proposition 4.1 at each time $t = kT_\delta$. The last one is on the assumption of $\max_{j,k} |\alpha_t^k - \alpha_t^j| < D_\infty$, which will be treated by the assumptions (4). In conclusion, we build a recurrence inequality

$$\mathbb{P}\{\tau_L \geq (\ell+1)T_\delta\} \geq (1 - P_\infty)\mathbb{P}\{\tau_L \geq \ell T_\delta\}, \tag{6}$$

for the stopping time τ_L which represent the left-hand side of (5),

$$\tau_L := \inf\{s > 0 : \ \mathcal{L}(\Theta_s) > L(s)\}.$$

□

Remark 5.1. *Note that the estimate* (5) *can be rewritten as follows.*

$$\mathbb{P}\left\{\max_{0 \leq s \leq \ell T_\delta} \mathcal{L}(\Theta_s) \leq L(s)\right\} > (1 - P_\infty)^\ell.$$

Thus, in a finite-time interval $[0, \ell T_\delta]$, at least with the probability $(1 - P_\infty)^\ell$, stochastic J-K particles will stay in a region where $\mathcal{L}(\Theta)$ is less than $L(s)$. Since the estimate becomes trivial as $\ell \to \infty$, our estimate (5) *provides a useful information only in a finite-time interval. Moreover, from how we defined P_∞ in* (3)*, one can see that choosing the constant δ arbitrarily small makes P_∞ larger than* 1*. Therefore, in Theorem 5.1, it is more likely to consider δ as a fixed positive constant, and P_∞ now becomes small when $\frac{\kappa}{\sigma}$ is large.*

5.2. **Stochastic persistency of nematic alignment state.** From now on, we provide the detailed proof of Theorem 5.1.

5.2.1. *Step A (Initial time-zone estimates).* We first consider the time $t \in [0, T_\delta]$. From the initial data, we choose extremal indices M_0, m_0 as in (3). Then, we need to study the evolution of $\alpha_t^{M_0 m_0} := \alpha_t^{M_0} - \alpha_t^{m_0}$, where the phases are in a half circle, $\max_{j,k} |\alpha_t^k - \alpha_t^j| < 2D_\infty$.

For the condition on the half circle, we define the first hitting time of $\alpha_t^{M_0 m_0} + 2\delta$ to $2D_\infty$ as follows:

$$\tau_{D_\infty}^0(\delta) := \tau_{D_\infty}^0(\delta, X_{in}, \Theta_{in}) := \inf\{t > 0 : \ \underbrace{\alpha_t^{M_0 m_0} + 2\delta}_{\approx \max_{j,k} |\alpha_t^k - \alpha_t^j|} > 2D_\infty\}.$$

Note that for $t < \tau^0(\delta)$,

$$\alpha_t^{m_0} - \delta \leq \alpha_t^k \leq \alpha_t^{M_0} + \delta, \quad k = 1, \cdots, N.$$

This yields

$$0 \leq \alpha_t^{k m_0} + \delta, \qquad \alpha_t^{k M_0} - \delta \leq 0. \tag{7}$$

On the other hand, for $t < \tau_{D_\infty}^0(\delta)$,

$$\alpha_t^{M_0 m_0} + 2\delta = (\alpha_t^{M_0 k} + \delta) + (\alpha_t^{k m_0} + \delta) \leq 2D_\infty. \tag{8}$$

Finally, we combine (7) and (8) to see that for $t < \tau^0(\delta) \wedge \tau_{D_\infty}^0(\delta)$,

$$0 \leq \alpha_t^{k m_0} + \delta \leq 2D_\infty < \pi \quad \text{and} \quad -\pi < -2D_\infty \leq \alpha_t^{k M_0} - \delta \leq 0, \quad k = 1, \cdots, N.$$

Now, we use the inequalities $|\sin x - \sin y| \le |x-y|$ to see

$$|\sin\alpha_t^{kM_0} - \sin(\alpha_t^{kM_0}-\delta)| \le \delta, \quad |\sin\alpha_t^{km_0} - \sin(\alpha_t^{km_0}+\delta)| \le \delta. \tag{9}$$

Then, we use (9) to see that for $t < \tau^0(\delta) \wedge \tau^0_{D_\infty}(\delta)$,

$$\begin{aligned}
&d\alpha_t^{M_0m_0}\\
&= \frac{2\kappa}{N}\sum_{k=1}^N \Big[\psi_t^{kM_0}\sin\alpha_t^{kM_0} - \psi_t^{km_0}\sin\alpha_t^{km_0}\Big]dt + 2\sqrt{2\sigma}d(B_t^{M_0}-B_t^{m_0})\\
&\le \frac{2\kappa}{N}\sum_{k=1}^N \Big[\psi_t^{kM_0}\Big(\sin(\alpha_t^{kM_0}-\delta)+\delta\Big) - \psi_t^{km_0}\Big(\sin(\alpha_t^{km_0}+\delta)-\delta\Big)\Big]dt\\
&+ 2\sqrt{2\sigma}d(B_t^{M_0}-B_t^{m_0})\\
&= \frac{2\kappa}{N}\sum_{k=1}^N \Big[-\psi_t^{kM_0}\Big(\sin(\alpha_t^{M_0k}+\delta)-\delta\Big) - \psi_t^{km_0}\Big(\sin(\alpha_t^{km_0}+\delta)-\delta\Big)\Big]dt\\
&+ 2\sqrt{2\sigma}d(B_t^{M_0}-B_t^{m_0})\\
&\le \frac{2\kappa}{N}\sum_{k=1}^N \Big[-\psi_t^{kM_0}\Big(R_\infty(\alpha_t^{M_0k}+\delta)-\delta\Big) - \psi_t^{km_0}\Big(R_\infty(\alpha_t^{km_0}+\delta)-\delta\Big)\Big]dt\\
&\quad + 2\sqrt{2\sigma}d(B_t^{M_0}-B_t^{m_0})\\
&\le \frac{2\kappa}{N}\sum_{k=1}^N \Big[2\delta\psi_M - \psi_m R_\infty(\alpha_t^{M_0m_0}+2\delta)\Big]dt + 2\sqrt{2\sigma}d(B_t^{M_0}-B_t^{m_0})\\
&= 2\kappa[2\delta\psi_M - \psi_m R_\infty(\alpha_t^{M_0m_0}+2\delta)]dt + 2\sqrt{2\sigma}d(B_t^{M_0}-B_t^{m_0}).
\end{aligned} \tag{10}$$

Here in the fourth equality, we used the relations:

$$\begin{aligned}
&R_\infty(\alpha_t^{M_0k}+\delta) := \frac{\sin 2D_\infty}{2D_\infty}(\alpha_t^{M_0k}+\delta) \le \sin(\alpha_t^{M_0k}+\delta),\\
&\text{and}\quad R_\infty(\alpha_t^{km_0}+\delta) \le \sin(\alpha_t^{km_0}+\delta).
\end{aligned}$$

Thus, relation (10) can be rewritten as

$$d(\alpha_t^{M_0m_0}+2\delta) \le 2\kappa[2\delta\psi_M - \psi_m R_\infty(\alpha_t^{M_0m_0}+2\delta)]dt + 2\sqrt{2\sigma}d(B_t^{M_0}-B_t^{m_0}). \tag{11}$$

We apply Ito's lemma to $(\alpha_t^{M_0m_0}+2\delta)e^{2\kappa\psi_m\bar{R}_\infty t}$ using (11) to see

$$d\Big((\alpha_t^{M_0m_0}+2\delta)e^{2\kappa\psi_m R_\infty t}\Big) \le 4\kappa\delta\psi_M e^{2\kappa\psi_m R_\infty t}dt + 2\sqrt{2\sigma}e^{2\kappa\psi_m R_\infty t}d(B_t^{M_0}-B_t^{m_0}).$$

This yields that for $t \le \tau^0(\delta)\wedge\tau^0_{D_\infty}$,

$$\frac{1}{2}\Big(\alpha_t^{M_0m_0}+2\delta\Big) \le \frac{1}{2}\Big(\alpha_0^{M_0m_0}+2\delta\Big)e^{-2\kappa\psi_m R_\infty t} + \frac{\delta\psi_M}{\psi_m R_\infty}(1-e^{-2\kappa\psi_m R_\infty t}) + \sqrt{2\sigma}\tilde{Z}_t^{(0)},$$

where the O-U process $\tilde{Z}_t^{(0)}$ is given by the following relation:

$$\tilde{Z}_t^{(0)} := \int_0^t e^{-2\kappa\psi_m R_\infty(t-s)}d(B_s^{M_1}-B_s^{m_1}).$$

Next, we also define the zeroth barrier function and stopping times:

$$\begin{aligned} &L_0(s) := (\mathcal{L}(\Theta_0)+\delta)e^{-2\kappa\psi_m R_\infty s} + \frac{\delta\psi_M}{\psi_m R_\infty}(1-e^{-2\kappa\psi_m R_\infty s}) + \delta, \\ &\tau^*_{L_0} := \inf\left\{ s>0 : \frac{1}{2}\alpha_s^{M_0 m_0} + \delta > L_0(s) \right\}, \\ &\tau_{L_0} := \inf\left\{ s>0 : \mathcal{L}(\Theta_s) > L_0(s) \right\}. \end{aligned} \tag{12}$$

Below, we provide some stochastic estimates for $\tilde{Z}_t^{(0)}$ and τ_{L_0}. For notational simplicity, we set

$$P_1 := A\left(\frac{\delta}{\sqrt{2\sigma}}, 2\kappa\psi_m R_\infty, T_\delta\right)e^{-\frac{\kappa\psi_m R_\infty \delta^2}{2\sigma}}, \qquad P_2 := 4N\sqrt{\frac{32\sigma}{\pi\delta\kappa C_\delta}}e^{-\frac{\delta\kappa C_\delta}{32\sigma}}.$$

Lemma 5.1. *Suppose that the coupling strength κ is sufficiently large such that*

$$r\left(\delta\sqrt{\frac{\kappa\psi_m R_\infty}{\sigma}}, 2\kappa\psi_m R_\infty T_\delta\right) \le r_0.$$

Then, we have

$$\begin{aligned} &(i)\ \mathbb{P}\left\{ \sup_{0\le t\le T_\delta} \sqrt{2\sigma}\tilde{Z}_t^{(0)} > \delta \right\} \le P_1, \\ &(ii)\ \mathbb{P}\left\{ \tau_{L_0} < T_\delta \right\} \le P_1 + P_2 =: P_\infty. \end{aligned}$$

Proof. (i) We choose κ sufficiently large such that

$$r(\delta\sqrt{\kappa\psi_m R_\infty/\sigma}, 2\kappa\psi_m R_\infty T_\delta) \le r_0.$$

Then, we have

$$\mathbb{P}\left\{ \sup_{0\le t\le T_\delta} \sqrt{2\sigma}\tilde{Z}_t^{(0)} > \delta \right\} \le A(\delta/\sqrt{2\sigma}, 2\kappa\psi_m R_\infty, T_\delta)e^{-\frac{\kappa\psi_m R_\infty\delta^2}{2\sigma}} = P_1.$$

(ii) Note that the relations:

$$\mathcal{L}(\Theta_0) = \frac{1}{2}\alpha_0^{M_0 m_0} \quad \text{and} \quad \mathcal{L}(\Theta_t) \le \frac{1}{2}\alpha_t^{M_0 m_0} + \delta \quad \text{for } t \le \tau^0(\delta),$$

imply

$$\begin{aligned} &\mathbb{P}\left\{ \exists s \le T_\delta \wedge \tau^0(\delta) \wedge \tau^0_{D_\infty} : \frac{1}{2}\alpha_s^{M_0 m_0} + \delta > L_0(s) \right\} \\ &\qquad\qquad \le \mathbb{P}\left\{ \sup_{0\le t\le T} \sqrt{2\sigma}\tilde{Z}_t^{(0)} > \delta \right\} \le P_1. \end{aligned} \tag{13}$$

Next, we bound the probability for the event $\{\tau_{L_0} < T_\delta\}$ as follows.

$$\begin{aligned} &\mathbb{P}\left\{\tau_{L_0} < T_\delta\right\} \le \mathbb{P}\left\{\tau_{L_0} \wedge \tau^0(\delta) < T_\delta\right\} \le \mathbb{P}\left\{\tau^*_{L_0} \wedge \tau^0(\delta) < T_\delta\right\} \\ &\le \mathbb{P}\left\{\tau^*_{L_0} \le T_\delta \wedge \tau^0(\delta) \wedge \tau^0_{D_\infty}\right\} + \mathbb{P}\left\{\tau^0(\delta) < T_\delta\right\} + \mathbb{P}\left\{\tau^0_{D_\infty} \le T_\delta \wedge \tau^0(\delta) \wedge \tau^*_{L_0}\right\} \\ &=: \mathcal{I}_{11} + \mathcal{I}_{12} + \mathcal{I}_{13}. \end{aligned} \tag{14}$$

Here we used the following relations:

(1) $\tau_{L_0} \wedge \tau^0(\delta) \le \tau_{L_0}$: This is clear.

(2) $\tau^*_{L_0} \wedge \tau^0(\delta) \le \tau_{L_0} \wedge \tau^0(\delta)$: This is clear when $\tau_{L_0} \ge \tau^0(\delta)$.
If $\tau_{L_0} < \tau^0(\delta)$, the set

$$\big\{ t > 0 : \mathcal{L}(\Theta_t) > L_0(t) \text{ and } t < \tau^0(\delta) \big\}$$

is nonempty and contained in

$$\left\{ t > 0 : \frac{1}{2}\alpha_t^{M_0 m_0} + \delta > L_0(t) \text{ and } t < \tau^0(\delta) \right\},$$

since $\mathcal{L}(\Theta_t) \le \frac{1}{2}\alpha_t^{M_0 m_0} + \delta$ for $t \le \tau^0(\delta)$.
Therefore, we have

$$\begin{aligned}\tau^*_{L_0} &= \inf \left\{ t > 0 : \frac{1}{2}\alpha_t^{M_0 m_0} + \delta > L_0(t) \text{ and } t < \tau^0(\delta) \right\} \\ &\le \inf \big\{ t > 0 : \mathcal{L}(\Theta_t) > L_0(t) \text{ and } t < \tau^0(\delta) \big\} = \tau_{L_0},\end{aligned}$$

and

$$\tau^*_{L_0} \wedge \tau^0(\delta) = \tau^*_{L_0} \le \tau_{L_0} = \tau_{L_0} \wedge \tau^0(\delta).$$

(3) The last inequality comes from the following relations:

$$\begin{aligned}\big\{\tau^*_{L_0} \wedge \tau^0(\delta) < T_\delta\big\} - \big\{\tau^0(\delta) < T_\delta\big\} &= \big\{\tau^*_{L_0} < T_\delta \le \tau^0(\delta)\big\} \subset \big\{\tau^*_{L_0} < T_\delta \wedge \tau^0(\delta)\big\} \\ &\subset \Big(\big\{\tau^*_{L_0} \le T \wedge \tau^0(\delta) \wedge \tau^0_{D_\infty}\big\} \cup \big\{\tau^0_{D_\infty} \le T_\delta \wedge \tau^0(\delta) \wedge \tau^*_{L_0}\big\} \Big).\end{aligned}$$

Below, we estimate the terms $\mathcal{I}_{1i},\ i = 1, 2, 3$ separately.

◇ (Estimate of $\mathcal{I}_{11}$): We use (13) to get

$$\mathcal{I}_{11} \le P_1.$$

◇ (Estimate of $\mathcal{I}_{12}$): We use Lemma 4.1 to obtain

$$\mathcal{I}_{12} \le 4N\sqrt{\frac{32\sigma}{\pi\delta\kappa C_\delta}}\, e^{-\frac{\delta\kappa C_\delta}{32\sigma}} =: P_2.$$

◇ (Estimate of $\mathcal{I}_{13}$): In this case, it follows from $(4)_1$ and (12) that

$$\begin{aligned}L_0(s) &= (\mathcal{L}(\Theta_0) + \delta)e^{-2\kappa\psi_m R_\infty s} + \frac{\delta\psi_M}{\psi_m R_\infty}(1 - e^{-2\kappa\psi_m R_\infty s}) + \delta \\ &\le \max\left\{\mathcal{L}(\Theta_0) + \delta, \frac{\delta\psi_M}{\psi_m R_\infty}\right\} + \delta < D_\infty, \qquad \forall s \le T_\delta.\end{aligned}$$

This implies

$$\mathcal{I}_{13} = 0.$$

Finally, in (14), we combine all the estimates $\mathcal{I}_{1i},\ i = 1, 2, 3$ to derive the desired estimate. □

5.2.2. *Step B (iterative time-zone estimates).* In this part, we consider the time interval $[(\ell-1)T_\delta, \ell T_\delta]$ to see how the iterative estimate works. First, we set $\ell = 2$. If we assume that τ_{L_0} is larger than or equal to T_δ, then at the instant T_δ, we have

$$\mathcal{L}(\Theta_{T_\delta}) \le L_0(T_\delta) = (\mathcal{L}(\Theta_0) + \delta)e^{-2\kappa\psi_m R_\infty T_\delta} + \frac{\delta\psi_M}{\psi_m R_\infty}(1 - e^{-2\kappa\psi_m R_\infty T_\delta}) + \delta < D_\infty.$$

From the data at T_δ, we can define new indices M_1 and m_1 satisfying

$$\mathcal{L}(\Theta_{T_\delta}) := \frac{1}{2}\Big(\alpha_{T_\delta}^{M_1} - \alpha_{T_\delta}^{m_1}\Big).$$

We define the first barrier function $L_1(s)$ as follows:

$$\begin{aligned} L_1(s) &:= L_0(s)\chi_{(0,T_\delta]} \\ &\quad + \left[(L_0(T_\delta)+\delta)e^{-2\kappa\psi_m R_\infty(s-T_\delta)} + \frac{\delta\psi_M}{\psi_m R_\infty}(1-e^{-2\kappa\psi_m R_\infty(s-T_\delta)}) + \delta\right]\chi_{(T_\delta,\infty)} \\ &= L_0(s) + \chi_{(T_\delta,\infty)}\cdot 2\delta e^{-2\kappa\psi_m R_\infty(s-T_\delta)}, \end{aligned}$$

where we used the defining relation of $L_0(T_\delta)$ in the second identity.

Next, we define four new stopping times analogous to (3) - (6):

$$\begin{aligned} \tau^1(\delta) &:= \inf\left\{s > T_\delta : \alpha_s^i \notin [\alpha_s^{m_1}-\delta, \alpha_s^{M_1}+\delta] \quad \text{for some } i\right\}, \\ \tau^1_{D_\infty} &:= \inf\left\{s > T_\delta : \frac{1}{2}\alpha_s^{M_1m_1} + \delta > D_\infty\right\}, \\ \tau^*_{L_1} &:= \inf\left\{s > T_\delta : \frac{1}{2}\alpha_s^{M_1m_1} + \delta > L_1(s)\right\}, \\ \tau_{L_1} &:= \inf\left\{s > 0 : D_{\mathbb{T}}(\alpha_s) > L_1(s)\right\}. \end{aligned}$$

For $T_\delta \le t \le \tau^1(\delta)\wedge\tau^1_{D_\infty}$, we can formulate the copy of (10):

$$d(\alpha_t^{M_1m_1}+2\delta) \le 2\kappa[2\delta\psi_M - \psi_m R_\infty(\alpha_t^{M_1m_1}+2\delta)]dt + 2\sqrt{2\sigma}d(B_t^{M_1}-B_t^{m_1}).$$

By the same argument as in Section 5.2.1, we have

$$\frac{1}{2}\Big(\alpha_t^{M_1m_1}+2\delta\Big) \le \frac{1}{2}(\alpha_{T_\delta}^{M_1m_1}+2\delta)e^{-2\kappa\psi_m R_\infty(t-T_\delta)} + \frac{\delta\psi_M}{\psi_m R_\infty}\Big(1-e^{-2\kappa\psi_m R_\infty(t-T_\delta)}\Big) + \sqrt{2\sigma}\tilde{Z}_t^{(1)},$$

where the O-Z process $\tilde{Z}_t^{(1)}$ is given by

$$\tilde{Z}_t^{(1)} := \int_0^{t-T_\delta} e^{-2\kappa\psi_m R_\infty(t-T_\delta-s)}d(B^{M_1}_{s+T_\delta} - B^{m_1}_{s+T_\delta}).$$

Similar to Lemma 5.1, we have the following lemma.

Lemma 5.2. *The following estimates hold.*

(i) $\mathbb{P}\left\{\sup_{T_\delta\le s\le 2T_\delta}\sqrt{2\sigma}\tilde{Z}_s^{(1)} > \delta\right\} \le P_1.$

(ii) $\mathbb{P}\left\{\exists s : T_\delta < s \le 2T_\delta\wedge\tau^1(\delta)\wedge\tau^1_{D_\infty},\ \frac{1}{2}\alpha_s^{M_1m_1}+\delta > L_1(s) \,\middle|\, \tau_{L_0}\ge T_\delta\right\} \le P_1.$

Proof. (i) The first estimate can be done as in Lemma 5.1:

$$\mathbb{P}\left\{\sup_{T_\delta\le t\le 2T_\delta}\sqrt{2\sigma}\tilde{Z}_t^{(1)} > \delta\right\} \le A(\delta/\sqrt{2\sigma}, 2\kappa\psi_m R_\infty, T_\delta)e^{-\frac{\kappa\psi_m R_\infty\delta^2}{2\sigma}} = P_1.$$

(ii) We use the first estimate to get

$$\begin{aligned} &\mathbb{P}\left\{\exists s : T_\delta < s \le 2T_\delta\wedge\tau^1(\delta)\wedge\tau^1_{D_\infty},\ \frac{1}{2}\alpha_s^{M_1m_1}+\delta > L_1(s) \,\middle|\, \tau_{L_0}\ge T_\delta\right\} \\ &\qquad \le \mathbb{P}\left\{\sup_{T_\delta\le s\le 2T_\delta}\sqrt{2\sigma}\tilde{Z}_s^{(1)} > \delta\right\} \le P_1. \end{aligned} \tag{15}$$

□

Therefore, we get the required estimations for $[T_\delta, 2T_\delta]$. For the general step, $[(\ell-1)T_\delta, \ell T_\delta]$, we may proceed similar estimates by considering desired data at the time $(\ell-1)T_\delta$.

Suppose that $\ell T_\delta \le \tau_{L_{\ell-1}}$, and set indices M_ℓ and m_ℓ such that

$$\mathcal{L}(\Theta_{\ell T_\delta}) = \frac{1}{2}\left(\alpha_{\ell T_\delta}^{M_\ell} - \alpha_{\ell T_\delta}^{m_\ell}\right),$$

and introduce ℓ-th barrier function and ℓ-th stopping times as follows:

$$\begin{aligned}
&\tau_{D_\infty}^{\ell} := \inf\left\{ s > \ell T_\delta : \frac{1}{2}\alpha_s^{M_\ell m_\ell} + \delta > D_\infty \right\},\\
&\tau^\ell(\delta) := \inf\left\{ s > \ell T_\delta : \alpha_s^i \notin \left[\alpha_s^{m_\ell} - \delta,\ \alpha_s^{M_\ell} + \delta\right] \text{ for some } i \right\},\\
&L_\ell(s) := L_{\ell-1}(s)\chi_{(0,\ell T_\delta]}\\
&\quad + \left[(L_{\ell-1}(\ell T_\delta) + \delta)e^{-2\kappa\psi_m R_\infty(s-\ell T_\delta)} + \frac{\delta\psi_M}{\psi_m R_\infty}(1 - e^{-2\kappa\psi_m R_\infty(s-\ell T_\delta)}) + \delta\right]\chi_{(\ell T_\delta,\infty)}\\
&\qquad = L_{\ell-1}(s) + 2\delta e^{-2\kappa\psi_m R_\infty(s-\ell T_\delta)}\chi_{(\ell T_\delta,\infty)}\\
&\qquad \vdots\\
&\qquad = L_0(s) + 2\delta\sum_{r=1}^{\ell} e^{-2\kappa\psi_m R_\infty(s-rT_\delta)}\chi_{(rT_\delta,\infty)},\\
&\tau_{L_\ell}^* := \inf\left\{ s > \ell T_\delta : \frac{1}{2}(\alpha_s^{M_\ell m_\ell} + 2\delta) > L_\ell(s) \right\},\\
&\tau_{L_\ell} := \inf\left\{ s > 0 : \mathcal{L}(\Theta_s) > L_\ell(s) \right\}.
\end{aligned}$$

For $\ell T_\delta \le t \le \tau^\ell(\delta) \wedge \tau_{D_\infty}^\ell$, we have

$$\frac{1}{2}(\alpha_t^{M_\ell m_\ell} + 2\delta) \le \frac{1}{2}(\alpha_{\ell T_\delta}^{M_\ell m_\ell} + 2\delta)e^{-2\kappa\psi_m R_\infty(t-\ell T_\delta)} + \frac{\delta\psi_M}{\psi_m R_\infty}(1 - e^{-2\kappa\psi_m R_\infty(t-\ell T_\delta)}) + \sqrt{2\sigma}\tilde{Z}_t^{(\ell)},$$

where the O-U process $\tilde{Z}_t^{(\ell)}$ is given as

$$\tilde{Z}_t^{(\ell)} := \int_0^{t-\ell T_\delta} e^{-2\kappa\psi_m R_\infty(t-\ell T_\delta - s)} d(B_{s+\ell T_\delta}^{M_\ell} - B_{s+\ell T_\delta}^{m_\ell}).$$

Then, we have

$$\begin{aligned}
&\mathbb{P}\left\{ \exists s : \ell T_\delta < s \le (\ell+1)T_\delta \wedge \tau^\ell(\delta) \wedge \tau_{D_\infty}^\ell,\ \frac{1}{2}\alpha_s^{M_\ell m_\ell} + \delta > L_\ell(s) \middle| \tau_{L_{\ell-1}} \ge \ell T_\delta \right\}\\
&\qquad = \mathbb{P}\left\{ \tau_{L_\ell}^* < (\ell+1)T_\delta \wedge \tau^\ell(\delta) \wedge \tau_{D_\infty}^\ell \middle| \tau_{L_{\ell-1}} \ge \ell T_\delta \right\},\\
&\mathbb{P}\left\{ \tau_{L_\ell}^* < (\ell+1)T_\delta \wedge \tau^\ell(\delta) \wedge \tau_{D_\infty}^\ell \middle| \tau_{L_{\ell-1}} \ge \ell T_\delta \right\} \le \mathbb{P}\left\{ \sup_{\ell T_\delta \le t \le (\ell+1)T_\delta} \sqrt{2\sigma}\tilde{Z}_t^{(\ell)} > \delta \right\} \le P_1.
\end{aligned}$$

5.2.3. *Derivation of the recursive inequality.* Now, we are ready to derive the recursive relation (6). First, we define a global barrier function and stopping time:

$$\begin{aligned}
L(s) &:= L_0(s) + 2\delta\sum_{r=1}^{\infty} e^{-2\kappa\psi_m R_\infty(s-rT_\delta)}\chi_{(rT_\delta,\infty)},\\
\tau_L &:= \inf\left\{ s > 0 : \mathcal{L}(\Theta_s) > L(s) \right\}.
\end{aligned}$$

Note that, for any integer ℓ, we have

$$L(s) = L_\ell(s) \quad \text{for } 0 \le s \le (\ell+1)T_\delta,$$

so that we have the equivalence between events:

$$\{\tau_L < (\ell+1)T_\delta\} \Longleftrightarrow \{\tau_{L_\ell} < (\ell+1)T_\delta\}. \tag{16}$$

Therefore, we get

$$\begin{aligned}
&\mathbb{P}\{\tau_L < (\ell+1)T_\delta\} \\
&\quad = \mathbb{P}\{\tau_L < \ell T_\delta\} + \mathbb{P}\{\ell T_\delta \le \tau_L < (\ell+1)T_\delta\} \\
&\quad = \mathbb{P}\{\tau_L < \ell T_\delta\} + \underbrace{\mathbb{P}\left\{\tau_L < (\ell+1)T_\delta \middle| \tau_L \ge \ell T_\delta\right\}}_{=:\Delta} \mathbb{P}\{\tau_L \ge \ell T_\delta\}.
\end{aligned} \tag{17}$$

In the above equality, the conditional probability Δ can be bounded using (16) as follows.

(18)

$$\begin{aligned}
\Delta &= \mathbb{P}\left\{\tau_{L_\ell} < (\ell+1)T_\delta \middle| \tau_{L_{\ell-1}} \ge \ell T_\delta\right\} \\
&\le \mathbb{P}\left\{\tau_{L_\ell} \wedge \tau^\ell(\delta) < (\ell+1)T_\delta \middle| \tau_{L_{\ell-1}} \ge \ell T_\delta\right\} \le \mathbb{P}\left\{\tau^*_{L_\ell} \wedge \tau^\ell(\delta) < (\ell+1)T_\delta \middle| \tau_{L_{\ell-1}} \ge \ell T_\delta\right\} \\
&\le \mathbb{P}\left\{\tau^*_{L_\ell} \le (\ell+1)T_\delta \wedge \tau^\ell(\delta) \wedge \tau^\ell_{D_\infty} \middle| \tau_{L_{\ell-1}} \ge \ell T_\delta\right\} + \mathbb{P}\left\{\tau^\ell(\delta) < (\ell+1)T_\delta \middle| \tau_{L_{\ell-1}} \ge \ell T_\delta\right\} \\
&\quad + \mathbb{P}\left\{\tau^\ell_{D_\infty} \le (\ell+1)T_\delta \wedge \tau^*_{L_\ell} \wedge \tau^\ell(\delta) \middle| \tau_{L_{\ell-1}} \ge \ell T_\delta\right\} \\
&=: \mathcal{I}_{21} + \mathcal{I}_{22} + \mathcal{I}_{23}.
\end{aligned}$$

Next, we estimate the terms $\mathcal{I}_{2i}$, $i = 1, 2, 3$, one by one.

$\diamond$ (Estimate of $\mathcal{I}_{2i}$, $i = 1, 2$): We use similar arguments as in (15) or Lemma 5.1 to get

$$\mathcal{I}_{21} \le P_1, \quad \mathcal{I}_{22} \le P_2. \tag{19}$$

$\diamond$ (Estimate of $\mathcal{I}_{23}$): For $\mathcal{I}_{23}$, we use the condition (3) in (4) to deduce

$$\mathcal{I}_{23} = 0,$$

where we used the relation:

$$L_\ell(s) \le L(s) \le \max\left\{\mathcal{L}(\Theta_0) + 2\delta,\ \frac{\delta\psi_M}{\psi_m R_\infty} + \delta + \frac{2\delta}{1 - e^{-\frac{\psi_m R_\infty \delta}{2C_\delta}}}\right\} < D_\infty.$$

Finally, we combine (17)-(19) to obtain

(20)

$$\begin{aligned}
\mathbb{P}\{\tau_L < (\ell+1)T_\delta\} &= \mathbb{P}\{\tau_L < \ell T_\delta\} + \mathbb{P}\left\{\tau_L < (\ell+1)T_\delta \middle| \tau_L \ge \ell T_\delta\right\} \mathbb{P}\{\tau_L \ge \ell T_\delta\} \\
&\le \mathbb{P}\{\tau_L < \ell T_\delta\} + P_\infty \mathbb{P}\{\tau_L \ge \ell T_\delta\},
\end{aligned}$$

or equivalently, we get (7):

$$\mathbb{P}\{\tau_L \ge (\ell+1)T_\delta\} \ge (1 - P_\infty)\mathbb{P}\{\tau_L \ge \ell T_\delta\}.$$

From the induction on ℓ and Lemma 5.1, we conclude Theorem 5.1:

$$\mathbb{P}\left\{\tau_L \geq \ell T_\delta\right\} \geq (1-P_\infty)^{\ell-1}\mathbb{P}\left\{\tau_L \geq T_\delta\right\} \geq (1-P_\infty)^{\ell}.$$

Remark 5.2. *For completeness, we need to check whether the condition* (4) *can be achievable. If we take a limit* $\delta \to 0$*, the left-hand side of condition* (4)*(i) becomes*

$$\max\left\{\mathcal{L}(\Theta_0), \frac{8D_\infty(\psi_M-\psi_m)}{\psi_m \sin 2D_\infty}\right\}.$$

Therefore, if $\mathcal{L}(\Theta_0) < D_\infty$ *and* $\frac{\psi_M}{\psi_m} < 1+\frac{\sin 2D_\infty}{8}$*, there exists a positive* δ *satisfying condition* (4)*(i). In addition, we require* $\frac{\delta}{\sqrt{2\sigma}} \gg 1$ *and* $\kappa \gg 1$ *for* (4)*(ii). The probability estimate* P_∞ *also gets meaningful values (that is,* $P_\infty < 1$*) only if* $\frac{\delta}{\sqrt{2\sigma}}$ *and* κ *are sufficiently large for fixed* T_δ*.*

6. Numerical simulations

In this section, we present several numerical simulations to the deterministic and stochastic models that demonstrate the ability of the models for simulating both alignment and nematic alignment. We here used the forward Euler method to the time evolution, and the stochastic model is simulated using the Box-Muller transform to generate the noise terms at each time step, i.e.,

$$\text{The noise term } = \sqrt{2\sigma\Delta t}\xi^j,$$

where ξ^j is the random number given by the Box-Muller transform, and Δt is the time step. For the communication weight function $\psi(s)$, we employ the following Cucker-Smale type:

$$\psi_{CS}(s) := \frac{1}{(1+s^2)^\beta} \quad \text{for } \beta = \frac{1}{2} \text{ or } 1.$$

We assume all the particles move with a constant speed $v_0 = 1$ inside a square domain $[0, 10] \times [0, 10]$, and periodic boundary conditions are imposed. Initially, all particle's positions X^j's are uniformly distributed in the domain. We employed the following parameters for simulations:

$$N = 500, \quad \kappa = 20 \quad \text{and} \quad \beta = \frac{1}{2} \text{ or } 1,$$

where some of them adopt different parameters to perform detailed analysis, which will be specified in the context. Moreover, the moving orientations at $t = 0$ are tuned to serve different purposes of the simulations.

Each simulation is characterized by two pairs of order parameters (R_1, ϕ_1) and (R_2, ϕ_2) introduced in Section 2:

$$R_\ell(t)e^{\mathrm{i}\ell\phi_\ell(t)} = \frac{1}{N}\sum_{k=1}^N e^{\mathrm{i}\ell\theta_t^k}.$$

Note that $R_1 \approx 1$ and $R_2 \approx 1$ indicates that the heading angle configuration Θ is close to alignment and nematic alignment, respectively. In the following figures, the temporal profiles of $\big(R_1, \phi_1\big)$ are shown with blue circles, and $\big(R_2, \phi_2\big)$ with red solid lines.

6.1. **Deterministic models.** In this subsection, we consider the deterministic model given by the system (1). To simulate the alignment, the initial data are given by a group of randomly located particles moving upwards, i.e. their orientation angles θ_0^j are contained in a quarter circle centered around $\frac{\pi}{2}$. Similarly, to simulate the nematic alignment, the initial orientation angles θ_0^j are contained in two quarter circles centered around $\frac{\pi}{2}$ and $\frac{3\pi}{2}$, respectively.

Fig. 1 - Fig. 2 show the simulations of the alignment with $\beta = \frac{1}{2}$ and 1, respectively. The left figures show the order parameters $\big(R_\ell, \phi_\ell\big)$ for $\ell = 1$ (blue circles) and 2 (red solid lines). Here, the order parameters R_1 and R_2 both converge to 1 due to the absence of noise. Meanwhile, the diameter $D(\Theta)$ and $D(2\Theta)$ converge to zero, as seen in the middle figures. The right figures show the Lyapunov function $\mathcal{L}(\Theta_t)$ (green crosses) and the upper bound (red solid lines) given by Theorem 3.1.

Since we considered a bounded (periodic) domain here, the interaction ψ_{CS} has a positive lower bound:

$$\psi_{CS}(\|x^i - x^j\|) \geq \frac{1}{(1 + (10\sqrt{2})^2)^\beta} = \frac{1}{201^\beta}.$$

This lower bound for ψ is used to make an upper bound (red solid lines) in the right figure of Fig. 1 - Fig. 2.

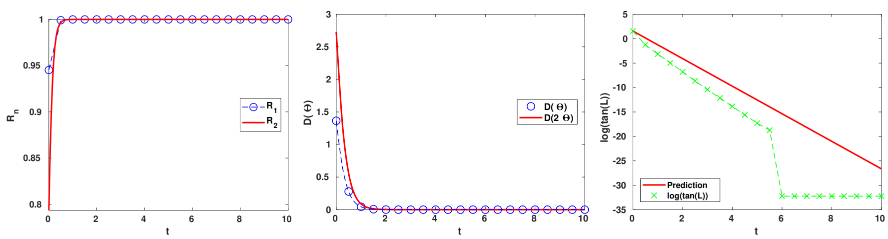

FIGURE 1. The deterministic simulation of the alignment with $\beta = \frac{1}{2}$.

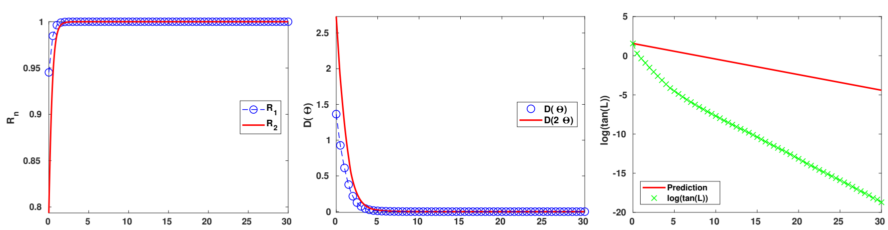

FIGURE 2. The deterministic simulation of the alignment with $\beta = 1$.

Fig. 3 - Fig. 4 show the deterministic simulations of the nematic alignment. R_2 converge to 1 in the left figures and $D(2\Theta)$ converge to 0 in the middle figures, both plotted with red solid lines. The two sets of order parameters indicates the formulation of the nematic alignment. From the time evolution of the Lyapunov functions in the right figures, we can observe the asymptotic vanishing behavior of $\mathcal{L}(\Theta_t)$ for both $\beta = 1$ and $\beta = \frac{1}{2}$.

On the other hand, the integral of ψ_{CS} for $\beta = 1$ and $\beta = \frac{1}{2}$ are given as follows:

$$\int_a^b \psi_{CS}(s)\,ds = \begin{cases} \arctan b - \arctan a & (\beta = 1) \\ \log \frac{b+\sqrt{1+b^2}}{a+\sqrt{1+a^2}}, & (\beta = \frac{1}{2}) \end{cases}.$$

Hence, Remark 3.2 assures that $\mathcal{L}(\Theta)$ vanishes asymptotically when $\beta = \frac{1}{2}$ and the particles are moving in unbounded domain, while it is not clear for $\beta = 1$. Still, since we considered a bounded periodic domain for x variables, the configuration Θ shows a nematic alignment as in Fig. 2.

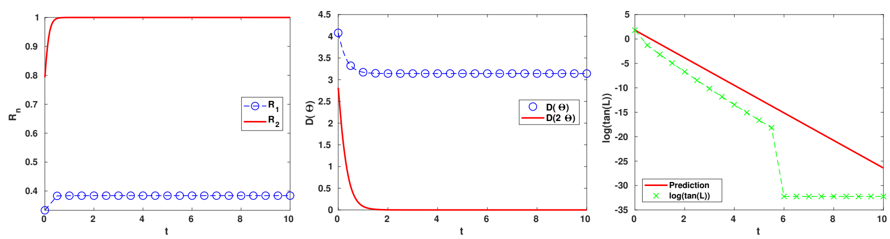

FIGURE 3. The deterministic simulation of the nematic alignment with $\beta = \frac{1}{2}$.

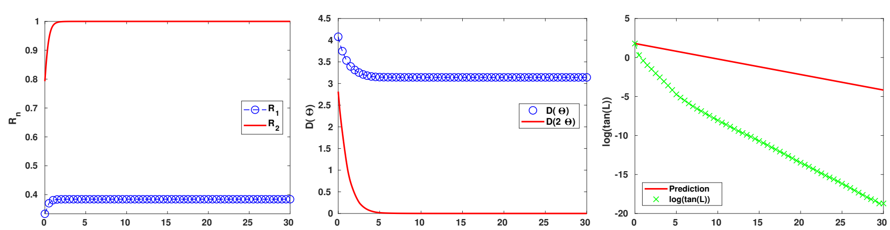

FIGURE 4. The deterministic simulation of the nematic alignment with $\beta = 1$.

6.2. **Stochastic models.** In this subsection, we consider the stochastic model given by the system (1). We here assume that the particles are randomly distributed in space, and $\beta = \frac{1}{2}$ in the communicate weight function $\psi_{CS}(s)$. Then, we compare the dynamics of the models with different values of σ in the noise term. For each σ, the average of 20 simulations resulting from the same initial data is taken and plotted in Fig. 5 - 6. One can observe that the development of the ordering states is closely related to the values of σ. When $\sigma = 0.5$, both tests become disordered with $R_1 \approx 0$ or $R_2 \approx 0$. We used a cutoff $\mathcal{L}(\Theta) < 0.4998\pi$ for this large σ to plot the right figures of Fig. 5 - 6, since $\log(\tan(\mathcal{L}(\Theta)))$ is not well defined after $\mathcal{L}(\Theta) \geq \frac{\pi}{2}$. Smaller values of σ resulted to alignment or nematic alignment, and these empirical average of Lyapunov functions $\mathcal{L}(\Theta_t)$ oscillate from some positive equilibrium due to the presence of noise.

Note that the barrier function $L(s)$ is close to $\mathcal{L}(\Theta_0)$ for all time, but this 'expected' $\mathcal{L}(\Theta)$ becomes significantly smaller than its intial value. From this point of view, it is reasonable that the probability of the event

$$\left\{ \mathcal{L}(\Theta_s) \text{ exceeds } L(s) \text{ at least once for } s \in [0,t] \right\}$$

is small for each fixed time t.

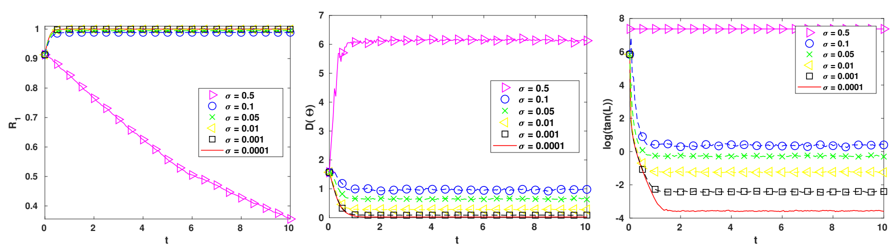

FIGURE 5. The stochastic simulation of the alignment with various σ s.

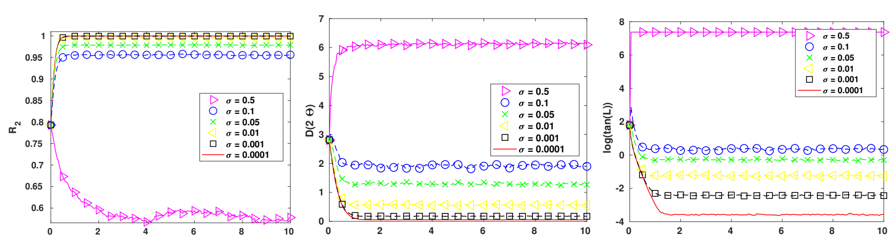

FIGURE 6. The stochastic simulation of the nematic alignment with various σ s.

7. Conclusion

In this paper, we provided a mathematical model for nematic alignment and its emergent dynamics in a random environment. In the same spirit of the Vicsek model, the generalized Justh-Krishnaprasad model describes the collective behavior of particles with unit speed in a planar domain.

In the absence of noise, the deterministic model describes the formation of nematic alignment asymptotically. In contrast, when there is the noise effect, we can calculate some upper bound of probability that the nematic alignment breaks down. In detail, the fluctuation of heading angles mainly follow the decreasing property, but the probability of escaping nematic state also grows in time. In principle, our modeling methodology and analysis can be applied to other phenomena consisting of multiple clusters, although our analysis is restricted to the bi-polar configurations.

Of course, there are several issues that were not discussed in our paper. For example, our analytical framework deals with the stability of nematic alignment by restricting the case that the initial configurations is initially close to the nematic alignment states. Thus, a natural question is whether we can verify our analysis for generic initial data or not. Numerical simulations suggest that this might be true. Moreover, in our probabilistic estimate, we assume that the communication weight has a positive lower bound, which simplifies synchronization estimates. We leave these interesting issues for a future work.

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