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Affine walled Brauer algebras and super Schur–Weyl duality [☆]

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ABSTRACT

A new class of associative algebras referred to as affine walled Brauer algebras are introduced. These algebras are free with infinite rank over a commutative ring containing 1. Then level two walled Brauer algebras over \mathbb{C} are defined, which are some cyclotomic quotients of affine walled Brauer algebras. We establish a super Schur–Weyl duality between affine walled Brauer algebras and general linear Lie superalgebras, and realize level two walled Brauer algebras as endomorphism algebras of tensor modules of Kac modules with mixed tensor products of the natural module and its dual over general linear Lie superalgebras, under some conditions. We also prove the weakly cellularity of level two walled Brauer algebras, and give a classification of their simple modules over \mathbb{C} . This in turn enables us to classify the indecomposable direct summands of the said tensor modules.

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1. Introduction

Schur–Weyl reciprocities set up close relationship between polynomial representations of general linear groups GL_n over \mathbb{C} and representations of symmetric groups \mathfrak{S}_r [15]. Such results have been generalized to several other cases. Brauer [4] studied similar problems for symplectic groups and orthogonal groups. A class of associative algebras $\mathcal{B}_r(\delta)$, called Brauer algebras, came into the picture, which play the same important role as that of symmetric groups in Schur’s work.

Walled Brauer algebras $\mathcal{B}_{r,t}(\delta)$ (cf. Definition 2.2) are subalgebras of Brauer algebras $\mathcal{B}_{r+t}(\delta)$. They first appeared independently in Koike’s work [22] and Turaev’s work [35], which were partially motivated by Schur–Weyl dualities between walled Brauer algebras and general linear groups arising from mutually commuting actions on mixed tensor modules $V^{\otimes r} \otimes (V^*)^{\otimes t}$ of the r -th power of the natural module V and the t -th power of the dual natural module V^* of GL_n for various $r, t \in \mathbb{Z}^{\geq 0}$. Benkart et al. [3] used walled Brauer algebras to study decompositions of mixed tensor modules of GL_n . Since then, walled Brauer algebras have been intensively studied, e.g., [8–11,26,32], etc. In particular, blocks and decomposition numbers of walled Brauer algebras over \mathbb{C} were determined in [9,10]. Recently, Brundan and Stroppel [8] obtained \mathbb{Z} -gradings on $\mathcal{B}_{r,t}(\delta)$, proved the Koszulity of $\mathcal{B}_{r,t}(\delta)$ and established Morita equivalences between $\mathcal{B}_{r,t}(\delta)$ and idempotent truncations of certain infinite dimensional versions of Khovanov’s arc algebras [19].

In 2002, by studying mixed tensor modules of general linear Lie superalgebras $\mathfrak{gl}_{m|n}$, Shader and Moon [32] set up super Schur–Weyl dualities between walled Brauer algebras and general linear Lie superalgebras. By studying tensor modules $K_\lambda \otimes V^{\otimes r}$ of Kac modules K_λ with the r -th power $V^{\otimes r}$ of the natural module V of $\mathfrak{gl}_{m|n}$, Brundan and Stroppel [7] further established super Schur–Weyl dualities between level two Hecke algebras $H_r^{p,q}$ (also denoted as $\mathcal{H}_{2,r}$) and general linear Lie superalgebras $\mathfrak{gl}_{m|n}$, which provide powerful tools enabling them to obtain various results including a spectacular one on Morita equivalences between blocks of categories of finite dimensional $\mathfrak{gl}_{m|n}$ -modules and categories of finite-dimensional left modules over some generalized Khovanov’s diagram algebras. A natural question is, what kind of algebras may come into the play if one replaces the tensor modules $K_\lambda \otimes V^{\otimes r}$ by the tensor modules $M^{r,t} := V^{\otimes r} \otimes K_\lambda \otimes (V^*)^{\otimes t}$ of Kac modules K_λ with the r -th power of the natural module V and the t -th power of the dual natural module V^* of $\mathfrak{gl}_{m|n}$. This is one of our motivations to introduce a new class of associative algebras $\mathcal{B}_{r,t}^{\text{aff}}$ (cf. Definition 2.7), referred to as affine walled Brauer algebras, over a commutative ring containing 1.

The new algebra $\mathcal{B}_{r,t}^{\text{aff}}$ (defined with parameters ω_a ’s and $\bar{\omega}_a$ ’s satisfying Corollary 4.3) can be realized as the free R -module $R[\mathbf{x}_r] \otimes \mathcal{B}_{r,t}(\omega_0) \otimes R[\bar{\mathbf{x}}_t]$ (the tensor product of the walled Brauer algebra $\mathcal{B}_{r,t}(\omega_0)$ with two polynomial algebras $R[\mathbf{x}_r] := R[x_1, x_2, \dots, x_r]$ and $R[\bar{\mathbf{x}}_t] := R[\bar{x}_1, \bar{x}_2, \dots, \bar{x}_t]$), such that $R[\mathbf{x}_r] \otimes R\mathfrak{S}_r$ and $R\bar{\mathfrak{S}}_t \otimes R[\bar{\mathbf{x}}_t]$ are isomorphic to the degenerate affine Hecke algebras $\mathcal{H}_r^{\text{aff}}$ and $\mathcal{H}_t^{\text{aff}}$ respectively (where \mathfrak{S}_r and $\bar{\mathfrak{S}}_t$

are symmetric groups contained in $\mathcal{B}_{r,t}(\omega_0)$ generated by s_i 's and \bar{s}_j 's respectively), and further, the following relations are satisfied (cf. [Theorem 4.16](#) and [Proposition 4.18](#))

- (1) $e_1(x_1 + \bar{x}_1) = (x_1 + \bar{x}_1)e_1 = 0, s_1e_1s_1x_1 = x_1s_1e_1s_1, \bar{s}_1e_1\bar{s}_1\bar{x}_1 = \bar{x}_1\bar{s}_1e_1\bar{s}_1,$
- (2) $s_i\bar{x}_1 = \bar{x}_1s_i, \bar{s}_ix_1 = x_1\bar{s}_i, x_1(e_1 + \bar{x}_1) = (e_1 + \bar{x}_1)x_1,$
- (3) $e_1x_1^ke_1 = \omega_k e_1, e_1\bar{x}_1^ke_1 = \bar{\omega}_k e_1, \forall k \in \mathbb{Z}^{\geq 0}.$

In this sense, the appearing of affine walled algebras $\mathcal{B}_{r,t}^{\text{aff}}$ is very natural.

Surprisingly, there is a super Schur–Weyl duality (SSWD) between affine walled algebras $\mathcal{B}_{r,t}^{\text{aff}}$ and general linear Lie superalgebras $\mathfrak{gl}_{m|n}$ over \mathbb{C} . In this case, level two walled Brauer algebras $\mathcal{B}_{r,t}^{p,q}(m, n)$ (defined as cyclotomic quotients of affine walled Brauer algebras $\mathcal{B}_{r,t}^{\text{aff}}$ with special parameters, cf. [\(5.37\)](#) and [Definition 5.11](#)) come into the play (cf. [Theorem 5.12](#)) under the assumption that $r + t \leq \min\{m, n\}$ (we remark that the general case will be discussed in [\[30\]](#)): they can be realized as endomorphism algebras of $M^{r,t}$ under some “reasonable” conditions (see the next paragraph). Moreover, they have weakly cellular structures (cf. [Theorem 6.12](#)), which enable us to give a classification of their simple modules (cf. [Theorem 7.6](#)). The result in turn allows us to classify the indecomposable direct summands of $M^{r,t}$ (cf. [Theorem 7.7](#)). We would like to mention that the SSWD is in fact the first and second fundamental theorems of invariant theory for the endomorphism algebra $\text{End}_{\mathfrak{g}}(M^{r,t})$ (see, e.g., [\[24\]](#)). In a sequel [\[30\]](#), we will show that the SSWD can provide us a powerful tool to tackle various problems associated with level two walled Brauer algebras, such as the determinations of decomposition numbers. On the other hand, the SSWD in turn enables us to study the category of finite dimensional $\mathfrak{gl}_{m|n}$ -modules, in particular we can use it to determine all $\mathfrak{gl}_{m|n}$ -singular vectors (or highest weight vectors) in $M^{r,t}$ (we remark that the determination of singular vectors in Lie superalgebra modules is an interesting but cumbersome problem).

At this point, we wish to mention that in establishing the SSWD, although we choose the highest weights λ of the Kac modules K_λ to be very special (cf. [\(5.6\)](#)), this is enough and reasonable for our purpose as in this case $M^{r,t}$ is a tilting module and every indecomposable projective module can be realized as a direct summand of $M^{r,t}$ for various r and t . With these special choices of λ 's, level two walled Brauer algebras appear very naturally. We would also like to mention that in contrast to level two Hecke algebras $H_r^{p,q}$ (or $\mathcal{H}_{2,r}$) in [\[7, IV\]](#), which only depend on $p - q$ and r , level two walled Brauer algebras $\mathcal{B}_{r,t}^{p,q}(m, n)$ heavily depend on parameters $p - q, r, t, m, n$ (cf. [Remark 5.14](#)). Nevertheless, one may expect that level two walled Brauer algebras will play the role similar to that of level two Hecke algebras and that there should be some close connections between level two walled Brauer algebras and cyclotomic Khovanov–Lauda–Rouquier algebras [\[20,27,28\]](#) (cf. [\[6,7\]](#) for level two Hecke algebras). We remark that higher level walled Brauer algebras will appear if we consider arbitrary typical integral dominant weight λ of $\mathfrak{gl}_{m|n}$. However, in this case, we cannot expect anything new from the viewpoint of representation theory of $\mathfrak{gl}_{m|n}$ (see [Remark 5.8](#)). In fact, it is a simple routine to generalize level two walled-Brauer algebras to higher level walled-Brauer algebras.

Another motivation of introducing affine walled Brauer algebras comes from Nazarov’s work [25] on the Jucys–Murphy elements of Brauer algebras and affine Wenzl algebras. We construct a family of Jucys–Murphy-like elements of walled Brauer algebras (cf. Definition 3.2), which have close relationship with certain central elements. By studying properties of these elements in details, we are not only able to give the precise definition of affine walled Brauer algebras $\mathcal{B}_{r,t}^{\text{aff}}$ (which can also be regarded as analogues of Nazarov’s affine Wenzl algebras [25] in this sense), but also able to set up a family of homomorphisms ϕ_k from affine walled Brauer algebras $\mathcal{B}_{r,t}^{\text{aff}}$ to walled Brauer algebras $\mathcal{B}_{k+r,k+t}(\omega_0)$ for any $k \in \mathbb{Z}^{\geq 1}$ (cf. Theorem 3.12). This then enables us to use the freeness of walled Brauer algebras to prove that affine walled Brauer algebras $\mathcal{B}_{r,t}^{\text{aff}}$ are free with infinite rank over a commutative ring containing 1 (cf. Theorems 4.14 and 4.16).

We organize the paper as follows. In Section 2, after recalling the notion of walled Brauer algebras, we introduce affine walled Brauer algebras over a commutative ring R containing 1. In Section 3, we introduce a family of Jucys–Murphy-like elements of walled Brauer algebras and establish a family of homomorphisms from affine walled Brauer algebras to walled Brauer algebras. Using these homomorphisms and the freeness of walled Brauer algebras, we prove the freeness of affine walled Brauer algebras in Section 4. In Section 5, we study the super Schur–Weyl duality between affine walled Brauer algebras (more precisely, level two walled Brauer algebras $\mathcal{B}_{r,t}^{p,q}(m,n)$) with special parameters and general linear Lie superalgebras $\mathfrak{gl}_{m|n}$ under the assumption that $r+t \leq \min\{m,n\}$. In Section 6, we construct a weakly cellular basis of level two walled Brauer algebras. Finally in Section 7, we give a classification of their simple modules, and a classification of the indecomposable direct summands of the aforementioned tensor modules.

2. The walled Brauer algebra $\mathcal{B}_{r,t}$ and its affinization

Throughout this section, let R be a commutative ring containing 1. The *walled Brauer algebra* is an associative algebra over R spanned by so-called *walled Brauer diagrams* as follows.

Fix two positive integers r and t . A *walled (r,t) -Brauer diagram* is a diagram with $(r+t)$ vertices on the top and bottom rows, and vertices on both rows are labeled from left to right by $r, \dots, 2, 1, \bar{1}, \bar{2}, \dots, \bar{t}$. Every vertex $i \in \{1, 2, \dots, r\}$ (resp., $\bar{i} \in \{\bar{1}, \bar{2}, \dots, \bar{t}\}$) on each row must be connected to a unique vertex \bar{j} (resp., j) on the same row or a unique vertex j (resp., \bar{j}) on the other row. In this way, we obtain 4 types of pairs $[i, j]$, $[i, \bar{j}]$, $[\bar{i}, j]$ and $[\bar{i}, \bar{j}]$. The pairs $[i, j]$ and $[\bar{i}, \bar{j}]$ are called *vertical edges*, and the pairs $[\bar{i}, j]$ and $[i, \bar{j}]$ are called *horizontal edges*. If we imagine that there is a wall which separates the vertices $1, \bar{1}$ on both top and bottom rows, then a walled (r,t) -Brauer diagram is a diagram with $(r+t)$ vertices on both rows such that each vertical edge cannot cross the wall and each horizontal edge has to cross the wall. For convenience, we call a walled (r,t) -Brauer diagram a *walled Brauer diagram* if there is no confusion.

Example 2.1. The following are (r, t) -Brauer diagrams:

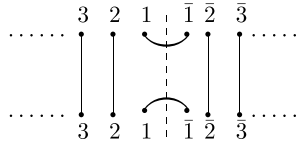


Fig. 1. e_1 .

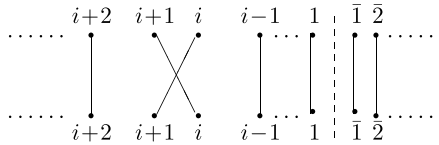


Fig. 2. s_i .

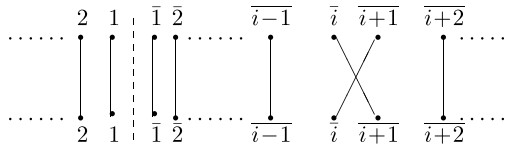


Fig. 3. \bar{s}_i .

Throughout, we denote by e_1, s_i, \bar{s}_i the diagrams in Figs. 1–3, respectively.

In order to define the product of two walled Brauer diagrams, we consider the *composition* $D_1 \circ D_2$ of two walled Brauer diagrams D_1 and D_2 , which is obtained by putting D_1 above D_2 and connecting each vertex on the bottom row of D_1 to the corresponding vertex on the top row of D_2 . If we remove all circles of $D_1 \circ D_2$, we will get a walled Brauer diagram, say D_3 . Let $n(D_1, D_2)$ be the number of circles appearing in $D_1 \circ D_2$. Then the *product* $D_1 D_2$ of D_1 and D_2 is defined to be $\delta^{n(D_1, D_2)} D_3$, where δ is a fixed element in R . For instance, in Example 2.1, $e_1^2 = \delta e_1$.

Definition 2.2. (See [22,35].) The walled Brauer algebra $\mathcal{B}_{r,t}(\delta)$ with respect to the parameter δ is the associative algebra over R spanned by all walled (r, t) -Brauer diagrams with product defined as above.

Remark 2.3. If we allow vertical edges can cross the wall and allow horizontal edges may not cross the wall (namely, a vertex can be connected to any other vertex), then we obtain $(r + t)$ -Brauer diagrams. The Brauer algebra $\mathcal{B}_{r+t}(\delta)$ [4] is the free R -modules spanned by all $(r + t)$ -Brauer diagrams with product defined as above. Thus a walled Brauer diagram is a Brauer diagram, and the walled Brauer algebra $\mathcal{B}_{r,t}(\delta)$ is a subalgebra of the Brauer algebra $\mathcal{B}_{r+t}(\delta)$.

The following result can be found in [16, Corollary 4.5] for a special case and [26, Theorem 4.1] in general.

Theorem 2.4. *Let R be a commutative ring containing 1 and δ . Then $\mathcal{B}_{r,t}(\delta)$ is an associative R -algebra generated by e_1, s_i, \bar{s}_j with $1 \leq i \leq r - 1$ and $1 \leq j \leq t - 1$ subject to the following relations*

- | | |
|--|---|
| (1) $s_i^2 = 1, 1 \leq i < r,$ | (8) $\bar{s}_i^2 = 1, 1 \leq i < t,$ |
| (2) $s_i s_j = s_j s_i, i - j > 1,$ | (9) $\bar{s}_i \bar{s}_j = \bar{s}_j \bar{s}_i, i - j > 1,$ |
| (3) $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, 1 \leq i < r - 1,$ | (10) $\bar{s}_i \bar{s}_{i+1} \bar{s}_i = \bar{s}_{i+1} \bar{s}_i \bar{s}_{i+1}, 1 \leq i < t - 1,$ |
| (4) $s_i e_1 = e_1 s_i, 2 \leq i < r,$ | (11) $\bar{s}_i e_1 = e_1 \bar{s}_i, 2 \leq i < t,$ |
| (5) $e_1 s_1 e_1 = e_1,$ | (12) $e_1 \bar{s}_1 e_1 = e_1,$ |
| (6) $e_1^2 = \delta e_1,$ | (13) $e_1 s_1 \bar{s}_1 e_1 s_1 = e_1 s_1 \bar{s}_1 e_1 \bar{s}_1,$ |
| (7) $s_i \bar{s}_j = \bar{s}_j s_i,$ | (14) $s_1 e_1 s_1 \bar{s}_1 e_1 = \bar{s}_1 e_1 s_1 \bar{s}_1 e_1.$ |

In particular, the rank of $\mathcal{B}_{r,t}(\delta)$ is $(r + t)!$.

We remark that Jung and Kang gave a presentation of walled Brauer superalgebras in [17, Theorem 5.1], and the presentation of walled Brauer algebras in Theorem 2.4 can be obtained from those of walled Brauer superalgebras by removing the generators of Clifford algebras inside walled Brauer superalgebras.

The following two results can be deduced from Theorem 2.4, easily.

Lemma 2.5. *There is an R -linear anti-involution $\sigma : \mathcal{B}_{r,t}(\delta) \rightarrow \mathcal{B}_{r,t}(\delta)$ fixing generators s_i, \bar{s}_j and e_1 for all possible i, j 's.*

Proof. The result follows from the symmetry of relations in Theorem 2.4, immediately. In particular, the image of a walled Brauer diagram D under the map σ is the diagram which is obtained from D by reflecting along a horizontal line. \square

Lemma 2.6. *We have $\mathcal{B}_{r,t}(\delta) \cong \mathcal{B}_{t,r}(\delta)$. In particular, the corresponding isomorphism sends s_i, e_1, \bar{s}_j of $\mathcal{B}_{r,t}(\delta)$ to \bar{s}_i, e_1, s_j of $\mathcal{B}_{t,r}(\delta)$.*

Proof. One can easily observe that the automorphism can be obtained by the reflection with respect to the imaginary wall in the middle. \square

In the present paper, we shall introduce a new class of associative algebras called affine walled Brauer algebras. Such algebras can also be considered as the analogues of Nazarov's affine Wenzl algebras in [25]. This is one of our motivations to introduce these algebras. Another motivation originates from super Schur–Weyl dualities in [7,32] and ours in Section 5.

Definition 2.7. (cf. Theorem 4.16) Let R be a commutative ring containing 1, ω_0, ω_1 . Fix $r, t \in \mathbb{Z}^{\geq 0}$. The affine walled Brauer algebra $\mathcal{B}_{r,t}^{\text{aff}}(\omega_0, \omega_1)$ is the associative R -algebra generated by $e_1, x_1, \bar{x}_1, s_i (1 \leq i \leq r - 1), \bar{s}_j (1 \leq j \leq t - 1)$, and two families of central elements $\omega_k (k \in \mathbb{Z}^{\geq 2}), \bar{\omega}_k (k \in \mathbb{Z}^{\geq 0})$, subject to the following relations

- | | |
|---|---|
| (1) $s_i^2 = 1, 1 \leq i < r,$ | (14) $\bar{s}_i^2 = 1, 1 \leq i < t,$ |
| (2) $s_i s_j = s_j s_i, i - j > 1,$ | (15) $\bar{s}_i \bar{s}_j = \bar{s}_j \bar{s}_i, i - j > 1,$ |
| (3) $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, 1 \leq i < r - 1,$ | (16) $\bar{s}_i \bar{s}_{i+1} \bar{s}_i = \bar{s}_{i+1} \bar{s}_i \bar{s}_{i+1}, 1 \leq i < t - 1,$ |
| (4) $s_i e_1 = e_1 s_i, 2 \leq i < r,$ | (17) $\bar{s}_i e_1 = e_1 \bar{s}_i, 2 \leq i < t,$ |
| (5) $e_1 s_1 e_1 = e_1,$ | (18) $e_1 \bar{s}_1 e_1 = e_1,$ |
| (6) $e_1^2 = \omega_0 e_1,$ | (19) $e_1 s_1 \bar{s}_1 e_1 s_1 = e_1 s_1 \bar{s}_1 e_1 \bar{s}_1,$ |
| (7) $s_i \bar{s}_j = \bar{s}_j s_i,$ | (20) $s_1 e_1 s_1 \bar{s}_1 e_1 = \bar{s}_1 e_1 s_1 \bar{s}_1 e_1,$ |
| (8) $e_1(x_1 + \bar{x}_1) = (x_1 + \bar{x}_1)e_1 = 0,$ | (21) $x_1(e_1 + \bar{x}_1) = (e_1 + \bar{x}_1)x_1,$ |
| (9) $e_1 s_1 x_1 s_1 = s_1 x_1 s_1 e_1,$ | (22) $e_1 \bar{s}_1 \bar{x}_1 \bar{s}_1 = \bar{s}_1 \bar{x}_1 \bar{s}_1 e_1,$ |
| (10) $s_i x_1 = x_1 s_i, 2 \leq i < r,$ | (23) $\bar{s}_i \bar{x}_1 = \bar{x}_1 \bar{s}_i, 2 \leq i < t,$ |
| (11) $s_i \bar{x}_1 = \bar{x}_1 s_i, 1 \leq i < r,$ | (24) $\bar{s}_i x_1 = x_1 \bar{s}_i, 1 \leq i < t,$ |
| (12) $e_1 x_1^k e_1 = \omega_k e_1, \forall k \in \mathbb{Z}^{\geq 0},$ | (25) $e_1 \bar{x}_1^k e_1 = \bar{\omega}_k e_1, \forall k \in \mathbb{Z}^{\geq 0},$ |
| (13) $x_1(s_1 x_1 s_1 - s_1) = (s_1 x_1 s_1 - s_1)x_1,$ | (26) $\bar{x}_1(\bar{s}_1 \bar{x}_1 \bar{s}_1 - \bar{s}_1) = (\bar{s}_1 \bar{x}_1 \bar{s}_1 - \bar{s}_1)\bar{x}_1.$ |

Remark 2.8. Later on we shall be mainly interested in the case when all central elements $\omega_a, \bar{\omega}_b$ with $a \in \mathbb{Z}^{\geq 2}, b \in \mathbb{Z}^{\geq 0}$ are specialized to some elements in R (cf. [Theorem 4.16](#)).

Remark 2.9. The reason we put ω_a 's into generators is that in order to be able to prove the freeness of $\mathcal{B}_{r,t}^{\text{aff}}$ (cf. [Theorem 4.14](#)), we need to construct a family of homomorphisms ϕ_k (cf. [Theorem 3.12](#)), which requires ω_a 's to be generators.

For simplicity, we use $\mathcal{B}_{r,t}^{\text{aff}}$ instead of $\mathcal{B}_{r,t}^{\text{aff}}(\omega_0, \omega_1)$ later on. In other words, we always assume that $\mathcal{B}_{r,t}^{\text{aff}}$ is the affine walled Brauer algebra with respect to the parameters ω_0 and ω_1 .

In the next two sections, we shall prove that $\mathcal{B}_{r,t}^{\text{aff}}$ is a free R -algebra with infinite rank.

3. Homomorphisms from $\mathcal{B}_{r,t}^{\text{aff}}$ to $\mathcal{B}_{k+r,k+t}(\omega_0)$

The purpose of this section is to establish a family of algebra homomorphisms ϕ_k from $\mathcal{B}_{r,t}^{\text{aff}}$ to $\mathcal{B}_{k+r,k+t}(\omega_0)$ for all $k \in \mathbb{Z}^{\geq 1}$. Then in the next section, we will use these homomorphisms and the freeness of walled Brauer algebras to prove the freeness of $\mathcal{B}_{r,t}^{\text{aff}}$. We remark that Nazarov [25] used the freeness of Brauer algebras to prove the freeness of affine Wenzl algebras.

Unless otherwise indicated, all elements considered in this section are in the walled Brauer algebra $\mathcal{B}_{r,t}(\delta)$ for some $r, t \in \mathbb{Z}^{\geq 0}$ with parameter $\delta = \omega_0$.

Denote by \mathfrak{S}_r (resp., $\bar{\mathfrak{S}}_t$) the symmetric group in r letters $1, 2, \dots, r$ (resp., t letters $\bar{1}, \bar{2}, \dots, \bar{t}$). It is well-known that the subalgebra of $\mathcal{B}_{r,t}(\delta)$ generated by $\{s_i \mid 1 \leq i < r\}$ (resp., $\{\bar{s}_j \mid 1 \leq j < t\}$) is isomorphic to the group algebra $R\mathfrak{S}_r$ (resp., $R\bar{\mathfrak{S}}_t$) of \mathfrak{S}_r (resp., $\bar{\mathfrak{S}}_t$).

Let $(i, j) \in \mathfrak{S}_r$ (resp., $(\bar{i}, \bar{j}) \in \bar{\mathfrak{S}}_t$) be the transposition which switches i and j (resp., \bar{i} and \bar{j}) and fixes others. Then s_i and \bar{s}_j can be identified with

$$s_i = (i, i + 1) \quad \text{and} \quad \bar{s}_j = (\bar{j}, \bar{j} + 1).$$

Set $L_1 = \bar{L}_1 = 0$ and

$$L_i = \sum_{j=1}^{i-1} (j, i), \quad \bar{L}_i = \sum_{\bar{j}=1}^{i-1} (\bar{j}, \bar{i}) \quad \text{for } i \geq 2. \tag{3.1}$$

Then L_i for $1 \leq i \leq r$ are known as the *Jucys–Murphy elements* of $R\mathfrak{S}_r$, and \bar{L}_j for $1 \leq j \leq t$ are the Jucys–Murphy elements of $R\bar{\mathfrak{S}}_t$. We will need the following well known result.

Lemma 3.1. *In $R\mathfrak{S}_r$ and $R\bar{\mathfrak{S}}_t$, for all possible i, j 's, we have*

- (1) $L_i s_j = s_j L_i, \quad \bar{L}_i \bar{s}_j = \bar{s}_j \bar{L}_i \quad \text{if } i \neq j, j + 1.$
- (2) $s_i L_i = L_{i+1} s_i - 1, \quad \bar{s}_i \bar{L}_i = \bar{L}_{i+1} \bar{s}_i - 1.$
- (3) $(L_i + L_{i+1}) s_i = s_i (L_i + L_{i+1}), (\bar{L}_i + \bar{L}_{i+1}) \bar{s}_i = \bar{s}_i (\bar{L}_i + \bar{L}_{i+1}).$

For convenience, we define the following cycles in \mathfrak{S}_r , where $1 \leq i, j \leq r$,

$$s_{i,j} = s_i s_{i+1} \cdots s_{j-1} = (j, j - 1, \dots, i) \quad \text{for } i < j, \tag{3.2}$$

and $s_{i,i} = 1$. If $i > j$, we set $s_{i,j} = s_{j,i}^{-1} = (j, j + 1, \dots, i)$. Similarly, for $1 \leq i, j \leq t$, we define $\bar{s}_{i,j} = (\bar{j}, \bar{j} - 1, \dots, \bar{i}) \in \bar{\mathfrak{S}}_t$ if $i < j$, or 1 if $i = j$, or $\bar{s}_{j,i}^{-1}$ else. Let $e_{i,j}$ be the element whose corresponding diagram is the walled Brauer diagram such that any of its edge is of form $[k, k]$ or $[\bar{k}, \bar{k}]$ except two horizontal edges $[i, \bar{j}]$ on both top and bottom rows. Namely,

$$e_{i,j} = \bar{s}_{j,1} s_{i,1} e_{1,1} s_{1,i} \bar{s}_{1,j} \quad \text{for } i, j \text{ with } 1 \leq i \leq r \text{ and } 1 \leq j \leq t. \tag{3.3}$$

We also simply denote $e_i = e_{i,i}$ for $1 \leq i \leq \min\{r, t\}$.

It follows from [8, Lemma 2.1] and [29, Proposition 2.5] that

$$c_{r,t} = \sum_{1 \leq i \leq r, 1 \leq j \leq t} e_{i,j} - \sum_{i=1}^r L_i - \sum_{j=1}^t \bar{L}_j, \tag{3.4}$$

is a central element in $\mathcal{B}_{r,t}(\delta)$. Such a central element has already been used in [10, Lemma 4.1] to study blocks of $\mathcal{B}_{r,t}(\delta)$ over \mathbb{C} . Motivated by (3.4), we define *Jucys–Murphy-like elements* y_i, \bar{y}_l below such that for any $k \in \mathbb{Z}^{\geq 1}$, elements y_{k+1}, \bar{y}_{k+1} in the image of the homomorphism ϕ_k (to be defined in Theorem 3.12) will play the same roles as those of x_1, \bar{x}_1 in $\mathcal{B}_{r,t}^{\text{aff}}$.

Definition 3.2. Fix an element $\delta_1 \in R$. For $1 \leq i \leq r$ and $1 \leq \ell \leq t$, let

$$y_i = \delta_1 + \sum_{j=1}^{i-1} e_{i,j} - L_i, \quad \text{and} \quad \bar{y}_\ell = -\delta_1 + \sum_{j=1}^{\ell-1} e_{j,\ell} - \bar{L}_\ell. \tag{3.5}$$

Lemma 3.3. Let $i \in \mathbb{Z}$ with $1 \leq i \leq \min\{r, t\}$.

- (1) $e_i y_i = e_i(\delta_1 + \bar{L}_i - L_i), \quad e_i \bar{y}_i = e_i(-\delta_1 + L_i - \bar{L}_i).$
- (2) $e_i(y_i + \bar{y}_i) = 0, \quad (y_i + \bar{y}_i)e_i = 0.$
- (3) $e_i s_i y_i s_i = s_i y_i s_i e_i, \quad e_i \bar{s}_i \bar{y}_i \bar{s}_i = \bar{s}_i \bar{y}_i \bar{s}_i e_i.$
- (4) $y_i(e_i + \bar{y}_i) = (e_i + \bar{y}_i)y_i.$
- (5) $y_i(s_i y_i s_i - s_i) = (s_i y_i s_i - s_i)y_i, \quad \bar{y}_i(\bar{s}_i \bar{y}_i \bar{s}_i - \bar{s}_i) = (\bar{s}_i \bar{y}_i \bar{s}_i - \bar{s}_i)\bar{y}_i.$
- (6) $s_j y_i = y_i s_j, \quad \bar{s}_j \bar{y}_i = \bar{y}_i \bar{s}_j \quad \text{if } j \neq i - 1, i.$
- (7) $s_j \bar{y}_i = \bar{y}_i s_j, \quad \bar{s}_j y_i = y_i \bar{s}_j \quad \text{if } j \neq i - 1.$
- (8) $e_{i+1} y_i = y_i e_{i+1}, \quad e_{i+1} \bar{y}_i = \bar{y}_i e_{i+1} \quad \text{if } i < \min\{r, t\}.$
- (9) $y_i y_{i+1} = y_{i+1} y_i, \quad \bar{y}_i \bar{y}_{i+1} = \bar{y}_{i+1} \bar{y}_i \quad \text{if } i < \min\{r, t\}.$

Proof. We remark that the second assertion of (2) follows from the first assertion of (2) by applying the anti-involution σ in Lemma 2.5. By Lemma 2.6, we need only check (4) and the first assertions of others.

Since $e_i e_{i,j} = e_i(\bar{j}, \bar{i})$ and $e_i e_{j,i} = e_i(j, i)$ for $j \neq i$, we have (1) and (2). Further, (3) follows from the equalities $e_i e_{k,j} = e_{k,j} e_i, e_i(k, j) = (k, j)e_i$ if $i \notin \{k, j\}$ together with the following:

$$s_i y_i s_i = \sum_{j=1}^{i-1} e_{i+1,j} - \sum_{j=1}^{i-1} (j, i + 1) + \delta_1 = y_{i+1} - s_i e_i s_i + s_i. \tag{3.6}$$

By Definition 3.2, we have

$$\begin{aligned} y_i y_{i+1} - y_{i+1} y_i &= \sum_{j=1}^{i-1} e_{i,j} \sum_{k=1}^i e_{i+1,k} - \sum_{k=1}^i e_{i+1,k} \sum_{j=1}^{i-1} e_{i,j} - \sum_{j=1}^{i-1} e_{i,j} L_{i+1} + L_{i+1} \sum_{j=1}^{i-1} e_{i,j} \\ &= \sum_{j=1}^{i-1} e_{i,j} s_i - s_i \sum_{j=1}^{i-1} e_{i,j} - \sum_{j=1}^{i-1} e_{i,j} s_i + s_i \sum_{j=1}^{i-1} e_{i,j}, \end{aligned}$$

which is equal to zero, proving (9).

Recall that σ is the anti-involution on $\mathcal{B}_{r,t}(\delta)$ in Lemma 2.5. We have $\sigma(y_j) = y_j$ and $\sigma(s_j) = s_j$. Using (3.6) and σ , we have

$$y_i(s_i y_i s_i - s_i) = y_i y_{i+1} - y_i s_i e_i s_i, \quad (s_i y_i s_i - s_i)y_i = y_{i+1} y_i - s_i e_i s_i y_i. \tag{3.7}$$

By (3), we have $y_i s_i e_i s_i = s_i e_i s_i y_i$. So, (5) follows from (9).

We remark that (6) and (7) can be checked easily by using [Theorem 2.4\(2\)–\(4\)](#). Since $e_{i+1} e_{i,j} = e_{i,j} e_{i+1}$ and $e_{i+1}(j, i) = (j, i) e_{i+1}$ for $1 \leq j \leq i - 1$, we have (8).

Finally, we check (4). We have $(y_1 + e_1)\bar{y}_{i+1} = \bar{y}_{i+1}(y_1 + e_1)$ by $e_1 \bar{y}_{i+1} = \bar{y}_{i+1} e_1$. By induction on j , we have $(y_j + e_j)\bar{y}_{i+1} = \bar{y}_{i+1}(y_j + e_j)$ and $e_j \bar{y}_{i+1} = \bar{y}_{i+1} e_j$ for all j with $1 \leq j \leq i$. So,

$$y_i \bar{y}_{i+1} = \bar{y}_{i+1} y_i. \tag{3.8}$$

By (3.6) and [Lemma 2.6](#), $e_i + \bar{y}_i = \bar{s}_i \bar{y}_{i+1} \bar{s}_i + \bar{s}_i$. So, (4) follows from (3.8) and (7). \square

The following result is a special case of [\[10, Proposition 2.1\]](#).

Proposition 3.4. *Let $\mathcal{B}_{r,t}(\delta)$ be defined over a field F . For any fixed k with $2 \leq k \leq \min\{r, t\}$, let $e = e_k$ if $\delta \neq 0$ or $e = e_k s_{k-1}$ otherwise. Let $\mathcal{B}_{k,k}(\delta)$ be the subalgebra of $\mathcal{B}_{r,t}(\delta)$ generated by $e_1, s_i, \bar{s}_i, 1 \leq i \leq k$. Then $e \mathcal{B}_{k,k}(\delta) e = e \mathcal{B}_{k-1,k-1}(\delta)$, which is isomorphic to $\mathcal{B}_{k-1,k-1}(\delta)$ as an F -algebra.*

The following result immediately follows from [Proposition 3.4](#), where elements $\omega_{a,k}, \bar{\omega}_{a,k}$ will be crucial in obtaining the homomorphisms ϕ_k in [Theorem 3.12](#).

Corollary 3.5. *Assume $\omega_0 \neq 0$. For $a \in \mathbb{Z}^{\geq 0}$, there exist unique $\omega_{a,k}, \bar{\omega}_{a,k} \in \mathcal{B}_{k-1,k-1}$ such that*

$$e_k y_k^a e_k = \omega_{a,k} e_k, \quad e_k \bar{y}_k^a e_k = \bar{\omega}_{a,k} e_k.$$

Furthermore, $\omega_{1,k} = -\bar{\omega}_{1,k} = \delta \delta_1$ and $\omega_{0,k} = \bar{\omega}_{0,k} = \delta$.

Lemma 3.6. *For any $k \in \mathbb{Z}^{\geq 1}$, we have $e_i \bar{y}_i^k = \sum_{j=0}^k a_{k,j}^{(i)} e_i y_i^j$ for some $a_{k,j}^{(i)} \in \mathcal{B}_{r,t}$ such that*

- (1) $a_{k,k}^{(i)} = (-1)^k$,
- (2) $a_{k,j}^{(i)} = \omega_{0,i} a_{k-1,j}^{(i)} - a_{k-1,j-1}^{(i)}, 1 \leq j \leq k - 1$,
- (3) $a_{k,0}^{(i)} = -\sum_{j=1}^{k-1} a_{k-1,j}^{(i)} \omega_{j,i}$.

In particular, $a_{k,j}^{(i)} \in R[\omega_{2,i}, \omega_{3,i}, \dots, \omega_{k-1,i}]$ for any j with $1 \leq j \leq k$ such that each monomial of $a_{k,j}^{(i)}$ is of form $\omega_{j_1,i} \cdots \omega_{j_\ell,i}$ with $\sum_{i=1}^{\ell} j_i \leq k - 1$.

Proof. By [Lemma 3.3\(2\)](#), the result holds for $k = 1$. In general, by [Lemma 3.3\(4\)](#),

$$e_i \bar{y}_i^j \bar{y}_i = e_i (e_i - y_i) y_i^j - \omega_{j,i} e_i = \omega_{0,i} e_i y_i^j - e_i y_i^{j+1} - \omega_{j,i} e_i.$$

Now, the result follows from induction on k . \square

Lemma 3.7. For $k, a \in \mathbb{Z}^{\geq 2}$, we have $\bar{\omega}_{a,k} \in R[\omega_{2,k}, \omega_{3,k}, \dots, \omega_{a,k}]$. Furthermore, both $\omega_{a,k}$ and $\bar{\omega}_{a,k}$ are central in $\mathcal{B}_{k-1, k-1}$.

Proof. The first assertion follows from Lemma 3.6. To prove the second, note that any $h \in \{e_1, s_i \mid 1 \leq i \leq k-2\}$ commutes with e_k, y_k . So, $e_k(h\omega_{a,k}) = e_k(\omega_{a,k}h)$. By Proposition 3.4, $h\omega_{a,k} = \omega_{a,k}h$. Finally, we need to check $e_k(h\omega_{a,k}) = e_k(\omega_{a,k}h)$ for any $h \in \{\bar{s}_1, \bar{s}_2, \dots, \bar{s}_{k-2}\}$. In this case, we use Lemma 3.6. More explicitly, we can use \bar{y}_k instead of y_k in $e_k y_k^a e_k$. Therefore, $h\omega_{a,k} = \omega_{a,k}h$, as required. \square

The following result follows from (3.6) and induction on a .

Lemma 3.8. For $k, a \in \mathbb{Z}^{\geq 1}$, we have

$$s_k y_{k+1}^a = (y_k + e_k)^a s_k - \sum_{b=0}^{a-1} (y_k + e_k)^{a-1-b} y_{k+1}^b.$$

The elements $z_{j,k}, \bar{z}_{j,k}$ defined below will be crucial in the description of $\omega_{a,k}$ (cf. Lemma 3.10). For $1 \leq j \leq k-1$, let

$$z_{j,k} = s_{j,k-1}(y_{k-1} + e_{k-1})s_{k-1,j}, \quad \bar{z}_{j,k} = \bar{s}_{j,k-1}(\bar{y}_{k-1} + e_{k-1})\bar{s}_{k-1,j}. \tag{3.9}$$

Then the following result can be verified, easily.

Lemma 3.9. For $1 \leq j \leq k-1$, we have

- (1) $z_{j,k} = \sum_{\ell=1}^{k-1} e_{j,\ell} - \sum_{1 \leq s \leq k-1, s \neq j} (s, j)$,
- (2) $\bar{z}_{j,k} = \sum_{\ell=1}^{k-1} e_{\ell,j} - \sum_{1 \leq \bar{s} \leq k-1, \bar{s} \neq \bar{j}} (\bar{s}, \bar{j})$.

Note that $\omega_{0,k} = \delta$ and $\omega_{1,k} = \delta\delta_1$, and $e_k h = 0$ for $h \in \mathcal{B}_{k-1, k-1}$ if and only if $h = 0$. We will use this fact freely in the proof of the following lemma, where we use the terminology that a monomial in $z_{j,k+1}$'s and $\bar{z}_{j,k+1}$'s is a *leading term* in an expression if it has the highest degree by defining $\deg z_{i,j} = \deg \bar{z}_{i,j} = 1$.

Lemma 3.10. Suppose $a \in \mathbb{Z}^{\geq 2}$. Then $\omega_{a,k+1}$ can be written as an R -linear combination of monomials in $z_{j,k+1}$'s and $\bar{z}_{j,k+1}$'s for $1 \leq j \leq k$ such that the leading terms of $\omega_{a,k+1}$ are $\sum_{j=1}^k (-z_{j,k+1}^{a-1} + (-1)^{a-1} \bar{z}_{j,k+1}^{a-1})$.

Proof. By Corollary 3.5 and Lemma 3.3(1), we have (cf. Lemma 3.1(1))

$$\omega_{a,k+1} e_{k+1} = e_{k+1} y_{k+1}^a e_{k+1} = e_{k+1} (\bar{L}_{k+1} - L_{k+1}) y_{k+1}^{a-1} e_{k+1} + \delta_1 e_{k+1} y_{k+1}^{a-1} e_{k+1}. \tag{3.10}$$

Note that the second term in the right-hand side of (3.10) does not contribute to the leading term. Considering the first term in the right-hand side of (3.10) and expressing L_{k+1} by (3.1), using $(j, k+1) = s_{j,k} s_k s_{k,j}$ (cf. (3.2)) and the fact that $s_{j,k}, s_{k,j}$ commute with y_{k+1}, e_{k+1} (cf. (3.3) and Lemma 3.3(6)), we see that a term in the right-hand side of (3.10) becomes

$$\begin{aligned}
 & -s_{j,k}e_{k+1}s_ky_{k+1}^{a-1}e_{k+1}s_{k,j} \\
 & = s_{j,k}e_{k+1} \left(- (y_k + e_k)^{a-1} + \sum_{b=0}^{a-2} (y_k + e_k)^{a-2-b} \omega_{b,k+1} \right) s_{k,j}, \tag{3.11}
 \end{aligned}$$

where the equality follows from Lemma 3.8 and Corollary 3.5. By Lemma 3.7, $\omega_{b,k+1}$ commutes with $s_{k,j}$. Now by inductive assumption, the right-hand side of (3.11) can be written as an R -linear combination of monomials with the required form such that the leading term is $-z_{j,k+1}^{a-1}$.

Now we consider terms in (3.10) concerning \bar{L}_{k+1} , namely we need to deal with $e_{k+1}(\bar{j}, \bar{k} + 1)y_{k+1}^{a-1}e_{k+1}$. We remark that it is hard to compute it directly. However, by Lemma 3.6 and induction on a , we can use $(-1)^{a-1}\bar{y}_{k+1}^{a-1}e_{k+1}$ to replace $y_{k+1}^{a-1}e_{k+1}$ in $e_{k+1}(\bar{j}, \bar{k} + 1)y_{k+1}^{a-1}e_{k+1}$ (by forgetting lower terms). This enables us to consider $(-1)^{a-1}e_{k+1}(\bar{j}, \bar{k} + 1)\bar{y}_{k+1}^{a-1}e_{k+1}$ instead. As above, this term can be written as the required form with leading term $(-1)^{a-1}\bar{z}_{j,k+1}^{a-1}$. The proof is completed. \square

Lemma 3.11. For $a \in \mathbb{Z}^{\geq 0}, k \in \mathbb{Z}^{\geq 1}$, both $\omega_{a,k+1}$ and $\bar{\omega}_{a,k+1}$ commute with y_{k+1} and \bar{y}_{k+1} .

Proof. By Lemmas 2.6, 3.7 and 3.10, it suffices to prove that both $z_{j,k+1}$ and $\bar{z}_{j,k+1}$ for $1 \leq j \leq k$, commute with y_{k+1} . By Lemma 3.3(9) and $y_{k+1}e_k = e_ky_{k+1}$, we have $y_{k+1}(e_k + y_k) = (e_k + y_k)y_{k+1}$. Note that $z_{k,k+1} = y_k + e_k$, we have $y_{k+1}z_{k,k+1} = z_{k,k+1}y_{k+1}$. In general, by Lemma 3.3(6), $y_{k+1}z_{j,k+1} = z_{j,k+1}y_{k+1}$. By (3.8) and Lemma 2.6, $y_{k+1}\bar{y}_k = \bar{y}_ky_{k+1}$. Since $\bar{y}_k + e_k = \bar{z}_{k,k+1}$ (cf. (3.9)), $y_{k+1}\bar{z}_{k,k+1} = \bar{z}_{k,k+1}y_{k+1}$. So, by Lemma 3.3(7), $y_{k+1}\bar{z}_{j,k+1} = \bar{z}_{j,k+1}y_{k+1}$. The result follows. \square

The following is the main result of this section. It follows from Theorem 2.4, Lemmas 3.3, 3.11 and Corollary 3.5.

Theorem 3.12. Let F be a field containing ω_0, ω_1 with $\omega_0 \neq 0$. For any $k \in \mathbb{Z}^{> 0}$, let $\mathcal{B}_{r+k,t+k}(\omega_0)$ be the walled Brauer algebra over F . Then there is an F -algebra homomorphism $\phi_k : \mathcal{B}_{r,t}^{\text{aff}} \rightarrow \mathcal{B}_{r+k,t+k}(\omega_0)$ sending

$$s_i, \bar{s}_j, e_1, x_1, \bar{x}_1, \omega_a, \bar{\omega}_a \mapsto s_{i+k}, \bar{s}_{j+k}, e_{k+1}, y_{k+1}, \bar{y}_{k+1}, \omega_{a,k+1}, \bar{\omega}_{a,k+1}, \tag{3.12}$$

respectively such that $\delta_1 = \omega_0^{-1}\omega_1$.

4. A basis of an affine walled Brauer algebra

Throughout this section, we assume that R is a commutative ring containing 1, ω_0 and ω_1 . The main purpose of this section is to prove that $\mathcal{B}_{r,t}^{\text{aff}}$ is free over R with infinite rank.

Lemma 4.1. There is an R -linear anti-involution $\sigma : \mathcal{B}_{r,t}^{\text{aff}} \rightarrow \mathcal{B}_{r,t}^{\text{aff}}$ fixing generators $s_i, \bar{s}_j, e_1, x_1, \bar{x}_1, \omega_a$ and $\bar{\omega}_b$ for all possible a, b, i, j 's.

Proof. This follows from the symmetry of the defining relations in [Definition 2.7](#) (cf. [Lemma 2.5](#)). \square

The following can be proven by arguments similar to those for [Lemma 3.6](#).

Lemma 4.2. *For any $k \in \mathbb{Z}^{\geq 1}$, we have $e_1 \bar{x}_1^k = \sum_{i=0}^k a_{k,i} e_1 x_1^i$ for some $a_{k,i} \in \mathcal{B}_{r,t}^{\text{aff}}$ such that*

- (1) $a_{k,k} = (-1)^k$,
- (2) $a_{k,i} = \omega_0 a_{k-1,i} - a_{k-1,i-1}$, $1 \leq i \leq k-1$,
- (3) $a_{k,0} = -\sum_{i=1}^{k-1} a_{k-1,i} \omega_i$.

In particular, $a_{k,i} \in R[\omega_2, \omega_3, \dots, \omega_{k-1}]$ for all i with $1 \leq i \leq k$ such that each monomial of $a_{k,i}$ is of form $\omega_{j_1} \cdots \omega_{j_\ell}$ with $\sum_{i=1}^\ell j_i \leq a-1$.

Recall that an element $a \in \mathcal{B}_{r,t}^{\text{aff}}$ is called $R[\omega_2, \omega_3, \dots, \bar{\omega}_0, \bar{\omega}_1, \dots]$ -torsion-free if $ab \neq 0$ for all non-zero $b \in R[\omega_2, \omega_3, \dots, \bar{\omega}_0, \bar{\omega}_1, \dots]$.

Corollary 4.3. *Assume e_1 is $R[\omega_2, \omega_3, \dots, \bar{\omega}_0, \bar{\omega}_1, \dots]$ -torsion-free. Then $\bar{\omega}_0 = \omega_0$, $\bar{\omega}_1 = -\omega_1$ and $\bar{\omega}_k = \sum_{i=0}^k a_{k,i} \omega_i \in R[\omega_2, \omega_3, \dots, \omega_k]$ for $k \geq 2$.*

Proof. Applying e_1 on the right hand side of $e_1 \bar{x}_1^k$ and using [Lemma 4.2](#) yield the result as required. \square

Remark 4.4. By [Corollary 4.3](#), $\mathcal{B}_{r,t}^{\text{aff}}$ can be generated by $s_i, \bar{s}_j, e_1, x_1, \bar{x}_1, \omega_a$ for all possible i, j, a if e_1 is $R[\omega_2, \omega_3, \dots, \bar{\omega}_0, \bar{\omega}_1, \dots]$ -torsion-free. In fact, when we prove the freeness of $\mathcal{B}_{r,t}^{\text{aff}}$, we do not need to assume that e_1 is $R[\omega_2, \omega_3, \dots, \bar{\omega}_0, \bar{\omega}_1, \dots]$ -torsion-free. What we need is that $\bar{\omega}_k$'s are determined by $\omega_2, \dots, \omega_k$ in [Corollary 4.3](#) with $a_{k,i}$ being determined in [Lemma 4.2](#). If so, $\mathcal{B}_{r,t}^{\text{aff}}$ is free over R , forcing e_1 to be $R[\omega_2, \omega_3, \dots]$ -torsion-free, automatically.

The elements defined below will play similar roles to that of x_1 and \bar{x}_1 :

$$x_i = s_{i-1} x_{i-1} s_{i-1} - s_{i-1}, \quad \bar{x}_j = \bar{s}_{j-1} \bar{x}_{j-1} \bar{s}_{j-1} - \bar{s}_{j-1}, \tag{4.1}$$

for $2 \leq i \leq r, 2 \leq j \leq t$. The following result can be checked easily.

Lemma 4.5. *We have*

- (1) $s_i x_i = x_{i+1} s_i + 1, x_i x_j = x_j x_i$ for $1 \leq i < j \leq r$.
- (2) $\bar{s}_i \bar{x}_i = \bar{x}_{i+1} \bar{s}_i + 1, \bar{x}_i \bar{x}_j = \bar{x}_j \bar{x}_i$ for $1 \leq i < j \leq t$.
- (3) Let $\phi_k : \mathcal{B}_{r,t}^{\text{aff}} \rightarrow \mathcal{B}_{r+k,t+k}(\omega_0)$ be the homomorphism in [Theorem 3.12](#). Then (recall notation $e_{i,j}$ in [\(3.3\)](#))

- (i) $\phi_k(x_\ell) = \sum_{j=1}^k e_{k+\ell,j} - L_{k+\ell} + \omega_0^{-1}\omega_1,$
- (ii) $\phi_k(\bar{x}_\ell) = \sum_{j=1}^k e_{j,k+\ell} - \bar{L}_{k+\ell} - \omega_0^{-1}\omega_1.$

Lemma 4.6. For $1 \leq i \leq r$ and $1 \leq j \leq t$, we have

- (1) $x_i(\bar{x}_j + e_{i,j}) = (\bar{x}_j + e_{i,j})x_i, \bar{x}_j(x_i + e_{i,j}) = (x_i + e_{i,j})\bar{x}_j.$
- (2) $e_{i,j}(x_i + \bar{x}_j) = -e_{i,j}(\bar{L}_j + L_i), (x_i + \bar{x}_j)e_{i,j} = -(\bar{L}_j + L_i)e_{i,j}.$

Proof. By symmetry and Lemma 4.1, we need only check the first assertions of (1)–(2). In fact, if $i = 1$, then (1) follows from Definition 2.7(21), (24). In general, it follows from induction on i . By Definition 2.7(8), $e_{1,2}(x_1 + \bar{x}_2) = -e_{1,2}\bar{L}_2$. Using Definition 2.7(24) and induction on j yields $e_{1,j}(x_1 + \bar{x}_j) = -e_{1,j}\bar{L}_j$. This is (2) for $i = 1$. The general case follows from induction on i . \square

Lemma 4.7. Suppose $1 \leq i, j \leq r$ and $1 \leq k, \ell \leq t$.

- (1) If $i \neq j$, then $e_{i,k}(x_j + L_j) = (x_j + L_j)e_{i,k}.$
- (2) If $k \neq \ell$, then $e_{i,k}(\bar{x}_\ell + \bar{L}_\ell) = (\bar{x}_\ell + \bar{L}_\ell)e_{i,k}.$

Proof. By symmetry, we need only to check (1). By Definition 2.7(9), we have $e_1(x_2 + L_2) = (L_2 + x_2)e_1$. Using induction on j yields $e_1(x_j + L_j) = (x_j + L_j)e_1$ for $j \geq 3$. This is (1) for $i = k = 1$. By induction on k , $e_{1,k}(x_j + L_j) = (x_j + L_j)e_{1,k}$. If $i < j$, by Lemmas 3.1 and 4.5, we have $e_{i,k}(x_j + L_j) = (x_j + L_j)e_{i,k}$.

In order to prove (1) for $i > j$, we need $e_{2,1}x_1 = x_1e_{2,1}$, which follows from Definition 2.7(9). By Definition 2.7(4), (24), we have $e_{i,k}x_1 = x_1e_{i,k}$. By induction on j , we have $e_{i,k}(x_j + L_j) = (x_j + L_j)e_{i,k}$ for all j with $j < i$, as required. \square

Lemma 4.8. Suppose $1 \leq i \leq r - 1$ and $1 \leq j \leq t - 1$.

- (1) $s_i(x_i + L_i) = (x_{i+1} + L_{i+1})s_i, \bar{s}_j(\bar{x}_j + \bar{L}_j) = (\bar{x}_{j+1} + \bar{L}_{j+1})\bar{s}_j.$
- (2) $e_{i,j}(x_i + L_i)^a e_{i,j} = \omega_a e_{i,j}, e_{i,j}(\bar{x}_j + \bar{L}_j)^a e_{i,j} = \bar{\omega}_a e_{i,j}$ for $a \in \mathbb{Z}^{\geq 0}$.

Proof. By symmetry, it suffices to check the first assertions of (1) and (2). We remark that (1) follows from Lemmas 3.1 and 4.5, and (2) follows from (1) together with induction on i . \square

We consider $\mathcal{B}_{r,t}^{\text{aff}}$ as a filtered algebra defined as follows. Set

$$\deg s_i = \deg \bar{s}_j = \deg e_1 = \deg \omega_a = 0 \quad \text{and} \quad \deg x_k = \deg \bar{x}_\ell = 1,$$

for all possible a, i, j, k, ℓ 's. Let $(\mathcal{B}_{r,t}^{\text{aff}})^{(k)}$ be the R -submodule spanned by monomials with degrees less than or equal to k for $k \in \mathbb{Z}^{\geq 0}$. Then we have the following filtration

$$\mathcal{B}_{r,t}^{\text{aff}} \supset \dots \supset (\mathcal{B}_{r,t}^{\text{aff}})^{(1)} \supset (\mathcal{B}_{r,t}^{\text{aff}})^{(0)} \supset (\mathcal{B}_{r,t}^{\text{aff}})^{(-1)} = 0. \tag{4.2}$$

Let $\text{gr}(\mathcal{B}_{r,t}^{\text{aff}}) = \bigoplus_{i \geq 0} (\mathcal{B}_{r,t}^{\text{aff}})^{[i]}$, where $(\mathcal{B}_{r,t}^{\text{aff}})^{[i]} = (\mathcal{B}_{r,t}^{\text{aff}})^{(i)} / (\mathcal{B}_{r,t}^{\text{aff}})^{(i-1)}$. Then $\text{gr}(\mathcal{B}_{r,t}^{\text{aff}})$ is a \mathbb{Z} -graded algebra associated to $\mathcal{B}_{r,t}^{\text{aff}}$. We use the same symbols to denote elements in $\text{gr}(\mathcal{B}_{r,t}^{\text{aff}})$. We remark that we will work with $\text{gr}(\mathcal{B}_{r,t}^{\text{aff}})$ when we prove the freeness of $\mathcal{B}_{r,t}^{\text{aff}}$.

Fix $r, t, f \in \mathbb{Z}^{>0}$ with $f \leq \min\{r, t\}$. We define the following subgroups of $\mathfrak{S}_r, \mathfrak{S}_r \times \mathfrak{S}_t$ and $\tilde{\mathfrak{S}}_t$ respectively,

$$\begin{aligned} \mathfrak{S}_{r-f} &= \langle s_j \mid f+1 \leq j < r \rangle, \\ \mathfrak{G}_f &= \langle \bar{s}_i s_i \mid 1 \leq i < f \rangle, \\ \tilde{\mathfrak{S}}_{t-f} &= \langle \bar{s}_j \mid f+1 \leq j < t \rangle. \end{aligned} \tag{4.3}$$

Observe that \mathfrak{G}_f is isomorphic to the symmetric group in f letters. The following result has been given in [29] without a detailed proof. We remark that $\mathcal{D}_{r,t}^f$ in (4.4) was defined in [12, Proposition 6.1] via certain *row-standard tableaux*.

Lemma 4.9. (See [29, Lemma 2.6].) *The following (recall notation $s_{i,j}$ in (3.2))*

$$\mathcal{D}_{r,t}^f = \{s_{f,i_f} \bar{s}_{f,j_f} \dots s_{1,i_1} \bar{s}_{1,j_1} \mid 1 \leq i_1 < \dots < i_f \leq r, k \leq j_k\}, \tag{4.4}$$

is a complete set of right coset representatives for $\mathfrak{S}_{r-f} \times \mathfrak{G}_f \times \tilde{\mathfrak{S}}_{t-f}$ in $\mathfrak{S}_r \times \tilde{\mathfrak{S}}_t$.

Proof. We denote by $\tilde{\mathcal{D}}_{r,t}^f$ the right-hand side of (4.4), and by $\mathcal{D}_{r,t}^f$ a complete set of right coset representatives. Then obviously $\tilde{\mathcal{D}}_{r,t}^f \subset \mathcal{D}_{r,t}^f$. In order to verify the inverse inclusion, it suffices to prove that $|\tilde{\mathcal{D}}_{r,t}^f|$, the cardinality of $\tilde{\mathcal{D}}_{r,t}^f$, is $\frac{r!t!}{(r-f)!(t-f)!f!} = C_r^f C_t^f f!$, which is clearly the cardinality of $\mathcal{D}_{r,t}^f$, where C_r^f is the binomial number. This will be done by induction on f as follows.

If $f = 0$, there is nothing to be proven. Assume $f \geq 1$. For any element in (4.4), we have $i_f \geq f$. For each fixed $i := i_f$, there are $t - f + 1$ choices of j_f with $j_f \geq f$, and further, conditions for other indices are simply conditions for $\mathcal{D}_{i-1,t}^{f-1}$. So,

$$\begin{aligned} |\tilde{\mathcal{D}}_{r,t}^f| &= (t - f + 1) \sum_{i=f}^r |\mathcal{D}_{i-1,t}^{f-1}| \\ &= (t - f + 1) \sum_{i=f}^r C_{i-1}^{f-1} C_t^{f-1} (f - 1)! \\ &= \sum_{i=f}^r C_{i-1}^{f-1} C_t^f f! = C_r^f C_t^f f!, \end{aligned}$$

where the second equality follows from inductive assumption on f , and the last follows from the well-known combinatorics formula $C_r^i = C_{r-1}^i + C_{r-1}^{i-1}$. \square

We denote

$$e^f = e_1 e_2 \cdots e_f \quad \text{for any } f \text{ with } 1 \leq f \leq \min\{r, t\}. \tag{4.5}$$

If $f = 0$, we set $e^0 = 1$. In [12, Theorem 6.13], Enyang constructed a cellular basis for q -walled Brauer algebras. The following result follows from this result immediately.

Theorem 4.10. (See [12].) *The following is an R -basis of $\mathcal{B}_{r,t}(\omega_0)$,*

$$\mathcal{M} = \{c^{-1}e^f wd \mid 1 \leq f \leq \min\{r, t\}, w \in \mathfrak{S}_{r-f} \times \bar{\mathfrak{S}}_{t-f}, c, d \in \mathcal{D}_{r,t}^f\}.$$

Definition 4.11. We say that

$$\mathbf{m} := \prod_{i=1}^r x_i^{\alpha_i} c^{-1} e^f w d \prod_{j=1}^t \bar{x}_j^{\beta_j} \prod_{k \in \mathbb{Z}^{\geq 2}} \omega_k^{a_k} \tag{4.6}$$

is a *regular monomial* if $c, d \in \mathcal{D}_{r,t}^f$, $\alpha_i, \beta_j \in \mathbb{Z}^{\geq 0}$ and $a_k \in \mathbb{Z}^{\geq 0}$ for $k \geq 2$ such that $a_k = 0$ for all but finitely many k 's.

Proposition 4.12. *Suppose R is a commutative ring which contains 1, ω_0, ω_1 . As an R -module, $\mathcal{B}_{r,t}^{\text{aff}}$ is spanned by all regular monomials.*

Proof. Let M be the R -submodule of $\mathcal{B}_{r,t}^{\text{aff}}$ spanned by all regular monomials $\mathbf{m} \in \mathcal{B}_{r,t}^{\text{aff}}$ given in (4.6). We want to prove

$$h\mathbf{m} = h \prod_{i=1}^r x_i^{\alpha_i} c^{-1} e^f w d \prod_{i=1}^t \bar{x}_i^{\beta_i} \prod_{i \in \mathbb{Z}^{\geq 2}} \omega_i^{a_i} \in M \quad \text{for any generator } h \text{ of } \mathcal{B}_{r,t}^{\text{aff}}. \tag{4.7}$$

If so, then M is a left $\mathcal{B}_{r,t}^{\text{aff}}$ -module, and thus $M = \mathcal{B}_{r,t}^{\text{aff}}$ by the fact that $1 \in M$.

We prove (4.7) by induction on $|\alpha| := \sum_{i=1}^r \alpha_i$. If $|\alpha| = 0$, i.e., $\alpha_i = 0$ for all possible i 's, then by Theorem 4.10, we have (4.7) unless $h = \bar{x}_1$.

If $h = \bar{x}_1$, by Lemma 4.5, we need to compute $\bar{x}_k e^f$ for all k with $1 \leq k \leq t$. If $k \in \{1, 2, \dots, f\}$, by Lemma 4.6(3), we can use $-x_k$ instead of \bar{x}_k . So, $h\mathbf{m} \in M$. Otherwise, by Lemma 4.7(2), we can use $e^f \bar{x}_k$ instead of $\bar{x}_k e^f$. So, (4.7) follows from Lemma 4.5 and Theorem 4.10.

Suppose $|\alpha| > 0$. By Lemma 4.5 and Theorem 4.10, we see that (4.7) holds for $h \in \{s_1, \dots, s_{r-1}, \bar{s}_1, \dots, \bar{s}_{t-1}, x_1\}$. If $h = \bar{x}_1$, then (4.7) follows from Lemma 4.6(1), and inductive assumption.

Finally, we assume $h = e_1$. If $\alpha_i \neq 0$ for some i with $2 \leq i \leq r$, then (4.7) follows from Lemma 4.7(1) and inductive assumption. Suppose $x^\alpha = x_1^{\alpha_1}$ with $\alpha_1 > 0$. We need to verify

$$e_1 x_1^{\alpha_1} c^{-1} e^f w d \prod_{i=1}^t \bar{x}_i^{\beta_i} \in M \quad \text{for } \alpha_1 > 0. \tag{4.8}$$

Note that $ce_1 c^{-1} = e_{i,j}$ for some i, j . By Lemma 4.5 and inductive assumption on $|\alpha|$, we can use $c^{-1} x_i^{\alpha_1}$ to replace $x_1^{\alpha_1} c^{-1}$ in (4.8). So, we need to verify

$$e_{i,j} x_i^{\alpha_1} e^f w d \prod_{i=1}^t \bar{x}_i^{\beta_i} \in M. \tag{4.9}$$

In fact, by Lemma 4.6(2) and inductive assumption, it is equivalent to verifying

$$e_{i,j} \bar{x}_j^{\alpha_1} e^f w d \prod_{i=1}^t \bar{x}_i^{\beta_i} \in M. \tag{4.10}$$

If $j \geq f + 1$, (4.10) follows from Lemma 4.7(2) and Theorem 4.10. Otherwise, $j \leq f$.

If $i = j$, by inductive assumption, we use $(x_i + L_i)^{\alpha_1}$ instead of $x_i^{\alpha_1}$ in $e_{i,j} x_i^{\alpha_1} e_j$. So, (4.9) follows from Lemma 4.8(2). If $i \neq j$, we have

$$e_{i,j} x_i^{\alpha_1} e_j = e_{i,j} e_j x_i^{\alpha_1} = (i, j) x_i^{\alpha_1} e_j = x_j^{\alpha_1} (i, j) e_j,$$

which holds in $\text{gr}(\mathcal{B}_{r,t}^{\text{aff}})$. By inductive assumption and our previous result on $h \in \{s_1, \dots, s_{r-1}, x_1\}$, we have (4.9) and hence (4.8). This completes the proof. \square

Assumption 4.13. In the remaining part of this paper, we always assume that $\bar{\omega}_i$'s are given in Corollary 4.3 with $a_{k,i}$ being determined in Lemma 4.2.

Now we are able to prove the main result of this section. We remark that the idea of the proof is motivated by Nazarov's work on affine Wenzl algebras in [25].

Theorem 4.14. *Suppose R is a commutative ring which contains $1, \omega_0, \omega_1$. Then $\mathcal{B}_{r,t}^{\text{aff}}$ is free over R spanned by all regular monomials in (4.6). In particular, $\mathcal{B}_{r,t}^{\text{aff}}$ is of infinite rank.*

Proof. Let \mathbf{M} be the set of all regular monomials of $\mathcal{B}_{r,t}^{\text{aff}}$. First, we prove that \mathbf{M} is F -linearly independent where F is the quotient field of $\mathbb{Z}[\omega_0, \omega_1]$ with ω_0, ω_1 being indeterminates.

Suppose conversely there is a finite subset \mathcal{S} of \mathbf{M} such that $\sum_{\mathbf{m} \in \mathcal{S}} r_{\mathbf{m}} \mathbf{m} = 0$ with $r_{\mathbf{m}} \neq 0$ for all $\mathbf{m} \in \mathcal{S}$. Recall from Definition 4.11 that each regular monomial is of the

form in (4.6). For each $\mathbf{m} \in \mathcal{S}$ as in (4.6), we set

$$k_{\mathbf{m}} = \max \left\{ |\alpha| + \sum_j j a_j, |\beta| + \sum_j j a_j \right\}, \quad k = \max \{ k_{\mathbf{m}} \mid \mathbf{m} \in \mathcal{S} \}, \quad (4.11)$$

where $|\alpha| = \sum_{i=1}^r \alpha_i, |\beta| = \sum_{i=1}^t \beta_i$. Consider the homomorphism $\phi_k : \mathcal{B}_{r,t}^{\text{aff}} \rightarrow \mathcal{B}_{r+k,t+k}(\omega_0)$ in Theorem 3.12. Then $\phi_k(\mathbf{m})$ can be written as a linear combination of $(r+k, t+k)$ -walled Brauer diagrams.

Using Lemma 4.5(3) to express $\phi_k(x_\ell)$ and $\phi_k(\bar{x}_\ell)$, and using Lemma 3.10 to express $\omega_{a,k+1}$ for $a \in \mathbb{Z}^{\geq 2}$, we see that some terms of $\phi_k(\mathbf{m})$ are of forms (we will see in the next paragraph that other terms of $\phi_k(\mathbf{m})$ will not contribute to our computations)

$$\prod_{i=1}^r (k+i, i_1) \cdots (k+i, i_{\alpha_i}) \phi_k(c^{-1} e^f w d) \prod_{j=1}^t (\bar{k}+i, \bar{j}_1) \cdots (\bar{k}+i, \bar{j}_{\beta_j}) \prod_{i \geq 2} \mathbf{c}_i, \quad (4.12)$$

where \mathbf{c}_i ranges over products of some disjoint cycles in \mathfrak{S}_k (or $\bar{\mathfrak{S}}_k$) with total length $i a_i$. We remark that such \mathbf{c}_i 's come from $\omega_{i,k+1}$. Further, the walled Brauer diagram corresponding to $\phi_k(c^{-1} e^f w d)$ has vertical edges $[i, i]$ and $[\bar{j}, \bar{j}]$ for all i, j with $1 \leq i, j \leq k$. We call the terms of the form (4.12) the *leading terms* if

- (i) either $k = |\alpha| + \sum_j j a_j$ or $k = |\beta| + \sum_j j a_j$ (cf. (4.11)), and
- (ii) the corresponding f in (4.12) is minimal among all terms satisfying (i), and
- (iii) in the first case of (i), the juxtaposition of the sequences $i_1, i_2, \dots, i_{\alpha_i}$ for $1 \leq i \leq r$ and $\mathbf{c}_i, i \geq 2$ run through all permutations of the sequences in $1, 2, \dots, k$; while in the second case of (i), the juxtaposition of the sequences $j_1, j_2, \dots, j_{\beta_j}$ for $1 \leq j \leq r$ and $\mathbf{c}_i, i \geq 2$ run through all permutations of the sequences in $\bar{1}, \bar{2}, \dots, \bar{k}$.

If we identify the factor $\phi_k(c^{-1} e^f w d)$ in the leading terms with the corresponding walled Brauer diagrams, we have

- (1) there are exactly f horizontal edges in both top and bottom rows,
- (2) no vertical edge of form $[i, i], 1 \leq i \leq k$ in the first case,
- (3) no vertical edge of form $[\bar{i}, \bar{i}], 1 \leq i \leq k$ in the second case,
- (4) no horizontal edge of form $[i, \bar{j}], 1 \leq i \leq k, \bar{1} \leq \bar{j} \leq \bar{k}$ in both rows.

These leading terms exactly appear in $\phi_k(\mathbf{m})$ when conditions (i)–(iii) are satisfied.

Other terms in $\phi_k(\sum_{\mathbf{m} \in \mathcal{S}} r_{\mathbf{m}} \mathbf{m})$ are non-leading terms, which are terms obtained by (4.12) by using some $e_{k+i, j}$'s (resp., $e_{j, k+i}$'s) or scalars instead of some $(k+i, i_j)$'s (resp., $(\bar{k}+i, \bar{i}_j)$'s) or using certain product of $e_{i, j}$'s, $1 \leq i, j \leq k$ instead of some factors of some cycles \mathbf{c}_i 's. Thus such terms cannot be proportional to any leading terms. Therefore \mathcal{S} is F -linearly independent. By Proposition 4.12, \mathbf{M} is a $\mathbb{Z}[\omega_0, \omega_1]$ -basis.

Now, for an arbitrary commutative ring R containing $1, \omega_0, \omega_1$, we can regard R as a left $\mathbb{Z}[\omega_0, \omega_1]$ -module such that the indeterminates $\omega_0, \omega_1 \in \mathbb{Z}[\omega_0, \omega_1]$ act on R as the scalars $\omega_0, \omega_1 \in R$ respectively. By standard arguments on specialization, $\mathcal{B}_{r,t}^{\text{aff}}$, which is defined over R , is isomorphic to $A \otimes_{\mathbb{Z}[\omega_0, \omega_1]} R$, where A is the algebra $\mathcal{B}_{r,t}^{\text{aff}}$ defined over $\mathbb{Z}[\omega_0, \omega_1]$. So, $\mathcal{B}_{r,t}^{\text{aff}}$ is free over R with infinite rank. \square

Let R be a commutative ring containing $1, \hat{\omega}_a$ for $a \in \mathbb{Z}^{\geq 2}$. Let I be the two-sided ideal of $\mathcal{B}_{r,t}^{\text{aff}}$ generated by $\omega_a - \hat{\omega}_a, a \in \mathbb{Z}^{\geq 2}$. Then there is an epimorphism $\psi : \mathcal{B}_{r,t}^{\text{aff}} \rightarrow \mathcal{B}_{r,t}^{\text{aff}}/I$. Let $\widehat{\mathcal{B}}_{r,t} = \mathcal{B}_{r,t}^{\text{aff}}/I$, namely $\widehat{\mathcal{B}}_{r,t}$ is the specialization of $\mathcal{B}_{r,t}^{\text{aff}}$ with ω_a being specialized to $\hat{\omega}_a$ for $a \in \mathbb{Z}^{\geq 2}$. Without confusion, we will simply denote elements $\hat{\omega}_a$ of R as ω_a .

Definition 4.15. We say that the image of a regular monomial \mathbf{m} of $\mathcal{B}_{r,t}^{\text{aff}}$ (cf. (4.6)) is a regular monomial of $\widehat{\mathcal{B}}_{r,t}$ if \mathbf{m} does not contain factors ω_i 's for $i \geq 2$.

In the rest part of the paper, we will be interested in the above specialized algebra $\widehat{\mathcal{B}}_{r,t}$. Without confusion, we will use $\mathcal{B}_{r,t}^{\text{aff}}$ to denote it.

The following result follows from Theorem 4.14, immediately.

Theorem 4.16. Suppose R is a commutative ring which contains $1, \omega_i$ with $i \in \mathbb{Z}^{\geq 0}$ and keep Assumption 4.13. Then $\mathcal{B}_{r,t}^{\text{aff}}$ is free over R spanned by all regular monomials. In particular, $\mathcal{B}_{r,t}^{\text{aff}}$ is of infinite rank.

We close this section by giving some relationship between affine walled Brauer algebras and degenerate affine Hecke algebras [21], walled Brauer algebras, etc.

Definition 4.17. The degenerate affine Hecke algebra $\mathcal{H}_n^{\text{aff}}$ is the unital R -algebra generated by $S_1, \dots, S_{n-1}, Y_1, \dots, Y_n$ and relations

$$\begin{aligned} S_i S_j &= S_j S_i, & Y_i Y_k &= Y_k Y_i, \\ S_i Y_i - Y_{i+1} S_i &= -1, & Y_i S_i - S_i Y_{i+1} &= -1, \\ S_j S_{j+1} S_j &= S_{j+1} S_j S_{j+1}, & S_i^2 &= 1, \end{aligned}$$

for $1 \leq i < n, 1 \leq j < n - 1$ with $|i - j| > 1$, and $1 \leq k \leq n$.

Proposition 4.18. Let R be a commutative ring containing $1, \omega_i$ with $i \in \mathbb{Z}^{\geq 0}$. Let $\mathcal{B}_{r,t}^{\text{aff}}$ be the affine walled Brauer algebra over R . Let I (resp., J) be the two-sided ideal of $\mathcal{B}_{r,t}^{\text{aff}}$ generated by x_1 and \bar{x}_1 (resp., e_1).

- (1) $\mathcal{B}_{r,t}^{\text{aff}}/I \cong \mathcal{B}_{r,t}$.
- (2) $\mathcal{B}_{r,t}^{\text{aff}}/J \cong \mathcal{H}_r^{\text{aff}} \otimes \mathcal{H}_t^{\text{aff}}$.
- (3) The subalgebra of $\mathcal{B}_{r,t}^{\text{aff}}$ generated by $e_1, s_1, \dots, s_{r-1}, \bar{s}_1, \dots, \bar{s}_{t-1}$ is isomorphic to the walled Brauer algebra $\mathcal{B}_{r,t}$ over R .

- (4) The subalgebra of $\mathcal{B}_{r,t}^{\text{aff}}$ generated by s_1, \dots, s_{r-1} and x_1 (resp., $\bar{s}_1, \dots, \bar{s}_{t-1}$ and \bar{x}_1) is isomorphic to the degenerate affine Hecke algebra $\mathcal{H}_r^{\text{aff}}$ (resp., $\mathcal{H}_t^{\text{aff}}$).

Note that the isomorphism in (4) sends x_1 (resp., \bar{x}_1) to $-Y_1$.

5. Super Schur–Weyl duality

The main purpose of this section is to set up the relationship between affine walled Brauer algebras $\mathcal{B}_{r,t}^{\text{aff}}$ with special parameters and general linear Lie superalgebras $\mathfrak{gl}_{m|n}$. Throughout the section, we assume the ground field is \mathbb{C} .

Denote $\mathfrak{g} = \mathfrak{gl}_{m|n}$. Let $V = \mathbb{C}^{m|n}$ be the natural \mathfrak{g} -module. As a \mathbb{C} -vector superspace $V = V_0 \oplus V_1$ with $\dim V_0 = m$ and $\dim V_1 = n$. Take a natural basis $\{v_i \mid i \in I\}$ of V , where $I = \{1, 2, \dots, m+n\}$. For convenience we define the map $[\cdot] : I \rightarrow \mathbb{Z}_2$ by $[i] = \bar{0}$ if $i \leq m$ and $[i] = \bar{1}$ if $i > m$. Then v_i has the parity $[v_i] = [i]$. Denote by E_{ij} the matrix unit, which has parity $[E_{ij}] = [i] + [j]$. The Lie bracket on \mathfrak{g} is defined by

$$[E_{ij}, E_{k\ell}] = \delta_{jk}E_{i\ell} - (-1)^{([i]+[j])([k]+[\ell])} \delta_{\ell i}E_{kj}, \tag{5.1}$$

where $\delta_{jk} = 1$ if $j = k$ and 0, otherwise. Let V^* be the dual space of V with dual basis $\{\bar{v}_i \mid i \in I\}$. Then V^* is a left \mathfrak{g} -module with action

$$E_{ab}\bar{v}_i = -(-1)^{[a]([a]+[b])} \delta_{ia}\bar{v}_b. \tag{5.2}$$

Let $\mathfrak{h} = \text{span}\{E_{ii} \mid i \in I\}$ be a Cartan subalgebra of \mathfrak{g} , and \mathfrak{h}^* the dual space of \mathfrak{h} with $\{\varepsilon_i \mid i \in I\}$ being the dual basis of $\{E_{ii} \mid i \in I\}$. Then an element $\lambda \in \mathfrak{h}^*$ (called a *weight*) can be written as

$$\lambda = \sum_{i \in I} \lambda_i \varepsilon_i = (\lambda_1, \dots, \lambda_m \mid \lambda_{m+1}, \dots, \lambda_{m+n}) \quad \text{with } \lambda_i \in \mathbb{C}. \tag{5.3}$$

Take

$$\rho = \sum_{i=1}^m (1-i)\varepsilon_i + \sum_{j=1}^n (m-j)\varepsilon_{m+j} = (0, -1, \dots, 1-m \mid m-1, m-2, \dots, m-n),$$

and denote

$$|\lambda| := \sum_{i \in I} \lambda_i \quad (\text{called the size of } \lambda), \tag{5.4}$$

$$\begin{aligned} \lambda^\rho &= \lambda + \rho = (\lambda_1^\rho, \dots, \lambda_m^\rho \mid \lambda_{m+1}^\rho, \dots, \lambda_{m+n}^\rho), \quad \text{where,} \\ \lambda_i^\rho &= \lambda_i + 1 - i \quad \text{if } i \leq m, \quad \text{and} \quad \lambda_i^\rho = \lambda_i + 2m - i \quad \text{if } i > m. \end{aligned} \tag{5.5}$$

A weight λ is called *integral dominant* if $\lambda_i - \lambda_{i+1} \in \mathbb{Z}^{\geq 0}$ for $i \in I \setminus \{m, m+n\}$. It is called *typical* if $\lambda_i^\rho + \lambda_j^\rho \neq 0$ for any $i \leq m < j$ (otherwise it is called *atypical*) [18].

Example 5.1. For any $p, q \in \mathbb{C}$,

$$\lambda_{pq} = (p, \dots, p \mid -q, \dots, -q), \tag{5.6}$$

is a typical integral dominant weight if and only if

$$p - q \notin \mathbb{Z} \quad \text{or} \quad p - q \leq -m \quad \text{or} \quad p - q \geq n. \tag{5.7}$$

(Note that the λ_{pq} defined in [7, IV] is the $\lambda_{p,q+m}$ defined here.) In this case, the finite-dimensional simple \mathfrak{g} -module $L_{\lambda_{pq}}$ with highest weight λ_{pq} coincides with the Kac-module $K_{\lambda_{pq}}$ [7, IV], [18].

Let M be any \mathfrak{g} -module. For any $r, t \in \mathbb{Z}^{\geq 0}$, set $M^{r,t} = V^{\otimes r} \otimes M \otimes (V^*)^{\otimes t}$. For convenience we denote the ordered set

$$J = J_1 \cup \{0\} \cup J_2, \quad \text{where } J_1 = \{r, \dots, 1\}, \quad J_2 = \{\bar{1}, \dots, \bar{t}\}, \tag{5.8}$$

and $r \prec \dots \prec 1 \prec 0 \prec \bar{1} \prec \dots \prec \bar{t}$. We write $M^{r,t}$ as

$$M^{r,t} = \bigotimes_{i \in J} V_i, \quad \text{where } V_i = V \text{ if } i \prec 0, \quad V_0 = M \text{ and } V_i = V^* \text{ if } i \succ 0, \tag{5.9}$$

(hereafter all tensor products will be taken according to the order in J), which is a left $U(\mathfrak{g})^{\otimes(r+t+1)}$ -module (where $U(\mathfrak{g})$ is the universal enveloping algebra of \mathfrak{g}), with the action given by

$$\left(\bigotimes_{i \in J} g_i \right) \left(\bigotimes_{i \in J} x_i \right) = (-1)^{\sum_{i \in J} [g_i] \sum_{j \prec i} [x_j]} \bigotimes_{i \in J} (g_i x_i) \quad \text{for } g_i \in U(\mathfrak{g}), \quad x_i \in V_i.$$

In particular, if we delete the tensor M (or take $M = \mathbb{C}$), then $M^{r,t}$ is the mixed tensor product studied in [32], and if $t = 0$ and $M = K_{\lambda_{pq}}$, then $M^{r,t}$ is the tensor module studied in [7, IV].

We denote

$$\Omega = \frac{1}{2} (\Delta(C) - C \otimes 1 - 1 \otimes C) = \sum_{i,j \in I} (-1)^{[j]} E_{ij} \otimes E_{ji} \in \mathfrak{g}^{\otimes 2}, \tag{5.10}$$

where Δ is the comultiplication of $U(\mathfrak{g})$, and $C = \sum_{a,b \in I} (-1)^{[b]} E_{ab} E_{ba}$ is the quadratic Casimir, which is a central element of $U(\mathfrak{g})$. Thus obviously,

$$[\Delta(E_{ab}), \Omega] = 0 \quad \text{for all } a, b \in I. \tag{5.11}$$

Because of this well-known property, Ω is also called a *Casimir element*.

For $a, b \in J$ with $a \prec b$, we define $\pi_{ab} : U(\mathfrak{g})^{\otimes 2} \rightarrow U(\mathfrak{g})^{\otimes(r+t+1)}$ by

$$\pi_{ab}(x \otimes y) = 1 \otimes \cdots \otimes 1 \otimes x \otimes 1 \otimes \cdots \otimes 1 \otimes y \otimes 1 \otimes \cdots \otimes 1, \tag{5.12}$$

where x and y are in the a -th and b -th tensors respectively.

Notation 5.2. From now on, we always suppose $M = K_\lambda$ is the Kac-module with highest weight $\lambda = \lambda_{pq}$ in (5.6) for $p, q \in \mathbb{C}$ (at this moment, we do not impose any condition on p, q) and a highest weight vector v_λ defined to have parity $\bar{0}$.

Note that

$$E_{ij}v_\lambda = \begin{cases} pv_\lambda & \text{if } 1 \leq i = j \leq m, \\ -qv_\lambda & \text{if } m < i = j \leq m + n, \\ 0 & \text{if } 1 \leq i \neq j \leq m \text{ or } m \leq i \neq j \leq m + n, \end{cases} \tag{5.13}$$

and K_λ is 2^{mn} -dimensional with a basis

$$B = \left\{ b^\sigma := \prod_{i=1}^n \prod_{j=1}^m E_{m+i,j}^{\sigma_{ij}} v_\lambda \mid \sigma = (\sigma_{ij})_{i,j=1}^{n,m} \in \{0, 1\}^{n \times m} \right\}, \tag{5.14}$$

where the products are taken in any fixed order (changing the order only changes the vectors by ± 1). Then $M^{r,t}$ is $2^{mn}(m+n)^{r+t}$ -dimensional with a basis

$$B_M = \left\{ b_M = \bigotimes_{i \in J_1} v_{k_i} \otimes b \otimes \bigotimes_{i \in J_2} \bar{v}_{k_i} \mid b \in B, k_i \in I \right\}. \tag{5.15}$$

Due to (5.11), the elements defined below are in the endomorphism algebra $\text{End}_{\mathfrak{g}}(M^{r,t})$ of the \mathfrak{g} -module $M^{r,t}$, which will be used throughout the section. We remark that the quadratic Casimir was also used to generate endomorphism algebras for some Lie algebras and their quantum groups (see e.g. [23]).

Definition 5.3. By (5.11), we can use (5.12) to define the following elements of the endomorphism algebra $\text{End}_{\mathfrak{g}}(M^{r,t})$,

$$\begin{aligned} s_i &= \pi_{i+1,i}(\Omega)|_{M^{r,t}} \quad (1 \leq i < r), & \bar{s}_j &= \pi_{\bar{j},\bar{j}+1}(\Omega)|_{M^{r,t}} \quad (1 \leq j < t), \\ x_1 &= -\pi_{10}(\Omega)|_{M^{r,t}}, & \bar{x}_1 &= -\pi_{0\bar{1}}(\Omega)|_{M^{r,t}}, & e_1 &= -\pi_{1\bar{1}}(\Omega)|_{M^{r,t}}. \end{aligned} \tag{5.16}$$

Note that the tensor space $M^{r,t}$ already admits a left action by the Lie superalgebra \mathfrak{g} . Our purpose is to establish a super Schur–Weyl duality (cf. Theorem 5.12) such that $M^{r,t}$ also admits a right action by some associative algebra (namely, the affine walled-Brauer algebra $\mathcal{B}_{r,t}^{\text{aff}}$ or its cyclotomic quotient, the level two walled Brauer algebra $\mathcal{B}_{r,t}^{p,q}(m, n)$,

cf. **Definition 5.11**). Thus as in literature (e.g., [7, IV]), elements defined above when regarded as elements in $\mathcal{B}_{r,t}^{\text{aff}}$ or in $\mathcal{B}_{r,t}^{p,q}(m,n)$ are usually written as right actions on $M^{r,t}$. However one shall always keep in mind that all elements are defined by left multiplication of the Casimir element Ω such that the first and second tensors in Ω act on some appropriate tensor positions in $M^{r,t}$, and also one shall always do bookkeeping on the sign change whenever the order of two elements (factors) in a term are exchanged.

Lemma 5.4. *We have the following.*

(1) For $w_1 \in V^{\otimes(r-1)}$, $w_2 \in (V^*)^{\otimes t}$ and $i \in I$,

$$(w_1 \otimes v_i \otimes v_\lambda \otimes w_2)x_1 = \begin{cases} -pw_1 \otimes v_i \otimes v_\lambda \otimes w_2 & \text{if } i \leq m, \\ -qw_1 \otimes v_i \otimes v_\lambda \otimes w_2 - \sum_{a=1}^m w_1 \otimes v_a \otimes E_{ia}v_\lambda \otimes w_2 & \text{if } i > m. \end{cases} \tag{5.17}$$

(2) For $w_1 \in V^{\otimes r}$, $w_2 \in (V^*)^{\otimes(t-1)}$ and $i \in I$,

$$(w_1 \otimes v_\lambda \otimes \bar{v}_i \otimes w_2)\bar{x}_1 = \begin{cases} pw_1 \otimes v_\lambda \otimes \bar{v}_i \otimes w_2 + \sum_{a=m+1}^{m+n} w_1 \otimes E_{ai}v_\lambda \otimes \bar{v}_a \otimes w_2 & \text{if } i \leq m, \\ qw_1 \otimes v_\lambda \otimes \bar{v}_i \otimes w_2 & \text{if } i > m. \end{cases} \tag{5.18}$$

(3) For $w_1 \in V^{\otimes(r-1)}$, $w_2 \in V^{\otimes(t-1)}$, $i, j \in I$,

$$(w_1 \otimes v_i \otimes v_\lambda \otimes \bar{v}_j \otimes w_2)e_1 = (-1)^{[i]}\delta_{ij} \sum_{a \in I} w_1 \otimes v_a \otimes v_\lambda \otimes \bar{v}_a \otimes w_2. \tag{5.19}$$

(4) For $a, b \in I$, and $w_1 \in V^{\otimes(r-1-i)}$, $w_2 \in M^{i-1,t}$, $w'_1 \in M^{r,j-1}$, $w'_2 \in (V^*)^{\otimes(t-1-j)}$,

$$\begin{aligned} (w_1 \otimes v_a \otimes v_b \otimes w_2)s_i &= (-1)^{[a][b]}w_1 \otimes v_b \otimes v_a \otimes w_2, \\ (w'_1 \otimes \bar{v}_a \otimes \bar{v}_b \otimes w'_2)\bar{s}_i &= (-1)^{[a][b]}w'_1 \otimes \bar{v}_b \otimes \bar{v}_a \otimes w'_2. \end{aligned} \tag{5.20}$$

Proof. The results can be easily verified. We prove (1) as an example. Since the two tensors of Ω in $x_1 = -\pi_{10}(\Omega)|_{M^{r,t}}$ act on v_i and v_λ respectively, the left-hand side of (5.17) is equal to

$$-w_1 \otimes \sum_{a,b \in I} (-1)^{[b]+[i]([a]+[b])} E_{ab}v_i \otimes E_{ba}v_\lambda \otimes w_2$$

$$= -w_1 \otimes \sum_{a \in I, a \leq i} (-1)^{[a][i]} v_a \otimes E_{ia} v_\lambda \otimes w_2,$$

which is equal to the right-hand side of (5.17) by (5.13). \square

Proposition 5.5. *There exists a \mathbb{C} -algebra homomorphism $\Phi : \mathcal{B}_{r,t}(n - m) \rightarrow \text{End}_{\mathfrak{g}}(M^{r,t})^{\text{op}}$, which sends generators $e_1, s_i, \bar{s}_j, 1 \leq i \leq r - 1, 1 \leq j \leq t - 1$ of $\mathcal{B}_{r,t}(n - m)$ in Theorem 2.4 to elements with the same symbols in Definition 5.3.*

Proof. By Definition 5.3, e_1, s_i, \bar{s}_j 's act only on $V^{\otimes r} \otimes (V^*)^{\otimes t}$ and have no relationship with the Kac module K_λ . So, our result follows from [32, Proposition 3.2]. \square

Lemma 5.6. *Let x_1, \bar{x}_1, e_1 be defined in (5.16). Then*

$$(x_1 + \bar{x}_1)e_1 = 0 = e_1(x_1 + \bar{x}_1), \tag{5.21}$$

$$x_1(e_1 + \bar{x}_1) = (e_1 + \bar{x}_1)x_1, \tag{5.22}$$

$$e_1 x_1^k e_1 = \omega_k e_1, \tag{5.23}$$

for $k \in \mathbb{Z}^{\geq 0}$ and for some $\omega_k \in \mathbb{C}$ such that $\omega_0 = m - n, \omega_1 = nq - mp$.

Proof. To prove the results, it suffices to consider the case $r = t = 1$. By (5.17), we have

$$\begin{aligned} (v_i \otimes v_\lambda \otimes \bar{v}_j)x_1 e_1 &= -\left(\sum_{a \in I} (-1)^{[i][a]} v_a \otimes E_{ia} v_\lambda \otimes \bar{v}_j\right) e_1 \\ &= (-1)^{1+[i][j]} (v_j \otimes E_{ij} v_\lambda \otimes \bar{v}_j) e_1 \\ &= (-1)^{[i][j]} \sum_{a,b \in I} (-1)^{[b]+([a]+[b])[i]} E_{ab} v_j \otimes E_{ij} v_\lambda \otimes E_{ba} \bar{v}_j \\ &= (-1)^{[i][j]} \sum_{a \in I} (-1)^{[j]+([a]+[j])[i]+1+[j]([j]+[a])} v_a \otimes E_{ij} v_\lambda \otimes \bar{v}_a \\ &= \sum_{a \in I} (-1)^{1+[a]([i]+[j])} v_a \otimes E_{ij} v_\lambda \otimes \bar{v}_a, \end{aligned} \tag{5.24}$$

and by (5.18),

$$\begin{aligned} (v_i \otimes v_\lambda \otimes \bar{v}_j)\bar{x}_1 e_1 &= -\left(\sum_{a \in I} (-1)^{1+[a][j]} v_i \otimes E_{aj} v_\lambda \otimes \bar{v}_a\right) e_1 \\ &= (-1)^{[i][j]} (v_i \otimes E_{ij} v_\lambda \otimes \bar{v}_i) e_1 \\ &= -(-1)^{[i][j]} \sum_{a,b \in I} (-1)^{[b]+([a]+[b])[j]} E_{ab} v_i \otimes E_{ij} v_\lambda \otimes E_{ba} \bar{v}_i \\ &= (-1)^{1+[i][j]} \sum_{a \in I} (-1)^{[i]+([a]+[i])[j]+1+[i]([i]+[a])} v_a \otimes E_{ij} v_\lambda \otimes \bar{v}_a \end{aligned}$$

$$= \sum_{a \in I} (-1)^{[a]([i]+[j])} v_a \otimes E_{ij} v_\lambda \otimes \bar{v}_a. \tag{5.25}$$

Thus $(x_1 + \bar{x}_1)e_1 = 0$. Similarly, one has $e_1(x_1 + \bar{x}_1) = 0$. This proves (5.21). To prove (5.22), by (5.17),

$$\begin{aligned} (v_i \otimes v_\lambda \otimes \bar{v}_j)x_1\bar{x}_1 &= - \sum_{b \in I} (-1)^{[i][b]} (v_b \otimes E_{ib} v_\lambda \otimes \bar{v}_j)\bar{x}_1 \\ &= \sum_{a, b \in I} (-1)^{[i][b]+[j]+([i]+[b])([j]+[a])} v_b \otimes E_{aj} E_{ib} v_\lambda \otimes E_{ja} \bar{v}_j \\ &= \sum_{a, b \in I} (-1)^{[i][b]+[j]+([i]+[b])([j]+[a])+1+[j]([j]+[a])} v_b \otimes E_{aj} E_{ib} v_\lambda \otimes \bar{v}_a \\ &= \sum_{a, b \in I} (-1)^{1+[i][b]+[a][j]+([i]+[b])([j]+[a])} v_b \otimes E_{aj} E_{ib} v_\lambda \otimes \bar{v}_a, \end{aligned} \tag{5.26}$$

and by (5.18),

$$\begin{aligned} (v_i \otimes v_\lambda \otimes \bar{v}_j)\bar{x}_1x_1 &= - \sum_{a \in I} (-1)^{1+[a][j]} (v_i \otimes E_{aj} v_\lambda \otimes \bar{v}_a)x_1 \\ &= \sum_{a, b \in I} (-1)^{1+[a][j]+[i]+[i]([i]+[b])} E_{bi} v_i \otimes E_{ib} E_{aj} v_\lambda \otimes \bar{v}_a \\ &= \sum_{a, b \in I} (-1)^{1+[a][j]+[b][i]} v_b \otimes E_{ib} E_{aj} v_\lambda \otimes \bar{v}_a. \end{aligned} \tag{5.27}$$

Using $E_{ib}E_{aj} = (-1)^{([i]+[b])([a]+[j])} E_{aj}E_{ib} + \delta_{ab}E_{ij} + (-1)^{1+([i]+[b])([a]+[j])} \delta_{ij}E_{ab}$ in (5.27), we obtain

$$\begin{aligned} (v_i \otimes v_\lambda \otimes \bar{v}_j)\bar{x}_1x_1 &= (v_i \otimes v_\lambda \otimes \bar{v}_j)x_1\bar{x}_1 + \sum_{a \in I} (-1)^{1+[a]([i]+[j])} v_a \otimes E_{ij} v_\lambda \otimes \bar{v}_a \\ &\quad + \delta_{ij} \sum_{a \in I} (-1)^{[a][i]+[b][i]+([i]+[b])([a]+[i])} v_b \otimes E_{ab} v_\lambda \otimes \bar{v}_a. \end{aligned}$$

By (5.19), we have

$$\begin{aligned} (v_i \otimes v_\lambda \otimes \bar{v}_j)e_1x_1 &= -\delta_{ij} \sum_{a, b \in I} (-1)^{[i]+[a]+[a]([a]+[b])} E_{ba} v_a \otimes E_{ab} v_\lambda \otimes \bar{v}_a \\ &= \delta_{ij} \sum_{a, b \in I} (-1)^{1+[i]+[a][b]} v_b \otimes E_{ab} v_\lambda \otimes \bar{v}_a. \end{aligned}$$

Comparing the above, we obtain $x_1(e_1 + \bar{x}_1) = (e_1 + \bar{x}_1)x_1$. To prove (5.23), for any $k \in \mathbb{Z}_+$, we have

$$\begin{aligned}
 & (v_i \otimes v_\lambda \otimes \bar{v}_j) e_1 x_1^k e_1 \\
 &= (-1)^{[i]} \delta_{ij} \sum_{\ell_0 \in I} (v_{\ell_0} \otimes v_\lambda \otimes \bar{v}_{\ell_0}) x_1^k e_1 \\
 &= \delta_{ij} \sum_{\ell_0, \ell_1, \dots, \ell_k \in I} (-1)^{k+[i]+\sum_{p=0}^{k-1} [\ell_p][\ell_{p+1}]} (v_{\ell_k} \otimes E_{\ell_{k-1}, \ell_k} E_{\ell_{k-2}, \ell_{k-1}} \cdots E_{\ell_0, \ell_1} v_\lambda \otimes \bar{v}_{\ell_0}) e_1 \\
 &= \omega_k (v_i \otimes v_\lambda \otimes \bar{v}_j) e_1 \quad \text{for some } \omega_k \in \mathbb{C},
 \end{aligned} \tag{5.28}$$

where the last equality is obtained as follows: if $\ell_k \neq \ell_0$, the corresponding terms become zero after applying e_1 by (5.19); otherwise $E_{\ell_{k-1}, \ell_k} E_{\ell_{k-2}, \ell_{k-1}} \cdots E_{\ell_0, \ell_1} v_\lambda$ is a weight vector in K_λ with weight λ , thus a multiple, denoted by ω_k , of v_λ .

In particular, if $k = 0$, from the first equality of (5.28), we immediately obtain $\omega_0 = m - n$ by (5.19). If $k = 1$, from the second equality of (5.28) and the above arguments, there is only one factor E_{ℓ_1, ℓ_0} with $\ell_1 = \ell_0$ we need to consider in the summand. Using (5.13), we obtain $\omega_1 = nq - mp$. \square

Now we can prove the following.

Theorem 5.7. *Let $M = K_\lambda$ be the Kac-module with highest weight $\lambda = \lambda_{pq}$ in (5.6) for $p, q \in \mathbb{C}$, and let $s_i, \bar{s}_j, x_1, e_1, \bar{x}_1 \in \text{End}_{\mathfrak{g}}(M^{r,t})$ be defined as in (5.16)–(5.20). Then all relations in Definition 2.7 hold with ω_a 's being specialized to the complex numbers*

$$\begin{aligned}
 \omega_0 &= m - n, & \omega_1 &= nq - mp, \\
 \omega_a &= (m - p - q)\omega_{a-1} - p(q - m)\omega_{a-2} \quad \text{for } a \geq 2.
 \end{aligned} \tag{5.29}$$

Furthermore, x_1, \bar{x}_1 satisfy

$$(x_1 + p)(x_1 + q - m) = 0, \quad (\bar{x}_1 - p + n)(\bar{x}_1 - q) = 0. \tag{5.30}$$

Proof. Note that those relations in Definition 2.7 which do not involve x_1 and \bar{x}_1 are relations of the walled Brauer algebra $\mathcal{B}_{r,t}(m - n)$ in Theorem 2.4, thus hold by Proposition 5.5.

Definition 2.7(9)–(11), (22)–(24) can be verified, easily. By (5.21) and (5.22), we have Definition 2.7(8) and (21). Definition 2.7(12) and the first two equations of (5.29) follow from (5.23). Similarly by symmetry, one can prove Definition 2.7(25).

The last equation of (5.29) follows from (5.30) by induction on a . Note that the first equation of (5.30) is [7, IV, Corollary 3.2] (here x_1 is the $-x_1$ in [7, IV, Corollary 3.2]), which can also be obtained directly by (5.17) by noting that x_1 has two eigenvalues $-p$ and $m - q$ as the summand in the second case of (5.17) is equal to $-\sum_{a=1}^m w_1 \otimes (E_{ia}(v_a \otimes v_\lambda) - v_i \otimes v_\lambda) \otimes w_2$. Similarly, we have the second equation of (5.30).

To prove Definition 2.7(13), let x_i with $2 \leq i \leq r$ be defined as in (4.1). Then x_i defined here are $-x_i$ defined in [7, IV, Lemma 3.1], in particular, they commute with

each other, i.e., we have Definition 2.7(13) as x_2 coincides with $s_1 x_1 s_1 - s_1$. Similarly we have Definition 2.7(26). \square

Remark 5.8. If we use arbitrary typical integral dominant weight λ of $\mathfrak{gl}_{m|n}$ instead of λ_{pq} , then higher level walled Brauer algebras appear. For example, let $k = k_1 + k_2$ and $m_1, \dots, m_{k_1}, n_1, \dots, n_{k_2} \in \mathbb{Z}^{\geq r+t}$ and $\sum_{i=1}^{k_1} m_i = m, \sum_{j=1}^{k_2} n_j = n$. Assume $p_1, \dots, p_{k_1}, q_1, \dots, q_{k_2} \in \mathbb{C}$ such that $p_i - p_{i+1}, q_{j+1} - q_j \in \mathbb{Z}^{\geq r+t}$ for all possible i, j . Let

$$\lambda = (\overbrace{p_1, \dots, p_1}^{m_1}, \dots, \overbrace{p_{k_1}, \dots, p_{k_1}}^{m_{k_1}} \mid \overbrace{-q_1, \dots, -q_1}^{n_1}, \dots, \overbrace{-q_{k_2}, \dots, -q_{k_2}}^{n_{k_2}}).$$

Then λ is a typical integral dominant weight if and only if

$$p_1 - q_1 \notin \mathbb{Z} \text{ or } p_1 - q_1 \leq -m \text{ or } p_{k_1} - q_{k_2} \geq n.$$

In this case a level k walled Brauer algebra appears, and x_1, \bar{x}_1 satisfy $\prod_{c=1}^k (x_1 - u_c) = 0$ and $\prod_{c=1}^k (\bar{x}_1 - \bar{u}_c) = 0$ with

$$u_c = \begin{cases} -(p_c - m'_{c-1}) & \text{if } 1 \leq c \leq k_1, \\ -(q_{c-k_1} - m + n'_{c-k_1-1}) & \text{if } k_1 < c \leq k, \end{cases}$$

$$\bar{u}_c = \begin{cases} p_c - n + \sum_{j=c+1}^{k_1} m_j & \text{if } 1 \leq c \leq k_1, \\ q_{c-k_1} - \sum_{j=c-k_1+1}^{k_2} n_j & \text{if } k_1 < c \leq k, \end{cases}$$

where, $m'_i = \sum_{\ell \leq i} m_\ell, n'_j = \sum_{\ell \leq j} n_\ell$, and $m'_{k_1+j} = m + n'_j$. We remark as stated in the Introduction that in this case, we cannot expect anything new from the viewpoint of representation theory of $\mathfrak{gl}_{m|n}$.

We need to introduce the following notion in order to prove the next result.

Definition 5.9. For an element $b = b^\sigma \in B$ as in (5.14), we denote $|b| = \sum_{i,j} \sigma_{i,j}$, called the degree of b . If $\sigma_{ij} \neq 0$, we say $E_{i+m,j}$ is a factor of b . For $b_M \in B_M$, we define its degree $|b_M|$ to be $|b|$, where $b \in B$ is a unique tensor factor of b_M .

For any $\alpha = (\alpha_1, \dots, \alpha_r) \in \{0, 1\}^r, \beta = (\beta_1, \dots, \beta_t) \in \{0, 1\}^t$, we define the following elements of $\text{End}_{\mathfrak{g}}(M^{r,t})$:

$$x^\alpha = \prod_{i=1}^r x_i^{\alpha_i}, \quad \bar{x}^\beta = \prod_{j=1}^t \bar{x}_j^{\beta_j}, \tag{5.31}$$

where x_i, \bar{x}_j are elements of $\text{End}_{\mathfrak{g}}(M^{r,t})$ defined as in (4.1).

Theorem 5.10. *We keep the assumption of Theorem 5.7, and assume $r + t \leq \min\{m, n\}$. Then the monomials*

$$\mathbf{m} := c^{-1}x^\alpha e^f \bar{x}^\beta wd, \tag{5.32}$$

with $\alpha \in \{0, 1\}^r$, $\beta \in \{0, 1\}^t$ and c, e^f, w, d as in Theorem 4.10 and Definition 4.11, are \mathbb{C} -linearly independent endomorphisms of $M^{r,t}$.

Proof. First we remark that for convenience we arrange factors of the monomial \mathbf{m} in (5.32) in a different order from the corresponding monomial \mathbf{m} in (4.6) (without factors ω_a 's, cf. Definition 4.15). Note that changing the order only differs an element by some element with lower degree, where the degree of \mathbf{m} is defined to be $\deg \mathbf{m} := |\alpha| + |\beta|$, and $|\alpha| = \sum_{i=1}^r \alpha_i$, $|\beta| = \sum_{i=1}^t \beta_i$.

Suppose there is a nonzero \mathbb{C} -combination $\mathbf{c} := \sum_{\mathbf{m}} r_{\mathbf{m}} \mathbf{m}$ of monomials (5.32) being zero. We fix a monomial $\mathbf{m}' := c'^{-1}x^{\alpha'} e^{f'} \bar{x}^{\beta'} w' d'$ in \mathbf{c} with nonzero coefficient $r_{\mathbf{m}'} \neq 0$ which satisfies the following conditions:

- (i) $|\alpha'| + |\beta'|$ is maximal;
- (ii) f' is minimal among all monomials satisfying (i).

We take the basis element $v = \otimes_{i \in J_1} v_{k_i} \otimes v_\lambda \otimes \otimes_{i \in J_2} \bar{v}_{k_i} \in B_M$ (cf. (5.15)) such that (note that here is the place where we require condition $r + t \leq \min\{m, n\}$)

- (1) $k_i = i + \alpha'_i m$ if $i < 0$;
- (2) $k_{\bar{i}} = i$ for $1 \leq i \leq f'$;
- (3) $k_{\bar{i}} = r + i + (1 - \beta'_i) m$ if $f' < i \leq t$.

We define p_v to be the maximal integer such that there exist p_v pairs $(i, \bar{j}) \in J_1 \times J_2$ satisfying $k_i - k_{\bar{j}} \in \{0, \pm m\}$. Then from the choice of v , we have the following fact:

$$p_v = f'. \tag{5.33}$$

Now take

$$u := (v)c'cd'^{-1}w'^{-1} \in M^{r,t}, \quad \text{and} \quad b' = \prod_{i=1}^r E_{i+m,i}^{\alpha'_i} \prod_{i=1}^t E_{r+i+m,i+r}^{\beta'_i} v_\lambda \in K_\lambda, \tag{5.34}$$

such that b' is a basis element in B with degree $|\alpha'| + |\beta'|$ by noting that $0 \leq \alpha'_i, \beta'_i \leq 1$. We denote $B_M^{b'}$ to be the subset of B_M consisting of elements with b' being a tensor factor. We define the projection $\hat{\pi}_{b'} : M^{r,t} \rightarrow \otimes_{i \in J \setminus \{0\}} V_i$ (cf. (5.9)) by mapping a basis element $b_M \in B_M$ to zero if $b_M \notin B_M^{b'}$, or else to the element obtained from b_M by deleting the tensor factor b' . Motivated by [7, IV, Corollary 3.3], we refer to $\hat{\pi}_{b'}(u)$ as the b' -component of u . We want to prove $\hat{\pi}_{b'}(u) \neq 0$.

Assume a monomial \mathbf{m} in (5.32) appears in the expression of \mathbf{c} with $r_{\mathbf{m}} \neq 0$. Consider the following element of $M^{r,t}$ which contributes to u in (5.34),

$$\begin{aligned} u_1 &:= (v)c' \mathbf{m} d'^{-1} w'^{-1} = (v)c' c^{-1} x^\alpha e^f \bar{x}^\beta w d d'^{-1} w'^{-1} \\ &= \left(\bigotimes_{i \in J_1} v_{k_{(i)c'c^{-1}}} \otimes v_\lambda \otimes \bigotimes_{i \in J_2} \bar{v}_{k_{(i)c'c^{-1}}} \right) x^\alpha e^f \bar{x}^\beta w d d'^{-1} w'^{-1}, \end{aligned} \tag{5.35}$$

where the last equality follows by noting that elements in $\mathfrak{S}_r \times \tilde{\mathfrak{S}}_t$ have natural right actions on $J_1 \cup J_2$ by permutations. Write u_1 as a \mathbb{C} -combination of basis B_M , and for $b_M \in B_M$, if b_M appears as a term with a nonzero coefficient in the combination, then we say that u_1 produces b_M . By (5.17), (5.18), Definition 5.9 and condition (i), u_1 cannot produce a basis element with degree higher than $|\alpha| + |\beta|$. Thus the b' -component of u_1 is zero if $|\alpha| + |\beta| < |\alpha'| + |\beta'|$. So we can assume $|\alpha| + |\beta| = |\alpha'| + |\beta'|$ by condition (i). Then $f \geq f'$ by condition (ii).

Note from definitions (4.1) and (5.12) that

$$x_i = -\pi_{i0}(\Omega)|_{M^{r,t}} + \text{some element of degree zero,}$$

and $e^f = e_1 \cdots e_f$ and $e_i = -\pi_{i\bar{i}}(\Omega)|_{M^{r,t}}$ (cf. (5.16)). By (5.19), we see that in order for u_1 in (5.35) to produce a basis element b_M in $B_M^{b'}$ (note that $b_M \in B_M^{b'}$ has tensor factor b' and all factors of b' have the form $E_{i+m,i}$ by (5.34)), we need at least f pairs $(i, \bar{j}) \in J_1 \times J_2$ with $k_i - k_{\bar{j}} \in \{0, \pm m\}$ by (5.19). Thus we can suppose $f = f'$ by (5.33) and the fact that $f \geq f'$.

Set $J_{f'} = (J_1 \cup J_2) \cap \{i \mid f' \preceq i \preceq \bar{f}'\}$ (cf. (5.8)). If $c \neq c'$, then by definition (4.4), we have

$$j' := (j)c'c^{-1} \notin J_{f'} \text{ for some } j \in J_{f'}. \tag{5.36}$$

Say $j' \in J_1$ (the proof is similar if $j' \in J_2$), then $f' < j' \leq r$. Condition (1) shows that either $f' < k_{j'} = j' \leq r$ or else $f' + m < k_{j'} = j' + m \leq r + m$. Then conditions (2) and (3) show that there is no $\bar{\ell} \in J_2$ with $k_{j'} - k_{\bar{\ell}} \in \{0, \pm m\}$. Since all factors of b' have the form $E_{i+m,i}$, we see that u_1 cannot produce a basis element in $B_M^{b'}$. Thus we can suppose $c = c'$.

By conditions (1) and (2), we see that if $\alpha_i \neq \alpha'_i$ for some i with $1 \leq i \leq f$, or $\alpha_i = 1 \neq \alpha'_i$ for some $i \in J_1$, then again u_1 cannot produce a basis element in $B_M^{b'}$. Thus we suppose: $\alpha_i = \alpha'_i$ if $1 \leq i \leq f$, and $\alpha'_i = 0$ implies $\alpha_i = 0$ for $i \in J$.

Consider the coefficient $\chi_{\tilde{b}_M}^{u_1}$ of the basis element $\tilde{b}_M := \otimes_{i \in J_1} v_i \otimes b' \otimes \otimes_{i \in J_2} \bar{v}_{i+m}$ in u_1 . If $\alpha'_i = 1$ but $\alpha_i = 0$ for some $i \in J$, then u_1 can only produce some basis elements which have at least a tensor factor, say v_ℓ , with $\ell > m$, and thus \tilde{b}_M cannot be produced. Thus we can suppose $\alpha = \alpha'$. Dually, we can suppose $\beta' = \beta$.

Now rewrite $w d d'^{-1} w'^{-1}$ as $w d \tilde{d}'^{-1} w''$, where $\tilde{d} = w d w^{-1}$, $\tilde{d}' = w d' w^{-1}$ and $w'' = w w'^{-1}$. Note that $w'' \in \mathfrak{S}_{r-f'} \times \tilde{\mathfrak{S}}_{t-f'}$, which only permutes elements of

$(J_1 \cup J_2) \setminus J_{f'}$. We see that if $\tilde{d} \neq \tilde{d}'$, then as in (5.36), there exists some $j \in J_{f'}$ with $j' := (j)\tilde{d}\tilde{d}'^{-1}w'' \notin J_{j'}$, thus \tilde{b}_M cannot be produced. So assume $\tilde{d} = \tilde{d}'$. Similarly we can suppose $w'' = 1$.

The above has in fact proved that if the coefficient $\chi_{\tilde{b}_M}^{u_1}$ is nonzero then u_1 in (5.35) must satisfy $(c, \alpha, f, \beta, d, w) = (c', \alpha', f', \beta', d', w')$, i.e., $u_1 = (v)x^{\alpha'}e^{f'}\bar{x}^{\beta'}$. In this case, one can easily verify that $\chi_{\tilde{b}_M}^{u_1} = \pm 1$. This proves that u defined in (5.34) is nonzero, a contradiction. The theorem is proven. \square

Motivated by Theorems 5.7 and 5.10, we give the notion of level two walled Brauer algebras as follows.

Definition 5.11. For any $m, n, r, t \in \mathbb{Z}^{\geq 1}$, $p, q \in \mathbb{C}$, let $\mathcal{B}_{r,t}^{\text{aff}}$ be the affine walled Brauer algebra defined over \mathbb{C} with specialized parameters (5.29). The level two walled Brauer algebra $\mathcal{B}_{r,t}^{p,q}(m, n)$ is defined to be the cyclotomic quotient associative \mathbb{C} -algebra

$$\mathcal{B}_{r,t}^{p,q}(m, n) = \mathcal{B}_{r,t}^{\text{aff}} / \langle (x_1 + p)(x_1 - m + q), (\bar{x}_1 - p + n)(\bar{x}_1 - q) \rangle. \tag{5.37}$$

Without confusion, we use the same symbol to denote the corresponding element in $\mathcal{B}_{r,t}^{p,q}(m, n)$. Using arguments on the degree, we can easily see that the monomials in (5.32) span $\mathcal{B}_{r,t}^{p,q}(m, n)$ (as a space). Thus

$$\dim_{\mathbb{C}} \mathcal{B}_{r,t}^{p,q}(m, n) \leq 2^{r+t}(r+t)!, \tag{5.38}$$

where the right-hand side is the total number of all monomials in (5.32).

As in [7, IV], we shall be mainly interested in the case when the Kac module K_{λ} is typical, namely, either $p - q \notin \mathbb{Z}$ or $p - q \leq -m$ or $p - q \geq n$. In this case, the tensor module $M^{r,t}$ is a tilting module. Using the vector space isomorphism $\text{Hom}_{\mathfrak{g}}(M_1 \otimes V, M_2) \cong \text{Hom}_{\mathfrak{g}}(M_1, M_2 \otimes V^*)$ for any two \mathfrak{g} -modules M_1, M_2 , one can easily obtain the vector space isomorphism: $\text{End}_{\mathfrak{g}}(M^{r,t}) \cong \text{End}_{\mathfrak{g}}(K_{\lambda} \otimes V^{\otimes(r+t)})$. Thus $\dim_{\mathbb{C}} \text{End}_{\mathfrak{g}}(M^{r,t}) = 2^{r+t}(r+t)!$ by [7, IV].

Now we can state the main result of this section.

Theorem 5.12 (*Super Schur–Weyl duality*). Assume $r+t \leq \min\{m, n\}$, and $p - q \notin \mathbb{Z}$ or $p - q \leq -m$ or $p - q \geq n$. The map $\phi : \mathcal{B}_{r,t}^{p,q}(m, n) \rightarrow \text{End}_{\mathfrak{g}}(M^{r,t})^{\text{op}}$ sending an element to the element with the same name is an algebra isomorphism. In particular, $\mathcal{B}_{r,t}^{p,q}(m, n)$ is of dimension $2^{r+t}(r+t)!$ over \mathbb{C} with a basis consisting of monomials in (5.32).

Proof. Denote by \mathcal{A} the subalgebra of $\text{End}_{\mathfrak{g}}(M^{r,t})$ generated by $s_i, \bar{s}_j, x_1, \bar{x}_1, e_1$. By Theorem 5.10, $\dim_{\mathbb{C}} \mathcal{A} \geq \dim_{\mathbb{C}} \text{End}_{\mathfrak{g}}(M^{r,t}) = 2^{r+t}(r+t)!$, forcing $\text{End}_{\mathfrak{g}}(M^{r,t}) = \mathcal{A}$. Using Theorem 5.7 yields an epimorphism from $\mathcal{B}_{r,t}^{\text{aff}}$ to \mathcal{A}^{op} killing the two-sided ideal $\langle (x_1 + p)(x_1 - m + q), (\bar{x}_1 - p + n)(\bar{x}_1 - q) \rangle$ of $\mathcal{B}_{r,t}^{\text{aff}}$, thus induces an epimorphism $\phi : \mathcal{B}_{r,t}^{p,q}(m, n) \rightarrow \mathcal{A}^{\text{op}}$. Thus $\dim_{\mathbb{C}} \mathcal{B}_{r,t}^{p,q}(m, n) \geq \dim_{\mathbb{C}} \mathcal{A}$. This together with (5.38) implies that ϕ is an isomorphism. Thus monomials in (5.32) form a basis of $\mathcal{B}_{r,t}^{p,q}(m, n)$. \square

Remark 5.13. Theorem 5.12 is in fact the first and second fundamental theorems of invariant theory for the endomorphism algebra $\text{End}_{\mathfrak{g}}(M^{r,t})$ for the case $r+t \leq \min\{m, n\}$ (see, e.g., [24]).

Remark 5.14. Level two walled Brauer algebras $\mathcal{B}_{r,t}^{p,q}(m, n)$ heavily depend on parameters $p - q, r, t, m, n$, in sharp contrast to level two Hecke algebras $H_r^{p,q}$ (or $\mathcal{H}_{2,r}$) in [7, IV], which only depend on $p - q$ and r .

We are now going to determine when $\mathcal{B}_{r,t}^{p,q}(m, n)$ is semisimple. For this purpose, we need the following result, which is a slight generalization of [33, Lemma 5.2] and [34, Lemma 3.6], where a \mathfrak{g} -highest weight of $M^{r,t}$ means a weight $\mu \in \mathfrak{h}^*$ such that there exists a nonzero \mathfrak{g} -highest weight vector $v \in M^{r,t}$ with weight μ (i.e., v is a vector satisfying $E_{ii}v = \mu_i v, E_{ji}v = 0$ for $1 \leq i < j \leq m + n$).

Lemma 5.15. *We keep the assumption of Theorem 5.10, and assume $\mu \in \mathfrak{h}^*$ is a \mathfrak{g} -highest weight of $M^{r,t}$. Then $|\mu| = |\lambda| + r - t$ and $-t \leq \sum_{i \in S} (\mu_i - \lambda_i) \leq r$ for any subset $S \subset I$, where $|\lambda|, |\mu|$ are sizes of λ, μ (cf. (5.4)).*

Proof. Let w_μ be a \mathfrak{g} -highest weight vector with weight μ , and write w_μ in terms of basis B_M in (5.15). As in the proofs of [33, Lemma 5.2] and [34, Lemma 3.6], w_μ must contain a basis element, say b_M , with degree 0 (cf. Definition 5.9), i.e., b_M has the form $w_1 \otimes v_\lambda \otimes w_2$ for some $w_1 \in V^{\otimes r}$ and $w_2 \in (V^*)^{\otimes t}$ such that w_1 (resp., w_2) is a weight vector with some weight η (resp., ζ) of size r (resp., $-t$) satisfying $\eta_i \in \mathbb{Z}^{\geq 0}$ (resp., $\zeta_i \in \mathbb{Z}^{\leq 0}$) for all $i \in I$. The result follows. \square

Theorem 5.16. *We keep the assumption of Theorem 5.12, then $\mathcal{B}_{r,t}^{p,q}(m, n)$ is semisimple if and only if $p - q \notin \mathbb{Z}$ or $p - q \leq -m - r$ or $p - q \geq n + t$.*

Proof. First assume $p - q \in \mathbb{Z}$ and $p - q \leq -m - r$. Let $\mu \in \mathfrak{h}^*$ be a \mathfrak{g} -highest weight of $M^{r,t}$. For $1 \leq i \leq m < j \leq m + n$, by definition of (5.5), we have (hereafter we define the partial order on \mathbb{C} such that $a \leq b$ if and only if $b - a$ is a nonnegative real number)

$$\begin{aligned} \mu_i^p + \mu_j^p &= \mu_i + \mu_j + 1 + 2m - i - j \\ &\leq \lambda_i + \lambda_j + r + 1 + 2m - 1 - (m + 1) \\ &= p + m + r - q - 1, \end{aligned}$$

which is strictly less than zero, i.e., μ is a typical integral dominant weight, where the inequality follows from Lemma 5.15 and $i \geq 1, j \geq m + 1$. By [18], $M^{r,t}$ is a completely reducible module which can be decomposed as a direct sum of typical finite dimensional simple modules: $M^{r,t} = \bigoplus_{\mu \in T} L_\mu^{\oplus k_\mu}$, where T is a finite set consisting of typical integral dominant weights, and $k_\mu \in \mathbb{Z}^{\geq 1}$. Thus $\mathcal{B}_{r,t}^{p,q}(m, n) \cong \text{End}_{\mathfrak{g}}(M^{r,t})^{\text{op}} \cong \bigoplus_{\mu \in T} M_{k_\mu}$ is a

semisimple associative algebra, where M_{k_μ} is the algebra of matrices of rank k_μ . The case $p - q \notin \mathbb{Z}$ or $p - q \geq n + t$ can be proven similarly.

Now suppose $p - q \in \mathbb{Z}$ and $q - m - r < p < q + n + t$. This together with condition (5.7) shows that either $|p - q + m|$, the absolute value of difference of two eigenvalues of x_1 , is an integer $< r$, or, $|p - q - n|$, the absolute value of difference of two eigenvalues of \bar{x}_1 , is an integer $< t$. Thus by [2, Theorem 6.1], either the level two degenerate Hecke algebra $\mathcal{H}_{2,r} := \mathcal{H}_r^{\text{aff}} / \langle (x_1 + p)(x_1 - m + q) \rangle$ (cf. Proposition 4.18(3)) is not semisimple or else $\mathcal{H}_{2,t} := \mathcal{H}_t^{\text{aff}} / \langle (\bar{x}_1 - p + n)(\bar{x}_2 - q) \rangle$ is not semisimple. In any case, $\mathcal{H}_{2,r} \otimes \mathcal{H}_{2,t} = \mathcal{B}_{r,t}^{p,q}(m, n) / \langle e_1 \rangle$ is not semisimple. As a result, $\mathcal{B}_{r,t}^{p,q}(m, n)$, the preimage of $\mathcal{H}_{2,r} \otimes \mathcal{H}_{2,t}$, cannot be semisimple. \square

In the next two sections, we shall study $\mathcal{B}_{r,t}^{p,q}(m, n)$ for given m, n (with the assumption $r + t \leq \min\{m, n\}$), thus we omit (m, n) from the notation, and simply denote it by $\mathcal{B}_{r,t}^{p,q}$. When there is no confusion, the notation is further simplified to \mathcal{B} .

6. Weakly cellular basis of level two walled Brauer algebras

In this section, we shall use Theorem 5.12 to construct a weakly cellular basis of $\mathcal{B} = \mathcal{B}_{r,t}^{p,q}(m, n)$ over \mathbb{C} for $m, n, r, t \in \mathbb{Z}^{\geq 1}$, $p, q \in \mathbb{C}$ such that $r + t \leq \min\{m, n\}$.

First recall that a *partition* of $k \in \mathbb{Z}^{\geq 0}$ is a sequence of non-negative integers $\lambda = (\lambda_1, \lambda_2, \dots)$ such that $\lambda_i \geq \lambda_{i+1}$ for all positive integers i and $|\lambda| := \lambda_1 + \lambda_2 + \dots = k$. Let $\Lambda^+(k)$ be the set of all partitions of k . A *bipartition* of k is an ordered 2-tuple of partitions $\lambda = (\lambda^{(1)}, \lambda^{(2)})$ such that $|\lambda| := |\lambda^{(1)}| + |\lambda^{(2)}| = k$.

Let $\Lambda_2^+(k)$ be the set of all bipartitions of k . Then $\Lambda_2^+(k)$ is a poset with the *dominance order* \triangleright as the partial order on it. More explicitly, we say $\lambda = (\lambda^{(1)}, \lambda^{(2)})$ is dominated by $\mu = (\mu^{(1)}, \mu^{(2)})$ and write $\mu \triangleright \lambda$ if

$$\sum_{j=1}^i \lambda_j^{(1)} \leq \sum_{j=1}^i \mu_j^{(1)} \quad \text{and} \quad |\lambda^{(1)}| + \sum_{j=1}^\ell \lambda_j^{(2)} \leq |\mu^{(1)}| + \sum_{j=1}^\ell \mu_j^{(2)}, \tag{6.1}$$

for all possible i, ℓ 's. We write $\mu \triangleright \lambda$ if $\mu \triangleright \lambda$ and $\lambda \neq \mu$.

For each partition λ of k , the *Young diagram* $[\lambda]$ is a collection of boxes arranged in left-justified rows with λ_i boxes in the i -th row of $[\lambda]$. If $\lambda = (\lambda^{(1)}, \lambda^{(2)}) \in \Lambda_2^+(k)$, then the corresponding Young diagram $[\lambda]$ is $([\lambda^{(1)}], [\lambda^{(2)}])$. In this case, a λ -*tableau* $\mathfrak{s} = (\mathfrak{s}_1, \mathfrak{s}_2)$ is obtained by inserting $i, 1 \leq i \leq k$ into $[\lambda]$ without repetition.

A λ -tableau \mathfrak{s} is said to be *standard* if the entries in \mathfrak{s}_1 and \mathfrak{s}_2 increase both from left to right in each row and from top to bottom in each column. Let $\mathcal{F}^{std}(\lambda)$ be the set of all standard λ -tableaux.

Definition 6.1. We define

- t^λ to be the λ -tableau obtained from the Young diagram $[\lambda]$ by inserting $1, 2, \dots, k$ from left to right along the rows of $[\lambda^{(1)}]$ and then $[\lambda^{(2)}]$;
- t_λ to be the λ -tableau obtained from $[\lambda]$ by inserting $1, 2, \dots, k$ from top to bottom along the columns of $[\lambda^{(2)}]$ and then $[\lambda^{(1)}]$.

For example, if $\lambda = ((3, 2, 1), (2, 1))$, then

$$t^\lambda = \left(\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 5 & \\ \hline 6 & & \\ \hline \end{array}, \begin{array}{|c|c|} \hline 7 & 8 \\ \hline 9 & \\ \hline \end{array} \right) \quad \text{and} \quad t_\lambda = \left(\begin{array}{|c|c|c|} \hline 4 & 7 & 9 \\ \hline 5 & 8 & \\ \hline 6 & & \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \hline \end{array} \right). \tag{6.2}$$

The symmetric group \mathfrak{S}_k acts on a λ -tableau \mathfrak{s} by permuting its entries. For $w \in \mathfrak{S}_k$, if $t^\lambda w = \mathfrak{s}$, we write $d(\mathfrak{s}) = w$. Then $d(\mathfrak{s})$ is uniquely determined by \mathfrak{s} .

Given a $\lambda \in \Lambda_2^+(k)$, let \mathfrak{S}_λ be the row stabilizer of t^λ . Then \mathfrak{S}_λ (sometimes denoted as $\mathfrak{S}_{\bar{\lambda}}$) is the Young subgroup of \mathfrak{S}_k with respect to the composition $\bar{\lambda}$, which is obtained from λ by concatenation. For example, $\bar{\lambda} = (3, 2, 1, 2, 1)$ if $\lambda = ((3, 2, 1), (2, 1))$.

Recall that $\mathcal{H}_r^{\text{aff}}$ is the degenerate affine Hecke algebra generated by S_i 's and Y_j 's (cf. Definition 4.17). Let $\mathcal{H}_{2,r} = \mathcal{H}_r^{\text{aff}}/I$, where $I = \langle (Y_1 - u)(Y_1 - v) \rangle$ is the two-sided ideal of $\mathcal{H}_r^{\text{aff}}$ generated by $(Y_1 - u)(Y_1 - v)$ for $u, v \in \mathbb{C}$. Then $\mathcal{H}_{2,r}$ is known as the level two degenerate Hecke algebra with parameters u, v . As mentioned in Proposition 4.18, our current elements x_1, \bar{x}_1 , which are two generators of $\mathcal{B}_{r,t}^{\text{aff}}$, correspond to $-Y_1$ in Definition 4.17. Thus, when we use the construction of cellular basis for $\mathcal{H}_{2,r}$ in [2], we need to use $-Y_1, -u, -v$ instead of Y_1, u, v , respectively. By abusing of notations, we will not distinguish between them. The following definition on $\mathfrak{m}_{\mathfrak{s}\mathfrak{t}}^\lambda$ is a special case of that in [2] for degenerate Hecke algebra $\mathcal{H}_{r,m}$ of type $G(r, 1, m)$.

Suppose $\lambda = (\lambda^{(1)}, \lambda^{(2)}) \in \Lambda_2^+(r)$ with $a = |\lambda^{(1)}|$. We set $\pi_0 = 1$ if $a = 0$, and $\pi_a = (x_1 - v)(x_2 - v) \cdots (x_a - v)$ for $a \geq 1$. Let

$$\mathfrak{m}_{\mathfrak{s}\mathfrak{t}}^\lambda = d(\mathfrak{s})^{-1} \pi_a \mathfrak{m}_{\bar{\lambda}} d(\mathfrak{t}), \tag{6.3}$$

where $\mathfrak{s}, \mathfrak{t} \in \mathcal{F}^{\text{std}}(\lambda)$ and $\mathfrak{m}_{\bar{\lambda}} = \sum_{w \in \mathfrak{S}_{\bar{\lambda}}} w$. In the following, we shall always omit the λ from notations $\mathfrak{m}_{\mathfrak{s}\mathfrak{t}}^\lambda, C_{\mathfrak{s}\mathfrak{t}}^\lambda$, etc., and simply denote them as $\mathfrak{m}_{\mathfrak{s}\mathfrak{t}}, C_{\mathfrak{s}\mathfrak{t}}$, etc.

Definition 6.2. (See [14].) Let A be an algebra over a commutative ring R containing 1. Fix a partially ordered set $\Lambda = (\Lambda, \triangleright)$, and for each $\lambda \in \Lambda$, let $T(\lambda)$ be a finite set. Further, fix $C_{\mathfrak{s}\mathfrak{t}} \in A$ for all $\lambda \in \Lambda$ and $\mathfrak{s}, \mathfrak{t} \in T(\lambda)$. Then the triple (Λ, T, C) is a *cell datum* for A if:

- (1) $\mathcal{C} := \{C_{\mathfrak{s}\mathfrak{t}} \mid \lambda \in \Lambda, \mathfrak{s}, \mathfrak{t} \in T(\lambda)\}$ is an R -basis for A ;
- (2) the R -linear map $*$: $A \rightarrow A$ determined by $(C_{\mathfrak{s}\mathfrak{t}})^* = C_{\mathfrak{t}\mathfrak{s}}$ for all $\lambda \in \Lambda$ and all $\mathfrak{s}, \mathfrak{t} \in T(\lambda)$ is an anti-involution of A ;

(3) for all $\lambda \in \Lambda$, $\mathfrak{s} \in T(\lambda)$ and $a \in A$, there exist scalars $r_{\mathfrak{t}\mathfrak{u}}(a) \in R$ such that

$$C_{\mathfrak{s}\mathfrak{t}}a = \sum_{\mathfrak{u} \in T(\lambda)} r_{\mathfrak{t}\mathfrak{u}}(a)C_{\mathfrak{s}\mathfrak{u}} \pmod{A^{\triangleright\lambda}},$$

where $A^{\triangleright\lambda} = \text{span}_R\{C_{\mathfrak{u}\mathfrak{v}}^\mu \mid \mu \triangleright \lambda, \mathfrak{u}, \mathfrak{v} \in T(\mu)\}$. Furthermore, each scalar $r_{\mathfrak{t}\mathfrak{u}}(a)$ is independent of \mathfrak{s} .

An algebra A is a *cellular algebra* if it has a cell datum. We call \mathcal{C} a *cellular basis* of A .

The notion of *weakly cellular algebras* in [13, Definition 2.9] is obtained from Definition 6.2 with condition (2) replaced by: there exists an anti-involution $*$ of A satisfying

$$(C_{\mathfrak{s}\mathfrak{t}})^* \equiv C_{\mathfrak{t}\mathfrak{s}} \pmod{A^{\triangleright\lambda}}. \tag{6.4}$$

The results and proofs of [14] are equally valid for weakly cellular algebras, so in the remainder of the paper we will not distinguish between cellular algebras and weakly cellular algebras.

We remark that [2, Theorem 6.3] holds over any commutative ring containing 1. In this paper, we need its special case below.

Theorem 6.3. (See [2].) *The set $\{\mathfrak{m}_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s}, \mathfrak{t} \in \mathcal{T}^{std}(\lambda), \lambda \in \Lambda_2^+(r)\}$ with $\mathfrak{m}_{\mathfrak{s}\mathfrak{t}}$ defined in (6.3) is a cellular basis of $\mathcal{H}_{2,r}$ over \mathbb{C} .*

Now, we construct a weakly cellular basis of \mathcal{B} over \mathbb{C} . Fix $r, t, f \in \mathbb{Z}^{>0}$ with $f \leq \min\{r, t\}$. We need to introduce more notations. In contrast to (4.3), we define the following subgroups of \mathfrak{S}_r , $\mathfrak{S}_r \times \bar{\mathfrak{S}}_t$ and $\bar{\mathfrak{S}}_t$ respectively,

$$\begin{aligned} S_{r-f} &= \langle s_j \mid 1 \leq j < r-f \rangle, \\ \mathcal{G}_f &= \langle \bar{s}_{t-i} s_{r-i} \mid 1 \leq i < f \rangle, \\ \bar{S}_{t-f} &= \langle \bar{s}_j \mid 1 \leq j < t-f \rangle. \end{aligned} \tag{6.5}$$

Let $\mathcal{D}_{r,t}^f$ be the set consisting of the following elements:

$$c := s_{r-f+1, i_{r-f+1}} \bar{s}_{t-f+1, j_{t-f+1}} \cdots s_{r, i_r} \bar{s}_{t, j_t} \text{ with } r \geq i_r > \cdots > i_{r-f+1} \text{ and } j_k \leq t-k. \tag{6.6}$$

Then by arguments similar to those for Lemma 4.9, $\mathcal{D}_{r,t}^f$ is a complete set of right coset representatives for $S_{r-f} \times \mathcal{G}_f \times \bar{S}_{t-f}$ in $\mathfrak{S}_r \times \bar{\mathfrak{S}}_t$. Let

$$\Lambda_{2,r,t} = \{(f, (\lambda, \mu)) \mid (\lambda, \mu) \in \Lambda_2^+(r-f) \times \Lambda_2^+(t-f), 0 \leq f \leq \min\{r, t\}\}. \tag{6.7}$$

Definition 6.4. For $(f, \lambda, \mu), (\ell, \alpha, \beta) \in \Lambda_{2,r,t}$, we define

$$(f, (\lambda, \mu)) \succeq (\ell, (\alpha, \beta)) \iff \text{either } f > \ell \text{ or } f = \ell \text{ and } \lambda \succeq_1 \alpha, \mu \succeq_2 \beta,$$

where in case $f = \ell$, the orders \succeq_1 and \succeq_2 are dominance orders on $\Lambda_2^+(r - f)$ and $\Lambda_2^+(t - f)$ respectively (cf. (6.1)). Then $(\Lambda_{2,r,t}, \succeq)$ is a poset.

For each $c \in \mathcal{D}_{r,t}^f$, as in (6.6), let κ_c be the r -tuple

$$\kappa_c = (k_1, \dots, k_r) \in \{0, 1\}^r \text{ such that } k_i = 0 \text{ unless } i = i_r, i_{r-1}, \dots, i_{r-f+1}. \tag{6.8}$$

Note that κ_c may have more than one choice for a fixed c , and it may be equal to κ_d although $c \neq d$ for $c, d \in \mathcal{D}_{r,t}^f$. We set $x^{\kappa_c} = \prod_{i=1}^r x_i^{k_i}$. By Lemma 4.5,

$$cx^{\kappa_c} = s_{r-f+1, i_{r-f+1}} x_{i_{r-f+1}}^{k_{i_{r-f+1}}} \cdots s_{r-1, i_{r-1}} x_{i_{r-1}}^{k_{i_{r-1}}} s_{r, i_r} x_{i_r}^{k_{i_r}} \bar{s}_{t-f+1, j_{t-f+1}} \cdots \bar{s}_{t, j_t}. \tag{6.9}$$

For each $(f, \lambda) \in \Lambda_{2,r,t}$ (thus λ is now a pair of bipartitions), let

$$\delta(f, \lambda) = \{(\mathbf{t}, c, \kappa_c) \mid \mathbf{t} \in \mathcal{T}^{std}(\lambda), c \in \mathcal{D}_{r,t}^f \text{ and } \kappa_c \in \mathbf{N}_f\}, \tag{6.10}$$

where $\mathbf{N}_f = \{\kappa_c \mid c \in \mathcal{D}_{r,t}^f\}$. We remark that in (6.10), $\lambda = (\lambda^{(1)}, \lambda^{(2)})$ with $\lambda^{(1)} \in \Lambda_2^+(r - f)$ and $\lambda^{(2)} \in \Lambda_2^+(t - f)$, and $\mathbf{t} = (\mathbf{t}^{(1)}, \mathbf{t}^{(2)})$ with $\mathbf{t}^{(i)}$ being a $\lambda^{(i)}$ -tableau for $i = 1, 2$. In contrast to (4.5), we define

$$\mathbf{e}^f = e_{r,t} e_{r-1, t-1} \cdots e_{r-f+1, t-f+1} \text{ if } f \geq 1, \text{ and } \mathbf{e}^0 = 1. \tag{6.11}$$

Definition 6.5. For each $(f, \lambda) \in \Lambda_{2,r,t}^+$ and $(\mathfrak{s}, \kappa_d, d), (\mathbf{t}, \kappa_c, c) \in \delta(f, \lambda)$, we define

$$C_{(\mathfrak{s}, \kappa_d, d)(\mathbf{t}, \kappa_c, c)} = x^{\kappa_d} d^{-1} \mathbf{e}^f \mathbf{m}_{\mathfrak{s}\mathbf{t}} c x^{\kappa_c}, \tag{6.12}$$

where $\mathbf{m}_{\mathfrak{s}\mathbf{t}}$ is a product of cellular basis elements for $\mathcal{H}_{2,r-f}$ and $\mathcal{H}_{2,t-f}$ described in Theorem 6.3.

We remark that an element in $\mathcal{H}_{2,r-f}$ (generated by s_1, \dots, s_{r-f-1} and x_1) may not commute with an element of $\mathcal{H}_{2,t-f}$ (generated by $\bar{s}_1, \dots, \bar{s}_{t-f-1}$ and \bar{x}_1). So, we always fix $\mathbf{m}_{\mathfrak{s}\mathbf{t}}$ as the product ab , such that a (resp., b) is obtained from the corresponding cellular basis element of $\mathcal{H}_{2,r-f}$ (resp. $\mathcal{H}_{2,t-f}$) described in Theorem 6.3 by using $-x_1, p, q - m$ (resp. $-\bar{x}_1, -q, n - p$) instead of Y_1, u, v , respectively.

Let I be the two-sided ideal of \mathcal{B} generated by e_1 . By Proposition 4.18(2), there is a \mathbb{C} -algebra isomorphism $\varepsilon_{r,t} : \mathcal{H}_{2,r} \times \mathcal{H}_{2,t} \cong \mathcal{B}/I$ such that

$$\varepsilon_{r,t}(s_i) = s_i + I, \quad \varepsilon_{r,t}(\bar{s}_j) = \bar{s}_j + I, \quad \varepsilon_{r,t}(x_k) = x_k + I, \quad \varepsilon_{r,t}(\bar{x}_\ell) = \bar{x}_\ell + I, \tag{6.13}$$

for all possible i, j, k, ℓ 's.

For each f with $0 \leq f \leq \min\{r, t\}$, let $\mathcal{B}(f)$ be the two-sided ideal of \mathcal{B} generated by \mathfrak{e}^f . Then there is a filtration of two-sided ideals of \mathcal{B} as follows:

$$\mathcal{B} = \mathcal{B}(0) \supset \mathcal{B}(1) \supset \dots \supset \mathcal{B}(k) \supset \mathcal{B}(k + 1) = 0, \text{ where } k = \min\{r, t\}.$$

Definition 6.6. Suppose $0 \leq f \leq \min\{r, t\}$ and $\lambda \in \Lambda_2^+(r - f) \times \Lambda_2^+(t - f)$. Define $\mathcal{B}^{\geq(f, \lambda)}$ to be the two-sided ideal of \mathcal{B} generated by $\mathcal{B}(f + 1)$ and S , where

$$S = \{\mathfrak{e}^f \mathfrak{m}_{\mathfrak{s}\mathfrak{t}} | \mathfrak{s}, \mathfrak{t} \in \mathcal{T}^{std}(\mu) \text{ and } \mu \in \Lambda_2^+(r - f) \times \Lambda_2^+(t - f) \text{ with } \mu \geq \lambda\}.$$

We also define $\mathcal{B}^{>(f, \lambda)} = \sum_{\mu \triangleright \lambda} \mathcal{B}^{\geq(f, \mu)}$, where $\mu \in \Lambda_2^+(r - f) \times \Lambda_2^+(t - f)$.

By [Theorem 5.12](#), $\mathcal{B}_{r-f, t-f}^{p, q}$ can be embedded into \mathcal{B} , thus we regard it as a subalgebra of \mathcal{B} .

Lemma 6.7. Suppose $d \in \mathcal{D}_{r, t}^f$ with $0 \leq f < \min\{r, t\}$. Then $\mathfrak{e}^f \langle e_1 \rangle \subset \mathcal{B}(f + 1)$, where $\langle e_1 \rangle$ is the two-sided ideal of $\mathcal{B}_{r-f, t-f}^{p, q}$ generated by e_1 .

Proof. By assumption, we have $r - f \geq 1$ and $t - f \geq 1$. It is easy to check that \mathfrak{e}^f commutes with any element in $\mathcal{B}_{r-f, t-f}^{p, q}$. Since $e_{r-f, t-f} = \bar{s}_{t-f, 1} s_{r-f, 1} e_1 s_{1, r-f} \bar{s}_{1, t-f}$, we have $\mathfrak{e}^f e_1 \in \mathcal{B}(f + 1)$, proving the result. \square

For $0 \leq f \leq \min\{r, t\}$, let $\pi_{f, r, t} : \mathcal{B}(f) \rightarrow \mathcal{B}(f)/\mathcal{B}(f + 1)$ be the canonical epimorphism. Since both $\mathcal{B}(f)$ and $\mathcal{B}(f + 1)$ are \mathcal{B} -bimodules, $\pi_{f, r, t}$ is a homomorphism as \mathcal{B} -bimodules. The following result follows from [\(6.13\)](#) and [Lemma 6.7](#), immediately.

Lemma 6.8. For each $f \in \mathbb{Z}^{\geq 0}$ with $f < \min\{r, t\}$, there is a well-defined \mathbb{C} -homomorphism $\sigma_f : \mathcal{H}_{2, r-f} \times \mathcal{H}_{2, t-f} \rightarrow \mathcal{B}(f)/\mathcal{B}(f + 1)$ such that

$$\sigma_f(h) = \mathfrak{e}^f \varepsilon_{r-f, t-f}(h)' + \mathcal{B}(f + 1) \text{ for } h \in \mathcal{H}_{2, r-f} \times \mathcal{H}_{2, t-f},$$

where $\varepsilon_{r-f, t-f}(h)'$ is the preimage of the element $\varepsilon_{r-f, t-f}(h) \in \mathcal{B}_{r-f, t-f}^{p, q}/I$ in $\mathcal{B}_{r-f, t-f}^{p, q}$, where I is the two-sided ideal of $\mathcal{B}_{r-f, t-f}^{p, q}$ generated by e_1 .

Lemma 6.9. Suppose $\lambda \in \Lambda_{2, r, t}$ and $0 \leq f \leq \min\{r, t\}$. For any $\mathfrak{s}, \mathfrak{t} \in \mathcal{T}^{std}(\lambda)$,

- (1) $\mathfrak{e}^f \mathfrak{m}_{\mathfrak{s}\mathfrak{t}} = \mathfrak{m}_{\mathfrak{s}\mathfrak{t}} \mathfrak{e}^f \in \mathcal{B}(f)$.
- (2) $\sigma_f(\mathfrak{m}_{\mathfrak{s}\mathfrak{t}}) = \pi_{f, r, t}(\mathfrak{e}^f \mathfrak{m}_{\mathfrak{s}\mathfrak{t}})$.
- (3) $\sigma(\mathfrak{e}^f \mathfrak{m}_{\mathfrak{s}\mathfrak{t}}) \equiv \mathfrak{e}^f \mathfrak{m}_{\mathfrak{s}\mathfrak{t}} \pmod{\mathcal{B}^{>(f, \lambda)}}$, where σ is the anti-involution on \mathcal{B} induced from that in [Lemma 4.1](#).

Proof. By [Lemma 4.7\(1\)](#), $e_{i, j}(x_k + L_k) = (x_k + L_k)e_{j, n}$ if $i \neq k$. Furthermore, $e_{i, j}(\ell, k) = (\ell, k)e_{i, j}$ if $1 \leq \ell < k < i$. So, $e_{i, j}L_k = L_k e_{i, j}$, forcing $e_{i, j}x_k = x_k e_{i, j}$. Similarly, $e_{i, j}\bar{x}_k =$

$\bar{x}_k e_{i,j}$ for $k < j$. So, $\epsilon^f \mathbf{m}_{st} = \mathbf{m}_{st} \epsilon^f$. The second assertion is trivial. By Lemma 4.6(3), $x_i \bar{x}_j \equiv \bar{x}_j x_i \pmod{J}$, where J is the two-sided ideal of $\mathcal{B}_{r-f,t-f}^{p,q}$ generated by e_1 . Now, (3) follows from Lemma 6.7 and (1). \square

Recall that the degree of a monomial $\mathbf{m} \in \mathcal{B}$ in (5.32) is $|\alpha| + |\beta| = \sum_{i=1}^r \alpha_i + \sum_{j=1}^t \beta_j$. So, \mathcal{B} is a filtered algebra, which associates to a \mathbb{Z} -graded algebra $\text{gr}(\mathcal{B})$ defined the same as in (4.2).

The following is motivated by Song and one of authors' work on q -walled Brauer algebras [29].

Proposition 6.10. *Fix $r, t, f \in \mathbb{Z}^{>0}$ with $f \leq \min\{r, t\}$. Let M_f be the left $\mathcal{B}_{r-f,t-f}^{p,q}$ -module generated by*

$$V_{r,t}^f = \{\epsilon^f dx^{\kappa_d} \mid (d, \kappa_d) \in \mathcal{D}_{r,t}^f \times \mathbf{N}_f\}. \tag{6.14}$$

Then M_f is a right \mathcal{B} -module.

Proof. We prove the result by induction on the degree of $\epsilon^f dx^{\kappa_d}$. If the degree is 0, then $\epsilon^f dx^{\kappa_d} = \epsilon^f d$. By the result on the walled Brauer algebra (which is the special case of [29, Proposition 2.9]), we have $\epsilon^f dh \in M_f$ for any $h \in \mathcal{B}_{r,t}(\omega_0)$. Note that $\mathcal{B}_{r,t}(\omega_0)$ is a subalgebra of \mathcal{B} .

Now, we consider $\epsilon^f dx_1$, where d has the form in (6.6). If $i_j = 1$ for some $j \geq r - f + 1$, then $j = r - f + 1$ and $\epsilon^f dx_1 \in V_{r,t}^f$. Otherwise, we have $(1)d = 1$, and $dx_1 = x_1 d$. Note that $r - f + 1 \geq i_{r-f+1} > 1$, we have $\epsilon^f x_1 d = x_1 \epsilon^f d \in M_f$.

We have $\epsilon^f d\bar{x}_1 = \epsilon^f \bar{x}_k d + \epsilon^f w$ for some $k, 1 \leq k \leq t$ and some $w \in \mathbb{C}\mathfrak{S}_r \times \mathbb{C}\bar{\mathfrak{S}}_t$. By corresponding result for walled Brauer algebras, we have $\epsilon^f w \in M_f$. If $k \leq r - f$, by Lemma 4.7, $\epsilon^f \bar{x}_k d$ can be replaced by $\bar{x}_k \epsilon^f d \in M_f$. If $k \geq r - f + 1$, by Lemma 4.6(2), we can use x_k instead of \bar{x}_k in $\epsilon^f \bar{x}_k d$. So, the required result follows from our previous arguments on s_i, \bar{s}_j and x_1 . This completes the proof when the degree of $\epsilon^f dx^{\kappa}$ is 0.

Suppose the degree of $\epsilon^f dx^{\kappa_d}$ is not 0. We want to prove $\epsilon^f dx^{\kappa_d} h \in M_f$ for any generators h of \mathcal{B} .

Case 1: $h \in \bar{\mathfrak{S}}_t$. We have $x^{\kappa_d} h = hx^{\kappa_d}$. By our previous result on degree 0, we have $\epsilon^f dh \in M_f$. Therefore, we need to check $\epsilon^f (dh)x^{\kappa_d} \in M_f$. If x_j is a term of x^{κ_d} , by induction on the degree, we have $\epsilon^f dhx \in M_f$, where x is obtained from x^{κ_d} by removing the factor x_j . So, $\epsilon^f dhx^{\kappa_d} \in M_f$ by inductive assumption on $\text{deg}(\epsilon^f dhx^{\kappa_d}) - 1$.

Case 2: $h \in \mathfrak{S}_r$. We have $x^{\kappa_d} h = hx$ in $\text{gr}(\mathcal{B})$, where x is obtained from x^{κ_d} by permuting some indices. By inductive assumption, it suffices to verify $\epsilon^f dhx \in M_f$ with $\text{deg}(x) = \text{deg}(x^{\kappa_d})$. This has already been verified in Case 1.

Case 3: $h = x_1$. If x_1 is a factor of x^{κ_d} , we have $x_1^2 = (p+q-m)x_1 - p(q-m)$ (cf. (5.30)). So, $\epsilon^f dx^{\kappa_d} x_1 \in M_f$ by inductive assumption on $\text{deg}(\epsilon^f dx^{\kappa_d}) - 1$. If x_1 is not a factor of x^{κ_d} , and if $i_{r-f+1} = 1$, where d has the form in (6.6), then there is nothing to be proven. Otherwise, $i_{r-f+1} > 1$ and $\epsilon^f dx^{\kappa_d} x_1 = x_1 \epsilon^f dx^{\kappa_d} \in M_f$.

Case 4: $h = \bar{x}_1$. By Lemma 4.6(1), $x^{\kappa_d}\bar{x}_1 = \bar{x}_1x^{\kappa_d}$ in $\text{gr}(\mathcal{B})$. So, the result follows from inductive assumption on degree and our previous results in Cases 2–3.

Case 5: $h = e_1$. We can assume $x^{\kappa_d} = x_1$. Otherwise, the result follows from Lemma 4.7, inductive assumption and our previous results in Cases 1–3. In this case, $i_{r-f+1} = 1$ and $e_{r-f+1,t-f+1}d = de_{1,j}$ for some j , $1 \leq j \leq t$. So, $\mathfrak{e}^f dx_1 e_1 = \mathfrak{e}^{f-1} de_{1,j} x_1 e_1$.

If $j = 1$, the required result follows from the equality $e_1 x_1 e_1 = (nq - mp)e_1$ (cf. (5.29)). Otherwise, by Lemmas 4.6 and 4.7,

$$e_{1,j} x_1 e_1 = -\bar{x}_1 e_{1,j} e_1 = -e_{1,j} \bar{x}_1 (\bar{1}, \bar{j}).$$

So, we need to verify $\mathfrak{e}^{f-1} de_{1,j} \bar{x}_1 (\bar{1}, \bar{j}) = \mathfrak{e}^f d\bar{x}_1 (\bar{1}, \bar{j}) \in M_f$, which follows from our previous results in Cases 1, 2 and 4. This completes the proof of Proposition 6.10. \square

Proposition 6.11. *Suppose $(f, \lambda) \in \Lambda_{2,r,t}$. Then $\Delta^R(f, \lambda)$ (resp., $\Delta^L(f, \lambda)$) is a right (resp., left) \mathcal{B} -module, where*

- $\Delta^R(f, \lambda)$ is \mathbb{C} -spanned by $\{\mathfrak{e}^f \mathfrak{m}_{\mathfrak{t}\lambda} d x^{\kappa_d} + \mathcal{B}^{\triangleright(f,\lambda)} \mid (\mathfrak{s}, d, \kappa_d) \in \delta(f, \lambda), \text{ and}$
- $\Delta^L(f, \lambda)$ is \mathbb{C} -spanned by $\{d^{-1} \mathfrak{e}^f \mathfrak{m}_{\mathfrak{s}\mathfrak{t}\lambda} + \mathcal{B}^{\triangleright(f,\lambda)} \mid (\mathfrak{s}, d, \kappa_d) \in \delta(f, \lambda)\}$.

Proof. We remark that $x_i \bar{x}_j = \bar{x}_j x_i$ in $\Delta^R(f, \lambda)$ (resp., $\Delta^L(f, \lambda)$) for all possible i, j 's. So, the result follows from Proposition 6.10 and Theorem 6.3 on the cellular basis of level two degenerate Hecke algebras $\mathcal{H}_{2,r-f} \times \mathcal{H}_{2,t-f}$. \square

Theorem 6.12. *Let $m, n, r, t \in \mathbb{Z}^{\geq 1}$, $p, q \in \mathbb{C}$ such that $r + t \leq \min\{m, n\}$. The set*

$$\mathcal{C} = \{C_{(\mathfrak{s}, \kappa_c, c)(\mathfrak{t}, \kappa_d, d)} \mid (\mathfrak{s}, \kappa_c, c), (\mathfrak{t}, \kappa_d, d) \in \delta(f, \lambda), \forall (f, \lambda) \in \Lambda_{2,r,t}\},$$

is a weakly cellular basis $\mathcal{B} = \mathcal{B}_{r,t}^{p,q}(m, n)$ over \mathbb{C} .

Proof. Suppose $0 \leq f \leq \min\{r, t\}$. By Proposition 6.11, $\mathcal{B}(f)/\mathcal{B}(f+1)$ is spanned by $C_{(\mathfrak{s}, c, \kappa_c)(\mathfrak{t}, d, \kappa_d)} + \mathcal{B}(f+1)$ for all $(\mathfrak{s}, c, \kappa_c), (\mathfrak{t}, d, \kappa_d) \in \delta(f, \lambda)$ and $\lambda \in \Lambda_{2,r-f,t-f}$. So, \mathcal{B} is spanned by \mathcal{C} . Counting the cardinality of \mathcal{C} yields $|\mathcal{C}| = 2^{r+t}(r+t)!$, which is the dimension of \mathcal{B} , stated in Theorem 5.12. So, \mathcal{C} is a \mathbb{C} -basis of \mathcal{B} . By Lemma 6.9(3) and Proposition 6.11, it is a weakly cellular basis in the sense of (6.4). \square

Remark 6.13. If we consider level two walled Brauer algebras over a commutative ring containing 1, and if we know its rank is equal to $2^{r+t}(r+t)!$, then all results in this section hold. We will prove it in [30].

7. Irreducible modules for \mathcal{B}

In this section, we classify simple \mathcal{B} -modules over \mathbb{C} via Theorem 6.12. So, we assume $r + t \leq \min\{m, n\}$.

First, we briefly recall the representation theory of cellular algebras [14]. At moment, we keep the notations in Definition 6.2. So, R is a commutative ring R containing 1 and A is a (weakly) cellular algebra over R with a weakly cellular basis $\{C_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s}, \mathfrak{t} \in T(\lambda), \lambda \in \Lambda\}$. We consider the right A -module in this section.

Recall that each cell module $C(\lambda)$ of A is the free R -module with basis $\{C_{\mathfrak{s}} \mid \mathfrak{s} \in T(\lambda)\}$, and every simple A -module arises in a unique way as the simple head of some cell module [14]. More explicitly, each $C(\lambda)$ comes equipped with the invariant form ϕ_λ which is determined by the equation

$$C_{\mathfrak{s}\mathfrak{t}}C_{\mathfrak{v}\mathfrak{s}} \equiv \phi_\lambda(C_{\mathfrak{t}}, C_{\mathfrak{v}}) \cdot C_{\mathfrak{s}\mathfrak{s}} \pmod{A^{\triangleright\lambda}}.$$

Consequently,

$$\text{Rad } C(\lambda) = \{x \in C(\lambda) \mid \phi_\lambda(x, y) = 0 \text{ for all } y \in C(\lambda)\},$$

is an A -submodule of $C(\lambda)$ and $D^\lambda = C(\lambda)/\text{Rad } C(\lambda)$ is either zero or absolutely simple. Graham and Lehrer [14] proved the following result.

Theorem 7.1. (See [14].) *Let (A, Λ) be a (weakly) cellular algebra over a field F . The set $\{D^\lambda \mid D^\lambda \neq 0, \lambda \in \Lambda\}$ consists of a complete set of pairwise non-isomorphic simple A -modules.*

By Theorem 6.12, we have cell modules $C(f, \lambda)$ with $(f, \lambda) \in \Lambda_{2,r,t}$ for \mathcal{B} . In fact, it is $\Delta^R(f, \lambda)$ in Proposition 6.11 up to an isomorphism. Let $\phi_{f,\lambda}$ be the corresponding invariant form on $C(f, \lambda)$. We use Theorem 7.1 to classify the simple \mathcal{B} -module over \mathbb{C} .

Let $\mathcal{H}_{2,r-f}$ (resp., $\mathcal{H}_{2,t-f}$) be the level two Hecke algebra which is isomorphic to the subalgebra of $\mathcal{B}_{r-f,t-f}^{p,q}$ generated by $s_1, s_2, \dots, s_{r-f-1}$ and x_1 (resp., $\bar{s}_1, \bar{s}_2, \dots, \bar{s}_{t-f-1}$ and \bar{x}_1). So, the eigenvalues of x_1 (resp., \bar{x}_1) are given in (5.30). By Theorem 6.3,

$$\{\mathfrak{m}_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s}, \mathfrak{t} \in \mathcal{T}^{std}(\lambda), \lambda \in \Lambda_2^+(r-f) \times \Lambda_2^+(t-f)\}$$

is a cellular basis of $\mathcal{H}_{2,r-f} \times \mathcal{H}_{2,t-f}$. We remark that $\mathfrak{m}_{\mathfrak{s}\mathfrak{t}}$ is a product of cellular basis elements of $\mathcal{H}_{2,r-f}$ and $\mathcal{H}_{2,t-f}$ described in Theorem 6.3.

Let $C(\lambda)$ be the cell module with respect to $\lambda \in \Lambda_{2,r-f} \times \Lambda_{2,t-f}$ for $\mathcal{H}_{2,r-f} \times \mathcal{H}_{2,t-f}$. Let ϕ_λ be the invariant form on $C(\lambda)$. For simplicity, we use $\mathbf{H}(2, f)$ to denote $\mathcal{H}_{2,r-f} \times \mathcal{H}_{2,t-f}$.

Proposition 7.2. *Suppose $r, t \in \mathbb{Z}^{\geq 2}$. We have $e_{r,t}\mathcal{B}_{r,t}^{p,q}e_{r,t} \subseteq e_{r,t}\mathcal{B}_{r-1,t-1}^{p,q}$.*

Proof. Recall that $\mathcal{B}_{r,t}^{p,q}$ is a (weakly) cellular algebra with cellular basis given in Theorem 6.12. So, it suffices to verify

$$e_{r,t}C_{(\mathfrak{s}, \kappa_d, d)(\mathfrak{t}, \kappa_c, c)}e_{r,t} \in e_{r,t}\mathcal{B}_{r-1,t-1}^{p,q}, \tag{7.1}$$

where $C_{(\mathfrak{s}, \kappa_d, d)(\mathfrak{t}, \kappa_c, c)} = x^{\kappa_d} d^{-1} \mathfrak{e}^f \mathfrak{m}_{\mathfrak{st}} c x^{\kappa_c}$ (cf. (6.12)).

Let $f = 0$. Since $\mathfrak{m}_{\mathfrak{st}}$ is a combination of monomials of form $\prod_{i=1}^r x_i^{\alpha_i} w_1 \bar{w}_1 \prod_{i=1}^t \bar{x}_i^{\beta_i}$, it suffices to verify

$$e_{r,t} \prod_{i=1}^r x_i^{\alpha_i} w_1 \bar{w}_1 \prod_{i=1}^t \bar{x}_i^{\beta_i} e_{r,t} \in e_{r,t} \mathcal{B}_{r-1,t-1}^{p,q} \tag{7.2}$$

We prove (7.2) by induction on the degree of $\prod_{i=1}^r x_i^{\alpha_i} w_1 \bar{w}_1 \prod_{i=1}^t \bar{x}_i^{\beta_i}$. The case for degree 0 follows from [10, Proposition 2.1]. In general, we assume that $\alpha_i = 0$ for $1 \leq i \leq r - 1$ and $\beta_j = 0$ for $1 \leq j \leq t - 1$. Otherwise, (7.2) follows from inductive assumption and the equalities $e_{r,t} x_i = x_i e_{r,t}$ and $e_{r,t} \bar{x}_j = \bar{x}_j e_{r,t}$ for $i \neq r$ and $j \neq t$.

By symmetry, we assume $\alpha_r = 1$. Write $w_1 = s_{r,k} w_2$ for some k , $1 \leq k \leq r$ and some $w_2 \in \mathfrak{S}_{r-1}$. Since any element in \mathfrak{S}_{r-1} commutes with $\bar{s}_j \in \tilde{\mathfrak{S}}_t$, \bar{x}_t and $e_{r,t}$, we can assume $w_1 \in \{1, s_{r-1}\}$.

If $w_1 = 1$, by Lemma 4.6 (2), it suffices to verify $e_{r,t} \bar{x}_t \bar{w}_1 \bar{x}_t^{\beta_t} e_{r,t} \in e_{r,t} \mathcal{B}_{r-1,t-1}^{p,q}$. In this case, we have $\bar{x}_t \bar{w}_1 = \bar{w}_1 \bar{x}_k + h$ for some $h \in \mathbb{C} \tilde{\mathfrak{S}}_t$ and some k with $(\bar{t}) \bar{w}_1 = \bar{k}$. By induction on degree, we need to verify

$$e_{r,t} \bar{w}_1 \bar{x}_k \bar{x}_t^{\beta_t} e_{r,t} \in e_{r,t} \mathcal{B}_{r-1,t-1}^{p,q}. \tag{7.3}$$

By Lemma 4.7 (2), and inductive assumption on degree, we have (7.3) if $k \neq t$. Otherwise, we have $k = t$ and $\bar{w}_1 \in \tilde{\mathfrak{S}}_{t-1}$. So, $e_{r,t} \bar{w}_1 = \bar{w}_1 e_{r,t}$. By induction on degree, we use $(\bar{x}_t + \bar{L}_t)^{1+\beta_t}$ instead of $\bar{x}_t \bar{x}_t^{\beta_t}$ in $e_{r,t} \bar{x}_t \bar{x}_t^{\beta_t} e_{r,t}$. So, the result follows from Lemma 4.8 (2). This verifies (7.2) provided $f = 0$.

Suppose $f > 0$. By (6.12), we rewrite (7.1) as follows:

$$e_{r,t} x^{\kappa_d} d^{-1} \mathfrak{e}^f \mathfrak{m}_{\mathfrak{st}} c x^{\kappa_c} e_{r,t} \in e_{r,t} \mathcal{B}_{r-1,t-1}^{p,q}. \tag{7.4}$$

Applying our previous result for $f = 0$ on $e_{r,t} x^{\kappa_d} d^{-1} \mathfrak{e}^f$ and $\mathfrak{e}^f \mathfrak{m}_{\mathfrak{st}} c x^{\kappa_c} e_{r,t}$ and noting that $\mathfrak{e}^f = e_{r,t} \cdots e_{r-f+1,t-f+1}$, we have (7.4) as required. \square

By recalling the definitions of ω_0 and ω_1 in (5.29), we see that $\omega_0 = \omega_1 = 0$ if and only if $m = n$ and $p = q$. For any $\lambda \in \Lambda_2^+(r - f) \times \Lambda_2^+(t - f)$ and $\mathfrak{t} \in \mathcal{F}^{std}(\lambda)$, we define $\mathfrak{m}_{\mathfrak{t}} = \mathfrak{m}_{\mathfrak{t}\lambda} + \mathbf{H}(2, f)^{\triangleright\lambda}$, where $\mathbf{H}(2, f)$ is given above Proposition 7.2.

Lemma 7.3. *Let \mathcal{B} be the level two walled Brauer algebra defined over \mathbb{C} . Suppose $(f, \lambda) \in \Lambda_{2,r,t}$ and $f \neq r$ if $r = t$. Then $\phi_{f,\lambda} \neq 0$ if and only if $\phi_\lambda \neq 0$.*

Proof. If $r \neq t$ or if $r = t$ and $f \neq r$, then either $s_{r,r-f}$ or $\bar{s}_{t,t-f}$ is well-defined. We denote such an element by w . So, $\mathfrak{e}^f w \mathfrak{e}^f = \mathfrak{e}^f$.

If $\phi_\lambda \neq 0$, then $\phi_\lambda(\mathfrak{m}_\mathfrak{s}, \mathfrak{m}_\mathfrak{t}) \neq 0$ for some $\mathfrak{s}, \mathfrak{t} \in \mathcal{T}^{std}(\lambda)$. We have $\phi_{f,\lambda} \neq 0$ since

$$\mathfrak{m}_{\mathfrak{t}^\lambda \mathfrak{s}} \mathbf{e}^f w \mathbf{e}^f \mathfrak{m}_{\mathfrak{t}^\lambda} \equiv \phi_\lambda(\mathfrak{m}_\mathfrak{s}, \mathfrak{m}_\mathfrak{t}) \mathbf{e}^f \mathfrak{m}_{\mathfrak{t}^\lambda \mathfrak{t}^\lambda} \pmod{\mathcal{B}^{\triangleright(f,\lambda)}}.$$

We remark that $x_i \bar{x}_j = \bar{x}_j x_i$ in $C(f, \lambda)$ for $1 \leq i \leq r-f$ and $1 \leq j \leq t-f$ (cf. Lemma 4.6 and Lemma 6.7).

Conversely, if $\phi_{f,\lambda} \neq 0$, then

$$\phi_{f,\lambda}(\mathbf{e}^f \mathfrak{m}_\mathfrak{s} dx^{\kappa_d}, \mathbf{e}^f \mathfrak{m}_\mathfrak{t} cx^{\kappa_c}) \neq 0,$$

for some $(\mathfrak{s}, d, \kappa_d), (\mathfrak{t}, c, \kappa_c) \in \delta(f, \lambda)$. We have $\phi_\lambda \neq 0$. Otherwise,

$$\mathfrak{m}_{\mathfrak{t}^\lambda \mathfrak{s}} h \mathfrak{m}_{\mathfrak{t}^\lambda} \equiv 0 \pmod{\mathbf{H}(2, f)^{\triangleright \lambda}},$$

for all $h \in \mathbf{H}(2, f)$. Using Proposition 7.2 repeatedly, we have

$$\mathbf{e}^f \mathfrak{m}_{\mathfrak{t}^\lambda \mathfrak{s}} cd^{-1} \mathbf{e}^f \mathfrak{m}_{\mathfrak{t}^\lambda} \equiv \mathfrak{m}_{\mathfrak{t}^\lambda \mathfrak{s}} h \mathfrak{m}_{\mathfrak{t}^\lambda} \mathbf{e}^f \pmod{\mathcal{B}(f+1)},$$

for some $h \in \mathbf{H}(2, f)$, forcing $\phi_{f,\lambda}(\mathbf{e}^f \mathfrak{m}_\mathfrak{s} dx^{\kappa_d}, \mathbf{e}^f \mathfrak{m}_\mathfrak{t} cx^{\kappa_c}) = 0$, a contradiction. \square

Lemma 7.4. *Let \mathcal{B} be the level two walled Brauer algebra defined over \mathbb{C} with $r = t$. Then $\phi_{r,0} \neq 0$ if at least one of ω_0 and ω_1 is non-zero. Otherwise, $\phi_{r,0} = 0$.*

Proof. Suppose $\omega_0 \neq 0$. We have $\phi_{r,0} \neq 0$ since $\mathbf{e}^f \mathbf{e}^f = \omega_0^f \mathbf{e}^f$. Otherwise, $\omega_0 = m - n = 0$, forcing $m = n$ and $\omega_1 = n(q - p)$. We consider $e_r e_{r-1} \cdots e_1 \prod_{i=1}^r (x_i + L_i)$ (where the product is in any order) and $e_r e_{r-1} \cdots e_1$ in the cell module $C(r, 0)$. By Lemma 4.8(2),

$$e_r e_{r-1} \cdots e_1 \prod_{i=1}^r (x_i + L_i) e_r e_{r-1} \cdots e_1 = \omega_1^r e_r e_{r-1} \cdots e_1.$$

We have $\phi_{r,0} \neq 0$ if $\omega_1 \neq 0$.

Finally, we assume $\omega_0 = \omega_1 = 0$ and $r = t$. In this case, we have $m = n$ and $p = q$. We claim that $\phi_{r,0} = 0$.

In fact, for any two basis elements $\mathbf{e}^r cx^{\kappa_c}$ and $\mathbf{e}^r dx^{\kappa_d}$ in $C(r, 0)$ with $c, d \in \mathcal{D}_{r,r}^r$, by using Proposition 7.2 repeatedly, we have

$$\mathbf{e}^r cx^{\kappa_c} x^{\kappa_d} d^{-1} \mathbf{e}^r \in \mathbf{e}^{r-1} e_1 \mathcal{B}_{1,1}^{p,q} e_1. \tag{7.5}$$

However, since we are assuming that $\omega_0 = \omega_1 = 0$, it is routine to check $e_1 \mathcal{B}_{1,1}^{p,q} e_1 = 0$. So, $\phi_{r,0} = 0$, as required. \square

In [21], Kleshchev classified the simple modules for degenerate cyclotomic Hecke algebra $\mathcal{H}_{r,n}$ over an arbitrary field via Kleshchev multipartitions of n . As mentioned in

[2, page 130], one could not say that $\phi_\lambda \neq 0$ if and only if λ is a Kleshchev multipartition. In our case, since level two walled Brauer algebras are only related to representations of cyclotomic degenerate Hecke algebras (more explicitly, level two Hecke algebras) over \mathbb{C} , we can use Vazirani’s result [36, Theorem 3.4]. In fact, it is not difficult to prove that there is an epimorphism from a standard module in [36, Theorem 3.4] to a cell module for degenerate cyclotomic Hecke algebra. So, $\phi_\lambda \neq 0$ if and only if λ is Kleshchev in the sense of [36, p273]. We recall the definition as follows. For our purpose, we only consider bipartitions.

Definition 7.5. Fix $u_1, u_2 \in \mathbb{C}$ with $u_1 - u_2 \in \mathbb{Z}$. A bipartition $\lambda = (\lambda^{(1)}, \lambda^{(2)}) \in \Lambda_2^+(n)$ of n is called a *Kleshchev bipartition* with respect to u_1, u_2 if

$$\lambda_{u_1 - u_2 + i}^{(1)} \leq \lambda_i^{(2)} \text{ for all possible } i.$$

If $u_1 - u_2 \notin \mathbb{Z}$, then we say that all bipartitions of n are Kleshchev bipartitions.

Since we consider a pair of bipartitions $(\lambda^{(1)}, \lambda^{(2)})$, where $\lambda^{(1)} \in \Lambda_2^+(r - f)$ and $\lambda^{(2)} \in \Lambda_2^+(t - f)$ for all $f, 0 \leq f \leq \min\{r, t\}$, we say that λ is *Kleshchev* if both $\lambda^{(1)}$ and $\lambda^{(2)}$ are Kleshchev with respect to the parameters $u_1 = p, u_2 = q - m$ and $u_1 = -q, u_2 = n - p$ respectively. The following result follows from Lemmas 7.3, 7.4 and our previous arguments immediately.

Theorem 7.6. Let $\mathcal{B} = \mathcal{B}_{r,t}^{p,q}(m, n)$ be the level two walled Brauer algebra over \mathbb{C} with condition $r + t \leq \min\{m, n\}$.

- (1) Suppose either $r \neq t$ or $r = t$ and one of ω_0, ω_1 is non-zero. Then the set of pairwise non-isomorphic simple \mathcal{B} -modules are indexed by

$$S = \{(f, \lambda) \in \Lambda_{2,r,t} \mid 0 \leq f \leq \min\{r, t\}, \lambda \text{ being Kleshchev}\}.$$

- (2) If $r = t$ and $\omega_0 = \omega_1 = 0$, then the set of pairwise non-isomorphic simple \mathcal{B} -modules are indexed by $\{(f, \lambda) \in \Lambda_{2,r,t} \mid 0 \leq f < r, \lambda \text{ being Kleshchev}\}$.

We close the paper by giving a classification of non-isomorphic indecomposable direct summands of $\mathfrak{gl}_{m|n}$ -modules $M^{r,t}$ (cf. (5.9)) provided that $M = K_\lambda$ is typical. Such direct summands are called *indecomposable tilting modules* of $\mathfrak{gl}_{m|n}$.

Theorem 7.7. Assume $r + t \leq \min\{m, n\}$.

- (1) If $p - q \in \mathbb{Z}$ with either $p - q \leq -m$ or $p - q \geq n$, then $M^{r,t}$ (cf. (5.9)) is a tilting module and the non-isomorphic indecomposable direct summands of $M^{r,t}$ are indexed by $\{(f, \mu) \in \Lambda_{2,r,t} \mid 0 \leq f \leq \min\{r, t\}, \mu \text{ being Kleshchev}\}$.
- (2) If $p - q \notin \mathbb{Z}$, then the non-isomorphic indecomposable direct summands of $M^{r,t}$ are simple and indexed by $\Lambda_{2,r,t}$.

Proof. Under the assumptions in (1) and (2), the Kac module $K_{\lambda_{pq}}$ is typical and at least one of ω_0 and ω_1 is non-zero. In this case, $M^{r,t}$ is a tilting module (see, e.g., [7, IV] for the case $t = 0$, from which one sees that it holds in general). By Theorem 5.12 and arguments similar to those in [5, §2] or [1, §5], there is a bijection between the set of non-isomorphic indecomposable direct summands of $M^{r,t}$ and the simple modules of \mathcal{B} . So, (1)–(2) follows from Theorem 7.6(1). (In fact, we will give an explicit relationship between indecomposable direct summands of $M^{r,t}$ and simple modules for $\mathcal{B}_{r,t}^{p,q}$ in [30], and further we will compute decomposition numbers of $\mathcal{B}_{r,t}^{p,q}$ arising from the super Schur–Weyl duality.) In particular, if $p - q \notin \mathbb{Z}$, all partitions $\lambda \in \Lambda_2^+(r - f) \times \Lambda_2^+(t - f)$ are Kleshchev. We remark that (2) also follows from Theorem 6.12 and Graham–Lehrer’s result in [14], which says that a cellular algebra is semisimple if and only if each cell module is equal to its simple head. \square

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After we finished the paper and posted it to arXiv.org, Professor C. Stroppel informed us that affine walled Brauer algebras are independently defined via affine walled Brauer categories by Sartori [31].

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