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Xuecheng WANG

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Annales Scientifiques de l'École Normale Supérieure,
45, rue d'Ulm, 75230 Paris Cedex 05, France.
Tél. : (33) 1 44 32 20 88. Fax : (33) 1 44 32 20 80.
annales@ens.fr

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Société Mathématique de France
Case 916 - Luminy
13288 Marseille Cedex 09
Tél. : (33) 04 91 26 74 64
Fax : (33) 04 91 41 17 51
email : abonnements@smf.emath.fr

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GLOBAL REGULARITY FOR THE 3D FINITE DEPTH CAPILLARY WATER WAVES

BY XUECHENG WANG

ABSTRACT. – In this paper, we prove global regularity, scattering, and the non-existence of small traveling waves for the $3D$ capillary waves system in the flat bottom setting for smooth localized small initial data.

To construct global solutions, we highly exploit the symmetric structures inside the capillary waves system and control both a low order weighted norm and a high order weighted norm of the profile of a good substitution variable over time to show that, although the nonlinear solution itself doesn't decay sharply at rate $1/(1+t)$ over time, the "$1+\alpha$" derivatives of the nonlinear solution indeed decays sharply, where α is some fixed positive number.

RÉSUMÉ. – Dans cet article, on démontre la régularité globale, la dispersion des solutions et la non-existence des petites ondes progressives pour un système d'equations des ondes capillaires en dimension 3 avec des petites données initiales régulières et localisées, dans le cas des fonds flats.

Pour construire des solutions globales, on exploite les structures symétriques du système d'ondes capillaires et contrôle à la fois les évolutions des deux normes avec poids du profil d'une bonne variable substitutive, l'une d'ordre petit et l'autre d'ordre grand. En conséquence, on montre que les dérivées d'ordre $1+\alpha$ de la solution non-linéaire décroissent rapidement au taux de $1/(1+t)$, bien que la solution elle-même ne décroît pas aussi rapidement, où α est un nombre positif fixé.

1. Introduction

1.1. The set-up of problem and previous results

We study the evolution of a constant density irrotational inviscid fluid, e.g., water, inside a time dependent domain $\Omega(t) \subset \mathbb{R}^3$, which has a fixed flat bottom Σ and a moving interface $\Gamma(t)$. Above the water region $\Omega(t)$ is vacuum. We neglect the gravity effect and only consider the surface tension effect in this paper. The problem under consideration is also known as the capillary waves system.

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After normalizing the depth of $\Omega(t)$ to be "1," we can represent $\Omega(t)$, $\Gamma(t)$, and Σ in the Eulerian coordinates as follows,

$$\Omega(t) := \{(x, y) : x \in \mathbb{R}^2, -1 \le y \le h(t, x)\},$$
$$\Gamma(t) := \{(x, h(t, x) : x \in \mathbb{R}^2\}, \quad \Sigma := \{(x, -1) : x \in \mathbb{R}^2\},$$

where $h(t, x)$ represents the height of interface, which will be a small perturbation of zero.

Let "u" and "p" denote the velocity and the pressure of the fluid respectively. Then the evolution of fluid can be described by the free boundary Euler equation as follows,

$$\partial_t u + u \cdot \nabla u = -\nabla p, \quad \nabla \cdot u = 0, \quad \nabla \times u = 0, \quad \text{in } \Omega(t). \tag{1.1}$$

The free surface $\Gamma(t)$ moves with the normal component of the velocity according to the kinematic boundary condition as follows,

$$\partial_t + u \cdot \nabla \text{ is tangent to } \cup_t \Gamma(t).$$

The pressure p satisfies the Young-Laplace equation as follows,

$$p = \sigma H(h), \quad \text{on } \Gamma(t),$$

where "σ" denotes the surface tension coefficient, which will be normalized to be one, and $H(h)$ represents the mean curvature of the interface, which is given as follows,

$$H(h) = \nabla \cdot \Big(\frac{\nabla h}{\sqrt{1 + |\nabla h|^2}} \Big).$$

Lastly, the following Neumann type boundary condition holds on the bottom Σ,

$$u \cdot \vec{\mathbf{n}} = 0, \quad \text{on } \Sigma.$$

Because the bottom is assumed to be fixed, the fluid cannot go through the bottom. This explains why the above boundary condition holds.

Since the velocity field is irrotational, we can represent it in terms of a velocity potential ϕ. Let ψ be the restriction of the velocity potential on the boundary $\Gamma(t)$, more precisely, $\psi(t, x) := \phi(t, x,\, h(t, x))$. From the divergence free condition and the boundary conditions, we can derive the Laplace equation with two boundary conditions as follows,

$$(\partial_y^2 + \Delta_x)\phi = 0, \quad \frac{\partial \phi}{\partial \vec{\mathbf{n}}}\Big|_\Sigma = 0, \quad \phi|_{\Gamma(t)} = \psi. \tag{1.2}$$

Hence, we can reduce the study of the motion of fluid in $\Omega(t)$ to the study of the evolution of the height function "$h(t, x)$" and the restricted velocity potential "$\psi(t, x)$" as follows,

$$\begin{cases} \partial_t h = G(h)\psi, \\ \partial_t \psi = H(h) - \frac{1}{2}|\nabla \psi|^2 + \dfrac{(G(h)\psi + \nabla h \cdot \nabla \psi)^2}{2(1 + |\nabla h|^2)}, \end{cases} \tag{1.3}$$

where $G(h)\psi = \sqrt{1 + |\nabla h|^2}\, \mathcal{N}(h)\psi$ and $\mathcal{N}(h)\psi$ is the Dirichlet-Neumann operator at the interface $\Gamma(t)$. See e.g., [42] for the derivation of the system (1.3).

The capillary waves system (1.3) has the conserved energy and the conserved momentum as follows, see e.g., [7],

(1.4)
$$\mathcal{H}(h(t), \psi(t)) := \Big[\int_{\mathbb{R}^2} \frac{1}{2}\psi(t) G(h(t))\psi(t) + \frac{|\nabla h(t)|^2}{1 + \sqrt{1 + |\nabla h(t)|^2}} dx \Big] = \mathcal{H}(h(0), \psi(0)),$$

$$\int_{\mathbb{R}^2} h(t,x)dx = \int_{\mathbb{R}^2} h(0,x)dx. \tag{1.5}$$

From [34, Lemma 3.4], we know that

(1.6) (Flat bottom setting) :

$$\Lambda_{\le 2}[G(h)\psi] = |\nabla|\tanh|\nabla|\psi - \nabla\cdot(h\nabla\psi) - |\nabla|\tanh|\nabla|(h|\nabla|\tanh|\nabla|\psi),$$

$$\text{(Flat bottom setting)}: \quad \Lambda_{\le 2}[\partial_t\psi] = \Delta h - \frac{1}{2}|\nabla\psi|^2 + \frac{1}{2}(|\nabla|\tanh|\nabla|\psi)^2, \tag{1.7}$$

where $\Lambda_{\le 2}[\mathscr{N}]$ denotes the linear terms and the quadratic terms of the nonlinearity $\mathscr{N}$.

From the above Taylor expansions, in the small solution regime, the conserved Hamiltonian in (1.4) tells us that the L^2-norm of $(\nabla_x h, |\nabla|\sqrt{\tanh|\nabla|}\psi)$ doesn't change much over time. More precisely, the following approximation holds,

$$\frac{1}{4}\big(\|\nabla h(t)\|_{L^2}^2 + \||\nabla|P_{\le 1}[\psi(t)]\|_{L^2}^2 + \||\nabla|^{1/2}P_{\ge 1}[\psi(t)]\|_{L^2}^2\big) \le \mathscr{H}(h(t),\psi(t))$$

$$= \mathscr{H}(h(0),\psi(0)) \le 4\big(\|\nabla h(t)\|_{L^2}^2 + \||\nabla|P_{\le 1}[\psi(t)]\|_{L^2}^2 + \||\nabla|^{1/2}P_{\ge 1}[\psi(t)]\|_{L^2}^2\big). \tag{1.8}$$

There is an extensive literature on the study of the water waves system. Without being exhaustive, we only discuss some previous works here and refer readers to the references therein.

Previous results on the local existence of the water waves system. – Due to the quasilinear nature of the water waves systems, to obtain the local existence, it is very important to get around the losing derivatives issue. Early works of Nalimov [32] and Yosihara [41] considered the local well-posedness of the small perturbation of a flat interface such that the Rayleigh-Taylor sign condition holds. It was first discovered by Wu [37, 38] that the Rayleigh-Taylor sign condition holds without the smallness assumptions in the infinite depth setting. She showed the local existence for arbitrary size of initial data in Sobolev spaces. After the breakthrough of Wu's work, there are many important works devote to improve the understanding of local well-posedness of the full water waves system and the free boundary Euler equations. Christodoulou-Lindblad [10] and Lindblad [31] considered the gravity waves with vorticity. Beyer-Gunter [8] considered the effect of surface tension. Lannes [30] considered the finite depth setting. See also Shatah-Zeng [33], and Coutand-Shkoller [11]. It turns out that local well-posedness also holds even if the curvature of the interface is unbounded and the bottom is very rough even without regularity assumption, only a finite separation condition is required, see the works of Alazard-Burq-Zuily [1, 2] for more detailed and precise description of this result.

Previous results on the long time behavior of the water waves system. – The long time behavior of the water waves system is more difficult and challenging. To study the long time behavior, the low frequency part of the solution plays an essential role. It is very interesting to see that the water waves systems in different settings have very different behavior at the low frequency part. Even for a small perturbation of static solution and flat interface, we only have few results so far. Note that, it is possible to develop the so-called "splash-singularity" for a large perturbation, see [9] and references therein for more details.

We first discuss previous results in the infinite depth setting. The first long-time result for the water waves system is due to the work of Wu [39], where she proved the almost global existence of the $2D$ gravity waves for small initial data. Subsequently, Germain-Masmoudi-Shatah [17] and Wu [40] proved the global existence for the $3D$ gravity waves system, which is the first global regularity result for the water waves system. Global existence of the $3D$ capillary waves was also obtained, see Germain-Masmoudi-Shatah [18]. For the $2D$ gravity waves system, it is highly nontrivial to bypass the almost global existence. As first pointed out by Ionescu-Pusateri [27] and independently by Alazard-Delort [3, 4], we have to modify the profile appropriately first to prove the global existence. The nonlinear solution possesses the modified scattering property instead of the usual scattering. Later, a different interesting proof of the almost global existence was obtained in the holomorphic coordinates by Hunter-Ifrim-Tataru [22], then Ifrim-Tataru [23] improved this result and gave another interesting proof of the global existence. The author [35] considered the infinite energy solution of the gravity waves in $2D$, which removed the momentum assumption assumed in previous results. Global existence of the capillary waves system in $2D$ was also obtained. See Ionescu-Pusateri [28, 29] and Ifrim-Tataru [23]. For the $3D$ gravity-capillary waves with any possible positive gravity effect constant and positive surface tension coefficient, Deng-Ionescu-Pausader-Pusateri [14] proved global existence for small localized initial data in the infinite depth setting.

Now, we move on to the finite depth setting. The behavior of the water waves system in the finite depth setting is more delicate than the infinite depth setting due to three factors listed as follows, the presence of small traveling waves, the more complicated structure at low frequencies, and less favorable quadratic terms.

Roughly speaking, the existence of small (in L^2 sense) traveling waves for the water waves system in different settings can be summarized as follows. From previous results [40, 17, 14] on the $3D$ water waves system in the infinite depth setting, we know that there is no small traveling waves regardless the size of σ/g. However, we do know the existence of small traveling waves for the $3D$ gravity-capillary waves system in the flat bottom setting as long as $\sigma/g > 1/3$, see [12]. From the recent work of the author [36], we know that there is no small traveling wave for the 3D gravity waves system in the flat bottom setting, i.e., $\sigma/g = 0$. So far, it is still not clear whether there exist small traveling waves for the $3D$ gravity-capillary waves system in the flat bottom setting if $0 < \sigma/g \leq 1/3$.

On the long time behavior side. Only results on the gravity waves system were obtained. The large time existence was obtained by Alvarez-Samaniego-Lannes [6] for the $3D$ finite depth gravity waves system. Recently, the author [34, 36] showed that the $3D$ gravity waves system admits global solutions for small smooth localized initial data in the flat bottom setting. For the $2D$ gravity waves system in the flat bottom setting, Harrop-Griffiths-Ifrim-Tataru [21] showed that the lifespan of the solution is at least of size $1/\epsilon^2$ if the small initial data is of size ϵ.

1.2. Main difficulties for the capillary waves system in the flat bottom settings

Note that the linear operator of the Dirichlet-Neumann operator changes with respect to the depth of water region. To help readers understand the main difficulties of the capillary waves in the finite depth setting, we compare the capillary waves system in the infinite depth setting and the flat bottom setting with the depth of water region normalized to be one. Intuitively speaking, we have the following two types of dispersive equations,

$$(1.9) \qquad \text{(Infinite depth setting)} \qquad (\partial_t + i|\nabla|^{3/2})u = \mathcal{N}_1(u),$$

$$(1.10) \qquad \text{(Flat bottom setting)} \qquad (\partial_t + i|\nabla|^{3/2}\sqrt{\tanh|\nabla|})u = \mathcal{N}_2(u).$$

The main new difficulties of the $3D$ capillary waves in the flat bottom setting, which are caused by the difference of linear operators in two settings at low frequencies, can be summarized by the following two facts.

(i) The nonlinearity of (1.10) doesn't have null structure at low frequencies, which does appear in the infinite depth setting. Intuitively speaking, the presence of null structure stabilizes the nonlinear effect. Hence, we expect a stronger nonlinear effect at low frequencies, which makes the global regularity problem more delicate in the flat bottom setting.

(ii) A new type of time resonance set appears for the capillary waves system in the flat bottom setting. The long time accumulated effect caused by the new time resonance set has not been carefully studied before. Given the fact that there exists finite time blow up solution for a similar equation but with a different nonlinearity, we expect that the nonlinear effect caused by the new type of time resonance set is very delicate.

For the sake of readers, we provide more detailed discussion about the existence of null structure at low frequencies in two different settings here. Note that

(1.11)

$$\text{(Infinite depth setting)}: \quad \Lambda_{\leq 2}[\partial_t h] = \Lambda_{\leq 2}[G(h)\psi] = |\nabla|\psi - \nabla\cdot(h\nabla\psi) - |\nabla|(h|\nabla|\psi),$$

(1.12)

$$\text{(Infinite depth setting)}: \quad \Lambda_{\leq 2}[\partial_t \psi] = \Delta h - \frac{1}{2}|\nabla\psi|^2 + \frac{1}{2}(|\nabla|\psi)^2.$$

From (1.11) and (1.12), it is easy to check that the symbols of quadratic terms vanish if the output frequency of quadratic terms is zero. Moreover, if the frequency of the height function "$h(t)$ " is zero, then the symbol of quadratic terms in "$\partial_t h(t)$" also vanishes. Unfortunately, we lose all these favorable cancelations for the capillary waves system (1.3) in the flat bottom setting. From (1.6) and (1.7), it is easy to check that the symbols of quadratic terms in the corresponding scenarios don't vanish in the flat bottom setting.

Due to the lack of null structures at low frequencies in the flat bottom setting, we expect much stronger nonlinear effect for the finite depth capillary waves. One way to capture the nonlinear effect is to study the growth of the profile of the solution, which is the pull back of the nonlinear solution along the linear flow, with respect to time.

For simplicity and also for intuitive purpose, we study a relevant toy model of the capillary waves system (1.3). More precisely, we consider the long time behavior of the following toy model,

$$(1.13) \quad \text{(Toy model)}: \quad (\partial_t - i\Delta)v = Q_1(v,\bar{v}) + Q_2(v,v) + Q_3(\bar{v},\bar{v}), \quad v : \mathbb{R}_t \times \mathbb{R}_x^2 \longrightarrow \mathbb{C},$$

where the symbols $q_i(\xi-\eta,\eta)$ of the quadratic terms $Q_i(\cdot,\cdot)$, $i\in\{1,2,3\}$, satisfy the following estimate,

(1.14)
$$\|\mathscr{F}^{-1}[q_i(\xi-\eta,\eta)\psi_k(\xi)\psi_{k_1}(\xi-\eta)\psi_{k_2}(\eta)]\|_{L^1}\leq C\min\{2^{2\max\{k_1,k_2\}},1\},\quad i\in\{1,2,3\},$$

where C is some absolute constant.

The toy model (1.13) is derived by only keeping the quadratic terms of (1.3), which are expected to be the leading terms in the small data regime, and replacing the linear operator $|\nabla|^{3/2}\sqrt{\tanh|\nabla|}$ by the leading operator $|\nabla|^2$ at low frequencies. The estimate of symbol in (1.14) captures the facts that there are at least two derivatives inside (1.3) and the size of symbol is "1" in both 1×1 (sizes of two input frequencies) $\to 0$ (size of the output frequency) type interaction and the $1\times 0\to 1$ type interaction.

It turns out that the toy model (1.13), which is a $2D$ quadratic Schrödinger equation, is already a very delicate problem due to the presence of $v\bar{v}$ type nonlinearity. Even the quadratic Schrödinger equation in $3D$ is not completely solved.

If without the $v\bar{v}$ type quadratic term, then the $1\times 1\to 0$ type interaction is actually not bad. Note that the phases are all of size 1 in the $1\times 1\to 0$ type interaction if there is no $v\bar{v}$ type quadratic term. The high oscillation of phase in time will also stabilize the growth of the profile in a neighborhood of zero frequency even without the smallness arose from the symbol. We refer readers to the works of Germain-Masmoudi-Shatah [15, 16] for more detailed discussion.

To capture the nonlinear effect of $v\bar{v}$ type quadratic term in the toy model (1.13), we study the growth of profile $g(t):=e^{-it\Delta}v(t)$ over time, which gives us a sense of what the dispersion of the nonlinear solution "$v(t)$" will be. From the Duhamel's formula, we have

(1.15)
$$\begin{aligned}\widehat{g}(t,\xi)=\widehat{g}(0,\xi)+\int_0^t\int_{\mathbb{R}^2}\big(&e^{i2s\xi\cdot\eta}q_1(\xi-\eta,\eta)\widehat{g}(s,\xi-\eta)\widehat{\bar{g}}(s,\eta)\\&+e^{i2s\eta\cdot(\xi-\eta)}q_2(\xi-\eta,\eta)\widehat{g}(s,\xi-\eta)\widehat{g}(s,\eta)\\&+e^{is(|\xi|^2+|\xi-\eta|^2+|\eta|^2)}q_3(\xi-\eta,\eta)\widehat{\bar{g}}(s,\xi-\eta)\widehat{\bar{g}}(s,\eta)\big)d\eta ds.\end{aligned}$$

We start from the first iteration by replacing "$g(s)$" on the right side of (1.15) with the initial data $g(0)$, whose frequency is localized around "1". As a result, intuitively speaking, the following rough estimate holds in a small neighborhood of zero,

(1.16)
$$ct\leq|\widehat{g}(t,\xi)|\leq Ct,\qquad \text{when } |\xi|\leq c/t,$$

where c and C are some absolute constants and the time "t" is very large.

Due to the nonlinear nature of the problem, the growth of profile at low frequencies will trigger the growth of profile at other modes of frequencies. Therefore, it is reasonable to expect that certain instability could possibly happen. Recently, Ikeda and Inui [24] showed that there exists a class of small L^2 initial data such that the solution of the quadratic Schrödinger equation with $v\bar{v}$ type nonlinearity blows up within a polynomial time in both $2D$ and $3D$.

Although this intuition, which comes from the first Picard iteration in (1.16), says that the nonlinear solution behaves differently from a linear flow. It says few precise information about the nonlinear solution itself. In this paper, our goal is not trying to classify all possible outcomes for different types of nonlinearities inside the toy model (1.13). Because it is a very delicate problem, we should not expect a universal answer. Instead, our goal is to exploit some hidden structures inside the capillary water waves (1.3) and show that the $3D$ capillary waves system (1.3) admits global solution for small localized initial data.

1.3. Main result

In this paper, we show that the solution of the capillary waves system (1.3) globally exists and scatters to a linear solution in a weak normed space for small initial data. More precisely, our main theorem is stated as follows,

THEOREM 1.1. – *Let $N_0 = 2000, \delta \in (0, 10^{-9}]$, and $\alpha = 1/10$. Assume that the initial data $(h_0, \psi_0) \in H^{N_0+1/2}(\mathbb{R}^2) \times H^{N_0+1/2}(\mathbb{R}^2)$ satisfies the following smallness condition,*

$$\|(h_0, \psi_0)\|_{H^{N_0+1/2}} + \sum_{\Gamma \in \{L,\Omega\}} \|(\Gamma h_0, \Gamma \psi_0)\|_{H^{10+1/2}} + \sum_{\Gamma^1,\Gamma^2 \in \{L,\Omega\}} \|(\Gamma^1\Gamma^2 h_0, \Gamma^1\Gamma^2\psi_0)\|_{H^{1/2}} \leq \epsilon_0,$$

where ϵ_0 is a sufficiently small constant, $\Omega := x^{\perp} \cdot \nabla_x$ and $L := x \cdot \nabla_x + 2$. Then there exists a unique global solution for the capillary water waves system (1.3) with initial data (h_0, ψ_0). Moreover, the solution scatters to a corresponding linear solution in a homogeneous Sobolev space $\dot{H}^{\alpha+\delta}$ and the following estimate holds,

(1.17)
$$\sup_{t\in[0,T]} (1+t)^{-\delta} \|(\tilde{\Lambda} h, \psi)(t)\|_{H^{N_0}} + (1+t)\Big[\sum_{k\in\mathbb{Z}} 2^{(1+\alpha)k+6k_+} \|P_k[(h,\psi)(t)]\|_{L^\infty}\Big] \leq C\epsilon_0,$$

where C is some absolute constant and $\tilde{\Lambda} := |\nabla|^{1/2}(\tanh|\nabla|)^{-1/2}$.

REMARK 1.1. – From (1.17), we know that the solution decays over time. This fact implies that there is no small traveling waves for the $3D$ capillary waves system (1.3) in the flat bottom setting, i.e., $\sigma/g = \infty$.

1.4. Main ideas of proof

The idea of proving global existence for the $3D$ finite depth capillary waves system (1.3) is classic, which is iterating the local existence result by controlling both the energy and the dispersion of the nonlinear solution over time.

The whole argument depends on the dispersion estimate of the nonlinear solution, which is very delicate. The main difficulty and the delicacy come from the complicated and large time resonance set associated with the quadratic terms. More precisely, the time resonance sets of the quadratic terms are defined as follows,

$$\mathcal{T}_{\mu,\nu} := \{(\xi,\eta) : \quad \Lambda(|\xi|) - \mu\Lambda(|\xi-\eta|) - \nu\Lambda(|\eta|) = 0\}, \quad \mu,\nu \in \{+,-\}.$$

As a typical example, the following approximation holds at low frequencies for the case when $\mu = +$ and $\nu = -$,

(1.18)
$$\mathscr{T}_{+,-} \cap \{(\xi,\eta) : |\xi|, |\eta| \le 2^{-10}\} \approx \{(\xi,\eta) : |\xi|, |\eta| \le 2^{-10}, \Lambda(|\xi|) - \Lambda(|\xi-\eta|) + \Lambda(|\eta|) \approx 2\xi\cdot\eta \approx 0\}.$$

Note that the time resonance set is almost everywhere since it is possible that "$\xi\cdot\eta = 0$" no matter what the sizes of $|\xi|$ and $|\eta|$ are.

Recall (1.16). Since the growth mode happens at a small neighborhood of zero, it is reasonable to expect that the spatial derivatives, which provide smallness at low frequencies, compensate the L^∞_x decay rate of the nonlinear solution. To capture this expectation, we aim to prove the sharp decay rate for certain derivatives of the nonlinear solution instead of the nonlinear solution itself.

Now, the first question is how many derivatives we associate with the nonlinear solution to obtain the sharp decay rate. To answer this question, we need to keep a basic principle in mind. Generally speaking, the more derivatives we associate with the solution the less information we can tell about the solution itself. Recall (1.16). Intuitively speaking, because of the accumulated effect of the $1\times 1 \to 0$ type interaction, it is unlikely that the "1−" derivatives of the profile of the nonlinear solution doesn't grow over time. Therefore, in practice, we expect that $1+\alpha$ derivatives of solution decay sharply, where α is a small positive number.

Now, the real question is whether we can close the argument and show that our expectation indeed holds globally in time. Despite the argument that we will present is very complicated and technical. There are two main ingredients that are very essential to the validity to the argument: (i) there are requisite symmetric structures inside the finite depth capillary wave system (1.3); (ii) the bulk scenario, which is nontrivial to justify and will be clear later, is the accumulated effect of the $t^{-1/2}\times t^{-1/2} \to t^{-1/2}$ type interaction. Recall (1.14), there are two derivatives in total at low frequencies. Hence, the accumulated effect of the $t^{-1/2}\times t^{-1/2} \to t^{-1/2}$ type interaction is compensated by the symbol of quadratic terms. As a result, the bulk scenario is not an issue.

We discuss some main ideas and strategies used in the bootstrap argument with more details as follows.

1.4.1. *Energy estimate: controlling the high frequency part of solution.* – We first point out that the difference of the high frequency part between the infinite depth setting and the flat bottom setting is very little. Thanks to the works of Alazard-Métivier [5] and Alazard-Burq-Zuilly [1, 2], by using the method of paralinearization and symmetrization, we can find a pair of good unknown variables, such that the equations satisfied by the good unknown variables have symmetries inside, which help us to avoid losing derivatives in the energy estimate.

Recall that we expect that the decay rate of $1+\alpha$ derivatives of solution is sharp. However, within our expectation, the L^∞_x-norm of the nonlinear solution itself in the worst scenario is only $(1+t)^{-1/2+\delta}$. As a result, a rough $L^2 - L^\infty$ type energy estimate is not sufficient to close the energy estimate. Hence, we need to pay special attention to the low frequency part of the input putted in L^∞-type space. To this end, an important step is to understand the structure of the low frequency part of the Dirichlet-Neumann operator, which has been studied in details in [34].

We first state our desired energy estimate and then explain the main intuitions behind. We expect that the following new type of energy estimate holds,

$$\Big|\frac{d}{dt}E(t)\Big| \leq CE(t)\big(\|(h(t),\psi(t))\|_{W^{6,1+\alpha}} + \|(h(t),\psi(t))\|_{W^{6,1}}\|(h(t),\psi(t))\|_{W^{6,0}}\big), \tag{1.19}$$

where C is some absolute constant and the $W^{\gamma,b}$ type function space is defined as follows,

$$\|f\|_{W^{\gamma,b}} := \sum_{k\in\mathbb{Z}}(2^{\gamma k}+2^{bk})\|P_k f\|_{L^\infty}, \quad b<\gamma. \tag{1.20}$$

Note that the desired new type of energy estimate (1.19) is sufficient to show that the energy only grows sub-polynomially as long as the nonlinear solution decays sharply in $W^{6,1+\alpha}$. To derive the new type energy estimate (1.19), besides the quadratic terms, we also need to pay special attention to the low frequency part of the cubic terms.

Now, we provide an intuitive explanation about why the desired estimate (1.19) holds. Note that the following three facts hold: (i) there are at least two derivatives in total inside the quadratic terms; (ii) we don't lose derivatives after utilizing symmetries during the energy estimates; (iii) the total number of derivatives doesn't decrease in this process. As a result, intuitively speaking, there are only two possible scenarios, which are listed as follows: (i) Including the High $\times$ High type interaction, there are at least two derivatives associated with the input with relatively smaller frequency; (ii) Smooth error terms. In other words, the high order Sobolev-norm of those terms can be controlled by their L^2-norms. Therefore, we can put the input with larger frequency in L^∞ and put the other input in L^2. In whichever scenario, the input putted in L^∞ type space always associates with two spatial derivatives, which explains the first estimate in (1.19). A very similar intuition also holds for cubic and higher order terms, which leads to the second part of (1.19).

1.4.2. *The dispersion estimate: sharp decay rate of the* $1+\alpha$ *derivatives of solution*. – To carry out the analysis of decay estimate, we first identify a good substitution variable, which has the same decay rate as the original solution. Instead of proving the dispersion estimate for the original variable, our goal is reduced to prove the sharp decay estimate for the good substitution variable over time.

We divide the rest of this subsubsection into three parts. (i) In the first part, we explain how to find such a good substitution variable. (ii) In the second part, we explain some main ideas in the estimate of the lower order weighted norm. Our goal is to prove that, under the assumption that the high order weighted norm only grows sub-polynomially over time, the low order weighted norm of the profile doesn't grow over time, which implies that the decay rate of $1+\alpha$ derivatives of the nonlinear solution is sharp. (iii) In the third part, we explain main ideas behind the estimate of high order weighted norm and show that it indeed grows only sub-polynomially over time.

A good substitution variable. – The good substitution variable is obtained by using the normal form transformation that removes some nonlinearities, which associate with phases that are highly oscillating in time. As a result, the equation satisfied by the good substitution variable has less terms, which simplify the whole argument.

For simplicity, we consider the toy model (1.13) to illustrate the main idea behind. Define the profile of $u(t)$ as $f(t) := e^{it\Lambda}u(t)$, as a result of direct computation, we have

$$\widehat{f}(t,\xi) = \widehat{f}(0,\xi) + \sum_{\mu,\nu\in\{+,-\}} \int_0^t \int_{\mathbb{R}^2} e^{is\Phi^{\mu,\nu}(\xi,\eta)} q_{\mu,\nu}(\xi-\eta,\eta)\widehat{f^{\mu}}(s,\xi-\eta)\widehat{f^{\nu}}(s,\eta)\,d\eta\,ds,$$

where $f^{+} := f =: P_{+}[f]$, $f^{-} := \bar{f} =: P_{-}[f]$, $q_{\mu,\nu}(\xi-\eta,\eta)$ is the symbol of $u^{\mu}u^{\nu}$ type quadratic term, and the phases $\Phi^{\mu,\nu}(\xi,\eta)$, $\mu,\nu\in\{+,-\}$, are defined as follows,

$$\Phi^{\mu,\nu}(\xi,\eta) = |\xi|^2 - \mu|\xi-\eta|^2 - \nu|\eta|^2, \quad \mu,\nu\in\{+,-\}.$$

Note that

$$\nabla_\eta \Phi^{+,+}(\xi,\eta) = -(\eta-\xi) - \eta \Longrightarrow \nabla_\eta\Phi^{+,+}(\xi,\xi/2) = 0.$$

Therefore, we can't do integration by parts in "η" around a small neighborhood of $(\xi,\xi/2)$ (space resonance set) . Fortunately, $(\xi,\xi/2)$ doesn't belong to the time resonance set. From the explicit formula, it is easy to check the validity of the following estimate,

$$\Phi^{+,+}(\xi,\xi/2) = |\xi|^2 - 2(|\xi|/2)^2 = |\xi|^2/2.$$

Very similarly, we can verify that the following estimate holds when $|\eta| \leq 2^{-10}|\xi|$ and $\mu = -$ or $|\xi|\leq 2^{-10}|\eta|$, $\mu\nu = +$,

$$2^{-2}\max\{|\xi|^2,|\eta|^2\} \leq |\Phi^{\mu,\nu}(\xi,\eta)| \leq 2^2\max\{|\xi|^2,|\eta|^2\}.$$

Since the associated phases are relatively large, we refer those cases as the high-oscillation-in-time cases.

To take the advantage of the high oscillation in time for these scenarios, we can use a normal form transformation to remove the high oscillation in time cases as follows,

$$v := u + \sum_{\mu,\nu\in\{+,-\}} A_{\mu,\nu}(u^{\mu},u^{\nu}), \quad a_{\mu,\nu}(\xi-\eta,\eta) = \sum_{k\in\mathbb{Z}} \frac{iq_{\mu,\nu}(\xi-\eta,\eta)}{\Phi^{\mu,\nu}(\xi,\eta)} \tag{1.21}$$

$$\times\big(\psi_{\leq k-10}(\eta-\xi/2)\psi_k(\xi) + \frac{1-\mu}{2}\psi_k(\xi)\psi_{\leq k-10}(\eta) + \frac{1+\mu\nu}{2}\psi_k(\xi)\psi_{\geq k+10}(\eta)\big),$$

where $a_{\mu,\nu}(\xi-\eta,\eta)$, $\mu,\nu\in\{+,-\}$, are the symbol of quadratic terms $A_{\mu,\nu}(\cdot,\cdot)$. Note that there are at least two derivatives inside the symbol, which cover the loss of dividing the phase. As a result, the normal form transformation is not singular.

Although the discussion so far is restricted to the toy model (1.13). For the capillary waves system (1.3), we use similar ideas not only for quadratic terms, but also for cubic terms and quartic terms, see (4.20). Please refer to Subsection 4.1 for more details.

The low order weighted norm. – We first define the low order weighted norm Z_1-norm and the high order weighted norm Z_2-norm as follows,

$$\|g\|_{Z_1} := \sum_{k\in\mathbb{Z}}\sum_{j\geq -k_-} \|g\|_{B_{k,j}}, \quad \|g\|_{B_{k,j}} := (2^{(1+\alpha)k} + 2^{10k_+})2^j\|\varphi_j^k(x)P_k g(x)\|_{L^2}, \tag{1.22}$$

$$\|g\|_{Z_2} := \sum_{\Gamma^1,\Gamma^2\in\{L,\Omega\}} \|\Gamma^1\Gamma^2 g\|_{L^2} + \|\Gamma^1 g\|_{L^2}, \tag{1.23}$$

where $\varphi_j^k(x)$ is defined in (2.1), which is first introduced in the work of Ionescu-Pausader [25]. An advantage of using this type of space is that it not only localizes the frequency but also localizes the spatial concentration. The atomic space of this type has been successfully used in many dispersive PDEs, see [14, 13, 19, 25, 26, 36].

Define the profile of the good substitution variable $v(t)$ as $g(t) := e^{it\Lambda}v(t)$. From the linear dispersion estimates (2.10) and (2.11) in Lemma 2.7, to prove the sharp decay rate, it would be sufficient to prove that the Z_1-norm of the profile $g(t)$ doesn't grow over time. Now, under the assumption that the Z_2-norm only grows polynomially, we explain some main ideas of how to prove that Z_1-norm doesn't grow over time.

Note that the High $\times$ High type interaction is not an issue because we put a very high order weighted (i.e., $2^{(1+\alpha)k}$) in the definition of Z_1-norm, see (1.22). It remains to consider the High $\times$ Low type interaction, e.g., $|\eta| \leq 2^{-10}|\xi|$. As a typical example of the bulk threshold case in the High $\times$ Low type interaction, we consider the case when $|\eta| \in [2^{-10}/t, 2^{10}/t], |\xi| \in [2^{-10}, 2^{10}]$. To get around the difficulty caused by the lack of null structure and the growth of profile around the small neighborhood of zero frequency (see(1.16)), we analyze more carefully about the source of the growth mode inside the nonlinearity of the capillary waves system (1.3).

Recall (1.5). We know that $\widehat{h}(t,0)$ is conserved over time. Moreover, a simple Fourier analysis shows that $|\widehat{\psi}(t,0)| \leq 2^{10}|t|\epsilon_0$, where ϵ_0 is the size of initial data. These two facts motivate us to expect that the source of trouble should be the restricted velocity potential $\widehat{\psi}(t,\eta)$ instead of the height function $\widehat{h}(t,\eta)$, i.e., the size of $\widehat{h}(t,\eta)$ should be much smaller than $\widehat{g}(t,\eta)$ when $|\eta| \leq 2^{10}/t$, where time "t" is very large. As a matter of fact, we do have a better estimate for $\widehat{h}(t,\eta)$, see (5.15) in Lemma 5.3, which says that $\widehat{h}(t,\eta)$ grows at most at rate "$t^{2\delta}$" with respect to time if $|\eta| \leq 2^{10}/t$. Recall again (1.6) and (1.7), we know that there is at least one spatial derivative associated with the velocity potential "$\psi(t)$". Therefore, if the velocity potential "$\psi(t)$" has the small frequency "η," the associated spatial derivative contributes the smallness of $|\eta|$. To sum up, either the symbol contributes the smallness of "$|\eta|$" or the input with smaller frequency is the height function "$h(t)$". In whichever case, the bulk threshold case $|\eta| \in [2^{-10}/t, 2^{10}/t], |\xi| \in [2^{-10}, 2^{10}]$ is not an issue.

For the non-threshold case, we do integration by parts in "η" once to take the advantage of the gap between the threshold case and the non-threshold case. For the High $\times$ High type interaction, the gap is created by the extra "$2^{\alpha k}$" we put in the definition of Z_1-norm. For the High $\times$ Low type and Low $\times$ High type interactions, the gap is created by the observation on the source of growth mode that we made in the above discussion. The gain of decay rate from the gap between the threshold case and the non-threshold case is more than the loss from the growth rate of the high order weighted norm. This fact leads to the conclusion that the Z_1-norm of the profile doesn't grow in time.

The high order weighted norm. – Now, we explain some essential ideas that make it possible to conclude that the high order weighted norm (Z_2-norm) of the profile grows at most rate "t^{δ}" with respect to time.

The losing derivatives issue. – Note that the high frequency part of the nonlinear solution is controlled by the high order energy (H^{N_0}-norm) of the nonlinear solution, which only grows at most at rate t^{δ} with respect to time. As a result, we only have to consider the case when the sizes of all frequencies are less than $t^{5/1900}$, which is only a minor growth rate. Therefore, the losing derivatives issue can be reduced to the losing time decay rate issue. For the cubic and higher order terms, we use the extra decay rate to cover the loss of losing derivatives. Since the "$1/t$" decay rate for the quadratic terms is critical to close the argument, we can't afford the loss of time decay rate.

To get around this issue, we notice that it would be sufficient to avoid losing derivatives at the quadratic level if the losing derivatives issue is only relevant at the quadratic level. After carefully studying the explicit quadratic terms of the capillary waves system (1.3), we study the system of equations satisfied by $(h, \psi - T_{|\nabla|\tanh|\nabla|\psi} h)$, which is the truncation up to quadratic level of the good unknown variable found in the paralinearization and symmetrization process, instead of the system of equations satisfied by (h, ψ). As a result, after utilizing the symmetric structure for the good substitution variable, the quadratic term doesn't lose derivatives.

The insufficient decay rates issue. – Now, we explain some main ideas used for two typical scenarios that it is not obvious to obtain the critical "$1/t$" decay rate over time.

Recall that we applied the vector field "$L := x \cdot \nabla + 2$" (equivalently, "$-\xi \cdot \nabla_\xi$" on the Fourier side) on the profile of the nonlinear solution in the definition of Z_2-norm. A drawback of using the "L" vector field is that we face a loss of "t" when "L" hits the phases of nonlinearities. To be more precise and as a typical example, we consider the High $\times$ Low type interaction of the quadratic term. Note that (see (6.17) and (6.19) for more details), the following decomposition holds if the vector field "L" hits the phase.

$$\xi \cdot \nabla_\xi \Phi^{+,\nu}(\xi, \eta) = O(1)\Phi^{+,\nu}(\xi, \eta) + O(|\eta|^2), \quad \text{when } |\eta| \leq 2^{-10}|\xi|. \tag{1.24}$$

For the first term on the right hand side of (1.24), which is comparable with the phase function $\Phi^{+,\nu}(\xi, \eta)$, we can take the advantage of oscillation in time by doing integration by parts in time once first and then take the advantage of the space oscillation in "η". For the second term of (1.24), the smallness of "$|\eta|^2$" acts like null structures, which allow us to obtain sharp L^∞_x decay rate even after taking the advantage of the space oscillation in "η".

Another typical scenario with insufficient time decay rate is the case when all the vector fields hit the input with the largest frequency. Since the L^∞_x-norm of the nonlinear solution itself only decays at rate $(1+t)^{-1/2+\delta}$, a rough $L^2 - L^\infty$ type estimate is not sufficient to close the argument. To get around this issue, we use the hidden symmetric structure inside the capillary waves system. As a result, a similar decomposition as in (1.24) holds for the symbol of quadratic terms after utilizing the symmetric structure inside the capillary waves system. Therefore, the aforementioned strategy is also applicable for the case we are considering. For the cubic terms, similar to the desired energy estimate (1.19), the symmetric structure inside the capillary waves system (1.3) also plays an essential role.

1.5. The outline of this paper

In Section 2, we introduce notations and some basic lemmas that will be used constantly. In Section 3, we prove a new type of energy estimate by using the method of paralinearization and symmetrization and paying special attention to the low frequency part. In Section 4, we identify a good substitution variable to carry out the estimate of weighted norms. In Section 5, we prove that the low order weighted norm doesn't grow over time under the assumptions that the high order weight norm only grows appropriately and a good control of the quintic and higher order remainder term is available. In Section 6, we prove that the high order weighted norm only grows appropriately under the assumption that we have a good control on the quintic and higher order remainder term. In Section 7, we first prove some fixed time weighted norm estimates, which were took for granted in Section 5 and 6, and then estimate the quintic and higher order reminder terms by using a fixed point type argument.

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2. Preliminaries

For $p \in \mathbb{N}_+$, then we use $\Lambda_p(\mathcal{N})$ to denote the p-th order terms of the nonlinearity $\mathcal{N}$. Also, we use notation $\Lambda_{\geq p}[\mathcal{N}]$ to denote the p-th and higher orders terms, i.e., $\Lambda_{\geq p}[\mathcal{N}] := \sum_{q\geq p} \Lambda_q[\mathcal{N}]$. For example, $\Lambda_2[\mathcal{N}]$ denotes the quadratic term of $\mathcal{N}$ and $\Lambda_{\geq 2}[\mathcal{N}]$ denotes the quadratic and higher order terms of $\mathcal{N}$. If there is no special annotation, Taylor expansions are in terms of the height function "$h(t)$" and the restricted velocity potential "$\psi(t)$".

We fix an even smooth function $\tilde{\psi} : \mathbb{R} \to [0,1]$ supported in $[-3/2, 3/2]$ and equals to 1 in $[-5/4, 5/4]$. For any $k \in \mathbb{Z}$, we define

$$\psi_k(x) := \tilde{\psi}(x/2^k) - \tilde{\psi}(x/2^{k-1}), \qquad \psi_{\leq k}(x) := \tilde{\psi}(x/2^k) = \sum_{l\leq k} \psi_l(x),$$
$$\psi_{\geq k}(x) := 1 - \psi_{\leq k-1}(x),$$

and use P_k, $P_{\leq k}$ and $P_{\geq k}$ to denote the projection operators by the Fourier multipliers ψ_k, $\psi_{\leq k}$ and $\psi_{\geq k}$ respectively. We use $f_k(x)$ to abbreviate $P_k f(x)$. We use both $\widehat{f}(\xi)$ and $\mathcal{F}(f)(\xi)$ to denote the Fourier transform of f, which is defined as follows,

$$\mathcal{F}(f)(\xi) = \int e^{-ix\cdot\xi} f(x) dx.$$

We use $\mathcal{F}^{-1}(g)$ to denote the inverse Fourier transform of $g(\xi)$. For an integer $k \in \mathbb{Z}$, we use k_+ to denote $\max\{k, 0\}$ and use k_- to denote $\min\{k, 0\}$.

Recall the Z_1-normed space and the Z_2-normed space we defined in (1.22) and (1.23). The spatial localization function $\varphi_j^k(x)$ used there is defined as follows,

$$\varphi_j^k(x) := \begin{cases} \psi_{(-\infty,-k]}(x) & \text{if } k+j = 0 \text{ and } k \leq 0, \\ \psi_{(-\infty,0]}(x) & \text{if } j = 0 \text{ and } k \geq 0, \\ \psi_j(x) & \text{if } k+j \geq 1 \text{ and } j \geq 1. \end{cases} \tag{2.1}$$

For any $k \in \mathbb{Z}$, $j \geq -k_-$, we define

$$f_{k,j} := P_{[k-2,k+2]}[\varphi_j^k(x) P_k f].$$

For two localized function $f(x), g(x) \in L^2$, we use the convention that the symbol $q(\cdot,\cdot)$ of a bilinear form $Q(\cdot,\cdot)$ is defined in the following sense throughout this paper,

$$\mathscr{F}[Q(f,g)](\xi) = \frac{1}{4\pi^2}\int_{\mathbb{R}^2} \widehat{f}(\xi-\eta)\widehat{g}(\eta)q(\xi-\eta,\eta)d\eta. \tag{2.2}$$

Very similarly, the symbol $c(\cdot,\cdot,\cdot)$ of a trilinear form $C(f,g,h)$ is defined in the following sense,

$$\mathscr{F}[C(f,g,h)](\xi) = \frac{1}{16\pi^4}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \widehat{f}(\xi-\eta)\widehat{g}(\eta-\sigma)\widehat{h}(\sigma)c(\xi-\eta,\eta-\sigma,\sigma)d\eta d\sigma.$$

Define a class of symbol and its associated norms as follows,

$$\begin{aligned}
\mathcal{S}^\infty &:= \{m : \mathbb{R}^4 \text{ or } \mathbb{R}^6 \to \mathbb{C}, m \text{ is continuous and } \quad \|\mathscr{F}^{-1}(m)\|_{L^1} < \infty\},\\
\|m\|_{\mathcal{S}^\infty} &:= \|\mathscr{F}^{-1}(m)\|_{L^1},\\
\|m(\xi,\eta)\|_{\mathcal{S}^\infty_{k,k_1,k_2}} &:= \|m(\xi,\eta)\psi_k(\xi)\psi_{k_1}(\xi-\eta)\psi_{k_2}(\eta)\|_{\mathcal{S}^\infty},\\
\|m(\xi,\eta,\sigma)\|_{\mathcal{S}^\infty_{k,k_1,k_2,k_3}} &:= \|m(\xi,\eta,\sigma)\psi_k(\xi)\psi_{k_1}(\xi-\eta)\psi_{k_2}(\eta-\sigma)\psi_{k_3}(\sigma)\|_{\mathcal{S}^\infty}.
\end{aligned}$$

Lemma 2.1. – *For $i \in \{1,2,3\}$, $f \in W^{i+1,\infty}(\mathbb{R}^{2i})$, there exists an absolute constant $C \in \mathbb{R}_+$ such that the following estimate holds,*

(2.3)
$$\Big\|\int_{\mathbb{R}^{2i}} f(\xi_1,\dots,\xi_i)\prod_{j=1}^{i} e^{ix_j\cdot\xi_j}\psi_{k_j}(\xi_j)d\xi_1\cdots d\xi_i\Big\|_{L^1_{x_1,\dots,x_i}} \le \sum_{m=0}^{i+1}\sum_{j=1}^{i} C2^{mk_j}\|\partial^m_{\xi_j} f\|_{L^\infty}.$$

Lemma 2.2. – *Assume that $m, m' \in S^\infty$, $f,g,h \in L^\infty(\mathbb{R}^2)\cap L^1(\mathbb{R}^2)$, $p,q,r,s \in [1,\infty]$, then there exists an absolute constant $C \in \mathbb{R}_+$, such that the following estimates hold,*

$$\|m\cdot m'\|_{S^\infty} \le C\|m\|_{S^\infty}\|m'\|_{S^\infty}, \tag{2.4}$$

$$\Big\|\mathscr{F}^{-1}\Big[\int_{\mathbb{R}^2} m(\xi,\eta)\widehat{f}(\xi-\eta)\widehat{g}(\eta)d\eta\Big]\Big\|_{L^p} \le C\|m\|_{\mathcal{S}^\infty}\|f\|_{L^q}\|g\|_{L^r}, \tag{2.5}$$

if $1/p = 1/q + 1/r$,

(2.6)
$$\Big\|\mathscr{F}^{-1}\Big[\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} m'(\xi,\eta,\sigma)\widehat{f}(\xi-\eta)\widehat{h}(\sigma)\widehat{g}(\eta-\sigma)d\eta d\sigma\Big]\Big\|_{L^p} \le C\|m'\|_{\mathcal{S}^\infty}\|f\|_{L^q}\|g\|_{L^r}\|h\|_{L^s},$$

where $1/p = 1/q + 1/r + 1/s$.

Definition 2.3. – Given $\rho \in \mathbb{N}_+, \rho \ge 0$ and $m \in \mathbb{R}$, we use $\Gamma^m_\rho(\mathbb{R}^2)$ to denote the space of locally bounded functions $a(x,\xi)$ on $\mathbb{R}^2\times(\mathbb{R}^2/\{0\})$, which are C^∞ with respect to ξ for $\xi \ne 0$. Moreover, for any $\alpha \in \mathbb{N}^2_+$, there exists a constant $C_\alpha(a)$, which only depends on "α" and the symbol $a(x,\xi)$ itself, such that the following estimate holds for the symbol $a(x,\xi)$,

$$\forall|\xi| \ge 1/2, \|\partial^\alpha_\xi a(\cdot,\xi)\|_{W^{\rho,\infty}} \le C_\alpha(a)(1+|\xi|)^{m-|\alpha|},$$

where $W^{\rho,\infty}$ is the usual Sobolev space. For a symbol $a \in \Gamma^m_\rho$, we define its norm as follows,

$$M^m_\rho(a) := \sup_{|\alpha|\le 2+\rho}\sup_{|\xi|\ge 1/2}\|(1+|\xi|)^{|\alpha|-m}\partial^\alpha_\xi a(\cdot,\xi)\|_{W^{\rho,\infty}}.$$

DEFINITION 2.4. – (i) We use $\dot{\Gamma}^m_\rho(\mathbb{R}^2)$ to denote the subspace of $\Gamma^m_\rho(\mathbb{R}^2)$, which consists of symbols that are homogeneous of degree m in ξ.

(ii) If $a = \sum_{0\le j<\rho} a^{(m-j)}$, where $a^{(m-j)} \in \dot{\Gamma}^{m-j}_{\rho-j}(\mathbb{R}^2)$, then we call $a^{(m)}$ and $a^{(m-1)}$ as the principal symbol and the subprincipal symbol of a respectively.

(iii) An operator T is said to be of order m, $m \in \mathbb{R}$, if for all $\mu \in \mathbb{R}$, it's bounded from $H^\mu(\mathbb{R}^2)$ to $H^{\mu-m}(\mathbb{R}^2)$. We use S^m to denote the set of all operators of order m.

For $a, f \in L^2$ and a pseudo differential operator $\tilde{a}(x,\xi)$, we define the operator $T_a f$ and $T_{\tilde{a}} f$ as follows,

$$T_a f = \mathscr{F}^{-1}[\int_{\mathbb{R}} \widehat{a}(\xi-\eta)\theta(\xi-\eta,\eta)\widehat{f}(\eta)d\eta], \quad T_{\tilde{a}} f = \mathscr{F}^{-1}[\int_{\mathbb{R}} \mathscr{F}_x(\tilde{a})(\xi-\eta,\eta)\theta(\xi-\eta,\eta)\widehat{f}(\eta)d\eta], \tag{2.7}$$

where the cut-off function is defined as follows,

$$\theta(\xi-\eta,\eta) = \begin{cases} 1 \text{ when } |\xi-\eta| \le 2^{-10}|\eta|, \\ 0 \text{ when } |\xi-\eta| \ge 2^{10}|\eta|. \end{cases} \tag{2.8}$$

LEMMA 2.5. – *Let $m \in \mathbb{R}$ and $\rho > 0$ and let $a \in \Gamma^m_\rho(\mathbb{R}^d)$, if we denote $(T_a)^*$ as the adjoint operator of T_a and denote $\bar{a}$ as the complex conjugate of a, then we know that, $(T_a)^* - T_{a^*}$ is of order $m-\rho$, where*

$$a^* = \sum_{|\alpha|<\rho} \frac{1}{i^{|\alpha|}\alpha!}\partial^\alpha_\xi \partial^\alpha_x \bar{a}.$$

Moreover, the operator norm of $(T_a)^* - T_{a^*}$ is bounded by $M^m_\rho(a)$.

Proof. – See [1, Theorem 3.10]. □

LEMMA 2.6. – *Let $m \in \mathbb{R}$ and $\rho > 0$, if given symbols $a \in \Gamma^m_\rho(\mathbb{R}^d)$ and $b \in \Gamma^{m'}_\rho(\mathbb{R}^d)$, we define*

$$a\sharp b = \sum_{|\alpha|<\rho} \frac{1}{i^{|\alpha|}\alpha!}\partial^\alpha_\xi a \partial^\alpha_x b,$$

then for all $\mu \in \mathbb{R}$, there exists a constant K such that

$$\|T_a T_b - T_{a\sharp b}\|_{H^\mu \to H^{\mu-m-m'+\rho}} \le K M^m_\rho(a) M^{m'}_\rho(b). \tag{2.9}$$

We have the following lemma on the L^∞_x decay estimate of the linear solution associated with the capillary wave system (1.3).

LEMMA 2.7. – *For $f \in L^1(\mathbb{R}^2)$ and any $\theta \in [0,1]$, there exists an absolute constant C and a constant C_θ which only depends on θ such that the following L^∞_x-type estimates hold,*

$$\|e^{it\Lambda}P_k f\|_{L^\infty_x} \le C(1+|t|)^{-1}2^{k/2}\|f\|_{L^1}, \quad \text{if } k \ge 0. \tag{2.10}$$

$$\|e^{it\Lambda}P_k f\|_{L^\infty_x} \le C_\theta(1+|t|)^{-\frac{1+\theta}{2}}2^{\frac{(1-\theta)k}{2}}\|f\|_{L^1}, \quad \text{if } k \le 0. \tag{2.11}$$

Proof. – After checking the expansion of the phase, see (6.13), we can apply the main result in [20, Theorem 1:(a)&(b)] directly to derive above results. □

3. The energy estimate

The goal of this section is to prove that the energy of solution grows at most at rate $(1+t)^{\delta}$ over time. We first state our bootstrap assumption as follows,

$$\sup_{t\in[0,T]}(1+t)^{-\delta}\|(\tilde{\Lambda}h(t),\psi(t)\|_{H^{N_0}}+(1+t)\|(h(t),\psi)(t)\|_{W^{6,1+\alpha}}\leq\epsilon_1:=\epsilon_0^{5/6}, \tag{3.1}$$

where $\tilde{\Lambda}:=|\nabla|^{1/2}(\tanh|\nabla|)^{-1/2}$ and the function space $W^{6,1+\alpha}$ was defined in (1.20).

The main result of this section is summarized as the following proposition.

Proposition 3.1. – *Under the bootstrap assumption* (3.1)*, there exists an absolute constant C such that the following energy estimate holds for any $t\in[0,T]$,*

$$\begin{aligned}&\|(\tilde{\Lambda}h(t),\psi(t))\|_{H^{N_0}}^2\\ &\leq C\big[\epsilon_0^2+\int_0^t\|(\tilde{\Lambda}h(s),\psi(s))\|_{H^{N_0}}^2\big(\|(h,\psi)\|_{W^{6,1+\alpha}}+\|(h,\psi)\|_{W^{6,0}}\|(h,\psi)\|_{W^{6,1}}\big)ds\big].\end{aligned} \tag{3.2}$$

We separate this section into three parts: (i) Firstly, we introduce main results and briefly explain main ideas of the paralinearization process for the capillary waves system (1.3). (ii) Secondly, with the highlighted structures of losing derivative inside the system (1.3), we symmetrize the system (1.3) such that it doesn't lose derivatives during the energy estimate. (iii) Lastly, we use the symmetrized system to prove the desired new energy estimate (3.2).

3.1. Paralinearization of the full system

Most of this section has been studied in details in [34]. Here we only briefly introduce related main results and main ideas behind those results. Please refer to [34] for more detailed discussions.

To perform the paralinearization process, we need some basic estimates of the Dirichlet-Neumann operator, which are obtained from analyzing the velocity potential inside the water region "$\Omega(t)$".

Recall that the velocity potential $\phi(t,x)$ satisfies the following Laplace equation with two boundary conditions as follows,

$$\Delta\phi=0,\quad \phi|_{\Gamma(t)}=\psi(t),\quad \partial_{\vec{n}}\phi|_{\Sigma}=0. \tag{3.3}$$

To simplify analysis, we map the water region "$\Omega(t)$" into the strap $\mathcal{S}:=\mathbb{R}\times[-1,0]$ by doing change of coordinates as follows,

$$(x,y)\longrightarrow(x,z),\quad z:=\frac{y-h(t,x)}{1+h(t,x)}.$$

We define $\varphi(t,z):=\phi(t,z+h(t,x))$. From (3.3), we have

$$P\varphi:=[\Delta_x+\tilde{a}\partial_z^2+\tilde{b}\cdot\nabla\partial_z+\tilde{c}\partial_z]\varphi=0,\quad \varphi|_{z=0}=\psi,\quad \partial_z\varphi|_{z=-1}=0, \tag{3.4}$$

where

$$\tilde{a} = \frac{(y+1)^2|\nabla h|^2}{(1+h)^4} + \frac{1}{(1+h)^2} = \frac{1+(z+1)^2|\nabla h|^2}{(1+h)^2}, \tag{3.5}$$

$$\tilde{b} = -2\frac{(y+1)\nabla h}{(1+h)^2} = \frac{-2(z+1)\nabla h}{1+h}, \quad \tilde{c} = \frac{-(z+1)\Delta_x h}{(1+h)} + 2\frac{(z+1)|\nabla h|^2}{(1+h)^2}, \tag{3.6}$$

$$G(h)\psi = [-\nabla h\cdot\nabla\phi + \partial_y\phi]|_{y=h} = \frac{1+|\nabla h|^2}{1+h}\partial_z\varphi|_{z=0} - \nabla\psi\cdot\nabla h. \tag{3.7}$$

Hence, to study the Dirichlet-Neumann operator, it is sufficient to study the only nontrivial part of $G(h)\psi$, which is $\partial_z\varphi|_{z=0}$.

From (3.4), we can derive the following fixed point type formulation for $\nabla_{x,z}\varphi$, which provides a good way to analyze and estimate the Dirichlet-Neumann operator in the small data regime. More precisely, we have

$$\begin{aligned}\nabla_{x,z}\varphi = &\Big[\big[\frac{e^{-(z+1)|\nabla|}+e^{(z+1)|\nabla|}}{e^{-|\nabla|}+e^{|\nabla|}}\big]\nabla\psi, \frac{e^{(z+1)|\nabla|}-e^{-(z+1)|\nabla|}}{e^{-|\nabla|}+e^{|\nabla|}}|\nabla|\psi\Big]\\ &+ [\mathbf{0}, g_1(z)] + \int_{-1}^{0}[K_1(z,s)-K_2(z,s)-K_3(z,s)](g_2(s)+\nabla\cdot g_3(s))ds\\ &+ \int_{-1}^{0} K_3(z,s)|\nabla|\mathrm{sign}(z-s)g_1(s) - |\nabla|[K_1(z,s)+K_2(z,s)]g_1(s)\,ds,\end{aligned} \tag{3.8}$$

where

$$\begin{aligned}K_1(z,s) := \Big[&\frac{\nabla}{2|\nabla|}\frac{e^{-z|\nabla|}-e^{z|\nabla|}}{e^{-|\nabla|}+e^{|\nabla|}}e^{(s-1)|\nabla|} + \frac{\nabla}{2|\nabla|}e^{(z+s)|\nabla|},\\ &-\frac{1}{2}\frac{e^{z|\nabla|}+e^{-z|\nabla|}}{e^{-|\nabla|}+e^{|\nabla|}}e^{(s-1)|\nabla|} + \frac{1}{2}e^{(z+s)|\nabla|}\Big],\end{aligned} \tag{3.9}$$

$$K_2(z,s) := \Big[\frac{\nabla}{2|\nabla|}\frac{e^{-z|\nabla|}-e^{z|\nabla|}}{e^{-|\nabla|}+e^{|\nabla|}}e^{-(s+1)|\nabla|}, -\frac{1}{2}\frac{e^{z|\nabla|}+e^{-z|\nabla|}}{e^{-|\nabla|}+e^{|\nabla|}}e^{-(s+1)|\nabla|}\Big], \tag{3.10}$$

$$K_3(z,s) = \Big[\frac{\nabla}{2|\nabla|}e^{-|z-s||\nabla|}, \frac{1}{2}e^{-|z-s||\nabla|}\mathrm{sign}(s-z)\Big], \tag{3.11}$$

$$g_1(z) = \frac{2h+h^2-(z+1)^2|\nabla h|^2}{(1+h)^2}\partial_z\varphi + \frac{(z+1)\nabla h\cdot\nabla\varphi}{1+h}, \quad g_1(-1)=0, \tag{3.12}$$

$$g_2(z) = \frac{(z+1)|\nabla h|^2\partial_z\varphi}{(1+h)^2} - \frac{\nabla h\cdot\nabla\varphi}{1+h}, \quad g_3(z) = \frac{(z+1)\nabla h\partial_z\varphi}{1+h}. \tag{3.13}$$

In the small data regime, the fixed point type formulation (3.8) is sufficient to derive the L^2-type and L^∞-type estimates for $\nabla_{x,z}\varphi$ as summarized in the following lemma.

Lemma 3.2. – *Assume that* $(h,\psi)\in H^{N_0+1/2}(\mathbb{R}^2)\times H^{N_0+1/2}(\mathbb{R}^2)$ *and given any* $k,\gamma\in\mathbb{R}$ *s.t.,* $k\le N_0-1$ *and* $\gamma\le N_0-3$. *Under the bootstrap assumption* (3.1), *there exists an absolute constant C such that the following estimates hold for the derivative of velocity potential "*$\nabla_{x,z}\varphi$*"*

inside the mapped water region,

(3.14)
$$\|\nabla_{x,z}\varphi\|_{L^\infty_z H^k} \le C\big[\|\nabla\psi\|_{H^k} + \|h\|_{H^{k+1}}\|\nabla\psi\|_{\widetilde{W^0}}\big],$$

(3.15)
$$\|\nabla_x\varphi\|_{L^\infty_z\widetilde{W^\gamma}} \le C\|\nabla\psi\|_{\widetilde{W^\gamma}}, \quad \|\partial_z\varphi\|_{L^\infty_z\widetilde{W^\gamma}} \le C\big[\|\psi\|_{W^{\gamma,1+\alpha}} + \|h\|_{\widetilde{W^{\gamma+1}}}\|\nabla\psi\|_{\widetilde{W^\gamma}}\big],$$

(3.16)
$$\|\Lambda_{\ge 2}[\nabla_{x,z}\varphi]\|_{L^\infty_z\widetilde{W^\gamma}} \le C\|\nabla\psi\|_{\widetilde{W^\gamma}}\|h\|_{\widetilde{W^{\gamma+1}}},$$

(3.17)
$$\|\Lambda_{\ge 2}[\nabla_{x,z}\varphi]\|_{L^\infty_z H^k} \le C\big[\|h\|_{\widetilde{W^1}}\||\nabla|\psi\|_{H^k} + \|\nabla\psi\|_{\widetilde{W^0}}\|h\|_{H^{k+1}}\big],$$
$$\|\Lambda_{\ge 2}[\nabla_{x,z}\varphi]\|_{L^\infty_z L^2} \le C\big[\big(\|(h,\psi)\|_{W^{6,1+\alpha}} + \|(h,\psi)\|_{W^{6,1}}\|(h,\psi)\|_{W^{6,0}}\big)\|(h,\psi)\|_{H^2}\big],$$

where the L^∞_x-type function space $\widetilde{W^\gamma}$ is defined as follows,

$$\widetilde{W^\gamma} := \{f : \|f\|_{\widetilde{W^\gamma}} := \|P_{\le 0}[f](x)\|_{L^\infty_x} + \sum_{k\in\mathbb{Z},k\ge 1} 2^{\gamma k}\|P_k(x)\|_{L^\infty_x} < \infty\}.$$

Proof. – Thanks to the small data regime, above estimates can be obtained from the fixed point type formulation in (3.8) by using a fixed point type argument. With minor modifications, the proof of above estimates are almost same as the proof of Lemma 3.3 in [34]. □

During the paralinearization process, we usually omit good error terms, which do not lose derivatives. For simplicity, we define the equivalence relation "≈" as follows,

$$A \approx B, \quad \text{if and only if } A - B \text{ is a good error term in the sense of (3.18)},$$

(3.18)
$$\|\text{good error term}\|_{H^k} \le C\big[\|(h,\psi)\|_{W^{6,1+\alpha}} + \|(h,\psi)\|_{W^{6,0}}\|(h,\psi)\|_{W^{6,1}}\big]\big(\|h\|_{H^k} + \|\psi\|_{H^{(k-1/2)_+}}\big),$$

where C is an absolute constant and $0 \le k \le N_0$.

As a result of paralinearization in Alazard-Burq-Zuily [1], modulo the good error terms, we can identify the principal part of the Dirichlet-Neumann operator as in the following lemma.

Lemma 3.3. – *Under the smallness condition* (4.49)*, the following equivalence relation in the sense of* (3.18) *holds,*

(3.19)
$$G(h)\psi \approx T_\lambda\omega - T_V\cdot\nabla h, \quad \omega := \psi - T_B h,$$
$$B \overset{\text{abbr}}{=} B(h)\psi = \frac{G(h)\psi + \nabla h\cdot\nabla\psi}{1+|\nabla h|^2}, \quad V \overset{\text{abbr}}{=} V(h)\psi = \nabla\psi - B\nabla h,$$
$$\lambda = \lambda^{(1)} + \lambda^{(0)}, \quad \lambda^{(1)} := \sqrt{(1+|\nabla h|^2)|\xi|^2 - (\nabla h\cdot\xi)^2},$$
$$\lambda^{(0)} = \frac{1+|\nabla h|^2}{2\lambda^{(1)}}\Big(\nabla\cdot\big(\frac{\lambda^{(1)} + i\nabla h\cdot\xi}{1+|\nabla h|^2}\nabla h\big) + i\nabla_\xi\lambda^{(1)}\cdot\nabla\big(\frac{\lambda^{(1)} + i\nabla h\cdot\xi}{1+|\nabla h|^2}\big)\Big),$$

where "ω" is the so-called good unknown variable and $\lambda^{(1)}$ and $\lambda^{(0)}$ are the principal symbol and sub-principal of the Dirichlet-Neumann operator respectively.

Proof. – The detailed proof of above lemma can be found in Alazard-Burq-Zuily [1, Proposition 3.14]. Only minor modifications are required. □

As a result of paralinearization in [1], modulo the good error terms, we can identify the bulk terms of the nonlinearity of "$\partial_t \psi$" in the capillary wave system (1.3) as in the following lemma.

LEMMA 3.4. – *Under the bootstrap assumption* (3.1)*, the following equivalence relation in the sense of* (3.18) *holds,*

$$H(h) \approx -T_l h, \qquad l = l^{(2)} + l^{(1)}, \qquad l^{(2)} = (1+|\nabla h|^2)^{-1/2}\Big(|\xi|^2 - \frac{(\nabla h\cdot\xi)^2}{1+|\nabla h|^2}\Big),$$
$$l^{(1)} = \frac{-i}{2}(\nabla_x\cdot\nabla_\xi)l^{(2)},$$

(3.20)
$$\frac{1}{2}|\nabla\psi|^2 - \frac{1}{2}\frac{(\nabla h\cdot\nabla\psi + G(h)\psi)^2}{1+|\nabla h|^2} \approx T_V\cdot\nabla\omega - T_B G(h)\psi.$$

Proof. – See [1, Lemma 3.25 & Lemma 3.26]. □

3.2. Symmetrization of the full system

In this subsection, we use the results obtained in the paralinearization process to find out the good substitution variables such that the system of equations satisfied by the good substitution variables has requisite symmetries inside.

Recall (1.3) and results in Lemma 3.3 and Lemma 3.4, we have

(3.21)
$$\begin{cases} \partial_t h \approx T_\lambda \omega - T_V\cdot\nabla h \\ \partial_t\psi \approx -T_l h + T_B G(h)\psi - T_V\cdot\nabla\omega. \end{cases}$$

The symmetrization process, which is only relevant at the high frequency part, is same as what Alazard-Burq-Zuily did in [1]. We first state the main results and then briefly explain main ideas behind.

Intuitively speaking, the symmetrization process can be summarized as seeking two good substitution variables $(U_1, U_2) = (T_p h, T_q w)$ with good unknown symbols $p(x,\xi)$ and $q(x,\xi)$ to be determined such that the system of equations satisfied by (U_1, U_2) has requisite symmetries such that it doesn't lose derivatives during energy estimate. As a result of the symmetrization process in Alazard-Burq-Zuily [1], the good substitution variables and their associated symbols are given as follows,

(3.22)
$$U_1 = \tilde\Lambda(h + T_{p|\xi|^{-1/2}-1}h), \quad U_2 = \omega + T_{q-1}\omega,$$
$$\omega = \psi - T_B h, \qquad \tilde\Lambda = |\nabla|^{1/2}(\tanh|\nabla|)^{-1/2},$$

where

(3.23)
$$p = p^{(1/2)} + p^{(-1/2)}, \quad q = (1+|\nabla h|^2)^{-1/2},$$

(3.24)
$$p^{(1/2)} = (1+|\nabla h|^2)^{-5/4}\sqrt{\lambda^{(1)}}, \quad p^{(-1/2)} = \frac{1}{\gamma^{(3/2)}}\big[q l^{(1)} - \gamma^{(1/2)}p^{(1/2)} + i\nabla_\xi\gamma^{(3/2)}\cdot\nabla_x p^{(1/2)}\big],$$

$$\gamma = \sqrt{l^{(2)}\lambda^{(1)}} + \sqrt{\frac{l^{(2)}}{\lambda^{(1)}}}\frac{\mathrm{Re}\lambda^{(0)}}{2} - \frac{i}{2}\big(\nabla_\xi \cdot \nabla_x\big)\sqrt{l^{(2)}\lambda^{(1)}} - |\xi|^{3/2}. \tag{3.25}$$

Note that, in the sense of losing derivatives, U_1 and U_2 are equivalent to $T_p h$ and $T_q \omega$. Here, we pulled out and emphasized the leading linear terms.

From (3.21) and (3.22), we can derive the system of equations satisfied by U_1 and U_2 as follows,

$$\begin{cases} \partial_t U_1 = \Lambda U_2 + T_\gamma U_2 - T_V \cdot \nabla U_1 + \mathfrak{R}_1, \\ \partial_t U_2 = -\Lambda U_1 - T_\gamma U_1 - T_V \cdot \nabla U_2 + \mathfrak{R}_2, \end{cases} \tag{3.26}$$

where $\mathfrak{R}_1$ and $\mathfrak{R}_2$ are good error terms in the sense of (3.18), i.e., the following estimate holds for the error terms for some absolute constant C,

$$\|\mathfrak{R}_1\|_{H^{N_0}} + \|\mathfrak{R}_2\|_{H^{N_0}} \leq C\big(\|(h,\psi)\|_{W^{6,1+\alpha}} + \|(h,\psi)\|_{W^{6,1}}\|(h,\psi)\|_{W^{6,0}}\big)\|(\tilde{\Lambda}h,\psi)\|_{H^{N_0}}. \tag{3.27}$$

Very importantly, the symbol "$\gamma(x,\xi)$" satisfies the following equivalence relation,

$$T_\gamma \sim (T_\gamma)^*, \tag{3.28}$$

where the equivalence relation "$\sim$" is defined in the following sense,

$$T_{a_1} \sim T_{a_2}, \text{ iff } \|T_{a_1} f - T_{a_2} f\|_{H^k} \leq C_k\big(\|(h,\psi)\|_{W^{6,1+\alpha}} + \|(h,\psi)\|_{W^{6,1}}\|(h,\psi)\|_{W^{6,0}}\big)\|f\|_{H^k},$$

where $k \in \mathbb{R}_+$ and C_k is some constant that only depends on "k". From the above equivalence relation, we can verify that the system (3.26) indeed has requisite symmetries for avoiding losing derivatives.

Now, we explain why the good unknown symbols $p(x,\xi)$ and $q(x,\xi)$ are given as (3.23) and (3.24). Recall that $\lambda \in \Gamma_5^1$ and $l \in \Gamma_5^2$. To obtain the system (3.26) from the system (3.21), naturally, we are seeking $p \in \Gamma_5^{1/2}$, $q \in \Gamma_5^0$ and $\lambda \in \Gamma_5^{3/2}$ such that the equivalence relation (3.28) and the following two equivalence relations hold at the same time,

$$T_p T_\lambda \sim T_{\gamma+|\xi|^{3/2}} T_q, \quad T_q T_l \sim T_{\gamma+|\xi|^{3/2}} T_p. \tag{3.29}$$

From Lemma 2.5, we have

$$\big(T_\gamma\big)^* \sim T_{\lambda^*}, \quad \lambda^* = \gamma^{(3/2)} + \overline{\gamma^{(1/2)}} + \frac{1}{i}\nabla_\xi \cdot \nabla_x \gamma^{(3/2)}. \tag{3.30}$$

Hence, (3.28) can be reformulated as follows,

$$T_\gamma \sim T_{\gamma^{(3/2)} + \overline{\gamma^{(1/2)}} + \frac{1}{i}\nabla_\xi \cdot \nabla_x \gamma^{(3/2)}}. \tag{3.31}$$

By using Lemma 2.6, we can derive six equations about the principal symbols and sub-principal symbols of $p(x,\xi)$, $q(x,\xi)$, and $\gamma(x,\xi)$ from the three equivalence relations in (3.29) and (3.31). After solving those equations, one can see that the principal symbols and sub-principal symbols of $p(x,\xi)$, $q(x,\xi)$, and $\gamma(x,\xi)$ are given as in (3.23), (3.24), and (3.25). For more detailed computations, please refer to [1, Subsection 4.2].

From the bootstrap assumption (3.1) and estimates in Lemma 3.2, the following estimate holds,

$$\|U_1 - \tilde{\Lambda}h\|_{H^{N_0}} + \|U_2 - \psi\|_{H^{N_0}} \leq C(\|h\|_{W^{6,1}} + \|\psi\|_{W^{6,1}})\|(\tilde{\Lambda}h,\psi)\|_{H^{N_0}} \leq C\epsilon_1^2. \tag{3.32}$$

where C is an absolute constant. From the above estimate (3.32), we know that the difference of energy between (U_1, U_2) and $(\tilde{\Lambda}h, \psi)$ is a higher order smallness. Therefore, to control the energy of $(\tilde{\Lambda}h, \psi)$ over time, it would be sufficient to control the energy of (U_1, U_2) over time.

3.3. Energy estimate

We define the energy as follows,

$$E_{N_0}(t) := \|U_1(t)\|_{L^2}^2 + \|U_2(t)\|_{L^2}^2 + \|U_1^{N_0}(t)\|_{L^2}^2 + \|U_2^{N_0}(t)\|_{L^2}^2, \tag{3.33}$$

where

$$U_1^{N_0}(t) = T_\beta U_1(t), \quad U_2^{N_0}(t) = T_\beta U_2(t), \quad \beta := \big(\gamma^{(3/2)} + |\xi|^{3/2}\big)^{2N_0/3}, \tag{3.34}$$

where $\gamma^{(3/2)}(x,\xi)$ is the principal symbol of $\gamma(x,\xi)$, which is defined in (3.25). Note that, from the above definition, the following equality holds,

$$\partial_\xi \beta \partial_x \big(\gamma^{(3/2)} + |\xi|^{3/2}\big) = \partial_\xi \big(\gamma^{(3/2)} + |\xi|^{3/2}\big) \partial_x \beta.$$

Hence, very importantly, the operator as follows is an operator of order zero,

$$T_\beta T_{\gamma+|\xi|^{3/2}} - T_{\gamma+|\xi|^{3/2}} T_\beta.$$

REMARK 3.1. – To estimate the high order Sobolev norm, we use the variable $T_\beta U_i$ instead of using $|\nabla|^{N_0} U_i$ because the commutator $[T_{|\xi|^{N_0}}, T_\gamma]$ is of order $1/2$, which causes the loss of derivatives. The idea of using the good variables $T_\beta U_1$ and $T_\beta U_2$ comes from the work of Alazard-Burq-Zuily [1].

Recall the definition (3.34) and the system (3.26). As a result of direct computation, we can derive the system of equations satisfied by $U_1^{N_0}$ and $U_2^{N_0}$ as follows,

$$\begin{cases} \partial_t U_1^{N_0} = \Lambda U_1^{N_0} + T_\gamma U_2^{N_0} - T_V \cdot \nabla U_1^{N_0} + \mathfrak{R}_1^{N_0}, \\ \partial_t U_2^{N_0} = -\Lambda U_1^{N_0} - T_\gamma U_2^{N_0} - T_V \cdot \nabla U_2^{N_0} + \mathfrak{R}_2^{N_0}, \end{cases} \tag{3.35}$$

where the good remainder terms $\mathfrak{R}_1^{N_0}$ and $\mathfrak{R}_1^{N_0}$ satisfy the following estimate,

$$\|\mathfrak{R}_1^{N_0}\|_{L^2} + \|\mathfrak{R}_2^{N_0}\|_{L^2} \le C\big(\|(h,\psi)\|_{W^{6,1+\alpha}} + \|(h,\psi)\|_{W^{6,1}} \|(h,\psi)\|_{W^{6,0}}\big) \|(\tilde{\Lambda}h,\psi)\|_{H^{N_0}}, \tag{3.36}$$

where C is some absolute constant. Recall (3.33). From the bootstrap assumption (3.1), it is easy to see that the following estimate holds,

$$\begin{aligned} c_2\big(\|\tilde{\Lambda}h(t)\|_{H^{N_0}}^2 + \|\psi(t)\|_{H^{N_0}}^2\big) &\le c_1\big(\|U_1(t)\|_{H^{N_0}}^2 + \|U_2(t)\|_{H^{N_0}}^2\big) \le E_{N_0}(t) \\ &\le C_1\big(\|U_1(t)\|_{H^{N_0}}^2 + \|U_2(t)\|_{H^{N_0}}^2\big) \\ &\le C_2\big(\|\tilde{\Lambda}h(t)\|_{H^{N_0}}^2 + \|\psi(t)\|_{H^{N_0}}^2\big), \end{aligned} \tag{3.37}$$

where c_i and C_i, $i \in \{1,2\}$, are some absolute constants.

Recall the systems of equations in (3.26) and (3.27). From the estimates (3.35) and (3.36) and the $L^2 - L^\infty$ type bilinear estimate, we have

$$
\begin{aligned}
\Big|\frac{d}{dt}E_{N_0}(t)\Big| &\leq C_1\Big[\|(U_1(t),U_2(t))\|_{H^{N_0}}\|(\mathfrak{R}_1(t),\mathfrak{R}_2(t),\mathfrak{R}_1^{N_0}(t),\mathfrak{R}_2^{N_0}(t))\|_{L^2} \\
&\qquad + \Big|\int_{\mathbb{R}^2} U_1\big(-T_V\cdot\nabla U_1\big) + U_2\big(-T_V\cdot\nabla U_2\big) \\
&\qquad\quad + U_1^{N_0}\big(-T_V\cdot\nabla U_1^{N_0}\big) + U_2^{N_0}\big(-T_V\cdot\nabla U_2^{N_0}\big)dx\Big| \\
&\qquad + \Big|\int_{\mathbb{R}^2} U_1(T_\lambda U_2) - U_2(T_\lambda U_1) + U_1^{N_0}(T_\lambda U_2^{N_0}) - U_2^{N_0}(T_\lambda U_1^{N_0})\Big|\Big] \\
&\leq C_2\Big[\big(\|(h,\psi)\|_{W^{6,1+\alpha}} + \|(h,\psi)\|_{W^{6,1}}\|(h,\psi)\|_{W^{6,0}}\big)\|(U_1,U_2)\|_{H^{N_0}}^2 \\
&\qquad + \Big|\int_{\mathbb{R}^2} U_1(T_\lambda - (T_\lambda)^*)U_2 + U_1^{N_0}(T_\lambda - (T_\lambda)^*)U_2^{N_0}dx\Big|\Big] \\
&\leq C_3\big(\|(h,\psi)\|_{W^{6,1+\alpha}} + \|(h,\psi)\|_{W^{6,1}}\|(h,\psi)\|_{W^{6,0}}\big)\|(U_1,U_2)\|_{H^{N_0}}^2,
\end{aligned} \tag{3.38}
$$

where $C_i, i \in \{1,2,3\}$, are some absolute constants. Note that, in the above estimate, we used the following facts, which are direct results from (3.30),

$$
\Lambda_1[\gamma] = |\xi|^{1/2}\big(\frac{1}{2}\Delta h - \frac{\xi}{|\xi|}\cdot\nabla_x(\nabla h\cdot\frac{\xi}{|\xi|})\big), \quad M_5^0(\Lambda_{\geq 2}[\gamma] - \Lambda_{\geq 2}[\gamma^*]) \leq C\|h\|_{W^{6,1}}^2, \tag{3.39}
$$

where C is some absolute constant. The first equality in the above equality (3.39) is derived from the explicit formula of γ in (3.25). Note that $\Lambda_1[\gamma]$ only depends on the second derivative of h, which explains why we can gain $(1+\alpha)$ derivatives at the low frequency part for the input putted in L^∞-type space.

Combining the estimates (3.38) and (3.37), it is easy to see that the desired estimate (3.2) in Proposition 3.1 holds. Hence finishing the proof of Proposition 3.1.

4. The set-up of the weighted norm estimates

By using the linear dispersion estimates in Lemma 2.7, we reduce the study of the dispersion estimate of the nonlinear solution to the study of the weighted norms of the profile of the nonlinear solution.

In this section, we mainly introduce the set-up of the weighted norms (the Z_1-norm and the Z_2-norm) estimates, which includes two main steps as follows: (i) We identify a good substitution variable, which allows us to study and control properly the evolution of the weighted norms of the good substitution variable over time. (ii) We reduce our goal of proving the sharp dispersion estimate to two desired estimates inside a fixed dyadic time interval.

4.1. A good substitution variable

To avoid losing derivatives at the quadratic level, we use the following variable instead of the velocity potential "ψ" itself,

$$
\tilde{\psi} := \psi - T_{|\nabla|\tanh|\nabla|\psi}h,
$$

which is the linear and quadratic terms of the good unknown variable "ω" defined in (3.22). Hence, instead of working on the system of equations satisfied by (h,ψ), we work on the system of equations satisfied by $(h,\tilde{\psi})$.

From (1.6) and (1.7), as a result of direct computations, we obtain the following equalities,

$$\Lambda_{\leq 2}[\partial_t h] = |\nabla|\tanh|\nabla|\tilde{\psi} + |\nabla|\tanh|\nabla|\big(T_{|\nabla|\tanh|\nabla|\tilde{\psi}}h\big) - \nabla\cdot(h\nabla\tilde{\psi}) - |\nabla|\tanh|\nabla|(h|\nabla|\tanh|\nabla|\tilde{\psi}) \tag{4.1}$$

$$\Lambda_{\leq 2}[\partial_t \tilde{\psi}] = \Delta h - \frac{1}{2}|\nabla\tilde{\psi}|^2 + \frac{1}{2}||\nabla|\tanh|\nabla|\tilde{\psi}|^2 - T_{|\nabla|\tanh|\nabla|\tilde{\psi}}|\nabla|\tanh|\nabla|\tilde{\psi} - T_{|\nabla|\tanh|\nabla|\Delta h}h. \tag{4.2}$$

We remark that the Taylor expansions in (4.1), (4.2) and also in the rest of paper are all in terms of $(h,\tilde{\psi})$.

Next, we reduce the system of equations satisfied by h and $\tilde{\psi}$ into a quasilinear equation satisfied by $u = \tilde{\Lambda}h + i\tilde{\psi}$, where $\tilde{\Lambda} = |\nabla|^{1/2}\big(\tanh|\nabla|\big)^{-1/2}$. Very naturally, we have

$$h = \tilde{\Lambda}^{-1}\Big(\frac{u+\bar{u}}{2}\Big), \quad \tilde{\psi} = c_+u + c_-\bar{u}, \quad c_\mu := -\mu i/2. \tag{4.3}$$

There, from (1.3), (4.1), and (4.2), we can derive the equation satisfied by u as follows,

$$(\partial_t + i\Lambda)u = \sum_{\mu,\nu\in\{+,-\}} Q_{\mu,\nu}(u^\mu,u^\nu) + \sum_{\tau,\kappa,\iota\in\{+,-\}} C_{\tau,\kappa,\iota}(u^\tau,u^\kappa,u^\iota) + \sum_{\mu_1,\mu_2,\nu_1,\nu_2\in\{+,-\}} D_{\mu_1,\mu_2,\nu_1,\nu_2}(u^{\mu_1},u^{\mu_2},u^{\nu_1},u^{\nu_2}) + \mathscr{R}, \tag{4.4}$$

where $\mathscr{R}$ denotes the quintic and higher order terms. From (4.1), (4.2), and (4.3), we can obtain the detailed formulas of quadratic terms as follows,

$$\begin{aligned} Q_{\mu,\nu}(u^\mu,u^\nu) = & -\frac{c_\nu}{2}\tilde{\Lambda}\partial_x(\tilde{\Lambda}^{-1}u^\mu\partial_x u^\nu) \\ & - \frac{c_\nu}{2}\tilde{\Lambda}|\nabla|\tanh|\nabla|(\tilde{\Lambda}^{-1}u^\mu|\nabla|\tanh|\nabla|u^\nu - T_{|\nabla|\tanh|\nabla|u^\nu}\tilde{\Lambda}^{-1}u^\mu) \\ & + \frac{ic_\mu c_\nu}{2}\big[-\nabla u^\mu\cdot\nabla u^\nu + |\nabla|\tanh|\nabla|u^\mu|\nabla|\tanh|\nabla|u^\nu \\ & - T_{|\nabla|\tanh|\nabla|u^\mu}|\nabla|\tanh|\nabla|u^\nu - T_{|\nabla|\tanh|\nabla|u^\nu}|\nabla|\tanh|\nabla|u^\mu\big] \\ & - \frac{i}{4}T_{|\nabla|\tanh|\nabla|\Delta u^\nu}u^\mu, \quad \mu,\nu\in\{+,-\}. \end{aligned} \tag{4.5}$$

We gave the detailed formulas of quadratic terms $Q_{\mu,\nu}(\cdot,\cdot)$, $\mu,\nu \in \{+,-\}$, because the precise detailed formulas help us to verify a symmetric structure that we will reveal later.

Define the profile of the solution $u(t)$ as $f(t) := e^{it\Lambda}u(t)$. From (4.4), we have

$$\partial_t\widehat{f}(t,\xi) = \sum_{(\mu,\nu)\in\{+,-\}}\int_{\mathbb{R}^2} e^{it\Phi^{\mu,\nu}(\xi,\eta)}q_{\mu,\nu}(\xi-\eta,\eta)\widehat{f^\mu}(t,\xi-\eta)\widehat{f^\nu}(\eta)d\eta$$

$$+ \sum_{\tau,\kappa,\iota\in\{+,-\}}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{it\Phi^{\tau,\kappa,\iota}(\xi,\eta,\sigma)}c_{\tau,\kappa,\iota}(\xi-\eta,\eta-\sigma,\sigma)\widehat{f^\tau}(t,\xi-\eta)\widehat{f^\kappa}(t,\eta-\sigma)\widehat{f^\iota}(t,\sigma)d\eta d\sigma$$

$$+ \sum_{\mu_1,\mu_2,\nu_1,\nu_2\in\{+,-\}}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{it\Phi^{\mu_1,\mu_2,\nu_1,\nu_2}(\xi,\eta,\sigma,\kappa)}d_{\mu_1,\mu_2,\nu_1,\nu_2}(\xi-\eta,\eta-\sigma,\sigma-\kappa,\kappa)\widehat{f^{\mu_1}}(t,\xi-\eta)$$

$$\times \widehat{f^{\mu_2}}(t,\eta-\sigma)\widehat{f^{\nu_1}}(t,\sigma-\kappa)\widehat{f^{\nu_2}}(t,\kappa)d\eta d\sigma d\kappa + e^{it\Lambda(\xi)}\widehat{\mathcal{R}}(t,\xi), \tag{4.6}$$

where the phases $\Phi^{\mu,\nu}(\xi,\eta)$, $\Phi^{\tau,\kappa,\iota}(\xi,\eta,\sigma)$, and $\Phi^{\mu_1,\mu_2,\nu_1,\nu_2}(\xi,\eta,\sigma,\kappa)$ are defined as follows,

$$\Phi^{\mu,\nu}(\xi,\eta) = \Lambda(|\xi|) - \mu\Lambda(|\xi-\eta|) - \nu\Lambda(|\eta|), \quad \Lambda(|\xi|) := |\xi|^{3/2}\sqrt{\tanh|\xi|}, \tag{4.7}$$

$$\Phi^{\tau,\kappa,\iota}(\xi,\eta,\sigma) = \Lambda(|\xi|) - \tau\Lambda(|\xi-\eta|) - \kappa\Lambda(|\eta-\sigma|) - \iota\Lambda(|\sigma|), \tag{4.8}$$

(4.9)

$$\Phi^{\mu_1,\mu_2,\nu_1,\nu_2}(\xi,\eta,\sigma,\kappa) = \Lambda(|\xi|) - \mu_1\Lambda(|\xi-\eta|) - \mu_2\Lambda(|\eta-\sigma|) - \nu_1\Lambda(|\sigma-\kappa|) - \nu_2\Lambda(|\kappa|).$$

From (4.5), we write explicitly the symbol $q_{\mu,\nu}(\xi-\eta,\eta)$ of $Q_{\mu,\nu}(u^\mu,u^\nu)$ in the sense of (2.2) as follows,

(4.10)

$$\begin{aligned} q_{\mu,\nu}(\xi-\eta,\eta) = \Big(&\frac{c_\nu\tilde{\lambda}(|\xi|^2)}{2\tilde{\lambda}(|\xi-\eta|^2)}\big(\xi\cdot\eta - |\xi||\eta|\tanh(|\xi|)\tanh(|\eta|)\big) \\ &+ \frac{ic_\mu c_\nu}{2}\big((\xi-\eta)\cdot\eta + |\xi-\eta||\eta|\times\tanh(|\xi-\eta|)\tanh(|\eta|)\big)\Big)\tilde{\theta}(\eta,\xi-\eta) \\ + \Big(&\frac{c_\mu\tilde{\lambda}(|\xi|^2)}{2\tilde{\lambda}(|\eta|^2)}\big((\xi-\eta)\cdot\xi - |\xi-\eta||\xi|\tanh(|\xi|)\tanh(|\xi-\eta|)\big) \\ &+ \frac{c_\nu\tilde{\lambda}(|\xi|^2)}{2\tilde{\lambda}(|\xi-\eta|^2)}\xi\cdot\eta + ic_\mu c_\nu(\xi-\eta)\cdot\eta + \frac{i}{4}|\eta|^2(\tanh|\eta|)^2\Big)\theta(\eta,\xi-\eta), \end{aligned}$$

where

$$\tilde{\lambda}(\xi) := |\xi|^{1/4}(\tanh(\sqrt{|\xi|}))^{-1/2}, \quad \tilde{\lambda}(\xi) = 1 + \frac{|\xi|}{6} + O(|\xi|^2), \quad \text{if } |\xi| \le 2^{-10},$$

$$\tilde{\theta}(\eta,\xi-\eta) := 1 - \theta(\eta,\xi-\eta) - \theta(\xi-\eta,\eta). \tag{4.11}$$

Note that, in (4.10), we switched the roles of $\xi-\eta$ and η when $|\xi-\eta| \le 2^{-10}|\eta|$. As a result, the following estimate holds inside the support of the symbol $q_{\mu,\nu}(\xi-\eta,\eta)$, $\mu,\nu\in\{+,-\}$,

$$k_2 \le k_1 + 10, \quad \text{where } \eta \in \mathrm{supp}(\psi_{k_2}(x)), \xi-\eta \in \mathrm{supp}(\psi_{k_1}(\xi-\eta)). \tag{4.12}$$

From the estimate (2.3) in Lemma 2.1 and the detailed formula of the symbol $q_{\mu,\nu}(\xi-\eta,\eta)$ in (4.10), the following rough estimate holds for some absolute constant C,

$$\|q_{\mu,\nu}(\xi-\eta,\eta)\psi_k(\xi)\psi_{k_1}(\xi-\eta)\psi_{k_2}(\eta)\|_{\mathcal{S}^\infty} \le C2^{2k_1}, \quad \mu,\nu\in\{+,-\}. \tag{4.13}$$

Moreover, from the explicit formula in (4.10), we can identify the leading part "$c(\xi)$" of $q_{+,\nu}(\xi-\eta,\eta)$ for the case when $|\eta| \le 2^{-10}|\xi|$ as follows,

$$c(\xi) := \frac{c_+}{2}\tilde{\lambda}(|\xi|^2)|\xi|^2\big(1-\tanh(|\xi|)^2\big). \tag{4.14}$$

After subtracting $c(\xi)$ from the symbol $q_{+,\nu}(\cdot,\cdot)$, from the estimate (2.3) in Lemma 2.1, the following improved estimate holds for some absolute constant C,

$$\|\big(q_{+,\nu}(\xi-\eta,\eta) - c(\xi)\big)\psi_{k_1}(\xi-\eta)\psi_{k_2}(\eta)\|_{\mathcal{S}^\infty} \le C2^{k_2+k_1}, \quad \text{if } k_2 \le k_1 - 10. \tag{4.15}$$

Since it is also very essential to identify the symmetric structure inside the cubic terms, which will play an important role in the Z_2-norm estimate of the cubic terms, we summarize some properties of the symbols of cubic terms of the Dirichlet-Neumann operator as in the following lemma.

LEMMA 4.1. – *After writing the cubic term* $\Lambda_3[B(h)\psi]$ *in terms of* u *and* $\bar{u}$ *via the equality* (4.3)*, we do dyadic decompositions for all inputs and rearrange inputs such that the following unique decomposition holds*

$$\Lambda_3[B(h)\psi] = \sum_{\mu,\nu,\tau\in\{+,-\}} C'_{\mu,\nu,\tau}(u^{\mu}, u^{\nu}, u^{\tau}),$$

where the first input u^{μ} *of cubic term* $C'_{\mu,\nu,\tau}(u^{\mu}, u^{\nu}, u^{\tau})$ *has the largest scale of dyadic localization among three inputs. Then there exists an absolute constant* C *such that the following estimates hold for the symbol* $c'_{\mu,\nu,\tau}(\xi,\eta,\sigma)$ *of the cubic term* $C'_{\mu,\nu,\tau}(u^{\mu}, u^{\nu}, u^{\tau})$,

$$\|c'_{\mu,\nu,\tau}(\xi,\eta,\sigma)\psi_{k_1}(\xi-\eta)\psi_{k_2}(\eta-\sigma)\psi_{k_3}(\sigma)\|_{\mathcal{S}^\infty} \leq C2^{2k_1+2k_{1,+}}. \tag{4.16}$$

(4.17)

$$\|\big(c'_{\mu,\nu,\tau}(\xi,\eta,\sigma) - \frac{c_\mu}{4}d(\xi)\big)\psi_{k_1}(\xi-\eta)\psi_{k_2}(\eta-\sigma)\psi_{k_3}(\sigma)\|_{\mathcal{S}^\infty} \leq C2^{\max\{k_2,k_3\}+3k_{1,+}},$$

if $k_2, k_3 \leq k_1 - 10$*, where the detailed formula of* $d(\xi)$ *is given in* (4.19)*. Moreover, there exists an absolute constant* C *such that the following rough estimate holds for the symbol of quartic terms* $\Lambda_4[B(h)\psi]$,

$$\|d_{\mu_1,\nu_1,\mu_2,\nu_2}(\xi,\eta,\sigma,\kappa)\psi_{k_1}(\xi-\eta)\psi_{k_2}(\eta-\sigma)\psi_{k_3}(\sigma-\kappa)\psi_{k_4}(\kappa)\|_{\mathcal{S}^\infty} \leq C2^{2\max\{k_1,\ldots,k_4\}+3\max\{k_1,\ldots,k_4\}_+}. \tag{4.18}$$

Proof. – Note that the detailed formulas of symbols of cubic terms and quartic terms can be derived from iterating the fixed point type formulation of $\nabla_{x,z}\varphi$ in (3.8). To prove (4.16) and (4.18), it is sufficient to prove that the corresponding estimates hold for $\Lambda_3[g_i(z)]$ and $\Lambda_4[g_i(z)]$, $i \in \{1,2,3\}$. From (3.12) and (3.13), we have

$$\Lambda_2[g_1(z)] = 2h\Lambda_1[\partial_z\varphi] + (z+1)\nabla h\cdot\Lambda_1[\nabla\varphi],$$
$$\Lambda_2[g_2(z)] = -\nabla h\cdot\Lambda_1[\nabla\varphi], \quad \Lambda_2[g_3(z)] = (z+1)\nabla h\Lambda_1[\partial_z\varphi].$$

Recall that

$$\Lambda_1[\nabla_{x,z}\varphi] = \Big[\Big[\frac{e^{-(z+1)|\nabla|}+e^{(z+1)|\nabla|}}{e^{-|\nabla|}+e^{|\nabla|}}\Big]\nabla\psi, \frac{e^{(z+1)|\nabla|}-e^{-(z+1)|\nabla|}}{e^{-|\nabla|}+e^{|\nabla|}}|\nabla|\psi\Big].$$

From the above formula and the formulation (3.8), we know that there are two derivatives inside $\Lambda_2[\nabla_{x,z}\varphi(z)]$ at low frequencies. We can keep doing the iteration process to check the minimal number and the maximal number of derivatives inside $\Lambda_3[\nabla_{x,z}\varphi]$. For example, from (3.12) and (3.13), we obtain the cubic terms of $g_i(z)$, $i \in \{1,2,3\}$, as follows,

$$\Lambda_3[g_1(z)] = 2h\Lambda_2[\partial_z\varphi] + \big(-3h^2 - (z+1)^2|\nabla h|^2\big)\Lambda_1[\partial_z\varphi] - (z+1)h\nabla h\cdot\nabla\varphi,$$
$$\Lambda_3[g_2(z)] = (z+1)|\nabla h|^2\Lambda_1[\partial_z\varphi] + h\nabla h\cdot\Lambda_1[\nabla\varphi] - \nabla h\cdot\Lambda_2[\nabla\varphi],$$
$$\Lambda_3[g_3(z)] = (z+1)\nabla h\Lambda_2[\partial_z\varphi] - (z+1)h\nabla h\Lambda_1[\partial_z\varphi].$$

Recall that there are at least two derivatives inside $\Lambda_2[\nabla_{x,z}\varphi]$. Hence, we know that there are at least two derivatives and at most four derivatives in total inside $\Lambda_3[\nabla_{x,z}\varphi]$. Following the same strategy, we know that there are at least two derivatives and at most five derivatives inside $\Lambda_4[\nabla_{x,z}\varphi]$. There two facts imply that our desired estimates (4.16) and (4.18) hold.

Next, we prove our desired estimate (4.17). We first identify the bulk term, in which all derivatives act on the input that has the largest scale of dyadic localization of frequencies.

With this principle in mind, recall (3.12) and (3.13), we know that the bulk term only appears in $g_1(z)$, which is $T_{(2h+h^2)/(1+h)^2}\partial_z\varphi$. Recall the fixed point formulation (3.8). We know that the bulk term of $\Lambda_2[\partial_z\varphi(z)]$, in which all derivatives act on the input with the largest scale of dyadic localization of frequencies, is given as follows,

$$\int_{-1}^{0}\big(-e^{-|z-s||\nabla|}-e^{(z+s)|\nabla|}+\frac{e^{z|\nabla|}+e^{-z|\nabla|}}{e^{-|\nabla|}+e^{|\nabla|}}(e^{(s-1)|\nabla|}+e^{-(s+1)|\nabla|})\big)$$
$$\times\frac{e^{s+1}|\nabla|-e^{-(s+1)|\nabla|}}{e^{|\nabla|}+e^{-|\nabla|}}|\nabla|^2\big(T_h\psi\big)ds+2\frac{e^{(z+1)|\nabla|}-e^{-(z+1)|\nabla|}}{e^{-|\nabla|}+e^{|\nabla|}}|\nabla|(T_h\psi).$$

In the same spirit, we can derive the bulk term of $\Lambda_3[\partial_z\varphi(z)|_{z=0}]$, in which all derivatives act on the input with the largest scale of dyadic localization of frequencies, is given as follows,

$$C(h,h,\psi)=\mathcal{F}^{-1}\big[\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\widehat{\psi}(\xi-\eta)\widehat{h}(\eta-\sigma)\widehat{h}(\sigma)d(\xi)\theta(\sigma,\xi)\theta(\eta-\sigma,\xi)d\eta d\sigma\big],$$

where

$$\begin{aligned}d(\xi):=2\int_{-1}^{0}\int_{-1}^{0}&\big(\frac{(e^{-(z+1)|\xi|}-e^{(z+1)|\xi|})}{e^{-|\xi|}+e^{|\xi|}}\big)\frac{e^{(s+1)|\xi|}-e^{-(s+1)|\xi|}}{e^{|\xi|}+e^{-|\xi|}}\\ &\times\big(\frac{(e^{z|\xi|}+e^{-z|\xi|})(e^{(s-1)|\xi|}+e^{-(s+1)|\xi|})}{e^{-|\xi|}+e^{|\xi|}}-e^{-|z-s||\xi|}-e^{(z+s)|\xi|}\big)|\xi|^3dsdz\\ &-\int_{-1}^{0}\big(\frac{(e^{-(s+1)|\xi|}-e^{(s+1)|\xi|})}{e^{-|\xi|}+e^{|\xi|}}\big)^2|\xi|^2ds+\tanh(|\xi|)|\xi|.\end{aligned}\tag{4.19}$$

After removing the bulk term, by definition, there is at least one derivative acts on the input, which doesn't have the largest scale of dyadic localization of frequencies, for the rest of terms. This fact implies that our desired estimate (4.17) holds. □

With the previous preparation, which improves our understanding of the equation satisfied by u in (4.4), we are now ready to find a good substitution variable. We seek a good substitution variable as follows,

$$\begin{aligned}v(t)=u(t)&+\sum_{\mu,\nu\in\{+,-\}}A_{\mu,\nu}(u^{\mu}(t),u^{\nu}(t))+\sum_{\tau,\kappa,\iota\in\{+,-\}}B_{\tau,\kappa,\iota}(u^{\tau}(t),u^{\kappa}(t),u^{\iota}(t))\\ &+\sum_{\mu_1,\mu_2,\nu_1,\nu_2\in\{+,-\}}E_{\mu_1,\mu_2,\nu_1,\nu_2}(u^{\mu_1}(t),u^{\mu_2}(t),u^{\nu_1}(t),u^{\nu_2}(t)),\end{aligned}\tag{4.20}$$

where quadratic terms $A_{\mu,\nu}(\cdot,\cdot)$, cubic terms $B_{\tau,\kappa,\iota}(\cdot,\cdot,\cdot)$, and quartic terms $E_{\mu_1,\mu_2,\nu_1,\nu_2}(\cdot,\cdot,\cdot,\cdot)$ are to be determined. From the equation satisfied by $u(t)$ in (4.4) and definition of $v(t)$ in (4.20), as a result of direct computation, we have

(4.21)
$$\begin{aligned}(\partial_t+i\Lambda)v=&\sum_{\mu,\nu\in\{+,-\}}\tilde{Q}_{\mu,\nu}(v^{\mu}(t),v^{\nu}(t))+\sum_{\tau,\kappa,\iota\in\{+,-\}}\tilde{C}_{\tau,\kappa,\iota}(v^{\tau}(t),v^{\kappa}(t),v^{\iota}(t))\\ &+\sum_{\mu_1,\mu_2,\nu_1,\nu_2\in\{+,-\}}\tilde{D}_{\mu_1,\mu_2,\nu_1,\nu_2}(v^{\mu_1}(t),v^{\mu_2}(t),v^{\nu_1}(t),v^{\nu_2}(t))+\mathcal{R}_1(t),\end{aligned}$$

where $\mathscr{R}_1(t)$ is the quintic and higher order terms. The quadratic terms and cubic terms are given as follows,

$$\tilde{Q}_{\mu,\nu}(v^{\mu}, v^{\nu}) = Q_{\mu,\nu}(v^{\mu}, v^{\nu}) + i\Lambda(A_{\mu,\nu}(v^{\mu}, v^{\nu})) - i\mu A_{\mu,\nu}(\Lambda v^{\mu}, v^{\nu}) - i\nu A_{\mu,\nu}(v^{\mu}, \Lambda v^{\nu}), \tag{4.22}$$

$$\begin{aligned}\tilde{C}_{\tau,\kappa,\iota}(v^{\tau}, v^{\kappa}, v^{\iota}) := \widehat{C}_{\tau,\kappa,\iota}(v^{\tau}, v^{\kappa}, v^{\iota}) + i\Lambda(B_{\tau,\kappa,\iota}(v^{\tau}, v^{\kappa}, v^{\iota})) - i\tau B_{\tau,\kappa,\iota}(\Lambda v^{\tau}, v^{\kappa}, v^{\iota})\\ - i\kappa B_{\tau,\kappa,\iota}(v^{\tau}, \Lambda v^{\kappa}, v^{\iota}) - i\iota B_{\tau,\kappa,\iota}(v^{\tau}, v^{\kappa}, \Lambda v^{\iota}),\end{aligned} \tag{4.23}$$

where the cubic term $\widehat{C}_{\tau,\kappa,\iota}(v^{\tau}, v^{\kappa}, v^{\iota})$ is the unique cubic term associated with the following equality, such that the scales of dyadic localized frequencies of inputs v^{τ}, v^{κ}, and v^{ι} are ordered in a descending manner after we rearrange the inputs,

$$\begin{aligned}\sum_{\tau,\kappa,\iota\in\{+,-\}} \widehat{C}_{\tau,\kappa,\iota}(v^{\tau}, v^{\kappa}, v^{\iota}) = \sum_{\tau,\kappa,\iota\in\{+,-\}} C_{\tau,\kappa,\iota}(v^{\tau}, v^{\kappa}, v^{\iota})\\ + \sum_{\mu,\nu,\mu_1,\nu_1\in\{+,-\}} A_{\mu,\nu}(P_{\mu}[Q_{\mu_1,\nu_1}(v^{\mu_1}, v^{\nu_1})], v^{\nu})\\ + A_{\mu,\nu}(v^{\nu}, P_{\nu}[Q_{\mu_1,\nu_1}(v^{\mu_1}, v^{\nu_1})])\\ - \tilde{Q}_{\mu,\nu}\big(P_{\mu}\big(A_{\mu_1,\nu_1}(v^{\mu_1}, u^{\nu_1})\big), v^{\nu}\big)\\ - \tilde{Q}_{\mu,\nu}\big(v^{\mu}, P_{\nu}\big(A_{\mu_1,\nu_1}(v^{\mu_1}, v^{\nu_1})\big)\big).\end{aligned} \tag{4.24}$$

More precisely, the following estimate holds inside the support of symbol $\hat{c}_{\tau,\kappa,\iota}(\xi-\eta, \eta-\sigma, \sigma)$ of the trilinear operator $\widehat{C}_{\tau,\kappa,\iota}(\cdot,\cdot,\cdot)$,

$$k_3 \le k_2 \le k_1, \qquad \text{where } \sigma \in \mathrm{supp}(\psi_{k_3}(x)), \eta-\sigma \in \mathrm{supp}(\psi_{k_2}(x)), \xi-\eta \in \mathrm{supp}(\psi_{k_1}(x)). \tag{4.25}$$

Similarly, for any $\mu_1, \mu_2, \nu_1, \nu_2 \in \{+,-\}$, the quartic term $\tilde{D}_{\mu_1,\mu_2,\nu_1,\nu_2}(v^{\mu_1}(t), v^{\mu_2}(t), v^{\nu_1}(t), v^{\nu_2}(t))$ in (4.21) is given as follows,

$$\begin{aligned}\tilde{D}_{\mu_1,\mu_2,\nu_1,\nu_2}(v^{\mu_1}(t), v^{\mu_2}(t), v^{\nu_1}(t), v^{\nu_2}(t)) = \widehat{D}_{\mu_1,\mu_2,\nu_1,\nu_2}(v^{\mu_1}(t), v^{\mu_2}(t), v^{\nu_1}(t), v^{\nu_2}(t))\\ + i\Lambda(E_{\mu_1,\mu_2,\nu_1,\nu_2}(v^{\mu_1}(t), v^{\mu_2}(t), v^{\nu_1}(t), v^{\nu_2}(t)))\\ - i\mu_1 E_{\mu_1,\mu_2,\nu_1,\nu_2}(\Lambda v^{\mu_1}(t), v^{\mu_2}(t), v^{\nu_1}(t), v^{\nu_2}(t))\\ - i\mu_2 E_{\mu_1,\mu_2,\nu_1,\nu_2}(v^{\mu_1}(t), \Lambda v^{\mu_2}(t), v^{\nu_1}(t), v^{\nu_2}(t))\\ - i\nu_1 E_{\mu_1,\mu_2,\nu_1,\nu_2}(v^{\mu_1}(t), v^{\mu_2}(t), \Lambda v^{\nu_1}(t), v^{\nu_2}(t))\\ - i\nu_2 E_{\mu_1,\mu_2,\nu_1,\nu_2}(v^{\mu_1}(t), v^{\mu_2}(t), v^{\nu_1}(t), \Lambda v^{\nu_2}(t)),\end{aligned} \tag{4.26}$$

where $\widehat{D}_{\mu_1,\mu_2,\nu_1,\nu_2}(v^{\mu_1}, v^{\mu_2}, v^{\nu_1}, v^{\nu_2})$ is the unique decomposition associated with the quartic terms such that the scales of dyadic localization of frequencies of all inputs are ordered in a descending manner. More precisely, the following estimate holds inside the support of symbol $\hat{d}_{\mu_1,\mu_2,\nu_1,\nu_2}(\xi-\eta, \eta-\sigma, \sigma-\kappa, \kappa)$ of the multilinear operator $\widehat{D}_{\mu_1,\mu_2,\nu_1,\nu_2}(\cdot,\cdot,\cdot,\cdot)$,

$$\begin{aligned}k_4 \le k_3 \le k_2 \le k_1, \qquad \text{where } \kappa \in \mathrm{supp}(\psi_{k_4}(x)), \sigma-\kappa \in \mathrm{supp}(\psi_{k_3}(x)),\\ \eta-\sigma \in \mathrm{supp}(\psi_{k_2}(x)), \qquad \xi-\eta \in \mathrm{supp}(\psi_{k_1}(x)),\end{aligned} \tag{4.27}$$

The detail formula of $\widehat{D}_{\mu_1,\mu_2,\nu_1,\nu_2}(\cdot,\cdot,\cdot,\cdot)$ can be obtained explicitly from $A_{\mu,\nu}(\cdot,\cdot)$, $B_{\tau,\kappa,\iota}(\cdot,\cdot,\cdot)$, $Q_{\mu,\nu}(\cdot,\cdot,\cdot)$, $C_{\tau,\kappa,\iota}(\cdot,\cdot,\cdot)$, and the quartic terms $D_{\mu_1,\mu_2,\nu_1,\nu_2}(\cdot,\cdot,\cdot,\cdot)$ in (4.4). Since its detailed formula is not necessary in later argument, for simplicity, we omit it here.

Now, we are ready to determine the normal formal transformation defined in (4.20).

Firstly, we consider the quadratic terms. Recall (4.7). Note that, if $|\eta| \leq 2^{-10}|\xi|$, $\mu = -$ or if $|\xi| \leq 2^{-10}|\eta|$, $\mu\nu = +$, the size of phase "$\Phi^{\mu,\nu}(\xi,\eta)$" is comparable to "$\max\{\Lambda(|\eta|), \Lambda(|\xi|)\}$," which is relatively big. Moreover, if (ξ,η) lies inside a small neighborhood of $(\xi, \xi/2)$(the space resonance set), the size of phase is also relatively big. More precisely, the following estimate holds,

$$c\Lambda(|\xi|) \leq |\Phi^{\mu,\nu}(\xi,\eta)| \leq C\Lambda(|\xi|), \quad \text{if} \quad |\eta - \xi/2| \leq 2^{-10}|\xi|,$$

where c and C are some absolute constants.

To take the advantage of the fact that the phase is highly oscillating with respect to time in the aforementioned scenarios, we use the normal form transformation $A_{\mu,\nu}(\cdot,\cdot)$ by choosing the symbol $a_{\mu,\nu}(\cdot,\cdot)$ defined as follows,

(4.28)
$$\begin{aligned} a_{\mu,\nu}(\xi-\eta,\eta) = \sum_{k_2\in\mathbb{Z}} \frac{i q_{\mu,\nu}(\xi-\eta,\eta)}{\Phi^{\mu,\nu}(\xi,\eta)} \psi_{k_2}(\eta)\big(\psi_{\leq k_2-10}(\eta-\xi/2)\psi_{\leq k_2+9}(\xi-\eta)\psi_{\geq k_2-9}(\xi) \\ + \mathbf{1}_{\{-\}}(\mu)\psi_{\geq k_2+10}(\xi-\eta) + \mathbf{1}_{\{+\}}(\mu\nu)\psi_{\leq k_2-10}(\xi)\psi_{\leq k_2+9}(\xi-\eta)\big), \end{aligned}$$

where $\mathbf{1}_S(\cdot)$ denotes the characteristic function of set S and the phase $\Phi^{\mu,\nu}(\xi,\eta)$ satisfies the following estimate inside the support of $a_{\mu,\nu}(\xi-\eta,\eta)$,

(4.29)
$$\begin{aligned} c\max\{|\xi|,|\eta|\}^2(1+\max\{|\xi|,|\eta|\})^{-1/2} &\leq |\Phi^{\mu,\nu}(\xi,\eta)| \\ &\leq C\max\{|\xi|,|\eta|\}^2(1+\max\{|\xi|,|\eta|\})^{-1/2}, \end{aligned}$$

where c and C are some absolute constants.

Next, we proceed to consider the cubic terms. Recall (4.8). Note that, for $\tau,\kappa,\iota \in \{+,-\}$, the phase $\Phi^{\tau,\kappa,\iota}(\xi,\eta,\sigma)$ is relatively big in the scenarios listed as follows,

- If $\tau = -$ and $|\eta|, |\sigma| \leq 2^{-10}|\xi|$.
- If $|\eta - \xi/2| \leq 2^{-10}|\xi|$ and $\sigma \leq 2^{-10}|\xi|$.
- If $|\eta - 2\xi/3| \leq 2^{-10}|\xi|$ and $|\sigma - \xi/3| \leq 2^{-10}|\xi|$, i.e., $(\xi-\eta, \eta-\sigma, \sigma)$ is close to $(\xi/3, \xi/3, \xi/3)$, which is the space resonance in η and σ set.
- If $|\xi - \eta + \tau\xi| \leq 2^{-10}|\xi|$, $|\eta - \sigma + \kappa\xi| \leq 2^{-10}|\xi|$, and $|\sigma + \iota\xi| \leq 2^{-10}|\xi|$, i.e., $(\xi-\eta, \eta-\sigma, \sigma)$ is very close to $(-\tau\xi, -\kappa\xi, -\iota\xi)$, which is the space resonance in η and σ set, where $(\tau,\kappa,\iota) \in \widetilde{S} := \{(+,-,-), (-,+,-), (-,-,+)\}$. See the proof of Lemma 5.7 for more details.

To take the advantage of the high oscillation of phase with respect to time in aforementioned scenarios, we use the normal form transformation $B_{\tau,\kappa,\iota}(\cdot,\cdot,\cdot)$ by choosing the symbol

$b_{\tau,\kappa,\iota}(\cdot,\cdot,\cdot)$ defined as follows,

(4.30)
$$b_{\tau,\kappa,\iota}(\xi-\eta,\eta-\sigma,\sigma)=\frac{i\widehat{c}_{\tau,\kappa,\iota}(\xi-\eta,\eta-\sigma,\sigma)}{\Phi^{\tau,\kappa,\iota}(\xi,\eta,\sigma)}\times\sum_{k\in\mathbb{Z}}\psi_k(\xi)\big(\mathbf{1}_{\widetilde{S}}((\tau,\kappa,\iota))\psi_{\le k-10}((1+\tau)\xi-\eta)\psi_{\le k-10}(\sigma+\iota\xi)$$
$$+\psi_{\le k-10}(\eta-2\xi/3)\psi_{\le k-10}(\sigma-\xi/3)$$
$$+\psi_{\le k-10}(\eta-\xi/2)\psi_{\le k-10}(\sigma)$$
$$+\mathbf{1}_{\{-\}}(\tau)\psi_{\le k-10}(\eta-\sigma)\psi_{\le k-10}(\sigma)\big),$$

where $\widehat{c}_{\tau,\kappa,\iota}(\cdot,\cdot,\cdot)$ is the associated symbol of cubic term $\widehat{C}_{\tau,\kappa,\iota}(\cdot,\cdot,\cdot)$ which is defined in (4.24) and the phase "$\Phi^{\tau,\kappa,\iota}(\xi,\eta,\sigma)$" satisfies the following estimate inside the support of the symbol $\widehat{c}_{\tau,\kappa,\iota}(\cdot,\cdot,\cdot)$,

(4.31)
$$|\Phi^{\tau,\kappa,\iota}(\xi,\eta,\sigma)| \in \max\{|\xi|,|\eta-\sigma|,|\sigma|\}^2(1+\max\{|\xi|,|\eta-\sigma|,|\sigma|\})^{-1/2}[c,C],$$

where where c and C are some absolute constants.

Lastly, we consider the quartic terms. Note that the phase $\Phi^{\mu_1,\mu_2,\nu_1,\nu_2}(\xi,\eta,\sigma,\kappa)$, which is defined in (4.9), is relatively big if $|\eta|,|\sigma|,|\kappa|\le 2^{-10}|\xi|$, $\mu_1=-$ or if $|\eta-\xi/2|,|\sigma|,|\kappa|\le 2^{-10}|\xi|$. Hence, we use the normal form transformation $E_{\mu_1,\mu_2,\nu_1,\nu_2}(\cdot,\cdot,\cdot,\cdot)$ by choosing its symbol $e_{\mu_1,\mu_2,\nu_1,\nu_2}(\cdot,\cdot,\cdot)$ defined as follows,

(4.32)
$$e_{\mu_1,\mu_2,\nu_1,\nu_2}(\xi-\eta,\eta-\sigma,\sigma-\kappa,\kappa)=\frac{i\widehat{d}_{\mu_1,\mu_2,\nu_1,\nu_2}(\xi-\eta,\eta-\sigma,\sigma-\kappa,\kappa)}{\Phi^{\mu_1,\mu_2,\nu_1,\nu_2}(\xi,\eta,\sigma,\kappa)}\sum_{k\in\mathbb{Z}}\psi_k(\xi)$$
$$\times\big(\psi_{\le k-10}(\eta-\xi/2)\psi_{\le k-10}(\sigma-\kappa)\psi_{\le k-10}(\kappa)+\mathbf{1}_{\{-\}}(\mu_1)\psi_{\le k-10}(\eta)\psi_{\le k-10}(\sigma-\kappa)\psi_{\le k-10}(\kappa)\big),$$

where $\widehat{d}_{\mu_1,\mu_2,\nu_1,\nu_2}(\cdot,\cdot,\cdot,\cdot)$ is the associated symbol of quartic term $\widehat{D}_{\mu_1,\mu_2,\nu_1,\nu_2}(\cdot,\cdot,\cdot,\cdot)$ and the phase $\Phi^{\mu_1,\mu_2,\nu_1,\nu_2}(\xi,\eta,\sigma,\kappa)$ satisfies the following estimate inside the support of the symbol $\widehat{d}_{\mu_1,\mu_2,\nu_1,\nu_2}(\cdot,\cdot,\cdot,\cdot)$,

(4.33)
$$|\Phi^{\mu_1,\mu_2,\nu_1,\nu_2}(\xi,\eta,\sigma,\kappa)|\in\max\{|\xi|,|\eta-\sigma|,|\sigma-\kappa|,|\kappa|\}^2(1+\max\{|\xi|,|\eta-\sigma|,|\sigma-\kappa|,|\kappa|\})^{-1/2}[c,C],$$

where c and C are some absolute constants.

From the estimates (4.29), (4.31), and (4.33, the estimate (2.3) in Lemma 2.1, and the estimate (4.13), the following estimate holds for some absolute constant C,

(4.34)
$$\begin{aligned}&\|a_{\mu,\nu}(\xi-\eta,\eta)\psi_k(\xi)\psi_{k_1}(\xi-\eta)\psi_{k_2}(\eta)\|_{\mathcal{S}^\infty}\\&\quad+\|b_{\tau,\kappa,\iota}(\xi-\eta,\eta)\psi_k(\xi)\psi_{k_1}(\xi-\eta)\psi_{k_2}(\eta-\sigma)\psi_{k_3}(\sigma)\|_{\mathcal{S}^\infty}\\&\quad+\|e_{\mu_1,\mu_2,\nu_1,\nu_2}(\xi-\eta,\eta-\sigma,\sigma-\kappa,\kappa)\psi_k(\xi)\psi_{k_1}(\xi-\eta)\\&\qquad\times\psi_{k_2}(\eta-\sigma)\psi_{k_3}(\sigma-\kappa)\psi_{k_4}(\kappa)\|_{\mathcal{S}^\infty}\\&\le C2^{k_{1,+}}.\end{aligned}$$

With the above determined normal form transformation, now we are ready to study the time evolution of the profile $g(t) := e^{it\Lambda}v(t)$ associated with $v(t)$. Recall (4.21). As a result of direct computations, we obtain the following equality,

(4.35)
$$\partial_t g(t,\xi)\psi_k(\xi) = \sum_{\mu,\nu\in\{+,-\}}\sum_{k_1,k_2\in\mathbb{Z},k_2\le k_1+10} B^{\mu,\nu}_{k,k_1,k_2}(t,\xi) + \sum_{\tau,\kappa,\iota\in\{+,-\}}\sum_{k_3\le k_2\le k_1} T^{\tau,\kappa,\iota}_{k,k_1,k_2,k_3}(t,\xi) + \sum_{\mu_1,\mu_2,\nu_1,\nu_2\in\{+,-\}}\sum_{k_4\le k_3\le k_2\le k_1} K^{\mu_1,\mu_2,\nu_1,\nu_2}_{k,k_1,k_2,k_3,k_4}(t,\xi) + e^{it\Lambda(\xi)}\widehat{\mathcal{R}_1}(t,\xi)\psi_k(\xi),$$

where

$$B^{\mu,\nu}_{k,k_1,k_2}(t,\xi) := \int_{\mathbb{R}^2} e^{it\Phi^{\mu,\nu}(\xi,\eta)}\tilde{q}_{\mu,\nu}(\xi-\eta,\eta)\widehat{g^{\mu}_{k_1}}(t,\xi-\eta)\widehat{g^{\nu}_{k_2}}(\eta)\psi_k(\xi)d\eta, \tag{4.36}$$

(4.37)
$$T^{\tau,\kappa,\iota}_{k,k_1,k_2,k_3}(t,\xi) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{it\Phi^{\tau,\kappa,\iota}(\xi,\eta,\sigma)}\tilde{d}_{\tau,\kappa,\iota}(\xi-\eta,\eta-\sigma,\sigma)\widehat{g^{\tau}_{k_1}}(t,\xi-\eta)\widehat{g^{\kappa}_{k_2}}(t,\eta-\sigma) \times \widehat{g^{\iota}_{k_3}}(t,\sigma)\psi_k(\xi)d\eta d\sigma,$$

$$K^{\mu_1,\mu_2,\nu_1,\nu_2}_{k,k_1,k_2,k_3,k_4}(t,\xi) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{it\Phi^{\mu_1,\mu_2,\nu_1,\nu_2}(\xi,\eta,\sigma,\kappa)}\tilde{e}_{\mu_1,\mu_2,\nu_1,\nu_2}(\xi-\eta,\eta-\sigma,\sigma-\kappa,\kappa) \times \widehat{g^{\mu_1}_{k_1}}(t,\xi-\eta)\widehat{g^{\mu_2}_{k_2}}(t,\eta-\sigma)\widehat{g^{\nu_1}_{k_3}}(t,\sigma-\kappa)\widehat{g^{\nu_2}_{k_4}}(t,\kappa)\psi_k(\xi)d\eta d\sigma d\kappa, \tag{4.38}$$

where

$$\tilde{q}_{\mu,\nu}(\xi-\eta,\eta) = \sum_{k_2\in\mathbb{Z}} q_{\mu,\nu}(\xi-\eta,\eta)\psi_{k_2}(\eta)\big(\psi_{\ge k_2-9}(\xi-2\eta)\psi_{\le k_2+4}(\xi-\eta)\psi_{\ge k_2-5}(\xi) + \frac{1+\mu}{2}\psi_{\ge k_2+5}(\xi-\eta) + \frac{(1-\mu\nu)}{2}\psi_{\le k_2-5}(\xi)\psi_{\le k_2+4}(\xi-\eta)\big), \tag{4.39}$$

(4.40)
$$\tilde{d}_{\tau,\kappa,\iota}(\xi-\eta,\eta-\sigma,\sigma) = \hat{c}_{\tau,\kappa,\iota}(\xi-\eta,\eta-\sigma,\sigma) + ib_{\tau,\kappa,\iota}(\xi-\eta,\eta-\sigma,\sigma)\Phi^{\tau,\kappa,\iota}(\xi,\eta,\sigma),$$

$$\tilde{e}_{\mu_1,\mu_2,\nu_1,\nu_2}(\xi-\eta,\eta-\sigma,\sigma-\kappa,\kappa) = \hat{d}_{\mu_1,\mu_2,\nu_1,\nu_2}(\xi-\eta,\eta-\sigma,\sigma-\kappa,\kappa) + ie_{\mu_1,\mu_2,\nu_1,\nu_2}(\xi-\eta,\eta-\sigma,\sigma-\kappa,\kappa)\Phi^{\mu_1,\mu_2,\nu_1,\nu_2}(\xi,\eta,\sigma,\kappa). \tag{4.41}$$

Recall the fact that we rearranged inputs such that the scales of dyadic localization of inputs are ordered in a descending manner, see (4.12), (4.25) and (4.27). It explains why we have $k_2 \le k_1+10$, $k_3 \le k_2 \le k_1$ and $k_4 \le k_3 \le k_2 \le k_1$ in (4.35).

Note that, from (4.39), the following equalities hold if $|\eta| \le 2^{-10}|\xi|$,

$$\tilde{q}_{-,\nu}(\xi-\eta,\eta) = 0, \quad \tilde{q}_{+,\nu}(\xi-\eta,\eta) = q_{+,\nu}(\xi-\eta,\eta). \tag{4.42}$$

Moreover, if $|\xi| \le 2^{-10}|\eta|$, we have

$$\tilde{q}_{\mu,\mu}(\xi-\eta,\eta) = 0, \qquad \mu\in\{+,-\}. \tag{4.43}$$

Recall (4.40) and (4.41). From the rough estimates of the symbols of the cubic terms in (4.16) and the quartic terms in (4.18), the following rough estimates hold for some absolute constant C,

$$\|\tilde{d}_{\tau,\kappa,\iota}(\xi-\eta,\eta-\sigma,\sigma)\|_{\mathcal{S}^{\infty}_{k,k_1,k_2}} \leq C 2^{2k_1+3k_{1,+}}, \tag{4.44}$$

$$\|\tilde{e}_{\mu_1,\mu_2,\nu_1,\nu_2}(\xi-\eta,\eta-\sigma,\sigma-\kappa,\kappa)\psi_{k_1}(\xi-\eta)\psi_{k_2}(\eta-\sigma)\psi_{k_1'}(\sigma-\kappa)\psi_{k_2'}(\kappa)\|_{\mathcal{S}^\infty} \leq C 2^{2k_1+4k_{1,+}}. \tag{4.45}$$

In later high order weighted norm estimate, we will also need to use the hidden symmetry inside the symbol $\tilde{d}_{\tau,\kappa,\iota}(\xi-\eta,\eta-\sigma,\sigma)$ when $|\sigma|,|\eta| \leq 2^{-10}|\xi|$. To this end, we identify the leading symbol inside $\tilde{d}_{\tau,\kappa,\iota}(\xi-\eta,\eta-\sigma,\sigma)$ first. From (4.30) and (4.40), we know that we only have to consider the case when $\tau=+$ and the leading part of $\tilde{d}_{\tau,\kappa,\iota}(\xi-\eta,\eta-\sigma,\sigma)$ is same as the leading part of $\tilde{c}_{\tau,\kappa,\iota}(\xi-\eta,\eta-\sigma,\sigma)$. Recall (4.24) and (4.28).If $k_2,k_3\leq k_1-10$, then the following estimate holds,

$$\|(\tilde{d}_{+,\kappa,\iota}(\xi-\eta,\eta-\sigma,\sigma)-e(\xi))\psi_{k_1}(\xi-\eta)\psi_{k_2}(\eta-\sigma)\psi_{k_3}(\sigma)\|_{\mathcal{S}^\infty} \leq C 2^{\max\{k_2,k_3\}+k_1+4k_{1,+}}. \tag{4.46}$$

where C is some absolute constant and $e(\xi)$ is given as follows,

$$e(\xi) := \frac{c_+}{4}\tilde{\lambda}(|\xi|^2)d(\xi) - \frac{i c(\xi)^2}{\Lambda(|\xi|)}, \tag{4.47}$$

where "$d(\xi)$" is defined in (4.19). We remark that the first part of $e(\xi)$ comes from the cubic term $C_{\tau,\kappa,\iota}(u^\tau,u^\kappa,u^\iota)$ in (4.24), see (4.17) in Lemma 4.1 and the second part of $e(\xi)$ comes from the composition of quadratic terms and the normal form transformation in (4.24).

4.2. Further reduction of the dispersion estimate

In this subsection, we first show that the dispersion rate of the nonlinear solution $v(t)$ and $u(t)$ are comparable in $W^{6,1+\alpha}$ space and then reduce the control of the dispersion rate of $v(t)$ into the control of weighted norms for the profile $g(t)$ of $v(t)$ in a fixed dyadic time interval.

LEMMA 4.2. – *Under the bootstrap assumption* (3.1)*, the following estimate holds,*

$$\sup_{t\in[0,T]} (1+t)\|v(t)-u(t)\|_{W^{6,1+\alpha}} + \|v(t)-u(t)\|_{H^{N_0-10}} \leq \epsilon_0. \tag{4.48}$$

Proof. – From the $L^\infty - L^\infty$ type bilinear estimate in (2.5), the estimate of symbols in (4.34) and the $L^\infty \to L^2$ type Sobolev embedding, the following estimate holds for some absolute constant C,

$$\|v(t)-u(t)\|_{W^{6,1+\alpha}} \leq C\|u(t)\|^{4/3}_{W^{6,1+\alpha}}\|u(t)\|^{2/3}_{H^{N_0}} \leq C(1+t)^{-6/5}\epsilon_1^2 \leq (1+t)^{-6/5}\epsilon_0.$$

From the $L^2 - L^\infty$ type bilinear estimate, the following estimate holds for some absolute constant C,

$$\|v(t)-u(t)\|_{H^{N_0-10}} \leq C\|u(t)\|_{H^{N_0}}\|u(t)\|_{W^{4,0}} \leq C\epsilon_1^2 \leq \epsilon_0. \qquad \square$$

Therefore, to control the dispersion rate of the nonlinear solution $u(t)$, now it would be sufficient to control the weighted norms of the profile $g(t)$ of the nonlinear solution $v(t)$. Recall the definitions of Z_1-norm and Z_2-norm in (1.22) and (1.23), we expect that the Z_1-norm of the profile $g(t)$ doesn't grow and the Z_2-norm of the profile only grows appropriately, which leads us to the following bootstrap assumption for some $T' \in (0, T]$,

$$\sup_{t\in[0,T']} (1+t)\|e^{-it\Lambda} g(t)\|_{W^{6,1+\alpha}} + \|g(t)\|_{Z_1} + (1+t)^{-\tilde{\delta}}\|g(t)\|_{Z_2} \leq \epsilon_1 = \epsilon_0^{5/6}, \tag{4.49}$$

where $\tilde{\delta} := 400\delta$.

To close the bootstrap argument, it would be sufficient to prove that there exists some absolute constant "C" such that the following estimates hold for any $t_1, t_2 \in [2^{m-1}, 2^m] \subset [0, T']$, $m \in \mathbb{Z}_+$,

$$\|g(t_2) - g(t_1)\|_{Z_1} \leq C 2^{-\delta m}\epsilon_0, \tag{4.50}$$

$$\|g(t_2)\|_{Z_2}^2 - \|g(t_1)\|_{Z_2}^2 \leq C 2^{2\tilde{\delta} m}\epsilon_0. \tag{4.51}$$

The proof of the desired estimate (4.50) is postponed to the Section 5 and the proof of the desired estimate (4.51) is postponed to the Section 6.

5. The low order weighted norm estimate

In this section, we mainly prove (4.50) under the bootstrap assumption (4.49). Recall (4.35). Note that, from the estimate (7.13) in Lemma 7.4, the low order weighted norm of the quintic and higher order remainder term $\widehat{\mathcal{R}_1}(t,\xi)$ is controlled. In the first subsection, we estimate the low order weight norm (Z_1-norm) of the quadratic terms $B^{\mu,\nu}_{k,k_1,k_2}(\xi,\eta)$ in details. In the last subsection, we estimate the Z_1-norm of the cubic terms $T^{\tau,\kappa,\iota}_{k,k_1,k_2,k_3}(t,\xi)$ and quartic terms $K^{\mu_1,\mu_2,\nu_1,\nu_2}_{k,k_1,k_2,k_3,k_4}(t,\xi)$ at the same time because the methods we will use for cubic terms and quartic terms are very similar.

5.1. The Z_1-norm estimate of quadratic terms

Recall (4.35). Based on the possible size of k_1 and k_2, we separate into two cases, which are the High-High type interaction and the High-Low type interaction.

The main result for the High-High type interaction is summarized in the following lemma.

LEMMA 5.1. – *Under the bootstrap assumption* (4.49)*, the following estimate holds for any* $\mu, \nu \in \{+,-\}$*, and any* $t_1, t_2 \in [2^{m-1}, 2^m]$,

$$\sum_{k\in\mathbb{Z}} \sum_{j\geq -k_-} \sum_{k_1,k_2\in\mathbb{Z}, |k_1-k_2|\leq 10} \left\| \mathcal{F}^{-1}\left[\int_{t_1}^{t_2} B^{\mu,\nu}_{k,k_1,k_2}(t,\xi)dt\right]\right\|_{B_{k,j}} \leq C 2^{-\delta m}\epsilon_0, \tag{5.1}$$

where C is some absolute constant.

Proof. – Recall (1.22) and (4.36). Note that, from the $L^2 \to L^1$ type Sobolev embedding and $L^2 - L^2$ type estimate, the following rough estimate holds for any $\mu, \nu \in \{+, -\}$,

$$\|\mathcal{F}^{-1}[\int_{t_1}^{t_2} B^{\mu,\nu}_{k,k_1,k_2}(t,\xi)dt]\|_{B_{k,j}} \leq \sup_{t\in[2^{m-1},2^m]} C2^{(2+\alpha)k+m+j+2k_1+10k_{1,+}}\|g_{k_1}(t)\|_{L^2}\|g_{k_2}(t)\|_{L^2}$$
$$\leq C2^{(2+\alpha)k+m+j+(2-2\alpha)k_1-(N_0-12)k_{1,+}}\epsilon_0,$$

where C is some absolute constant. From the above estimate, we can first rule out the case when $k \leq -(1+\delta)(m+j)/(2+\alpha)$ or $k_1 \leq -(1+\delta)(m+j)/(4-\alpha)$ or $k_1 \geq (m+j)/(N_0-30)$. As a result, it is sufficient to consider the case when k and k_1 are restricted in the following range,

(5.2)
$$-(1+\delta)(m+j)/(2+\alpha) \leq k \leq k_1 \leq (m+j)/(N_0-30), \quad k_1 \geq -(1+\delta)(m+j)/(4-\alpha).$$

Recall again (4.36). After doing spatial localizations for two inputs, the following decomposition holds,

$$\mathcal{F}^{-1}[B^{\mu,\nu}_{k,k_1,k_2}(t,\xi)](x) = \sum_{j_1\geq -k_{1,-}, j_2\geq -k_{2,-}} \mathcal{F}^{-1}[B^{\mu,\nu,j_1,j_2}_{k,k_1,k_2}(t,\xi)](x), \tag{5.3}$$

$$\mathcal{F}^{-1}[B^{\mu,\nu,j_1,j_2}_{k,k_1,k_2}(t,\xi)](x) = \int_{\mathbb{R}^2\times\mathbb{R}^2} e^{ix\cdot\xi+it\Phi^{\mu,\nu}(\xi,\eta)}\tilde{q}_{\mu,\nu}(\xi-\eta,\eta)\widehat{g^{\mu}_{k_1,j_1}}(t,\xi-\eta)$$
$$\times \widehat{g^{\nu}_{k_2,j_2}}(\eta)\psi_k(\xi)d\eta d\xi. \tag{5.4}$$

From the linear decay estimates in Lemma 2.7, the bootstrap assumptions (3.1) and (4.49), and the estimate (4.48) in Lemma 4.2, we obtain the following estimates for any $t \in [2^{m-1}, 2^m] \subset [0, T']$,

$$\|g_{k,j}(t)\|_{L^2} \leq \|\varphi^k_j(x)g_k(t)\|_{L^2} \leq C\min\{2^{-j-(1+\alpha)k-8k_+}, 2^{-2j-2k+\tilde{\delta}m}\}\epsilon_1,$$
$$\|e^{-it\Lambda}g_k(t)\|_{L^\infty} \leq C\min\{2^{-m-(1+\alpha)k-6k_+}, 2^{-m+\tilde{\delta}m-k}\}\epsilon_1,$$
$$\|g_k(t)\|_{L^2} \leq C2^{-(N_0-10)k_++\delta m}\epsilon_0,$$

where C is some absolute constant.

Based on the possible size of j, we separate into two cases as follow.

Case 1. – If $j \geq (1+\delta)\max\{m+k_1, -k_-\} + 2\tilde{\delta}m$. We first consider the case when $\min\{j_1, j_2\} \geq j - \delta j - \delta m$, the following estimate holds,

$$\sum_{\min\{j_1,j_2\}\geq j-\delta j-\delta m} \|\mathcal{F}^{-1}[\int_{t_1}^{t_2} B^{\mu,\nu}_{k_1,j_1,k_2,j_2}(t,\xi)dt]\|_{B_{k,j}}$$
$$\leq \sup_{t\in[2^{m-1},2^m]} \sum_{\min\{j_1,j_2\}\geq j-\delta j-\delta m} C2^{(2+\alpha)k}$$
$$\times 2^{m+j+10k_++2k_1}\|g_{k_1,j_1}(t)\|_{L^2}\|g_{k_2,j_2}(t)\|_{L^2}$$
$$\leq C2^{(2+\alpha)k+m+\tilde{\delta}m+10\delta m-(2-2\delta)j+(2-2\alpha)k_1-6k_{1,+}}\epsilon_0$$
$$\leq C2^{-2\delta m-2\delta j}\epsilon_0,$$

where C is some absolute constant.

Now we proceed to consider the case $\min\{j_1, j_2\} \le j - \delta j - \delta m$. Note that, when η is not very close to $\xi/2$ (space resonance set), e.g., $|\eta - \xi/2| \ge 2^{-10}|\xi|$, the following estimates hold,

$$|\nabla_\eta \Phi^{\mu,\nu}(\xi,\eta)| = 2\big|\mu\lambda'(|\xi-\eta|^2)(\xi-\eta) - \nu\lambda'(|\eta|^2)\eta\big| \ge 2^{-10}|\xi|\big(|\xi-\eta| + |\eta| + 1\big)^{-1/2}, \tag{5.5}$$

$$|\nabla_\eta \Phi^{\mu,\nu}(\xi,\eta)| + |\nabla_\xi \Phi^{\mu,\nu}(\xi,\eta)| \le 2^{10}\max\{|\xi|,|\eta|\}(|\xi| + |\eta| + 1)^{-1/2}, \tag{5.6}$$

where $\lambda(|x|) := \Lambda(\sqrt{|x|})$. Therefore, from (5.6), we know that the following estimate holds if $|x| \in [2^{j-2}, 2^{j+2}]$,

$$|\nabla_\xi\big(x\cdot\xi + t\Phi^{\mu,\nu}(\xi,\eta)\big)| = \big|x + t\nabla_\xi\Phi^{\mu,\nu}(\xi,\eta)\big| \in [2^{j-4}, 2^{j+4}]. \tag{5.7}$$

If $j_2 = \min\{j_1, j_2\}$, then we can do change of variables first to switch the role of $\xi-\eta$ and η. As a result, the following estimate holds if $|x| \in [2^{j-2}, 2^{j+2}]$,

$$|\nabla_\xi\big(x\cdot\xi + t\Phi^{\mu,\nu}(\xi,\xi-\eta)\big)| = \big|x + t\nabla_\xi\Phi^{\mu,\nu}(\xi,\xi-\eta)\big| \in [2^{j-4}, 2^{j+4}].$$

To sum up, in whichever case, by doing integration by parts in ξ once, we gain 2^{-j} by paying the price of at most $\max\{2^{\min\{j_1,j_2\}}, 2^{-k}\}$. Hence, the net gain of doing integration by parts in "ξ" once is at least $2^{-\delta m-\delta j}$. After doing this process many times, we can see rapidly decay.

Case 2. – If $j \le (1+\delta)\max\{m+k_1, -k_-\} + 2\tilde{\delta}m$. As j is bounded from above now, from (5.2), we have the following upper bound and lower bound for k and k_1,

$$-m/(1+\alpha/3) \le k \le k_1 \le 2\beta m, \quad j \le \max\{m+k_1, -k_-\} + 3\tilde{\delta}m, \quad \beta := 1/(N_0 - 50), \tag{5.8}$$

Hence, it would be sufficient to consider fixed k and k_1 inside the range (5.8), as there are at most m^3 cases to consider, which is only a logarithmic loss.

After doing integration by parts in η many times, we can rule out the case when $\max\{j_1, j_2\} \le m + k_- - 3\beta m$. It remains to consider the case when $\max\{j_1, j_2\} \ge m + k_- - 3\beta m$. From $L^2 - L^\infty$ type bilinear estimate in Lemma 2.2, the following estimate holds after putting the input with the maximum spatial concentration in L^2 and the other input in L^∞,

$$\begin{aligned}
\sum_{\max\{j_1,j_2\}\ge m+k_--3\beta m} &\|\mathcal{F}^{-1}[\int_{t_1}^{t_2} B^{\mu,\nu}_{k_1,j_1,k_2,j_2}(t,\xi)dt]\|_{B_{k,j}} \\
&\le C 2^{(1+\alpha)k+m+j+2k_1+10k_+-m-(1+\alpha)k_1} \\
&\quad\times \min\{2^{-m-k_--(1+\alpha)k_1+6\beta m}, 2^{-2k_1-2(m+k_--3\beta m)+\tilde{\delta}m}\}\epsilon_1^2 \\
&\le C\min\{2^{\alpha k+12k_++(1-2\alpha)k_1+10\beta m}, 2^{-(1-\alpha)k+12k_+-\alpha k_1-m+10\beta m}\}\epsilon_0 \\
&\le C 2^{-10\delta m}\epsilon_0,
\end{aligned} \tag{5.9}$$

where C is some absolute constant. To sum up, from the above estimate and the previous discussion, it is easy to see that the desired estimate (5.1) holds. □

The main result of the High-Low type interaction is summarized in the following lemma.

LEMMA 5.2. – *Under the bootstrap assumption* (4.49)*, the following estimate holds for any* $\mu \in \{+,-\}$*, and any* $t_1, t_2 \in [2^{m-1}, 2^m]$*,*

$$\sum_{k\in\mathbb{Z}} \sum_{j\geq -k_-} \sum_{k_1,k_2\in\mathbb{Z}, k_2\leq k_1-10} \| \sum_{\nu\in\{+,-\}} \mathscr{F}^{-1}[\int_{t_1}^{t_2} B^{\mu,\nu}_{k,k_1,k_2}(t,\xi)dt]\|_{B_{k,j}} \leq C 2^{-\delta m}\epsilon_0, \tag{5.10}$$

where C *is some absolute constant.*

Proof. – Recall (4.42). Note that $\mu = +$ for the case we are considering. Recall (4.14) and (4.15). Motivated from the improved estimate (4.15), we split the symbol "$\tilde{q}_{+,\nu}(\xi,\eta)$" into two parts as follows,

$$\tilde{q}_{+,\nu}(\xi-\eta,\eta) = q^1_{+,\nu}(\xi-\eta,\eta) + q^2_{+,\nu}(\xi-\eta,\eta),$$

$$q^1_{+,\nu}(\xi-\eta,\eta) = c(\xi), \quad q^2_{+,\nu}(\xi-\eta,\eta) = q_{+,\nu}(\xi-\eta,\eta) - c(\xi). \tag{5.11}$$

Hence, motivated from the above decomposition of the symbol $\tilde{q}_{+,\nu}(\xi-\eta,\eta)$, we do decomposition as follows,

$$\sum_{\nu\in\{+,-\}} \int_{t_1}^{t_2} B^{+,\nu}_{k,k_1,k_2}(t,\xi)dt = \sum_{i=1,2} I^i_{k,k_1,k_2},$$

$$I^i_{k,k_1,k_2} = \sum_{\nu\in\{+,-\}} \int_{t_1}^{t_2}\int_{\mathbb{R}^2} e^{it\Phi^{+,\nu}(\xi,\eta)} q^i_{+,\nu}(\xi-\eta,\eta)\widehat{g_{k_1}}(t,\xi-\eta)\widehat{g^{\nu}_{k_2}}(t,\eta)\psi_k(\xi)d\eta dt, \; i=1,2.$$

Recall (5.11). Note that $q^1_{+,\nu}(\xi-\eta,\eta)$ doesn't depend on the sign "ν". Hence, we have

$$I^1_{k,k_1,k_2} = 2\int_{t_1}^{t_2}\int_{\mathbb{R}^2} e^{it(\Lambda(|\xi|)-\Lambda(|\xi-\eta|))} c(\xi)\widehat{g_{k_1}}(t,\xi-\eta)\widehat{\mathrm{Re}(v)}(t,\eta)\psi_{k_2}(\eta)\psi_k(\xi)d\eta dt.$$

From (4.20) and the estimate (5.15) in Lemma 5.3, the following estimate holds after using the volume of the support of "η,"

$$\begin{aligned}
\|I^1_{k_1,k_2}\|_{B_{k,j}} &\leq \sup_{t\in[2^{m-1},2^m]} C_1 2^{(3+\alpha)k+m+j+10k_+}\|g_{k_1}(t)\|_{L^2}2^{2k_2}\|\widehat{\mathrm{Re}(v)}(t,\xi)\psi_{k_2}(\xi)\|_{L^\infty_\xi}\\
&\leq C_2 2^{(3+\alpha)k+m+\delta m+j+2k_2-(N_0-30)k_+}(\|\widehat{h}(t,\xi)\psi_{k_2}(\xi)\|_{L^\infty_\xi} + \|u\|^2_{H^{10}} + \|u\|^3_{H^{10}} + \|u\|^4_{H^{10}})\\
&\leq C_3\big(2^{(3+\alpha)k+2m+10\delta m+j+3k_2-(N_0-30)k_+}\epsilon_0 + 2^{(3+\alpha)k+3m+10\delta m+j+4k_2-(N_0-30)k_+}\epsilon_0\big),
\end{aligned} \tag{5.12}$$

where C_1, C_2, and C_3 are some absolute constants.

Now we proceed to estimate $I^2_{k_1,k_2}$. Recall (5.11) and (4.15). From the $L^2 - L^\infty$ type bilinear estimate (2.5) in Lemma 2.2 and $L^\infty \to L^2$ type Sobolev embedding, we have

$$\begin{aligned}
\|I^2_{k_1,k_2}\|_{B_{k,j}} &\leq \sup_{t\in[2^{m-1},2^m]} C 2^{(2+\alpha)k+m+j+k_2+k_1+10k_+}\|g_{k_1}(t)\|_{L^2}\|e^{it\Lambda}g_{k_2}(t)\|_{L^\infty}\\
&\leq C 2^{(3+\alpha)k-(N_0-10)k_++m+j+2k_2+2\delta m}\epsilon_0,
\end{aligned} \tag{5.13}$$

where C is some absolute constant. To sum up, from (5.12) and (5.13), we can rule out the case when $k_2 \leq -(1+5\delta)\max\{(m+j)/2, (3m+j)/4\}$ or $k \geq 4(m+j)/(N_0-40)$. Now, we only need to consider the case when k_2 is restricted in the following range,

$$-(1+5\delta)\max\{(m+j)/2, (3m+j)/4\} \leq k_2 \leq k \leq (3m+j)/(N_0-40). \tag{5.14}$$

Similar to the idea used in the proof of Lemma 5.1, we also separate into two cases based on the size of " j " as follows.

Case 1. – If $j \geq (1+\delta)\max\{m+k, -k_-\} + 10\delta m$. We first consider the case when $\min\{j_1, j_2\} \leq j - \delta j - \delta m$. Same as we considered in the High × High type interaction, we can also do integration by parts in "ξ" many times to see rapidly decay. Now, we proceed to consider the case when $\min\{j_1, j_2\} \geq j - \delta j - \delta m$. From $L^2 - L^\infty$ type bilinear estimate and $L^\infty \to L^2$ type Sobolev embedding, we have

$$
\begin{aligned}
\|I^1_{k_1,k_2}\|_{B_{k,j}} \leq &\sup_{t\in[2^{m-1},2^m]} \sum_{\min\{j_1,j_2\}\geq j-\delta j-\delta m} C 2^{(1+\alpha)k+10k_+ + m + j + 2k_1 + k_2} \\
&\|g_{k_1,j_1}(t)\|_{L^2} \|g_{k_2,j_2}(t)\|_{L^2} \\
&\leq C 2^{(1+\alpha)k + k_2 + (1+50\beta)m - (1-50\beta)j} 2^{-j/2 - k_2/2} \epsilon_1^2 \leq C 2^{-\beta m} \epsilon_0,
\end{aligned}
$$

where C is some absolute constant.

Case 2. – If $j \leq (1+\delta)\max\{m+k, -k_-\} + 10\delta m$. For this case, whether j_1 is less than j_2 makes a difference. Hence, we separate into two cases based on whether j_1 is smaller than j_2 as follows.

If $j_1 \leq j_2$. – For this case, we don't need to do change of coordinates to switch the role between $\xi - \eta$ and η. Note that $|\nabla_\xi \Phi^{+,\nu}(\xi,\eta)| \leq C|\eta|$ holds for some absolute constant. Since this upper bound is better than the one used in the rough estimate, which leads to expect that the upper bound of " j " can be improved. More precisely, we can rule out the case when $j \geq \max\{m+k_2, -k_-\} + 100\beta m$ and $j_1 \leq j - \delta m$ by doing integration by parts in ξ many times. If $j \geq \max\{m+k_2, -k_-\} + 100\beta m$ and $j - \delta m \leq j_1 \leq j_2$, then the following estimate holds after using the $L^2 - L^\infty$ type bilinear estimate and $L^\infty \to L^2$ type Sobolev embedding,

$$
\begin{aligned}
&\sum_{j-\delta m \leq j_1 \leq j_2} \|\mathcal{F}^{-1}[\int_{t_1}^{t_2} B^{+,\nu}_{k_1,j_1,k_2,j_2}(t,\xi)dt]\|_{B_{k,j}} \\
&\quad \leq \sup_{t\in[2^{m-1},2^m]} \sum_{j-\delta m\leq j_1\leq j_2} C 2^{(1+\alpha)k + 10k_+ + m + j + 2k_1} \|g_{k_1,j_1}(t)\|_{L^2} 2^{k_2} \|g_{k_2,j_2}(t)\|_{L^2} \\
&\quad \leq C 2^{(1+\alpha)k + k_2 + (1+50\beta)m - (1-50\beta)j} 2^{-25\beta j - 25\beta k_2} \epsilon_1^2 \\
&\quad \leq C 2^{-\beta m} \epsilon_0,
\end{aligned}
$$

where C is some absolute constant.

It remains to consider the case when $j \leq \max\{m+k_2, -k_-\} + 100\beta m$. If moreover $k_- + k_2 \leq -m + \beta m$, it is easy to see our desired estimate holds from (5.12) and (5.13). Hence, we only have to consider the case when $k_- + k_2 \geq -m + \beta m$. For this case, we have $j \leq m + k_2 + 100\beta m$. Recall (5.14), we know that $k_2 \geq -4m/5 - 30\beta m$.

Same as in the decomposition (5.3), we also do spatial localizations for two inputs. After doing integration by parts in "η" many times, we can rule out the case when $j_2 \leq m + k_{1,-} - 10\delta m$. Therefore, it remains to consider the case when $j_2 \geq m + k_{1,-} - 10\delta m$.

After putting g_{k_2,j_2} in L^2 and putting g_{k_1,j_1} in L^∞, we have

$$\begin{aligned}\sum_{j_2\geq \max\{m+k_{1,-}-10\delta m, j_1\}} &\|\mathcal{F}^{-1}[\int_{t_1}^{t_2} B^{+,\nu}_{k_1,j_1,k_2,j_2}(t,\xi)dt]\|_{B_{k,j}}\\ &\leq \sum_{j_2\geq \max\{m+k_{1,-}-10\delta m, j_1\}} C2^{(1+\alpha)k+10k_+}\\ &\quad\times 2^{2k_1+m+j}\sup_{t\in[2^m,2^{m+1}]}\|e^{-it\Lambda}g_{k_1,j_1}(t)\|_{L^\infty}\|g_{k_2,j_2}(t)\|_{L^2}\\ &\leq C2^{-m-k_2+150\beta m}\epsilon_1^2\leq C2^{-\beta m}\epsilon_0,\end{aligned}$$

where C is some absolute constant.

If $-k_2\leq j_2\leq j_1$. – We first consider the case when $k_1+k_2\leq -4m/5$. From the L^2-L^∞ type bilinear estimate and $L^\infty\to L^2$ type Sobolev embedding, the following estimate holds for some absolute constant C,

$$\begin{aligned}\sum_{j_2\leq j_1}&\|\mathcal{F}^{-1}[\int_{t_1}^{t_2} B^{+,\nu}_{k_1,j_1,k_2,j_2}(t,\xi)dt]\|_{B_{k,j}}\\ &\leq \sup_{t\in[2^{m-1},2^m]}\sum_{j_2\leq j_1} C2^{(1+\alpha)k+10k_++m+j+2k_1}\|g_{k_1,j_1}(t)\|_{L^2}2^{k_2}\|g_{k_2,j_2}(t)\|_{L^2}\\ &\leq \sum_{-k_2\leq j_1} C2^{2m+(4+\alpha)k_1+k_2}2^{-2k_1-2j_1+50\beta m}\epsilon_1^2\\ &\leq C2^{(2+\alpha)k+3k_2+2m+50\beta m}\epsilon_1^2\\ &\leq C2^{-\beta m}\epsilon_0.\end{aligned}$$

Lastly, it remains to consider the case when $k_1+k_2\geq -4m/5$. For this case, we do integration by parts in η many times to rule out the case when $j_1\leq m+k_{1,-}-10\delta m$. For the case when $j_1\geq m+k_{1,-}-10\delta m$, the following estimate holds from the L^2-L^∞ type bilinear estimate,

$$\begin{aligned}&\sum_{j_1\geq \max\{j_2,m+k_{1,-}-10\delta m\}}\left\|\mathcal{F}^{-1}\left[\int_{t_1}^{t_2} B^{+,\nu}_{k_1,j_1,k_2,j_2}(t,\xi)dt\right]\right\|_{B_{k,j}}\\ &\leq \sum_{j_1\geq \max\{j_2,m+k_{1,-}-10\delta m\}} C2^{(1+\alpha)k+10k_+}2^{m+j+2k_1}\sup_{t\in[2^{m-1},2^m]}\|g_{k_1,j_1}(t)\|_{L^2}\|e^{-it\Lambda}g_{k_2,j_2}(t)\|_{L^\infty}\\ &\leq C2^{-m-(1+\alpha)k_2+50\beta m}\epsilon_1^2\leq C2^{-\beta m}\epsilon_0,\end{aligned}$$

where C is some absolute constant. Hence finishing the proof of the desired estimate (5.10). □

LEMMA 5.3. – *Under the bootstrap assumption* (3.1)*, the following estimate holds for* $t\in[2^{m-1},2^m]\subset[0,T]$*,* $m\in\mathbb{N}$ *and* $k\in\mathbb{Z}, k\leq 0$*,*

$$\|\widehat{h}(t,\xi)\psi_k(\xi)\|_{L^\infty_\xi}\leq C2^{2\delta m}\big(2^{2k+2m}+2^{k+m}\big)\epsilon_0, \tag{5.15}$$

where C is some absolute constant.

Proof. – Recall (4.6). It is easy to see the following estimate holds for any $t \in [2^{m-1}, 2^m]$ and $k \leq 0$,

$$\|\widehat{f}(t,\xi)\psi_k(\xi)\|_{L^\infty_\xi} \leq \epsilon_0 + C_1 \int_0^t \|f(s)\|^2_{H^{10}} ds \leq C_2 2^{m+2\delta m}\epsilon_0, \tag{5.16}$$

where C_1 and C_2 are some absolute constants. Recall the equation satisfied by the height function "$h(t)$" in (1.3) and the Taylor expansion of the Dirichlet-Neumann operator in (1.6), we have

$$\partial_t \widehat{h}(t,\xi) = |\xi| \tanh(|\xi|)\widehat{\psi}(t,\xi) + \mathscr{F}[\Lambda_2[G(h)\psi]](\xi) + \mathscr{F}[\Lambda_{\geq 3}[G(h)\psi]](\xi).$$

Hence, from $L^2 - L^2$ type bilinear estimate (2.5) in Lemma 2.2 and the estimate (5.16), the following estimate holds for any $k \leq 0$,

$$\|\widehat{h}(t,\xi)\psi_k(\xi)\|_{L^\infty_\xi} \leq \epsilon_0 + C_1\Big(\int_0^t 2^{2k}\|\widehat{\psi}(s,\xi)\psi_k(\xi)\|_{L^\infty_\xi} ds + \int_0^t 2^k \|h(s)\|_{H^{10}}\|\psi(s)\|_{H^{10}} ds\Big)$$

$$\leq \epsilon_0 + C_2\Big(\int_0^t 2^{2k}\|\widehat{f}(s,\xi)\psi_k(\xi)\|_{L^\infty_\xi} ds + \int_0^t 2^k\|f(s)\|^2_{H^{10}} ds\Big) \leq C_3 2^{2\delta m}\big(2^{2k+2m} + 2^{k+m}\big)\epsilon_0, \tag{5.17}$$

where C_1, C_2, and C_3 are some absolute constants. Hence finishing the proof of the desired estimate (5.15). □

5.2. The Z_1 estimates of cubic terms and quartic terms.

The main goal of this subsection is to prove the following proposition,

PROPOSITION 5.4. – *Under the bootstrap assumption* (4.49)*, the following estimate holds for some absolute constant* C *and any* $t \in [2^{m-1}, 2^m]$,

$$\sum_{k\in\mathbb{Z}}\sum_{j\geq -k_-}\Big[\sum_{k_3\leq k_2\leq k_1}\|\mathscr{F}^{-1}[T^{\tau,\kappa,\iota}_{k,k_1,k_2,k_3}(t,\xi)]\|_{B_{k,j}} + \sum_{k_4\leq k_3\leq k_2\leq k_1}\|\mathscr{F}^{-1}[K^{\mu_1,\mu_2,\nu_1,\nu_2}_{k,k_1,k_2,k_3,k_4}(t,\xi)]\|_{B_{k,j}}\Big]$$

$$\leq C 2^{-m-\beta m}\epsilon_0, \tag{5.18}$$

where $T^{\tau,\kappa,\iota}_{k,k_1,k_2,k_3}(t,\xi)$ *and* $K^{\mu_1,\mu_2,\nu_1,\nu_2}_{k,k_1,k_2,k_3,k_4}(t,\xi)$ *are defined in* (4.37) *and* (4.38) *respectively.*

Proof. – Same as the strategy used in the estimate of quadratic terms, we can do integration by parts in "ξ" many times to rule out the case when $j \geq (1+\delta)\max\{m+k_1, -k_-\}+2\tilde{\delta}m$. Hence, in the rest of this section, we restrict ourself to the case when

$$j \leq (1+\delta)\max\{m+k_1, -k_-\} + 2\tilde{\delta}m.$$

From the $L^2 - L^\infty - L^\infty$ type trilinear estimate in Lemma 2.2, the following estimate holds for some absolute constant C,

$$\begin{aligned}
&\| \mathcal{F}^{-1}[T^{\tau,\kappa,\iota}_{k,k_1,k_2,k_3}(t,\xi)]\|_{B_{k,j}} \\
&\quad \le C 2^{(1+\alpha)k+j+2k_1+2k_{1,+}+10k_+}\|e^{-it\Lambda}g_{k_1}\|_{L^\infty}\|g_{k_2}(t)\|_{L^2}\|e^{-it\Lambda}g_{k_3}\|_{L^\infty} \\
&\quad \le C \min\{2^{(1+\alpha)k+2k_1+k_3+20\beta m}, 2^{(1+\alpha)k+3k_1-(N_0-30)k_{1,+}+k_3+m+\beta m}\}\epsilon_0,
\end{aligned} \tag{5.19}$$

$$\begin{aligned}
&\| \mathcal{F}^{-1}[K^{\mu_1,\mu_2,\nu_1,\nu_2}_{k,k_1,k_2,k_3,k_4}(t,\xi)]\|_{B_{k,j}} \\
&\quad \le C 2^{(1+\alpha)k+10k_++j+2k_1+2k_{1,+}}\|e^{-it\Lambda}g_{k_1}\|_{L^\infty}\|e^{-it\Lambda}g_{k_2}\|_{L^2}\|g_{k_3}(t)\|_{L^2} \\
&\qquad \times \|e^{-it\Lambda}g_{k_4}(t)\|_{L^\infty} \le C 2^{(1+\alpha)k+k_4+20\beta m}\min\{2^{2k_1-m/2}, 2^{3k_1-(N_0-30)k_{1,+}+m/2}\}\epsilon_0.
\end{aligned} \tag{5.20}$$

From the rough estimate (5.19), we can rule out the case when $k_3 \le -m - 30\beta m$, or $k_1 \ge 2\beta m$ or $k \le -m/(1+\alpha/2)$ for the cubic terms. From the rough estimate (5.20), we can rule out the case when $k_4 \le -m/2 - 30\beta m$ or $k_1 \ge 2\beta m$ or $k \le -m/(2+\alpha)$ for the quartic terms.

Therefore, for the cubic terms, it would be sufficient to obtain the following desired estimate

$$\sup_{t\in[2^{m-1},2^m]} \| \mathcal{F}^{-1}[T^{\tau,\kappa,\iota}_{k,k_1,k_2,k_3}(t,\xi)](x)\|_{B_{k,j}} \le C 2^{-m-\beta m}\epsilon_0, \tag{5.21}$$

where integers k, k_1, k_2, k_3 are fixed inside the following range

$$\text{(Cubic terms)} \quad -m-30\beta m \le k_3 \le k_2 \le k_1 \le 2\beta m, \quad -m/(1+\alpha/2) \le k \le 2\beta m. \tag{5.22}$$

For the quartic terms, it would be sufficient to obtain the following estimate,

$$\sup_{t\in[2^{m-1},2^m]} \| \mathcal{F}^{-1}[K^{\mu_1,\mu_2,\nu_1,\nu_2}_{k,k_1,k_2,k_3,k_4}(t,\xi)](x)\|_{B_{k,j}} \le C 2^{-m-\beta m}\epsilon_0, \tag{5.23}$$

where integers k, k_1, k_2, k_3, k_4 are fixed inside the following range,

$$\text{(Quartic terms)} \quad -m/2-30\beta m \le k_4 \le k_3 \le k_2 \le k_1 \le 2\beta m, \quad -m/(2+\alpha) \le k \le 2\beta m. \tag{5.24}$$

From the results in Lemma 5.5, Lemma 5.6, and Lemma 5.7, we know that the desired estimates (5.21) and (5.23)) indeed hold. Hence finishing the desired estimate (5.18. □

Lemma 5.5. – *Under the bootstrap assumption* (4.49) *and the assumption that* $k_2 \le k_1 - 10$, *the desired estimate* (5.21) *for the cubic terms holds for fixed* k, k_1, k_2, *and* k_3 *inside the range listed in* (5.22) *and the desired estimate* (5.23) *holds for fixed* k, k_1, k_2, k_3 *and* k_4 *inside the range listed in* (5.24).

Proof. – Recall the normal form transformation we did in Subsection 4.1. Note that the case when "$\tau = -$" is removed by the normal form transformation when $k_2 \le k_1 - 10$. Hence, we can restrict ourself to the case "$\tau = +$". Recall (4.8). Note that the following estimates hold for the case we are considering,

$$|\nabla_\xi \Phi^{+,\kappa,\iota}(\xi,\eta,\sigma)| = |\nabla_\xi \Phi^{+,\mu_2,\nu_1,\nu_2}(\xi,\eta,\sigma)| = \big|\Lambda'(|\xi|)\frac{\xi}{|\xi|} - \Lambda'(|\xi-\eta|)\frac{\xi-\eta}{|\xi-\eta|}\big|$$

$$\le 2\max\{2^{k_1}\angle(\xi,\xi-\eta), |\xi|-|\xi-\eta|\} \le 4|\eta| \le 2^{k_2+3}. \tag{5.25}$$

$$2^{k_1-k_{1,+}/2-10} \le |\nabla_\eta \Phi^{+,\kappa,\iota}(\xi,\eta,\sigma)| = \big|\Lambda'(|\xi-\eta|)\frac{\xi-\eta}{|\xi-\eta|} + \kappa\Lambda'(|\eta-\sigma|)\frac{\eta-\sigma}{|\eta-\sigma|}\big| \le 2^{k_1-k_{1,+}/2+10}. \tag{5.26}$$

After doing spatial localizations for the inputs $\widehat{g_{k_1}}(\cdot)$ and $\widehat{g_{k_2}}(\cdot)$, we have the decomposition as follows,

(5.27)

$$T^{\tau,\kappa,\iota}_{k,k_1,k_2,k_3}(t,\xi) = \sum_{j_1\ge -k_{1,-}, j_2\ge -k_{2,-}} T^{\tau,\kappa,\iota}_{k_1,j_1,k_2,j_2}(t,\xi),$$

$$T^{\tau,\kappa,\iota}_{k_1,j_1,k_2,j_2}(t,\xi) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{it\Phi^{\tau,\kappa,\iota}(\xi,\eta,\sigma)}\tilde{d}_{\tau,\kappa,\iota}(\xi-\eta,\eta-\sigma,\sigma)\widehat{g^{\tau}_{k_1,j_1}}(t,\xi-\eta) \times \widehat{g^{\kappa}_{k_2,j_2}}(t,\eta-\sigma)\widehat{g^{\iota}_{k_3}}(t,\sigma)d\sigma d\eta,$$

$$K^{\mu_1,\mu_2,\nu_1,\nu_2}_{k,k_1,k_2,k_3,k_4}(t,\xi) = \sum_{j_1\ge -k_{1,-}, j_2\ge -k_{2,-}} K^{\mu_1,\mu_2,\nu_1,\nu_2}_{k_1,j_1,k_2,j_2}(t,\xi),$$

$$K^{\mu_1,\mu_2,\nu_1,\nu_2}_{k_1,j_1,k_2,j_2}(t,\xi) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{it\Phi^{\mu_1,\mu_2,\nu_1,\nu_2}(\xi,\eta,\sigma,\kappa)}\tilde{e}_{\mu_1,\mu_2,\nu_1,\nu_2}(\xi-\eta,\eta-\sigma,\sigma-\kappa,\kappa) \times \widehat{g^{\mu_1}_{k_1,j_1}}(t,\xi-\eta)\widehat{g^{\mu_2}_{k_2,j_2}}(t,\eta-\sigma)\widehat{g^{\nu_1}_{k_3}}(t,\sigma-\kappa)\widehat{g^{\nu_2}_{k_4}}(t,\kappa)d\kappa d\sigma d\eta.$$

Based on the possible size of j, we separate into two cases as follows.

If $j \ge \max\{m+k_2, -k_{1,-}\} + \beta m$. – Recall (5.25). By doing integration by parts in "ξ" many times, we can rule out the case $j_1 \le j-\delta m$. If $j_1 \ge j-\delta m$, then from $L^2-L^\infty-L^\infty$ type multilinear estimate in Lemma 2.2, the following estimates hold,

$$\|\sum_{j_1\ge j-\delta m}\mathcal{F}^{-1}[T^{\tau,\kappa,\iota}_{k_1,j_1,k_2,j_2}(t,\xi)]\|_{B_{k,j}}$$
$$\le \sum_{j_1\ge j-\delta m} C2^{(1+\alpha)k+j+2k_1+2k_{1,+}+10k_+}2^{k_2}\|g_{k_2}(t)\|_{L^2}\|e^{-it\Lambda}g_{k_3}(t)\|_{L^\infty}\|g_{k_1,j_1}(t)\|_{L^2}$$
$$\le C2^{-m/2+30\beta m}2^{k_2-j}\epsilon_0 \le C2^{-3m/2+40\beta m}\epsilon_0.$$

$$\|\sum_{j_1\ge j-\delta m}\mathcal{F}^{-1}[K^{\mu_1,\mu_2,\nu_1,\nu_2}_{k_1,j_1,k_2,j_2}(t,\xi)]\|_{B_{k,j}}$$
$$\le \sum_{j_1\ge j-\delta m} C2^{(1+\alpha)k+j+2k_1+2k_{1,+}+10k_+}2^{k_2}\|g_{k_2}(t)\|_{L^2}\|e^{-it\Lambda}g_{k_3}(t)\|_{L^\infty}$$
$$\times\|e^{-it\Lambda}g_{k_4}(t)\|_{L^\infty}\|g_{k_1,j_1}(t)\|_{L^2}$$
$$\le C2^{-2m+40\beta m}\epsilon_0.$$

If $j \leq \max\{m+k_2, -k_{1,-}\}+\beta m$. – From the $L^2-L^\infty-L^\infty-L^\infty$ type multilinear estimate, the following estimate holds for some absolute constant C,

$$\begin{aligned}\|\mathcal{F}^{-1}[K^{\mu_1,\mu_2,\nu_1,\nu_2}_{k,k_1,k_2,k_3,k_4}(t,\xi)]\|_{B_{k,j}} &\leq C2^{(1+\alpha)k+10k_+ +j+2k_1+2k_{1,+}}\|e^{-it\Lambda}g_{k_1}\|_{L^\infty}\|e^{-it\Lambda}g_{k_2}\|_{L^\infty}\|e^{-it\Lambda}g_{k_3}\|_{L^\infty}\|g_{k_4}\|_{L^2}\\ &\leq C2^{-3m/2+40\beta m}\epsilon_0.\end{aligned}$$

Hence finishing the proof of the desired estimate (5.23) for the quartic terms.

Now we proceed to estimate the cubic terms "$T^{+,\kappa,\iota}_{k,k_1,k_2,k_3}(t,\xi)$". If moreover $k_1+k_2 \leq -m/2-12\beta m$, then the following estimate holds from the $L^2-L^\infty-L^\infty$ type trilinear estimate (2.6) in Lemma 2.2 and $L^\infty \to L^2$ type Sobolev embedding,

$$\begin{aligned}\|\mathcal{F}^{-1}[T^{+,\kappa,\iota}_{k,k_1,k_2,k_3}(t,\xi)]\|_{B_{k,j}} &\leq C2^{(1+\alpha)k+10k_+ +j+2k_1+2k_{1,+}}\|e^{-it\Lambda}g_{k_1}(t)\|_{L^\infty}2^{k_2}\|g_{k_2}(t)\|_{L^2}\|g_{k_3}(t)\|_{L^2}\\ &\leq C2^{2k_1+2k_2+20\beta m}\epsilon_0 + 2^{k_1+2k_2+20\beta m}\epsilon_0 \leq C2^{-m-\beta m}\epsilon_0,\end{aligned}$$

where C is some absolute constant. If $k_1+k_2 \geq -m/2-12\beta m$. Recall (5.26). By doing integration by parts in "η" many times, we can rule out the case when $\max\{j_1,j_2\} \leq m+k_{1,-}-\beta m$. For the case when $\max\{j_1,j_2\} \geq m+k_{1,-}-\beta m$, the following estimate holds from $L^2-L^\infty-L^\infty$ type trilinear estimate (2.6) in Lemma 2.2,

$$\begin{aligned}&\sum_{\max\{j_1,j_2\}\geq m+k_{1,-}-\beta m}\|\mathcal{F}^{-1}[T^{\tau,\kappa,\iota}_{k_1,j_1,k_2,j_2}(t,\xi)]\|_{B_{k,j}}\\ &\leq \sum_{j_1\geq\max\{m+k_{1,-}-\beta m,j_2\}} C2^{(1+\alpha)k+10k_+ +2k_1+j}\|g_{k_1,j_1}(t)\|_{L^2}\|e^{-it\Lambda}g_{k_2,j_2}(t)\|_{L^\infty}\|e^{-it\Lambda}g_{k_3}(t)\|_{L^\infty}\\ &+ \sum_{j_2\geq\max\{m+k_{1,-}-\beta m,j_1\}} C2^{(1+\alpha)k+10k_+ +j+2k_1}\|g_{k_2,j_2}(t)\|_{L^2}\|e^{-it\Lambda}g_{k_1,j_1}(t)\|_{L^\infty}\|e^{-it\Lambda}g_{k_3}(t)\|_{L^\infty}\\ &\leq C2^{-5m/2+50\beta m-k_2}\epsilon_0 \leq C2^{-m-\beta m}\epsilon_0,\end{aligned} \tag{5.28}$$

where C is some absolute constant. Hence finishing the proof of desired estimates (5.21) and (5.23) for the case when $k_2 \leq k_1-10$. □

Lemma 5.6. – *Under the bootstrap assumption* (4.49) *and the assumption that* $k_1-10 \leq k_2 \leq k_1+1$ *and* $k_2 \leq k_3-10 \leq k_2+1$*, the desired estimate* (5.21) *for the cubic terms holds for fixed* $k, k_1, k_2,$ *and* k_3 *inside the range listed in* (5.22) *and the desired estimate* (5.23) *holds for fixed* k, k_1, k_2, k_3 *and* k_4 *inside the range listed in* (5.24).

Proof. – From $L^2-L^\infty-L^\infty-L^\infty$ type mutilinear estimate, we have

$$\begin{aligned}\|\mathcal{F}^{-1}[K^{\mu_1,\mu_2,\nu_1,\nu_2}_{k,k_1,k_2,k_3,k_4}(t,\xi)]\|_{B_{k,j}} &\leq C2^{(1+\alpha)k+10k_+ +j+2k_1}\|e^{-it\Lambda}g_{k_1}(t)\|_{L^\infty}\|e^{-it\Lambda}g_{k_2}(t)\|_{L^\infty}\\ &\quad\times\|e^{-it\Lambda}g_{k_3}(t)\|_{L^\infty}\|g_{k_4}(t)\|_{L^2}\\ &\leq C2^{-3m/2+40\beta m}\epsilon_0.\end{aligned} \tag{5.29}$$

Hence finishing the proof of the quartic terms. Now, it remains to estimate the cubic terms "$T^{\tau,\kappa,\iota}_{k,k_1,k_2,k_3}(t,\xi)$". After putting g_{k_3} in L^2 and the other two inputs in L^∞, from the

$L^2 - L^\infty - L^\infty$ type trilinear estimate (2.6) in Lemma 2.2 when $k \le -2\beta m$, the following estimate holds for some absolute constant C,

$$\begin{aligned}\|\mathcal{F}^{-1}[T^{\tau,\kappa,\iota}_{k,k_1,k_2,k_3}(t,\xi)]\|_{B_{k,j}} &\le C2^{(1+\alpha)k+j+2k_1+2k_{1,+}}\|e^{-it\Lambda}g_{k_1}(t)\|_{L^\infty}\|e^{-it\Lambda}g_{k_2}(t)\|_{L^2}\|g_{k_3}(t)\|_{L^2}\\ &\le C\max\{2^{\alpha k-2m+2\beta m}, 2^{(1+\alpha)k-m+\beta m}\}\epsilon_1^3 \le C2^{-m-\beta m}\epsilon_0.\end{aligned}$$

Hence, it remains to consider the case when $k \ge -2\beta m$. Recall the normal form transformation we did in Subsection 4.1. Note that the case when η is close to $\xi/2$ is removed, see (4.30). Hence, the following estimate always holds for the case we are considering,

$$|\nabla_\eta \Phi^{\tau,\kappa,\iota}(\xi,\eta,\sigma)| \ge 2^{k-k_{1,+}/2-10}. \tag{5.30}$$

From the above estimate, after doing integration by parts in "η" many times, we can rule out the case when $\max\{j_1, j_2\} \le m + k_- - 3\beta m$. Hence, we only have to consider the case when $\max\{j_1, j_2\} \ge m + k_- - 3\beta m$. From the $L^2 - L^\infty - L^\infty$ type estimate (2.6) in Lemma 2.2, the following estimate holds,

$$\begin{aligned}&\sum_{\max\{j_1,j_2\}\ge m+k_- -3\beta m}\|\mathcal{F}^{-1}[T^{\tau,\kappa,\iota}_{k_1,j_1,k_2,j_2}(t,\xi)]\|_{B_{k,j}}\\ &\le \sum_{\max\{j_1,j_2\}\ge m+k_- -3\beta m} C2^{(1+\alpha)k+10k_+ +j+2k_1}\|g_{k_1,j_1}(t)\|_{L^2}2^{k_2}\|g_{k_2,j_2}(t)\|_{L^2}\|e^{-it\Lambda}g_{k_3}(t)\|_{L^\infty}\\ &\le C2^{-3m/2+50\beta m}\epsilon_0,\end{aligned}$$

where C is some absolute constant. Hence finishing the proof. □

LEMMA 5.7. – *Under the bootstrap assumption* (4.49) *and the assumption that* $k_1 - 10 \le k_2 \le k_1 + 1$ *and* $k_2 \le k_3 - 10 \le k_2 + 1$*, the desired estimate* (5.21) *for the cubic terms holds for fixed* k, k_1, k_2*, and* k_3 *inside the range listed in* (5.22) *and the desired estimate* (5.23) *holds for fixed* k, k_1, k_2, k_3 *and* k_4 *inside the range listed in* (5.24)*.*

Proof. – Note that, because the size of "k_3" plays little role in (5.29), the estimate (5.29) still holds for the quartic terms. Hence, we only have to estimate the cubic term "$T^{\tau,\kappa,\iota}_{k,k_1,k_2,k_3}(t,\xi)$". Define

$$\mathcal{S}_1 := \{(+,-,-),(-,+,+)\}, \qquad \mathcal{S}_2 := (+,-,+),(-,+,-)\}, \tag{5.31}$$

$$\mathcal{S}_3 := \{(+,+,-),(-,-,+)\}, \qquad \mathcal{S}_4 := \{(+,+,+),(-,-,-)\}. \tag{5.32}$$

Recall (4.8). Note that the space resonance in both "η" and "σ" set is given as follows,

$$\begin{aligned}\mathcal{R}_{\tau,\kappa,\iota} &:= \{(\xi,\eta,\sigma) : \nabla_\eta\Phi^{\tau,\kappa,\iota}(\xi,\eta,\sigma) = \nabla_\sigma\Phi^{\tau,\kappa,\iota}(\xi,\eta,\sigma) = 0\}\\ &= \{(\xi,\eta,\sigma) : \xi = \big((1+\tau\kappa)(1+\kappa\iota) - \tau\kappa\big)\sigma, \eta = (1+\kappa\iota)\sigma\}, \quad \tau,\kappa,\iota \in \{+,-\}.\end{aligned}$$

More specifically, we have

$$\begin{aligned}\mathcal{R}_{\tau,\kappa,\iota} &= \{(\xi,\eta,\sigma) : \xi = \sigma, \eta = 2\sigma\}, \quad (\xi-\eta,\eta-\sigma,\sigma)|_{\mathcal{R}_{\tau,\kappa,\iota}} = (-\xi,\xi,\xi), \ (\tau,\kappa,\iota) \in \mathcal{S}_1,\\ \mathcal{R}_{\tau,\kappa,\iota} &= \{(\xi,\eta,\sigma) : \xi = \sigma, \eta = 0\}, \quad (\xi-\eta,\eta-\sigma,\sigma)|_{\mathcal{R}_{\tau,\kappa,\iota}} = (\xi,-\xi,\xi), \ (\tau,\kappa,\iota) \in \mathcal{S}_2,\\ \mathcal{R}_{\tau,\kappa,\iota} &= \{(\xi,\eta,\sigma) : \xi = -\sigma, \eta = 0\}, \quad (\xi-\eta,\eta-\sigma,\sigma)|_{\mathcal{R}_{\tau,\kappa,\iota}} = (\xi,\xi,-\xi), \ (\tau,\kappa,\iota) \in \mathcal{S}_3,\\ \mathcal{R}_{\tau,\kappa,\iota} &= \{(\xi,\eta,\sigma) : \xi = 3\sigma, \eta = 2\sigma\}, \quad (\xi-\eta,\eta-\sigma,\sigma)|_{\mathcal{R}_{\tau,\kappa,\iota}} = (\xi/3,\xi/3,\xi/3), \ (\tau,\kappa,\iota) \in \mathcal{S}_4.\end{aligned}$$

When $(\tau,\kappa,\iota)\in\mathcal{S}_1\cup\mathcal{S}_2\cup\mathcal{S}_3$. – Note that, after changing of variables, those three cases are symmetric. Hence, it would be sufficient to estimate the case when $(\tau,\kappa,\iota)\in\mathcal{S}_1$ in details.

We first do change of variables and then localize around the space resonance set with a well chosen threshold. As a result, we can decompose the cubic term $T^{\tau,\kappa,\iota}_{k,k_1,k_2,k_3}(t,\xi)$ as follows,

(5.33)
$$T^{\tau,\kappa,\iota}_{k,k_1,k_2,k_3}(t,\xi)=\sum_{l_1,l_2\geq \bar{l}_\tau} C^{\tau,l_1,l_2}(t,\xi),\quad C^{\tau,l_1,l_2}(t,\xi)=\sum_{j_1\geq -k_{1,-},j_2\geq -k_{2,-}} C^{\tau,l_1,l_2}_{j_1,j_2}(t,\xi),$$

(5.34)
$$C^{\tau,l_1,l_2}_{j_1,j_2}(t,\xi):=\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{it\widetilde{\Phi}^{\tau,\kappa,\iota}(\xi,\eta,\sigma)}\tilde{d}_{\tau,\kappa,\iota}(\xi,2\xi+\eta+\sigma,\xi+\sigma)\widehat{g^{\tau}_{k_1,j_1}}(t,-\xi-\eta-\sigma)$$
$$\times\widehat{g^{\kappa}_{k_2,j_2}}(t,\xi+\eta)\widehat{g^{\iota}_{k_3}}(t,\xi+\sigma)\varphi_{l_1;\bar{l}_\tau}(\eta)\varphi_{l_2;\bar{l}_\tau}(\sigma)\,d\sigma\, d\eta,$$

where the phase $\widetilde{\Phi}^{\tau,\kappa,\iota}(\xi,\eta,\sigma)$ is defined as follows,

$$\widetilde{\Phi}^{\tau,\kappa,\iota}(\xi,\eta,\sigma):=\Lambda(|\xi|)-\tau\Lambda(|\xi+\eta+\sigma|)-\kappa\Lambda(|\xi+\eta|)-\iota\Lambda(|\xi+\sigma|),\quad(\tau,\kappa,\iota)\in\mathcal{S}_1,\tag{5.35}$$

the thresholds $\bar{l}_-:=-2m/5-10\beta m$ and $\bar{l}_+:=k_--10$ and the cutoff function $\varphi_{l;\bar{l}}(\cdot)$ with the threshold $\bar{l}$ is defined as follows,

$$\varphi_{l;\bar{l}}(x):=\begin{cases}\psi_{\leq\bar{l}}(|x|) & \text{if } l=\bar{l}\\ \psi_l(|x|) & \text{if } l>\bar{l}.\end{cases}\tag{5.36}$$

If $\tau=+$, *i.e.*, $(\tau,\kappa,\iota)=(+,-,-)$. – Recall the normal form transformation that we did in Subsection 4.1 , see (4.20) and (4.30). For the case we are considering, i.e., $(\tau,\kappa,\iota)\in\widetilde{S}$, we already canceled out the case when $\max\{l_1,l_2\}=\bar{l}_+$. Hence it would be sufficient to consider the case when $\max\{l_1,l_2\}>\bar{l}_-$. By the symmetry between inputs, without loss of generality, we assume that $l_2=\max\{l_1,l_2\}>\bar{l}_+:=k_--10$. For this case, we take the advantage of the fact that $\nabla_\eta\widetilde{\Phi}^{\tau,\kappa,\iota}(\xi,\eta,\sigma)$ is relatively big, i.e., we are away from the space resonance in "η" set. More precisely, we have

$$\big|\nabla_\eta\widetilde{\Phi}^{+,-,-}(\xi,\eta,\sigma)\big|=\Big|\Lambda'(|\xi+\eta+\sigma|)\frac{\xi+\eta+\sigma}{|\xi+\eta+\sigma|}-\Lambda'(|\xi+\eta|)\frac{\xi+\eta}{|\xi+\eta|}\Big|\geq 2^{l_2-10}.\tag{5.37}$$

Hence, we can do integration by parts in "η" many times to rule out the case when $\max\{j_1,j_2\}\leq m+k_--\beta m$. From the $L^2-L^\infty-L^\infty$ type trilinear estimate (2.6) in Lemma 2.2 and the $L^\infty\to L^2$ type Sobolev embedding, the following estimate holds,

$$\sum_{\max\{j_1,j_2\}\geq m+k_--\beta m}\|\mathcal{F}^{-1}[C^{+,l_1,l_2}_{j_1,j_2}(t,\xi)](x)\|_{B_{k,j}}\leq\sum_{\max\{j_1,j_2\}\geq m+k_--\beta m} C2^{(1+\alpha)k+10k_++j+2k_1}$$
$$\times\|e^{-it\Lambda}g_{k_3}(t)\|_{L^\infty}\|g_{k_2,j_2}(t)\|_{L^2}2^{k_2}\|g_{k_1,j_1}(t)\|_{L^2}\leq C2^{-3m/2+40\beta m}\epsilon_0,$$

where C is some absolute constant.

If $\tau=-$, *i.e.*, $(\tau,\kappa,\iota)=(-,+,+)$. – By the symmetry between l_1 and l_2, without loss of generality, we assume that $l_2=\max\{l_1,l_2\}$. Recall (5.35). We have

$$\big|\nabla_\xi\widetilde{\Phi}^{-,+,+}(\xi,\eta,\sigma)\big|$$
$$=\Big|\Lambda'(|\xi|)\frac{\xi}{|\xi|}+\Lambda'(|\xi+\eta|)\frac{\xi+\eta+\sigma}{|\xi+\eta+\sigma|}-\Lambda'(|\xi+\eta|)\frac{\xi+\eta}{|\xi+\eta|}-\Lambda'(|\xi+\sigma|)\frac{\xi+\sigma}{|\xi+\sigma|}\Big|.$$

From the above equality, we know that the following estimate holds,

$$\big|\nabla_\xi \widetilde{\Phi}^{-,+,+}(\xi,\eta,\sigma)\big|\varphi_{l_1;\bar{l}_\tau}(\eta)\varphi_{l_2;\bar{l}_\tau}(\sigma) \leq 2^{l_2+10}. \tag{5.38}$$

Hence, we can first rule out the case when $j \geq m + l_2 + 2\beta m$ by doing integration by parts in "ξ" many times. From now on, it would be sufficient to consider the case when $j \leq m + l_2 + 2\beta m$.

We first consider the case when $l_2 = \bar{l}_- = -2m/5 - 10\beta m$. After using the volume of supports in "η" and "σ," the following estimate holds,

$$\begin{aligned}\|\mathscr{F}^{-1}[C^{-,\bar{l}_-,\bar{l}_-}(t,\xi)](x)\|_{B_{k,j}} &\leq C2^{(1+\alpha)k+10k_+ +j+2k_1}2^{4\bar{l}}\|g_{k_1}(t)\|_{L^2}\|g_{k_2}(t)\|_{L^1}\|g_{k_3}(t)\|_{L^1}\\ &\leq C2^{5\bar{l}+m+30\beta m}\epsilon_1^3 \leq C2^{-m-\beta m}\epsilon_0,\end{aligned}$$

where C is some absolute constant. Now, we proceed to consider the case when $l_2 > \bar{l}_- = -2m/5 - 10\beta m$. Note that the following estimate holds for the case we are considering,

$$\big|\nabla_\eta \widetilde{\Phi}^{-,+,+}(\xi,\eta,\sigma)\big| \geq 2^{l_2-k_+/2-10}. \tag{5.39}$$

Therefore, we can do integration by parts in η many times to rule out the case when $\max\{j_1, j_2\} \leq m + l_2 - 4\beta m$. From the $L^2 - L^\infty - L^\infty$ type trilinear estimate (2.6) in Lemma 2.2, the following estimate holds for some absolute constant C when $\max\{j_1, j_2\} \geq m + l_2 - 4\beta m$,

$$\begin{aligned}&\sum_{\max\{j_1,j_2\}\geq m+l_2-4\beta m} \|\mathscr{F}^{-1}[C^{-,l_1,l_2}_{j_1,j_2}(t,\xi)](x)\|_{B_{k,j}}\\ &\leq C2^{(1+\alpha)k+10k_+ +j+2k_1}\|e^{-it\Lambda}g_{k_3}(t)\|_{L^\infty}\\ &\quad\times\Big[\sum_{j_2\geq\max\{m+l_2-4\beta m,j_1\}}\|g_{k_2,j_2}(t)\|_{L^2}\|e^{-it\Lambda}g_{k_1,j_1}(t)\|_{L^\infty}\\ &\quad+\sum_{j_1\geq\max\{m+l_2-4\beta m,j_2\}}\|e^{-it\Lambda}g_{k_2,j_2}(t)\|_{L^\infty}\\ &\quad\times\|g_{k_1,j_1}(t)\|_{L^2}\Big] \leq C2^{-2m-l_2-m/2+40\beta m}\epsilon_0 \leq C2^{-m-\beta m}\epsilon_0.\end{aligned} \tag{5.40}$$

When $(\tau,\kappa,\iota) \in \mathscr{S}_4$. – Very similarly, we localize around the space resonance set "$(\xi/3,\xi/3,\xi/3)$" by doing change of variables for "$T^{\tau,\kappa,\iota}_{k_1,k_2,k_3}(t,\xi)$" as follows,

$$\begin{aligned}T^{\tau,\kappa,\iota}_{k_1,k_2,k_3}(t,\xi) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{it\widehat{\Phi}^{\tau,\kappa,\iota}(\xi,\eta,\sigma)}\tilde{d}_{\tau,\kappa,\iota}(\xi,2\xi/3+\eta+\sigma,\xi/3+\sigma)\widehat{g^{\tau}_{k_1}}(t,\xi/3-\eta-\sigma)\\ \times\widehat{g^{\kappa}_{k_2}}(t,\xi/3+\eta)\widehat{g^{\iota}_{k_3}}(t,\xi/3+\sigma)\,d\sigma\, d\eta,\end{aligned}$$

where the phase $\widehat{\Phi}^{\tau,\kappa,\iota}(\xi,\eta,\sigma)$ is defined as follows,

$$\widehat{\Phi}^{\tau,\kappa,\iota}(\xi,\eta,\sigma) := \Lambda(|\xi|)-\tau\Lambda(|\xi/3-\eta-\sigma|)-\kappa\Lambda(|\xi/3+\eta)-\iota\Lambda(|\xi/3+\sigma|), \quad (\tau,\kappa,\iota)\in\mathscr{S}_4.$$

Recall the normal form transformation that we did in Subsection 4.1. The symbol around a neighborhood of $(\xi/3,\xi/3,\xi/3)$ has been canceled, see (4.30) and (4.40). Hence, the following decomposition holds,

$$T^{\tau,\kappa,\iota}_{k,k_1,k_2,k_3}(t,\xi) = \sum_{i=1,2} T^{\tau,\kappa,\iota}_{k_1,k_2,k_3;i}(t,\xi),$$

$$T^{\tau,\kappa,\iota}_{k_1,k_2,k_3;1}(t,\xi) = \sum_{j_1\geq -k_{1,-}, j_2\geq -k_{2,-}} T^{\tau,\kappa,\iota}_{k_1,j_1,k_2,j_2;1}(t,\xi),$$

$$T^{\tau,\kappa,\iota}_{k_1,k_2,k_3;2}(t,\xi) = \sum_{j_1\geq -k_{1,-}, j_3\geq -k_{3,-}} T^{\tau,\kappa,\iota}_{k_1,j_1,k_3,j_3;2}(t,\xi)$$

$$T^{\tau,\kappa,\iota}_{k_1,j_1,k_2,j_2;1}(t,\xi) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{it\widehat{\Phi}^{\tau,\kappa,\iota}(\xi,\eta,\sigma)}\tilde{d}_{\tau,\kappa,\iota}(\xi, 2\xi/3+\eta+\sigma, \xi/3+\sigma)\widehat{g^{\tau}_{k_1,j_1}}(t,\xi/3-\eta-\sigma)$$

$$\times \widehat{g^{\kappa}_{k_2,j_2}}(t,\xi/3+\eta)\widehat{g^{\iota}_{k_3}}(t,\xi/3+\sigma)\psi_{\geq k-20}(2\eta+\sigma)d\sigma d\eta, \tag{5.41}$$

$$T^{\tau,\kappa,\iota}_{k_1,j_1,k_3,j_3;2}(t,\xi) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{it\widehat{\Phi}^{\tau,\kappa,\iota}(\xi,\eta,\sigma)}\tilde{d}_{\tau,\kappa,\iota}(\xi, 2\xi/3+\eta+\sigma, \xi/3+\sigma)\psi_{\geq k-20}(2\sigma+\eta)$$

$$\times \psi_{\leq k-20}(2\eta+\sigma)\widehat{g^{\tau}_{k_1,j_1}}(t,\xi/3-\eta-\sigma)\widehat{g^{\kappa}_{k_2}}(t,\xi/3+\eta) \tag{5.42}$$

$$\times \widehat{g^{\iota}_{k_3,j_3}}(t,\xi/3+\sigma)d\sigma d\eta.$$

The estimates of "$T^{\tau,\kappa,\iota}_{k_1,k_2,k_3;1}(t,\xi)$" and "$T^{\tau,\kappa,\iota}_{k_1,k_2,k_3;2}(t,\xi)$" are very similar. For simplicity, we only estimate $T^{\tau,\kappa,\iota}_{k_1,k_2,k_3;1}(t,\xi)$ in details here. Note that "$2\eta+\sigma$" is bounded from below by 2^{k-10} for the case we are considering, which implies that the size of $\nabla_\eta\widehat{\Phi}^{\tau,\kappa,\iota}(\xi,\eta,\sigma)$ is bounded from blow by $2^{k-k_+/2-20}$. Therefore, after doing integration by parts many times in "η," we can rule out the case when $\max\{j_1,j_2\}\leq m+k_- -2\beta m$. For the case when $\max\{j_1,j_2\}\geq m+k_- -2\beta m$, a similar estimate as in (5.40) holds for some absolute constant C as follows,

$$\sum_{\max\{j_1,j_2\}\geq m+k_- -2\beta m} \|\mathscr{F}^{-1}[T^{\tau,\kappa,\iota}_{k_1,j_1,k_2,j_2;1}(t,\xi)]\|_{B_{k,j}} \leq \text{R.H.S. of (5.40)} \leq C2^{-m-\beta m}\epsilon_0.$$

Hence finishing the proof. □

6. The high order weighted norm estimate

In this section, our main goal is to prove (4.51) under the smallness assumption (4.49). The plan of this section is listed as follows. (i) In Subsection 6.1, we first classify different scenarios when estimating the left hand side of (4.51) and then show a key decomposition, e.g., (6.17), holds when the vector field $\hat{L}_\xi$ hits the phases $\Phi^{\mu,\nu}(\xi,\eta)$. (ii) In Subsection 6.2 and Subsection 6.3, we finish the Z_2-estimate of the quadratic terms for the High-High type interaction and the High-Low type interaction respectively; (iii) In Subsection 6.4, we finish the Z_2-estimate of the cubic terms; (iv) In Subsection 6.5, we finish the Z_2-estimate of the quartic terms. Therefore, combining the aforementioned estimates with the estimate (7.13) of quintic and higher order reminder terms "$\mathscr{R}_1$" in Lemma 7.4 in Section 7, we finish the high order weighted norm estimate.

6.1. The set-up of the Z_2-norm estimate

Define

$$\hat{\Omega}_\xi := -\xi^{\perp}\cdot\nabla_\xi, \quad d_\Omega := 0, \quad \xi_\Omega := -\xi^{\perp}, \quad \hat{L}_\xi := -\xi\cdot\nabla_\xi, \quad d_L := -2, \quad \xi_L := -\xi. \tag{6.1}$$

(6.2)

$$\chi_k^1 := \{(k_1,k_2) : |k_1-k_2| \le 10, k \le k_1+10\}, \quad \chi_k^2 := \{(k_1,k_2) : k_2 \le k_1-10, |k_1-k| \le 10\}.$$

Recall that $L := x\cdot\nabla + 2$ and $\Omega := x^{\perp}\cdot\nabla$ and the Z_2 norm is defined in (1.23). We have

$$\hat{\Omega}_\xi \widehat{g}(t,\xi) = \widehat{\Omega g}(t,\xi), \quad \hat{L}_\xi \widehat{g}(t,\xi) = \widehat{Lg}(t,\xi),$$

$$\|g(t)\|_{Z_2} \in \Big(\sum_{\Gamma_\xi^1,\Gamma_\xi^2\in\{\hat{\Omega}_\xi,\hat{L}_\xi\}} \|\Gamma_\xi^1\Gamma_\xi^2\widehat{g}(t,\xi)\|_{L^2} + \|\Gamma_\xi^1\widehat{g}(t,\xi)\|_{L^2}\Big)[c,C], \tag{6.3}$$

where c and C are some absolute constants.

Since the estimate of the second part of the right hand side of (6.3) is similar and also much easier than the first part, for simplicity, we only estimate the first part in details here. Therefore, to prove the desired estimate (4.51), it would be sufficient to prove the following desired estimate for any $\Gamma_\xi^1,\Gamma_\xi^2 \in \{\hat{L}_\xi,\hat{\Omega}_\xi\}$(correspondingly, $\Gamma^1,\Gamma^2 \in \{L,\Omega\}$) and any $t_1,t_2\in[2^{m-1},2^m]$,

$$\Big|\mathrm{Re}\Big[\int_{t_1}^{t_2}\int_{\mathbb{R}^2} \overline{\Gamma_\xi^1\Gamma_\xi^2\widehat{g}(t,\xi)}\Gamma_\xi^1\Gamma_\xi^2\partial_t\widehat{g}(t,\xi)\,d\xi\,dt\Big]\Big| \le C2^{2\tilde{\delta}m}\epsilon_0^2, \tag{6.4}$$

where C is some absolute constant.

Recall (4.35). We first classify the quadratic terms. Recall (4.36). From the direct computations, we have the following identity for the quadratic terms,

$$\begin{aligned}
&\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\overline{\Gamma_\xi^1\Gamma_\xi^2\widehat{g_k}(t,\xi)}\Gamma_\xi^1\Gamma_\xi^2 B^{\mu,\nu}_{k,k_1,k_2}(t,\xi)\,d\xi\,dt \\
&\quad= \int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\overline{\Gamma_\xi^1\Gamma_\xi^2\widehat{g_k}(t,\xi)}e^{it\Phi^{\mu,\nu}(\xi,\eta)}\Big[\Gamma_\xi^1\Gamma_\xi^2\big(\tilde{q}_{\mu,\nu}(\xi-\eta,\eta)\widehat{g^{\mu}_{k_1}}(t,\xi-\eta)\big)\widehat{g^{\nu}_{k_2}}(t,\eta) \\
&\qquad+ \sum_{l,n=\{1,2\}} it\big(\Gamma_\xi^l\Phi^{\mu,\nu}(\xi,\eta)\big)\Gamma_\xi^n\big(\tilde{q}_{\mu,\nu}(\xi-\eta,\eta)\widehat{g^{\mu}_{k_1}}(t,\xi-\eta)\big)\widehat{g^{\nu}_{k_2}}(t,\eta) \\
&\qquad- t^2\Gamma_\xi^1\Phi^{\mu,\nu}(\xi,\eta)\Gamma_\xi^2\Phi^{\mu,\nu}(\xi,\eta)\tilde{q}_{\mu,\nu}(\xi-\eta,\eta)\widehat{g^{\mu}_{k_1}}(t,\xi-\eta)\widehat{g^{\nu}_{k_2}}(t,\eta)\Big]d\eta\,d\xi\,dt.
\end{aligned} \tag{6.5}$$

To make the formulation (6.5) symmetric, we separate $\Gamma_\xi^i\widehat{g}(t,\xi-\eta)$, $i\in\{1,2\}$, into two parts as follows,

$$\Gamma_\xi^i\widehat{g}(t,\xi-\eta) = \Gamma_{\xi-\eta}^i\widehat{g}(t,\xi-\eta) - \Gamma_\eta^i\widehat{g}(t,\xi-\eta).$$

After applying the above decomposition to the equality (6.5), we do integration by parts in "η" in (6.5) to move the derivative in front of $\Gamma_\eta^i\widehat{g}(t,\xi-\eta)$ around, see (6.1). As a result, the following equality holds,

$$\mathrm{Re}\Big[\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\overline{\Gamma_\xi^1\Gamma_\xi^2\widehat{g}_k(t,\xi)}\Gamma_\xi^1\Gamma_\xi^2 B^{\mu,\nu}_{k,k_1,k_2}(t,\xi)\,d\xi\,dt\Big] = \sum_{i=1,2,3,4}\mathrm{Re}[P^i_{k,k_1,k_2}], \tag{6.6}$$

where

$$P^1_{k,k_1,k_2} := \sum_{\{l,n\}=\{1,2\}} \int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \overline{\widehat{\Gamma^1\Gamma^2 g_k}(t,\xi)}\, e^{it\Phi^{\mu,\nu}(\xi,\eta)}\, it(\Gamma^l_\xi+\Gamma^l_\eta)\Phi^{\mu,\nu}(\xi,\eta)$$

$$\times\big[\tilde q_{\mu,\nu}(\xi-\eta,\eta)\big(\widehat{g^{\nu}_{k_2}}(t,\eta)\widehat{\Gamma^n g^{\mu}_{k_1}}(t,\xi-\eta)+\widehat{g^{\mu}_{k_1}}(t,\xi-\eta)\widehat{\Gamma^n g^{\nu}_{k_2}}(t,\eta)\big) \tag{6.7}$$

$$+(\Gamma^n_\xi+\Gamma^n_\eta+d_{\Gamma^n})\tilde q_{\mu,\nu}(\xi-\eta,\eta)\widehat{g^{\mu}_{k_1}}(t,\xi-\eta)\widehat{g^{\nu}_{k_2}}(t,\eta)\big]d\eta\, d\xi\, dt,$$

$$P^2_{k,k_1,k_2} := -\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \overline{\widehat{\Gamma^1\Gamma^2 g_k}(t,\xi)}\, e^{it\Phi^{\mu,\nu}(\xi,\eta)}\, t^2(\Gamma^1_\xi+\Gamma^1_\eta)\Phi^{\mu,\nu}(\xi,\eta)(\Gamma^2_\xi+\Gamma^2_\eta)$$

$$\times\Phi^{\mu,\nu}(\xi,\eta)\tilde q_{\mu,\nu}(\xi-\eta,\eta)\widehat{g^{\mu}_{k_1}}(t,\xi-\eta)\widehat{g^{\nu}_{k_2}}(t,\eta)\,d\eta\, d\xi\, dt, \tag{6.8}$$

$$P^3_{k,k_1,k_2} := \int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \overline{\widehat{\Gamma^1\Gamma^2 g_k}(t,\xi)}\, e^{it\Phi^{\mu,\nu}(\xi,\eta)}\Big(\tilde q_{\mu,\nu}(\xi-\eta,\eta)\big(\widehat{\Gamma^1\Gamma^2 g^{\mu}_{k_1}}(t,\xi-\eta)\widehat{g^{\nu}_{k_2}}(t,\eta)$$

$$+\widehat{g^{\mu}_{k_1}}(t,\xi-\eta)\widehat{\Gamma^1\Gamma^2 g^{\nu}_{k_2}}(t,\eta)\big)$$

$$+(\Gamma^1_\xi+\Gamma^1_\eta+d_{\Gamma^1})(\Gamma^2_\xi+\Gamma^2_\eta+d_{\Gamma^2})\tilde q_{\mu,\nu}(\xi-\eta,\eta)\widehat{g^{\mu}_{k_1}}(t,\xi-\eta)\widehat{g^{\nu}_{k_2}}(t,\eta)$$

$$+(\Gamma^l_\xi+\Gamma^l_\eta+d_{\Gamma^l})\tilde q_{\mu,\nu}(\xi-\eta,\eta) \tag{6.9}$$

$$\times\big(\widehat{\Gamma^n g^{\mu}_{k_1}}(t,\xi-\eta)\widehat{g^{\nu}_{k_2}}(t,\eta)+\widehat{g^{\mu}_{k_1}}(t,\xi-\eta)\widehat{\Gamma^n g^{\nu}_{k_2}}(t,\eta)\big)\Big)d\eta\, d\xi\, dt,$$

$$P^4_{k,k_1,k_2} := \sum_{j_1\geq -k_{1,-},\, j_2\geq -k_{2,-}} P^{4,j_1,j_2}_{k,k_1,k_2},\quad P^{4,j_1,j_2}_{k,k_1,k_2} := \sum_{\{l,n\}=\{1,2\}}\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\overline{\widehat{\Gamma^1\Gamma^2 g_k}(t,\xi)}$$

$$\times e^{it\Phi^{\mu,\nu}(\xi,\eta)}\tilde q_{\mu,\nu}(\xi-\eta,\eta)\widehat{\Gamma^l g^{\mu}_{k_1,j_1}}(t,\xi-\eta)\widehat{\Gamma^n g^{\nu}_{k_2,j_2}}(t,\eta)\,d\eta\, d\xi\, dt. \tag{6.10}$$

Now we reveal a subtle structure inside the symbol "$(\Gamma_\xi+\Gamma_\eta)\Phi^{\mu,\nu}(\xi,\eta)$," which appears in $P^i_{k,k_1,k_2}, i\in\{1,2\}$, see (6.7) and (6.8). Note that, the following equalities hold when $|\eta|\leq 2^{-10}|\xi|$ and $\mu=+$,

$$\big(\hat L_\xi+\hat L_\eta\big)\Phi^{\mu,\nu}(\xi,\eta) = -2\xi\cdot\big(\lambda'(|\xi|^2)\xi-\lambda'(|\xi-\eta|^2)(\xi-\eta)\big)-2\eta\cdot\big(-\lambda'(|\xi-\eta|^2)(\eta-\xi)$$

$$-\nu\lambda'(|\eta|)\eta\big) = -4\big(\lambda'(|\xi|^2)+\lambda''(|\xi|^2)|\xi|^2\big)\xi\cdot\eta+O(|\eta|^2), \tag{6.11}$$

$$(\hat\Omega_\xi+\hat\Omega_\eta)\Phi^{\mu,\nu}(\xi,\eta) = -2\xi^{\perp}\cdot\big(\lambda'(|\xi|^2)\xi-\mu\lambda(|\xi-\eta|^2)(\xi-\eta)\big)$$

$$-2\eta^{\perp}\cdot\big(-\mu\lambda(|\xi-\eta|^2)(\eta-\xi)-\nu\lambda'(|\eta|^2)\eta\big) = -2\mu\lambda'(|\xi-\eta|^2)\big(\xi^{\perp}\cdot\eta+\eta^{\perp}\cdot\xi\big)=0, \tag{6.12}$$

where $\lambda(|x|):=\Lambda(\sqrt{|x|})$. The following approximation holds when $|\xi|$ is very close to zero,

$$\Lambda(|\xi|) = |\xi|^2-\frac{1}{6}|\xi|^4+O(|\xi|^6),\quad \lambda(|\xi|)=|\xi|-\frac{1}{6}|\xi|^2+O(|\xi|^3),\quad |\xi|\leq 2^{-10}. \tag{6.13}$$

Moreover, the following equalities hold when $|\xi|\leq 2^{-10}|\eta|$ and $\mu\nu=-$,

$$\big(\hat L_\xi+\hat L_\eta\big)\Phi^{\mu,\nu}(\xi,\eta) = -2\lambda'(|\xi|^2)|\xi|^2+\mu 2\lambda'(|\xi-\eta|^2)\xi\cdot(\xi-\eta)+2\mu\lambda'(|\xi-\eta|^2)\eta\cdot(\eta-\xi)$$

$$+2\nu\lambda'(|\eta|^2)|\eta|^2 = -4\mu\big(\lambda'(|\eta|^2)+\lambda''(|\eta|^2)|\eta|^2\big)\xi\cdot\eta+O(|\xi|^2), \tag{6.14}$$

$$(\hat\Omega_\xi+\hat\Omega_\eta)\Phi^{\mu,\nu}(\xi,\eta) = -2\mu\lambda'(|\xi-\eta|^2)\big(\xi^{\perp}\cdot\eta+\eta^{\perp}\cdot\xi\big)=0. \tag{6.15}$$

Now, we show that similar decompositions also hold for the phase $\Phi^{\mu,\nu}(\xi,\eta)$ in two different scenarios so that we can link the symbol $(\Gamma_\xi + \Gamma_\eta)\Phi^{\mu,\nu}(\xi,\eta)$ with the phase $\Phi^{\mu,\nu}(\xi,\eta)$. Note that the following expansion holds when $|\eta| \le 2^{-10}|\xi|$ and $\mu = +$,

$$\Phi^{\mu,\nu}(\xi,\eta) = \lambda(|\xi|^2) - \lambda(|\xi|^2 - 2\xi\cdot\eta + |\eta|^2) - \nu\lambda(|\eta|^2) = 2\lambda'(|\xi|^2)\xi\cdot\eta + O(|\eta|^2). \tag{6.16}$$

Hence, from (6.16) and (6.11), the following identity holds when $|\eta| \le 2^{-10}|\xi|$ and $\mu = +$,

$$\big(\hat{L}_\xi + \hat{L}_\eta\big)\Phi^{\mu,\nu}(\xi,\eta) = \tilde{c}(\xi-\eta)\Phi^{\mu,\nu}(\xi,\eta) + O(|\eta|^2), \quad \tilde{c}(\xi) := -\frac{2\lambda''(|\xi|^2)|\xi|^2 + 2\lambda'(|\xi|^2)}{\lambda'(|\xi|^2)}. \tag{6.17}$$

Moreover, the following approximation holds for the phase $\Phi^{\mu,\nu}(\xi,\eta)$ when $|\xi| \le 2^{-10}|\eta|$ and $\mu\nu = -$,

$$\Phi^{\mu,\nu}(\xi,\eta) = \lambda(|\xi|^2) - \mu\big(\lambda(|\xi|^2 - 2\xi\cdot\eta + |\eta|^2) - \lambda(|\eta|^2)\big) = 2\mu\lambda'(|\eta|^2)\xi\cdot\eta + O(|\xi|^2). \tag{6.18}$$

Therefore, from (6.18) and (6.14), the following identity holds when $|\xi| \le 2^{-10}|\eta|$ and $\mu\nu = -$,

$$\big(\hat{L}_\xi + \hat{L}_\eta\big)\Phi^{\mu,\nu}(\xi,\eta) = \tilde{c}(\xi-\eta)\Phi^{\mu,\nu}(\xi,\eta) + O(|\xi|^2). \tag{6.19}$$

6.2. Z_2-norm estimate of the quadratic terms: if $|k_1 - k_2| \le 10$

Recall the decomposition (6.6). We know that the Z_2-norm estimate of the quadratic terms in the High-High type interaction follows from the estimate (6.20) in Lemma 6.1 and the estimate (6.22) in Lemma 6.2.

LEMMA 6.1. – *Under the bootstrap assumption* (4.49)*, the following estimate holds for some absolute constant C,*

$$\Big|\sum_{|k_1-k_2|\le 10, k\le k_1+20} P^3_{k,k_1,k_2}\Big| + \Big|\sum_{|k_1-k_2|\le 10, k\le k_1+20} P^4_{k,k_1,k_2}\Big| \le C2^{2\tilde{\delta}m}\epsilon_0. \tag{6.20}$$

Proof. – Recall (6.6). From the $L^2 - L^\infty$ type bilinear estimate (2.5) in Lemma 2.2, we have

$$\begin{aligned}
&\Big|\sum_{|k_1-k_2|\le 10, k\le k_1+20} P^3_{k,k_1,k_2}\Big| \\
&\le \sup_{t_1,t_2\in[2^{m-1},2^m]} \sum_{|k_1-k_2|\le 10} C2^{m+2k_1}\|P_{\le k_1+20}\Gamma^1\Gamma^2 g(t)\|_{L^2}\big(\|e^{-it\Lambda}g_{k_1}(t)\|_{L^\infty} \\
&\quad + \|e^{-it\Lambda}g_{k_2}(t)\|_{L^\infty}\big)\Big(\sum_{l,m\in\{1,2\}}\|\Gamma^1\Gamma^2 g_{k_m}(t)\|_{L^2} + \|\Gamma^l g_{k_m}\|_{L^2} + \|g_{k_m}(t)\|_{L^2}\Big) \qquad (6.21)\\
&\le C2^{2\tilde{\delta}m}\epsilon_0^2,
\end{aligned}$$

where C is some absolute constant. The estimate of P^4_{k,k_1,k_2} is similar but slightly different. The spatial concentrations of inputs play a role. Note that, from the definition of Z_i-norms, $i \in \{1,2\}$ in (1.22) and (1.23) and the linear decay estimate (2.11) in Lemma 2.7, the following estimate holds for some absolute constant C,

$$\|\Gamma^l g_{k,j}\|_{L^2} \le C2^{-k-j+\tilde{\delta}m}\epsilon_1, \quad \|e^{-it\Lambda}\Gamma^l g_{k,j}\|_{L^\infty} \le C2^{-m+k+2j}\|\varphi^k_j(x)P_k g(t)\|_{L^2}.$$

After first doing spatial localizations for the inputs $g_{k_1}(t)$ and $g_{k_2}(t)$ and then put the input with smaller spatial concentration in L^∞ and the other input in L^2, the following estimate holds for some constants C_1 and C_2,

$$\begin{aligned}
&\Big|\sum_{|k_1-k_2|\le 10, k\le k_1+20} P^4_{k,k_1,k_2}\Big| \\
&\le \sup_{t_1,t_2\in[2^{m-1},2^m]} \sum_{|k_1-k_2|\le 10} \sum_{\{l,m\}=\{1,2\}} C_1 2^{m+2k_1} \|P_{\le k_1+20}\Gamma^1\Gamma^2 g(t)\|_{L^2} \\
&\quad \times \Big(\sum_{j_1\ge j_2} \|\Gamma^l g_{k_1,j_1}\|_{L^2}\|e^{-it\Lambda}\Gamma^n g_{k_2,j_2}\|_{L^\infty} + \sum_{j_2\ge j_1} \|e^{-it\Lambda}\Gamma^l g_{k_1,j_1}\|_{L^\infty}\|\Gamma^n g_{k_2,j_2}\|_{L^2}\Big) \\
&\le \sum_{j_2} C_2 2^{2k_1+2j_2}\|\varphi^{k_2}_{j_2}(x) P_{k_2} g(t)\|_{L^2}\Big(\sum_{j_1\ge j_2} 2^{2\tilde{\delta}m - j_1}\epsilon_1\Big) \\
&\quad + \sum_{j_1} C_2 2^{2k_1+2j_1}\|\varphi^{k_1}_{j_1}(x) P_{k_1} g(t)\|_{L^2}\Big(\sum_{j_2\ge j_1} 2^{2\tilde{\delta}m - j_2}\epsilon_1\Big) \\
&\le C_2 2^{2\tilde{\delta}m}\epsilon_0^2.
\end{aligned}$$

From the above estimate and the estimate (6.21), we know that the desired estimate (6.20) holds. □

LEMMA 6.2. – *Under the bootstrap assumption* (4.49)*, the following estimate holds for some absolute constant* C*,*

$$\Big|\sum_{|k_1-k_2|\le 10, k\le k_1+20} P^1_{k,k_1,k_2}\Big| + \Big|\sum_{|k_1-k_2|\le 10, k\le k_1+20} P^2_{k,k_1,k_2}\Big| \le C 2^{2\tilde{\delta}m}\epsilon_0. \tag{6.22}$$

Proof. – Note that, from the $L^2 - L^\infty$ type bilinear estimate (2.5) in Lemma 2.2, the following estimate holds for some absolute constant C,

$$\begin{aligned}
&|P^1_{k,k_1,k_2}| + |P^2_{k,k_1,k_2}| \\
&\qquad \le \sup_{t\in[2^{m-1},2^m]} C(2^{2m+k+3k_1} + 2^{3m+2k+4k_1}) \sum_{i,j=1,2} \big(\|g_{k_i}(t)\|_{L^2} + 2^{k_1}\|\nabla_\xi \widehat{g}_{k_i}(t,\xi)\|_{L^2}\big) \\
&\qquad\quad \times \|\Gamma^1\Gamma^2 g_k\|_{L^2}\|e^{-it\Lambda} g_{k_j}(t)\|_{L^\infty} \qquad (6.23) \\
&\qquad \le C 2^{\tilde{\delta}m+\delta m}(2^{m+k+k_{1,-}-15k_{1,+}} + 2^{2m+2k+2k_{1,-}-14k_{1,+}})\epsilon_0^2. \qquad (6.24)
\end{aligned}$$

From the above rough estimate (6.23), we can rule out the case when $k+k_{1,-}\le -m+\tilde{\delta}m/3$ or $k_1\ge m/5$. From now on, we restrict ourself to the case when $k+k_{1,-}\ge -m+\tilde{\delta}m/3$ and $k_1\le m/5$.

Recall (6.7) and (6.15). We know that the integral inside P^1_{k,k_1,k_2} actually vanishes when $\Gamma^l = \hat{\Omega}_\xi$. Hence, we only need to consider the case when $\Gamma^l_\xi = \hat{L}_\xi$. Based on the possible size of k, we separate into two cases as follows.

Case 1: if $k \le k_1 - 10$. – Recall the normal form transformation that we did in Subsection 4.1. For the case we are considering, we have $\mu\nu = -$. Recall (6.19). To take the advantage of this decomposition, we decompose P^1_{k,k_1,k_2} and P^2_{k,k_1,k_2} into two parts respectively as follows,

$$|P^1_{k,k_1,k_2}| \le \sum_{\Gamma\in\{L,\Omega\}} |\Gamma^{1,1}_{k,k_1,,k_2}| + |\Gamma^{1,2}_{k,k_1,,k_2}|, \quad |P^2_{k,k_1,k_2}| \le |\widetilde{P}^1_{k,k_1,k_2}| + |\widetilde{P}^2_{k,k_1,k_2}|, \tag{6.25}$$

where

$$\Gamma^{1,i}_{k,k_1,,k_2} := \int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \overline{\widehat{\Gamma^1\Gamma^2 g_k}(t,\xi)}\, e^{it\Phi^{\mu,\nu}(\xi,\eta)}\, it\tilde{q}^i_{\mu,\nu}(\xi-\eta,\eta)\big[\tilde{q}_{\mu,\nu}(\xi-\eta,\eta)\big(\widehat{\Gamma g^\mu_{k_1}}(t,\xi-\eta)$$

$$\times \widehat{g^\nu_{k_2}}(t,\eta) + \widehat{g^\mu_{k_1}}(t,\xi-\eta)\widehat{\Gamma g^\nu_{k_2}}(t,\eta)\big) + (\Gamma_\xi+\Gamma_\eta+d_\Gamma)\tilde{q}_{\mu,\nu}(\xi-\eta,\eta)\widehat{g^\mu_{k_1}}(t,\xi-\eta)\widehat{g^\nu_{k_2}}(t,\eta)\big]\, d\eta\, d\xi\, dt, \tag{6.26}$$

$$\widetilde{P}^i_{k,k_1,k_2} = -\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \overline{\widehat{LLg_k}(t,\xi)}\, e^{it\Phi^{\mu,\nu}(\xi,\eta)}\, t^2 \widehat{q}^i_{\mu,\nu}(\xi,\eta)\widehat{g^\mu_{k_1}}(t,\xi-\eta)\widehat{g^\nu_{k_2}}(t,\eta)\, d\eta\, d\xi\, dt, \tag{6.27}$$

where

$$\tilde{q}^1_{\mu,\nu}(\xi-\eta,\eta) = \tilde{c}(\xi-\eta)\Phi^{\mu,\nu}(\xi,\eta), \quad \tilde{q}^2_{\mu,\nu}(\xi-\eta,\eta) := (\hat{L}_\xi+\hat{L}_\eta)\Phi^{\mu,\nu}(\xi,\eta) - \tilde{c}(\xi-\eta)\Phi^{\mu,\nu}(\xi,\eta). \tag{6.28}$$

$$\widehat{q}^1_{\mu,\nu}(\xi,\eta) = \widehat{p}^1_{\mu,\nu}(\xi,\eta)\Phi^{\mu,\nu}(\xi,\eta), \quad \widehat{p}^1_{\mu,\nu}(\xi,\eta) := \tilde{q}_{\mu,\nu}(\xi-\eta,\eta)(\hat{L}_\xi+\hat{L}_\eta)\Phi^{\mu,\nu}(\xi,\eta)\tilde{c}(\xi-\eta), \tag{6.29}$$

$$\widehat{q}^2_{\mu,\nu}(\xi,\eta) = \tilde{q}_{\mu,\nu}(\xi-\eta,\eta)(\hat{L}_\xi+\hat{L}_\eta)\Phi^{\mu,\nu}(\xi,\eta)\big((\hat{L}_\xi+\hat{L}_\eta)\Phi^{\mu,\nu}(\xi,\eta) - \tilde{c}(\xi-\eta)\Phi^{\mu,\nu}(\xi,\eta)\big). \tag{6.30}$$

From the estimate (2.3) Lemma 2.1, the following estimates hold for some absolute constant C,

$$\begin{aligned} \|\tilde{q}^2_{\mu,\nu}(\xi-\eta,\eta)\|_{\mathcal{S}^\infty_{k,k_1,k_2}} \le C2^{2k}, \quad \|\widehat{p}^1_{\mu,\nu}(\xi,\eta)\|_{\mathcal{S}^\infty_{k,k_1,k_2}} &\le C2^{k+3k_1}, \\ \|\widehat{q}^2_{\mu,\nu}(\xi,\eta)\|_{\mathcal{S}^\infty_{k,k_1,k_2}} &\le C2^{3k+3k_1}. \end{aligned} \tag{6.31}$$

After doing integration by parts in "η" once for $\Gamma^{1,2}_{k,k_1,,k_2}$ and doing integration by parts in "η" twice for $\widetilde{P}^2_{k,k_1,k_2}$, the following estimates hold for some absolute constants C_1 and C_2,

$$\begin{aligned}\sum_{k\le k_1+20,|k_1-k_2|\le 10} &|\Gamma^{1,2}_{k,k_1,,k_2}| + |\widetilde{P}^2_{k,k_1,k_2}| \\ &\le \sup_{t\in[2^{m-1},2^m]} \sum_{k\le k_1+20,|k_1-k_2|\le 10} \Big[C_1 \|\Gamma^1\Gamma^2 g_k(t)\|_{L^2} 2^{m+k+k_1+k_{1,+}} \\ &\quad \times \Big(\sum_{i=0,1,2} 2^{ik_1}\|\nabla^i_\xi \widehat{g_{k_1}}(t,\xi)\|_{L^2} + 2^{ik_1}\|\nabla^i_\xi \widehat{g_{k_2}}(t,\xi)\|_{L^2}\Big)\Big(\sum_{i=1,2}\|e^{-it\Lambda} g_{k_i}\|_{L^\infty}\Big) \\ &\quad + \sum_{j_1\ge j_2} C_1 2^{k+3k_1+k_{1,+}} \\ &\quad \times \|\Gamma^1\Gamma^2 g_k(t)\|_{L^2} 2^{2j_2}\|\varphi^{k_2}_{j_2}(x)P_{k_2}g(t)\|_{L^2} 2^{j_1}\|\varphi^{k_1}_{j_1}(x)P_{k_1}g(t)\|_{L^2} \\ &\quad + \sum_{j_2\ge j_1} C_1 2^{k+3k_1+k_{1,+}}\|\Gamma^1\Gamma^2 g_k(t)\|_{L^2} \\ &\quad \times 2^{2j_1}\|\varphi^{k_2}_{j_2}(x)P_{k_2}g(t)\|_{L^2} 2^{j_2}\|\varphi^{k_1}_{j_1}(x)P_{k_1}g(t)\|_{L^2}\Big] \le C_2 2^{2\tilde{\delta}m}\epsilon_0^2. \end{aligned} \tag{6.32}$$

To sum up, from the estimate (6.25), the estimate (6.32), the estimate (6.33) in Lemma 6.3 and the estimate (6.45) in Lemma 6.4, we finish the estimate of P^i_{k,k_1,k_2}, $i \in \{1,2\}$, for the case we are considering.

Case 2: If $k \ge k_1 - 10$ and $|k_1 - k_2| \le 10$. – For the case we are considering, the sizes of all frequencies are comparable, which implies that the estimate (6.32) also holds for P^1_{k,k_1,k_2} and P^2_{k,k_1,k_2} without decomposing the symbols of quadratic terms as in the estimate (6.25).

Hence finishing the proof of the desired estimate (6.22). □

LEMMA 6.3. – *Under the bootstrap assumption* (4.49) *and the assumption that $k + k_{1,-} \ge -m + \tilde{\delta}m/3$, $k \le k_1 - 10$ and $k_1 \le m/5$, the following estimate holds for some absolute constant C,*

$$|\Gamma^{1,1}_{k,k_1,k_2}| \le C 2^{9/5\tilde{\delta}m}\epsilon_0^2. \tag{6.33}$$

Proof. – Recall the associated symbol $\tilde{q}^1_{\mu,\nu}(\xi-\eta,\eta)$ of $\Gamma^{1,1}_{k,k_1,k_2}$ in (6.28). To take the advantage of smallness of symbol near the time resonance set, we do integration by parts in time once. As a result, we have

$$\Gamma^{1,1}_{k,k_1,k_2} = \sum_{i=1,2} \widetilde{\Gamma}^{1,i}_{k,k_1,k_2}, \quad \widetilde{\Gamma}^{1,1}_{k,k_1,k_2} = \sum_{j_1\ge -k_{1,-},j_2\ge -k_{2,-}} \widetilde{\Gamma}^{j_1,j_2,1,1}_{k,k_1,k_2}, \tag{6.34}$$

$$\widetilde{\Gamma}^{j_1,j_2,1,1}_{k,k_1,k_2} := -\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \overline{\widehat{\Gamma^1\Gamma^2 g_k}(t,\xi)} e^{it\Phi^{\mu,\nu}(\xi,\eta)}\tilde{c}(\xi-\eta)\big[\tilde{q}_{\mu,\nu}(\xi-\eta,\eta)$$

$$\times\big(\widehat{g^{\nu}_{k_2,j_2}}(t,\eta)\widehat{\Gamma g^{\mu}_{k_1,j_1}}(t,\xi-\eta)+\widehat{g^{\mu}_{k_1,j_1}}(t,\xi-\eta)\widehat{\Gamma g^{\nu}_{k_2,j_2}}(t,\eta)\big)$$

$$+(\Gamma_\xi+\Gamma_\eta+d_\Gamma)\tilde{q}_{\mu,\nu}(\xi,\eta)\widehat{g^{\mu}_{k_1,j_1}}(t,\xi-\eta)\widehat{g^{\nu}_{k_2,j_2e}}(t,\eta)\big]d\eta d\xi dt$$

$$+\sum_{i=1,2}(-1)^i\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\overline{\widehat{\Gamma^1\Gamma^2 g_k}(t_i,\xi)}e^{it_i\Phi^{\mu,\nu}(\xi,\eta)}t_i\tilde{c}(\xi-\eta)\big[\tilde{q}_{\mu,\nu}(\xi-\eta,\eta)$$

$$\times\big(\widehat{\Gamma g^{\mu}_{k_1,j_1}}(t_i,\xi-\eta)\widehat{g^{\nu}_{k_2 j_2}}(t_i,\eta)+\widehat{g^{\mu}_{k_1,j_1}}(t_i,\xi-\eta)\widehat{\Gamma g^{\nu}_{k_2,j_2}}(t_i,\eta)\big)$$

$$+(\Gamma_\xi+\Gamma_\eta+d_\Gamma)\tilde{q}_{\mu,\nu}(\xi-\eta,\eta)\widehat{g^{\mu}_{k_1,j_1}}(t_i,\xi-\eta)\widehat{g^{\nu}_{k_2,j_2}}(t_i,\eta)\big]d\eta d\xi, \tag{6.35}$$

$$\widetilde{\Gamma}^{1,2}_{k,k_1,k_2} = -\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{it\Phi^{\mu,\nu}(\xi,\eta)}t\tilde{c}(\xi-\eta)\big[\tilde{q}_{\mu,\nu}(\xi-\eta,\eta)\partial_t\big(\overline{\widehat{\Gamma^1\Gamma^2 g_k}(t,\xi)}$$

$$\times\big(\widehat{\Gamma g^{\mu}_{k_1}}(t,\xi-\eta)\widehat{g^{\nu}_{k_2}}(t,\eta)+\widehat{g^{\mu}_{k_1}}(t,\xi-\eta)\widehat{\Gamma g^{\nu}_{k_2}}(t,\eta)\big)\big) \tag{6.36}$$

$$+(\Gamma_\xi+\Gamma_\eta+d_\Gamma)\tilde{q}_{\mu,\nu}(\xi-\eta,\eta)\partial_t\big(\overline{\widehat{\Gamma^1\Gamma^2 g_k}(t,\xi)}\widehat{g^{\mu}_{k_1}}(t,\xi-\eta)\widehat{g^{\nu}_{k_2}}(t,\eta)\big)\big]d\eta d\xi dt.$$

For $\widetilde{\Gamma}^{1,1}_{k,k_1,k_2}$, we do integration by parts in "η" once. As a result, from the $L^2 - L^\infty$ type bilinear estimate (2.5) in Lemma 2.2, the following estimate holds for some absolute constants C ,

$$|\widetilde{\Gamma}^{1,1}_{k,k_1,k_2}| \le \sup_{t\in[2^{m-1},2^m]} C2^{k_{1,+}}\|\Gamma^1\Gamma^2 g_k\|_{L^2}$$

$$\times\Big[2^{-k+k_1}\big(\sum_{i=0,1,2}2^{ik_1}\|\nabla^i_\xi\widehat{g}_{k_1}(t,\xi)\|_{L^2}+2^{ik_1}\|\nabla^i_\xi\widehat{g}_{k_2}(t,\xi)\|_{L^2}\big)$$

$$\times\big(\sum_{i=1,2}\|e^{-it\Lambda}g_{k_i}(t)\|_{L^\infty}\big)+\sum_{j_1\ge j_2}2^{-k+3k_1+j_1}\|e^{-it\Lambda}\mathscr{F}^{-1}[\nabla_\xi\widehat{g}_{k_2,j_2}]\|_{L^\infty}\|g_{k_1,j_1}\|_{L^2}$$

$$+\sum_{j_2\ge j_1}2^{-k+3k_1+j_2}\|e^{-it\Lambda}\mathscr{F}^{-1}[\nabla_\xi\widehat{g}_{k_1,j_1}]\|_{L^\infty}\|g_{k_2,j_2}\|_{L^2}\Big]$$

$$\le C2^{-m-k-k_1+2\tilde{\delta}m+\delta m}\epsilon_0^2 \le C2^{9\tilde{\delta}m/5}\epsilon_0^2.$$

Now, we proceed to estimate $\widetilde{\Gamma}^{1,2}_{k,k_1,k_2}$ in (6.36). Since "∂_t" can hit every input inside $\widetilde{\Gamma}^{1,2}_{k,k_1,k_2}$, which creates many terms. We put terms that have similar structures together and split $\widetilde{\Gamma}^{1,2}_{k,k_2,k_2}$ into five parts as follows,

$$\widetilde{\Gamma}^{1,2}_{k,k_1,k_2} = \sum_{i=1,2,3,4,5}\widehat{\Gamma}^{i}_{k,k_1,k_2}, \tag{6.37}$$

$$\widehat{\Gamma}^{1}_{k,k_1,k_2} = -\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{it\Phi^{\mu,\nu}(\xi,\eta)}t\tilde{c}(\xi-\eta)\big[\tilde{q}_{\mu,\nu}(\xi-\eta,\eta)$$

$$\times\big(\widehat{\Gamma g^{\mu}_{k_1}}(t,\xi-\eta)\widehat{g^{\nu}_{k_2}}(t,\eta)+\widehat{g^{\mu}_{k_1}}(t,\xi-\eta)\widehat{\Gamma g^{\nu}_{k_2}}(t,\eta)\big)$$

$$+(\Gamma_\xi+\Gamma_\eta+d_\Gamma)\tilde{q}_{\mu,\nu}(\xi-\eta,\eta)\widehat{g^{\mu}_{k_1}}(t,\xi-\eta)\widehat{g^{\nu}_{k_2}}(t,\eta)\big]$$

$$\times \overline{\big(\partial_t \widehat{\Gamma^1\Gamma^2 g_k}(t,\xi) - \sum_{\nu'\in\{+,-\}}\sum_{(k_1',k_2')\in\chi_k^2} \widetilde{B}^{+,\nu'}_{k,k_1',k_2'}(t,\xi)\big)} d\eta d\xi dt, \tag{6.38}$$

where $\widetilde{B}^{+,\nu'}_{k,k_1',k_2'}(t,\xi)$ is defined in (7.8),

$$\widehat{\Gamma}^2_{k,k_1,k_2} = \sum_{\nu'\in\{+,-\}}\sum_{(k_1',k_2')\in\chi_k^2} -\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \overline{e^{it\Phi^{+,\nu'}(\xi,\kappa)}\widehat{\Gamma^1\Gamma^2 g_{k_1'}}(t,\xi-\kappa)\widehat{g^{\nu'}_{k_2'}}(t,\kappa)\tilde{q}_{+,\nu'}(\xi-\kappa,\kappa)}$$

$$\times t e^{it\Phi^{\mu,\nu}(\xi,\eta)}\tilde{c}(\xi-\eta)\big[\tilde{q}_{\mu,\nu}(\xi-\eta,\eta)\big(\widehat{\Gamma g^{\mu}_{k_1}}(t,\xi-\eta)\widehat{g^{\nu}_{k_2}}(t,\eta) + \widehat{g^{\mu}_{k_1}}(t,\xi-\eta)\widehat{\Gamma g^{\nu}_{k_2}}(t,\eta)\big)$$

$$+ (\Gamma_\xi + \Gamma_\eta + d_\Gamma)\tilde{q}_{\mu,\nu}(\xi-\eta,\eta)\widehat{g^{\mu}_{k_1}}(t,\xi-\eta)\widehat{g^{\nu}_{k_2}}(t,\eta)\big] d\kappa d\eta d\xi dt, \tag{6.39}$$

$$\widehat{\Gamma}^3_{k,k_1,k_2} = -\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \overline{\widehat{\Gamma^1\Gamma^2 g_k}(t,\xi)} e^{it\Phi^{\mu,\nu}(\xi,\eta)} t\tilde{c}(\xi-\eta)\big[\tilde{q}_{\mu,\nu}(\xi-\eta,\eta)\big(\widehat{\Gamma g^{\mu}_{k_1}}(t,\xi-\eta)\partial_t\widehat{g^{\nu}_{k_2}}(t,\eta)$$

$$+\partial_t\widehat{g^{\mu}_{k_1}}(t,\xi-\eta)\widehat{\Gamma g^{\nu}_{k_2}}(t,\eta) + \widehat{\Gamma\Lambda_{\geq 3}[\partial_t g^{\mu}_{k_1}]}(t,\xi-\eta)\widehat{g^{\nu}_{k_2}}(t,\eta) + \widehat{g^{\mu}_{k_1}}(t,\xi-\eta)\widehat{\Gamma\Lambda_{\geq 3}[\partial_t g^{\nu}_{k_2}]}(t,\eta)\big)$$

$$+ (\Gamma_\xi + \Gamma_\eta + d_\Gamma)\tilde{q}_{\mu,\nu}(\xi-\eta,\eta)\partial_t\big(\widehat{g^{\mu}_{k_1}}(t,\xi-\eta)\widehat{g^{\nu}_{k_2}}(t,\eta)\big)\big] d\eta d\xi dt, \tag{6.40}$$

$$\widehat{\Gamma}^i_{k,k_1,k_2} = \sum_{k_1',k_2'\in\mathbb{Z}} \Gamma^{k_1',k_2';i-3}_{k,k_1,k_2}, \quad \Gamma^{k_1',k_2';i-4}_{k,k_1,k_2} := \sum_{j_1'\geq -k_{1,-}',\, j_2'\geq -k_{2,-}'} \Gamma^{k_1',j_1',k_2',j_2';i-4}_{k,k_1,k_2}, \quad i\in\{4,5\}, \tag{6.41}$$

$$\Gamma^{k_1',j_1',k_2',j_2';1}_{k,k_1,k_2} := \sum_{\tau,\iota\in\{+,-\}} -\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \overline{\widehat{\Gamma^1\Gamma^2 g_k}(t,\xi)} e^{it\Phi^{\mu,\nu}(\xi,\eta)} t\tilde{c}(\xi-\eta)\tilde{q}_{\mu,\nu}(\xi-\eta,\eta)$$

$$\times\big(P_\mu[e^{it\Phi^{\tau,\iota}(\xi-\eta,\sigma)}\tilde{q}_{\tau,\iota}(\xi-\eta-\sigma,\sigma)\widehat{g^{\iota}_{k_2',j_2'}}(t,\sigma)\Gamma_{\xi-\eta}\widehat{g^{\tau}_{k_1',j_1'}}(t,\xi-\eta-\sigma)]\widehat{g^{\nu}_{k_2}}(t,\eta)$$

$$+ \widehat{g^{\mu}_{k_1}}(t,\xi-\eta)P_\nu[e^{it\Phi^{\tau,\iota}(\eta,\sigma)}\tilde{q}_{\tau,\iota}(\eta-\sigma,\sigma)\Gamma_\eta\widehat{g^{\tau}_{k_1',j_1'}}(t,\eta-\sigma)\widehat{g^{\iota}_{k_2',j_2'}}(t,\sigma)]\big) d\sigma d\eta d\xi dt, \tag{6.42}$$

$$\Gamma^{k_1',j_1',k_2',j_2';2}_{k,k_1,k_2} = \sum_{\tau,\iota\in\{+,-\}} -\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \overline{\widehat{\Gamma^1\Gamma^2 g_k}(t,\xi)} e^{it\Phi^{\mu,\nu}(\xi,\eta)} it^2\tilde{c}(\xi-\eta)\tilde{q}_{\mu,\nu}(\xi-\eta,\eta)$$

$$\times\big(P_\mu[e^{it\Phi^{\tau,\iota}(\xi-\eta,\sigma)}\Gamma_{\xi-\eta}\Phi^{\tau,\iota}(\xi-\eta,\sigma)\tilde{q}_{\tau,\iota}(\xi-\eta-\sigma,\sigma)\widehat{g^{\iota}_{k_2',j_2'}}(t,\sigma)\widehat{g^{\tau}_{k_1',j_1'}}(t,\xi-\eta-\sigma)]\widehat{g^{\nu}_{k_2}}(t,\eta)$$

$$+\widehat{g^{\mu}_{k_1}}(t,\xi-\eta)P_\nu[e^{it\Phi^{\tau,\iota}(\eta,\sigma)}\Gamma_\eta\Phi^{\tau,\iota}(\eta,\sigma)\tilde{q}_{\tau,\iota}(\eta-\sigma,\sigma)\widehat{g^{\tau}_{k_1',j_1'}}(t,\eta-\sigma)\widehat{g^{\iota}_{k_2',j_2'}}(t,\sigma)]\big) d\sigma d\eta d\xi dt. \tag{6.43}$$

Recall (6.38). For $\widehat{\Gamma}^1_{k,k_2,k_2}$, we do integration by parts in "η" once. From (7.7) in Lemma 7.2 and the $L^2 - L^\infty$ type bilinear estimate (2.5) in Lemma 2.2, the following estimate holds for some constant C,

$$|\widehat{\Gamma}^1_{k,k_1,k_2}| \leq \sup_{t\in[2^{m-1},2^m]} C 2^{-k+5k_{1,+}+k_1}\big(2^{\tilde{\delta}m+\delta m} + 2^{3\tilde{\delta}m+k}\big)$$
$$\times\big[\big(\sum_{i=0,1,2} 2^{ik_1}\|\nabla_\xi^i\widehat{g_{k_1}}(t,\xi)\|_{L^2}\big)\|e^{-it\Lambda}g_{k_2}(t)\|_{L^\infty}$$
$$+\big(\sum_{i=0,1,2} 2^{ik_1}\|\nabla_\xi^i\widehat{g_{k_2}}(t,\xi)\|_{L^2}\big)\|e^{-it\Lambda}g_{k_1}\|_{L^\infty}$$

$$+ \sum_{j_1\geq j_2} 2^{-m+2j_2+j_1+k_1+k_2} \|\varphi_{j_1}^{k_1}(x) g_{k_1}(t)\|_{L^2} \|\varphi_{j_2}^{k_2}(x) g_{k_2}(t)\|_{L^2}$$
$$+ \sum_{j_2\geq j_1} 2^{-m+2j_1+j_2+k_1+k_2} \|\varphi_{j_1}^{k_1}(x) g_{k_1}(t)\|_{L^2} \|\varphi_{j_2}^{k_2}(x) g_{k_2}(t)\|_{L^2}\big]$$
$$\leq C 2^{-m+2\tilde{\delta}m+\delta m-k-k_1}\epsilon_0^2 + 2^{-m+4\beta m}\epsilon_0^2 \leq C 2^{9\tilde{\delta}m/5}\epsilon_0^2.$$

Recall (6.39). For $\widehat{\Gamma}^2_{k,k_2,k_2}$, we do integration by parts in "η" once. Recall that $|k_1'-k| \leq 10$. The loss of 2^{-k} from integration by parts in "η" is compensated by the smallness of $2^{2k_1'}$ from the symbol $\tilde{q}_{+,\nu'}(\xi-\kappa,\kappa)$. As a result, from the $L^2 - L^\infty$ type bilinear estimate (2.5) in Lemma 2.2, the following estimate holds for some absolute constant C,

$$|\widehat{\Gamma}^2_{k,k_1,k_2}| \leq \sup_{t\in[2^{m-1},2^m]} \sum_{k_2'\leq k-10} C 2^{m+k+k_1} \|\Gamma^1\Gamma^2 g_{k_1'}\|_{L^2} \|e^{-it\Lambda} g_{k_2'}(t)\|_{L^\infty}$$
$$\times \Big(\sum_{i=0,1,2} 2^{ik_1}\|\nabla_\xi^i \widehat{g}_{k_1}(t,\xi)\|_{L^2} + 2^{ik_1}\|\nabla_\xi^i \widehat{g}_{k_2}(t,\xi)\|_{L^2}\Big)$$
$$\times \Big(\sum_{i=1,2} 2^{k_1}\|e^{-it\Lambda}\mathcal{F}^{-1}[\nabla_\xi \widehat{g}_{k_i}]\|_{L^\infty} + \|e^{-it\Lambda} g_{k_i}\|_{L^\infty}\Big)$$
$$\leq C 2^{-m/2+\beta m}\epsilon_0^2.$$

Now, we proceed to estimate $\widehat{\Gamma}^3_{k,k_2,k_2}$. Recall (6.40). From estimate (7.1) in Lemma 7.1, estimate (5.18) in Proposition 5.4, (7.13) in Lemma 7.4, and the $L^2 - L^\infty$ type bilinear estimate (2.5) in Lemma 2.2, the following estimate holds for some absolute constant C,

$$|\widehat{\Gamma}^3_{k,k_1,k_2}| \leq \sup_{t\in[2^{m-1},2^m]} \sum_{l=1,2} C 2^{2m+(2-\alpha)k_1}$$
$$\Big[\Big(\|\partial_t \widehat{g}_{k_l}(t,\xi) - \sum_{\mu,\nu\in\{+,-\}} \sum_{(k_1',k_2')\in\chi^1_{k_l}} B^{\mu,\nu}_{k_l,k_1',k_2'}(t,\xi)\|_{L^2} \|e^{-it\Lambda}\Gamma g_{k_{3-l}}(t)\|_{L^\infty}$$
$$+ \|\Gamma g_{k_{3-l}}(t)\|_{L^2} \sum_{(k_1',k_2')\in\chi^1_{k_l}} \|e^{-it\Lambda}\mathcal{F}^{-1}[B^{\mu,\nu}_{k_l,k_1',k_2'}(t,\xi)]\|_{L^\infty}$$
$$+ \|\Lambda_{\geq 3}[\partial_t g_{k_l}]\|_{Z_1} \|e^{-it\Lambda} g_{k_{3-l}}(t)\|_{L^\infty}\Big] \|\Gamma^1\Gamma^2 g_k(t)\|_{L^2} \leq C 2^{3\tilde{\delta}m/2}\epsilon_0^2.$$

Lastly, we estimate $\widehat{\Gamma}^4_{k,k_1,k_2}$ and $\widehat{\Gamma}^5_{k,k_1,k_2}$. Recall (6.41). Based on the size of difference between k_1' and k_2' and the size of $k_{1,-}' + k_2$, we split into three cases as follows,

If $|k_1' - k_2'| \leq 10$. – For this case, we know that $\nabla_\sigma \Phi^{\tau,\iota}(\cdot,\cdot)$ is bounded from blow by $2^{k_{1,-}-k_{1,+}'}$. Hence, to take advantage of this fact, we do integration by parts in "σ" once for $\Gamma^{k_1',k_2';1}_{k,k_1,k_2}$ and do integration by parts in "σ" twice for $\Gamma^{k_1',k_2';2}_{k,k_1,k_2}$. As a result, from the $L^2 - L^\infty$ type bilinear estimate (2.5) in Lemma 2.2, the following estimate holds for some absolute constant C,

$$\sum_{|k_1'-k_2'|\leq 10, |k_1-k_2|\leq 10} \sum_{i=1,2} |\Gamma^{k_1',k_2';i}_{k,k_1,k_2}| \leq \sup_{t\in[2^{m-1},2^m]} \sum_{|k_1'-k_2'|\leq 10, |k_1-k_2|\leq 10} C 2^{m+k_1+k_1'+2k_{1,+}'}$$
$$\times \Big(\sum_{i=0,1,2} 2^{ik_1'}\|\nabla_\xi^i \widehat{g_{k_1'}}(t,\xi)\|_{L^2} + 2^{ik_1'}\|\nabla_\xi^i \widehat{g_{k_2'}}(t,\xi)\|_{L^2}\Big)$$

$$\times \Big(\sum_{i=1,2} 2^{k_1} \| e^{-it\Lambda} \mathscr{F}^{-1}[\nabla_\xi \widehat{g}_{k'_i}] \|_{L^\infty} + \| e^{-it\Lambda} g_{k'_i} \|_{L^\infty} \Big)$$
$$\times \Big(\sum_{i=1,2} \| e^{-it\Lambda} g_{k_i} \|_{L^\infty} \Big) \| \Gamma^1 \Gamma^2 g_k(t) \|_{L^2} \leq C 2^{-\beta m} \epsilon_0^2.$$

If $k'_2 \leq k'_1 - 10$ *and* $k'_{1,-} + k'_2 \leq -19m/20$. – Note that $|k'_1 - k_1| \leq 10$. For this case, we use the same strategy that we used in the estimates (5.12) and (5.13). From the estimate (5.15) in Lemma 5.3, we know that the following estimate holds for some absolute constant C,

$$\sum_{k'_2 \leq k_1 - 10, k'_2 + k_{1,-} \leq -9m/10} \sum_{i=1,2} |\Gamma^{k'_1, k'_2; i}_{k, k_1, k_2}|$$
$$\leq \sup_{t \in [2^{m-1}, 2^m]} \sum_{k'_2 \leq k_1 - 10, k'_2 + k_{1,-} \leq -9m/10} C \big(2^{3k'_2} \| \widehat{g}_{k'_2}(t,\xi) \|_{L^\infty_\xi}$$
$$+ 2^{k'_1 + 2k'_2} \| \widehat{\mathrm{Re}[v]}(t,\xi) \psi_{k'_2}(\xi) \|_{L^\infty_\xi} \big) \Big(\sum_{i=1,2} \| e^{-it\Lambda} g_{k_i} \|_{L^\infty} \Big)$$
$$\times \big(2^{2m + 2k_1 + k'_1} \| \Gamma g_{k'_1} \|_{L^2} + 2^{3m + 2k_1 + 2k'_1 + k'_2} \| g_{k'_1}(t) \|_{L^2} \big) \| \Gamma^1 \Gamma^2 g_k(t) \|_{L^2}$$
$$\leq \sum_{k'_2 \leq k_1 - 10, k'_2 + k_{1,-} \leq -9m/10} C 2^{3\tilde{\delta} m + 2m + 2k_1 + 3k'_2} (1 + 2^{2m + 2k_{1,-} + 2k'_2}) \epsilon_0^2$$
$$\leq C 2^{-\beta m} \epsilon_0^2.$$

If $k'_2 \leq k'_1 - 10$ *and* $k_{1,-} + k'_2 \geq -19m/20$. – We first do integration by parts in "σ" many times to rule out the case when $\max\{j'_1, j'_2\} \leq m + k_{1,-} - \beta m$. If $\max\{j'_1, j'_2\} \geq m + k_{1,-} - \beta m$, from the $L^2 - L^\infty - L^\infty$ type trilinear estimate (2.6) in Lemma 2.2, the following estimate holds for some absolute constant C,

$$\sum_{i=1,2} \sum_{\max\{j'_1, j'_2\} \geq m + k_{1,-} - \beta m} |\Gamma^{k'_1, j'_1, k'_2, j'_2; i}_{k, k_1, k_2}|$$
$$\leq \sup_{t \in [2^{m-1}, 2^m]} C \Big[\sum_{j'_1 \geq \max\{j'_2, m + k_{1,-} - \beta m\}} \big(2^{m + j'_1 + 5k_1} + 2^{2m + 5k_1 + k'_2} \big)$$
$$\times \| g_{k'_1, j'_1}(t) \|_{L^2} \| g_{k'_2, j'_2}(t) \|_{L^1} \Big(\sum_{i=1,2} \| e^{-it\Lambda} g_{k_i} \|_{L^\infty} \Big)$$
$$+ \sum_{j'_2 \geq \max\{j'_1, m + k_{1,-} - \beta m\}} \times \big(2^{m + j'_2 + 5k_1} + 2^{2m + 5k_1 + k'_2} \big) \| g_{k'_2, j'_2}(t) \|_{L^2} \| g_{k'_1, j'_1}(t) \|_{L^1}$$
$$\times \Big(\sum_{i=1,2} \| e^{-it\Lambda} g_{k_i} \|_{L^\infty} \Big) \Big] \| \Gamma^1 \Gamma^2 g_k(t) \|_{L^2} \leq C 2^{-m - k'_2 + 10\beta m} \epsilon_0^2$$
$$\leq C 2^{-\beta m} \epsilon_0^2. \tag{6.44}$$

Hence finishing the proof. □

LEMMA 6.4. – *Under the bootstrap assumption* (4.49) *and the assumption that* $k + k_{1,-} \geq -m + \tilde{\delta} m/3$, $k \leq k_1 - 10$, *and* $k_1 \leq m/5$, *the following estimate holds for some absolute constant* C,

$$|\widetilde{P}^1_{k, k_1, k_2}| \leq C 2^{9/5 \tilde{\delta} m} \epsilon_0^2. \tag{6.45}$$

Proof. – Recall (6.27) and its associated symbol in (6.29). To take the advantage of the small symbol near the time resonance set, for $\widetilde{P}^1_{k,k_1,k_2}$, we do integration by parts in time once. As a result, we have

$$\widetilde{P}^1_{k,k_1,k_2} = \sum_{i=1,2,3,4,5} \widehat{P}^i_{k,k_1,k_2},$$

$$\widehat{P}^1_{k,k_1,k_2} = \sum_{i=1,2}(-1)^i \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \overline{\widehat{LLg_k}(t_i,\xi)} e^{it_i\Phi^{\mu,\nu}(\xi,\eta)} i t_i^2 \widehat{p}^1_{\mu,\nu}(\xi,\eta) \times \widehat{g^\nu_{k_2}}(t_i,\eta)\widehat{g^\mu_{k_1}}(t_i,\xi-\eta)\,d\eta\, d\xi$$

$$-\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \overline{\widehat{LLg_k}(t,\xi)} e^{it\Phi^{\mu,\nu}(\xi,\eta)} i2t \widehat{p}^1_{\mu,\nu}(\xi,\eta)\widehat{g^\mu_{k_1}}(t,\xi-\eta)\widehat{g^\nu_{k_2}}(t,\eta)\,d\eta\, d\xi\, dt. \tag{6.46}$$

$$\widehat{P}^2_{k,k_1,k_2} = -\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \overline{\big(\partial_t \widehat{LLg_k}(t,\xi) - \sum_{\nu\in\{+,-\}}\sum_{(k_1',k_2')\in\chi_k^2} \widehat{B}^{+,\nu}_{k,k_1',k_2'}(t,\xi)\big)}$$

$$\times e^{it\Phi^{\mu,\nu}(\xi,\eta)} i t^2 \widehat{p}^1_{\mu,\nu}(\xi,\eta)\widehat{g^\mu_{k_1}}(t,\xi-\eta)\widehat{g^\nu_{k_2}}(t,\eta)\,d\eta\, d\xi\, dt, \tag{6.47}$$

$$\widehat{P}^3_{k,k_1,k_2} := \sum_{j_1\geq -k_{1,-}, j_2\geq -k_{2,-}} \widehat{P}^{3,j_1,j_2}_{k,k_1,k_2}, \tag{6.48}$$

$$\widehat{P}^{3,j_1,j_2}_{k,k_1,k_2} = \sum_{\nu'\in\{+,-\}}\sum_{(k_1',k_2')\in\chi_k^2} -\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{it\Phi^{\mu,\nu}(\xi,\eta)} i t^2 e^{-it\Phi^{+,\nu'}(\xi,\kappa)}$$
$$\times \overline{\widehat{LLg_{k_1'}}(t,\xi-\kappa)\widehat{g^{\nu'}_{k_2'}}(t,\kappa)\tilde{q}_{+,\nu'}(\xi-\kappa,\kappa)}\, \widehat{p}^1_{\mu,\nu}(\xi,\eta)$$
$$\times \widehat{g^\mu_{k_1,j_1}}(t,\xi-\eta)\widehat{g^\nu_{k_2,j_2}}(t,\eta)\,d\kappa\, d\eta\, d\xi\, dt, \tag{6.49}$$

$$\widehat{P}^4_{k,k_1,k_2} := -\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{it\Phi^{\mu,\nu}(\xi,\eta)} i t^2 \widehat{p}^1_{\mu,\nu}(\xi,\eta)\overline{\widehat{\Gamma^1\Gamma^2 g_k}(t,\xi)}$$

$$\times \big(\widehat{\Lambda_{\geq 3}[\partial_t g^\mu_{k_1}]}(t,\xi-\eta)\widehat{g^\nu_{k_2}}(t,\eta) + \widehat{g^\mu_{k_1}}(t,\xi-\eta)\widehat{\Lambda_{\geq 3}[\partial_t g^\nu_{k_2}]}(t,\eta)\big)\,d\eta\, d\xi\, dt, \tag{6.50}$$

$$\widehat{P}^5_{k,k_1,k_2} = \sum_{k_1',k_2'\in\mathbb{Z}} \widehat{P'}^{k_1',k_2'}_{k,k_1,k_2}, \tag{6.51}$$

$$\widehat{P'}^{k_1',k_2'}_{k,k_1,k_2} = \sum_{j_1'\geq -k_{1,-}', j_2'\geq -k_{2,-}'} \widehat{P'}^{k_1',j_1',k_2',j_2'}_{k,k_1,k_2},$$

$$\widehat{P'}^{k_1',j_1',k_2',j_2'}_{k,k_1,k_2} := \sum_{\mu',\nu'\in\{+,-\}} -\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \overline{\widehat{\Gamma^1\Gamma^2 g_k}(t,\xi)} e^{it\Phi^{\mu,\nu}(\xi,\eta)} i t^2 \widehat{p}^1_{\mu,\nu}(\xi,\eta) \tag{6.52}$$
$$\times \big[P_\mu[e^{it\Phi^{\mu',\nu'}(\xi-\eta,\sigma)}\tilde{q}_{\mu',\nu'}(\xi-\eta-\sigma,\sigma)\widehat{g^{\mu'}_{k_1',j_1'}}(t,\xi-\eta-\sigma)\widehat{g^{\nu'}_{k_2',j_2'}}(t,\sigma)]$$
$$\times \psi_{k_1}(\xi-\eta)\widehat{g^\nu_{k_2}}(t,\eta) + \widehat{g^\mu_{k_1}}(t,\xi-\eta)P_\nu \tag{6.53}$$

$$\times [e^{it\Phi^{\mu',\nu'}(\eta,\sigma)}\tilde{q}_{\mu',\nu'}(\eta-\sigma,\sigma)\widehat{g^{\mu'}_{k'_1,j'_1}}(t,\eta-\sigma)\widehat{g^{\nu'}_{k'_2,j'_2}}(t,\sigma)]\psi_{k_2}(\eta)\big]d\sigma d\eta d\xi dt.$$

Recall (6.46) and (6.47). For $\widehat{P}^1_{k,k_1,k_2}$ and $\widehat{P}^2_{k,k_1,k_2}$, we do integration by parts in "η" twice. As a result, from the L^2-L^∞ type bilinear estimate (2.5) in Lemma 2.2 and estimate (7.7) in Lemma 7.2, the following estimate holds for some absolute constant C,

$$\begin{aligned}
&\sum_{i=1,2}|\widehat{P}^i_{k,k_1,k_2}|\\
&\quad\le \sup_{t\in[2^{m-1},2^m]} C2^{k_1+6k_{1,+}+\delta m}\big(2^{\tilde{\delta}m-k}+2^{3\tilde{\delta}m}\big)\\
&\qquad\times\Big[\Big(\sum_{i=0,1,2,j=1,2}2^{ik_1}\|\nabla^i_\xi\widehat{g}_{k_j}(t,\xi)\|_{L^2}\Big)\Big(\sum_{i=1,2}\|e^{-it\Lambda}g_{k_i}(t)\|_{L^\infty}\Big)\\
&\qquad+\sum_{j_1\ge j_2}2^{2k_1+j_1}\|e^{-it\Lambda}\mathscr{F}^{-1}\big[\nabla_\xi\widehat{g}_{k_2,j_2}\big]\|_{L^\infty}\|g_{k_1,j_1}(t)\|_{L^2}\\
&\qquad+\sum_{j_2\ge j_1}2^{2k_1+j_2}\|e^{-it\Lambda}\mathscr{F}^{-1}\big[\nabla_\xi\widehat{g}_{k_1,j_1}\big]\|_{L^\infty}\|g_{k_2,j_2}(t)\|_{L^2}\Big]\\
&\quad\le C2^{-m-k-k_1+2\tilde{\delta}m+\delta m}\epsilon_0^2+C2^{-m+4\beta m}\epsilon_0^2\le C2^{9\tilde{\delta}m/5}\epsilon_0^2.
\end{aligned}$$

Now, we proceed to estimate $\widehat{P}^3_{k,k_1,k_2}$. Recall (6.49). Note that $(k'_1,k'_2)\in\chi^2_k$, i.e., $|k'_1-k|\le 10$. Hence the symbol $\tilde{q}_{+,\nu'}(\xi-\kappa,\kappa)$ contributes the smallness of "2^{2k}". By doing integration by parts in "η" many times, we can rule out the case when $\max\{j_1,j_2\}\le m+k_--k_{1,+}-\beta m$. If $\max\{j_1,j_2\}\ge m+k_--k_{1,+}-\beta m$, from the L^2-L^∞ type bilinear estimate (2.5) in Lemma 2.2, the following estimate holds for some absolute constant C,

$$\begin{aligned}
&\sum_{\substack{k'_2\le k'_1-10\\ \max\{j_1,j_2\}\ge m+k_--k_{1,+}-\beta m}}|\widehat{P'}^3_{k,k_1,j_1,k_2,j_2}|\\
&\quad\le \sup_{t\in[2^{m-1},2^m]}\sum_{\substack{k'_2\le k'_1-10\\ \max\{j_1,j_2\}\ge m+k_--k_{1,+}-\beta m}}C2^{3m+3k+3k_1}\\
&\qquad\times\Big(\sum_{j_1\ge\max\{j_2,m+k_--k_{1,+}-\beta m\}}\|e^{-it\Lambda}g_{k_2,j_2}(t)\|_{L^\infty}\|g_{k_1,j_1}(t)\|_{L^2}\\
&\qquad+\sum_{j_2\ge\max\{j_1,m+k_--k_{1,+}-\beta m\}}\|g_{k_2,j_2}(t)\|_{L^2}\\
&\qquad\times\|e^{-it\Lambda}g_{k_1,j_1}(t)\|_{L^\infty}\Big)\|LLg_{k'_1}(t)\|_{L^2}\|e^{-it\Lambda}g_{k'_2}(t)\|_{L^\infty}\\
&\quad\le C2^{-m/2+10\beta m}\epsilon_0^2.
\end{aligned}$$

Now, we proceed to estimate $\widehat{P}^4_{k,k_1,k_2}$. Recall (6.50) and the estimate of symbol "$\widehat{p}^1_{\mu,\nu}(\xi,\eta)$" in (6.31). For this case, we do integration by parts in "η" once. As a result, from estimate (5.18) in Proposition (5.4), estimate (7.13) in Lemma (7.4), and L^2-L^∞ type bilinear estimate (2.5) in Lemma 2.2, the following estimate holds for some absolute constant C,

$$|\widehat{P}^4_{k,k_1,k_2}|\le \sup_{t\in[2^{m-1},2^m]}C2^{2m+(2-\alpha)k_1+k_{1,+}}\|\Gamma^1\Gamma^2 g_k(t)\|_{L^2}$$

$$\times \big(\|\Lambda_{\geq 3}[\partial_t g_{k_1}]\|_{Z_1}\|e^{-it\Lambda}g_{k_2}(t)\|_{L^\infty} + \|\Lambda_{\geq 3}[\partial_t g_{k_2}]\|_{Z_1}\|e^{-it\Lambda}g_{k_1}(t)\|_{L^\infty}\big)$$
$$\leq C2^{-\beta m+2\tilde{\delta}m}\epsilon_0^2 \leq C2^{-\delta m}\epsilon_0^2.$$

Lastly, we proceed to estimate $\widehat{P}^5_{k,k_1,k_2}$. Recall (6.51) and (6.53). We first consider the case when $|k_1'-k_2'|\leq 10$. By doing integration by parts in "σ" many times, we can rule out the case when $\max\{j_1',j_2'\}\leq m+k_{1,-}-k'_{1,+}-\beta m$. If $\max\{j_1',j_2'\}\geq m+k_{1,-}-k'_{1,+}-\beta m$, after using the $L^2-L^\infty-L^\infty$ type trilinear estimate (2.6) in Lemma 2.2, the following estimate holds for some absolute constant C,

$$\begin{aligned}\sum_{\substack{|k_1'-k_2'|\leq 10, k_1\leq k_1'+10\\ \max\{j_1',j_2'\}\geq m+k_{1,-}-k'_{1,+}-\beta m}} &|\widehat{P'}^{k_1',j_1',k_2',j_2'}_{k,k_1,k_2}|\\
&\leq \sup_{t\in[2^{m-1},2^m]}\sum_{\substack{i=1,2\\ |k_1'-k_2'|\leq 10,k_1\leq k_1'+10}} C2^{3m+k+3k_1+2k_1'}\\
&\quad\times\Big(\sum_{j_1'\geq \max\{j_2',m+k_{1,-}-k'_{1,+}-\beta m\}}\|g_{k_1',j_1'}(t)\|_{L^2}\|e^{-it\Lambda}g_{k_2',j_2'}\|_{L^\infty}\\
&\qquad+\sum_{j_2'\geq \max\{j_1',m+k_{1,-}-k'_{1,+}-\beta m\}}\|g_{k_2',j_2'}(t)\|_{L^2}\|e^{-it\Lambda}g_{k_1',j_1'}\|_{L^\infty}\Big)\\
&\quad\times\|e^{-it\Lambda}g_{k_i}(t)\|_{L^\infty}\|LLg_k(t)\|_{L^2}\\
&\leq C2^{-m+10\beta m}\epsilon_0^2.\end{aligned}$$

It remains to consider the case when $k_2'\leq k_1'-10$. We split it into four cases based on the size of $k_1'+k_2'$ and whether k is greater than k_2' as follows.

If $k'_{1,-}+k_2'\leq -19m/20$ and $k\leq k_2'+20$. – By using the same strategy that we used in the estimates (5.12) and (5.13), from estimate (5.15) in Lemma 5.3, the following estimate holds for some absolute constant C,

$$\begin{aligned}\sum_{k_2'\leq k_1'-10,|k_1-k_1'|\leq 20} &|\widehat{P'}^{k_1',k_2'}_{k,k_1,k_2}|\\
&\leq \sup_{t\in[2^{m-1},2^m]}\sum_{k_2'\leq k_1'-10,|k_1-k_1'|\leq 20} C2^{3m+k+4k_1}\|LLg_k(t)\|_{L^2}\|g_{k_1'}(t)\|_{L^2}\\
&\quad\times\big(\|e^{-it\Lambda}g_{k_1}(t)\|_{L^\infty}+\|e^{-it\Lambda}g_{k_2}(t)\|_{L^\infty}\big)\\
&\quad\times\big(2^{k_1'+2k_2'}\|\widehat{\mathrm{Re}[v]}(t,\xi)\psi_{k_2'}(\xi)\|_{L^\infty_\xi}+2^{3k_2'}\|\widehat{g}_{k_2'}(t,\xi)\|_{L^\infty_\xi}\big)\\
&\leq C2^{3m+2\tilde{\delta}m+4k_2'+3k_1-15k_{1,+}}(1+2^{m+k_1+k_2'})\leq C2^{-\beta m}\epsilon_0^2.\end{aligned}$$

If $k'_{1,-}+k_2'\leq -19m/20$ and $k\geq k_2'+20$. – For the case we are considering, we have $|\sigma|\leq 2^{-5}|\xi|\leq 2^{-10}|\eta|$. Hence, the following estimate holds,

$$\begin{aligned}(6.54)\quad |\nabla_\eta\big(\Phi^{\mu,\nu}(\xi,\eta)+\nu(\Phi^{\mu',\nu'}(\eta,\sigma))\big)|+|\nabla_\eta\big(\Phi^{\mu,\nu}(\xi,\eta)+\mu(\Phi^{\mu',\nu'}(\xi-\eta,\sigma))\big)|\\
\geq 2^{-10}|\xi-\sigma|(1+|\eta|)^{-1/2}\geq 2^{k-k_{1,+}/2-20}.\end{aligned}$$

To take advantage of this fact, we do integration by parts in "η" once. As a result, from estimate (5.15) in Lemma 5.3, the following estimate holds for some absolute constant C,

$$\begin{aligned}|\widehat{P'}^{k_1',k_2'}_{k,k_1,k_2}| \leq & \sup_{t\in[2^{m-1},2^m]} C2^{2m+3k_1+k_{1,+}} \\ & \times \Big(\sum_{i=1,2} \|e^{-it\Lambda} g_{k_i}(t)\|_{L^\infty} + 2^{k_1}\|e^{-it\Lambda}\mathscr{F}^{-1}[\nabla_\xi \widehat{g_{k_i}}(t,\xi)]\|_{L^\infty}\Big) \\ & \times \Big(\sum_{i=1,2} \|g_{k_1'}(t)\|_{L^2} + 2^{k_1}\|\nabla_\xi \widehat{g}_{k_i}(t,\xi)\|_{L^2} + 2^{k_1}\|\nabla_\xi \widehat{g}_{k_1'}(t,\xi)\|_{L^2}\Big) \\ & \times \Big(2^{k_1'+2k_2'}\|\widehat{\mathrm{Re}[v]}(t,\xi)\psi_{k_2'}(\xi)\|_{L^\infty_\xi} + 2^{3k_2'}\|\widehat{g}_{k_2'}(t,\xi)\|_{L^\infty_\xi}\Big)\|LLg_k(t)\|_{L^2} \\ \leq & C2^{-\beta m}\epsilon_0^2.\end{aligned}$$

If $k_{1,-}'+k_2' \geq -19m/20$ *and* $k \leq k_2'+20$. – By doing integration by parts in "σ" many times, we can rule out the case when $\max\{j_1', j_2'\} \leq m+k_{1,-}-\beta m$. If $\max\{j_1', j_2'\} \geq m+k_{1,-}-\beta m$, from the $L^2 - L^\infty - L^\infty$ type trilinear estimate (2.6) in Lemma 2.2, the following estimate holds for some absolute constant C,

$$\begin{aligned}\sum_{\max\{j_1',j_2'\}\geq m+k_{1,-}-\beta m} & |\widehat{P'}^{k_1',j_1',k_2',j_2'}_{k,k_1,k_2}| \\ \leq & \sup_{t\in[2^{m-1},2^m]} C2^{3m+k+3k_1+2k_1'}\Big(\sum_{i=1,2}\|e^{-it\Lambda}g_{k_i}(t)\|_{L^\infty}\Big) \\ & \times \Big(\sum_{j_1'\geq \max\{j_2',m+k_{1,-}-\beta m\}} \|g_{k_1',j_1'}(t)\|_{L^2}\|e^{-it\Lambda}g_{k_2',j_2'}(t)\|_{L^\infty} \\ & + \sum_{j_2'\geq \max\{j_1',m+k_{1,-}-\beta m\}} \|g_{k_2',j_2'}(t)\|_{L^2}\times\|e^{-it\Lambda}g_{k_1',j_1'}(t)\|_{L^\infty}\Big)\|LLg_k(t)\|_{L^2} \\ \leq & C2^{-m-k_2'+10\beta m}\epsilon_0^2 \leq C2^{-\beta m}\epsilon_0^2.\end{aligned}$$

If $k_{1,-}'+k_2' \geq -19m/20$ *and* $k \geq k_2'+20$. – By doing integration by parts in "σ" many times, we can rule out the case when $\max\{j_1', j_2'\} \leq m+k_{1,-}-\beta m$. Now, it remains to consider the case when $\max\{j_1', j_2'\} \geq m+k_{1,-}-\beta m$. As $k \geq k_2'+20$, it is easy to see that the estimate (6.54) still holds. For this case, we do integration by parts in "η" once. As a result, from the $L^2 - L^\infty - L^\infty$ type estimate, the following estimate holds for some absolute constant C,

$$\begin{aligned}\sum_{\max\{j_1',j_2'\}\geq m+k_{1,-}-\beta m} & |\widehat{P'}^{k_1',j_1',k_2',j_2'}_{k,k_1,k_2}| \\ \leq & C2^{2m+4k_1}\Big(\sum_{i=1,2} 2^{k_1}\|e^{-it\Lambda}\mathscr{F}^{-1}[\nabla_\xi\widehat{g_{k_i}}(t,\xi)]\|_{L^\infty} + \|e^{-it\Lambda}g_{k_i}(t)\|_{L^\infty}\Big) \\ & \times\Big(\sum_{j_1'\geq\max\{j_2',m+k_{1,-}-\beta m\}} 2^{j_1'}\|g_{k_1',j_1'}(t)\|_{L^2}\|e^{-it\Lambda}g_{k_2',j_2'}(t)\|_{L^\infty} \\ & + \sum_{j_2'\geq\max\{j_1',m+k_{1,-}-\beta m\}} 2^{-m+2j_1'}\|g_{k_1',j_1'}(t)\|_{L^2}\|g_{k_2',j_2'}\|_{L^2}\Big)\|LLg_k(t)\|_{L^2} \\ \leq & C2^{-m/2+10\beta m}\epsilon_0^2 + C2^{-m-k_2'+10\beta m}\epsilon_0^2 \leq C2^{-\beta m}\epsilon_0^2.\end{aligned}$$

Hence finishing the proof. □

6.3. Z_2-norm estimate of the quadratic terms: if $k_2 \leq k_1 - 10$.

Note that, for the case we are considering, we have $\mu = +$ (see (4.42)). To simplify the problem, we first rule out the very high frequency case and very low frequency case.

We first consider the case when $k_1 + k_2 \leq -19m/20$. By using the same strategy that we used in the estimates of (5.12) and (5.13), from estimate (5.15) in Lemma 5.3, the following estimate holds for some absolute constant C,

$$
\begin{aligned}
&\Big| \int_{t_1}^{t_2} \int_{\mathbb{R}^2} \overline{\Gamma_\xi^1 \Gamma_\xi^2 \widehat{g}(t,\xi)} \Gamma_\xi^1 \Gamma_\xi^2 B_{k,k_1,k_2}^{+,\nu}(t,\xi) d\xi dt \Big| \\
&\quad \leq \sup_{t \in [2^{m-1},2^m]} C \|\Gamma_1 \Gamma_2 g_k(t)\|_{L^2} \Big(\sum_{i=0,1,2} 2^{ik_1} \|\nabla_\xi^i \widehat{g}_{k_1}(t,\xi)\|_{L^2} \Big) 2^{m+k_1} \big(1 + 2^{2m+2k_2+2k_1}\big) \\
&\quad\quad \times \min\big\{ 2^{k_1+2k_2} \|\widehat{\mathrm{Re}[v]}(t,\xi)\psi_{k_2}(\xi)\|_{L_\xi^\infty} + 2^{3k_2} \|\widehat{g}_{k_2}(t,\xi)\|_{L_\xi^\infty}, 2^{k_1+k_2} \|g_{k_2}(t)\|_{L^2} \big\} \\
&\quad \leq C 2^{3\tilde{\delta} m} \min\{ 2^{m+2k_1+k_2}(1 + 2^{2m+2k_1+2k_2}), 2^{2m+k_1+3k_2}(1 + 2^{3m+3k_1+3k_2}) \} \\
&\quad \leq C 2^{-\beta m} \epsilon_0^2.
\end{aligned}
$$

Next, we consider the case when k_1 is relatively big. More precisely, we consider the case when $k_1 \geq 5\beta m$ and $k_1 + k_2 \geq -19m/20$. Recall (6.5). Note that $\Gamma_\xi \widehat{g_{k_1}}(t, \xi - \eta) = -\xi_\Gamma \cdot \nabla_\eta \widehat{g_{k_1}}(t, \xi - \eta)$. When Γ_ξ hits $\widehat{g_{k_1}}(t, \xi - \eta)$, we do integration by parts in "η "to move around the derivative ∇_η in front of $\widehat{g_{k_1}}(t, \xi - \eta)$. As a result, from the $L^2 - L^\infty$ type bilinear estimate, the following estimate holds for some absolute constant C,

$$
\begin{aligned}
&\sum_{k_1 \geq 5\beta m, k_2 \geq -m-k_1} \Big| \int_{t_1}^{t_2} \int_{\mathbb{R}^2} \overline{\Gamma_\xi^1 \Gamma_\xi^2 \widehat{g}(t,\xi)} \Gamma_\xi^1 \Gamma_\xi^2 B_{k,k_1,k_2}^{+,\nu}(t,\xi) d\xi dt \Big| \\
&\quad \leq \sum_{k_1 \geq 5\beta m, k_2 \geq -m-k_1} \sup_{t \in [2^{m-1},2^m]} C \|\Gamma_1 \Gamma_2 g_k(t)\|_{L^2} \|g_{k_1}(t)\|_{L^2} 2^{k_2} \\
&\quad\quad \times \big(2^{-2k_2} \|g_{k_2}(t)\|_{L^2} + 2^{-k_2} \|\nabla_\xi \widehat{g}_{k_2}(t)\|_{L^2} + \|\nabla_\xi^2 \widehat{g}_{k_2}(t)\|_{L^2} \big) \\
&\quad \leq \sum_{k_1 \geq 5\beta m, k_2 \geq -m-k_1} C 2^{3m+\beta m+4k_1-k_2-(N_0-30)k_{1,+}} \epsilon_0 \leq C 2^{-\beta m} \epsilon_0.
\end{aligned}
\tag{6.55}
$$

Hence, for the rest of this subsection, we restrict ourself to the case when $k_1 + k_2 \geq -19m/20$ and $k_1 \leq 5\beta m$. Recall the decomposition (6.6). We know that the desired estimate for the remaining cases follows from the estimate (6.56) in Lemma 6.5, the estimate (6.71) in Lemma 6.6, and the estimate (6.93) in Lemma 6.8. Hence finishing the Z_2-norm estimate of the quadratic terms for the High× Low type interaction.

LEMMA 6.5. – *Under the bootstrap assumption* (4.49)*, the following estimate holds for some absolute constant C,*

$$
\sum_{k_1,k_2 \in \mathbb{Z}, |k-k_1| \leq 10, k_2 \leq k_1 - 10, k_1+k_2 \geq -19m/20, k_1 \leq 5\beta m} |\mathrm{Re}[P_{k,k_1,k_2}^3]| + |P_{k,k_1,k_2}^4| \leq C 2^{2\tilde{\delta} m} \epsilon_0^2.
\tag{6.56}
$$

Proof. – We first estimate P^4_{k,k_1,k_2}. Recall (6.10). By doing integration by parts in "η" many times, we can rule out the case when $\max\{j_1,j_2\}\le m+k_{1,-}-\beta m$. If $\max\{j_1,j_2\}\ge m+k_{1,-}-\beta m$, from the L^2-L^∞ type bilinear estimate, the following estimate holds for some absolute constant C,

$$\sum_{\max\{j_1,j_2\}\ge m+k_{1,-}-\beta m}|P^{4,j_1,j_2}_{k,k_1,k_2}|$$

$$\le \sup_{t\in[2^{m-1},2^m]} C2^{m+2k_1}\Big[\sum_{j_1\ge\max\{j_2,m+k_{1,-}-\beta m\}}2^{-m+k_1+j_1+k_2+2j_2}$$
$$\times\|\varphi^{k_1}_{j_1}(x)g_{k_1}(t)\|_{L^2}\|\varphi^{k_2}_{j_2}(x)g_{k_2}(t)\|_{L^2}$$
$$+\sum_{j_2\ge\max\{j_1,m+k_{1,-}-\beta m\}}2^{-m+k_1+2j_1+k_2+j_2}\|\varphi^{k_2}_{j_2}(x)g_{k_2}(t)\|_{L^2}\Big]$$
$$\times\|\Gamma^1\Gamma^2 g_k(t)\|_{L^2}\le C2^{-m-k_2+20\beta m}\epsilon_0^2\le C2^{-\beta m}\epsilon_0^2. \tag{6.57}$$

It remains to estimate P^3_{k,k_1,k_2}, we decompose it into three parts as follows,

(6.58)
$$P^3_{k,k_1,k_2}=\sum_{i=1,2,3}Q^i_{k,k_1,k_2},$$

(6.59)
$$Q^1_{k,k_1,k_2}=\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\overline{\widehat{\Gamma^1\Gamma^2 g_k}(t,\xi)}e^{it\Phi^{+,\nu}(\xi,\eta)}\tilde q_{+,\nu}(\xi-\eta,\eta)\widehat{\Gamma^1\Gamma^2 g_{k_1}}(t,\xi-\eta)\widehat{g^\nu_{k_2}}(t,\eta)\,d\eta\, d\xi\, dt,$$

(6.60)
$$Q^2_{k,k_1,k_2}=\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\overline{\widehat{\Gamma^1\Gamma^2 g_k}(t,\xi)}e^{it\Phi^{+,\nu}(\xi,\eta)}\tilde q_{+,\nu}(\xi-\eta,\eta)\widehat{g_{k_1}}(t,\xi-\eta)\widehat{\Gamma^1\Gamma^2 g^\nu_{k_2}}(t,\eta)\,d\eta\, d\xi\, dt,$$

$$Q^3_{k,k_1,k_2}=\sum_{j_1\ge -k_{1,-},j_2\ge -k_{2,-}}Q^{j_1,j_2,3}_{k,k_1,k_2},\quad Q^{j_1,j_2,3}_{k,k_1,k_2}:=\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\overline{\widehat{\Gamma^1\Gamma^2 g_k}(t,\xi)}e^{it\Phi^{+,\nu}(\xi,\eta)}$$
$$\times\Big[\sum_{\{l,n\}=\{1,2\}}(\Gamma^l_\xi+\Gamma^l_\eta+d_{\Gamma^l})\tilde q_{+,\nu}(\xi-\eta,\eta)$$
$$\times\big(\widehat{\Gamma^n g_{k_1,j_1}}(t,\xi-\eta)\widehat{g^\nu_{k_2,j_2}}(t,\eta)+\widehat{g_{k_1,j_1}}(t,\xi-\eta)\widehat{\Gamma^n g^\nu_{k_2,j_2}}(t,\eta)\big)$$
$$+(\Gamma^1_\xi+\Gamma^1_\eta+d_{\Gamma^1})(\Gamma^2_\xi+\Gamma^2_\eta+d_{\Gamma^2})\tilde q_{+,\nu}(\xi-\eta,\eta)\widehat{g_{k_1,j_1}}(t,\xi-\eta)\widehat{g^\nu_{k_2,j_2}}(t,\eta)\Big]d\eta\, d\xi\, dt. \tag{6.61}$$

We first estimate Q^1_{k,k_1,k_2}. Note that after switching the role of ξ and $\xi-\eta$ inside Q^1_{k,k_1,k_2}, we have

$$\mathrm{Re}\big[Q^1_{k,k_1,k_2}\big]=\mathrm{Re}\big[\widetilde{Q}^1_{k,k_1,k_2}\big],$$

$$\widetilde{Q}^1_{k,k_1,k_2}:=\frac{1}{2}\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\overline{\widehat{\Gamma^1\Gamma^2 g}(t,\xi)}e^{it\Phi^{+,\nu}(\xi,\eta)}p^{+,\nu}_{k,k_1}(\xi-\eta,\eta)\widehat{\Gamma^1\Gamma^2 g}(t,\xi-\eta)\widehat{g^\nu_{k_2}}(t,\eta)\,d\eta\, d\xi\, dt,$$

where

$$p^{+,\nu}_{k,k_1}(\xi-\eta,\eta)=\tilde q_{+,\nu}(\xi-\eta,\eta)\psi_k(\xi)\psi_{k_1}(\xi-\eta)+\overline{\tilde q_{+,-\nu}(\xi,-\eta)}\psi_k(\xi-\eta)\psi_{k_1}(\xi). \tag{6.62}$$

From (4.10) and (4.39), we have

$$p_{k,k_1}^{+,\nu}(\xi-\eta,\eta) = p_{k,k_1}^{+,\nu,1}(\xi-\eta,\eta) + p_{k,k_1}^{+,\nu,2}(\xi-\eta,\eta) = O(1)\xi\cdot\eta + O(|\eta|^2).$$

Recall (6.16). From the above decomposition, we can decompose $p_{+,\nu}(\xi-\eta,\eta)$ into two parts as follows,

$$p_{k,k_1}^{+,\nu}(\xi-\eta,\eta) = \sum_{i=1,2} \tilde{p}_{k,k_1}^{+,\nu,i}(\xi-\eta,\eta), \quad \tilde{p}_{k,k_1}^{+,\nu,1}(\xi-\eta,\eta) = \frac{-i}{2} a_{k,k_1}(\xi)\Phi^{+,\nu}(\xi,\eta), \tag{6.63}$$

where uniquely determined symbols $a_{k,k_1}(\xi)$ and $\tilde{p}_{k,k_1}^{+,\nu,2}(\xi-\eta,\eta)$ satisfy the following estimate for some absolute constant C,

$$\|\tilde{p}_{k,k_1}^{+,\nu,2}(\xi-\eta,\eta)\|_{\mathcal{S}^\infty_{k,k_1,k_2}} \leq C2^{2k_2}, \|a_{k,k_1}(\xi)\|_{\mathcal{S}^\infty_{k,k_1,k_2}} \leq C. \tag{6.64}$$

Correspondingly, we decompose $\widetilde{Q}^1_{k,k_1,k_2}$ into two parts as follows,

$$\widetilde{Q}^1_{k,k_1,k_2} = \sum_{i=1,2} \widetilde{Q}^{1;i}_{k,k_1,k_2},$$

$$\widetilde{Q}^{1;i}_{k,k_1,k_2} := \int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \overline{\widehat{\Gamma^1\Gamma^2 g}(t,\xi)} e^{it\Phi^{+,\nu}(\xi,\eta)} \times \tilde{p}_{k,k_1}^{+,\nu,i}(\xi-\eta,\eta)\widehat{\Gamma^1\Gamma^2 g}(t,\xi-\eta)\widehat{g^\nu_{k_2}}(t,\eta)\, d\eta\, d\xi\, dt, \quad i\in\{1,2\}.$$

From the L^2-L^∞ type bilinear estimate (2.5) in Lemma 2.2, the following estimate holds,

$$\begin{aligned}\sum_{|k-k_1|\leq 10, k_2\leq k_1-10} |\widetilde{Q}^{1;2}_{k,k_1,k_2}| &\leq \sup_{t\in[2^{m-1},2^m]} \sum_{|k-k_1|\leq 10, k_2\leq k_1-10} C2^{2k_2}\|\Gamma^1\Gamma^2 g_{k_1}(t)\|_{L^2}\|e^{-it\Lambda}g_{k_2}(t)\|_{L^\infty} \\ &\quad \times \|\Gamma^1\Gamma^2 g_k(t)\|_{L^2} \leq C2^{2\tilde{\delta}m}\epsilon_0^2,\end{aligned} \tag{6.65}$$

where C is some absolute constant. For $\widetilde{Q}^{1;1}_{k,k_1,k_2}$, we do integration by parts in time. As a result, we have

$$\widetilde{Q}^{1;1}_{k,k_1,k_2} = \sum_{i=1,2} \widehat{Q}^{1;i}_{k,k_1,k_2},$$

$$\begin{aligned}\widehat{Q}^{1;1}_{k,k_1,k_2} :=& \sum_{i=1,2}(-1)^{i-1}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \overline{\widehat{\Gamma^1\Gamma^2 g}(t_i,\xi)} e^{it_i\Phi^{+,\nu}(\xi,\eta)} \widehat{\Gamma^1\Gamma^2 g}(t_i,\xi-\eta)\frac{a_{k,k_1}(\xi)}{2}\widehat{g^\nu_{k_2}}(t_i,\eta)\, d\eta\, d\xi \\ &+ \int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \overline{\widehat{\Gamma^1\Gamma^2 g}(t,\xi)} e^{it\Phi^{+,\nu}(\xi,\eta)}\frac{a_{k,k_1}(\xi)}{2}\widehat{\Gamma^1\Gamma^2 g}(t,\xi-\eta)\partial_t\widehat{g^\nu_{k_2}}(t,\eta)\, d\eta\, d\xi\, dt,\end{aligned}$$

$$\widehat{Q}^{1;2}_{k,k_1,k_2} := \frac{1}{2}\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{it\Phi^{+,\nu}(\xi,\eta)} a_{k,k_1}(\xi)\partial_t\big(\overline{\widehat{\Gamma^1\Gamma^2 g}(t,\xi)}\widehat{\Gamma^1\Gamma^2 g}(t,\xi-\eta)\big)\widehat{g^\nu_{k_2}}(t,\eta)\, d\eta\, d\xi\, dt.$$

From the L^2-L^∞ type bilinear estimate (2.5) in Lemma 2.2, the following estimate holds for some absolute constant C,

$$\begin{aligned}|\widehat{Q}^{1;1}_{k,k_1,k_2}| &\leq \sup_{t\in[2^{m-1},2^m]} C\|\Gamma^1\Gamma^2 g_k(t)\|_{L^2}\|\Gamma^1\Gamma^2 g_{k_1}(t)\|_{L^2}\big(\|e^{-it\Lambda}g_{k_2}\|_{L^\infty} + 2^m\|e^{-it\Lambda}\partial_t g_{k_2}(t)\|_{L^\infty}\big) \\ &\leq C2^{-m/2+\beta m}\epsilon_0^2.\end{aligned} \tag{6.66}$$

Recall the estimate (7.7) in Lemma 7.2. It motivates us to do the decomposition as follows,

$$\widehat{Q}^{1;2}_{k,k_1,k_2} = \widehat{Q}^{1;2,1}_{k,k_1,k_2} + \widehat{Q}^{1;2,2}_{k,k_1,k_2},$$

where

$$\begin{aligned}\widehat{Q}^{1;2,1}_{k,k_1,k_2} := &\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{it\Phi^{+,\nu}(\xi,\eta)}\frac{a_{k,k_1}(\xi)}{2}\widehat{g^{\nu}_{k_2}}(t,\eta)\\ &\times\Big[\overline{\big(\partial_t\widehat{\Gamma^1\Gamma^2 g_k}(t,\xi) - \sum_{(k_1',k_2')\in\chi_k^2}\sum_{\nu'\in\{+,-\}}\widetilde{B}^{+,\nu'}_{k,k_1',k_2'}(t,\xi)\big)}\widehat{\Gamma^1\Gamma^2 g}(t,\xi-\eta)\\ &+\overline{\widehat{\Gamma^1\Gamma^2 g}(t,\xi)}\big(\partial_t\widehat{\Gamma^1\Gamma^2 g_{k_1}}(t,\xi-\eta) - \sum_{(k_1',k_2')\in\chi_{k_1}^2}\sum_{\nu'\in\{+,-\}}\widetilde{B}^{+,\nu'}_{k_1,k_1',k_2'}(t,\xi-\eta)\big)\Big]d\eta\, d\xi\, dt,\end{aligned}$$

$$\begin{aligned}\widehat{Q}^{1;2,2}_{k,k_1,k_2} := &\sum_{k_2'\leq k_1'-10, |k_1-k_1'|\leq 10}\sum_{\nu'\in\{+,-\}}\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{it\Phi^{+,\nu}(\xi,\eta)}\frac{a_{k,k_1}(\xi)}{2}\Big[\widehat{\Gamma^1\Gamma^2 g}(t,\xi-\eta)\widehat{g^{\nu}_{k_2}}(t,\eta)\\ &\times\overline{e^{it\Phi^{+,\nu}(\xi,\kappa)}\widehat{\Gamma^1\Gamma^2 g_{k_1'}}(t,\xi-\kappa)\widehat{g^{\nu'}_{k_2'}}(t,\kappa)\tilde{q}_{+,\nu'}(\xi-\kappa,\kappa)} + \overline{\widehat{\Gamma^1\Gamma^2 g}(t,\xi)}\widehat{g^{\nu}_{k_2}}(t,\eta)\\ &\times e^{it\Phi^{+,\nu'}(\xi-\eta,\kappa)}\tilde{q}_{+,\nu'}(\xi-\eta-\kappa,\kappa)\widehat{\Gamma^1\Gamma^2 g_{k_1'}}(t,\xi-\eta-\kappa)\widehat{g^{\nu'}_{k_2'}}(t,\kappa)\Big]d\eta\, d\kappa\, d\xi\, dt.\end{aligned}$$

From estimate (7.7) in Lemma 7.2 and the $L^2 - L^\infty$ type bilinear estimate (2.5) in Lemma 2.2, the following estimate holds for some absolute constant C,

$$|\widehat{Q}^{1;2,1}_{k,k_1,k_2}| \leq \sup_{t\in[2^{m-1},2^m]} C2^{40\beta m}\epsilon_0\|\Gamma^1\Gamma^2 g_k(t)\|_{L^2}\|e^{-it\Lambda}g_{k_2}(t)\|_{L^\infty} \leq C2^{-m/2+60\beta m}\epsilon_0^2.$$

Now, we proceed to estimate $\widehat{Q}^{1;2,2}_{k,k_1,k_2}$. To utilize symmetry, we do change of variables for the second part of integration as follows $(\xi,\eta,\kappa)\longrightarrow(\xi-\kappa,\eta,-\kappa)$. As a result, we have

$$\begin{aligned}\widehat{Q}^{1;2,2}_{k,k_1,k_2} := &\sum_{k_2'\leq k_1'-10, |k_1-k_1'|\leq 10}\sum_{\nu'\in\{+,-\}}\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{it\Phi^{+,\nu}(\xi,\eta)-it\Phi^{+,\nu'}(\xi,\kappa)}\\ &\times\Big[\widehat{\Gamma^1\Gamma^2 g}(t,\xi-\eta)\widehat{g^{\nu}_{k_2}}(t,\eta)\frac{a_{k,k_1}(\xi)}{2}\overline{\widehat{\Gamma^1\Gamma^2 g_{k_1'}}(t,\xi-\kappa)\widehat{g^{\nu'}_{k_2'}}(t,\kappa)\tilde{q}_{+,\nu'}(\xi-\kappa,\kappa)}\\ &+\frac{a_{k,k_1}(\xi-\kappa)}{2}\overline{\widehat{\Gamma^1\Gamma^2 g}(t,\xi-\kappa)}\widehat{g^{\nu}_{k_2}}(t,\eta)\tilde{q}_{+,\nu'}(\xi-\eta,-\kappa)\\ &\times\widehat{\Gamma^1\Gamma^2 g_{k_1'}}(t,\xi-\eta)\widehat{g^{\nu'}_{k_2'}}(t,-\kappa)\Big]d\eta\, d\kappa\, d\xi\, dt\\ = &\sum_{k_2'\leq k_1'-10, |k_1-k_1'|\leq 10}\sum_{\nu'\in\{+,-\}}\frac{1}{2}\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{it\Phi^{+,\nu}(\xi,\eta)-it\Phi^{+,\nu'}(\xi,\kappa)}\\ &\times\widehat{\Gamma^1\Gamma^2 g}(t,\xi-\eta)\widehat{g^{\nu}_{k_2}}(t,\eta)\overline{\widehat{\Gamma^1\Gamma^2 g}(t,\xi-\kappa)\widehat{g^{\nu'}_{k_2'}}(t,-\kappa)}\tilde{r}^{k,k_1'}_{\nu,\nu'}(\xi,\eta,\kappa)\, d\eta\, d\kappa\, d\xi\, dt, \qquad (6.67)\end{aligned}$$

where

$$\tilde{r}^{k,k_1'}_{\nu,\nu'}(\xi,\eta,\kappa) := a_{k,k_1}(\xi)\overline{\tilde{q}_{+,-\nu'}(\xi-\kappa,\kappa)}\psi_{k_1'}(\xi-\kappa) + a_{k,k_1}(\xi-\kappa)\tilde{q}_{+,\nu'}(\xi-\eta,-\kappa)\psi_{k_1'}(\xi-\eta),$$

$$\Phi^{+,\nu}(\xi,\eta) - \Phi^{+,\nu'}(\xi,\kappa) = -\Lambda(\xi-\eta) - \nu\Lambda(\eta) + \Lambda(\xi-\kappa) - \nu'\Lambda(\kappa).$$

Recall (4.14) and (4.15). From the Lemma 2.1, we know that the following estimate holds

$$\|\tilde{r}_{\nu,\nu'}(\xi,\eta,\kappa)\psi_{k_2'}(\kappa)\psi_{k_1}(\xi-\eta)\psi_{k_2}(\eta)\|_{\mathcal{S}^\infty} \leq C 2^{\max\{k_2,k_2'\}+k_1}, \tag{6.68}$$

where C is some absolute constant. From (6.68), and the $L^2-L^\infty-L^\infty-L^\infty$ type multilinear estimate, the following estimate holds for some absolute constant C,

$$|\widehat{Q}^{1;2,2}_{k,k_1,k_2}| \leq \sup_{t\in[2^{m-1},2^m]} \sum_{k_2'\leq k_1-10} C 2^{m+\max\{k_2,k_2'\}+k_1} \|\Gamma^1\Gamma^2 g_{k_1}(t)\|_{L^2} \|e^{-it\Lambda} g_{k_2}(t)\|_{L^\infty} \times \|e^{-it\Lambda} g_{k_2'}(t)\|_{L^\infty} \|\Gamma^1\Gamma^2 g_k(t)\|_{L^2} \leq C 2^{-m/2+30\beta m}\epsilon_0^2.$$

Next, we estimate Q^2_{k,k_1,k_2}. Recall (6.60). From the L^2-L^∞ type bilinear estimate (2.5) in Lemma 2.1, (4.14) and (4.15), the following estimate holds for some absolute constant C,

$$\Big|\sum_{k_2\leq k_1+2,|k-k_1|\leq 10} Q^2_{k,k_1,k_2}\Big| \leq \sup_{t\in[2^{m-1},2^m]} \sum_{|k-k_1|\leq 10} C 2^{m+2k_1} \|P_{\leq k_1+2}[\Gamma^1\Gamma^2 g](t)\|_{L^2} \times \|e^{-it\Lambda} g_{k_1}(t)\|_{L^\infty} \|\Gamma^1\Gamma^2 g_k(t)\|_{L^2} \leq C 2^{2\tilde{\delta} m}\epsilon_0^2. \tag{6.69}$$

Lastly, we estimate Q^3_{k,k_1,k_2}. Recall (6.61). By doing integration by parts in "η" many times, we can rule out the case when $\max\{j_1,j_2\}\leq m+k_{1,-}-\beta m$. If $\max\{j_1,j_2\}\geq m+k_{1,-}-\beta m$, from the L^2-L^∞ type bilinear estimate, the following estimate holds for some absolute constant C,

$$\begin{aligned}
&\sum_{\max\{j_1,j_2\}\geq m+k_{1,-}-\beta m} |Q^{j_1,j_2,3}_{k,k_1,k_2}| \\
&\leq \sup_{t\in[2^{m-1},2^m]} \sum_{i=1,2} C 2^{m+2k_1} \Big(\sum_{j_1\geq \max\{j_2, m+k_{1,-}-\beta m\}} 2^{k_1+j_1} \|g_{k_1,j_1}(t)\|_{L^2} \\
&\quad \times (\|e^{-it\Lambda}\Gamma^i g_{k_2,j_2}(t)\|_{L^\infty} + \|e^{-it\Lambda} g_{k_2,j_2}(t)\|_{L^\infty}) \\
&\quad + \sum_{j_2\geq \max\{j_1, m+k_{1,-}-\beta m\}} C 2^{k_2+j_2} \|g_{k_2,j_2}(t)\|_{L^2} \\
&\quad \times (\|e^{-it\Lambda}\Gamma^i g_{k_1,j_1}(t)\|_{L^\infty} + \|e^{-it\Lambda} g_{k_1,j_1}(t)\|_{L^\infty})\Big) \|\Gamma^1\Gamma^2 g_k(t)\|_{L^2} \\
&\leq C 2^{-m+20\beta m-k_2}\epsilon_0^2 \leq C 2^{-\beta m}\epsilon_0^2.
\end{aligned} \tag{6.70}$$

Hence finishing the proof. □

LEMMA 6.6. – *Under the bootstrap assumption* (4.49)*, the following estimate holds for some absolute constant C,*

$$\sum_{k_1,k_2\in\mathbb{Z},|k-k_1|\leq 10,k_2\leq k_1-10,k_1+k_2\geq -19m/20,k_1\leq 5\beta m} |P^1_{k,k_1,k_2}| \leq C 2^{2\tilde{\delta} m}\epsilon_0^2. \tag{6.71}$$

Proof. – Recall (6.7) and (6.12). Same as in the High × High type interaction, we know that the integral inside P^1_{k,k_1,k_2} vanishes if $\Gamma^l=\Omega$. Hence, we only have to consider the case when $\Gamma^l = L$. Recall (6.17). We know that similar decompositions as in (6.25) and (6.28) also hold. Recall (6.28) and (6.17). From the estimate (6.8) in Lemma 2.1, the following estimate holds for some absolute constant C,

$$\|\tilde{q}^2_{+,\nu}(\xi,\eta)\psi_k(\xi)\psi_{k_1}(\xi-\eta)\psi_{k_2}(\eta)\|_{\mathcal{S}^\infty} \leq C 2^{2k_2}, \quad \text{if } k_2\leq k_1-10. \tag{6.72}$$

After doing integration by parts in "η" once, the following decomposition holds,

$$|\Gamma^{1,2}_{k,k_1,k_2}| \leq |\Gamma^{1,2;1}_{k,k_1,k_2}| + |\Gamma^{1,2;2}_{k,k_1,k_2}|, \tag{6.73}$$

where

$$\begin{aligned}
\Gamma^{1,2;1}_{k,k_1,,k_2} := & \int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \overline{\widehat{\Gamma^1\Gamma^2 g_k}(t,\xi)} e^{it\Phi^{+,\nu}(\xi,\eta)} \nabla_\eta \\
& \cdot \Big(\frac{\nabla_\eta \Phi^{+,\nu}(\xi,\eta)}{|\nabla_\eta \Phi^{+,\nu}(\xi,\eta)|^2} \tilde{q}^2_{+,\nu}(\xi-\eta,\eta)\big[\tilde{q}_{+,\nu}(\xi-\eta,\eta) \\
& \quad \times \big(\widehat{\Gamma g_{k_1}}(t,\xi-\eta)\widehat{g^\nu_{k_2}}(t,\eta) + \widehat{g_{k_1}}(t,\xi-\eta)\widehat{\Gamma g^\nu_{k_2}}(t,\eta)\big) \\
& \quad + (\Gamma_\xi + \Gamma_\eta + d_\Gamma)\tilde{q}_{+,\nu}(\xi-\eta,\eta)\widehat{g_{k_1}}(t,\xi-\eta)\widehat{g^\nu_{k_2}}(t,\eta)\big]\Big) \\
& - \overline{\widehat{\Gamma^1\Gamma^2 g_k}(t,\xi)} e^{it\Phi^{+,\nu}(\xi,\eta)} \nabla_\eta \\
& \cdot \Big(\frac{\nabla_\eta \Phi^{+,\nu}(\xi,\eta)}{|\nabla_\eta \Phi^{+,\nu}(\xi,\eta)|^2} \tilde{q}^2_{+,\nu}(\xi-\eta,\eta)\tilde{q}_{+,\nu}(\xi-\eta,\eta)\widehat{g^\nu_{k_2}}(t,\eta)\Big)\widehat{\Gamma g_{k_1}}(t,\xi-\eta)\, d\eta\, d\xi\, dt, \\
\Gamma^{1,2;2}_{k,k_1,k_2} := & \sum_{j_1\geq -k_{1,-}, j_2 \geq -k_{2,-}} \Gamma^{1,2;2}_{k,k_1,j_1,k_2,j_2}, \\
\Gamma^{1,2;2}_{k,k_1,j_1,k_2,j_2} := & \int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \overline{\widehat{\Gamma^1\Gamma^2 g_k}(t,\xi)} e^{it\Phi^{+,\nu}(\xi,\eta)} \widehat{\Gamma g_{k_1,j_1}}(t,\xi-\eta)\nabla_\eta \\
& \cdot \Big(\frac{\nabla_\eta \Phi^{+,\nu}(\xi,\eta)}{|\nabla_\eta \Phi^{+,\nu}(\xi,\eta)|^2} \tilde{q}^2_{+,\nu}(\xi-\eta,\eta)\tilde{q}_{+,\nu}(\xi-\eta,\eta)\widehat{g^\nu_{k_2,j_2}}(t,\eta)\Big) d\eta\, d\xi\, dt.
\end{aligned}$$

From the $L^2 - L^\infty$ type bilinear estimate, the following estimate holds for some absolute constant C,

$$\begin{aligned}
& \sum_{k_2 \leq k_1 - 10, |k_1 - k| \leq 10} |\Gamma^{1,2;1}_{k,k_1,,k_2}| \\
& \quad \leq \sup_{t\in[2^{m-1},2^m]} \sum_{k_2 \leq k_1 - 10, |k_1 - k| \leq 10} \sum_{i=1,2} C 2^{m+2k_2} \|\Gamma^1\Gamma^2 g_k(t)\|_{L^2} \\
& \quad \times \big[\big(2^{2k_1} \|\nabla^2_\xi \widehat{g}_{k_1}(t,\xi)\|_{L^2} + 2^{k_1}\|\nabla_\xi \widehat{g}_{k_1}(t,\xi)\|_{L^2}\big) \|e^{-it\Lambda} g_{k_2}(t)\|_{L^\infty} \\
& \quad + 2^{k_1}\|e^{-it\Lambda} g_{k_1}(t)\|_{L^\infty}\big(2^{k_2}\|\nabla^2_\xi \widehat{g}_{k_2}(t,\xi)\|_{L^2} \\
& \quad + \|\nabla_\xi \widehat{g}_{k_2}(t,\xi)\|_{L^2} + 2^{-k_2}\|g_{k_2}(t)\|_{L^2}\big) \\
& \quad + \sum_{j_1 \geq j_2} 2^{-m+k_2+k_1} 2^{2j_2} \|\varphi^{k_2}_{j_2}(x) g_{k_2}(t)\|_{L^2} 2^{j_1} \|\varphi^{k_1}_{j_1}(x) g_{k_1}(t)\|_{L^2} \\
& \quad + \sum_{j_2 \geq j_1} 2^{-m+k_2+k_1} 2^{2j_1} \|\varphi^{k_2}_{j_2}(x) g_{k_2}(t)\|_{L^2} 2^{j_2} \|\varphi^{k_1}_{j_1}(x) g_{k_1}(t)\|_{L^2}\big] \leq C 2^{2\tilde{\delta} m}\epsilon_0^2.
\end{aligned} \tag{6.74}$$

Now, we proceed to estimate $\Gamma^{1,2;2}_{k,k_1,,k_2}$. By doing integration by parts in "η" many times, we can rule out the case when $\max\{j_1, j_2\} \leq m + k_{1,-} - \beta m$. If $\max\{j_1, j_2\} \geq m + k_{1,-} - \beta m$, from the $L^2 - L^\infty$ type bilinear estimate, the following estimate holds for some absolute constant C,

$$\sum_{\max\{j_1,j_2\}\geq m + k_{1,-} - \beta m} |\Gamma^{1,2;2}_{k,k_1,j_1,k_2,j_2}|$$

$$
\begin{aligned}
&\leq \sup_{t\in[2^{m-1},2^m]} C 2^{m+2k_2+k_1}\Big[\sum_{j_1\geq \max\{j_2, m+k_{1,-}-\beta m\}} 2^{k_1+j_1}\|g_{k_1,j_1}(t)\|_{L^2} \\
&\quad \times \big(2^{-k_2}\|e^{-it\Lambda} g_{k_2,j_2}\|_{L^\infty} + \|e^{-it\Lambda}\mathcal{F}^{-1}[\nabla_\xi g_{k_2,j_2}(t)]\|_{L^\infty}\big) \\
&\quad + \sum_{j_2\geq \max\{j_1, m+k_{1,-}-\beta m\}} 2^{j_2}\|g_{k_2,j_2}(t)\|_{L^2}\|e^{-it\Lambda}\Gamma g_{k_1,j_1}(t)\|_{L^\infty}\Big]\|\Gamma^1\Gamma^2 g_k(t)\|_{L^2}
\end{aligned}
$$

$$
\leq C 2^{-m/2+20\beta m}\epsilon_0^2. \tag{6.75}
$$

Recall the estimates (6.25) and (6.73). From the estimates (6.74), (6.75) and the estimate (6.76) in Lemma 6.7, we know that our desired estimate (6.71) holds. □

LEMMA 6.7. – *Under the bootstrap assumption* (4.49) *and the assumption that* $k_1 + k_2 \geq -19m/20$ *and* $k_1 \leq 5\beta m$*, the following estimate holds for some absolute constant* C*,*

$$
|\Gamma^{1,1}_{k,k_1,k_2}| \leq C 2^{-\beta m}\epsilon_0^2. \tag{6.76}
$$

Proof. – Same as in the High × High interaction, we do integration by parts in time once. As a result, we have the same formulations as in (6.34), (6.35) and (6.36).

We first estimate $\widetilde{\Gamma}^{1,1}_{k,k_1,k_2}$. Recall (6.34). By doing integration by parts in "η" many times, we can rule out the case when $\max\{j_1, j_2\} \leq m+k_{1,-}-\beta m$. If $\max\{j_1, j_2\} \geq m+k_{1,-}-\beta m$, from the L^2-L^∞ type bilinear estimate (2.5) in Lemma 2.2, the following estimate holds for some absolute constant C,

$$
\begin{aligned}
&\sum_{\max\{j_1,j_2\}\geq m+k_{1,-}-\beta m} |\widetilde{\Gamma}^{j_1,j_2,1,1}_{k,k_1,k_2}| \\
&\leq \sup_{t\in[2^{m-1},2^m]} C 2^{m+2k_1}\|\Gamma^1\Gamma^2 g_k\|_{L^2} \\
&\quad \times \Big[\sum_{j_1\geq \max\{j_2, m+k_{1,-}-\beta m\}} 2^{k_1+j_1}\|g_{k_1,j_1}(t)\|_{L^2}\big(\|e^{-it\Lambda} g_{k_2,j_2}\|_{L^\infty} \\
&\quad + 2^{k_2}\|e^{-it\Lambda}\mathcal{F}^{-1}[\nabla_\xi \widehat{g}_{k_2,j_2}(t)]\|_{L^\infty}\big) \\
&\quad + \sum_{j_2\geq \max\{j_1, m+k_{1,-}-\beta m\}} 2^{k_2+j_2}\|g_{k_2,j_2}(t)\|_{L^2}\big(\|e^{-it\Lambda} g_{k_1,j_1}\|_{L^\infty} \\
&\quad + 2^{k_1}\|e^{-it\Lambda}\mathcal{F}^{-1}[\nabla_\xi \widehat{g}_{k_1,j_1}(t)]\|_{L^\infty}\big)\Big] \\
&\leq C 2^{-m-k_2+20\beta m}\epsilon_0^2 \leq C 2^{-\beta m}\epsilon_0^2.
\end{aligned}
$$

Now, we proceed to estimate $\widetilde{\Gamma}^{1,2}_{k,k_1,k_2}$. Recall (6.36). Since now k_1 and k_2 are not comparable, different from the decomposition we did in (6.37) in the High × High type interaction, we do decomposition as follows,

$$
\widetilde{\Gamma}^{1,2}_{k,k_1,k_2} = \sum_{i=1,\dots,7} \widetilde{\Gamma}^{1,2;i}_{k,k_1,k_2} \tag{6.77}
$$

$$
\widetilde{\Gamma}^{1,2;2}_{k,k_1,k_2} = \sum_{k_2'\leq k_1'+10} \widehat{\Gamma}^{k_1',k_2',1}_{k,k_1,k_2}, \quad \widehat{\Gamma}^{k_1',k_2',1}_{k,k_1,k_2} = \sum_{j_2\geq -k_{2,-}, j_1'\geq -k_{1,-}', j_2'\geq -k_{2,-}'} \widehat{\Gamma}^{k_1',j_1',k_2',j_2',1}_{k,k_1,k_2,j_2}, \tag{6.78}
$$

$$
\widetilde{\Gamma}^{1,2;3}_{k,k_1,k_2} = \sum_{k_2'\leq k_1'+10} \widehat{\Gamma}^{k_1',k_2',2}_{k,k_1,k_2}, \quad \widehat{\Gamma}^{k_1',k_2',2}_{k,k_1,k_2} = \sum_{j_1\geq -k_{1,-}, j_1'\geq -k_{1,-}', j_2'\geq -k_{2,-}'} \widehat{\Gamma}^{k_1',j_1',k_2',j_2',2}_{k,k_1,j_1,k_2}, \tag{6.79}
$$

$$\widetilde{\Gamma}^{1,2;i}_{k,k_1,k_2} = \sum_{j_1\geq -k_{1,-}, j_2\geq -k_{2,-}} \widetilde{\Gamma}^{1,2;i}_{k,k_1,j_1,k_2,j_2}, \quad i\in\{4,5\},$$

$$\widetilde{\Gamma}^{1,2;1}_{k,k_1,k_2} := -\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \overline{\widehat{\Gamma^1\Gamma^2 g_k}(t,\xi)} e^{it\Phi^{+,\nu}(\xi,\eta)}\tilde{c}(\xi-\eta)t\partial_t \widehat{g^{\nu}_{k_2}}(t,\eta) \\ \times\big(\tilde{q}_{+,\nu}(\xi,\eta)\widehat{\Gamma g_{k_1}}(t,\xi-\eta) + (\Gamma_\xi+\Gamma_\eta+d_\Gamma)\tilde{q}_{+,\nu}(\xi,\eta)\widehat{g_{k_1}}(t,\xi-\eta)\big)d\eta d\xi dt,$$

which results from the case when ∂_t hits the input "$\widehat{g_{k_2}}(t,\xi-\eta)$" in (6.36).

$$\widehat{\Gamma}^{k_1',j_1',k_2',j_2',1}_{k,k_1,k_2,j_2} := \sum_{\mu',\nu'\in\{+,-\}} -\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \overline{\widehat{\Gamma^1\Gamma^2 g_k}(t,\xi)} e^{it\Phi^{+,\nu}(\xi,\eta)} t\tilde{c}(\xi-\eta) e^{it\Phi^{\mu',\nu'}(\xi-\eta,\sigma)} \\ \times \tilde{q}_{\mu',\nu'}(\xi-\eta-\sigma,\sigma)\psi_{k_1}(\xi-\eta)\widehat{g^{\mu'}_{k_1',j_1'}}(t,\xi-\eta-\sigma)\widehat{g^{\nu'}_{k_2',j_2'}}(t,\sigma) \\ \times\big(\tilde{q}_{+,\nu}(\xi-\eta,\eta)\widehat{\Gamma g^{\nu}_{k_2,j_2}}(t,\eta) + (\Gamma_\xi+\Gamma_\eta+d_\Gamma)\tilde{q}_{+,\nu}(\xi-\eta,\eta)\widehat{g^{\nu}_{k_2,j_2}}(t,\eta)\big)d\eta d\xi dt, \tag{6.80}$$

which is resulted from the quartic terms when ∂_t hits the input "$\widehat{g_{k_1}}(t,\xi-\eta)$" in (6.36).

$$\widehat{\Gamma}^{k_1',j_1',k_2',j_2',2}_{k,k_1,j_1,k_2} := \sum_{\mu',\nu'\in\{+,-\}} -\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \overline{\widehat{\Gamma^1\Gamma^2 g_k}(t,\xi)} e^{it\Phi^{+,\nu}(\xi,\eta)}\tilde{c}(\xi-\eta)t\tilde{q}_{+,\nu}(\xi-\eta,\eta) \\ \times\widehat{g_{k_1,j_1}}(t,\xi-\eta)e^{it\Phi^{\mu',\nu'}(\eta,\sigma)}\big[\Gamma_\eta\big(\tilde{q}_{\mu',\nu'}(\eta-\sigma,\sigma)\widehat{g^{\mu'}_{k_1',j_1'}}(t,\eta-\sigma)\big) \\ + it\Gamma_\eta\Phi^{\mu',\nu'}(\eta,\sigma)\tilde{q}_{\mu',\nu'}(\eta-\sigma,\sigma)\widehat{g^{\mu'}_{k_1',j_1'}}(t,\eta-\sigma)\big] \\ \times\widehat{g^{\nu'}_{k_2',j_2'}}(t,\sigma)d\sigma d\eta d\xi dt, \tag{6.81}$$

which is resulted from the quartic terms when ∂_t hits the input "$\widehat{\Gamma g_{k_2}}(t,\eta)$" in (6.36).

(6.82)

$$\widetilde{\Gamma}^{1,2;4}_{k,k_1,j_1,k_2,j_2} := -\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \overline{\widehat{\Gamma^1\Gamma^2 g_k}(t,\xi)} e^{it\Phi^{+,\nu}(\xi,\eta)} t\tilde{c}(\xi-\eta) \\ \times\big[\big(\widehat{\Gamma\Lambda_{\geq 3}[\partial_t g]}_{k_1,j_1}(t,\xi-\eta)\widehat{g^{\nu}_{k_2,j_2}}(t,\eta) + \widehat{g_{k_1,j_1}}(t,\xi-\eta)\widehat{\Gamma\Lambda_{\geq 3}[\partial_t g^{\nu}]}_{k_2,j_2}(t,\eta) \\ + \widehat{\Lambda_{\geq 3}[\partial_t g]}_{k_1,j_1}(t,\xi-\eta)\widehat{\Gamma g^{\nu}_{k_2,j_2}}(t,\eta)\big)\tilde{q}_{+,\nu}(\xi-\eta,\eta) \\ + (\Gamma_\xi+\Gamma_\eta+d_\Gamma)\tilde{q}_{+,\nu}(\xi-\eta,\eta)\widehat{\Lambda_{\geq 3}[\partial_t g]}_{k_1,j_1}(t,\xi-\eta)\widehat{g^{\nu}_{k_2,j_2}}(t,\eta)\big)d\eta d\xi dt, \tag{6.83}$$

which is resulted from the quintic and higher order terms when ∂_t hits the inputs "$g_{k_1}(t)$," "$\Gamma g_{k_1}(t)$," and "$\Gamma g_{k_2}(t)$" in (6.36).

$$\widetilde{\Gamma}^{1,2;5}_{k,k_1,j_1,k_2,j_2} = -\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{it\Phi^{+,\nu}(\xi,\eta)} t\tilde{c}(\xi-\eta) \\ \overline{\times\big(\partial_t\widehat{\Gamma^1\Gamma^2 g_k}(t,\xi) - \sum_{\nu\in\{+,-\}}\sum_{(k_1',k_2')\in\chi_k^2}\widetilde{B}^{+,\nu}_{k,k_1',k_2'}(t,\xi)\big)} \\ \times\big(\tilde{q}_{+,\nu}(\xi-\eta,\eta)(\widehat{g_{k_1,j_1}}(t,\xi-\eta)\widehat{\Gamma g^{\nu}_{k_2,j_2}}(t,\eta) + \widehat{\Gamma g_{k_1,j_1}}(t,\xi-\eta)\widehat{g^{\nu}_{k_2,j_2}}(t,\eta)) \\ + (\Gamma_\xi+\Gamma_\eta)\tilde{q}_{+,\nu}(\xi-\eta,\eta)\widehat{g_{k_1,j_1}}(t,\xi-\eta)\widehat{g^{\nu}_{k_2,j_2}}(t,\eta)\big)d\eta d\xi dt, \tag{6.84}$$

which is resulted from the good error terms when ∂_t hits "$\Gamma^1\Gamma^2 g_k(t)$" in (6.36).

$$\widetilde{\Gamma}^{1,2;6}_{k,k_1,k_2} = \sum_{|k_1'-k_2'|\le 10} \widehat{\Gamma}^{k_1',k_2',3}_{k,k_1,k_2;1} + \sum_{k_2'\le k_1'-10} \widehat{\Gamma}^{k_1',k_2',3}_{k,k_1,k_2;2}, \tag{6.85}$$

$$\widehat{\Gamma}^{k_1',k_2',3}_{k,k_1,k_2;i} = \sum_{j_1'\ge -k'_{1,-}, j_2'\ge -k'_{2,-}, j_2\ge -k_{2,-}} \widehat{\Gamma}^{k_1',j_1',k_2',j_2',3}_{k,k_1,k_2,j_2;i}, \quad i\in\{1,2\}, \tag{6.86}$$

$$\begin{aligned}\widehat{\Gamma}^{k_1',j_1',k_2',j_2',3}_{k,k_1,k_2,j_2;1} := \sum_{\mu',\nu'\in\{+,-\}} -\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} &\overline{\widehat{\Gamma^1\Gamma^2 g_k}(t,\xi)} e^{it\Phi^{+,\nu}(\xi,\eta)}\tilde{c}(\xi-\eta)\\ &\times t\tilde{q}_{+,\nu}(\xi-\eta,\eta)\widehat{g^{\nu}_{k_2,j_2}}(t,\eta)\psi_{k_1}(\xi-\eta)e^{it\Phi^{\mu',\nu'}(\xi-\eta,\sigma)}\widehat{g^{\nu'}_{k_2',j_2'}}(t,\sigma)\\ &\times\Big[it\Gamma_{\xi-\eta}\Phi^{\mu',\nu'}(\xi-\eta,\sigma)\tilde{q}_{\mu',\nu'}(\xi-\eta-\sigma,\sigma)\widehat{g^{\mu'}_{k_1',j_1'}}(t,\xi-\eta-\sigma)\\ &\quad+\Gamma_{\xi-\eta}\big(\tilde{q}_{\mu',\nu'}(\xi-\eta-\sigma,\sigma)\widehat{g^{\mu'}_{k_1',j_1'}}(t,\xi-\eta-\sigma)\big)\Big]d\sigma d\eta d\xi dt,\end{aligned} \tag{6.87}$$

which is resulted from the quartic terms when ∂_t hits the input "$\widehat{\Gamma g_{k_1}}(t,\xi-\eta)$" in (6.36) and moreover two inputs inside $\Lambda_2[\partial_t\widehat{\Gamma g_{k_1}}(t,\xi-\eta)]$ have comparable sizes of frequencies, see (6.85).

$$\begin{aligned}\widehat{\Gamma}^{k_1',j_1',k_2',j_2',3}_{k,k_1,k_2,j_2;2} := \sum_{\nu'\in\{+,-\}} -\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} &\overline{\widehat{\Gamma^1\Gamma^2 g_k}(t,\xi)} e^{it\Phi^{+,\nu}(\xi,\eta)}\tilde{c}(\xi-\eta)t\tilde{q}_{+,\nu}(\xi-\eta,\eta)\widehat{g^{\nu}_{k_2,j_2}}(t,\eta)\\ \times\psi_{k_1}(\xi-\eta)e^{it\Phi^{+,\nu'}(\xi-\eta,\sigma)}\widehat{g^{\nu'}_{k_2',j_2'}}(t,\sigma)&\Big[it\Gamma_{\xi-\eta}\Phi^{+,\nu'}(\xi-\eta,\sigma)\tilde{q}_{+,\nu'}(\xi-\eta-\sigma,\sigma)\widehat{g_{k_1',j_1'}}(t,\xi-\eta-\sigma)\\ +\Gamma_{\xi-\eta}\big(\tilde{q}_{+,\nu'}(\xi-\eta-\sigma,\sigma)\widehat{g_{k_1',j_1'}}(t,\xi-\eta-\sigma)\big)&-\widehat{\Gamma g_{k_1',j_1'}}(t,\xi-\eta-\sigma)\tilde{q}_{+,\nu'}(\xi-\eta-\sigma,\sigma)\Big]d\sigma d\eta d\xi dt,\end{aligned} \tag{6.88}$$

which is resulted from the quartic terms when ∂_t hits the input "$\widehat{\Gamma g_{k_1}}(t,\xi-\eta)$" in (6.36) and moreover two inputs inside $\Lambda_2[\partial_t\widehat{\Gamma g_{k_1}}(t,\xi-\eta)]$ have different size of frequencies (see (6.85)) and the bulk term of this scenario is removed.

$$\begin{aligned}\widetilde{\Gamma}^{1,2;7}_{k,k_1,k_2} = \sum_{k_2'\le k_1'-10, |k_1-k_1'|\le 10}\sum_{\nu'\in\{+,-\}} -\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} &e^{it\Phi^{+,\nu}(\xi,\eta)-it\Phi^{+,\nu'}(\xi,\kappa)} t r^{\nu,\nu'}_{k_1,k_1'}(\xi,\eta,\kappa)\\ &\times\widehat{\Gamma g}(t,\xi-\eta)\widehat{g^{\nu}_{k_2}}(t,\eta)\overline{\widehat{\Gamma^1\Gamma^2 g}(t,\xi-\kappa)}\widehat{g^{\nu'}_{k_2'}}(t,-\kappa)d\kappa d\eta d\xi dt,\end{aligned} \tag{6.89}$$

which is resulted from putting the bulk term inside "$\Lambda_2[\partial_t\widehat{\Gamma g_{k_1}}(t,\xi-\eta)]$" and the bulk term inside "$\Lambda_2[\partial_t\widehat{\Gamma^1\Gamma^2 g_k}(t,\xi)]$" together, and the symbol $r^{\nu,\nu'}_{k_1,k_1'}(\xi,\eta,\kappa)$ is given as follows,

$$\begin{aligned}r^{\nu,\nu'}_{k_1,k_1'}(\xi,\eta,\kappa) &= \tilde{c}(\xi-\eta)\tilde{q}_{+,\nu}(\xi-\eta,\eta)\overline{\tilde{q}_{+,-\nu'}(\xi-\kappa,\kappa)}\psi_{k_1'}(\xi-\kappa)\psi_{k_1}(\xi-\eta)\psi_k(\xi)\\ &+\tilde{c}(\xi-\eta-\kappa)\tilde{q}_{+,\nu}(\xi-\kappa-\eta,\eta)\tilde{q}_{+,\nu'}(\xi-\eta,-\kappa)\psi_{k_1'}(\xi-\eta)\psi_k(\xi-\kappa)\psi_{k_1}(\xi-\eta-\kappa).\end{aligned}$$

Recall (4.14) and (4.15). From the estimate (2.3) in Lemma 2.1, the following estimate holds for some absolute constant C,

$$\|r^{\nu,\nu'}_{k_1,k_1'}(\xi,\eta,\kappa)\psi_{k_2}(\eta)\psi_{k_2'}(\kappa)\|_{\mathcal{S}^\infty} \le C2^{\max\{k_2,k_2'\}+3k_1}. \tag{6.90}$$

With the above preparation of classifying all terms inside $\widetilde{\Gamma}^{1,2}_{k,k_1,k_2}$, see the decomposition (6.77). Now, we are ready to estimate them one by one. From estimate (7.2) in Lemma 7.1 and the $L^2 - L^\infty$ type bilinear estimate, we have

$$\begin{aligned}|\widetilde{\Gamma}^{1,2;1}_{k,k_1,k_2}| \leq & \sup_{t\in[2^{m-1},2^m]} \sum_{i=1,2} C 2^{2m+2k_1} \|\partial_t \widehat{g_{k_2}}(t,\xi)\|_{L^2} \\ & \times \big(\|e^{-it\Lambda}\Gamma^i g_{k_1}(t)\|_{L^\infty} + \|e^{-it\Lambda} g_{k_1}(t)\|_{L^\infty}\big) \|\Gamma^1\Gamma^2 g_k(t)\|_{L^2} \\ & \leq C 2^{m+2\tilde{\delta}m}(2^{-21m/20} + 2^{-2m-k_2+2\tilde{\delta}m})\epsilon_0^2 \leq C 2^{-\beta m}\epsilon_0^2,\end{aligned}$$

where C is some absolute constant. Now, we proceed to estimate $\widetilde{\Gamma}^{1,2;2}_{k,k_1,k_2}$. Recall (6.78) and (6.80). We split into two cases as follows based on the size of difference between k_1' and k_2'.

If $|k_1' - k_2'| \leq 5$. – Note that $k_1' \geq k_1 - 5 \geq k_2 + 5$. By doing integration by parts in "η" many times, we can rule out the case when $\max\{j_1', j_2\} \leq m + k'_{1,-} - \beta m$. Hence, it would be sufficient to consider the case when $\max\{j_1', j_2\} \geq m + k'_{1,-} - \beta m$. From the $L^2 - L^\infty - L^\infty$ type trilinear estimate (2.6) in Lemma 2.2, the following estimate holds for some absolute constant C,

$$\begin{aligned}&\sum_{|k_1'-k_2'|\leq 5} \Big| \sum_{\max\{j_1',j_2\}\geq m+k'_{1,-}-\beta m} \widehat{\Gamma}^{k_1',j_1',k_2',j_2',1}_{k,k_1,k_2,j_2} \Big| \\ &\quad \leq \sup_{t\in[2^{m-1},2^m]} \sum_{|k_1'-k_2'|\leq 5} C 2^{2m+2k_1+2k_1'} \|\Gamma^1\Gamma^2 g_k(t)\|_{L^2} \\ &\quad \times \Big[\sum_{j_1'\geq \max\{j_2, m+k'_{1,-}-\beta m\}} \|g_{k_1',j_1'}\|_{L^2} \|e^{-it\Lambda} g_{k_2'}\|_{L^\infty} \\ &\quad \times (\|e^{-it\Lambda} g_{k_2,j_2}\|_{L^\infty} + \|e^{-it\Lambda}\Gamma^n g_{k_2,j_2}\|_{L^\infty}) \\ &\quad + \sum_{j_2\geq \max\{j_1', m+k'_{1,-}-\beta m\}} \|e^{-it\Lambda} g_{k_1',j_1'}\|_{L^\infty} 2^{k_2+j_2} \|g_{k_2,j_2}\|_{L^2} \|e^{-it\Lambda} g_{k_2'}(t)\|_{L^\infty} \Big] \\ &\quad \leq C 2^{-m-k_2+20\beta m}\epsilon_0^2 \leq C 2^{-2\beta m}\epsilon_0^2.\end{aligned}$$

If $k_2' \leq k_1' - 5$. – For this case we have $|k_1 - k_1'| \leq 2$ and $k_1' \geq k_2 + 5$. If moreover $k_1 + k_2' \leq -9m/10$, then from estimate (5.15) in Lemma 5.3, the following estimate holds for some absolute constant C,

$$\begin{aligned}&\sum_{k_2'\leq \min\{-9m/10-k_1, k_1-10\}} |\widehat{\Gamma}^{k_1',k_2',1}_{k,k_1,k_2}| \\ &\quad \leq \sup_{t\in[2^{m-1},2^m]} \sum_{k_2'\leq \min\{-9m/10-k_1,k_1-10\}} C 2^{2m+3k_1} \|\Gamma^1\Gamma^2 g_k(t)\|_{L^2} \\ &\quad \times \|e^{-it\Lambda} g_{k_1'}\|_{L^\infty} \big(\|g_{k_2}\|_{L^2} + \|\Gamma^n g_{k_2}\|_{L^2}\big) \\ &\quad \times \big(2^{3k_2'} \|\widehat{g}_{k_2'}(t,\xi)\|_{L^\infty_\xi} + 2^{k_1+2k_2'} \|\widehat{\mathrm{Re}[v]}(t,\xi)\psi_{k_2'}(\xi)\|_{L^\infty_\xi}\big) \\ &\quad \leq C 2^{-2\beta m}\epsilon_0^2.\end{aligned}$$

Lastly, if $k_1 + k_2' \geq -9m/10$, we can do integration by part in "σ" many times to rule out the case when $\max\{j_1', j_2'\} \leq m + k_{1,-} - \beta m$. Also, by doing integration by parts in "η" many times, we can rule out the case when $\max\{j_1', j_2\} \leq m + k_{1,-} - \beta m$. Hence, it would be sufficient to consider the case when $\max\{j_1', j_2\} \geq m + k_{1,-} - \beta m$ and $\max\{j_1', j_2'\} \geq m + k_{1,-} - \beta m$, which implies that one of the following two cases must holds: (i) $j_1' \geq m + k_{1,-} - \beta m$; (ii) $j_1' \leq m + k_{1,-} - \beta m$ and $j_2, j_2' \geq m + k_{1,-} - \beta m$. From the $L^2 - L^\infty - L^\infty$ type trilinear estimate (2.6) in Lemma 2.2, the following estimate holds for some absolute constant C,

$$\begin{aligned}
&\Big|\sum_{\max\{j_1',j_2'\},\max\{j_1',j_2\}\geq m+k_{1,-}-\beta m} \widehat{\Gamma}^{k_1',j_1',k_2',j_2',1}_{k,k_1,k_2,j_2}\Big| \\
&\leq \sup_{t\in[2^{m-1},2^m]} C2^{2m+4k_1}\Big[\sum_{j_1'\geq m+k_{1,-}-\beta m} \|g_{k_1',j_1'}\|_{L^2}\|e^{-it\Lambda}g_{k_2',j_2'}\|_{L^\infty} \\
&\quad \times \big(\|e^{-it\Lambda}g_{k_2,j_2}\|_{L^\infty} + \|e^{-it\Lambda}\Gamma g_{k_2,j_2}\|_{L^\infty}\big) \\
&\quad + \sum_{j_2',j_2\geq m+k_{1,-}-\beta m} 2^{2k_2+j_2}\|g_{k_2,j_2}\|_{L^2}\|g_{k_2',j_2'}\|_{L^2}\|e^{-it\Lambda}g_{k_1',j_1'}\|_{L^\infty}\Big]\|\Gamma^1\Gamma^2 g_k(t)\|_{L^2} \\
&\leq C2^{-m-k_2'+20\beta m}\epsilon_0^2 \leq C2^{-2\beta m}\epsilon_0^2.
\end{aligned}$$

Now, we proceed to estimate $\widetilde{\Gamma}^{1,2;3}_{k,k_1,k_2}$. Recall (6.79) and (6.81). We separate into two cases as follows based on the size of difference between k_1' and k_2'.

If $|k_1' - k_2'| \leq 10$. – Note that $k_1' \geq k_2 - 5$. By doing integration by parts in "σ," we can rule out the case when $\max\{j_1', j_2'\} \leq m + k_{2,-} - k_{1,+}' - \beta m$. If $\max\{j_1', j_2'\} \geq m + k_{2,-} - k_{1,+}' - \beta m$, from the $L^2 - L^\infty - L^\infty$ type trilinear estimate (2.6) in Lemma 2.2, the following estimate holds for some absolute constant C,

$$\begin{aligned}
&\sum_{|k_1'-k_2'|\leq 10}\Big|\sum_{\max\{j_1',j_2'\}\geq m+k_{2,-}-k_{1,+}'-\beta m} \widehat{\Gamma}^{k_1',j_1',k_2',j_2',2}_{k,k_1,j_1,k_2}\Big| \\
&\quad \leq \sup_{t\in[2^{m-1},2^m]} \sum_{|k_1'-k_2'|\leq 10} C2^{2m+2k+2k_1'}\|\Gamma^1\Gamma^2 g_k(t)\|_{L^2} \\
&\quad \times \Big(\sum_{j_1'\geq\{j_2',m+k_{2,-}-k_{1,+}'-\beta m\}} 2^{k_1'}(2^{j_1'}+2^{m+k_2})\|g_{k_1',j_1'}(t)\|_{L^2}\|e^{-it\Lambda}g_{k_2',j_2'}(t)\|_{L^\infty} \\
&\quad + \sum_{j_2'\geq\{j_1',m+k_{2,-}-k_{1,+}'-\beta m\}} 2^{k_1'}(2^{j_1'}+2^{m+k_2})\|g_{k_2',j_2'}(t)\|_{L^2}2^{-m}\|g_{k_1',j_1'}(t)\|_{L^1}\Big) \\
&\quad \times \|e^{-it\Lambda}g_{k_1}(t)\|_{L^\infty} \leq C2^{-m-k_2+20\beta m}\epsilon_0^2 \leq C2^{-2\beta m}\epsilon_0^2.
\end{aligned}$$

If $k_2' \leq k_1' - 10$. – For this case, we have $k_2 - 2 \leq k_1' \leq k_2 + 2 \leq k_1 - 5$. By doing integration by parts in "η," we can rule out the case when $\max\{j_1, j_1'\} \leq m + k_{1,-} - \beta m$. If $\max\{j_1, j_1'\} \geq m + k_{1,-} - \beta m$, from the $L^2 - L^\infty - L^\infty$ type trilinear estimate (2.6) in Lemma 2.2, the following estimate holds for some absolute constant C,

$$\sum_{k_2'\leq k_1'-10}\Big|\sum_{\max\{j_1,j_1'\}\geq m+k_{1,-}-\beta m} \widehat{\Gamma}^{k_1',j_1',k_2',j_2',2}_{k,k_1,j_1,k_2}\Big|$$

$$
\begin{aligned}
&\leq \sup_{t\in[2^{m-1},2^m]} \sum_{k_2'\leq k_1'-10} C2^{2m+2k+2k_1'}\|\Gamma^1\Gamma^2 g_k(t)\|_{L^2}\|e^{-it\Lambda}g_{k_2'}(t)\|_{L^\infty}\\
&\quad\times\Big(\sum_{j_1\geq\max\{j_1',m+k_{1,-}-\beta m\}} (2^{k_1'+j_1'}+2^{m+k_2+k_1'})\|g_{k_1,j_1}(t)\|_{L^2}2^{-m}\|g_{k_1',j_1'}(t)\|_{L^1}\\
&\quad+\sum_{j_1'\geq\max\{j_1,m+k_{1,-}-\beta m\}} (2^{k_1'+j_1'}+2^{m+k_2+k_1'})\|g_{k_1',j_1'}(t)\|_{L^2}\|e^{-it\Lambda}g_{k_1,j_1}(t)\|_{L^\infty}\Big)\\
&\leq C2^{-m/2+20\beta m}\epsilon_0^2.
\end{aligned}
$$

Now, we proceed to estimate $\widetilde{\Gamma}^{1,2;4}_{k,k_1,k_2}$ and $\widetilde{\Gamma}^{1,2;5}_{k,k_1,k_2}$. Recall (6.77), (6.83), and (6.84). By doing integration by parts in "η," we can rule out the case when $\max\{j_1,j_2\}\leq m+k_{1,-}-\beta m$. If $\max\{j_1,j_2\}\geq m+k_{1,-}-\beta m$, from estimate (7.7) in Lemma 7.2, (6.137) in Lemma 6.14, and the L^2-L^∞ type bilinear estimate (2.5) in Lemma 2.2, the following estimate holds for some absolute constant C,

$$
\begin{aligned}
&\sum_{i=4,5}\sum_{\max\{j_1,j_2\}\geq m+k_{1,-}-\beta m} |\widetilde{\Gamma}^{1,2;i}_{k,k_1,j_1,k_2,j_2}|\\
&\quad\leq \sup_{t\in[2^{m-1},2^m]} C2^{m+2k_1+\beta m}\Big[\sum_{j_1\geq\max\{j_2,m+k_{1,-}-\beta m\}} 2^{k_1+j_1}\\
&\qquad\times(\|e^{-it\Lambda}g_{k_2,j_2}(t)\|_{L^\infty}+\|e^{-it\Lambda}\Gamma g_{k_2,j_2}(t)\|_{L^\infty})\\
&\qquad\times\big(2^{6k_+}\|g_{k_1,j_1}(t)\|_{L^2}+2^m\|\Lambda_{\geq 3}[\partial_t g(t)]_{k_1,j_1}\|_{L^2}\big)\\
&\quad+2^{m+k_2}\|g_{k_1,j_1}\|_{L^2}2^{k_2+j_2}\|\Lambda_{\geq 3}[\partial_t g(t)]_{k_2,j_2}\|_{L^2}\\
&\quad+\sum_{j_2\geq\max\{j_1,m+k_{1,-}-\beta m\}} 2^{k_1+j_1}\big(2^{6k_+}\|g_{k_1,j_1}\|_{L^2}+2^m\|\Lambda_{\geq 3}[\partial_t g(t)]_{k_1,j_1}\|_{L^2}\big)2^{k_2}\|g_{k_2,j_2}\|_{L^2}\\
&\quad+2^{k_2+j_2}\big(2^{6k_+}\|g_{k_2,j_2}\|_{L^2}+2^m\|\Lambda_{\geq 3}[\partial_t g(t)]_{k_2,j_2}\|_{L^2}\big)\\
&\quad\times\|e^{-it\Lambda}g_{k_1,j_1}(t)\|_{L^\infty}\Big]\leq C2^{-m+40\beta m-k_2}\epsilon_0^2\leq C2^{-2\beta m}\epsilon_0^2.
\end{aligned}
$$

Now, we proceed to estimate $\widetilde{\Gamma}^{1,2;6}_{k,k_1,k_2}$. Recall (6.85) and (6.86). We split into three cases based on the difference between k_1' and k_2' and the size of $k_1'+k_2'$.

If $|k_1'-k_2'|\leq 10$, i.e., we are estimating $\widehat{\Gamma}^{k_1',k_2',3}_{k,k_1,k_2;1}$. – Note that we have $k_1'\geq k_1-5$. Recall (6.87). By doing integration by parts in "σ" many times, we can rule out the case when $\max\{j_1',j_2'\}\leq m+k_{1,-}-k_{1,+}'-\beta m$. If $\max\{j_1',j_2'\}\geq m+k_{1,-}-k_{1,+}'-\beta m$, from the $L^2-L^\infty-L^\infty$ type trilinear estimate (2.6) in Lemma 2.2, the following estimate holds for some absolute constant C,

$$
\begin{aligned}
&\Big|\sum_{\max\{j_1',j_2'\}\geq m+k_{1,-}-k_{1,+}'-\beta m} \widehat{\Gamma}^{k_1',j_1',k_2',j_2',3}_{k,k_1,k_2,j_2;1}\Big|\\
&\quad\leq \sup_{t\in[2^{m-1},2^m]} C2^{2m+2k+2k_1'}\|\Gamma^1\Gamma^2 g_k(t)\|_{L^2}\|e^{-it\Lambda}g_{k_2}(t)\|_{L^\infty}\\
&\qquad\times\Big(\sum_{j_1'\geq\max\{j_2',m+k_{1,-}-k_{1,+}'-\beta m\}} (2^{m+k_1+k_2'}+2^{k_2'+j_1'})\|g_{k_1',j_1'}\|_{L^2}\|e^{-it\Lambda}g_{k_2',j_2'}\|_{L^\infty}
\end{aligned}
$$

$$+\sum_{j_2'\geq \max\{j_1', m+k_{1,-}-k'_{1,+}-\beta m\}} (2^{m+k_1+k_2'}+2^{k_2'+j_1'})\|g_{k_2',j_2'}\|_{L^2}2^{-m}\|g_{k_1',j_1'}\|_{L^1})$$
$$\leq C2^{-m/2+20\beta m}\epsilon_0^2.$$

$\oplus$ If $k_2'\leq k_1'-10$ and $k_1'+k_2'\leq -19m/20$. For this case, we have $|k_1'-k_1|\leq 5$. Recall (6.85). From estimate (5.15) in Lemma 5.3, the following estimate holds for some absolute constant C,

$$|\widehat{\Gamma}^{k_1',k_2',3}_{k,k_1,k_2;2}|\leq \sup_{t\in[2^{m-1},2^m]} C2^{2m+3k_1}\|\Gamma^1\Gamma^2 g_k(t)\|_{L^2}\|e^{-it\Lambda}g_{k_2}(t)\|_{L^\infty}$$
$$\times\big((2^{m+k_2'+k_1}+1)\|g_{k_1'}\|_{L^2}+\sum_{i=1,2}2^{k_2'}\|\nabla_\xi\widehat{g}_{k_1'}(t,\xi)\|_{L^2}\big)$$
$$\times\big(2^{3k_2'}\|\widehat{g}_{k_2'}(t)\|_{L^\infty_\xi}+2^{k_1+2k_2'}\|\widehat{\mathrm{Re}[v]}(t,\xi)\psi_{k_2'}(\xi)\|_{L^\infty_\xi}\big)$$
$$\leq C2^{-2\beta m}\epsilon_0^2.$$

If $k_2'\leq k_1'-10$ *and* $k_1'+k_2'\geq -19m/20$. – Recall (6.88). By doing integration by parts in "σ" many times, we can rule out the case when $\max\{j_1',j_2'\}\leq m+k_{1,-}-\beta m$. By doing integration by parts in "η" many times, we can rule out the case when $\max\{j_1',j_2\}\geq m+k_{1,-}-\beta m$. Therefore, we only need to consider the case when $\max\{j_1',j_2'\}\geq m+k_{1,-}-\beta m$ and $\max\{j_1',j_2\}\geq m+k_{1,-}-\beta m$. In other words, either $j_1'\geq m+k_{1,-}-\beta m$ or $j_2',j_2\geq m+k_{1,-}-\beta m$. From the $L^2-L^\infty-L^\infty$ type trilinear estimate (2.6) in Lemma 2.2, the following estimate holds,

$$\Big|\sum_{\max\{j_1',j_2'\},\max\{j_1',j_2\}\geq m+k_{1,-}-\beta m}\widehat{\Gamma}^{k_1',j_1',k_2',j_2',3}_{k,k_1,k_2,j_2;2}\Big|$$
$$\leq \sup_{t\in[2^{m-1},2^m]} C2^{2m+4k}\Big(\sum_{j_1'\geq m+k_{1,-}-\beta m}(2^{m+k_1+k_2'}+2^{k_2'+j_1'})$$
$$\times\|g_{k_1',j_1'}\|_{L^2}\|e^{-it\Lambda}g_{k_2',j_2'}\|_{L^\infty}\|e^{-it\Lambda}g_{k_2,j_2}(t)\|_{L^\infty}$$
$$+\sum_{j_2',j_2\geq m+k_{1,-}-\beta m}(2^{m+k_1+k_2'}\|e^{-it\Lambda}g_{k_1',j_1'}\|_{L^\infty}+2^{k_2'}\|e^{-it\Lambda}\mathcal{F}^{-1}[\nabla_\xi\widehat{g_{k_1',j_1'}}]\|_{L^\infty})$$
$$\times\|g_{k_2',j_2'}\|_{L^2}2^{k_2}\|g_{k_2,j_2}(t)\|_{L^2}\Big)\|\Gamma^1\Gamma^2 g_k(t)\|_{L^2}$$
$$\leq C2^{-2\beta m}\epsilon_0^2.$$

Lastly, we estimate $\widetilde{\Gamma}^{1,2;7}_{k,k_1,k_2}$. Recall (6.89). After doing spatial localizations for inputs "Γg_{k_1}" and "g_{k_2}" inside $\widetilde{\Gamma}^{1,2;7}_{k,k_1,k_2}$, we have

(6.91) $$\widetilde{\Gamma}^{1,2;7}_{k,k_1,k_2}=\sum_{j_1\geq -k_{1,-},j_2\geq -k_{2,-}}\widetilde{\Gamma}^{1,2;7}_{k,k_1,j_1,k_2,j_2},$$

$$\widetilde{\Gamma}^{1,2;7}_{k,k_1,j_1,k_2,j_2}:=\sum_{k_2'\leq k_1'-10,|k_1-k_1'|\leq 10}\sum_{\nu'\in\{+,-\}}\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}e^{it\Phi^{+,\nu}(\xi,\eta)-it\Phi^{+,\nu'}(\xi,\kappa)}tr^{\nu,\nu'}_{k_1,k_1'}(\xi,\eta,\kappa)$$

(6.92) $$\times\widehat{\Gamma g_{k_1,j_1}}(t,\xi-\eta)\widehat{g^{\nu}_{k_2,j_2}}(t,\eta)\overline{\widehat{\Gamma^1\Gamma^2 g_{k_1'}}(t,\xi-\kappa)\widehat{g^{\nu'}_{k_2'}}(t,-\kappa)}d\eta d\kappa d\xi dt.$$

By doing integration by parts in "η" many times, we can rule out the case when $\max\{j_1, j_2\} \le m+k_{1,-}-\beta m$. If $\max\{j_1, j_2\} \ge m+k_{1,-}-\beta m$, from the $L^2-L^\infty-L^\infty$ type trilinear estimate (2.6) in Lemma 2.2 and (6.90), the following estimate holds for some absolute constant C,

$$\begin{aligned}
&\sum_{\max\{j_1,j_2\}\ge m+k_{1,-}-\beta m} |\widetilde{\Gamma}^{1,2;7}_{k,k_1,j_1,k_2,j_2}| \\
&\le \sup_{t\in[2^{m-1},2^m]} \sum_{k_2'\le k_1-10} C 2^{2m+\max\{k_2,k_2'\}+3k_1} \|e^{-it\Lambda} g_{k_2'}(t)\|_{L^\infty} \\
&\times \Big(\sum_{j_1\ge \max\{j_2, m+k_{1,-}-\beta m\}} \|\Gamma g_{k_1,j_1}(t)\|_{L^2} \|e^{-it\Lambda} g_{k_2,j_2}(t)\|_{L^\infty} \\
&+ \sum_{j_2\ge \max\{j_1, m+k_{1,-}-\beta m\}} \|e^{-it\Lambda}\Gamma g_{k_1,j_1}(t)\|_{L^\infty} \|g_{k_2,j_2}(t)\|_{L^2}\Big) \|\Gamma^1\Gamma^2 g_k(t)\|_{L^2} \\
&\le C 2^{-m/4+20\beta m}\epsilon_1^3 + C 2^{-m-k_2+20\beta m}\epsilon_0^2 \le C 2^{-2\beta m}\epsilon_0^2.
\end{aligned}$$

Hence finishing the proof. □

LEMMA 6.8. – *Under the bootstrap assumption* (4.49), *the following estimate holds for some absolute constant* C,

$$\sum_{k_1,k_2\in\mathbb{Z}, |k-k_1|\le 10, k_2\le k_1-10, k_1+k_2\ge -19m/20, k_1\le 5\beta m} |P^2_{k,k_1,k_2}| \le C 2^{2\tilde{\delta} m}\epsilon_0^2. \tag{6.93}$$

Proof. – Recall (6.8) and (6.12). Note that P^2_{k,k_1,k_2} vanishes except when $\Gamma^1 = \Gamma^2 = L$. Hence, we only have to consider the case when $\Gamma^1 = \Gamma^2 = L$. We decompose it into two parts as follows,

$$\begin{aligned}
P^2_{k,k_1,k_2} = \sum_{i=1,2} P^{2,i}_{k,k_1,k_2}, \quad P^{2,i}_{k,k_1,k_2} = -\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \overline{\widehat{LLg_k}(t,\xi)} e^{it\Phi^{+,\nu}(\xi,\eta)} t^2 \widehat{q}^{\,i}_{+,\nu}(\xi,\eta) \\
\times \widehat{g_{k_1}}(t,\xi-\eta)\widehat{g^{\nu}_{k_2}}(t,\eta)\, d\eta\, d\xi\, dt, \quad i\in\{1,2\},
\end{aligned}$$

where $\widehat{q}^{\,i}_{+,\nu}(\xi-\eta,\eta)$, $i\in\{1,2\}$, are defined (6.29) and (6.30). After doing integration by parts in "η" twice, from the estimate of the symbol $\widehat{q}^{\,2}_{+,\nu}(\xi-\eta,\eta)$ in (6.31), the following estimate holds,

$$\begin{aligned}
&\sum_{k_2\le k_1-10, |k_1-k|\le 10} |P^{2,2}_{k,k_1,,k_2}| \le \sup_{t\in[2^{m-1},2^m]} \sum_{k_2\le k_1-10, i=1,2} 2^{m+k_1+3k_2+k_{1,+}} \\
&\Big[\|e^{-it\Lambda} g_{k_2}(t)\|_{L^\infty}\big(\|\nabla^2_\xi \widehat{g}_{k_1}(t,\xi)\|_{L^2} + 2^{-k_2}\|\nabla_\xi \widehat{g}_{k_1}(t,\xi)\|_{L^2}\big) \\
&+ \|e^{-it\Lambda} g_{k_1}(t)\|_{L^\infty}\big(\|\nabla^2_\xi \widehat{g}_{k_2}(t,\xi)\|_{L^2} + 2^{-k_2}\|\nabla_\xi \widehat{g}_{k_2}(t,\xi)\|_{L^2} + 2^{-2k_2}\|g_{k_2}(t)\|_{L^2}\big) \\
&+ \sum_{j_1\ge j_2} 2^{-m+2j_2+j_1} \|\varphi^{k_2}_{j_2}(x) g_{k_2}(t)\|_{L^2} \|\varphi^{k_1}_{j_1}(x) g_{k_1}(t)\|_{L^2} \\
&+ \sum_{j_2\ge j_1} 2^{-m+2j_1+j_2} \|\varphi^{k_1}_{j_1}(x) g_{k_1}(t)\|_{L^2} \|\varphi^{k_2}_{j_2}(x) g_{k_2}(t)\|_{L^2}\Big] \|\Gamma^1\Gamma^2 g_k(t)\|_{L^2} \\
&\le C 2^{2\tilde{\delta} m}\epsilon_0^2.
\end{aligned}$$

For "$P^{2,1}_{k,k_1,k_2}$," we do integration by parts in time once. As a result, we have

$$P^{2,1}_{k,k_1,k_2} = \sum_{i=1,2,3,4,5} \widetilde{P}^i_{k,k_1,k_2}, \quad \widetilde{P}^1_{k,k_1,k_2} = \sum_{j_1\geq -k_{1,-},\, j_2\geq -k_{2,-}} \widetilde{P}^{j_1,j_2,1}_{k,k_1,k_2}, \tag{6.94}$$

$$\begin{aligned}\widetilde{P}^{j_1,j_2,1}_{k,k_1,k_2} &= \sum_{i=1,2}(-1)^i \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \overline{\widehat{\Gamma^1\Gamma^2 g_k}(t_i,\xi)}\, e^{it_i\Phi^{+,\nu}(\xi,\eta)}\, i t_i^2\, \widehat{p}^1_{+,\nu}(\xi,\eta) \\ &\quad \times \widehat{g_{k_1,j_1}}(t_i,\xi-\eta)\widehat{g^{\nu}_{k_2,j_2}}(t_i,\eta)\, d\eta\, d\xi \\ &\quad - \int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \overline{\widehat{\Gamma^1\Gamma^2 g_k}(t,\xi)}\, e^{it\Phi^{+,\nu}(\xi,\eta)}\, i 2t\, \widehat{p}^1_{+,\nu}(\xi,\eta) \\ &\quad \times \widehat{g_{k_1,j_1}}(t,\xi-\eta)\widehat{g^{\nu}_{k_2,j_2}}(t,\eta) - e^{it\Phi^{+,\nu}(\xi,\eta)}\, i t^2\, \widehat{p}^1_{+,\nu}(\xi,\eta)\widehat{g_{k_1,j_1}}(t,\xi-\eta) \\ &\quad \times \widehat{g^{\nu}_{k_2,j_2}}(t,\eta)\overline{\Big(\partial_t \widehat{\Gamma^1\Gamma^2 g_k}(t,\xi) - \sum_{\nu\in\{+,-\}}\sum_{(k_1',k_2')\in\chi_k^2} \widetilde{B}^{+,\nu}_{k,k_1',k_2'}(t,\xi)\Big)}\, d\eta\, d\xi\, dt,\end{aligned} \tag{6.95}$$

$$\widetilde{P}^2_{k,k_1,k_2} = \sum_{k_2'\leq k_1'+10} \widehat{P}^{2,k_1',k_2'}_{k,k_1,k_2}, \quad \widehat{P}^{2,k_1',k_2'}_{k,k_1,k_2} = \sum_{j_1\geq -k_{1,-},\, j_1'\geq -k'_{1,-},\, j_2'\geq -k'_{2,-}} \widehat{P}^{2,k_1',j_1',k_2',j_2'}_{k,k_1,j_1,k_2}, \tag{6.96}$$

$$\begin{aligned}\widehat{P}^{2,k_1',j_1',k_2',j_2'}_{k,k_1,j_1,k_2} &:= \sum_{\mu',\nu'\in\{+,-\}} -\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \overline{\widehat{\Gamma^1\Gamma^2 g_k}(t,\xi)}\, e^{it\Phi^{+,\nu}(\xi,\eta)}\, i t^2\, \widehat{p}^1_{+,\nu}(\xi,\eta)\widehat{g_{k_1,j_1}}(t,\xi-\eta) \\ &\quad \times P_\nu\big[e^{it\Phi^{\mu',\nu'}(\eta,\sigma)}\tilde{q}_{\mu',\nu'}(\eta-\sigma,\sigma)\widehat{g^{\mu'}_{k_1',j_1'}}(t,\eta-\sigma)\widehat{g^{\nu'}_{k_2',j_2'}}(t,\sigma)\big]\, d\eta\, d\xi\, dt,\end{aligned} \tag{6.97}$$

$$\widetilde{P}^3_{k,k_1,k_2} = \sum_{|k_1'-k_2'|\leq 10} \widehat{P}^{3,k_1',k_2'}_{k,k_1,k_2}, \quad \widehat{P}^{3,k_1',k_2'}_{k,k_1,k_2} = \sum_{j_1'\geq -k'_{1,-},\, j_2'\geq -k'_{2,-}} \widehat{P}^{3,k_1',j_1',k_2',j_2'}_{k,k_1,k_2}, \tag{6.98}$$

$$\begin{aligned}\widehat{P}^{3,k_1',j_1',k_2',j_2'}_{k,k_1,k_2} &= \sum_{\mu',\nu'\in\{+,-\}} -\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \overline{\widehat{\Gamma^1\Gamma^2 g_k}(t,\xi)}\, e^{it\Phi^{+,\nu}(\xi,\eta)}\, i t^2\, \widehat{p}^1_{+,\nu}(\xi,\eta)\, e^{it\Phi^{\mu',\nu'}(\xi-\eta,\sigma)} \\ &\quad \times \tilde{q}_{\mu',\nu'}(\xi-\eta-\sigma,\sigma)\widehat{g^{\mu'}_{k_1',j_1'}}(t,\xi-\eta-\sigma)\widehat{g^{\nu'}_{k_2',j_2'}}(t,\sigma)\widehat{g^{\nu}_{k_2}}(t,\eta)\, d\eta\, d\xi\, dt.\end{aligned} \tag{6.99}$$

$$\begin{aligned}\widetilde{P}^4_{k,k_1,k_2} &= \sum_{j_1\geq -k_{1,-},\, j_2\geq -k_{2,-}} \widetilde{P}^4_{k,k_1,j_1,k_2,j_2}, \\ \widetilde{P}^4_{k,k_1,j_1,k_2,j_2} &:= -\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{it\Phi^{+,\nu}(\xi,\eta)}\, i t^2\, \widehat{p}^1_{+,\nu}(\xi,\eta)\overline{\widehat{\Gamma^1\Gamma^2 g_k}(t,\xi)} \\ &\quad \times \Big(\widehat{\Lambda_{\geq 3}[\partial_t g]_{k_1,j_1}}(t,\xi-\eta)\widehat{g^{\nu}_{k_2,j_2}}(t,\eta) + \widehat{g_{k_1,j_1}}(t,\xi-\eta)\widehat{\Lambda_{\geq 3}[\partial_t g^{\nu}]_{k_2,j_2}}(t,\eta)\Big)\, d\eta\, d\xi\, dt,\end{aligned} \tag{6.100}$$

$$\begin{aligned}\widetilde{P}^5_{k,k_1,k_2} &= \sum_{k_2'\leq k_1'-10,\, |k_1-k_1'|\leq 10}\ \sum_{\nu'\in\{+,-\}} -\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{it\Phi^{+,\nu}(\xi,\eta)}\, i t^2\, \widehat{p}^1_{+,\nu}(\xi,\eta) \\ &\quad \Big[\widehat{g_{k_1}}(t,\xi-\eta)\widehat{g^{\nu}_{k_2}}(t,\eta)\overline{e^{it\Phi^{+,\nu'}(\xi,\kappa)}\widehat{\Gamma^1\Gamma^2 g_{k_1'}}(t,\xi-\kappa)\widehat{g^{\nu'}_{k_2'}}(t,\kappa)\tilde{q}_{+,\nu'}(\xi-\kappa,\kappa)}\end{aligned} \tag{6.101}$$

$$+\overline{\widehat{\Gamma^1\Gamma^2 g_k}(t,\xi)}\widehat{g^{\nu}_{k_2}}(t,\eta)e^{it\Phi^{+,\nu'}(\xi-\eta,\kappa)}\tilde{q}_{+,\nu'}(\xi-\eta-\kappa,\kappa)$$

$$\times\widehat{g_{k_1'}}(t,\xi-\eta-\kappa)\overline{\widehat{g^{\nu'}_{k_2'}}(t,\kappa)}\Big]d\eta d\kappa d\xi dt. \tag{6.102}$$

$$=\sum_{k_2'\leq k_1'-10,|k_1-k_1'|\leq 10}\sum_{\nu'\in\{+,-\}}-\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}e^{it\Phi^{+,\nu}(\xi,\eta)-it\Phi^{+,\nu'}(\xi,\kappa)}$$

$$\times it^2\widetilde{r}^{\nu,\nu'}_{k_1,k_1'}(\xi,\eta,\kappa)\widehat{g}(t,\xi-\eta)\widehat{g^{\nu}_{k_2}}(t,\eta)\overline{\widehat{\Gamma^1\Gamma^2 g}(t,\xi-\kappa)}\widehat{g^{\nu'}_{k_2'}}(t,-\kappa)d\eta d\kappa d\xi dt,$$

where the symbol "$\widehat{p}^1_{\mu,\nu}(\xi,\eta)$" is defined in (6.29) and the symbol $\widetilde{r}^{\nu,\nu'}_{k_1,k_1'}(\xi,\eta,\kappa)$ is defined as follows,

$$\widetilde{r}^{\nu,\nu'}_{k_1,k_1'}(\xi,\eta,\kappa)=\widehat{p}^1_{+,\nu}(\xi,\eta)\overline{\tilde{q}_{+,-\nu'}(\xi-\kappa,\kappa)}\psi_{k_1'}(\xi-\kappa)\psi_{k_1}(\xi-\eta)\psi_k(\xi)$$
$$+\widehat{p}^1_{+,\nu}(\xi-\kappa,\eta)\tilde{q}_{+,\nu'}(\xi-\eta,-\kappa)\psi_{k_1'}(\xi-\eta)\psi_{k_1}(\xi-\eta-\kappa)\psi_k(\xi-\kappa).$$

Recall (6.29), (4.14) and (4.15). From the estimate (2.3) in Lemma 2.1, the following estimate holds,

$$\|\widetilde{r}^{\nu,\nu'}_{k_1,k_1'}(\xi,\eta,\kappa)\psi_{k_2}(\eta)\psi_{k_2'}(\kappa)\|_{\mathcal{S}^\infty}\leq C2^{\max\{k_2,k_2'\}+k_2+4k_1}, \tag{6.103}$$

where C is some absolute constant. After doing spatial localizations for inputs $\widehat{g}_{k_1}(t)$ and $\widehat{g}_{k_2}(t)$ in $\widetilde{P}^5_{k,k_1,k_2}$, the following decomposition holds,

(6.104)

$$\widetilde{P}^5_{k,k_1,k_2}=\sum_{k_2'\leq k_1'-10,|k_1-k_1'|\leq 10}\widehat{P}^{5,k_1',k_2'}_{k,k_1,k_2},\quad \widehat{P}^{5,k_1',k_2'}_{k,k_1,k_2}=\sum_{j_1\geq -k_{1,-},j_2\geq -k_{2,-}}\widehat{P}^{5,k_1',k_2'}_{k,k_1,j_1,k_2,j_2},$$

$$\widehat{P}^{5,k_1',k_2'}_{k,k_1,j_1,k_2,j_2}=\sum_{\nu'\in\{+,-\}}-\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}e^{it\Phi^{+,\nu}(\xi,\eta)-it\Phi^{+,\nu'}(\xi,\kappa)}it^2\widetilde{r}^{\nu,\nu'}_{k_1,k_1'}(\xi,\eta,\kappa)\widehat{g_{k_1,j_1}}(t,\xi-\eta)$$

$$\times\widehat{g^{\nu}_{k_2,j_2}}(t,\eta)\overline{\widehat{\Gamma^1\Gamma^2 g_{k_1'}}(t,\xi-\kappa)}\widehat{g^{\nu'}_{k_2'}}(t,-\kappa)d\eta d\kappa d\xi dt. \tag{6.105}$$

With the above preparation, now we are ready to estimate $\tilde{P}^i_{k,k_1,k_2}$, $i\in\{1,\ldots,5\}$, one by one.

We first estimate $\tilde{P}^1_{k,k_1,k_2}$. Recall (6.94) and (6.95). By doing integration by parts in "η" many times, we can rule out the case when $\max\{j_1,j_2\}\leq m+k_{1,-}-\beta m$. If $\max\{j_1,j_2\}\geq m+k_{1,-}-\beta m$, from the L^2-L^∞ type bilinear estimate (2.5) in Lemma 2.2, (7.7) in Lemma 7.2,, the following estimate holds for some absolute constant C,

$$\sum_{\max\{j_1,j_2\}\geq m+k_{1,-}-\beta m}|\widetilde{P}^{j_1,j_2,1}_{k,k_1,k_2}|$$
$$\leq\sup_{t\in[2^{m-1},2^m]}C2^{2m+2\tilde{\delta}m+k_2+3k_1+6k_+}\Big(\sum_{j_1\geq\max\{j_2,m+k_{1,-}-\beta m\}}\|g_{k_1,j_1}(t)\|_{L^2}$$
$$\times\|e^{-it\Lambda}g_{k_2,j_2}(t)\|_{L^\infty}+\sum_{j_2\geq\max\{j_1,m+k_{1,-}-\beta m\}}\|g_{k_2,j_2}(t)\|_{L^2}\|e^{-it\Lambda}g_{k_1,j_1}(t)\|_{L^\infty}\Big)$$
$$\leq C2^{-2\beta m}\epsilon_0^2.$$

Now we proceed to estimate $\tilde{P}^2_{k,k_1,k_2}$. Recall (6.96) and (6.97). Based on the size of the difference between k_1' and k_1, we split into two cases as follows,

If $k_1' \geq k_1 - 5$. – For this case, we have $k_1' \geq k_2 + 5$ and $|k_1' - k_2'| \leq 5$. By doing integration by parts in "σ," we can rule out the case when $\max\{j_1', j_2'\} \leq m + k_{2,-} - k'_{1,+} - \beta m$. If $\max\{j_1', j_2'\} \geq m + k_{2,-} - k'_{1,+} - \beta m$, from the $L^2 - L^\infty - L^\infty$ type trilinear estimate (2.6) in Lemma 2.2, the following estimate holds for some constant C,

$$\begin{aligned}
&\sum_{k_1' \geq k_1 - 5} \Big| \sum_{\max\{j_1', j_2'\} \geq m + k_{2,-} - k'_{1,+} - \beta m} \widehat{P}^{2,k_1',j_1',k_2',j_2'}_{k,k_1,j_1,k_2} \Big| \\
&\quad \leq \sup_{t \in [2^{m-1}, 2^m]} \sum_{|k_1' - k_2'| \leq 5} C 2^{3m + k_2 + 3k_1 + 2k_1'} \\
&\qquad \times \Big(\sum_{j_1' \geq \max\{j_2', m + k_{2,-} - k'_{1,+} - \beta m\}} \| g_{k_1', j_1'}(t) \|_{L^2} \| e^{-it\Lambda} g_{k_2', j_2'} \|_{L^\infty} \\
&\qquad + \sum_{j_2' \geq \max\{j_1', m + k_{2,-} - k'_{1,+} - \beta m\}} \| g_{k_2', j_2'}(t) \|_{L^2} \| e^{-it\Lambda} g_{k_1', j_1'} \|_{L^\infty} \Big) \\
&\qquad \times \| \Gamma^1 \Gamma^2 g_k(t) \|_{L^2} \| e^{-it\Lambda} g_{k_1}(t) \|_{L^\infty} \leq C 2^{-m - k_2 + 30\beta m} \epsilon_0^2 \leq C 2^{-2\beta m} \epsilon_0^2.
\end{aligned}$$

If $k_1' \leq k_1 - 5$. – For this case, we do integration by parts in "η" many times to rule out the case when $\max\{j_1', j_1\} \leq m + k_{1,-} - \beta m$. If $\max\{j_1', j_1\} \geq m + k_{1,-} - \beta m$, from the $L^2 - L^\infty - L^\infty$ type trilinear estimate (2.6) in Lemma 2.2, the following estimate holds for some absolute constant C,

$$\begin{aligned}
&\sum_{k_2' \leq k_1' \leq k_1 - 5} \Big| \sum_{\max\{j_1', j_1\} \geq m + k_{1,-} - \beta m} \widehat{P}^{2,k_1',j_1',k_2',j_2'}_{k,k_1,j_1,k_2} \Big| \\
&\quad \leq \sup_{t \in [2^{m-1}, 2^m]} \sum_{k_2' \leq k_1' \leq k_1 - 5} C 2^{3m + k_2 + 3k_1 + 2k_1'} \\
&\qquad \times \Big(\sum_{j_1' \geq \max\{j_1, m + k_{1,-} - \beta m\}} \| g_{k_1', j_1'}(t) \|_{L^2} \| e^{-it\Lambda} g_{k_1, j_1} \|_{L^\infty} \\
&\qquad + \sum_{j_1 \geq \max\{j_1', m + k_{1,-} - \beta m\}} \| g_{k_1, j_1}(t) \|_{L^2} \\
&\qquad \times \| e^{-it\Lambda} g_{k_1', j_1'} \|_{L^\infty} \Big) \| e^{-it\Lambda} g_{k_2'}(t) \|_{L^\infty} \| \Gamma^1 \Gamma^2 g_k(t) \|_{L^2} \\
&\quad \leq C 2^{-m/2 + 30\beta m} \epsilon_0^2.
\end{aligned}$$

Now, we proceed to estimate $\tilde{P}^3_{k,k_1,k_2}$. Recall (6.98) and (6.99). Note that $|k_1' - k_2'| \leq 10$ and "$\nabla_\sigma \Phi^{\mu',\nu'}(\xi - \eta, \sigma)$" always has a lower bound, which is $2^{k_1 - k'_{1,+}}$. By doing integration by parts in "σ" many times, we can rule out the case when $\max\{j_1', j_2'\} \leq m + k_{1,-} - k'_{1,+} - \beta m$. If $\max\{j_1', j_2'\} \geq m + k_{1,-} - k'_{1,+} - \beta m$, from the $L^2 - L^\infty - L^\infty$ type trilinear estimate (2.6) in Lemma 2.2, the following estimate holds for some constant C,

$$\sum_{|k_1' - k_2'| \leq 10} \sum_{\max\{j_1', j_2'\} \geq m + k_{1,-} - k'_{1,+} - \beta m} |\widehat{P}^{3,k_1',j_1',k_2',j_2'}_{k,k_1,k_2}|$$

$$
\begin{aligned}
&\leq \sup_{t\in[2^{m-1},2^m]} \sum_{|k_1'-k_2'|\leq 10} C 2^{3m+k_2+3k_1+2k_1'} \\
&\quad \times \Big(\sum_{j_1'\geq \max\{j_2', m+k_{1,-}-k_{1,+}'-\beta m\}} \|g_{k_1',j_1'}(t)\|_{L^2} \|e^{-it\Lambda} g_{k_2',j_2'}\|_{L^\infty} \\
&\qquad + \sum_{j_2'\geq \max\{j_1', m+k_{1,-}-k_{1,+}'-\beta m\}} \|g_{k_2',j_2'}(t)\|_{L^2} \\
&\quad \times \|e^{-it\Lambda} g_{k_1',j_1'}\|_{L^\infty}\Big) \|e^{-it\Lambda} g_{k_2}(t)\|_{L^\infty} \|\Gamma^1\Gamma^2 g_k(t)\|_{L^2} \\
&\leq C 2^{-m/2+30\beta m}\epsilon_0^2.
\end{aligned}
$$

Now, we proceed to estimate $\tilde{P}^4_{k,k_1,k_2}$. Recall (6.100). By doing integration by parts in "η" many times, we can rule out the case when $\max\{j_1, j_2\} \leq m + k_{1,-} - \beta m$. If $\max\{j_1, j_2\} \geq m+k_{1,-}-\beta m$, from the L^2-L^∞ type bilinear estimate (2.5) in Lemma 2.2, estimate (6.137) in Lemma 6.14, and estimate (7.3) in Lemma 7.1, the following estimate holds for some absolute constant C,

$$
\begin{aligned}
&\sum_{\max\{j_1,j_2\}\geq m+k_{1,-}-\beta m} \|\widetilde{P}^4_{k,k_1,j_1,k_2,j_2}\|_{L^2} \\
&\qquad \leq \sup_{t\in[2^{m-1},2^m]} C 2^{3m+k_2+3k_1} \|\Gamma^1\Gamma^2 g_k(t)\|_{L^2} \\
&\qquad\quad \times \Big[\sum_{j_1\geq \max\{j_2, m+k_{1,-}-\beta m\}} \|\Lambda_{\geq 3}[\partial_t g^\mu]_{k_1,j_1}\|_{L^2} \|e^{-it\Lambda} g_{k_2,j_2}\|_{L^\infty} \\
&\qquad\quad + \|g_{k_1,j_1}\|_{L^2} 2^{k_2} \|\Lambda_{\geq 3}[\partial_t g_{k_2}]\|_{L^2} \\
&\qquad\quad + \sum_{j_2\geq \max\{j_1, m+k_{1,-}-\beta m\}} \|\Lambda_{\geq 3}[\partial_t g^\mu]_{k_2,j_2}\|_{L^2} \|e^{-it\Lambda} g_{k_1,j_1}\|_{L^\infty} \\
&\qquad\quad + 2^{k_2} \|g_{k_2,j_2}(t)\|_{L^2} \|\Lambda_{\geq 3}[\partial_t g^\mu_{k_1}]\|_{L^2}\Big] \\
&\qquad \leq C 2^{-m-k_2+40\beta m}\epsilon_0^2 + C 2^{-m/2+40\beta m}\epsilon_0^2 \leq C 2^{-2\beta m}\epsilon_0^2.
\end{aligned}
$$

Lastly, we estimate $\tilde{P}^5_{k,k_1,k_2}$. Recall (6.104) and (6.105). For the case we are considering, we have $k_2' \leq k_1' - 10$ and $|k_1' - k_1| \leq 10$. By doing integration by parts in "η" many times, we can rule out the case when $\max\{j_1, j_2\} \leq m + k_{1,-} - \beta m$. If $\max\{j_1, j_2\} \geq m + k_{1,-} - \beta m$, from the $L^2 - L^\infty - L^\infty$ type trilinear estimate (2.6) in Lemma 2.2 and estimate (6.103), the following estimate holds for some absolute constant C,

$$
\begin{aligned}
&\sum_{k_2'\leq k_1-10} \sum_{\max\{j_1,j_2\}\geq m+k_{1,-}-\beta m} |\widehat{P}^{5,k_1',k_2'}_{k,k_1,j_1,k_2,j_2}| \\
&\quad \leq \sup_{t\in[2^{m-1},2^m]} \sum_{k_2'\leq k_1-10} C 2^{3m+k_2+\max\{k_2,k_2'\}+4k_1} \\
&\quad \times \|\Gamma^1\Gamma^2 g_k(t)\|_{L^2} \Big(\sum_{j_1\geq \max\{j_2, m+k_{1,-}-\beta m\}} \|g_{k_1,j_1}\|_{L^2} \|e^{-it\Lambda} g_{k_2,j_2}\|_{L^\infty} \\
&\qquad + \sum_{j_2\geq \max\{j_1, m+k_{1,-}-\beta m\}} \|g_{k_2,j_2}\|_{L^2} \|e^{-it\Lambda} g_{k_1,j_1}\|_{L^\infty}\Big) \|e^{-it\Lambda} g_{k_2'}(t)\|_{L^\infty}
\end{aligned}
$$

$$\leq C2^{-m/2+30\beta m}\epsilon_0^2 + C2^{-m-k_2+30\beta m}\epsilon_0^2 \leq C2^{-2\beta m}\epsilon_0^2.$$

Hence finishing the proof. □

6.4. The Z_2 norm estimate of cubic terms

Recall (4.35) and (4.37). Note that we have $k_3 \leq k_2 \leq k_1$ for the case we are considering. For any $\Gamma_\xi^1, \Gamma_\xi^2 \in \{\hat{L}_\xi, \hat{\Omega}_\xi\}$, we have

$$\Gamma_\xi^1\Gamma_\xi^2\Lambda_3[\partial_t\widehat{g}(t,\xi)]\psi_k(\xi) = \sum_{\tau,\kappa,\iota\in\{+,-\}}\sum_{k_3\leq k_2+1\leq k_1+2}\sum_{i=1,2,3,4} T^{\tau,\kappa,\iota,i}_{k,k_1,k_2,k_3}(t,\xi),$$

$$T^{\tau,\kappa,\iota,i}_{k,k_1,k_2,k_3}(t,\xi) = \sum_{j_1\geq -k_{1,-}, j_2\geq -k_{2,-}, j_3\geq -k_{3,-}} T^{\tau,\kappa,\iota,i}_{k,k_1,j_1,k_2,j_2,k_3,j_3}(t,\xi), \quad i\in\{3,4\},$$

where

$$T^{\tau,\kappa,\iota,1}_{k,k_1,k_2,k_3}(t,\xi) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{it\Phi^{\tau,\kappa,\iota}(\xi,\eta,\sigma)}\tilde{d}_{\tau,\kappa,\iota}(\xi-\eta,\eta-\sigma,\sigma)\Gamma_\xi^1\Gamma_\xi^2\widehat{g^\tau_{k_1}}(t,\xi-\eta)$$
$$\times\widehat{g^\kappa_{k_2}}(t,\eta-\sigma)\widehat{g^\iota_{k_3}}(t,\sigma)\psi_k(\xi)d\eta d\sigma, \tag{6.106}$$

$$T^{\tau,\kappa,\iota,2}_{k,k_1,k_2,k_3}(t,\xi) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{it\Phi^{\tau,\kappa,\iota}(\xi,\eta,\sigma)}\big[\Gamma_\xi^1\Gamma_\xi^2\big(\tilde{d}_{\tau,\kappa,\iota}(\xi-\eta,\eta-\sigma,\sigma)\big)\widehat{g^\tau_{k_1}}(t,\xi-\eta)$$
$$+\sum_{\{l,n\}=\{1,2\}}\Gamma_\xi^l\tilde{d}_{\tau,\kappa,\iota}(\xi-\eta,\eta-\sigma,\sigma)\Gamma_\xi^n\widehat{g^\tau_{k_1}}(t,\xi-\eta)\big] \tag{6.107}$$
$$\times\widehat{g^\kappa_{k_2}}(t,\eta-\sigma)\widehat{g^\iota_{k_3}}(t,\sigma)\psi_k(\xi)d\eta d\sigma,$$

$$T^{\tau,\kappa,\iota,3}_{k,k_1,j_1,k_2,j_2,k_3,j_3}(t,\xi) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{it\Phi^{\tau,\kappa,\iota}(\xi,\eta,\sigma)}e^{it\Phi^{\tau,\kappa,\iota}(\xi,\eta,\sigma)}it\big(\Gamma_\xi^l\Phi^{\tau,\kappa,\iota}(\xi,\eta,\sigma)\big)$$
$$\times\Gamma_\xi^n\big(\tilde{d}_{\tau,\kappa,\iota}(\xi-\eta,\eta-\sigma,\sigma)\widehat{g^\tau_{k_1,j_1}}(t,\xi-\eta)\big)\widehat{g^\kappa_{k_2,j_2}}(t,\eta-\sigma) \tag{6.108}$$
$$\times\widehat{g^\iota_{k_3,j_3}}(t,\sigma)\psi_k(\xi)d\eta d\sigma,$$

$$T^{\tau,\kappa,\iota,4}_{k,k_1,j_1,k_2,j_2,k_3,j_3}(t,\xi) = -\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{it\Phi^{\tau,\kappa,\iota}(\xi,\eta,\sigma)}t^2\big(\Gamma_\xi^1\Phi^{\tau,\kappa,\iota}(\xi,\eta,\sigma)\Gamma_\xi^2\Phi^{\tau,\kappa,\iota}(\xi,\eta,\sigma)\big)$$
$$\times\tilde{d}_{\tau,\kappa,\iota}(\xi-\eta,\eta-\sigma,\sigma)\widehat{g^\tau_{k_1,j_1}}(t,\xi-\eta)\widehat{g^\kappa_{k_2,j_2}}(t,\eta-\sigma) \tag{6.109}$$
$$\times\widehat{g^\iota_{k_3,j_3}}(t,\sigma)\psi_k(\xi)d\eta d\sigma.$$

Therefore, we have

$$\mathrm{Re}\Big[\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\overline{\Gamma_\xi^1\Gamma_\xi^2\widehat{g}(t,\xi)}\Gamma_\xi^1\Gamma_\xi^2\Lambda_3[\partial_t\widehat{g}(t,\xi)]\psi_k(\xi)d\xi dt\Big] \tag{6.110}$$
$$= \sum_{\tau,\kappa,\iota\in\{+,-\}}\sum_{k_3\leq k_2\leq k_1}\sum_{i=1,2,3,4}\mathrm{Re}\big[T^{\tau,\kappa,\iota,i}_{k,k_1,k_2,k_3}\big],$$

$$T^{\tau,\kappa,\iota,i}_{k,k_1,k_2,k_3} = \int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\overline{\Gamma_\xi^1\Gamma_\xi^2\widehat{g}(t,\xi)}T^{\tau,\kappa,\iota,i}_{k,k_1,k_2,k_3}(t,\xi)d\xi dt. \tag{6.111}$$

The main goal of this subsection is to prove the following proposition,

PROPOSITION 6.9. – *Under the bootstrap assumption* (4.49)*, the following estimates hold for some absolute constant* C*,*

(6.112)
$$\sup_{t_1,t_2\in[2^{m-1},2^m]}\Big|\sum_{k\in\mathbb{Z}}\mathrm{Re}\Big[\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\overline{\Gamma^1_\xi\Gamma^2_\xi\widehat{g}(t,\xi)}\Gamma^1_\xi\Gamma^2_\xi\Lambda_3[\partial_t\widehat{g}(t,\xi)]\psi_k(\xi)d\xi dt\Big]\Big|\leq C2^{2\tilde{\delta}m}\epsilon_0^2,$$

(6.113)
$$\sup_{t\in[2^{m-1},2^m]}\sum_{i=1,2,3,4}\sum_{k\in\mathbb{Z}}\sum_{k_3\leq k_2\leq k_1}\|T^{\tau,\kappa,\iota,i}_{k,k_1,k_2,k_3}(t,\xi)\|_{L^2}\leq C2^{-m+\tilde{\delta}m}\big(1+2^{2\tilde{\delta}m+k+5k_+}\big)\epsilon_0.$$

Proof. – To simplify the problem, we first rule out the very high frequency case and the very low frequency case. Very similar to what we did in the estimate of quadratic terms (see (6.55)), we do integration by parts in η to move the derivatives ∇_η of $\nabla_\xi\widehat{g}_{k_1}(t,\xi-\eta)=-\nabla_\eta\widehat{g}_{k_1}(t,\xi-\eta)$ around such that there is no derivatives in front of $\widehat{g}_{k_1}(t,\xi-\eta)$. As a result, from the $L^2-L^\infty-L^\infty$ type trilinear estimate (2.6) in Lemma 2.2 and the $L^\infty\to L^2$ type Sobolev embedding, the following estimate holds for some absolute constant C,

$$\sum_{i=1,2,3,4}\|T^{\tau,\kappa,\iota,i}_{k,k_1,k_2,k_3}(t,\xi)\|_{L^2}\leq C2^{2m+2k_1+6k_{1,+}}\|g_{k_1}(t)\|_{L^2}2^{k_2+k_3}\|g_{k_3}(t)\|_{L^2}\big(2^{-2k_2}\|g_{k_2}(t)\|_{L^2}$$
$$+2^{-k_2}\|\nabla_\xi\widehat{g}_{k_2}(t)\|_{L^2}+\|\nabla^2_\xi\widehat{g}_{k_2}(t)\|_{L^2}\big)\leq C2^{2m+\beta m-(N_0-20)k_{1,+}}\epsilon_0. \tag{6.114}$$

From the above estimate, we can rule out the case when $k_1\geq 4\beta m$. It remains to consider the case when $k_1\leq 4\beta m$. Next, we proceed to rule out the very low frequency case. If either $k\leq -2m$ or $k_3\leq -3m-30\beta m$, then from the $L^2-L^\infty-L^\infty$ type trilinear estimate, the following estimate holds for some absolute constant C,

$$\sum_{i=1,2,3,4}\|T^{\tau,\kappa,\iota,i}_{k,k_1,k_2,k_3}(t,\xi)\|_{L^2}\leq C(1+2^{2m+2k})2^{k+k_3+4k_{1,+}}\big(2^{2k_1}\|\nabla^2_\xi\widehat{g}_{k_1}(t,\xi)\|_{L^2}$$
$$+2^{k_1}\|\nabla_\xi\widehat{g}_{k_1}(t,\xi)\|_{L^2}+\|g_{k_1}(t)\|_{L^2}\big)\|g_{k_2}(t)\|_{L^2}\|g_{k_3}(t)\|_{L^2}\leq C2^{-m-\beta m}\epsilon_0. \tag{6.115}$$

Therefore, from now on, we restrict ourself to the case when k,k_1,k_2, and k_3 are in the range listed as follows,

$$-3m-30\beta m\leq k_3\leq k_2\leq k_1\leq 4\beta m,\quad -2m\leq k\leq 4\beta m. \tag{6.116}$$

Recall (6.107). From the $L^2-L^\infty-L^\infty$ type trilinear estimate (2.6) in Lemma 2.2, the following estimate holds for some absolute constant C,

$$\|T^{\tau,\kappa,\iota,2}_{k,k_1,k_2,k_3}(t,\xi)\|_{L^2}\leq C2^{2k_1+4k_{1,+}}\big(\|e^{-it\Lambda}g_{k_1}\|_{L^\infty}+\sum_{i=1,2}\|e^{-it\Lambda}\Gamma^i g_{k_1}\|_{L^\infty}\big)\|e^{-it\Lambda}g_{k_2}\|_{L^\infty}$$
$$\times\|g_{k_3}\|_{L^2}\leq C2^{-3m/2+50\beta m}\epsilon_0^2\Longrightarrow|T^{\tau,\kappa,\iota,2}_{k,k_1,k_2,k_3}|\leq C2^{-m/2+50\beta m}\epsilon_0^2.$$

Since there are only at most "m^4 "cases in the range (6.116), to prove (6.112) and (6.113), it would be sufficient to prove the following estimate for any $i=1,3,4$, any $\tau,\kappa,\iota\in\{+,-\}$, and any fixed k,k_1,k_2,k_3 in the range (6.116),

(6.117)
$$|\mathrm{Re}\big[T^{\tau,\kappa,\iota,i}_{k,k_1,k_2,k_3}\big]|\leq C2^{3\tilde{\delta}m/2}\epsilon_0^2,\quad \|T^{\tau,\kappa,\iota,i}_{k,k_1,k_2,k_3}(t,\xi)\|_{L^2}\leq C2^{-m+\tilde{\delta}m/2}\big(1+2^{2\tilde{\delta}m+k+5k_+}\big)\epsilon_0,$$

where C is some absolute constant.

From the results in the next three lemmas, i.e., Lemma 6.10, Lemma 6.11, and Lemma 6.12, we know that our desired estimates in (6.117) indeed holds for fixed k, k_1, k_2, k_3 in the range listed in (6.116). Hence finishing the proof. □

Lemma 6.10. – *For* $i = 1, 3, 4$ *and fixed* k, k_1, k_2, k_3 *in the range* (6.116)*, our desired estimates listed in* (6.117) *hold if moreover* $k_2 \le k_1 - 10$.

Proof. – Recall the normal form transformation that we did in Subsection 4.1, see (4.30) and (4.40). For the case we are considering, which is $k_2 \le k_1 - 10$, we have "$\tau = +$" and the fact that the estimate $|\nabla_\xi \Phi^{+,\kappa,\iota}(\xi,\eta,\sigma)| \le 2^{k_2}$ holds for some absolute constant C.

We first estimate $T^{+,\kappa,\iota,1}_{k_1,k_2,k_3}$ and $T^{+,\kappa,\iota,1}_{k_1,k_2,k_3}(t,\xi)$. Recall (6.106) and (6.111). From the $L^2 - L^\infty - L^\infty$ type trilinear estimate (2.6) in Lemma 2.2, the following estimate holds for some absolute constant C,

$$\|T^{+,\kappa,\iota,1}_{k_1,k_2,k_3}(t,\xi)\|_{L^2} \le C 2^{2k_1+4k_{1,+}}\big(2^{2k_1}\|\nabla^2_\xi \widehat{g_{k_1}}(t,\xi)\|_{L^2} + 2^{k_1}\|\nabla_\xi \widehat{g_{k_1}}(t,\xi)\|_{L^2} + \|g_{k_1}(t)\|_{L^2}\big)$$
$$\times \|e^{-it\Lambda} g_{k_2}\|_{L^\infty}\|e^{-it\Lambda} g_{k_3}\|_{L^\infty} \le C 2^{-m+7\tilde{\delta}m/3+k+5k_+}\epsilon_0. \tag{6.118}$$

Since the L^∞_x decay rate of the nonlinear solution itself is slightly slower than $t^{-1/2}$, a rough $L^2 - L^\infty - L^\infty$ is not sufficient to close the estimate of $T^{\tau,\kappa,\iota,1}_{k_1,k_2,k_3}$. An essential ingredient is to utilize symmetry such that one of the inputs putted in L^∞ associates with a spatial derivative. To see the symmetric structure, we decompose $T^{+,\kappa,\iota,1}_{k_1,k_2,k_3}$ into three parts as follows,

$$T^{+,\kappa,\iota,1}_{k,k_1,k_2,k_3} = \sum_{i=1,2,3} T^{+,\kappa,\iota,1;i}_{k,k_1,k_2,k_3}, \quad T^{+,\kappa,\iota,1;1}_{k,k_1,k_2,k_3} = \int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \overline{\widehat{\Gamma^1\Gamma^2 g}(t,\xi)} e^{it\Phi^{+,\kappa,\iota}(\xi,\eta,\sigma)} e(\xi)$$
$$\times \widehat{\Gamma^1\Gamma^2 g}(t,\xi-\eta)\psi_k(\xi)\psi_{k_1}(\xi-\eta)\widehat{g^\kappa_{k_2}}(t,\eta-\sigma)\widehat{g^\iota_{k_3}}(t,\sigma)\,d\eta\, d\sigma\, d\xi\, dt,$$
$$T^{+,\kappa,\iota,1;2}_{k,k_1,k_2,k_3} = \int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \overline{\widehat{\Gamma^1\Gamma^2 g}(t,\xi)} e^{it\Phi^{+,\kappa,\iota}(\xi,\eta,\sigma)}\big(\tilde{d}_{+,\kappa,\iota}(\xi,\eta,\sigma) - e(\xi)\big)$$
$$\times \psi_k(\xi)\widehat{\Gamma^1\Gamma^2 g_{k_1}}(t,\xi-\eta)\widehat{g^\kappa_{k_2}}(t,\eta-\sigma)\widehat{g^\iota_{k_3}}(t,\sigma)\,d\eta\, d\sigma\, d\xi\, dt,$$
$$T^{+,\kappa,\iota,1;3}_{k,k_1,k_2,k_3} = \int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \overline{\widehat{\Gamma^1\Gamma^2 g}(t,\xi)} e^{it\Phi^{+,\kappa,\iota}(\xi,\eta,\sigma)}\tilde{d}_{+,\kappa,\iota}(\xi-\eta,\eta-\sigma,\sigma)\widehat{g^\kappa_{k_2}}(t,\eta-\sigma)$$
$$\times \psi_k(\xi)\widehat{g^\iota_{k_3}}(t,\sigma)\big(\Gamma^1_\xi\Gamma^2_\xi\widehat{g_{k_1}}(t,\xi-\eta) - \widehat{\Gamma^1\Gamma^2 g_{k_1}}(t,\xi-\eta)\big)\,d\eta\, d\sigma\, d\xi\, dt,$$

where $e(\xi)$ is defined in (4.47). After switching the role of ξ and $\xi-\eta$ inside $T^{+,\kappa,\iota,1;1}_{k_1,k_2,k_3}$, we have

$$\sum_{\tau,\kappa,\iota\in\{+,-\}} \mathrm{Re}[T^{+,\kappa,\iota,1;1}_{k_1,k_2,k_3}] = \sum_{\tau,\kappa,\iota\in\{+,-\}} \mathrm{Re}[\tilde{T}^{+,\kappa,\iota}_{k_1,k_2,k_3}], \quad \tilde{T}^{+,\kappa,\iota}_{k_1,k_2,k_3} := \frac{1}{2}\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\overline{\widehat{\Gamma^1\Gamma^2 g}(t,\xi)}$$
$$\times e^{it\Phi^{+,\kappa,\iota}(\xi,\eta,\sigma)}\tilde{d}_{k,k_1}(\xi,\eta,\sigma)\widehat{\Gamma^1\Gamma^2 g}(t,\xi-\eta)\widehat{g^\kappa_{k_2}}(t,\eta-\sigma)\widehat{g^\iota_{k_3}}(t,\sigma)\,d\eta\, d\sigma\, d\xi\, dt,$$

where

$$\tilde{d}_{k,k_1}(\xi,\eta,\sigma) := e(\xi)\psi_{k_1}(\xi-\eta)\psi_k(\xi) + \overline{e(\xi-\eta)}\psi_{k_1}(\xi)\psi_k(\xi-\eta).$$

Recall (4.47). From the estimate (2.3) in Lemma 2.1, the following estimate holds for some absolute constant C,

$$\|\tilde{d}_{k,k_1}(\xi,\eta,\sigma)\psi_{k_2}(\eta-\sigma)\psi_{k_3}(\sigma)\|_{\mathcal{S}^\infty} \le C 2^{\max\{k_2,k_3\}+k_1+4k_{1,+}}. \tag{6.119}$$

From (6.119), (4.46) and the $L^2 - L^\infty - L^\infty$ type trilinear estimate (2.5) in Lemma 2.2, we have

$$\sum_{i=1,2,3}\sum_{\tau,\kappa,\iota\in\{+,-\}} |\mathrm{Re}[T^{+,\kappa,\iota,1;i}_{k,k_1,k_2,k_3}]|$$
$$\leq \sup_{t\in[2^{m-1},2^m]} C2^{m+k_1+4k_{1,+}+\max\{k_2,k_3\}}\big(2^{2k_1}\|\nabla_\xi^2\widehat{g}_{k_1}(t,\xi)\|_{L^2}$$
$$+ 2^{k_1}\|\nabla_\xi\widehat{g}_{k_1}(t,\xi)\|_{L^2} + \|g_{k_1}(t)\|_{L^2}\big)\|\Gamma^1\Gamma^2 g_{k_1}\|_{L^2}\|e^{-it\Lambda}g_{k_2}\|_{L^\infty}\|e^{-it\Lambda}g_{k_3}\|_{L^\infty}$$
$$\leq C2^{-m/2+50\beta m}\epsilon_0^2,$$

where C is some absolute constant.

Therefore, now it would be sufficient to estimate $T^{+,\kappa,\iota,i}_{k_1,k_2,k_3}$ and $T^{+,\kappa,\iota,i}_{k_1,k_2,k_3}(t,\xi)$, $i\in\{3,4\}$. Recall (6.108), (6.109), and (6.111). By doing integration by parts in "η" many times, we can rule out the case when $\max\{j_1,j_2\}\leq m+k_{1,-}-\beta m$. If $\max\{j_1,j_2\}\geq m+k_{1,-}-\beta m$, from the $L^2 - L^\infty - L^\infty$ type trilinear estimate (2.6) in Lemma 2.2, the following estimate holds for some absolute constant C,

$$\sum_{i=3,4}\Big\|\sum_{\max\{j_1,j_2\}\geq m+k_{1,-}-\beta m} T^{+,\kappa,\iota,i}_{k,k_1,j_1,k_2,j_2,k_3,j_3}(t,\xi)\Big\|_{L^2}$$
$$\leq C2^{m+3k_1+k_2+4k_{1,+}}\|e^{-it\Lambda}g_{k_3}\|_{L^\infty}$$
$$\times\Big[\sum_{j_1\geq\max\{j_2,m+k_{1,-}-\beta m\}}(2^{k_1+j_1}+(1+2^{m+k_1+k_2}))\|g_{k_1,j_1}\|_{L^2}\|e^{-it\Lambda}g_{k_2,j_2}\|_{L^\infty}$$
$$+\sum_{j_2\geq\max\{j_1,m+k_{1,-}-\beta m\}}\big(2^{k_1}\|e^{-it\Lambda}\mathscr{F}^{-1}[\nabla_\xi\widehat{g_{k_1,j_1}}(t,\xi)]\|_{L^\infty}$$
$$+(1+2^{m+k_1+k_2})\|e^{-it\Lambda}g_{k_1,j_1}\|_{L^\infty}\big)\|g_{k_2,j_2}\|_{L^2}\Big]$$
$$\leq C2^{-3m/2+50\beta m}\epsilon_0.$$

Note that above estimate is sufficient to imply our second desired estimate in (6.117). Hence finishing the proof. □

LEMMA 6.11. – *For* $i=1,3,4$ *and fixed* k,k_1,k_2,k_3 *in the range* (6.116) *, our desired estimate* (6.117) *holds if either* $|k_1-k_2|\leq 10$ *and* $k_3\leq k_2-10$ *or* $|k_1-k_2|\leq 10$, $|k_3-k_2|\leq 10, k\leq k_1-10$.

Proof. – The estimate of $T^{\tau,\kappa,\iota,1}_{k,k_1,k_2,k_3}(t,\xi)$ is straightforward. As $|k_1-k_2|\leq 10$, the size of symbol compensates the decay rate of $e^{-it\Lambda}g_{k_2}(t)$. From the $L^2 - L^\infty - L^\infty$ type trilinear estimate (2.6) in Lemma 2.2, the following estimate holds for some absolute constant C,

$$\|T^{\tau,\kappa,\iota,1}_{k,k_1,k_2,k_3}(t,\xi)\|_{L^2}\leq C2^{2k_1+4k_{1,+}}\big(2^{2k_1}\|\nabla_\xi^2\widehat{g}_{k_1}(t,\xi)\|_{L^2}+2^{k_1}\|\nabla_\xi\widehat{g}_{k_1}(t,\xi)\|_{L^2}+\|g_{k_1}(t)\|_{L^2}\big)$$
$$\times\|e^{-it\Lambda}g_{k_2}\|_{L^\infty}\|e^{-it\Lambda}g_{k_3}\|_{L^\infty}\leq C2^{-3m/2+50\beta m}\epsilon_0. \tag{6.120}$$

Now, we proceed to estimate $T^{\tau,\kappa,\iota,3}_{k,k_1,k_2,k_3}(t,\xi)$ and $T^{\tau,\kappa,\iota,4}_{k,k_1,k_2,k_3}(t,\xi)$. Recall (6.108) and (6.109). Note that, if either $|k_1-k_2|\leq 10$ and $k_3\leq k_2-10$ or $|k_1-k_2|\leq 10$, $|k_3-k_2|\leq 10, k\leq k_1-10$, we know that $\nabla_\eta\Phi^{\tau,\kappa,\iota}(\xi,\eta,\kappa)$ has a lower bound, which is $2^{k-4\beta m}$. To take advantage of this fact, we do integration by parts in "η" many times to rule out the case when $\max\{j_1, j_2\} \leq m + k_- - 5\beta m$. If $\max\{j_1, j_2\} \geq m + k_- - 5\beta m$, from the $L^2 - L^\infty - L^\infty$ type trilinear estimate (2.6) in Lemma 2.2, the following estimate holds for some absolute constant C,

$$
\begin{aligned}
&\sum_{i=3,4} \Big\| \sum_{\max\{j_1,j_2\}\geq m+k_- -5\beta m} T^{+,\kappa,\iota,i}_{k,k_1,j_1,k_2,j_2,k_3,j_3}(t,\xi) \Big\|_{L^2} \\
&\quad \leq C 2^{m+2k+k_1+4k_{1,+}} \|e^{-it\Lambda} g_{k_3}\|_{L^\infty} \\
&\qquad \times \Big(\sum_{j_2 \geq \max\{j_1, m+k_- -5\beta m\}} \big((1+2^{m+2k_1})\|e^{-it\Lambda} g_{k_1,j_1}\|_{L^\infty} + 2^{k_1}\|e^{-it\Lambda} \\
&\qquad \times \mathcal{F}^{-1}[\nabla_\xi \widehat{g_{k_1,j_1}}(t,\xi)]\|_{L^\infty}\big)\|g_{k_2,j_2}\|_{L^2} \\
&\qquad + \sum_{j_1 \geq \max\{j_2, m+k_- -5\beta m\}} \big(2^{m+2k_1} + 2^{j_1+k_1}\big)\|e^{-it\Lambda} g_{k_2,j_2}\|_{L^\infty}\|g_{k_1,j_1}\|_{L^2}\Big) \qquad (6.121)\\
&\quad \leq C 2^{-3m/2+50\beta m}\epsilon_0.
\end{aligned}
$$

From (6.120) and (6.121), it is easy to see our desired estimates in (6.117) hold. □

LEMMA 6.12. – *For $i = 1, 3, 4$ and fixed k, k_1, k_2, k_3 in the range* (6.116) *, our desired estimate* (6.117) *holds if $|k_1 - k_2| \leq 10$, $|k_3 - k_2| \leq 10$, and $|k - k_1| \leq 10$.*

Proof. – Since we still have $|k_1 - k_2| \leq 10$, the estimate of $T^{\tau,\kappa,\iota,1}_{k,k_1,k_2,k_3}(t,\xi)$ in (6.120) still holds. It would be sufficient to estimate $T^{\tau,\kappa,\iota,3}_{k,k_1,k_2,k_3}(t,\xi)$ and $T^{\tau,\kappa,\iota,4}_{k,k_1,k_2,k_3}(t,\xi)$, which is more delicate. For those cases, we need to study the space resonance in "η" set and the space resonance in "σ" set carefully as we did in the Z_1-norm estimate of cubic terms in the proof of Lemma 5.7.

Recall that we already canceled out the case when $(\tau,\kappa,\iota) \in \mathcal{S}_4$ (see (5.32)) and $(\xi-\eta, \eta-\sigma, \sigma)$ is very close to $(\xi/3, \xi/3, \xi/3)$ in the normal form transformation. Therefore, for the case when $(\tau,\kappa,\iota) \in \mathcal{S}_4$, we only have to consider the case when $(\xi-\eta, \eta-\sigma, \sigma)$ is not close to $(\xi/3, \xi/3, \xi/3)$, in which case either $\nabla_\eta \Phi^{\tau,\kappa,\iota}(\xi,\eta,\kappa)$ or $\nabla_\sigma \Phi^{\tau,\kappa,\iota}(\xi,\eta,\kappa)$ has a good lower bound, which allows us to do integration by parts either in η or in σ. The estimate of this case is similar and also easier than the estimate of (6.121) in the proof of Lemma 6.11. We omit details here.

Now, we focus on the case when $(\tau,\kappa,\iota) \in \mathcal{S}_i$, $i \in \{1,2,3\}$. By the symmetries between inputs, it would be sufficient to consider the case when $(\tau,\kappa,\iota) \in \mathcal{S}_1$, i.e., $(\tau,\kappa,\iota) \in \{(+,-,-),(-,+,+)\}$. After changing the variables as follows $(\xi,\eta,\sigma) \longrightarrow (\xi, 2\xi+\eta+\sigma, \xi+\sigma)$, we have the following decomposition for $i \in \{3,4\}$,

$$
\begin{aligned}
&T^{\tau,\kappa,\iota,i}_{k,k_1,k_2,k_3}(t,\xi) := \sum_{l_1,l_2 \geq \bar{l}_\tau} H^{l_1,l_2,\tau,i-2}(t,\xi), \\
&H^{l_1,l_2,\tau,i-2}(t,\xi) = \sum_{j_1 \geq -k_{1,-}, j_2 \geq -k_{2,-}} H^{l_1,l_2,\tau,i-2}_{j_1,j_2}(t,\xi), \\
&H^{l_1,l_2,\tau,1}_{j_1,j_2}(t,\xi) := \sum_{\{l,n\}=\{1,2\}} \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{it\tilde{\Phi}^{\tau,\kappa,\iota}(\xi,\eta,\sigma)} it\big(\Gamma^l_\xi \Phi^{\tau,\kappa,\iota}(\xi, 2\xi+\eta+\sigma, \xi+\sigma)\big)\varphi_{l_1;\bar{l}_\tau}(\eta) \\
&\qquad \times \varphi_{l_2;\bar{l}_\tau}(\sigma)\psi_k(\xi)\Gamma^n_\xi\big(\tilde{d}_{\tau,\kappa,\iota}(-\xi-\eta-\sigma, \xi+\eta, \xi+\sigma)\widehat{g^\tau_{k_1,j_1}}(t,-\xi-\eta-\sigma)\big)
\end{aligned}
$$

$$\times \widehat{g^{\kappa}_{k_2,j_2}}(t,\xi+\eta)\widehat{g^{\iota}_{k_3,j_3}}(t,\xi+\sigma)d\eta d\sigma,$$

$$H^{l_1,l_2,\tau,2}_{j_1,j_2} := -\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{it\widetilde{\Phi}^{\tau,\kappa,\iota}(\xi,\eta,\sigma)}t^2\big(\Gamma^1_\xi\Phi^{\tau,\kappa,\iota}(\xi,2\xi+\eta+\sigma,\xi+\sigma)$$
$$\times\Gamma^2_\xi\Phi^{\tau,\kappa,\iota}(\xi,2\xi+\eta+\sigma,\xi+\sigma)\big)\tilde{d}_{\tau,\kappa,\iota}(-\xi-\eta-\sigma,\xi+\eta,\xi+\sigma)$$
$$\times\widehat{g^{\tau}_{k_1,j_1}}(t,-\xi-\eta-\sigma)\widehat{g^{\kappa}_{k_2,j_2}}(t,\xi+\eta)\widehat{g^{\iota}_{k_3}}(t,\xi+\sigma)\psi_k(\xi)$$
$$\times\varphi_{l_1;\bar{l}_\tau}(\eta)\varphi_{l_2;\bar{l}_\tau}(\sigma)d\eta d\sigma,$$

where $\widetilde{\Phi}^{\tau,\kappa,\iota}(\xi,\eta,\sigma)$ is defined in (5.35), the cutoff function $\varphi_{l;\bar{l}}(\cdot)$ is defined in (5.36) and the thresholds are chosen as follows, $\bar{l}_+ := k_1 - 10$ and $\bar{l}_- := -m/2 + 10\delta m + k_{1,+}/2$.

If $\tau = +$, i.e., $(\tau,\kappa,\iota) = (+,-,-)$. – Recall the normal form transformation that we did in (4.1), see (4.20) and (4.30). For the case we are considering, $(\tau,\kappa,\iota)\in\widetilde{S}$, we have already removed the case when $\max\{l_1,l_2\} = \bar{l}_+$. Hence it would be sufficient to consider the case when $\max\{l_1,l_2\} > \bar{l}_+$. Due to the symmetry between inputs, we assume that $l_2 = \max\{l_1,l_2\}$. As $l_2 > \bar{l}_+$, we can take the advantage of the fact that "$\nabla_\eta\widetilde{\Phi}^{\tau,\kappa,\iota}(\xi,\eta,\sigma)$" is big by doing integration by parts in "η". From (5.37), we can rule out the case when $\max\{j_1,j_2\}\le m+k_- -\beta m$ by doing integration by parts in "η" many times.

If $\max\{j_1,j_2\}\ge m+k_- -\beta m$, from the $L^2-L^\infty-L^\infty$ type trilinear estimate (2.6) in Lemma 2.2, the following estimates holds for some absolute constant C,

$$\sum_{\max\{j_1,j_2\}\ge m+k_- -\beta m}\sum_{i=1,2}\|H^{l_1,l_2,\tau,i}_{j_1,j_2}(t,\xi)\|_{L^2}$$
$$\le C2^{m+4k_1+4k_{1,+}}\Big(\sum_{j_1\ge\max\{j_2,m+k_- -\beta m\}}\big(2^{m+2k_1}+2^{k_1+j_1}\big)$$
$$\times\|g_{k_1,j_1}(t)\|_{L^2}\|e^{-it\Lambda}g_{k_2,j_2}(t)\|_{L^\infty}$$
$$+\sum_{j_2\ge\max\{j_1,m+k_- -\beta m\}}\big((2^{m+2k_1}+1)\|e^{-it\Lambda}g_{k_1,j_1}(t)\|_{L^\infty}$$
$$+2^{k_1}\|e^{-it\Lambda}\mathcal{F}^{-1}[\nabla_\xi\widehat{g_{k_1,j_1}}(t,\xi)]\|_{L^\infty}\big)\|g_{k_2,j_2}(t)\|_{L^2}\Big)\|e^{-it\Lambda}g_{k_3}(t)\|_{L^\infty} \tag{6.122}$$
$$\le C2^{-2m+50\beta m}\epsilon_0.$$

If $\tau = -$, i.e., $(\tau,\kappa,\iota) = (-,+,+)$. – Note that the estimates (5.38) and (5.39) hold for the case we are considering. Same as before, due to the symmetry between inputs, without loss of generality, we assume that $l_2 = \max\{l_1,l_2\}$.

We first consider the case when $l_2 > \bar{l}_-$. Recall (5.37), by doing integration by parts in "η" many times, we can rule out the case when $\max\{j_1,j_2\}\le m+l_2-4\beta m$. If $\max\{j_1,j_2\}\ge m+l_2-4\beta m$, from the $L^2-L^\infty-L^\infty$ type trilinear estimate, the following estimate holds for some absolute constant C,

$$\sum_{\max\{j_1,j_2\}\ge m+l_2-4\beta m}\sum_{i=1,2}\|H^{l_1,l_2,-,i}_{j_1,j_2}(t,\xi)\|_{L^2}\le C2^{m+3k_1+l_2+4k_{1,+}}$$
$$\times\Big(\sum_{j_1\ge\max\{j_2,m+l_2-4\beta m\}}\big(2^{m+k_1+l_2}+2^{j_1+k_1}\big)\|g_{k_1,j_1}(t)\|_{L^2}\|e^{-it\Lambda}g_{k_2,j_2}(t)\|_{L^\infty}$$

$$
\begin{aligned}
&+\sum_{j_2\geq \max\{j_1,m+l_2-4\beta m\}}\Big((1+2^{m+k_1+l_2})\|e^{-it\Lambda}g_{k_1,j_1}(t)\|_{L^\infty}\\
&+2^{k_1}\|e^{-it\Lambda}\mathscr{F}^{-1}[\nabla_\xi\widehat{g_{k_1,j_1}}(t,\xi)]\|_{L^\infty}\Big)\|g_{k_2,j_2}(t)\|_{L^2}\Big)\|e^{-it\Lambda}g_{k_3}(t)\|_{L^\infty} \quad (6.123)\\
&\leq C2^{-2m+50\beta m}\epsilon_0^2.
\end{aligned}
$$

Lastly, we consider the case when $l_2=\bar{l}_-=-m/2+10\delta m+k_{1,+}/2$. Recall the estimate (5.38). For this case, we use the volume of support in "η" and "σ". As a result, the following estimate holds for some absolute constant C,

$$
\begin{aligned}
&\sum_{i=1,2}\|H^{\bar{l}_-,\bar{l}_-,-,i}(t,\xi)\|_{L^2}\\
&\quad\leq C2^{4k_{1,+}}\big(2^{2m+6\bar{l}+4k_1}+2^{m+5\bar{l}+4k_1}\big)\big(2^{-k_1}\|g_{k_1}(t)\|_{L^2}+\|\nabla_\xi\widehat{g_{k_1}}(t,\xi)(t)\|_{L^2}\big)\\
&\quad\times\|g_{k_2}(t)\|_{L^1}\|g_{k_3}(t)\|_{L^1}\leq C2^{-m+100\delta m}\epsilon_0^2. \quad (6.124)
\end{aligned}
$$

Hence finishing the proof. □

6.5. The Z_2 norm estimate of the quartic terms

Recall (4.38). For any $\Gamma_\xi^1,\Gamma_\xi^2\in\{\hat{L}_\xi,\hat{\Omega}_\xi\}$, we have

$$
\Gamma_\xi^1\Gamma_\xi^2\Lambda_4[\partial_t\widehat{g}(t,\xi)]\psi_k(\xi)=\sum_{\mu_1,\mu_2,\nu_1,\nu_2\in\{+,-\}}\sum_{k_4\leq k_3\leq k_2\leq k_1}\sum_{i=1,2,3,4}K^{\mu_1,\mu_2,\nu_1,\nu_2,i}_{k,k_1,k_2,k_3,k_4}(t,\xi),
$$

$$
K^{\mu_1,\mu_2,\nu_1,\nu_2,i}_{k,k_1,k_2,k_3,k_4}(t,\xi)=\sum_{j_1\geq -k_{1,-},j_2\geq -k_{2,-}}K^{\mu_1,\mu_2,\nu_1,\nu_2,i}_{k,k_1,j_1,k_2,j_2,k_3,k_4}(t,\xi),\quad i\in\{3,4\},
$$

where

(6.125)

$$
\begin{aligned}
K^{\mu_1,\mu_2,\nu_1,\nu_2,1}_{k,k_1,k_2,k_3,k_4}(t,\xi)&:=\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}e^{it\Phi^{\mu_1,\mu_2,\nu_1,\nu_2}(\xi,\eta,\sigma,\kappa)}\\
&\times\tilde{e}_{\mu_1,\mu_2,\nu_1,\nu_2}(\xi-\eta,\eta-\sigma,\sigma-\kappa,\kappa)\Gamma_\xi^1\Gamma_\xi^2\widehat{g^{\mu_1}_{k_1}}(t,\xi-\eta) \quad (6.126)\\
&\times\widehat{g^{\mu_2}_{k_2}}(t,\eta-\sigma)\widehat{g^{\nu_1}_{k_3}}(t,\sigma-\kappa)\widehat{g^{\nu_2}_{k_4}}(t,\kappa)\psi_k(\xi)d\kappa d\sigma d\eta,\\
K^{\mu_1,\mu_2,\nu_1,\nu_2,2}_{k,k_1,k_2,k_3,k_4}(t,\xi)&:=\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}e^{it\Phi^{\mu_1,\mu_2,\nu_1,\nu_2}(\xi,\eta,\sigma,\kappa)}\psi_k(\xi)\\
&\times\big[\Gamma_\xi^1\Gamma_\xi^2\big(\tilde{e}_{\mu_1,\mu_2,\nu_1,\nu_2}(\xi-\eta,\eta-\sigma,\sigma-\kappa,\kappa)\big)\widehat{g^{\mu_1}_{k_1}}(t,\xi-\eta)\\
&+\sum_{\{l,n\}=\{1,2\}}\Gamma_\xi^l\tilde{e}_{\mu_1,\mu_2,\nu_1,\nu_2}(\xi-\eta,\eta-\sigma,\sigma-\kappa,\kappa)\Gamma_\xi^n\widehat{g^{\mu_1}_{k_1}}(t,\xi-\eta)\big]\\
&\times\widehat{g^{\mu_2}_{k_2}}(t,\eta-\sigma)\widehat{g^{\nu_1}_{k_3}}(t,\sigma-\kappa)\widehat{g^{\nu_2}_{k_4}}(t,\kappa)d\kappa d\sigma d\eta, \quad (6.127)\\
K^{\mu_1,\mu_2,\nu_1,\nu_2,3}_{k,k_1,j_1,k_2,j_2,k_3,k_4}(t,\xi)&:=\sum_{\{l,n\}=\{1,2\}}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\psi_k(\xi)e^{it\Phi^{\mu_1,\mu_2,\nu_1,\nu_2}(\xi,\eta,\sigma,\kappa)}it\\
&\times\big(\Gamma_\xi^l\Phi^{\mu_1,\mu_2,\nu_1,\nu_2}(\xi,\eta,\sigma,\kappa)\big)\Gamma_\xi^n\big(\tilde{e}_{\mu_1,\mu_2,\nu_1,\nu_2}(\xi-\eta,\eta-\sigma,\sigma-\kappa,\kappa)\\
&\times\widehat{g^{\mu_1}_{k_1,j_1}}(t,\xi-\eta)\big)\widehat{g^{\mu_2}_{k_2,j_2}}(t,\eta-\sigma)\widehat{g^{\nu_1}_{k_3}}(t,\sigma-\kappa)\widehat{g^{\nu_2}_{k_4}}(t,\kappa)d\kappa d\sigma d\eta,
\end{aligned}
$$

$$
\begin{aligned}
K^{\mu_1,\mu_2,\nu_1,\nu_2,4}_{k,k_1,j_1,k_2,j_2,k_3,k_4}(t,\xi) := -\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} & \psi_k(\xi) e^{it\Phi^{\mu_1,\mu_2,\nu_1,\nu_2}(\xi,\eta,\sigma,\kappa)} t^2 \Gamma^1_\xi \Phi^{\mu_1,\mu_2,\nu_1,\nu_2}(\xi,\eta,\sigma,\kappa) \\
& \times \Gamma^2_\xi \Phi^{\mu_1,\mu_2,\nu_1,\nu_2}(\xi,\eta,\sigma,\kappa) \tilde{e}_{\mu_1,\mu_2,\nu_1,\nu_2}(\xi-\eta,\eta-\sigma,\sigma-\kappa,\kappa) \\
& \times \widehat{g^{\mu_1}_{k_1,j_1}}(t,\xi-\eta)\widehat{g^{\mu_2}_{k_2,j_2}}(t,\eta-\sigma)\widehat{g^{\nu_1}_{k_3}}(t,\sigma-\kappa)\widehat{g^{\nu_2}_{k_4}}(t,\kappa)\, d\kappa\, d\eta\, d\sigma. \qquad (6.128)
\end{aligned}
$$

The main goal of this subsection is to prove the following proposition.

Proposition 6.13. – *Under the bootstrap assumption (4.49), the following estimates hold for some absolute constant C and any $t \in [2^{m-1}, 2^m]$,*

$$
\sup_{t_1,t_2\in[2^{m-1},2^m]} \Big| \sum_k \int_{t_1}^{t_2} \int_{\mathbb{R}^2} \overline{\Gamma^1_\xi \Gamma^2_\xi \widehat{g}(t,\xi)} \Gamma^1_\xi \Gamma^2_\xi \Lambda_4[\partial_t \widehat{g}(t,\xi)] \psi_k(\xi)\, d\xi\, dt \Big| \le C 2^{2\tilde{\delta} m} \epsilon_0^2. \qquad (6.129)
$$

$$
\sup_{t\in[2^{m-1},2^m]} \|\Gamma^1_\xi \Gamma^2_\xi \Lambda_4[\partial_t \widehat{g}(t,\xi)]\|_{L^2} \le C 2^{-m+\tilde{\delta} m} \epsilon_0^2. \qquad (6.130)
$$

Proof. – As usual, we first rule out the very high frequency case and the very low frequency case. Same as what we did in the estimate of cubic terms, we move the derivative $\nabla_\xi = -\nabla_\eta$ in front of $\widehat{g}_{k_1}(t,\xi-\eta)$ around by doing integration by parts in η such that there is no derivative in front of $\widehat{g}_{k_1}(t,\xi-\eta)$. As a result, the following estimate holds,

$$
\begin{aligned}
\sum_{i=1,2,3,4} \|K^{\mu_1,\mu_2,\nu_1,\nu_2,i}_{k,k_1,k_2,k_3,k_4}(t,\xi)\|_{L^2} \le & C(1+2^{2m+2k}) 2^{6k_{1,+}} \|g_{k_1}(t)\|_{L^2} \\
& \times \big(\|\nabla^2_\xi \widehat{g_{k_2}}(t,\xi)\|_{L^2} + 2^{-k_2} \|\nabla_\xi \widehat{g_{k_2}}(t,\xi)\|_{L^2} + 2^{-2k_2} \|g_{k_2}(t)\|_{L^2} \big) \\
& \times 2^{k_3+k_4} \|g_{k_3}(t)\|_{L^2} \|g_{k_4}(t)\|_{L^2} \le C 2^{2m+\beta m-(N_0-10)k_{1,+}} \epsilon_0^2, \qquad (6.132)
\end{aligned}
$$

where C is some absolute constant. Hence, we can rule out the case when $k_1 \ge 4\beta m$. It remains to consider the case when $k_1 \le 4\beta m$. We can also rule out the very low frequencies case. If either $k_4 \le -3m - 30\beta m$ or $k \le -2m$, then the following estimate holds for some absolute constant C,

$$
\begin{aligned}
\sum_{i=1,2,3,4} \|K^{\mu_1,\mu_2,\nu_1,\nu_2,1}_{k,k_1,k_2,k_3,k_4}(t,\xi)\|_{L^2} \le & C(1+2^{2m+2k}) 2^{k+k_4+4k_{1,+}} \\
& \times \big(2^{2k_1} \|\nabla^2_\xi \widehat{g}_{k_1}(t,\xi)\|_{L^2} + 2^{k_1} \|\nabla_\xi \widehat{g}_{k_1}(t,\xi)\|_{L^2} + \|g_{k_1}(t)\|_{L^2} \big) \\
& \times \|e^{-it\Lambda} g_{k_2}(t)\|_{L^\infty} \|g_{k_3}(t)\|_{L^2} \|g_{k_4}(t)\|_{L^2} \le C 2^{-m-\beta m} \epsilon_0^2.
\end{aligned}
$$

Now it would be sufficient to consider fixed k, k_1, k_2, k_3, and k_4 in the following range,

$$
-3m - 30\beta m \le k_4 \le k_3 \le k_2 \le k_1 \le 4\beta m, \quad -2m \le k \le 3\beta m. \qquad (6.133)
$$

From the $L^2 - L^\infty - L^\infty - L^\infty$ type multilinear estimate, the following estimate holds

$$
\begin{aligned}
\sum_{i=1,2} \|K^{\mu_1,\mu_2,\nu_1,\nu_2,i}_{k,k_1,k_2,k_3,k_4}(t,\xi)\|_{L^2} \le & C 2^{2k_1+4k_{1,+}} \\
& \times \big(2^{2k_1} \|\nabla^2_\xi \widehat{g}_{k_1}(t,\xi)\|_{L^2} + 2^{k_1} \|\nabla_\xi \widehat{g}_{k_1}(t,\xi)\|_{L^2} + \|g_{k_1}(t,\xi)\|_{L^2} \big) \\
& \times \|e^{-it\Lambda} g_{k_2}(t)\|_{L^\infty} \|e^{-it\Lambda} g_{k_3}(t)\|_{L^\infty} \|e^{-it\Lambda} g_{k_4}(t)\|_{L^\infty} \le C 2^{-3m/2+50\beta m} \epsilon_0^2, \qquad (6.134)
\end{aligned}
$$

where C is some absolute constant.

It remains to estimate the case when $i = 3, 4$. We first consider the case when $k_1 - 10 \le k_3$. For this case, the following estimate holds from the $L^2 - L^\infty - L^\infty - L^\infty$ type estimate, the following estimate holds for some absolute constant C,

$$
\begin{aligned}
\sum_{i=3,4} \| K^{\mu_1,\mu_2,\nu_1,\nu_2,i}_{k,k_1,k_2,k_3,k_4}(t,\xi)\|_{L^2} &\le C 2^{m+4k_1+4k_{1,+}} \\
&\times \big[\big(\|e^{-it\Lambda} g_{k_1}(t)\|_{L^\infty} + 2^{k_1}\|e^{-it\Lambda}\mathcal{F}^{-1}[\nabla_\xi \widehat{g}_{k_1}(t,\xi)]\|_{L^\infty}\big) + 2^{m+2k_1}\|e^{-it\Lambda} g_{k_1}(t)\|_{L^\infty}\big] \\
&\times \|e^{-it\Lambda} g_{k_2}(t)\|_{L^\infty}\|e^{-it\Lambda} g_{k_3}(t)\|_{L^\infty}\|g_{k_4}(t)\|_{L^2} \\
&\le C 2^{-m+\tilde{\delta}m/2}\epsilon_0^2.
\end{aligned} \tag{6.135}
$$

Lastly, we consider the case when $k_3 \le k_1 - 10$. Recall (4.32) and (4.41). Because of the construction of the normal form transformation we did in Subsection 4.1, we know that the case when η is very close to $\xi/2$ and $|\sigma|, |\kappa| \le 2^{-10}|\xi|$ is removed, which means that "$\nabla_\eta \Phi^{\mu_1,\mu_2,\nu_1,\nu_2}(\xi,\eta,\sigma,\kappa)$" has a lower bound, which is $2^{k-k_{1,+}}$. To take advantage of this fact, we do integration by parts in "η" many times to rule out the case when $\max\{j_1, j_2\} \le m + k_- - 5\beta m$. If $\max\{j_1, j_2\} \ge m + k_- - 5\beta m$, from the $L^2 - L^\infty - L^\infty - L^\infty$ type estimate, the following estimate holds for some absolute constant C,

$$
\begin{aligned}
&\sum_{i=3,4}\sum_{\max\{j_1,j_2\}\ge m+k_- -5\beta m} \| K^{\mu_1,\mu_2,\nu_1,\nu_2,i}_{k,k_1,j_1,k_2,j_2,k_3,k_4}(t,\xi)\|_{L^2} \\
&\quad\le \sum_{j_1\ge \max\{j_2, m+k_- -5\beta m\}} C 2^{m+k+k_2+2k_1+4k_{1,+}}\big(2^{m+k+k_1} + 2^{k_1+j_1}\big)\|g_{k_1,j_1}(t)\|_{L^2} \\
&\quad\quad\times \|e^{-it\Lambda} g_{k_2,j_2}(t)\|_{L^\infty}\|e^{-it\Lambda} g_{k_3}(t)\|_{L^\infty}\|e^{-it\Lambda} g_{k_4}(t)\|_{L^\infty} \\
&\quad\quad+ \sum_{j_2\ge \max\{j_1, m+k_- -5\beta m\}} C 2^{m+k+k_2+2k_1+4k_{1,+}}\big(2^{k_1}\|e^{-it\Lambda}\mathcal{F}^{-1}[\nabla_\xi \widehat{g_{k_1,j_1}}(t,\xi)]\|_{L^\infty} \\
&\quad\quad+ 2^{m+k+k_1}\|e^{-it\Lambda} g_{k_1,j_1}(t)\|_{L^\infty}\big) 2^{k_2}\|g_{k_2,j_2}(t)\|_{L^2}\|e^{-it\Lambda} g_{k_3}(t)\|_{L^\infty}\|g_{k_4}(t)\|_{L^2} \\
&\quad\le C 2^{-3m/2+50\beta m}\epsilon_0^2.
\end{aligned} \tag{6.136}
$$

To sum up, from the estimates (6.134), (6.135) and (6.136), we know that our desired estimates (6.129) and (6.130) hold. □

Lemma 6.14. – *Under the bootstrap assumption* (4.49)*, the following estimates hold for any* $t \in [2^{m-1}, 2^m]$ *and any* $\Gamma^1_\xi, \Gamma^2_\xi \in \{\hat{L}_\xi, \hat{\Omega}_\xi\}$,

$$
\|\Gamma^1_\xi \Gamma^2_\xi \Lambda_{\ge 3}[\partial_t \widehat{g}_k(t,\xi)]\|_{L^2} \le C 2^{-m+\tilde{\delta}m}\big(1 + 2^{2\tilde{\delta}m+k+5k_+}\big)\epsilon_0, \tag{6.137}
$$

Proof. – The desired estimate (6.137) follows straightforwardly from the estimate (6.113) in Proposition (6.9), the estimate (6.130) in Proposition (6.13), and the estimate (7.13) in Lemma 7.4. □

7. Fixed time weighted norm estimates

There are mainly two tasks to complete in this section. (i) Firstly, we prove some fixed time weighted norm estimates, which are stated in Lemma 7.1 and Lemma 7.2 and have been used in previous two sections. (ii) Lastly, we estimate both the low order weighted norm (Z_1-norm) and the high order weighted norm (Z_2-norm) of the profile of the quintic and higher order remainder term $\mathcal{R}_1$, see the equation satisfied by the good substitution variable v in (4.21). Therefore, finishing the bootstrap argument of the weighted norms of the profile $g(t) = e^{it\Lambda}v(t)$ over time.

LEMMA 7.1. – *Under the bootstrap assumption* (4.49)*, the following estimates hold,*

$$\sup_{t\in[2^{m-1},2^m]} \|\partial_t \widehat{g}_k(t,\xi) - \sum_{\mu,\nu\in\{+,-\}} \sum_{(k_1,k_1)\in\chi_k^1} B^{\mu,\nu}_{k,k_1,k_2}(t,\xi)\|_{L^2} \leq C2^{-21m/20}\epsilon_0, \tag{7.1}$$

$$\sup_{t\in[2^{m-1},2^m]} \|\partial_t \widehat{g}_k(t,\xi)\|_{L^2} \leq C\min\{2^{-2m-k+2\tilde{\delta}m}, 2^{-m+\delta m}\}\epsilon_0 + C2^{-21m/20}\epsilon_0, \tag{7.2}$$

$$\sup_{t\in[2^{m-1},2^m]} \|\Lambda_{\geq 3}[\partial_t \widehat{g}_k(t,\xi)]\|_{L^2} \leq C2^{-3m/2+\beta m}\epsilon_0, \tag{7.3}$$

where C is some absolute constant, χ_k^1 is defined in (6.2) *and $B^{\mu,\nu}_{k,k_1,k_2}(t,\xi)$ is defined in* (4.36).

Proof. – For the cubic and higher order terms, after putting the input with the smallest frequency in L^2 and all other inputs in L^∞, the decay rate of L^2 norm is at least $2^{-3m/2+\beta m}$, which gives us our desired estimate (7.3). Hence to prove (7.1) and (7.2), we only have to consider the quadratic terms "$B^{\mu,\nu}_{k,k_1,k_2}(t,\xi)$". Recall (4.36), after doing spatial localizations for two inputs, we have

$$B^{\mu,\nu}_{k,k_1,k_2}(t,\xi) = \sum_{j_1\geq -k_{1,-}, j_2\geq -k_{2,-}} B^{\mu,\nu,j_1,j_2}_{k,k_1,k_2}(t,\xi),$$

$$B^{\mu,\nu,j_1,j_2}_{k,k_1,k_2}(t,\xi) = \int_{\mathbb{R}^2} e^{it\Phi^{\mu,\nu}(\xi,\eta)} \tilde{q}_{\mu,\nu}(\xi,\eta)\widehat{g^{\mu}_{k_1,j_1}}(t,\xi-\eta)\widehat{g^{\nu}_{k_2,j_2}}(t,\eta)\psi_k(\xi)d\eta.$$

We first consider the case when $|k_1-k_2|\leq 10$. From the L^2-L^∞ type bilinear estimate (2.5) in Lemma 2.2, the following estimate holds for some absolute constant C,

$$\sum_{|k_1-k_2|\leq 10} \|B^{\mu,\nu}_{k,k_1,k_2}(t,\xi)\|_{L^2} \leq \sum_{|k_1-k_2|\leq 10} C2^{2k_1}\|g_{k_1}\|_{L^2}\|e^{-it\Lambda}g_{k_2}(t)\|_{L^\infty} \leq C2^{-m+\delta m}\epsilon_0. \tag{7.4}$$

Meanwhile, after doing integration by parts in "η" once, the following estimate also holds,

$$\sum_{|k_1-k_2|\leq 10} \|B^{\mu,\nu}_{k,k_1,k_2}(t,\xi)\|_{L^2} \leq \sum_{|k_1-k_2|\leq 10} C2^{2k_1}2^{-m-k+k_{1,+}}\big(\|e^{-it\Lambda}g_{k_1}\|_{L^\infty} + \|e^{-it\Lambda}g_{k_2}\|_{L^\infty}\big)$$
$$\times\big(\|\nabla_\xi \widehat{g}_{k_1}(t,\xi)\|_{L^2} + \|\nabla_\xi \widehat{g}_{k_2}(t,\xi)\|_{L^2} + 2^{-k_1}\|g_{k_1}(t)\|_{L^2}\big) \leq C2^{-2m-k+2\tilde{\delta}m}\epsilon_0. \tag{7.5}$$

Now, we consider the case when $k_2 \leq k_1 - 10$ and $k_{1,-} + k_2 \leq -18m/19$. Similar to the proof of the estimate (5.10) in Lemma 5.2, from the estimate (5.15) in Lemma 5.3, the following estimate holds for some absolute constant C,

$$\sum_{k_{1,-}+k_2\leq -18m/19} \Big\| \sum_{\nu\in\{+,-\}} B^{\mu,\nu}_{k,k_1,k_2}(t,\xi) \Big\|$$

$$\begin{aligned}
&\leq \sum_{k_{1,-}+k_2\leq -18m/19} C\|g_{k_1}(t)\|_{L^2}\min\{2^{2k_1+k_2}\|g_{k_2}(t)\|_{L^2},\\
&\quad\times 2^{k_1+3k_2}\|\widehat{g}_{k_2}(t,\xi)\|_{L^\infty_\xi}+2^{2k_1+2k_2}\|\widehat{\mathrm{Re}[v]}(t,\xi)\psi_{k_2}(\xi)\|_{L^\infty_\xi}\}\\
&\leq \sum_{k_{1,-}+k_2\leq -18m/19} C2^{3\tilde{\delta}m}\min\{2^{2k_{1,-}+k_2},2^{2k_2}\big(2^{k_{1,-}+k_2+m}+2^{2k_{1,-}+2k_2+2m}\big)\}\\
&\leq C2^{-21m/20}\epsilon_0.
\end{aligned}$$

Lastly, we consider the case when $k_2 \leq k_1 - 10$ and $k_{1,-} + k_2 \geq -18m/19$. After doing integration by parts in "η" many times, we can rule out the case when $\max\{j_1, j_2\} \leq m + k_{1,-} - \beta m$. If $\max\{j_1, j_2\} \geq m + k_{1,-} - \beta m$, from the $L^2 - L^\infty$ type bilinear estimate (2.5) in Lemma 2.2, the following estimate holds for some absolute constant C,

$$\begin{aligned}
&\sum_{\max\{j_1,j_2\}\geq m+k_{1,-}-\beta m}\|B^{\mu,\nu,j_1,j_2}_{k,k_1,k_2}(t,\xi)\|_{L^2}\\
&\leq \sum_{j_1\geq\max\{j_2,m+k_{1,-}-\beta m\}} C2^{2k_1}\|e^{-it\Lambda}g_{k_2,j_2}\|_{L^\infty}\|g_{k_1,j_1}\|_{L^2}\\
&\quad+\sum_{j_2\geq\max\{j_1,m+k_{1,-}-\beta m\}} C2^{2k_1}\|e^{-it\Lambda}g_{k_1,j_1}\|_{L^\infty}\|g_{k_2,j_2}\|_{L^2}\\
&\leq C2^{-3m-2k_2-k_{1,-}+3\beta m}\epsilon_0\leq C2^{-21m/20}\epsilon_0.
\end{aligned}\tag{7.6}$$

Combining the estimates (7.4), (7.5), and (7.6), it is easy to see that our desired estimate (7.2) holds. □

Lemma 7.2. – *Under the bootstrap assumption* (4.49)*, the following estimate holds for any* $t \in [2^{m-1}, 2^m]$,

$$\|\partial_t\widehat{\Gamma_1\Gamma_2 g_k}(t,\xi)-\sum_{\nu\in\{+,-\}}\sum_{(k_1,k_2)\in\chi^2_k}\widetilde{B}^{+,\nu}_{k,k_1,k_2}(t,\xi)\|_{L^2}\leq C2^{-m+\tilde{\delta}m+\delta m}\big(1+2^{2\tilde{\delta}m+k+5k_+}\big)\epsilon_0,\tag{7.7}$$

where C *is some absolute constant,* $\Gamma_1, \Gamma_2 \in \{L, \Omega\}$ *and* $\widetilde{B}^{+,\nu}_{k,k_1,k_2}(t,\xi)$ *is defined as follows,*

$$\widetilde{B}^{+,\nu}_{k,k_1,k_2}(t,\xi):=\int_{\mathbb{R}^2}e^{it\Phi^{+,\nu}(\xi,\eta)}\tilde{q}_{+,\nu}(\xi-\eta,\eta)\widehat{\Gamma_1\Gamma_2 g_{k_1}}(t,\xi-\eta)\widehat{g^{\nu}_{k_2}}(t,\eta)\psi_k(\xi)d\eta.\tag{7.8}$$

Proof. – From (6.113) in Proposition 6.9, (6.130) in Proposition 6.13, and (7.13) in Lemma 7.4, we know that all terms except quadratic terms inside $\partial_t\widehat{\Gamma_1\Gamma_2 g_k}(t,\xi)$ already satisfy the desired estimate 7.7. Hence, we only need to estimate the quadratic terms. Based on the possible size of k_1 and k_2, we separate into two cases as follows.

If $(k_1, k_2) \in \chi^1_k$*, i.e.,* $|k_1 - k_2| \leq 10$*.* – Note that the following equality holds,

$$\begin{aligned}
\Gamma^1_\xi\Gamma^2_\xi B^{\mu,\nu}_{k_1,k_2}(t,\xi)&=\sum_{i=1,2,3}K^{\mu,\nu,1;i}_{k_1,k_2},\\
K^{\mu,\nu,1;1}_{k_1,k_2}&:=\int_{\mathbb{R}^2}e^{it\Phi^{\mu,\nu}(\xi,\eta)}\Gamma^1_\xi\Gamma^2_\xi\big(\tilde{q}_{\mu,\nu}(\xi-\eta,\eta)\widehat{g^{\mu}_{k_1}}(t,\xi-\eta)\big)\widehat{g^{\nu}_{k_2}}(t,\eta)d\eta,\\
K^{\mu,\nu,1;2}_{k_1,k_2}&:=\sum_{l,m=\{1,2\}}\int_{\mathbb{R}^2}e^{it\Phi^{\mu,\nu}(\xi,\eta)}it\big(\Gamma^l_\xi\Phi^{\mu,\nu}(\xi,\eta)\big)\Gamma^n_\xi\big(\tilde{q}_{\mu,\nu}(\xi-\eta,\eta)
\end{aligned}$$

$$\times \widehat{g_{k_1}^{\mu}}(t,\xi-\eta)\big)\widehat{g_{k_2}^{\nu}}(t,\eta)d\eta,$$

$$K_{k_1,k_2}^{\mu,\nu,1;3} := -\int_{\mathbb{R}^2} e^{it\Phi^{\mu,\nu}(\xi,\eta)}t^2\big(\Gamma_\xi^1\Phi^{\mu,\nu}(\xi,\eta)\Gamma_\xi^2\Phi^{\mu,\nu}(\xi,\eta)\big)\tilde{q}_{\mu,\nu}(\xi-\eta,\eta)$$
$$\times \widehat{g_{k_1}^{\mu}}(t,\xi-\eta)\big)\widehat{g_{k_2}^{\nu}}(t,\eta)d\eta.$$

From the $L^2 - L^\infty$ type bilinear estimate (2.5) in Lemma 2.2, the following estimate holds for some absolute constant C,

$$\sum_{|k_1-k_2|\leq 10} \|K_{k_1,k_2}^{\mu,\nu,1;1}\|_{L^2} \leq C2^{2k_1}\big(2^{2k}\|\nabla_\xi^2\widehat{g}_{k_1}(t,\xi)\|_{L^2} + 2^k\|\nabla_\xi\widehat{g}_{k_1}(t,\xi)\|_{L^2} + \|\widehat{g}_{k_1}(t,\xi)\|_{L^2}\big)$$
$$\times \|e^{-it\Lambda}g_{k_2}(t)\|_{L^\infty} \leq C2^{-m+\tilde{\delta}m}\epsilon_0.$$

We do integration by parts in "η" once for $K_{k_1,k_2}^{\mu,\nu,1;2}$ and do integration by parts in "η" twice for $K_{k_1,k_2}^{\mu,\nu,1;3}$. As a result, the following estimate holds for some absolute constant C,

$$\sum_{|k_1-k_2|\leq 10}\sum_{i=2,3} \|K_{k_1,k_2}^{\mu,\nu,1;i}\|_{L^2}$$
$$\leq \sum_{|k_1-k_2|\leq 10} C2^{2k_1}\big(\sum_{i=0,1,2} 2^{ik_1}\|\nabla_\xi^i\widehat{g}_{k_1}(t,\xi)\|_{L^2} + 2^{ik_1}\|\nabla_\xi^i\widehat{g}_{k_2}(t,\xi)\|_{L^2}\big)$$
$$\times \big(\|e^{-it\Lambda}g_{k_1}(t)\|_{L^\infty} + \|e^{-it\Lambda}g_{k_2}(t)\|_{L^\infty}\big)$$
$$+ \sum_{|k_1-k_2|\leq 10}\sum_{j_1\geq\max\{-k_{1,-},j_2\}} C2^{4k_1}\|e^{-it\Lambda}\mathcal{F}^{-1}[\nabla_\xi\widehat{g}_{k_2,j_2}(t,\xi)]\|_{L^\infty}\|\nabla_\xi\widehat{g}_{k_1,j_1}(t,\xi)\|_{L^2}$$
$$+ \sum_{|k_1-k_2|\leq 10}\sum_{j_2\geq\max\{-k_{2,-},j_1\}} C2^{4k_1}\|e^{-it\Lambda}\mathcal{F}^{-1}[\nabla_\xi\widehat{g}_{k_1,j_1}(t,\xi)]\|_{L^\infty}\|\nabla_\xi\widehat{g}_{k_2,j_2}(t,\xi)\|_{L^2}$$
$$\leq C2^{-m+\tilde{\delta}m}\epsilon_0 + \sum_{j_1\geq -k_{1,-}} C2^{-m+4k_1+2j_1}\|\varphi_{j_1}^{k_1}(x)g_{k_1}(t)\|_{L^2}\big(\sum_{j_2\geq j_1} 2^{j_2}\|\varphi_{j_2}^{k_2}(x)g_{k_2}(t)\|_{L^2}\big)$$
$$+ \sum_{j_2\geq -k_{2,-}} C2^{-m+4k_1+2j_2}\|\varphi_{j_2}^{k_2}(x)g_{k_2}(t)\|_{L^2}\big(\sum_{j_1\geq j_2} 2^{j_1}\|\varphi_{j_1}^{k_1}(x)g_{k_1}(t)\|_{L^2}\big)$$
$$\leq C2^{-m+\tilde{\delta}m}\epsilon_0.$$

If $(k_1,k_2)\in\chi_k^2$*, i.e.,* $k_2\leq k_1-10$. – For this case we have $\mu=+$. We separate it into two cases based on the size of k_1+k_2. If $k_1+k_2\leq -18m/19$, the following estimate holds from estimates (5.15) in Lemma 5.3,

$$\sum_{i=1,2,3}\Big\|\sum_{\nu\in\{+,-\}} K_{k_1,k_2}^{+,\nu,1;i}\Big\|_{L^2} + \Big\|\sum_{\nu\in\{+,-\}} \widetilde{B}_{k,k_1,k_2}^{+,\nu}(t,\xi)\Big\|_{L^2}$$
$$\leq \Big(\big(\sum_{i=0,1,2} 2^{ik_1}\|\nabla_\xi^i\widehat{g}_{k_1}(t,\xi)\|_{L^2}\big)$$
$$+ 2^{m+k_1+k_2}\big(\sum_{i=0,1} 2^{ik_1}\|\nabla_\xi^i\widehat{g}_{k_1}(t,\xi)\|_{L^2}\big) + 2^{2m+2k_1+2k_2}\|g_{k_1}(t)\|_{L^2}\Big)$$
$$\times C\min\{2^{k_1+3k_2}\|\widehat{g}_{k_2}(t)\|_{L_\xi^\infty} + 2^{2k_1+2k_2}\|\widehat{\mathrm{Re}[v]}(t,\xi)\psi_{k_2}(\xi)\|_{L_\xi^\infty}, 2^{2k_1+k_2}\|g_{k_2}\|_{L^2}\}$$

$$
\begin{aligned}
&\le C(2^{k_1+\tilde{\delta}m} + 2^{2m+3k_1+2k_2+2\tilde{\delta}m}) \min\{2^{k_1+k_2}, 2^{3k_2+m} + 2^{k_1+4k_2+2m}\}\epsilon_0 \\
&\le C 2^{-m-\beta m}\epsilon_0,
\end{aligned}
$$

where C is some absolute constant.

Now, we will rule out the case when k_1 is relatively large. Same as before, we move the derivative $\nabla_\xi = -\nabla_\eta$ in front of $\widehat{g}_{k_1}(t, \xi-\eta)$ around by doing integration by parts in η such that there is no derivative in front of $\widehat{g}_{k_1}(t, \xi-\eta)$. As a result, if $k_1 + k_2 \ge -18m/19$ and $k_1 \ge 5\beta m$, the following estimate holds for some absolute constant C,

$$
\begin{aligned}
\sum_{k_1+k_2\ge -18m/19, k_1\ge 5\beta m} \sum_{i=1,2,3} &\|K^{+,\nu,1;i}_{k_1,k_2}\|_{L^2} + \|\widetilde{B}^{+,\nu}_{k,k_1,k_2}(t,\xi)\|_{L^2} \\
&\le C 2^{2m+2k_1+k_2+4k_{1,+}} \|g_{k_1}(t)\|_{L^2} \\
&\quad \times \big(\|\nabla_\xi^2 \widehat{g}_{k_2}(t,\xi)\|_{L^2} + 2^{-k_2}\|\nabla_\xi \widehat{g}_{k_2}(t,\xi)\|_{L^2} + 2^{-2k_2}\|g_{k_2}(t)\|_{L^2}\big) \\
&\le \sum_{k_1+k_2\ge -18m/19, k_1\ge 5\beta m} C 2^{2m+\beta m+2k_1-k_2-(N_0-10)k_{1,+}}\epsilon_1^2 \le C 2^{-m-\beta m}\epsilon_0.
\end{aligned}
$$

Lastly, we consider the case when $k_1 + k_2 \ge -18m/19$ and $k_1 \le 5\beta m$. Note that

$$
\Gamma^1_\xi \Gamma^2_\xi B^{+,\nu}_{k_1,k_2}(t,\xi) - \int_{\mathbb{R}^2} e^{it\Phi^{+,\nu}(\xi,\eta)} \tilde{q}_{+,\nu}(\xi-\eta,\eta) \widehat{\Gamma^1\Gamma^2 g_{k_1}}(t,\xi-\eta)\widehat{g^\nu_{k_2}}(t,\eta)d\eta = \sum_{i=1}^{4} K^{+,\nu,2;i}_{k_1,k_2},
$$

where

$$
\begin{aligned}
K^{+,\nu,2;1}_{k_1,k_2} &= \int_{\mathbb{R}^2} e^{it\Phi^{+,\nu}(\xi,\eta)} \tilde{q}_{+,\nu}(\xi-\eta,\eta)\widehat{g_{k_1}}(t,\xi-\eta)\widehat{\Gamma^1\Gamma^2 g^\nu_{k_2}}(t,\eta)d\eta, \\
K^{+,\nu,2;2}_{k_1,k_2} &= \sum_{j_1\ge k_{1,-}, j_2\ge -k_{2,-}} K^{+,\nu,2;2}_{k_1,j_1,k_2,j_2}, \\
K^{+,\nu,2;2}_{k_1,j_1,k_2,j_2} := \sum_{(l,n)\in\{(1,2),(2,1)\}} &\int_{\mathbb{R}^2} e^{it\Phi^{+,\nu}(\xi,\eta)} \Big[\tilde{q}_{+,\nu}(\xi-\eta,\eta)\widehat{\Gamma^l g_{k_1,j_1}}(t,\xi-\eta)\widehat{\Gamma^n g^\nu_{k_2,j_2}}(t,\eta) \\
&+ (\Gamma^l_\xi + \Gamma^l_\eta + d_{\Gamma^l})\tilde{q}_{+,\nu}(\xi-\eta,\eta)\big(\widehat{\Gamma^n g_{k_1,j_1}}(t,\xi-\eta)\widehat{g^\nu_{k_2,j_2}}(t,\eta) \\
&+ \widehat{g_{k_1,j_1}}(t,\xi-\eta)\widehat{\Gamma^n g^\nu_{k_2,j_2}}(t,\eta)\big) \\
&+ it(\Gamma^l_\xi + \Gamma^l_\eta)\Phi^{+,\nu}(\xi,\eta)(\Gamma^n_\xi + \Gamma^n_\eta + d_{\Gamma^n})\tilde{q}_{+,\nu}(\xi-\eta,\eta)\widehat{g_{k_1,j_1}}(t,\xi-\eta)\widehat{g^\nu_{k_2,j_2}}(t,\eta) \\
&+ (\Gamma^1_\xi + \Gamma^1_\eta + d_{\Gamma^1})(\Gamma^2_\xi + \Gamma^2_\eta + d_{\Gamma^2})\tilde{q}_{+,\nu}(\xi-\eta,\eta)\widehat{g_{k_1,j_1}}(t,\xi-\eta)\widehat{g^\nu_{k_2,j_2}}(t,\eta)d\eta. \\
K^{+\nu,2;3}_{k_1,k_2} &= \sum_{(l,n)\in\{(1,2),(2,1)\}} \int_{\mathbb{R}^2} e^{it\Phi^{+,\nu}(\xi,\eta)} it(\Gamma^l_\xi + \Gamma^l_\eta)\Phi^{+,\nu}(\xi,\eta)\tilde{q}_{+,\nu}(\xi-\eta,\eta) \\
&\quad \times \big(\widehat{g^\nu_{k_2}}(t,\eta)\widehat{\Gamma^n g_{k_1}}(t,\xi-\eta) + \widehat{g_{k_1}}(t,\xi-\eta)\widehat{\Gamma^n g^\nu_{k_2}}(t,\eta)\big)d\eta \\
K^{+,\nu,2;4}_{k_1,k_2} &= -\int_{\mathbb{R}^2} e^{it\Phi^{+,\nu}(\xi,\eta)} t^2(\Gamma^1_\xi + \Gamma^1_\eta)\Phi^{\mu,\nu}(\xi,\eta)(\Gamma^2_\xi + \Gamma^2_\eta)\Phi^{+,\nu}(\xi,\eta)\tilde{q}_{+,\nu}(\xi-\eta,\eta) \\
&\quad \times \widehat{g_{k_1}}(t,\xi-\eta)\widehat{g^\nu_{k_2}}(t,\eta)d\eta.
\end{aligned}
$$

From the $L^2 - L^\infty$ type estimate (2.5) in Lemma 2.2, the following estimate holds,

$$
\|K^{+,\nu,2;1}_{k_1,k_2}\|_{L^2} \le C 2^{2k_1}\|\Gamma^1\Gamma^2 g_{k_2}(t)\|_{L^2}\|e^{-it\Lambda}g_{k_1}(t)\|_{L^\infty} \le C 2^{-m+\tilde{\delta}m}\epsilon_0.
$$

where C is some absolute constant. Now, we proceed to estimate $K^{+,\nu,2;2}_{k_1,k_2}$. By doing integration by parts in η many times, we can rule out the case when $\max\{j_1,j_2\}\leq m+k_{1,-}-\beta m$. From the L^2-L^∞ type estimate (2.5) in Lemma 2.2, the following estimate holds when $\max\{j_1,j_2\}\geq m+k_{1,-}-\beta m$,

$$\sum_{\max\{j_1,j_2\}\geq m+k_{1,-}-\beta m}\|K^{+,\nu,2;2}_{k_1,j_1,k_2,j_2}\|_{L^2}$$
$$\leq \sum_{j_1\geq \max\{m+k_{1,-}-\beta m, j_2\}} C2^{2k_1}\big(2^{j_1+k_1+k_2+j_2}+2^{m+k_1+k_2}\big)$$
$$\times \|g_{k_1,j_1}(t)\|_{L^2}2^{-m}\|g_{k_2,j_2}(t)\|_{L^1}$$
$$+\sum_{j_2\geq \max\{m+k_{1,-}-\beta m, j_1\}} C2^{2k_1}\big(2^{j_1+k_1+k_2+j_2}+2^{m+k_1+k_2}\big)$$
$$\times \|g_{k_2,j_2}(t)\|_{L^2}2^{-m}\|g_{k_1,j_1}(t)\|_{L^1}$$
$$\leq C2^{-2m-k_2+20\beta m}\epsilon_0\leq C2^{-m-\beta m}\epsilon_0,$$

where C is some absolute constant.

Lastly, it remains to consider $K^{+,\nu,2;i}_{k_1,k_2}$, $i\in\{3,4\}$. We do integration by parts in "η" once for $K^{+,\nu,2;3}_{k_1,k_2}$ and do integration by parts in "η" twice for $K^{+,\nu,2;4}_{k_1,k_2}$. As a result, the following estimate holds,

$$\|K^{+,\nu,2;3}_{k_1,k_2}\|_{L^2}+\|K^{+,\nu,2;4}_{k_1,k_2}\|_{L^2}$$
$$\leq C\Big(\sum_{i=0,1,2}2^{ik_2}\|\nabla^i_\xi\widehat{g}_{k_2}(t,\xi)\|_{L^2}+2^{ik_1}\|\nabla^i_\xi\widehat{g}_{k_1}(t,\xi)\|_{L^2}\Big)$$
$$\times\big(2^{2k_1}\|e^{-it\Lambda}g_{k_1}\|_{L^\infty}+2^{k_1+k_2}\|e^{-it\Lambda}g_{k_2}\|_{L^\infty}\big)$$
$$+\sum_{j_1\geq\max\{-k_{1,-},j_2\}, j_2\geq -k_{2,-}} C2^{-m+3k_1+k_2+j_1+2j_2}\|\varphi^{k_1}_{j_1}(x)g_{k_1}(t)\|_{L^2}\|\varphi^{k_2}_{j_2}(x)g_{k_2}(t)\|_{L^2}$$
$$+\sum_{j_2\geq\max\{-k_{2,-},j_1\}, j_1\geq -k_{1,-}} C2^{-m+3k_1+k_2+j_2+2j_1}\|\varphi^{k_1}_{j_1}(x)g_{k_1}(t)\|_{L^2}\|\varphi^{k_2}_{j_2}(x)g_{k_2}(t)\|_{L^2}$$
$$\leq C2^{-m+2\tilde{\delta}m+\delta m/2+k}\epsilon_0,$$

where C is some absolute constant. Hence finishing the proof. □

The rest of this section is devoted to prove the weighted norm estimates for the remainder term $\mathcal{R}_1$ in (4.35), which will be done by using the fixed point type formulation (3.8). Before that, we first prove the weighted norm estimates for a very general multilinear form, which will be used as black boxes.

For $g_i\in H^{N_0-10}\cap Z_1\cap Z_2$, $i\in\{1,\ldots,5\}$, we define a multilinear form as follows,

$$Q^{\tau,\kappa,\iota}_{k,\mu,\nu}(g_1(t),g_2(t),g_3(t),g_4(t),g_5(t))(\xi)$$
$$:=\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{it\Phi^{\tau,\kappa,\iota}_{\mu,\nu}(\xi,\eta,\sigma,\eta',\sigma')}q^{\tau,\kappa,\iota}_{\mu,\nu}(\xi,\eta,\sigma,\eta',\sigma')$$
$$\times\widehat{g^\tau_1}(t,\xi-\eta)\widehat{g^\kappa_2}(t,\eta-\sigma)\widehat{g^\iota_3}(t,\sigma-\eta')\widehat{g^\mu_4}(t,\eta'-\sigma')\widehat{g^\nu_5}(t,\sigma')\psi_k(\xi)d\sigma' d\eta' d\eta d\sigma,$$

where the phase $\Phi^{\tau,\kappa,\iota}_{\mu,\nu}(\xi,\eta,\sigma,\eta',\sigma')$ is defined as follows,

$$\Phi^{\tau,\kappa,\iota}_{\mu,\nu}(\xi,\eta,\sigma,\eta',\sigma') = \Lambda(|\xi|)-\tau\Lambda(|\xi-\eta|)-\kappa\Lambda(|\eta-\sigma|)-\iota\Lambda(|\sigma-\eta'|)-\mu\Lambda(|\eta'-\sigma'|)-\nu\Lambda(|\sigma'|),$$

and the symbol $q^{\tau,\kappa,\iota}_{\mu,\nu}(\xi,\eta,\sigma,\eta',\sigma')$ satisfies the following estimate for some absolute constant C,

$$\|q^{\tau,\kappa,\iota}_{\mu,\nu}(\xi,\eta,\sigma,\eta',\sigma')\psi_k(\xi)\psi_{k_1}(\xi-\eta)\psi_{k_2}(\eta-\sigma)\psi_{k_3}(\sigma-\eta')\psi_{k_4}(\eta'-\sigma')\psi_{k_5}(\sigma')\|_{\mathcal{S}^\infty} \le C2^{2k_1+6\max\{k_1,\dots,k_5\}_+}.$$

For $i\in\{0,1,2\}$, we define auxiliary function spaces as follows,

(7.9)
$$\|f\|_{\widetilde{Z}_i} := \sup_{k\in\mathbb{Z}}\sup_{j\ge -k_-}\|f\|_{\widetilde{B}^i_{k,j}}, \quad \|f\|_{\widetilde{B}^i_{k,j}} := 2^{(1-\delta)k+k_++(20-5i)k_++ij+\delta j}\|\varphi^k_j(x)P_kf\|_{L^2}.$$

From the above definition and the definition of Z_i-norm, $i\in\{1,2\}$, in (1.22) and (1.23), we know that the following estimates hold for some absolute constant C,

$$\sum_{k\in\mathbb{Z}}2^{k+(20-5i)k_+}\|\nabla^i_\xi\widehat{f}_k(t,\xi)\|_{L^2}\le C\|f\|_{\widetilde{Z}_i}, \qquad \|f\|_{Z_l}\le C\|f\|_{\widetilde{Z}_l},$$

where $i\in\{0,1,2\}$, $l\in\{1,2\}$.

Lemma 7.3. – *Let $g_i(t)\in H^{N_0-10}\cap Z_1\cap Z_2$, $i\in\{1,\dots,5\}$. Assume that the following estimate holds for any $t\in[2^{m-1},2^m]$, $m\in\mathbb{Z}_+$,*

$$2^{-\delta m}\|g_i(t)\|_{H^{N_0-10}}+\|g_i(t)\|_{Z_1}+2^{-\tilde{\delta}m}\|g_i(t)\|_{Z_2}\le\epsilon_1:=\epsilon_0^{5/6}, \quad i\in\{1,\dots,5\},$$

then the following estimates hold for any $t\in[2^{m-1},2^m]$ and any $\mu,\nu,\kappa,\iota,\tau\in\{+,-\}$,

(7.10)
$$\sum_{i=0,1,2}2^{(3-i)m}\|\mathcal{F}^{-1}\big[Q^{\tau,\kappa,\iota}_{k,\mu,\nu}(g_1(t),g_2(t),g_3(t),g_4(t),g_5(t))(\xi)\big]\|_{\widetilde{Z}_i}\le C2^{-m/2+190\beta m}\epsilon_0^2,$$

where C is some absolute constant.

Proof. – As usual, we rule out the very high frequency case and the very low frequency case first. Without loss of generality, we assume that $k_5\le k_4\le k_3\le k_2\le k_1$. From the $L^2-L^\infty-L^\infty-L^\infty-L^\infty$ type multilinear estimate and the $L^\infty\to L^2$ type Sobolev estimate, the following estimate holds for some absolute constant C,

$$\sum_{i=0,1,2}2^{(3-i)m}\|\mathcal{F}^{-1}[Q^{\tau,\kappa,\iota}_{k,\mu,\nu}(g_{1,k_1}(t),g_{2,k_2}(t),g_{3,k_3}(t),g_{4,k_4}(t),g_{5,k_5}(t))(\xi)]\|_{\widetilde{B}^i_{k,j}}$$
(7.11)
$$\le C2^{3m+(2+\delta)j}2^{30k_{1,+}+(1-\delta)k+k_5}\|g_{k_1}\|_{L^2}\|e^{-it\Lambda}g_{k_2}\|_{L^\infty}\times\|e^{-it\Lambda}g_{k_3}\|_{L^\infty}\|e^{-it\Lambda}g_{k_4}\|_{L^\infty}\|g_{k_5}\|_{L^2}.$$

From estimate (7.11), we can rule out the case when $k_{1,+}\ge(3m+2j)/(N_0-45)$ or $k_5\le -3m-2(1+2\delta)j$, or $k\le -3m-2(1+2\delta)j$. Hence it would be sufficient to consider fixed k,k_1,k_2,k_3,k_4, and k_5 in the following range,

(7.12)
$$-3m-2(1+2\delta)j\le k_5,k\le k_1+2\le(3m+2j)/(N_0-45).$$

From now on, $k, k_i, i \in \{1, \ldots, 5\}$, are restricted inside the range (7.12). We first consider the case when $j \geq (1+\delta)(m+k_{1,+}) + \beta m$. For this case, we do spatial localization for inputs "g_{k_1}" and "g_{k_2}". Note that the following estimate holds for the case we are considering,

$$2^{j-10} \leq \left|\nabla_\xi\left[x\cdot\xi + t\,\Phi^{\tau,\kappa,\iota}_{\mu,\nu}(\xi,\eta,\sigma,\eta',\sigma')\right]\right|\varphi^k_j(x) \leq 2^{j+10}.$$

Therefore, by doing integration by parts in "ξ" many times, we can rule out the case when $\min\{j_1, j_2\} \leq j - \delta j - \delta m$, where j_1 and j_2 are the spatial concentrations of g_{k_1} and g_{k_2} respectively. For the case when $\min\{j_1, j_2\} \geq j - \delta j - \delta m$, the following estimate holds from the $L^2 - L^\infty - L^\infty - L^\infty - L^\infty$ type multilinear estimate,

$$\begin{aligned}
&\sum_{\min\{j_1,j_2\}\geq j-\delta j-\delta m}\ \sum_{i=0,1,2} 2^{(3-i)m}\|\mathscr{F}^{-1}\big[Q^{\tau,\kappa,\iota}_{k,\mu,\nu}(g_{1,k_1,j_1}(t), g_{2,k_2,j_2}(t),\\
&\qquad\qquad g_{3,k_3}(t), g_{4,k_4}(t), g_{5,k_5}(t))(\xi)\big]\|_{\widetilde{B}^i_{k,j}}\\
&\leq \sum_{i=0,1,2}\ \sum_{\min\{j_1,j_2\}\geq j-\delta j-\delta m} C2^{(3-i)m+ij+\delta j+3\beta m+(3-\delta)k_1+30k_{1,+}}\\
&\quad\times \|g_{1,k_1,j_1}\|_{L^2}2^{k_2}\|g_{2,k_2,j_2}\|_{L^2}\|e^{-it\Lambda}g_{3,k_3}\|_{L^\infty}\|e^{-it\Lambda}g_{4,k_4}\|_{L^\infty}\|e^{-it\Lambda}g_{5,k_5}\|_{L^\infty}\\
&\leq C2^{-m/2+50\beta m}\epsilon_0^2,
\end{aligned}$$

where C is some absolute constant.

It remains to consider the case when $j \leq (1+\delta)(m+k_{1,+}) + \beta m$. Recall (7.12). Note that j now is bounded, we have $-6m \leq k_5 \leq k_1 \leq 5\beta m$. We split into three cases based on sizes of the difference between k_1 and k_2 and the difference between k_2 and k_3 as follows.

If $k_2 \leq k_1 - 10$. – For this case, we have a good lower bound for $\nabla_\eta \Phi^{\tau,\kappa,\iota}_{\mu,\nu}(\xi,\eta,\sigma,\eta',\sigma')$. Hence, we can do integration by parts in "η" many times to rule out the case when $\max\{j_1, j_2\} \leq m + k_{1,-} - \beta m$. From the $L^2 - L^\infty - L^\infty - L^\infty - L^\infty$ type multilinear estimate, the following estimate holds for some absolute constant C,

$$\begin{aligned}
&\sum_{\max\{j_1,j_2\}\geq m+k_{1,-}-\beta m}\ \sum_{i=0,1,2} 2^{(3-i)m}\|\mathscr{F}^{-1}\big[Q^{\tau,\kappa,\iota}_{k,\mu,\nu}(g_{1,k_1,j_1}(t), g_{2,k_2,j_2}(t),\\
&\qquad\qquad g_{3,k_3}(t), g_{4,k_4}(t), g_{5,k_5}(t))(\xi)\big]\|_{\widetilde{B}^i_{k,j}}\\
&\leq \sum_{j_1\geq\max\{-k_{1,-},j_2,m+k_{1,-}-\beta m\}} C2^{3m+4\beta m+3k_1+30k_{1,+}}\|g_{1,k_1,j_1}\|_{L^2}\|e^{-it\Lambda}g_{2,k_2,j_2}\|_{L^\infty}\\
&\quad\times\|e^{-it\Lambda}g_{3,k_3}\|_{L^\infty}\|e^{-it\Lambda}g_{4,k_4}\|_{L^\infty}\|e^{-it\Lambda}g_{5,k_5}\|_{L^\infty}\\
&\quad+\sum_{j_2\geq\max\{-k_{2,-},j_1,m+k_{1,-}-\beta m\}} C2^{3m+4\beta m+3k_1+30k_{1,+}+k_4+k_5}\|g_{2,k_2,j_2}\|_{L^2}\\
&\quad\times\|e^{-it\Lambda}g_{1,k_1,j_1}\|_{L^\infty}\|e^{-it\Lambda}g_{3,k_3}\|_{L^\infty}\|g_{4,k_4}\|_{L^2}\|g_{5,k_5}\|_{L^2}\\
&\leq C2^{-m/2+180\beta m}\epsilon_0^2.
\end{aligned}$$

If $|k_1 - k_2| \leq 10$ *and* $k_3 \leq k_1 - 20$. – Note that, $\nabla_\sigma \Phi^{\tau,\kappa,\iota}_{\mu,\nu}(\xi,\eta,\sigma,\eta',\sigma')$ has a good lower bound for the case we are considering. Hence, by doing integration by parts in σ, we can rule out the case when $\max\{j_2, j_3\} \leq m + k_{2,-} - \beta m$, where j_2 and j_3 are the spatial concentrations of inputs g_{k_2} and g_{k_3} respectively. From the $L^2 - L^\infty - L^\infty - L^\infty - L^\infty$

type multilinear estimate, the following estimate holds for some absolute constant C,

$$
\begin{aligned}
\sum_{\max\{j_2,j_3\}\geq m+k_{2,-}-\beta m}\ \sum_{i=0,1,2} & 2^{(3-i)m}\|\mathcal{F}^{-1}\big[Q^{\tau,\kappa,\iota}_{k,\mu,\nu}(g_{1,k_1}(t), g_{2,k_2,j_2}(t), g_{3,k_3,j_3}(t),\\
& \qquad g_{4,k_4}(t), g_{5,k_5}(t))(\xi)\big]\|_{\widetilde{B}^i_{k,j}}\\
\leq & \sum_{j_2\geq\max\{-k_{2,-},j_3,m+k_{1,-}-\beta m\}} C2^{3m+4\beta m+3k_1+30k_{1,+}}\|g_{3,k_3,j_3}\|_{L^2}\\
& \times\|e^{-it\Lambda}g_{2,k_2,j_2}\|_{L^\infty}\|e^{-it\Lambda}g_{1,k_1}\|_{L^\infty}\|e^{-it\Lambda}g_{4,k_4}\|_{L^\infty}\|e^{-it\Lambda}g_{k_5}\|_{L^\infty}\\
& + \sum_{j_3\geq\max\{-k_{3,-},j_2,m+k_{1,-}-\beta m\}} C2^{3m+4\beta m+3k_1+30k_{1,+}+k_4+k_5}\|g_{3,k_3,j_3}\|_{L^2}\\
& \times\|e^{-it\Lambda}g_{2,k_2,j_2}\|_{L^\infty}\|e^{-it\Lambda}g_{1,k_1}\|_{L^\infty}\|g_{4,k_4}\|_{L^2}\|g_{5,k_5}\|_{L^2}\\
\leq & C2^{-m/2+180\beta m}\epsilon_0^2.
\end{aligned}
$$

If $|k_1-k_2|\leq 10$ *and* $|k_2-k_3|\leq 10$. – This case is straightforward. By the $L^2-L^\infty-L^\infty-L^\infty-L^\infty$ type multilinear estimate, the following estimate holds for some absolute constant C,

$$
\begin{aligned}
\sum_{i=0,1,2} & 2^{(3-i)m}\|\mathcal{F}^{-1}\big[Q^{\tau,\kappa,\iota}_{k,\mu,\nu}(g_{1,k_1}(t), g_{2,k_2}(t), g_{3,k_3}(t), g_{4,k_4}(t), g_{5,k_5}(t))(\xi)\big]\|_{\widetilde{B}^i_{k,j}}\\
& \leq C2^{3m+4\beta m+3k_1+30k_{1,+}}\|g_{5,k_5}(t)\|_{L^2}\|e^{-it\Lambda}g_{1,k_1}(t)\|_{L^\infty}\|e^{-it\Lambda}g_{2,k_2}(t)\|_{L^\infty}\\
& \quad\times\|e^{-it\Lambda}g_{3,k_3}(t)\|_{L^\infty}\|e^{-it\Lambda}g_{4,k_4}(t)\|_{L^\infty}\\
& \leq C2^{-m/2+180\beta m}\epsilon_0^2.
\end{aligned}
$$

To sum up, our desired estimate (7.10) holds, hence finishing the proof. □

With the multilinear estimate (7.10) in the above lemma, now we are ready to estimate the Z_i-norm of the quintic and higher order remainder term $\mathcal{R}_1$, $i\in\{1,2\}$.

LEMMA 7.4. – *Under the bootstrap assumption* (4.49), *there exists some absolute constant C such that the following estimate holds for the profile of the remainder term* $\mathcal{R}_1$,

$$\sup_{t\in[2^{m-1},2^m]}\sum_{i=1,2}\|e^{it\Lambda}\mathcal{R}_1\|_{Z_i}\leq C2^{-3m/2+200\beta m}\epsilon_0. \tag{7.13}$$

Proof. – Recall the definition of $u=\tilde{\Lambda}h+i\tilde{\psi}$ and the definition of v in (4.20). To estimate the weighted norms of the reminder term $e^{it\Lambda}\mathcal{R}_1$, from estimate (7.10) in Lemma 7.3, we know that it would be sufficient to estimate the weighted norms of $e^{it\Lambda}\Lambda_{\geq 5}[B(h)\psi]=e^{it\Lambda}\Lambda_{\geq 5}[\partial_z\varphi](t)|_{z=0}$.

Recall the fixed point type formulation for $\nabla_{x,z}\varphi$ in (3.8). We decompose $\Lambda_{\geq 5}[g_i(z)]$ into two parts: one of them doesn't depend on $\Lambda_{\geq 5}[\nabla_{x,z}\varphi]$ while the other part does depend (linearly depend) on $\Lambda_{\geq 5}[\nabla_{x,z}\varphi]$. For the first part, estimate (7.10) in Lemma 7.3 is very sufficient. Hence, it remains to estimate the second part. As usual, by doing integration by parts in ξ many times, we can rule out the case when $j\geq(1+\delta)\big(\max\{m+k_{1,+},-k_-\}\big)+\beta m$.

If $j \leq (1+\delta)\big(\max\{m+k_{1,+}, -k_-\}\big) + \beta m$, from the estimate (7.10) in Lemma 7.3 and the $L^2 - L^\infty$ type bilinear estimate, the following estimate holds for some absolute constant C,

$$\begin{aligned}
\sum_{i=1,2} \|e^{it\Lambda}\Lambda_{\geq 5}[\nabla_{x,z}\varphi](t)\|_{L^\infty_z Z_i} &\leq C2^{-3m/2+200\beta m}\epsilon_0 \\
&+ C2^{2m+3\beta m}\big(\|\Lambda_{\geq 6}[\nabla_{x,z}\varphi](t,\xi)\|_{L^\infty_z H^{15}}\|e^{-it\Lambda}g\|_{W^{20,0}} + \|g\|_{H^{20}}\||\nabla|\Lambda_5[\nabla_{x,z}\varphi]\|_{L^\infty_z H^{15}}\big) \\
&+ C2^{3\beta m}\|g\|_{H^{20}}\|\Lambda_{\geq 5}[\nabla_{x,z}\varphi](t,\xi)\|_{L^\infty_z H^{20}}.
\end{aligned} \tag{7.14}$$

Similar to the proof of (3.17) in Lemma 3.2, the following estimate holds for $i \in \{5,6\}$,

$$\begin{aligned}
&\|\Lambda_{\geq i}[\nabla_{x,z}\varphi]\|_{L^\infty_z H^{20}} \\
&\quad \leq C\big[\|(h,\psi)\|_{W^{30,1}}\|(h,\psi)\|^{i-2}_{W^{30,0}}\|(h,\psi)\|_{H^{30}} + \|(h,\psi)\|_{W^{30}}\|\Lambda_{\geq i}[\nabla_{x,z}\varphi]\|_{L^\infty_z H^{20}}\big],
\end{aligned}$$

where C is some absolute constant. Under the bootstrap assumption (4.49), the above estimate further implies the following estimate,

$$\begin{aligned}
&\|\Lambda_{\geq i}[\nabla_{x,z}\varphi]\|_{L^\infty_z H^{20}} \\
&\quad \leq 2C\|(h,\psi)\|_{W^{30,1}}\|(h,\psi)\|^{i-2}_{W^{30,0}}\|(h,\psi)\|_{H^{30}} \leq C2^{-im/2+\beta m}\epsilon_0^2, \quad i \in \{5,6\}.
\end{aligned} \tag{7.15}$$

Therefore, from the estimates (7.14) and (7.15) and the estimate (7.10) in Lemma 7.3, we obtain the following estimate,

$$\sum_{i=1,2} \|e^{it\Lambda}\Lambda_{\geq 5}[\nabla_{x,z}\varphi](t)\|_{L^\infty_z Z_i} \leq C2^{-3m/2+200\beta m}\epsilon_0, \tag{7.16}$$

where C is some absolute constant. Hence finishing the proof of the desired estimate (7.13). □

BIBLIOGRAPHY

[1] T. Alazard, N. Burq, C. Zuily, On the water-wave equations with surface tension, *Duke Math. J.* **158** (2011), 413–499.

[2] T. Alazard, N. Burq, C. Zuily, On the Cauchy problem for gravity water waves, *Invent. math.* **198** (2014), 71–163.

[3] T. Alazard, J.-M. Delort, Global solutions and asymptotic behavior for two dimensional gravity water waves, *Ann. Sci. Éc. Norm. Supér.* **48** (2015), 1149–1238.

[4] T. Alazard, J.-M. Delort, *Sobolev estimates for two dimensional gravity water waves*, Astérisque **374**, 2015.

[5] T. Alazard, G. Métivier, Paralinearization of the Dirichlet to Neumann operator, and regularity of three-dimensional water waves, *Comm. Partial Differential Equations* **34** (2009), 1632–1704.

[6] B. Alvarez-Samaniego, D. Lannes, Large time existence for 3D water-waves and asymptotics, *Invent. math.* **171** (2008), 485–541.

[7] T. B. Benjamin, P. J. Olver, Hamiltonian structure, symmetries and conservation laws for water waves, *J. Fluid Mech.* **125** (1982), 137–185.

[8] K. Beyer, M. Günther, On the Cauchy problem for a capillary drop. I. Irrotational motion, *Math. Methods Appl. Sci.* **21** (1998), 1149–1183.

[9] A. Castro, D. Córdoba, C. Fefferman, F. Gancedo, J. Gómez-Serrano, Finite time singularities for the free boundary incompressible Euler equations, *Ann. of Math.* **178** (2013), 1061–1134.

[10] D. Christodoulou, H. Lindblad, On the motion of the free surface of a liquid, *Comm. Pure Appl. Math.* **53** (2000), 1536–1602.

[11] D. Coutand, S. Shkoller, Well-posedness of the free-surface incompressible Euler equations with or without surface tension, *J. Amer. Math. Soc.* **20** (2007), 829–930.

[12] S. Deng, S.-M. Sun, Exact theory of three-dimensional water waves at the critical speed, *SIAM J. Math. Anal.* **42** (2010), 2721–2761.

[13] Y. Deng, A. D. Ionescu, B. Pausader, The Euler-Maxwell system for electrons: global solutions in 2D, *Arch. Ration. Mech. Anal.* **225** (2017), 771–871.

[14] Y. Deng, A. D. Ionescu, B. Pausader, F. Pusateri, Global solutions of the gravity-capillary water-wave system in three dimensions, *Acta Math.* **219** (2017), 213–402.

[15] P. Germain, N. Masmoudi, J. Shatah, Global solutions for 3D quadratic Schrödinger equations, *Int. Math. Res. Not.* **2009** (2009), 414–432.

[16] P. Germain, N. Masmoudi, J. Shatah, Global solutions for 2D quadratic Schrödinger equations, *J. Math. Pures Appl.* **97** (2012), 505–543.

[17] P. Germain, N. Masmoudi, J. Shatah, Global solutions for the gravity water waves equation in dimension 3, *Ann. of Math.* **175** (2012), 691–754.

[18] P. Germain, N. Masmoudi, J. Shatah, Global existence for capillary water waves, *Comm. Pure Appl. Math.* **68** (2015), 625–687.

[19] Y. Guo, A. D. Ionescu, B. Pausader, Global solutions of the Euler-Maxwell two-fluid system in 3D, *Ann. of Math.* **183** (2016), 377–498.

[20] Z. Guo, L. Peng, B. Wang, Decay estimates for a class of wave equations, *J. Funct. Anal.* **254** (2008), 1642–1660.

[21] B. Harrop-Griffiths, M. Ifrim, D. Tataru, Finite depth gravity water waves in holomorphic coordinates, *Ann. PDE* **3** (2017), Paper No. 4.

[22] J. K. Hunter, M. Ifrim, D. Tataru, Two dimensional water waves in holomorphic coordinates, *Comm. Math. Phys.* **346** (2016), 483–552.

[23] M. Ifrim, D. Tataru, The lifespan of small data solutions in two dimensional capillary water waves, *Arch. Ration. Mech. Anal.* **225** (2017), 1279–1346.

[24] M. Ikeda, T. Inui, Small data blow-up of L^2 or H^1-solution for the semilinear Schrödinger equation without gauge invariance, *J. Evol. Equ.* **15** (2015), 571–581.

[25] A. D. Ionescu, B. Pausader, The Euler-Poisson system in 2D: global stability of the constant equilibrium solution, *Int. Math. Res. Not.* **2013** (2013), 761–826.

[26] A. D. Ionescu, B. Pausader, Global solutions of quasilinear systems of Klein-Gordon equations in 3D, *J. Eur. Math. Soc. (JEMS)* **16** (2014), 2355–2431.

[27] A. D. Ionescu, F. Pusateri, Global solutions for the gravity water waves system in 2d, *Invent. math.* **199** (2015), 653–804.

[28] A. D. Ionescu, F. Pusateri, Global analysis of a model for capillary water waves in two dimensions, *Comm. Pure Appl. Math.* **69** (2016), 2015–2071.

[29] A. D. Ionescu, F. Pusateri, *Global regularity for 2D water waves with surface tension*, Mem. Amer. Math. Soc. **256**, 2018.

[30] D. Lannes, Well-posedness of the water-waves equations, *J. Amer. Math. Soc.* **18** (2005), 605–654.

[31] H. Lindblad, Well-posedness for the motion of an incompressible liquid with free surface boundary, *Ann. of Math.* **162** (2005), 109–194.

[32] V. I. Nalimov, The Cauchy-Poisson problem, *Dinamika Splošn. Sredy* **18** (1974), 104–210.

[33] J. Shatah, C. Zeng, Geometry and a priori estimates for free boundary problems of the Euler equation, *Comm. Pure Appl. Math.* **61** (2008), 698–744.

[34] X. Wang, On the 3-dimensional water waves system above a flat bottom, *Anal. PDE* **10** (2017), 893–928.

[35] X. Wang, Global infinite energy solutions for the 2D gravity water waves system, *Comm. Pure Appl. Math.* **71** (2018), 90–162.

[36] X. Wang, Global solution for the 3D gravity water waves system above a flat bottom, *Adv. Math.* **346** (2019), 805–886.

[37] S. Wu, Well-posedness in Sobolev spaces of the full water wave problem in 2-D, *Invent. math.* **130** (1997), 39–72.

[38] S. Wu, Well-posedness in Sobolev spaces of the full water wave problem in 3-D, *J. Amer. Math. Soc.* **12** (1999), 445–495.

[39] S. Wu, Almost global wellposedness of the 2-D full water wave problem, *Invent. math.* **177** (2009), 45–135.

[40] S. Wu, Global wellposedness of the 3-D full water wave problem, *Invent. math.* **184** (2011), 125–220.

[41] H. Yosihara, Gravity waves on the free surface of an incompressible perfect fluid of finite depth, *Publ. Res. Inst. Math. Sci.* **18** (1982), 49–96.

[42] V. Zakharov, Stability of periodic waves of finite amplitude on the surface of a deep fluid, *J. Appl. Mech. Tech. Phys.* **9** (1968), 190–194.

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Xuecheng Wang
YMSC
Tsinghua University
Beijing, China
E-mail: `xuecheng@tsinghua.edu.cn`, `xuecheng.wang.work@gmail.com`