ON THE RESIDUE METHOD FOR PERIOD INTEGRALS

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Abstract. By applying the residue method for period integrals and Langlands-Shahidi's theory for residues of Eisenstein series, we study the period integrals for six spherical varieties. For each spherical variety, we prove a relation between the period integrals and certain automorphic L-functions. In some cases, we also study the local multiplicity of the spherical varieties.

1. Introduction and main results

Let k be a number field, and $\mathbb{A}$ its ring of adeles. Let G be a reductive group defined over k, and let H be a closed subgroup of G. Assume that $X = H\backslash G$ is spherical variety of G (i.e. the Borel subgroup $B \subset G$ acts with a Zariski dense orbit). Let A_G be the maximal split torus of the center of G and $A_{G,H} = A_G \cap H$. Let π be a cuspidal automorphic representation of $G(\mathbb{A})$ whose central character is trivial on $A_{G,H}(\mathbb{A})$. For $\phi \in \pi$, we define the period integral $\mathcal{P}_H(\phi)$ to be

$$\mathcal{P}_H(\phi) := \int_{H(k)A_{G,H}(\mathbb{A})\backslash H(\mathbb{A})} \phi(h)dh.$$

One of the most fundamental problems in the relative Langlands program is to find the relation between the period integral $\mathcal{P}_{H,\chi}(\phi)$ and some automorphic L-function $L(s,\pi,\rho_X)$. Here $\rho_X : {}^LG \to \mathrm{GL}_n(\mathbb{C})$ is a finite dimensional representation of the L-group ${}^LG$ of G.

In this paper, by applying the residue method for period integrals and Langlands-Shahidi's theory for residues of Eisenstein series, we study the period integrals for six spherical varieties. For each spherical variety, we prove a relation between the period integrals and some automorphic L-functions. The L-functions that are related to these spherical varieties include the standard L-functions of the general linear group, orthogonal group, unitary group and GE_6, the exterior square L-function of GL_{2n}, and a degree 12 L-function of $\mathrm{GL}_4 \times \mathrm{GL}_2$. In some cases, we also study the local multiplicity of the spherical varieties.

1.1. **The main results.** In this subsection, we are going to summarize the main results of this paper. For simplicity, we will only state the main results when G is split (except for the model related to unitary group which is only quasisplit). We refer the readers to later sections for details about the quasi-split and non-split cases. All the L-functions that show up in this paper are the completed Langlands-Shahidi L-functions; in particular, they include local factors from the archimedean places.

Remark 1.1. *In this paper, we only consider the case when H is reductive. However, our method can also be applied to the non-reductive case. For instance, in our previous paper* [PWZ]*, we studied the period integral for the Ginzburg-Rallis model, which is non-reductive.*

1.1.1. *The model* $(\mathrm{SO}_{2n+1}, \mathrm{SO}_{n+1} \times \mathrm{SO}_n)$. Let $G = \mathrm{SO}_{2n+1}$ be the split odd orthogonal group, ρ_X be the standard representation of ${}^LG = \mathrm{Sp}_{2n}(\mathbb{C})$, and $H = \mathrm{SO}_{n+1} \times \mathrm{SO}_n$ be a closed subgroup of G (H does not need to be split or quasi-split).

Theorem 1.2. *Let π be a cuspidal generic automorphic representation of $G(\mathbb{A})$. If the period integral $\mathcal{P}_H(\phi)$ is nonzero for some $\phi \in \pi$, then the L-function $L(s,\pi,\rho_X)$ is nonzero at $s = 1/2$.*

Theorem 1.2 will be proved in Section 4.

1.1.2. *The model* $(\mathrm{SO}_{2n}, \mathrm{SO}_{n+1} \times \mathrm{SO}_{n-1})$. Let $G = \mathrm{SO}_{2n}$ be the split even orthogonal group, ρ_X be the standard representation of ${}^L G = \mathrm{SO}_{2n}(\mathbb{C})$, and $H = \mathrm{SO}_{n+1} \times \mathrm{SO}_{n-1}$ be a closed subgroup of G (H does not need to be split or quasi-split).

Theorem 1.3. *Let π be a cuspidal generic automorphic representation of $G(\mathbb{A})$. If the period integral $\mathcal{P}_H(\phi)$ is nonzero for some $\phi \in \pi$, then the L-function $L(s, \pi, \rho_X)$ has a pole at $s = 1$.*

Locally, let F be a p-adic field, and π be an irreducible smooth representation of $G(F)$. We define the multiplicity

$$m(\pi) := \dim(\mathrm{Hom}_{H(F)}(\pi, 1)).$$

Theorem 1.4. *Let π be an irreducible generic tempered representation of $G(F)$. If $m(\pi) \neq 0$, then the local L-function $L(s, \pi, \rho_X)$ has a pole at $s = 0$.*

Theorem 1.3 and 1.4 will be proved in Section 5.

1.1.3. *The model* $(\mathrm{U}_{2n}, \mathrm{U}_n \times \mathrm{U}_n)$. Let k'/k be a quadratic extension, $G = \mathrm{U}_{2n}$ be the quasi-split unitary group, and $H = \mathrm{U}_n \times \mathrm{U}_n$ be a closed subgroup of G (H does not need to be quasi-split). Let π be a generic cuspidal automorphic representation of $G(\mathbb{A})$ with trivial central character, and let Π be the base change of π to $\mathrm{GL}_{2n}(\mathbb{A}_{k'})$. Let $L(s, \pi)$ (resp. $L(s, \Pi)$) be the standard L-function of π (resp. Π). We have $L(s, \pi) = L(s, \Pi)$.

Theorem 1.5. *If the period integral $\mathcal{P}_H(\phi)$ is nonzero for some $\phi \in \pi$, then the standard L-function $L(s, \pi) = L(s, \Pi)$ is nonzero at $s = 1/2$. Moreover, if we assume that Π is cuspidal and there exists a local place $v \in |k|$ such that k'/k splits at v and π_v is a discrete series of $G(k_v) = \mathrm{GL}_{2n}(k_v)$, then the exterior square L-function $L(s, \Pi, \wedge^2)$ has a pole at $s = 1$ (i.e. Π is of symplectic type).*

Remark 1.6. *Combining Theorem 1.5 with the result in* [FJ93] *for the linear model* $(\mathrm{GL}_{2n}, \mathrm{GL}_n \times \mathrm{GL}_n)$*, we know that if the period integral $\mathcal{P}_H(\phi)$ is nonzero on the space of π, then the $\mathrm{GL}_n(\mathbb{A}_{k'}) \times \mathrm{GL}_n(\mathbb{A}_{k'})$-period integral is nonzero on the space of Π.*

Remark 1.7. *The model $(\mathrm{U}_{2n}, \mathrm{U}_n \times \mathrm{U}_n)$ was mentioned to the third author independently by L. Clozel, J. Getz and K. Prasanna during his stay at the IAS. We would like to thank them for suggesting it to us.*

In fact, our result may be viewed as a special case of a principle put forward by Getz and Wambach in [GW14]. *They conjectured that for any reductive group H and any involution σ of H, the non-vanishing of the period integrals of the model (H, H^σ) (H^σ being the group of fixed points of σ) for a cuspidal automorphic representation π of $H(\mathbb{A})$ should be (roughly) equivalent to the non-vanishing of the period integrals of the model (G, G^σ) for the base change of π to $G(\mathbb{A})$. Here $G = Res_{k'/k}H$ with k'/k quadratic. Our result in Theorem 1.5 confirms one direction of a special case of their conjecture.*

On the other hand, the model $(\mathrm{U}_4, \mathrm{U}_2 \times \mathrm{U}_2)$ and its twists also appear in the work of Ichino-Prasanna [IP18] *in the context of algebraic cycles on Shimura varieties.*

Theorem 1.5 will be proved in Section 6.

1.1.4. *The Jacquet-Guo model.* Let k'/k be a quadratic extension, $G = \mathrm{GL}_{2n}$, and $H = Res_{k'/k}\mathrm{GL}_n$ be a closed subgroup of G (in particular $H(\mathbb{A}) = \mathrm{GL}_n(\mathbb{A}_{k'})$). The model (G, H) is the so called Jacquet-Guo model, and it was first be studied in [Guo96]. Let $\rho_{X,1}$ (resp. $\rho_{X,2}$) be the standard representation (resp. exterior square representation) of ${}^L G = \mathrm{GL}_{2n}(\mathbb{C})$.

Theorem 1.8. *Let π be a cuspidal automorphic representation of $G(\mathbb{A})$ with trivial central character. If the period integral $\mathcal{P}_H(\phi)$ is nonzero for some $\phi \in \pi$, then the L-function $L(s, \pi, \rho_{X,1})$ is nonzero at $s = 1/2$ and the L-function $L(s, \pi, \rho_{X,2})$ has a pole at $s = 1$.*

Remark 1.9. *In* [FMW18]*, under some local requirements on π (i.e. π is supercuspidal at some split place and H-elliptic at another place), the authors prove Theorem 1.8 by the relative trace formula method.*

Locally, let F be a p-adic field, E/F be a quadratic extension, $G(F) = \mathrm{GL}_{2n}(F)$ and $H(F) = \mathrm{GL}_n(E)$. Let π be an irreducible smooth representation of $G(F)$ with trivial central character. We define the multiplicity

$$m(\pi) := \dim(\mathrm{Hom}_{H(F)}(\pi, 1)).$$

Theorem 1.10. *Let π be an irreducible tempered representation of $G(F)$ with trivial central character. If $m(\pi) \neq 0$, then the local L-function $L(s, \pi, \rho_{X,2})$ has a pole at $s = 0$.*

Theorems 1.8 and 1.10 will be proved in Section 7.

1.1.5. *The model* $(\mathrm{GE}_6, A_1 \times A_5)$. Let $G = \mathrm{GE}_6$ be the similitude group of the split exceptional group E_6. Fix a quaternion algebra B over k, and define $H = (B^\times \times \mathrm{GL}_3(B))^0 := \{(x, g) \in B^\times \times \mathrm{GL}_3(B) : n_B(x) = N_6(g)\}$; here n_B is the degree two reduced norm on B and N_6 is the degree six reduced norm on $M_3(B)$. One has a map $H \to \mathrm{GE}_6$ with μ_2 kernel. Let ρ_X be a 27 dimensional fundamental representation of ${}^L PGE_6 = \mathrm{E}_6^{\mathrm{sc}}(\mathbb{C})$.

Theorem 1.11. *Let π be a cuspidal generic automorphic representation of $G(\mathbb{A})$ with trivial central character. Assume that $L(2, \pi, \rho_X) \neq 0$ (this is always the case if π is tempered). If the period integral $\mathcal{P}_H(\phi)$ is nonzero for some $\phi \in \pi$, then the L-function $L(s, \pi, \rho_X)$ has a pole at $s = 1$.*

Locally, let F be a p-adic field. Given an irreducible smooth representation π of $G(F)$ with trivial central character, we define the multiplicity

$$m(\pi) := \dim(\mathrm{Hom}_{H(F)}(\pi, 1)).$$

Theorem 1.12. *Let π be an irreducible generic tempered representation of $G(F)$ with trivial central character. If $m(\pi) \neq 0$, then the local L-function $L(s, \pi, \rho_X)$ has a pole at $s = 0$.*

Theorem 1.11 and 1.12 will be proved in Section 8.

1.1.6. *The model* $(\mathrm{GL}_4 \times \mathrm{GL}_2, \mathrm{GL}_2 \times \mathrm{GL}_2)$. Let $G = \mathrm{GL}_4 \times \mathrm{GL}_2$, and $H = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \times \begin{pmatrix} a \end{pmatrix} | a, b \in \mathrm{GL}_2 \right\}$ be a closed subgroup of G. Let $\rho_X = \wedge^2 \otimes std$ be a 12 dimensional representation of ${}^L G = \mathrm{GL}_4(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C})$.

Theorem 1.13. *Let π be a cuspidal automorphic representation of $G(\mathbb{A})$ with trivial central character. Assume that $L(3/2, \pi, \rho_X) \neq 0$ (this is always the case if π is tempered). If the period integral $\mathcal{P}_H(\phi)$ is nonzero for some $\phi \in \pi$, then $L(1/2, \pi, \rho_X) \neq 0$.*

Remark 1.14. *Since π has trivial central character, by the exceptional isomorphism* $\mathrm{PGL}_4 \simeq \mathrm{PGSO}_6$*, we can view π as a cuspidal automorphic representation of* $\mathrm{GSO}_6(\mathbb{A})$ *with trivial central character. Then the L-function $L(s, \pi, \rho_X)$ becomes the tensor L-function of* $\mathrm{GSO}_6 \times \mathrm{GL}_2$.

Theorem 1.13 will be proved in Section 9. Locally, in Section 9.5, we will show that the summation of the multiplicities of the model (G, H) is always equal to 1 over every tempered local Vogan L-packet.

1.2. **Organization of the paper and remarks on the proofs.** The theorems on global L-functions are all proved by the residue method, together with the Langlands-Shahidi's theory for residues of Eisenstein series. Recall that in the residue method one relates the period integrals of cuspidal representations to the period integrals of certain residue representations. This method goes back to Jacquet-Rallis [JR92], and has been applied by Jiang [Jia98], Ginzburg-Jiang-Rallis [GJR04a],[GJR05],[GJR09], Ichino-Yamana [IY], Ginzburg-Lapid [GL07], and by us in a previous paper [PWZ]. In Section 3, which serves as an extended introduction, we explain our strategy of proof in more detail. We will also discuss the connection between the residue method and the dual groups of spherical varieties.

The paper is organized as follows. In Section 2, we set up notations relating to Eisenstein series and truncation operators. Then in Section 3, we explain the strategy of the proofs of the main theorems (i.e. the residue method). In Section 4-9, we prove the main theorems for all six spherical varieties.

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2. Eisenstein series and the truncation operators

2.1. **General notations.** Let G be a connected reductive algebraic group over k. We fix a maximal k-split torus A_0 of G. Let P_0 be a minimal parabolic subgroup of G defined over k containing A_0, M_0 be the Levi part of P_0 containing A_0 and U_0 be the unipotent radical of P_0. Let $\mathcal{F}(P_0)$ be the set of parabolic subgroups of G containing P_0. Elements in $\mathcal{F}(P_0)$ are called standard parabolic subgroups of G. We also use $\mathcal{F}(M_0)=\mathcal{F}(A_0)$ (resp. $\mathcal{L}(M_0)$) to denote the set of parabolic subgroups (resp. Levi subgroups) of G containing A_0; these are the semi-standard parabolic subgroups (resp. Levi subgroups).

For $P\in\mathcal{F}(M_0)$, we have the Levi decomposition $P=MN$ with N be the unipotent radical of P and M be the Levi subgroup containing A_0. We use $A_P\subset A_0$ to denote the maximal k-split torus of the center of M. Put

$$\mathfrak{a}_0^*=X(A_0)\otimes_{\mathbb{Z}}\mathbb{R}=X(M_0)\otimes_{\mathbb{Z}}\mathbb{R}$$

and let $\mathfrak{a}_0$ be its dual vector space. Here $X(H)$, for any k-group H, denotes the group of rational characters of H. The inclusions $A_P\subset A_0$ and $M_0\subset M$ identify $\mathfrak{a}_P$ as a direct factor of $\mathfrak{a}_0$, we use $\mathfrak{a}_0^P$ to denote its complement. Similarly, $\mathfrak{a}_P^*=X(A_P)\otimes_{\mathbb{Z}}\mathbb{R}$ is a direct factor of $\mathfrak{a}_0^*$ and we use $\mathfrak{a}_0^{P,*}$ to denote its complement. Let $\Delta_P\subset\mathfrak{a}_P^*$ be the set of simple roots for the action of A_P on N and we use Δ_0 to denote Δ_{P_0}. Similarly, for $P\subset Q$, we can also define the subset $\Delta_P^Q\subset\Delta_P$. Then we define the chamber

$$\mathfrak{a}_P^+=\{H\in\mathfrak{a}_P|\ \langle H,\alpha\rangle\ >0,\ \forall\alpha\in\Delta_P\}.$$

Let $\Delta_0^\vee\subset\mathfrak{a}_0^G$ be the set of simple coroots given by the theory of root systems. For $\alpha\in\Delta_0$ we denote $\alpha^\vee\in\Delta_0^\vee$ the corresponding coroot. We define $\widehat{\Delta}_0\subset\mathfrak{a}_0^{G,*}$ to be the dual basis of $\Delta_0^\vee$, i.e. the set of weights. In particular, we get a natural bijection between Δ_0 and $\widehat{\Delta}_0$ which we denote by $\alpha\mapsto\varpi_\alpha$. Let $\widehat{\Delta}_P\subset\widehat{\Delta}_0$ be the set corresponding to $\Delta_0\smallsetminus\Delta_0^P$.

For any subgroup $H\subset G$, let $H(\mathbb{A})^1$ denote the common kernel of all characters on $H(\mathbb{A})$ of the form $|\chi(\cdot)|_{\mathbb{A}}$ where $\chi\in X(H)$ and $|\cdot|_{\mathbb{A}}$ is the absolute value on the ideles of $\mathbb{A}$. Fix K a maximal compact subgroup of $G(\mathbb{A})$ adapted to M_0. We define the Harish-Chandra map $H_P: G(\mathbb{A})\to\mathfrak{a}_P$ via the relation

$$\langle\chi,H_P(x)\rangle=|\chi(p)|_{\mathbb{A}},\quad\forall\chi\in X(P)=\mathrm{Hom}(P,\mathbb{G}_m)$$

where $x=pk$ is the Iwasawa decomposition $G(\mathbb{A})=P(\mathbb{A})K$. Let A_P^∞ be the connected component of the identity of $\mathrm{Res}_{k/\mathbb{Q}}A_P(\mathbb{R})$. Then $M(\mathbb{A})^1$ is the kernel of H_P restricted to $M(\mathbb{A})$ and we have the direct product decomposition of commuting subgroups $M(\mathbb{A})=A_P^\infty M(\mathbb{A})^1$.

For any group H we use $[H]$ to denote $H(k)\backslash H(\mathbb{A})$ and $[H]^1$ to denote $H(k)\backslash H(\mathbb{A})^1$.

2.2. **Haar measures.** We fix compatible Haar measures on $G(\mathbb{A})$, $G(\mathbb{A})^1$ and A_G^∞. For all unipotent subgroups N of G, we fix a Haar measure on $N(\mathbb{A})$ so that $[N]$ is of volume one. On K we also fix a Haar measure of volume 1. For any $P = MN \in \mathcal{F}(A_0)$, let $\rho_P \in \mathfrak{a}_P^*$ be the half sum of the weights of the action of A_P on N. We choose compatible Haar measures on A_P^∞ and $M_P(\mathbb{A})^1$ such that

$$\int_{P(k)\backslash H(\mathbb{A})} f(h)\,dh = \int_K \int_{[M]^1} \int_{A_P^\infty} \int_{[U]} e^{\langle -2\rho_P, H_P(a)\rangle} f(uamk)\,dudadmdk$$

for $f \in C_c^\infty(P(k)\backslash G(\mathbb{A}))$.

2.3. **The computation of ρ_P when P is maximal.** Let $P \in \mathcal{F}(P_0)$ be the maximal parabolic subgroup that corresponds to the simple root α, i.e. $\{\alpha\} = \Delta_0 \smallsetminus \Delta_0^P$. Let ϖ be the corresponding weight. We have $\rho_P \in \mathfrak{a}_P^{G,*}$. Since P is maximal, $\mathfrak{a}_P^{G,*}$ is one dimensional. Hence there exists a constant $c \in \mathbb{R}$ such that $\rho_P = c\varpi$. In the following proposition, we write down the constant c in five cases. It will be used in later sections. The computation is easy and standard, and hence we will skip it.

Proposition 2.1. *(1) If $G = \mathrm{SO}_n$ and P is the parabolic subgroup whose Levi part is isomorphic to $\mathrm{SO}_{n-2} \times \mathrm{GL}_1$, then $c = \frac{n-2}{2}$.*

(2) If $G = \mathrm{Sp}_{2n}$ and P is the Siegel parabolic subgroup, then $c = \frac{n+1}{2}$.

(3) If $G = \mathrm{U}_n$ and P is the parabolic subgroup whose Levi part is isomorphic to $\mathrm{U}_{n-2} \times \mathrm{GL}_1$, then $c = \frac{n-1}{2}$.

(4) If $G = \mathrm{SO}_{10}$ and P is the parabolic subgroup whose Levi part is isomorphic to $\mathrm{SO}_6 \times \mathrm{GL}_2$, then $c = \frac{7}{2}$.

(5) If $G = \mathrm{E}_7$ is simply-connected and P is the parabolic subgroup whose Levi part is of type E_6, then $c = 9$.

2.4. **Eisenstein series.** Let $P = MN$ be a parabolic subgroup of G. Given a cuspidal automorphic representation π of $M(\mathbb{A})$, let $\mathcal{A}_\pi$ be the space of automorphic forms ϕ on $N(\mathbb{A})M(k)\backslash G(\mathbb{A})$ such that $M(\mathbb{A})^1 \ni m \mapsto \phi(mg) \in L^2_\pi([M]^1)$ for any $g \in G(\mathbb{A})$, where $L^2_\pi([M]^1)$ is the π-isotypic part of $L^2([M]^1)$, and such that

$$\phi(ag) = e^{\langle \rho_P, H_P(a)\rangle}\phi(g), \quad \forall g \in G(\mathbb{A}),\ a \in A_P^\infty.$$

Suppose that P is a maximal parabolic subgroup. Let $\varpi \in \widehat{\Delta}_P$ be the corresponding weight. We then define

$$E(g,\phi,s) = \sum_{\delta \in P(k)\backslash G(k)} \phi(\delta g) e^{\langle s\varpi, H_P(\delta g)\rangle}, \quad s \in \mathbb{C},\ g \in G(\mathbb{A}).$$

The series converges absolutely for $s \gg 0$ and admits a meromorphic continuation to all $s \in \mathbb{C}$.

Suppose moreover that M is stable for the conjugation by the simple reflection in the Weyl group of G corresponding to P. We have in this case the intertwining operator $M(s) : \mathcal{A}_\pi \to \mathcal{A}_\pi$ that satisfies $E(M(s)\phi, -s) = E(\phi, s)$ and

$$E(g,\phi,s)_P = \phi(g)e^{\langle s\varpi, H_P(g)\rangle} + e^{\langle -s\varpi, H_P(g)\rangle} M(s)\phi(g), \quad g \in G(\mathbb{A})$$

where $E(\cdot,\phi,s)_P$ is the constant term of $E(\cdot,\phi,s)$ along P

$$E(g,\phi,s)_P := \int_{[N]} E(ug,\phi,s)\,du.$$

When the Eisenstein series $E(g,\phi,s)$ has a pole at $s = s_0$, the intertwining operator also has a pole at $s = s_0$, we use $Res_{s=s_0}E(g,\phi,s)$ (resp. $Res_{s=s_0}M(s)$) to denote the residue of the Eisenstein series (resp. intertwining operator). The poles of Eisenstein series $E(g,\phi,s)$ are simple for $Re(s) > 0$ and their residues are square integrable automorphic forms. Also the Eisenstein series, their derivatives and residues are of moderate growth.

2.5. **Arthur-Langlands truncation operator.** We continue assuming that P is maximal. We identify the space $\mathfrak{a}_P^G$ with $\mathbb{R}$ so that $T \in \mathbb{R}$ corresponds to an element whose pairing with $\varpi \in \mathfrak{a}_P^*$ is T. We will assume this isomorphism is measure preserving. Let $\widehat{\tau}_P$ be the characteristic function of

$$\{H \in \mathfrak{a}_P \mid \varpi(H) > 0 \ \forall \varpi \in \widehat{\Delta}_P\}.$$

Given a locally integrable function F on $G(k)\backslash G(\mathbb{A})$ we define its truncation as

$$\Lambda^T F(g) = F(g) - \sum_{\delta \in P(k)\backslash G(k)} \widehat{\tau}_P(H_P(\delta g) - T) \int_{[N]} F(u\delta g)\, du, \quad g \in G(k)\backslash G(\mathbb{A}),$$

where $T \in \mathbb{R}$ and the sum is actually finite.

2.6. **The relative truncation operator and the regularized period integral.** For later applications, we also need the relative truncation operator which was recently introduced by the third author in [Zyd19]. Let $H \subset G$ be a closed connected reductive subgroup, and let $P = MN \subset G$ still be a maximal parabolic subgroup. With the same notation as in Section 2.3, let ϖ be the corresponding weight and $c \in \mathbb{R}$ be the constant such that $\rho_P = c\varpi$. Fix a maximal split torus $A_{0,H}$ (resp. A_0) of H (resp. G) such that $A_{0,H} \subset A_0$. Then $\mathfrak{a}_H := \mathfrak{a}_{0,H}$ is a subspace of $\mathfrak{a}_0$. For simplicity, we assume that G has trivial split center (i.e. $A_G = \{1\}$).

Remark 2.2. *In* [Zyd19]*, the author defined the relative truncation operator for general automorphic functions and also for a general pair (G, H) with H reductive (H does not need to be a spherical subgroup). But for our applications in this paper, we only consider the case when when the automorphic function is a cuspidal Eisenstein series induced from a maximal parabolic subgroup.*

We fix a minimal subgroup $P_{0,H}$ of H with $A_{0,H} \subset P_{0,H}$. This allows us to define the set of standard (resp. semi-standard) parabolic subgroups of H. We will use $\mathcal{F}_H(P_{0,H})$ (resp. $\mathcal{F}_H(A_{0,H})$) to denote this set. We can also define the chamber $\mathfrak{a}_H^+ = \mathfrak{a}_{P_{0,H}}^+$ of $\mathfrak{a}_H$. Let $\bar{\mathfrak{a}}_H^+$ be the closure of $\mathfrak{a}_H^+$.

Definition 2.3. *We use $\mathcal{F}^G(P_{0,H}, P)$ to denote the set of semi-standard parabolic subgroups $Q = LU \in \mathcal{F}(A_0)$ of G that satisfy the following two conditions.*

(1) Q is a conjugate of P.
(2) $\mathfrak{a}_Q^+ \cap \bar{\mathfrak{a}}_H^+ \neq \emptyset$.

The next proposition was proved in Proposition 3.1 of [Zyd19].

Proposition 2.4. *Let $Q = LU$ be a semi-standard parabolic subgroup of G that is conjugate to P. Then the following statements hold.*

(1) If $Q \in \mathcal{F}^G(P_{0,H}, P)$, then $Q_H = Q \cap H$ is a standard parabolic subgroup of H with the Levi decomposition $Q_H = L_H U_H$ such that $L_H = L \cap H$ and $U_H = U \cap H$.
(2) Let A_L be the maximal split torus of the center of L. If $A_L \subset A_{0,H}$ (this is always the case when $A_0 = A_{0,H}$), then the inverse of (1) holds. In other words, if $Q_H = Q \cap H$ is a standard parabolic subgroup of H with the Levi decomposition $Q_H = L_H U_H$ such that $L_H = L \cap H$ and $U_H = U \cap H$, then $Q \in \mathcal{F}^G(P_{0,H}, P)$.

Remark 2.5. *Let $F = k_v$ ($v \in |k|$) be a local field. Combining the proposition above and Corollary 2.10 in Section 2.8, we know that for all $Q \in \mathcal{F}^G(P_{0,H}, P)$, $Q(F)H(F)$ is closed in $G(F)$.*

For $Q \in \mathcal{F}^G(P_{0,H}, P)$, let $A_L \subset A_0$ be the maximal split torus of the center of L, and let ϖ_Q be the weight correspond to Q. By the definition of the constant c, we have

$$\rho_Q = c\varpi_Q.$$

Since $\mathfrak{a}_Q^+ \cap \bar{\mathfrak{a}}_H^+ \neq \emptyset$ and $\mathfrak{a}_Q$ is one-dimensional, we have $\mathfrak{a}_Q \subset \mathfrak{a}_H$. We can therefore restrict ρ_{Q_H} to $\mathfrak{a}_Q$ and we define the real number c_Q^H to satisfy

$$\rho_{Q_H}|_{\mathfrak{a}_Q} = c_Q^H \rho_Q.$$

Remark 2.6. *In the case when U is abelian, let $n_Q = \dim(U)$ and $n_{Q,H} = \dim(U_H)$. Then*

$$c_Q^H = \frac{n_{Q,H}}{n_Q}.$$

We fix a cuspidal automorphic representation π of $M(\mathbb{A})$ and let $E(g, \phi, s)$ be the Eisenstein series defined in the previous section. For $Q \in \mathcal{F}^G(P_{0,H}, P)$, let $W(P, Q)$ be the two element set of isometries between $\mathfrak{a}_P$ and $\mathfrak{a}_Q$. For $w \in W(P, Q)$ let $sgn(w) \in \{-1, 1\}$ be such that $w\varpi_P = sgn(w)\varpi_Q$. We have then

$$E(g, \phi, s)_Q = \sum_{w \in W(P,Q)} M(w, s)\phi(g) e^{\langle sgn(w)s\varpi_Q, H_Q(g)\rangle}$$

for some explicit intertwining operators $M(w, s)$, independent of s if $sgn(w) = 1$.

In [Zyd19], the author defined a relative truncation operator, denoted by $\Lambda^{T,H}$, on the space $\mathcal{A}(G)$ of autormorphic forms on G, where $T \in \mathfrak{a}_H$. It depends on a choice of a good maximal compact K_H of $H(\mathbb{A})$ which we fix now. For all $\varphi \in \mathcal{A}(G)$, the truncation $\Lambda^{T,H}\varphi$ is a rapidly decreasing function on $[H]$. The following is the consequence of Theorem 4.1 of [Zyd19] and the discussion in Paragraph 4.7 of loc. cit.

Theorem 2.7. *(1) For all $\phi \in \mathcal{A}_\pi$ and $T \in \mathfrak{a}_H^+$ sufficiently regular, the integral*

$$\int_{[H]} \Lambda^{T,H} E(h, \phi, s) dh$$

is absolutely convergent for all $s \in \mathbb{C}$ in the domain of holomorphy of the Eisenstein series $E(\phi, s)$. Moreover, it defines a meromorphic function on $\mathbb{C}$.

(2) Define the regularized period for $E(\phi, s)$ to be

$$\mathcal{P}_{H,reg}(E(\phi, s)) := \int_{[H]} \Lambda^{T,H} E(h, \phi, s) dh - \sum_{Q \in \mathcal{F}^G(P_{0,H},P)} \sum_{w \in W(P,Q)}$$
$$\frac{e^{\langle (sgn(w)s + c(1-2c_Q^H))\varpi_Q, T\rangle}}{sgn(w)s + c(1-2c_Q^H)} \int_{K_H} \int_{L_H(k)A_L^\infty \backslash L_H(\mathbb{A})} M(w, s)\phi(mk) dm dk.$$

Then the integrals defining $\mathcal{P}_{H,reg}(E(\phi, s))$ are absolutely convergent and the functional $\mathcal{P}_{H,reg}(\cdot)$ is right $H(\mathbb{A})$-invariant.

2.7. **Some nonvanishing results of automorphic L-functions.**

Proposition 2.8. *(1) Let π be a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A})$. Then the standard L-function $L(s, \pi)$ is holomorphic nonzero when $Re(s) > 1$.*

(2) Let π be a generic cuspidal automorphic representation of $\mathrm{SO}_n(\mathbb{A})$, $\mathrm{Sp}_{2n}(\mathbb{A})$ or $\mathrm{U}_n(\mathbb{A})$. Then the standard L-function $L(s, \pi)$ is holomorphic nonzero when $Re(s) > 1$.

(3) Let π be a cuspidal automorphic representation of $\mathrm{GL}_{2n}(\mathbb{A})$. Then the exterior square L-function $L(s, \pi, \wedge^2)$ is nonzero when $Re(s) > 1$.

Proof. By Theorem 5.3 of [JS81] and Theorem 2.4 of [CPS04], the tensor L-function $L(s, \tau \times \sigma)$ is holomorphic nonzero for $Re(s) > 1$ for any cuspidal automorphic representation τ (resp. σ) of $\mathrm{GL}_{m_1}(\mathbb{A})$ (resp. $\mathrm{GL}_{m_2}(\mathbb{A})$). This proves (1) by taking $\tau = \pi$ and $\sigma = 1$. (2) is a direct consequence of (1) together with the results in [CKPSS04] and [KK05].

For (3), by taking $\sigma = \tau = \pi$, we know that the L-function $L(s, \pi \times \pi) = L(s, \pi, Sym^2)L(s, \pi, \wedge^2)$ is holomorphic nonzero for $Re(s) > 1$. Hence it is enough to show that the symmetric square L-function $L(s, \pi, Sym^2)$ is holomorphic when $Re(s) > 1$. This follows from Corollary 5.8 of [Tak14] and Theorem 3.1(3) of [Kim00]. □

2.8. **A criterion for closed orbit.** Let F be a local field, G be a linear algebraic group defined over F and H, P be two closed subgroups of G.

Proposition 2.9. *Suppose that the morphism*

$$H/(H \cap P) \hookrightarrow G/P$$

is a closed immersion. Then $H(F)P(F)$ is closed in $G(F)$.

Proof. Let $P_H = H \cap P$, and $H \times^{P_H} P$ be the quotient on the right of $H \times P$ by the diagonally embedded subgroup P_H. Let $p : H \times^{P_H} P \to G$ be the morphism induced by the map $H \times P \to G$ given by $(h, p) \mapsto hp^{-1}$. We have the following cartesian diagram

$$\begin{array}{ccc} H \times^{P_H} P & \xrightarrow{p} & G \\ \downarrow & & \downarrow \\ H/P_H & \longrightarrow & G/P \end{array}$$

which shows that $H \times^{P_H} P$ is isomorphic to the fiber product $(H/P_H) \times_{G/P} G$. By assumption, $H/P_H \hookrightarrow G/P$ is a closed immersion. By the property of pullbacks, $p : H \times^{P_H} P \to G$ is also a closed immersion. Hence $(H \times^{P_H} P)(F)$ is closed in $G(F)$.

The set $H(F)P(F)$ is identified with the subset $(H(F) \times P(F))/P_H(F)$ of $(H \times^{P_H} P)(F)$. The inclusion $(H(F) \times P(F))/P_H(F) \hookrightarrow (H \times^{P_H} P)(F)$ is closed by Corollary A.1.6 of [AG09]. This proves the proposition. □

Corollary 2.10. *Let G be a connected reductive group, $H \subset G$ a closed connected reductive subgroup, and $P \subset G$ a parabolic subgroup (all defined over F). Assume that $P_H = H \cap P$ is a parabolic subgroup of H. Then $H(F)P(F)$ is closed in $G(F)$.*

Proof. By the proposition above, we only need to show that the morphism $H/P_H \hookrightarrow G/P$ is a closed immersion. Since P is a parabolic subgroup of G and P_H is a parabolic subgroup of H, we know that both G/P and H/P_H are projective. Hence the morphism $H/P_H \hookrightarrow G/P$ is a closed immersion. □

3. The strategy of the proof

3.1. **The first step.** Let (G, H) be one of the six spherical pairs in Section 1. Our goal is to prove a relation between the period integral $\mathcal{P}_H(\phi)$ and certain automorphic L-function $L(s, \phi, \rho_X)$. The first step of our method is to find another spherical variety $\underline{X} = \underline{H}\backslash\underline{G}$ that satisfies the following three conditions.

(1) G is isomorphic to the Levi component $\underline{M}$ of a maximal parabolic subgroup $\underline{P} = \underline{MU}$ of $\underline{G}$ (up to modulo the center).
(2) The L-function $L(s, \pi, \rho_X)$ appears in the Langlands-Shahidi L-function for $(\underline{G}, \underline{M})$.
(3) $\underline{H} \cap \underline{P} = (\underline{H} \cap \underline{M}) \ltimes (\underline{H} \cap \underline{N})$ is a maximal parabolic subgroup of $\underline{H}$ such that $\underline{H} \cap \underline{M}$ is isomorphic to the group H (up to modulo the center).

Such a spherical variety does not exist in general. But if it exists, we can use it to prove a relation between the period integral and the automorphic L-function. In particular, for all of the six spherical pairs in Section 1, we can find a spherical pair $(\underline{G}, \underline{H})$ that satisfies the conditions above. For simplicity, we assume that $\underline{G}$ has trivial split center.

Remark 3.1. *In* [KS17]*, Knop-Schalke have defined the dual group $G_X^\vee$ for every affine spherical varieties $X = H\backslash G$ together with a natural morphism $\iota_X : G_X^\vee \to G^\vee$ from the dual group of the spherical variety to the dual group of G. Let $Cent_{G^\vee}(Im(\iota_X))$ be the centralizer of the image of the map ι_X in $G^\vee$. Following the notation in* [KS17]*, we use $\mathfrak{l}_X^\wedge$ to denote the Lie algebra of $Cent_{G^\vee}(Im(\iota_X))$. We say the spherical variety X is tempered if $\mathfrak{l}_X^\wedge = 0$.*

For all the six spherical pairs (G,H) in Section 1 (as well as all the other known cases), the dual groups of the spherical varieties $X = H\backslash G$ and $\underline{X} = \underline{H}\backslash\underline{G}$ satisfy the following two conditions.

(4) $G_X^\vee = G_{\underline{X}}^\vee$.

(5) $\mathfrak{l}_X^\wedge = 0$, $\mathfrak{l}_{\underline{X}}^\wedge = \mathfrak{sl}_2$ *or* $\mathfrak{so}_3$.

In general, we believe that for a given spherical pair (G,H), if we can find another spherical pair $(\underline{G},\underline{H})$ that satisfies Conditions (1),(4) and (5), then it should also satisfies Conditions (2) and (3) (up to conjugating the parabolic subgroup $\underline{P}$).

In this paper, we have considered all the spherical pairs (G,H) with G simple and H reductive (see Table 3 of [KS17]*) such that there exists another spherical pair $(\underline{G},\underline{H})$ that satisfies Condition (1), (4) and (5) (except those pairs that have already been studied by other people). We also consider a case when G is not simple, i.e. the model* $(\mathrm{GL}_4 \times \mathrm{GL}_2, \mathrm{GL}_2 \times \mathrm{GL}_2)$.

After we find the spherical pair $(\underline{G},\underline{H})$, we consider the period integral

$$\mathcal{P}_{\underline{H}}(E(\phi,s)) := \int_{\underline{H}(k)\backslash\underline{H}(\mathbb{A})} E(\phi,s)(\underline{h})d\underline{h}$$

for a cuspidal Eisenstein series $E(\phi,s)$ of $\underline{G}(\mathbb{A})$ induced from the maximal parabolic subgroup $\underline{P} = \underline{MU}$ and the cuspidal automorphic representation π of $\underline{M}(\mathbb{A}) \simeq G(\mathbb{A})$. This period integral is not convergent in general, hence we need to truncate the Eisenstein series $E(\phi,s)$. As we explained in the previous section, we have two different truncation operators. In the next two subsections, we will explain how to study the truncated period integrals by using these two different truncation operators. Our goal is to prove a relation between the truncated $\underline{H}$-period integral of the residue of the Eisenstein series $E(\phi,s)$ $(\phi \in \mathcal{A}_\pi)$ and the H-period integral of the cusp forms in π.

To end this subsection, we give a list of $(\underline{G},\underline{H})$ for the six spherical pairs in Section 1. We refer the reader to Table 3 of [KS17] for the dual groups of spherical varieties.

- When $(G,H) = (\mathrm{SO}_{2n+1}, \mathrm{SO}_{n+1}\times\mathrm{SO}_n)$, we let $(\underline{G},\underline{H}) = (\mathrm{SO}_{2n+3}, \mathrm{SO}_{n+3}\times\mathrm{SO}_n)$. In this case, $G_X^\vee = G_{\underline{X}}^\vee = \mathrm{Sp}_{2n}(\mathbb{C})$. This model will be discussed in Section 4.
- When $(G,H) = (\mathrm{SO}_{2n}, \mathrm{SO}_{n+1}\times\mathrm{SO}_{n-1})$, we let $(\underline{G},\underline{H}) = (\mathrm{SO}_{2n+2}, \mathrm{SO}_{n+3}\times\mathrm{SO}_{n-1})$. In this case, $G_X^\vee = G_{\underline{X}}^\vee = \mathrm{SO}_{2n-1}(\mathbb{C})$. This model will be discussed in Section 5.
- When $(G,H) = (\mathrm{U}_{2n}, \mathrm{U}_n\times\mathrm{U}_n)$, we let $(\underline{G},\underline{H}) = (\mathrm{U}_{2n+2}, \mathrm{U}_{n+2}\times\mathrm{U}_n)$. In this case, $G_X^\vee = G_{\underline{X}}^\vee = \mathrm{Sp}_{2n}(\mathbb{C})$. This model will be discussed in Section 6.
- When $(G,H) = (\mathrm{GL}_{2n}, Res_{k'/k}\mathrm{GL}_n)$, we let $(\underline{G},\underline{H}) = (\mathrm{Sp}_{4n}, Res_{k'/k}\mathrm{Sp}_{2n})$. In this case, $G_X^\vee = G_{\underline{X}}^\vee = \mathrm{Sp}_{2n}(\mathbb{C})$. This model will be discussed in Section 7.
- When $(G,H) = (\mathrm{GE}_6, A_1\times A_5)$, we let $\underline{G} = \mathrm{E}_7$ and $\underline{H}$ be the symmetric subgroup of $\underline{G}$ of type $A_1\times D_6$. In this case, $G_X^\vee = G_{\underline{X}}^\vee = \mathrm{F}_4(\mathbb{C})$. This model will be discussed in Section 8.
- When $(G,H) = (\mathrm{GL}_4\times\mathrm{GL}_2, \mathrm{GL}_2\times\mathrm{GL}_2)$, we let $(\underline{G},\underline{H}) = (\mathrm{GSO}_{10}, \mathrm{GSpin}_7\times\mathrm{GL}_1)$. In this case, $G_X^\vee = G_{\underline{X}}^\vee = \mathrm{GL}_4(\mathbb{C})\times\mathrm{GL}_2(\mathbb{C})$. This model will be discussed in Section 9.

3.2. **Method 1: relative truncation operator.** In this subsection, we will use the relative truncation operator to study the period integral $\mathcal{P}_{\underline{H}}(E(\phi,s))$. We recall from Theorem 2.7 the definition of the regularized period integral:

$$\mathcal{P}_{\underline{H},reg}(E(\phi,s)) := \int_{[\underline{H}]} \Lambda^{T,\underline{H}} E(h,\phi,s)dh - \sum_{\underline{Q}\in\mathcal{F}^{\underline{G}}(P_{0,\underline{H}},\underline{P})} \sum_{w\in W(\underline{P},\underline{Q})}$$

$$\frac{e^{\langle (sgn(w)s+c(1-2c_{\underline{Q}}^{\underline{H}}))\varpi_{\underline{Q}},T\rangle}}{sgn(w)s+c(1-2c_{\underline{Q}}^{\underline{H}})}\int_{K_{\underline{H}}}\int_{L_{\underline{H}}(k)A_{\underline{L}}^{\infty}\backslash L_{\underline{H}}(\mathbb{A})} M(w,s)\phi(mk)dmdk.$$

Then we need to show that for $Re(s) >> 0$, the following two statements hold.

(1) The regularized period integral $\mathcal{P}_{\underline{H},reg}(E(\phi,s))$ is equal to 0.

(2) For all $\underline{Q}\in\mathcal{F}^{\underline{G}}(P_{0,\underline{H}},\underline{P})$ with $\underline{Q}\neq\underline{P}$, $w\in W(P,Q)$ and $\phi\in\mathcal{A}_{\pi}$, we have

$$\int_{L_{\underline{H}}(k)A_{\underline{L}}^{\infty}\backslash L_{\underline{H}}(\mathbb{A})} M(w,s)\phi(m)dm=0.$$

Assume that we have proved (1) and (2). Then the equation above implies that

$$\int_{[\underline{H}]}\Lambda^{T,\underline{H}}E(h,\phi,s)dh=\frac{e^{\langle (s+c(1-2c_{\underline{P}}^{\underline{H}}))\varpi_{\underline{P}},T\rangle}}{s+c(1-2c_{\underline{P}}^{\underline{H}})}\int_{K_{\underline{H}}}\int_{[H]}\phi(hk)dhdk + \frac{e^{\langle (-s+c(1-2c_{\underline{P}}^{\underline{H}}))\varpi_{\underline{P}},T\rangle}}{-s+c(1-2c_{\underline{P}}^{\underline{H}})}\int_{K_{\underline{H}}}\int_{[H]}M(s)\phi(hk)dhdk. \tag{3.1}$$

Here we have used Condition (3) of the pair $(\underline{G},\underline{H})$. (3.1) tells us that the truncated $\underline{H}$-period integral of the Eisenstein series $E(\phi,s)$ is equal to the H-period integrals of ϕ and $M(s)\phi$. Let $s_0=-c(1-2c_{\underline{P}}^{\underline{H}})$. By taking the residue at $s=s_0$ for the above equation, we have

$$\int_{[\underline{H}]}\Lambda^{T,\underline{H}}Res_{s=s_0}E(h,\phi,s)dh=\int_{K_{\underline{H}}}\int_{[H]}\phi(hk)dhdk+\frac{e^{\langle -2s_0\varpi_{\underline{P}},T\rangle}}{-2s_0}\int_{K_{\underline{H}}}\int_{[H]}Res_{s=s_0}M(s)\phi(hk)dhdk. \tag{3.2}$$

In particular, we get the following proposition.

Proposition 3.2. *If the period integral $\mathcal{P}_H(\phi)$ is non-zero for some $\phi\in\pi$, then the cuspidal Eisenstein series $E(\phi,s)$ and the intertwining operator $M(s)\phi$ has a pole at $s=s_0$.*

Remark 3.3. *When π is generic, the above proposition implies that if the period integral $\mathcal{P}_H(\phi)$ is non-zero for some $\phi\in\pi$, the Langlands-Shahidi L-function for $(\underline{G},\underline{M})$ has a pole at $s=s_0$. By assumption (2) of the pair $(\underline{G},\underline{H})$, the L-function $L(s,\pi,\rho_X)$ appears in the Langlands-Shahidi L-function for $(\underline{G},\underline{M})$. Hence the above proposition gives a relation between the period integral $\mathcal{P}_H(\phi)$ and the automorphic L-function $L(s,\pi,\rho_X)$.*

Remark 3.4. *A similar version of this method has been used by Ichino-Yamana (*[IY]*) for the unitary Gan-Gross-Prasad model case.*

Now we discuss the proof of (1) and (2). By Theorem 2.7, the regularized period integral $\mathcal{P}_{\underline{H},reg}(E(\phi,s))$ is right $\underline{H}(\mathbb{A})$-invariant. As a result, in order to prove (1), it is enough to prove the following local statement.

(3) Let $\pi=\otimes'_{v\in|k|}\pi_v$, and let $\Pi_s=I_{\underline{P}}^{\underline{G}}\pi_s=\otimes'_{v\in|k|}I_{\underline{P}}^{\underline{G}}\pi_{v,s}=\otimes'_{v\in|k|}\Pi_{v,s}$. Here $\pi_s=\pi\otimes\varpi_{\underline{P}}^{s}$, $\pi_{v,s}=\pi_v\otimes\varpi_{\underline{P}}^{s}$, and $I_{\underline{P}}^{\underline{G}}(\cdot)$ is the normalized parabolic induction. Then there exists $v\in|k|$ such that

$$\mathrm{Hom}_{\underline{H}(k_v)}(\Pi_{v,s},1)=\{0\}$$

for $Re(s)>>0$. In other words, $\Pi_{v,s}$ is not $\underline{H}(k_v)$-distinguished.

Obviously (3) is not true for arbitrary π. Hence we need to make some assumption.

Assumption 3.5. *There exists a non-archimedean place $v\in|k|$ such that π_v is generic.*

Now we fix $v\in|k|$ that satisfies the assumption above. In order to prove (3), it is enough to prove the following statement.

(4) $\mathrm{Hom}_{\underline{H}(k_v)}(\tau_v, 1) = \{0\}$ for all generic representations of $\underline{G}(k_v)$.

Remark 3.6. *By the same argument as above, in order to prove (2), it is enough to show that an analogue of statement (4) holds for the pair* $(\underline{L}, L_{\underline{H}})$*. In other words, it is enough to show that*

(5) $\mathrm{Hom}_{L_{\underline{H}}(k_v)}(\tau_v, 1) = \{0\}$ *for all generic representations of* $\underline{L}(k_v)$*.*

Under the relative local Langlands conjecture of Sakellaridis-Venkatesh in [SV17], (4) should hold for all spherical varieties that are not tempered (note that by Condition (5) of the pair $(\underline{G},\underline{H})$, the spherical variety $\underline{X} = \underline{H}\backslash\underline{G}$ is not tempered). However, only the symmetric pair case has been recently proved by Prasad in [Pra18]. We will briefly recall his result in Section 3.4. Hence in order to prove (4), we need to make a stronger assumption.

Assumption 3.7. *(1) H is a symmetric subgroup of G.*

(2) There exists a non-archimedean place $v \in |k|$ such that π_v is generic.

In general, if one can extend Prasad's result to all spherical varieties, then Assumption 3.5 is enough. We want to point out that Assumption 3.5 is also necessary for this method because without the generic assumption, there exist examples such that Condition (1) and (2) fail.

Remark 3.8. *For the six spherical pairs in Section 1, five of them are symmetric. The only exception is the model* $(\mathrm{GL}_4 \times \mathrm{GL}_2, \mathrm{GL}_2 \times \mathrm{GL}_2)$.

3.2.1. *Some remarks about the period integrals of the residue representations.* When the Eisenstein series $E(\phi, s)$ has a pole at $s = s_0$, the residue is a square integrable automorphic form with the only non-trivial exponent $-s_0$ along parabolic subgroups conjugate to $\underline{P}$. By Theorem 4.1 of [Zyd19], the regularized period integral of $Res_{s=s_0}E(\phi, s)$ is the constant term in T of the truncated period integral

$$\int_{[\underline{H}]} \Lambda^{T,\underline{H}} Res_{s=s_0} E(h, \phi, s) dh,$$

at least when $s_0 \neq c(1 - 2c^{H}_{\underline{Q}})$ for all $\underline{Q} \in \mathcal{F}^{\underline{G}}(P_{0,\underline{H}}, \underline{P})$ (this condition will be satisfied for all the spherical pairs that we consider). As a result, by taking the constant term in T of (3.2), we have

$$\mathcal{P}_{\underline{H},reg}(Res_{s=s_0}E(\phi, s)) = \int_{K_{\underline{H}}} \int_{[H]} \phi(hk) dh dk. \tag{3.3}$$

In other words, the regularized $\underline{H}$-period integral of $Res_{s=s_0}E(\phi, s)$ is equal to the H-period integral of ϕ, up to some compact integration. By the same argument as in Lemma 5.8 of [IY], we obtain that the H-period integral is nonzero on the space of π if and only if the regularized $\underline{H}$-integral is nonzero on the space $\{Res_{s=s_0}E(\phi, s)|\ \phi \in \mathcal{A}_\pi\}$. This matches the general conjecture of Sakellaridis-Venkatesh [SV17] for period integrals because of the conditions (4) and (5) of the pair $(\underline{G}, \underline{H})$ in Section 3.1.

Moreover, by Theorem 4.6 of [Zyd19], when

$$s_0 > c(1 - 2c^{H}_{\underline{Q}}),\ \forall \underline{Q} \in \mathcal{F}^{\underline{G}}(P_{0,\underline{H}}, \underline{P}), \tag{3.4}$$

the regularized period integral $\mathcal{P}_{\underline{H},reg}(Res_{s=s_0}E(\phi, s))$ is equal to the actual period integral

$$\int_{[\underline{H}]} Res_{s=s_0} E(h, \phi, s) dh.$$

In particular, the actual period integral is absolutely convergent. For all the cases we consider, $s_0 = -c(1-2c^{H}_{\underline{P}})$ is a positive real number (in fact, it is either 1 or $\frac{1}{2}$), hence the inequality in (3.4) is automatic when $\underline{Q} = \underline{P}$. In particular, if the set $\mathcal{F}^{\underline{G}}(P_{0,\underline{H}}, \underline{P})$ only contains one element $\underline{P}$, (3.4) holds. As a result, for those cases, the regularized period integral on the left hand side of (3.3) can be replaced by the actual period integral of $Res_{s=s_0}E(h,\phi,s)$. As we will see in later sections, for all the models we consider, the following are the cases when $\mathcal{F}^{\underline{G}}(P_{0,\underline{H}},\underline{P})=\{\underline{P}\}$.

- $\underline{G}=\mathrm{SO}_{2n+3}$, $\underline{H}=\mathrm{SO}_{n+3}\times\mathrm{SO}_n$ and the SO_n-part of $\underline{H}$ is anisotropic, discussed in Section 4.
- $\underline{G}=\mathrm{SO}_{2n+2}$, $\underline{H}=\mathrm{SO}_{n+3}\times\mathrm{SO}_{n-1}$ and the SO_{n-1}-part of $\underline{H}$ is anisotropic, discussed in Section 5.
- $\underline{G}=\mathrm{U}_{2n+2}$, $\underline{H}=\mathrm{U}_{n+2}\times\mathrm{U}_n$ and U_n-part of $\underline{H}$ is anisotropic, discussed in Section 6.
- $\underline{G}=\mathrm{Sp}_{4n}$ and $\underline{H}=Res_{k'/k}\mathrm{Sp}_{2n}$, discussed in Section 7.
- $\underline{G}=E_7^{sc}$ the semisimple, simply-connected group of type E_7, $\underline{H}$ the symmetric subgroup of type $D_6\times A_1$, and $\underline{H}$ not split. This will be discussed in Section 8.

For all the other cases we consider, the set $\mathcal{F}^{\underline{G}}(P_{0,\underline{H}},\underline{P})=\{\underline{P},\ \underline{P}'\}$ contains two elements. As we will see in later sections, in those cases, the inequality (3.4) will fail when $\underline{Q}=\underline{P}'$. By Theorem 4.6 of [Zyd19] again, the period integral of $Res_{s=s_0}E(h,\phi,s)$ is divergent and the regularization is necessary in those cases. This phenomenon has already been observed for the model $(\mathrm{Sp}_{4n},\mathrm{Sp}_{2n}\times\mathrm{Sp}_{2n})$ by Lapid and Offen in [LO18].

3.3. **Method 2: Arthur-Langlands truncation operator.** In this subsection, we will use the Arthur-Langlands truncation operator to study the period integral $\mathcal{P}_{\underline{H}}(E(\phi,s))$. We need one assumption.

Assumption 3.9. *The double coset $\underline{P}(k)\backslash\underline{G}(k)/\underline{H}(k)$ has finitely many orbits.*

Remark 3.10. *For the six spherical varieties in Section 1, three of them satisfy this assumption: the Jacquet-Guo model $(\mathrm{GL}_{2n},\mathrm{Res}_{k'/k}(\mathrm{GL}_n))$, $(\mathrm{GE}_6,A_1\times A_5)$, and $(\mathrm{GL}_4\times\mathrm{GL}_2,\mathrm{GL}_2\times\mathrm{GL}_2)$).*

Let $\{\gamma_i|\ 1\leq i\leq l\}$ be a set of representatives for the double coset $\underline{P}(k)\backslash\underline{G}(k)/\underline{H}(k)$. For $1\leq i\leq l$, let $\underline{H}_i=\underline{H}\cap\gamma_i^{-1}\underline{P}\gamma_i$. Without loss of generality, we assume that $\gamma_1=1$.

Consider the truncated period integral

$$\mathcal{P}_{\underline{H}}(\Lambda^T E(\phi,s)):=\int_{\underline{H}(k)\backslash\underline{H}(\mathbb{A})}\Lambda^T E(\phi,s)(\underline{h})d\underline{h}$$

where Λ^T is the Arthur-Langlands truncation operator. By unfolding the integral, we have

$$\mathcal{P}_{\underline{H}}(\Lambda^T E(\phi,s))=\sum_{i=1}^{l}I_i(\phi,s)+J_i(\phi,s) \tag{3.5}$$

with

$$I_i(\phi,s)=\int_{\underline{H_i}(k)\backslash\underline{H}(\mathbb{A})}(1-\hat{\tau}_{\underline{P}}(H_{\underline{P}}(\gamma_i\underline{h})-T))e^{<s\varpi_{\underline{P}},H_{\underline{P}}(\gamma_i\underline{h})>}\phi(\gamma_i\underline{h})\,d\underline{h},$$

$$J_i(\phi,s)=\int_{\underline{H_i}(k)\backslash\underline{H}(\mathbb{A})}\hat{\tau}_{\underline{P}}(H_{\underline{P}}(\gamma_i\underline{h})-T)e^{<-s\varpi_{\underline{P}},H_{\underline{P}}(\gamma_i\underline{h})>}M(s)\phi(\gamma_i\underline{h})\,d\underline{h}.$$

Here $M(s)$ is the intertwining operator, and the factors $\hat{\tau}_{\underline{P}}(H_{\underline{P}}(\gamma_i\underline{h})-T)$, $e^{<s\varpi_{\underline{P}},H_{\underline{P}}(\gamma_i\underline{h})>}$ come from the truncation operator Λ^T.

The first step is to show that the integrals $I_i(\phi,s)$ and $J_i(\phi,s)$ are absolutely convergent when $Re(s)>>0$. In Section 5 of our previous paper [PWZ], we have developed a general argument for proving the absolute convergence. The only thing we need to check is that (H,H_i) is a good pair. We refer the readers to Section 5.3 of [PWZ] for the definition of good pair.

After the first step, we need to show that when $Re(s)>>0$, we have

$$I_i(\phi,s)=J_i(\phi,s)=0$$

for $2\leq i\leq l$. For $i=1$, by Condition (3) of the pair $(\underline{G},\underline{H})$, we can show that

$$I_1(\phi,s)=\frac{e^{(s-s_0)T}}{s-s_0}\int_{K_{\underline{H}}}\int_{[H]}\phi(hk)dhdk,\ \ J_1(\phi,s)=\frac{e^{(-s-s_0)T}}{-s-s_0}\int_{K_{\underline{H}}}\int_{[H]}M(s)\phi(hk)dhdk$$

where $s_0 = -c(1 - c_{\underline{P}}^{\underline{H}})$ as in Method 1. This implies that

$$\mathcal{P}_{\underline{H}}(\Lambda^T E(\phi, s)) = \frac{e^{(s-s_0)T}}{s-s_0} \int_{K_{\underline{H}}} \int_{[H]} \phi(hk) dh dk + \frac{e^{(-s-s_0)T}}{-s-s_0} \int_{K_{\underline{H}}} \int_{[H]} M(s)\phi(hk) dh dk. \tag{3.6}$$

The above equation is an analogue of equation (3.1) for Method 1. Then we can use the same argument as in Method 1 to finish the proof.

Remark 3.11. *This method was first used by Jacquet-Rallis (*[JR92]*) to study the period integrals of the residue representations for the model* $(\mathrm{GL}_{2n}, \mathrm{Sp}_{2n})$*. Later in* [Jia98]*, Jiang used this method to study the trilinear* GL_2 *model (Jiang was the first one to use this method to study the period integrals of cusp forms). In our previous paper* [PWZ]*, we applied this method to the Ginzburg-Rallis model case. A similar version of this method has been used by Ginzburg-Jiang-Rallis (*[GJR04a],[GJR05],[GJR09]*) for the orthogonal Gan-Gross-Prasad model. See also* [GL07] *for a slightly different approach.*

Remark 3.12. *Compared with Method 1, Method 2 has two disadvantages and two advantages. The two disadvantages are*

- *We need to assume that the double coset* $\underline{P}(k)\backslash\underline{G}(k)/\underline{H}(k)$ *has finitely many orbit (i.e. Assumption 3.9). In particular, it cannot be applied to the spherical pairs* $(\mathrm{SO}_{2n+1}, \mathrm{SO}_{n+1} \times \mathrm{SO}_n)$*,* $(\mathrm{SO}_{2n}, \mathrm{SO}_{n+1} \times \mathrm{SO}_{n-1})$ *and* $(\mathrm{U}_{2n}, \mathrm{U}_n \times \mathrm{U}_n)$*.*
- *As we explained above, in Method 2, we need to show that* $I_i(\phi, s) = J_i(\phi, s) = 0$ *for* $2 \le i \le l$*. This requires us to study all the orbits in the double coset* $\underline{P}(k)\backslash\underline{G}(k)/\underline{H}(k)$*. On the other hand, for Method 1, we only need to study the closed orbits (see Remark 2.5). In some cases (e.g. the model* $(\mathrm{GE}_6, A_1 \times A_5)$*), those nonclosed orbits can be hard to study because one needs to compute explicitly the image in* $\underline{M} = \underline{P}/\underline{N}$ *of the intersection* $\underline{P} \cap \gamma_i \underline{H} \gamma_i^{-1}$*.*

The two advantages are

- *Method 2 can be applied to the case when* H *is not reductive while Method 1 can only be applied to the reductive case (this is due to the fact that the relative truncation operator was only defined in the reductive case). For example, in our previous paper* [PWZ]*, we used Method 2 to study the period integrals of the Ginzburg-Rallis model, which is not reductive.*
- *Even if* H *is reductive, as we explained in the previous section, unless one can extend Prasad's result to all the spherical varieties, we can only apply Method 1 when* H *is a symmetric subgroup. On the other hand, Method 2 can be applied to the non-symmetric case.*

Remark 3.13. *We will use Method 1 to study the following five spherical pairs:* $(\mathrm{SO}_{2n+1}, \mathrm{SO}_{n+1} \times \mathrm{SO}_n)$*,* $(\mathrm{SO}_{2n}, \mathrm{SO}_{n+1} \times \mathrm{SO}_{n-1})$*,* $(\mathrm{U}_{2n}, \mathrm{U}_n \times \mathrm{U}_n)$*,* $(\mathrm{GE}_6, A_1 \times A_5)$*, and the Jacquet-Guo model. These pairs are all symmetric. We use Method 2 to study the spherical pair* $(\mathrm{GL}_4 \times \mathrm{GL}_2, \mathrm{GL}_2 \times \mathrm{GL}_2)$*, which is not symmetric. Method 2 can also be used to study the Jacquet-Guo model and the pair* $(\mathrm{GE}_6, A_1 \times A_5)$*, but it will be more complicated than Method 1.*

3.4. **A local result of Prasad.** In this subsection, we recall a recent result of Prasad for the distinguished representations of symmetric pairs in [Pra18]. Let F be a p-adic field of characteristic 0. Let G be a quasi-split reductive group defined over F, θ be an involution automorphism of G defined over F. Let G^θ be the group of fixed points, H be the connected component of identity of G^θ, and $H_1 = [H, H]$ be the derived group of H.

We say (G, H) is quasi-split if there exists a Borel subgroup B of $G(\bar{F})$ such that $B \cap \theta(B)$ is a maximal torus of G.

Theorem 3.14 (Theorem 1 of [Pra18])**.** *If* (G, H) *is not quasi-split, then the Hom space*

$$\mathrm{Hom}_{H_1(F)}(\pi, 1)$$

is zero for all generic representations π of $G(F)$. In other words, there is no $H_1(F)$-distinguished generic representation of $G(F)$.

In Theorem 1(2) of [Pra18], the author also gives an easy criterion for one to check whether (G,H) is quasi-split by looking at the real form of $G(\mathbb{C})$ associated to the involution θ. We refer the reader to Section 1 of [Pra18] for details. By that criterion, we can easily prove the following corollary.

Corollary 3.15. *The following symmetric pairs are not quasi-split. In particular, there is no $H_1(F)$-distinguished generic representation of $G(F)$.*

(1) $G = \mathrm{GL}_{2n}$ and $H = \mathrm{GL}_{n+1} \times \mathrm{GL}_{n-1}$.
(2) $G = \mathrm{SO}_{2n+3}$ the split odd orthogonal group, and $H = \mathrm{SO}_{n+k} \times \mathrm{SO}_{n+3-k}$ with $k \geq 3$.
(3) $G = \mathrm{SO}_{2n+2}$ a quasi-split even orthogonal group, and $H = \mathrm{SO}_{n+k} \times \mathrm{SO}_{n+2-k}$ with $k \geq 3$.
(4) $G = \mathrm{SO}_{2n}$ the split even orthogonal group ($n \geq 1$), and $H = \mathrm{GL}_n$ be the Levi subgroup of the Siegel parabolic subgroup of G.
(5) $G = \mathrm{Sp}_{4n}$ and $H = \mathrm{Sp}_{2n} \times \mathrm{Sp}_{2n}$.
(6) $G = \mathrm{GE}_6$ the similitude group of the split exceptional group E_6, and H be symmetric subgroup of G of type $D_5 \times \mathrm{GL}_1$.
(7) $G = \mathrm{E}_7^{\mathrm{sc}}$ be the split, simply-connected exceptional group, and H be symmetric subgroup of G of type $D_6 \times A_1$.

4. The model $(\mathrm{SO}_{2n+1}, \mathrm{SO}_{n+1} \times \mathrm{SO}_n)$

4.1. **The result.** Let W_1 (resp. W_2) be a quadratic space defined over k of dimension $n+1$ (resp. n), and $W = W_1 \oplus W_2$. Let $G = \mathrm{SO}(W)$ and $H = \mathrm{SO}(W_1) \times \mathrm{SO}(W_2)$. Let $D = Span\{v_{0,+}, v_{0,-}\}$ be a two-dimensional quadratic space with $< v_{0,+}, v_{0,+} >=< v_{0,-}, v_{0,-} >= 0$ and $< v_{0,+}, v_{0,-} >= 1$, $V_1 = W_1 \oplus D$, $V_2 = W_2$, $V = V_1 \oplus V_2$, $\underline{G} = \mathrm{SO}(V)$, and $\underline{H} = \mathrm{SO}(V_1) \times \mathrm{SO}(V_2)$.

Let $D_+ = Span\{v_{0,+}\}$ and $D_- = Span\{v_{0,-}\}$. Then $D = D_+ \oplus D_-$ as a vector space. Let $\underline{P} = \underline{M}\underline{N}$ be the maximal parabolic subgroup of $\underline{G}$ that stabilizes the subspace D_+ with $\underline{M}$ be the Levi subgroup that stabilizes the subspaces D_+, W and D_-. Then $\underline{M} = \mathrm{SO}(W) \times \mathrm{GL}_1$.

Let $\pi = \otimes_{v \in |k|} \pi_v$ be a cuspidal automorphic representation of $G(\mathbb{A})$. Then $\pi \otimes 1$ is a cuspidal automorphic representation of $\underline{M}(\mathbb{A})$. To simplify the notation, we will still use π to denote this cuspidal automorphic representation. For $\phi \in \mathcal{A}_\pi$ and $s \in \mathbb{C}$, let $E(\phi, s)$ be the Eisenstein series on $\underline{G}(\mathbb{A})$. The goal of this section is to prove the following theorem.

Theorem 4.1. *Assume that there exists a local non-archimedean place $v \in |k|$ such that π_v is a generic representation of $G(k_v)$ (in particular, $G(k_v)$ is split). If the period integral $\mathcal{P}_H(\cdot)$ is nonzero on the space of π, then there exists $\phi \in \mathcal{A}_\pi$ such that the Eisenstein series $E(\phi, s)$ has a pole at $s = 1/2$.*

Theorem 4.1 will be proved in the last subsection of this section. The next proposition shows that Theorem 1.2 follows from Theorem 4.1.

Proposition 4.2. *Theorem 4.1 implies Theorem 1.2.*

Proof. We first recall the statement of Theorem 1.2. Let $G = \mathrm{SO}_{2n+1}$ be the split odd orthogonal group, $H = \mathrm{SO}_{n+1} \times \mathrm{SO}_n$ be a closed subgroup of G (not necessarily split), and π be a generic cuspidal automorphic representation of $G(\mathbb{A})$. If the period integral $\mathcal{P}_H(\cdot)$ is nonzero on the space of π, we need to show that the L-function $L(s, \pi, \rho_X)$ is nonzero at $s = 1/2$. Here ρ_X is the standard representation of ${}^L G = \mathrm{Sp}_{2n}(\mathbb{C})$.

By the Theorem 4.1, if the period integral $\mathcal{P}_H(\cdot)$ is nonzero on the space of π, there exists $\phi \in \mathcal{A}_\pi$ such that the Eisenstein series $E(\phi, s)$ has a pole at $s = 1/2$. In this case, the normalizing factor of the intertwining operator is

$$\frac{L(s, \pi, \rho_X)\zeta_k(2s)}{L(s+1, \pi, \rho_X)\zeta_k(2s+1)}$$

where $\zeta_k(s)$ is the Dedekind zeta function. By Theorem 4.7 of [KK11], the normalized intertwining operator is holomorphic at $s = 1/2$. By Proposition 2.8, $L(3/2, \pi, \rho_X) \neq 0$. It follows that the numerator $L(s, \pi, \rho_X)\zeta_k(2s)$ has a pole at $s = 1/2$, which implies that $L(\frac{1}{2}, \pi, \rho_X) \neq 0$. This proves Theorem 1.2. □

Remark 4.3. *For a generic representation π of SO_{2n+1}, the central value $L(1/2, \pi)$ is linked to the so called Bessel periods by the Gan-Gross-Prasad conjecture* [GGP12] *and has been studied in* [GJR04a, GJR05]*. In* [JS07b, JS07a] *it is linked on the other hand to the first occurrence problem in theta correspondence.*

4.2. **The parabolic subgroups.** For $i = 1, 2$, we fix a maximal hyperbolic subspace (resp. anisotropic subspace) $W_{i,h}$ (resp. $W_{i,an}$) of W_i such that $W_i = W_{i,h} \oplus W_{i,an}$. We fix a basis $\{w_{i,\pm 1}, \cdots, w_{i,\pm m_i}\}$ of $W_{i,h}$ such that

$$< w_{i,j}, w_{i,k} >=< w_{i,-j}, w_{i,-k} >= 0, \ < w_{i,-j}, w_{i,k} >= \delta_{jk}, \ \forall 1 \leq j \leq k \leq m_i.$$

Let $V_0 = W_{1,an} \oplus W_{2,an}$. Fix a maximal hyperbolic subspace (resp. anisotropic subspace) $V_{0,h}$ (resp. $V_{0,an}$) of V_0 such that $V_0 = V_{0,h} \oplus V_{0,an}$. We fix a basis $\{v_{0,\pm 1}, \cdots, v_{0,\pm l}\}$ of $V_{0,h}$ such that

$$< v_{0,j}, v_{0,k} >=< v_{0,-j}, v_{0,-k} >= 0, \ < v_{0,-j}, v_{0,k} >= \delta_{jk}, \ \forall 1 \leq j \leq k \leq l.$$

We use capital letters to denote the one-dimensional vector space spanned by vectors in small letters (e.g. $W_{i,1} = Span\{w_{i,1}\}$).

Remark 4.4. *m_1, m_2 and l may be zero.*

For $i = 1, 2$, let $P_{0,i} = M_{0,i}N_{0,i}$ be the parabolic subgroup of $\mathrm{SO}(W_i)$ that stabilizes the filtration

$$Span\{w_{i,1}\} \subset Span\{w_{i,1}, w_{i,2}\} \subset \cdots \subset Span\{w_{i,1}, \cdots, w_{i,m_i}\},$$

and $M_{0,i}$ be the subgroup of $\mathrm{SO}(W_i)$ that stabilizes the subspaces

$$W_{i,j}, \ W_{i,-j}, \ W_{i,an}, \ \forall 1 \leq j \leq m_i.$$

Let $A_{0,i}$ be the split center of $M_{0,i}$ which can be identified with $(\mathrm{GL}_1)^{m_i}$ under the natural isomorphism

$$A_{0,i} \simeq \mathrm{GL}(W_{i,1}) \times \cdots \times \mathrm{GL}_{W_{i,m_i}}.$$

Then $P_{0,i}$ is a minimal parabolic subgroup of $\mathrm{SO}(W_i)$ and $A_{0,i}$ is a maximal split torus of $\mathrm{SO}(W_i)$.

Let $\underline{P}_{0,1} = \underline{M}_{0,1}\underline{N}_{0,1}$ be the parabolic subgroup of $\mathrm{SO}(V_1)$ that stabilizes the filtration

$$Span\{v_{0,+}\} \subset Span\{v_{0,+}, w_{1,1}\} \subset \cdots \subset Span\{v_{0,+}, w_{1,1}, \cdots, w_{1,m_1}\},$$

and $\underline{M}_{0,1}$ be the subgroup of $\mathrm{SO}(V_1)$ that stabilizes the subspaces

$$D_+, W_{1,j}, \ W_{1,-j}, \ W_{1,an}, \ \forall 1 \leq j \leq m_1.$$

Let $\underline{A}_{0,1}$ be the split center of $\underline{M}_{0,1}$ which can be identified with $(\mathrm{GL}_1)^{m_1+1}$ under the natural isomorphism

$$\underline{A}_{0,1} \simeq \mathrm{GL}(D_+) \times \mathrm{GL}(W_{1,1}) \times \cdots \times \mathrm{GL}_{W_{1,m_i}}.$$

Then $\underline{P}_{0,1}$ is a minimal parabolic subgroup of $\mathrm{SO}(V_1)$ and $\underline{A}_{0,1}$ is a maximal split torus of $\mathrm{SO}(V_1)$ with $P_{0,1} \subset \underline{P}_{0,1}$ and $A_{0,1} \subset \underline{A}_{0,1}$.

On the other hand, let $\underline{P}_0 = \underline{M}_0 N_0$ be the parabolic subgroup of $\underline{G}$ that stabilizes the filtration

$$Span\{v_{0,+}\} \subset Span\{v_{0,+}, w_{1,1}\} \subset \cdots \subset Span\{v_{0,+}, w_{1,1}, \cdots, w_{1,m_1}\} \subset Span\{v_{0,+}, w_{1,1}, \cdots, w_{1,m_1}, w_{2,1}\}$$
$$\subset \cdots \subset Span\{v_{0,+}, w_{1,1}, \cdots, w_{1,m_1}, w_{2,1}, \cdots, w_{2,m_2}\} \subset Span\{v_{0,+}, w_{1,1}, \cdots, w_{1,m_1}, w_{2,1}, \cdots, w_{2,m_2}, v_{0,1}\}$$
$$\subset \cdots \subset Span\{v_{0,+}, w_{1,1}, \cdots, w_{1,m_1}, w_{2,1}, \cdots, w_{2,m_2}, v_{0,1}, \cdots v_{0,l}\}$$

and $\underline{M}_0$ be the subgroup of $\underline{G}$ that stabilizes the subspaces

$$D_+, W_{i,j}, \ W_{i,-j}, \ V_{0,k}, \ V_{0,an}, \ \forall 1 \leq i \leq 2, 1 \leq j \leq m_i, 1 \leq k \leq l.$$

Let $\underline{A}_0$ be the split center of $\underline{M}_0$ which can be identified with $(\mathrm{GL}_1)^{m_1+m_2+l+1}$ under the natural isomorphism

$$\underline{A}_0 \simeq \mathrm{GL}(D_+)\times \mathrm{GL}(W_{1,1})\times\cdots\times \mathrm{GL}_{W_{1,m_1}}\times \mathrm{GL}(W_{2,1})\times\cdots\times \mathrm{GL}(W_{2,m_2})\times \mathrm{GL}_{V_{0,1}}\times\cdots\times \mathrm{GL}(V_{0,l}).$$

Then $\underline{P}_0$ is a minimal parabolic subgroup of $\underline{G}$ and $\underline{A}_0$ is a maximal split torus of $\underline{G}$ with $\underline{P}_0\subset \underline{P}$ and $\underline{A}_{0,1}\times A_{0,2}\subset \underline{A}_0$.

Definition 4.5. *We use $\mathcal{F}(\underline{M}_0,\underline{P})$ to denote the set of semi-standard parabolic subgroups $\underline{Q}\in\mathcal{F}(\underline{M}_0)$ of $\underline{G}$ that are conjugated to $\underline{P}$.*

The following proposition is a direct consequence of the definitions above.

Proposition 4.6. *Consider the set*

$$X_{iso}=\{v_{0,\pm},\ w_{i,\pm j}, v_{0,\pm k}|\ 1\leq i\leq 2, 1\leq j\leq m_i, 1\leq k\leq l\}.$$

For any element $w\in X_{iso}$, let P_w be the stabilizers of $Span\{w\}$ in $\underline{G}$. Then $P_w\in\mathcal{F}(\underline{M}_0,\underline{P})$ and this gives us a natural bijection between the sets $\mathcal{F}(\underline{M}_0,\underline{P})$ and X_{iso}. Moreover, the parabolic subgroup $\underline{P}$ corresponds to the vector $v_{0,+}$ under this bijection.

Let $\underline{P}_{0,H}=\underline{P}_{0,1}\times P_{0,2}$ be a minimal parabolic subgroup of $\underline{H}=\mathrm{SO}(V_1)\times\mathrm{SO}(V_2)=\mathrm{SO}(V_1)\times\mathrm{SO}(W_2)$. The following corollary is a direct consequence of the discussions above together with the definition of the set $\mathcal{F}^{\underline{G}}(\underline{P}_{0,H},\underline{P})$ in Section 2.6.

Corollary 4.7. *If $m_2=0$, then $\mathcal{F}^{\underline{G}}(\underline{P}_{0,H},\underline{P})=\{\underline{P}\}$. If $m_2\neq 0$, then $\mathcal{F}^{\underline{G}}(\underline{P}_{0,H},\underline{P})=\{\underline{P},\underline{P}'\}$ where $\underline{P}'$ is the parabolic subgroup corresponds to the vector $w_{2,1}$.*

To end this subsection, we discuss the intersections $\underline{P}\cap\underline{H}$ and $\underline{P'}\cap\underline{H}$. Let $P_1=M_1N_1$ be the maximal parabolic subgroup of $\mathrm{SO}(V_1)$ that stabilizes the space D_+, and M_1 be the subgroup of $\mathrm{SO}(V_1)$ that stabilizes the subspaces D_+,D_- and W_1. Then $M_1\simeq\mathrm{SO}(W_1)\times\mathrm{GL}(D_+)$ and we have

$$\underline{P}\cap\underline{H}=P_1\times\mathrm{SO}(W_2),\ \underline{M}\cap\underline{H}=M_1\times\mathrm{SO}(W_2),\ \underline{N}\cap\underline{H}=N_1\times\{1\}.$$

For $\underline{P'}\cap\underline{H}$, let $\underline{P'}=\underline{M'N'}$ where $\underline{M'}$ is the subgroup of $\underline{G}$ that stabilizes the subspaces $W_{2,1}$, $W_{2,-1}$ and $V_1\oplus W_2'$ where

$$W_2'=Span\{w_{2,\pm 2},\cdots,w_{2,\pm m_2}\}.$$

Let $P_2=M_2N_2$ be the maximal parabolic subgroup of $\mathrm{SO}(V_2)=\mathrm{SO}(W_2)$ that stabilizes the space $W_{2,1}$, and M_2 be the subgroup of $\mathrm{SO}(V_2)$ that stabilizes the subspaces $W_{2,1},W_{2,-1}$ and W_2'. Then $M_2\simeq\mathrm{SO}(W_2')\times\mathrm{GL}(W_{2,1})$ and we have

$$\underline{P'}\cap\underline{H}=\mathrm{SO}(V_1)\times P_2,\ \underline{M'}\cap\underline{H}=\mathrm{SO}(V_1)\times M_2,\ \underline{N'}\cap\underline{H}=\{1\}\times N_2.$$

4.3. **The proof of Theorem 4.1.** In this section, we will prove Theorem 4.1. We assume that $m_2\neq 0$, the proof for the case when $m_2=0$ is similar and much easier (this is due to the fact that the set $\mathcal{F}(\underline{M}_0,\underline{P})$ only contains one element when $m_2=0$, see Corollary 4.7). We use Method 1 introduced in Section 3.2.

With the same notations as in Theorem 4.1 and Section 3.2, we want to study the regularized period integral $\mathcal{P}_{\underline{H},reg}(E(\phi,s))$ for $\phi\in\mathcal{A}_\pi$. First, let's prove statement (1) and (2) in Section 3.2 for the current case. For (1), by our assumptions on π together with the argument in Section 3.2, it is enough to show that statement (4) of Section 3.2 holds for the pair $(\underline{G},\underline{H})$. But this just follows from Corollary 3.15(1). For (2), as we discussed in Remark 3.6, it is enough to show that the pair

$$(\underline{M'},\underline{M'}\cap\underline{H'})=(\mathrm{SO}(2n+1)\times\mathrm{GL}_1,\mathrm{SO}(n+3)\times\mathrm{SO}(n-2)\times\mathrm{GL}_1)$$

satisfies statement (5) in Remark 3.6. This also follows from Corollary 3.15(1).

Then we compute the constant $s_0=-c(1-2c_{\underline{P}}^{H})$ for the current case. By Remark 2.6, we have

$$c_{\underline{P}}^{H}=\frac{\dim(N_1)}{\dim\underline{N}}=\frac{n+1}{2n+1}.$$

By Proposition 2.1, we have $c = \frac{2n+1}{2}$. This implies that

$$s_0 = -c(1 - 2c_{\underline{P}}^{\underline{H}}) = 1/2.$$

Combining the discussions above, equation (3.2) in Method 1 becomes

$$\int_{[\underline{H}]} \Lambda^{T,\underline{H}} Res_{s=1/2} E(h,\phi,s)dh = \int_{K_{\underline{H}}} \int_{[H]} \phi(hk)dhdk - e^{\langle -\varpi_{\underline{P}}, T\rangle} \int_{K_{\underline{H}}} \int_{[H]} Res_{s=1/2} M(s)\phi(hk)dhdk$$

for the current case. This finishes the proof of Theorem 4.1.

Remark 4.8. *When $m_2 \neq 0$ (i.e. when W_2 is not anisotropic), according to our discussion above, the set $\mathcal{F}^{\underline{G}}(\underline{P}_{0,H}, \underline{P})$ contains two elements $\underline{P}$ and $\underline{P}'$. It is easy to see that $c(1 - 2c_{\underline{P}'}^{\underline{H}}) = \frac{2n+1}{2}(1 - 2\frac{n-2}{2n+1}) = \frac{5}{2} > s_0 = \frac{1}{2}$. This confirms the discussion in Section 3.2.1.*

5. The model $(\mathrm{SO}_{2n}, \mathrm{SO}_{n+1} \times \mathrm{SO}_{n-1})$

5.1. **The global result.** Let W_1 (resp. W_2) be a quadratic space defined over k of dimension $n+1$ (resp. $n-1$), and $W = W_1 \oplus W_2$. Let $G = \mathrm{SO}(W)$ and $H = \mathrm{SO}(W_1) \times \mathrm{SO}(W_2)$. Let $D = Span\{v_{0,+}, v_{0,-}\}$ be a two-dimensional quadratic space with $< v_{0,+}, v_{0,+} >=< v_{0,-}, v_{0,-} >= 0$ and $< v_{0,+}, v_{0,-} >= 1$, $V_1 = W_1 \oplus D$, $V_2 = W_2$, $V = V_1 \oplus V_2$, $\underline{G} = \mathrm{SO}(V)$, and $\underline{H} = \mathrm{SO}(V_1) \times \mathrm{SO}(V_2)$.

Let $D_+ = Span\{v_{0,+}\}$ and $D_- = Span\{v_{0,-}\}$. Then $D = D_+ \oplus D_-$ as a vector space. Let $\underline{P} = \underline{M}\underline{N}$ be the maximal parabolic subgroup of G that stabilizes the subspace D_+ with $\underline{M}$ be the Levi subgroup that stabilizes the subspaces D_+, W and D_-. Then $\underline{M} = \mathrm{SO}(W) \times \mathrm{GL}_1$.

Let $\pi = \otimes_{v\in|k|}\pi_v$ be a cuspidal automorphic representation of $G(\mathbb{A})$. Then $\pi \otimes 1$ is a cuspidal automorphic representation of $\underline{M}(\mathbb{A})$. To simplify the notation, we will still use π to denote this cuspidal automorphic representation. For $\phi \in \mathcal{A}_\pi$ and $s \in \mathbb{C}$, let $E(\phi, s)$ be the Eisenstein series on $\underline{G}(\mathbb{A})$.

Theorem 5.1. *Assume that there exists a local non-archimedean place $v \in |k|$ such that π_v is a generic representation of $G(k_v)$ (in particular, $G(k_v)$ is quasi-split). If the period integral $\mathcal{P}_H(\cdot)$ is nonzero on the space of π, then there exists $\phi \in \mathcal{A}_\pi$ such that the Eisenstein series $E(\phi, s)$ has a pole at $s = 1$.*

Proof. The proof is very similar to the proof of Theorem 4.1, we will skip it here. The only thing worth to point out is that in the case of Theorem 4.1, the constant $-c(1-2c_{\underline{P}}^{\underline{H}}) = -\frac{2n+1}{2}(1-\frac{2(n+1)}{2n+1})$ is equal to 1/2 and this is why we can show that the Eisenstein series $E(\phi, s)$ has a pole at $s = 1/2$. For the current case, the constant $-c(1 - 2c_{\underline{P}}^{\underline{H}}) = -\frac{2n}{2}(1 - \frac{2(n+1)}{2n})$ is equal to 1. This is why we can show that the Eisenstein series $E(\phi, s)$ has a pole at $s = 1/2$. □

Remark 5.2. *As in the previous case, when W_2 is not anisotropic, the set $\mathcal{F}^{\underline{G}}(\underline{P}_{0,H}, \underline{P})$ will contain two elements $\underline{P}$ and $\underline{P}'$. And one can easily show that $c(1 - 2c_{\underline{P}'}^{\underline{H}}) = \frac{2n}{2}(1 - 2\frac{n-3}{2n}) = 3 > s_0 = 1$. This confirms the discussion in Section 3.2.1.*

Remark 5.3. *In* [GRS97] *the existence of pole at $s = 1$ of $L(s, \pi)$ for π a generic cuspidal representation of SO_{2n} is linked to a different (non-reductive period) and also to the so called first occurrence problem in theta correspondence.*

Proposition 5.4. *Theorem 5.1 implies Theorem 1.3.*

Proof. We first recall the statement of Theorem 1.3. Let $G = \mathrm{SO}_{2n}$ be the split even orthogonal group, $H = \mathrm{SO}_{n+1} \times \mathrm{SO}_{n-1}$ be a closed subgroup of G (not necessarily split), and π be a generic cuspidal automorphic representation of $G(\mathbb{A})$. If the period integral $\mathcal{P}_H(\cdot)$ is nonzero on the space of π, we need to show that the L-function $L(s, \pi, \rho_X)$ has a pole at $s = 1$. Here ρ_X is the standard representation of ${}^LG = \mathrm{SO}_{2n}(\mathbb{C})$.

By Theorem 5.1, if the period integral $\mathcal{P}_H(\cdot)$ is nonzero on the space of π, there exists $\phi \in \mathcal{A}_\pi$ such that the Eisenstein series $E(\phi, s)$ has a pole at $s = 1$. In this case, the normalizing factor of the intertwining operator is

$$\frac{L(s, \pi, \rho_X)}{L(s+1, \pi, \rho_X)}.$$

By Theorem 4.7 of [KK11], the normalized intertwining operator is holomorphic at $s = 1$. By Proposition 2.8, $L(2, \pi, \rho_X) \neq 0$. It follows that the numerator $L(s, \pi, \rho_X)$ has a pole at $s = 1$. This proves Theorem 1.3. □

Remark 5.5. *By the same argument, we can also prove Theorem 1.3 when G is quasi-split.*

5.2. **The local result.** Let F be a p-adic field, and $\underline{G}, \underline{H}, \underline{P} = \underline{MN}, G, H$ be the groups defined in the previous subsection. Let π be an irreducible smooth representation of $G(F)$. We can view π as an irreducible smooth representation of $\underline{M}(F) \simeq G(F) \times \mathrm{GL}_1(F)$ by making it trivial on $\mathrm{GL}_1(F)$. By abusing of notation, we still use π to denote such representation. We also extend π to $\underline{P}(F)$ by making it trivial on $\underline{N}(F)$. For $s \in \mathbb{C}$, we use π_s to denote the representation $\pi \otimes \varpi^s$. Here $\varpi = \varpi_{\underline{P}} \in \mathfrak{a}_{\underline{M}}^*$ is the fundamental weight associated to $\underline{P}$, and ϖ^s is the character of $\underline{M}(F)$ defined by

$$\varpi^s(m) = e^{\langle s\varpi, H_{\underline{M}}(m)\rangle},\ m \in \underline{M}(F).$$

Let $I_{\underline{P}}^{\underline{G}}(\cdot)$ be the normalized parabolic induction. In other words,

$$I_{\underline{P}}^{\underline{G}}(\pi) = \{f : \underline{G}(F) \to \pi |\ f \text{ locally constant},\ f(nmg) = \delta_{\underline{P}}(m)^{1/2} \cdot \pi(m) f(g), \\ \forall m \in \underline{M}(F), n \in \underline{N}(F),\ g \in \underline{G}(F)\},$$

and the $\underline{G}(F)$-action is the right translation. The goal of this section is to prove the following theorem.

Theorem 5.6. *If π is an irreducible representation of $G(F)$ such that the Hom space*

$$\mathrm{Hom}_{H(F)}(\pi, 1)$$

is nonzero, then the representation $I_{\underline{P}}^{\underline{G}}(\pi_1)$ is $\underline{H}(F)$-distinguished, i.e. $\mathrm{Hom}_{\underline{H}(F)}(I_{\underline{P}}^{\underline{G}}(\pi_1), 1) \neq \{0\}$.

Before we prove the theorem, we first show that Theorem 5.6 implies Theorem 1.4.

Proposition 5.7. *Theorem 5.6 implies Theorem 1.4.*

Proof. Assume that G is split. Let π be a generic tempered representation of $G(F)$ such that the Hom space

$$\mathrm{Hom}_{H(F)}(\pi, 1)$$

is nonzero, we need to show that the local L-function $L(s, \pi, \rho_X)$ has a pole at $s = 0$. Here ρ_X is the standard L-function of ${}^L G = \mathrm{SO}_{2n}(\mathbb{C})$.

By Theorem 5.6, we know that the induced representation $I_{\underline{P}}^{\underline{G}}(\pi_1)$ is $\underline{H}(F)$-distinguished. If $I_{\underline{P}}^{\underline{G}}(\pi_1)$ is irreducible, then it is generic since π is generic. On the other hand, by Corollary 3.15(1), we know that there is no $\underline{H}(F)$-distinguished generic representation of $\underline{G}(F)$. This is a contradiction and hence we know that $I_{\underline{P}}^{\underline{G}}(\pi_1)$ is reducible.

By Lemma B.2 of [GI16] and the Standard Module Conjecture [HO13], we have that $I_{\underline{P}}^{\underline{G}}(\pi_1)$ is reducible if and only if the local gamma factor $\gamma(s, \pi, \rho_X) = \epsilon(s, \pi, \rho_X) \frac{L(1-s, \pi, \rho_X)}{L(s, \pi, \rho_X)}$ has a pole at $s = 1$ (with respect to any non-trivial additive character ψ since it doesn't change the existence or not of a pole at $s = 1$). Since π is tempered, $L(s, \pi, \rho_X)$ is holomorphic and nonzero when $Re(s) > 0$ (Theorem 1.1 of [HO13]), which implies that the L-function $L(s, \pi, \rho_X)$ has a pole at $s = 0$. This proves the proposition. □

Remark 5.8. *By the same argument, we can also prove Theorem 1.4 when G is quasi-split.*

For the rest of this section, we will prove Theorem 5.6. Let $P_{\underline{H}} = \underline{H} \cap \underline{P}$, $M_{\underline{H}} = \underline{H} \cap \underline{M}$, and $N_{\underline{H}} = \underline{H} \cap \underline{N}$. Then $P_{\underline{H}} = M_{\underline{H}} N_{\underline{H}}$ is a maximal parabolic subgroup of $\underline{H}$ with $M_{\underline{H}} \simeq H \times \mathrm{GL}_1$. We need two lemmas.

Lemma 5.9. *$\underline{P}(F)\underline{H}(F)$ is a closed subset of $\underline{G}(F)$.*

Proof. This follows from Corollary 2.10. □

Lemma 5.10. *We have the following equality of characters of $M_{\underline{H}}(F)$.*

$$\delta_{\underline{P}}^{-1/2}\delta_{P_{\underline{H}}} = \varpi.$$

Here $\delta_{\underline{P}}$ and ϖ are characters of $\underline{M}(F)$, and we view them as characters of $M_{\underline{H}}(F)$ by restriction.

Proof. By the definition of the constants c and $c_{\underline{P}}^{H}$, we have

$$\delta_{\underline{P}} = \varpi^{2c},\ \delta_{P_{\underline{H}}} = \varpi^{2cc_{\underline{P}}^{H}}.$$

This implies that

$$\delta_{\underline{P}}^{-1/2}\delta_{P_{\underline{H}}} = \varpi^{-c+2cc_{\underline{P}}^{H}} = \varpi^{(-n)+2n\cdot\frac{n+1}{2n}} = \varpi.$$

□

Remark 5.11. *The statement in the lemma above is equivalent to the equality*

$$s_0 = -c(1-2c_{\underline{P}}^{H}) = 1.$$

Now we are ready to prove the theorem. Let $\underline{G}(F)_0 = \underline{G}(F) - \underline{P}(F)\underline{H}(F)$. By Lemma 5.9, it is an open subset of $\underline{G}(F)$. We realize the representation $I_{\underline{P}}^{G}(\pi_1)$ on the space

$$\begin{aligned} I_{\underline{P}}^{G}(\pi_1) &= \{f : \underline{G}(F) \to \pi |\ f \text{ locally constant},\ f(nmg) = \delta_{\underline{P}}(m)^{1/2}\varpi(m)\cdot\pi(m)f(g), \\ &\quad \forall m \in \underline{M}(F), n \in \underline{N}(F),\ g \in \underline{G}(F)\} \end{aligned}$$

with the $G(F)$-action given by the right translation. Let V' be the subspace of $I_{\underline{P}}^{G}(\pi_1)$ consisting of all the functions whose support is contained in $\underline{G}(F)_0$. Then we know that V' is $\underline{H}(F)$-invariant. Moreover, as a representation of $\underline{H}(F)$, we have

$$I_{\underline{P}}^{G}(\pi_1)/V' \simeq ind_{P_{\underline{H}}}^{H}(\delta_{\underline{P}}^{1/2}\varpi\pi)$$

where *ind* is the compact induction. By the reciprocity law and Lemma 5.10, we have

$$\begin{aligned} \mathrm{Hom}_{\underline{H}(F)}(I_{\underline{P}}^{G}(\pi_1)/V', 1) &\simeq \mathrm{Hom}_{P_{\underline{H}}(F)}(\delta_{\underline{P}}^{1/2}\varpi\pi, \delta_{P_{\underline{H}}}) = \mathrm{Hom}_{P_{\underline{H}}(F)}(\pi, 1) \\ &= \mathrm{Hom}_{M_{\underline{H}}(F)}(\pi, 1) = \mathrm{Hom}_{H(F)}(\pi, 1) \neq \{0\}. \end{aligned}$$

This implies that $\mathrm{Hom}_{\underline{H}(F)}(I_{\underline{P}}^{G}(\pi_1), 1) \neq \{0\}$ and finishes the proof of Theorem 5.6.

6. The model $(\mathrm{U}_{2n}, \mathrm{U}_n \times \mathrm{U}_n)$

Let k'/k be a quadratic extension, W_1 and W_2 be two Hermitian spaces of dimension n, and $W = W_1 \oplus W_2$. Let $G = \mathrm{U}(W)$ and $H = \mathrm{U}(W_1) \times \mathrm{U}(W_2)$. Let $D = Span\{v_{0,+}, v_{0,-}\}$ be a two-dimensional Hermitian space with $< v_{0,+}, v_{0,+} >=< v_{0,-}, v_{0,-} >= 0$ and $< v_{0,+}, v_{0,-} >= 1$, $V_1 = W_1 \oplus D$, $V_2 = W_2$, $V = V_1 \oplus V_2$, $\underline{G} = \mathrm{U}(V)$, and $\underline{H} = \mathrm{U}(V_1) \times \mathrm{U}(V_2)$.

Let $D_+ = Span\{v_{0,+}\}$ and $D_- = Span\{v_{0,-}\}$. Then $D = D_+ \oplus D_-$ as a vector space. Let $\underline{P} = \underline{M}\underline{N}$ be the maximal parabolic subgroup of $\underline{G}$ that stabilizes the subspace D_+ with $\underline{M}$ be the Levi subgroup that stabilizes the subspaces D_+, W and D_-. Then $\underline{M} = \mathrm{U}(W) \times Res_{k'/k}\mathrm{GL}_1$.

Let $\pi=\otimes_{v\in|k|}\pi_v$ be a cuspidal automorphic representation of $G(\mathbb{A})$ with trivial central character. Then $\pi\otimes 1$ is a cuspidal automorphic representation of $\underline{M}(\mathbb{A})$. To simplify the notation, we will still use π to denote this cuspidal automorphic representation. For $\phi\in\mathcal{A}_\pi$ and $s\in\mathbb{C}$, let $E(\phi,s)$ be the Eisenstein series on $\underline{G}(\mathbb{A})$. Let Π be the base change of π to $\mathrm{GL}_{2n}(\mathbb{A}_{k'})$.

Theorem 6.1. *Assume that there exists a local non-archimedean place $v\in|k|$ such that π_v is a generic representation of $G(k_v)$. If the period integral $\mathcal{P}_H(\cdot)$ is nonzero on the space of π, then there exists $\phi\in\mathcal{A}_\pi$ such that the Eisenstein series $E(\phi,s)$ has a pole at $s=1/2$.*

Proof. The proof is very similar to the orthogonal group case (Theorem 4.1), we will skip it here. The only thing worth to mention is the computation of the constant $s_0=-c(1-2c_{\underline{P}}^{H})$. By Proposition 2.1(3), we have $c=\frac{2n+1}{2}$. On the other hand, although the unipotent radical $\underline{N}$ is not abelian in this case, it is easy to see that $c_{\underline{P}}^{H}=\frac{n+1}{2n+1}$. As a result, we have

$$s_0=-c(1-2c_{\underline{P}}^{H})=-\frac{2n+1}{2}(1-\frac{2(n+1)}{2n+1})=1/2.$$

□

Remark 6.2. *As in the previous cases, when W_2 is not anisotropic, the set $\mathcal{F}^{\underline{G}}(\underline{P}_{0,H},\underline{P})$ will contain two elements $\underline{P}$ and $\underline{P}'$. And one can easily show that $c(1-2c_{\underline{P}'}^{H})=\frac{2n+1}{2}(1-2\frac{n-1}{2n+1})=\frac{3}{2}>s_0=\frac{1}{2}$. This confirms the discussion in Section 3.2.1.*

The next proposition proves the first part of Theorem 1.5.

Proposition 6.3. *Assume that G is quasi-split and π is generic. If the period integral $\mathcal{P}_H(\cdot)$ is nonzero on the space of π, then the standard L-function $L(s,\pi)$ is nonzero at $s=1/2$.*

Proof. By the Theorem 6.1, if the period integral $\mathcal{P}_H(\cdot)$ is nonzero on the space of π, there exists $\phi\in\mathcal{A}_\pi$ such that the Eisenstein series $E(\phi,s)$ has a pole at $s=1/2$. In this case, the normalizing factor of the intertwining operator is (Section 2.1 and 2.2 of [KK11])

$$\frac{L(s,\pi)\zeta_k(2s)}{L(s+1,\pi)\zeta_k(2s+1)}$$

where $\zeta_k(s)$ is the Dedekind zeta function. By Theorem 4.7 of [KK11], the normalized intertwining operator is holomorphic at $s=1/2$. By Proposition 2.8, $L(3/2,\pi)\neq 0$. It follows that the numerator $L(s,\pi)\zeta_k(2s)$ has a pole at $s=1/2$, which implies that $L(\frac{1}{2},\pi)\neq 0$. This proves Theorem 1.2. □

Now it remains to prove the second part of Theorem 1.5. We first recall the statement. Assume that Π is cuspidal. Also assume that there exists a local place $v_0\in|k|$ such that k'/k splits at v_0 and π_{v_0} is a discrete series of $G(k_{v_0})=\mathrm{GL}_{2n}(k_{v_0})$. We need to show that if the period integral $\mathcal{P}_H(\cdot)$ is nonzero on the space of π, then the exterior square L-function $L(s,\Pi,\wedge^2)$ has a pole at $s=1$ (i.e. Π is of symplectic type).

We first show that Π is self-dual. Let $\pi=\otimes_{v\in|k|}\pi_v$. By the automorphic Cebotarev density theorem proved in [Ram15], in order to show that Π is self-dual, it is enough to show that π_v is self-dual for all the non-archimedean places $v\in|k|$ such that the quadratic extension k'/k splits at v. We fix such a local place v. Then π_v is an irreducible smooth representation of $G(k_v)=\mathrm{GL}_{2n}(k_v)$. Since the period integral $\mathcal{P}_H(\cdot)$ is nonzero on the space of π, we know that locally the Hom space

$$\mathrm{Hom}_{H(k_v)}(\pi_v,1)$$

is nonzero. By Theorem 1.1 of [JR96], we know that π_v is self-dual. This proves that Π is self-dual. Since Π is cuspidal, this implies that Π is either of symplectic type or of orthogonal type.

Now in order to show that Π is of symplectic type, it is enough to show that at the split place $v_0\in|k|$, π_{v_0} is not of orthogonal type. By our assumption, π_{v_0} is a discrete series of

$G(k_{v_0}) = \mathrm{GL}_{2n}(k_{v_0})$, hence it is enough to show that π_{v_0} is of symplectic type (this is because a discrete series of GL_{2n} can not be of symplectic type and orthogonal type at the same time).

By the discussion above, we know that the Hom space $\mathrm{Hom}_{H(k_{v_0})}(\pi_{v_0}, 1)$ is nonzero. In other words, π_{v_0} is distinguished by the linear model. By Theorem 5.1 of [Mat14], we know that π_{v_0} is distinguished by the Shalika model. Then by Proposition 3.4 of [LM17], we know that π_{v_0} is of symplectic type. This finishes the proof of Theorem 1.5.

7. The Jacquet-Guo model

7.1. **The global result.** Let $k' = k(\sqrt{\alpha})$ be a quadratic extension of k with $i = \sqrt{\alpha}$. Let W be a k'-vector space of dimension $2n$. Fix a basis $\{w_1, \cdots, w_{2n}\}$ of W. We define a nondegenerate skew-symmetric k'-bilinear form B on W to be

$$B(w_j, w_k) = \delta_{j+k-1,2n},\ B(w_l, w_k) = -\delta_{l+k-1,2n},\ 1 \leq j \leq n, n+1 \leq l \leq 2n, 1 \leq k \leq 2n.$$

In other words, in terms of the basis $\{w_1, \cdots, w_{2n}\}$, B is defined by the skew-symmetric matrix

$$J_{2n} = \begin{pmatrix} 0 & w_n \\ -w_n & 0 \end{pmatrix}$$

where w_n is the longest Weyl element in GL_n. Then we define the symplectic group $\underline{H} = \mathrm{Sp}(W, B)$. In other words, $\underline{H} = Res_{k'/k}\mathrm{Sp}_{2n}$.

Now we define the group $\underline{G}$. View W as a k-vector space of dimension $4n$. Then $\{w_1, iw_1, \cdots, w_{2n}, iw_{2n}\}$ is a basis of W. We define a non-degenerate skew-symmetric k-bilinear form B_k on W to be

$$B_k(v_1, v_2) = \mathrm{tr}_{k'/k}(B(v_1, v_2)),\ v_1, v_2 \in W.$$

Then we define $\underline{G} = \mathrm{Sp}(W, B_F)$ (i.e. $\underline{G} = \mathrm{Sp}_{4n}$). We have $\underline{H} \subset \underline{G}$. For $1 \leq j \leq 2n$, let W_j be the k'-subspace of W spanned by w_j, $W_{j,+}$ (resp. $W_{j,-}$) be the k-subspace of W spanned by w_j (resp. iw_j).

Let $\underline{P} = \underline{MN}$ be the Siegel parabolic subgroup of $\underline{G}$ that stabilizes the k-subspace $Span_k\{w_j, iw_j|\ 1 \leq j \leq n\}$, and $\underline{M}$ be the Levi subgroup that stabilizes the k-subspaces $Span_k\{w_j, iw_j|\ 1 \leq j \leq n\}$ and $Span_k\{w_j, iw_j|\ n+1 \leq j \leq 2n\}$. On the mean time, let $P_{\underline{H}} = M_{\underline{H}}N_{\underline{H}}$ be the Siegel parabolic subgroup of $\underline{H}$ that stabilizes the k'-subspace $Span_{k'}\{w_1, \cdots, w_n\}$, and $M_{\underline{H}}$ be the Levi subgroup that stabilizes the k'-subspaces $Span_{k'}\{w_1, \cdots, w_n\}$ and $Span_{k'}\{w_{n+1}, \cdots, w_{2n}\}$. Then it is easy to see that $\underline{P} \cap \underline{H} = P_{\underline{H}}$, $\underline{M} \cap \underline{H} = M_{\underline{H}}$ and $\underline{N} \cap \underline{H} = N_{\underline{H}}$. We let $G = \underline{M} = \mathrm{GL}_{2n}$ and $H = M_{\underline{H}} = Res_{k'/k}\mathrm{GL}_n$.

Let $\pi = \otimes_{v\in|k|}\pi_v$ be a cuspidal automorphic representation of $G(\mathbb{A}) = \underline{M}(\mathbb{A})$ with trivial central character. For $\phi \in \mathcal{A}_\pi$ and $s \in \mathbb{C}$, let $E(\phi, s)$ be the Eisenstein series on $\underline{G}(\mathbb{A})$. The goal of this section is to prove the following theorem.

Theorem 7.1. *If the period integral $\mathcal{P}_H(\cdot)$ is nonzero on the space of π, then there exists $\phi \in \mathcal{A}_\pi$ such that the Eisenstein series $E(\phi, s)$ has a pole at $s = 1/2$.*

Remark 7.2. *Since $G = \mathrm{GL}_{2n}$, all the cuspidal automorphic representations of $G(\mathbb{A})$ are generic.*

Theorem 7.1 will be proved in Section 7.3. The next proposition shows that Theorem 1.8 follows from Theorem 7.1.

Proposition 7.3. *Theorem 7.1 implies Theorem 1.8.*

Proof. We need to show that if the period integral $\mathcal{P}_H(\cdot)$ is nonzero on the space of π, then the L-function $L(s, \pi, \rho_{X,1})$ is nonzero at $s = 1/2$ and the L-function $L(s, \pi, \rho_{X,2})$ has a pole at $s = 1$. Here $\rho_{X,1}$ (resp. $\rho_{X,2}$) is the standard representation (resp. exterior square representation) of ${}^L\underline{M} = {}^LG = \mathrm{GL}_{2n}(\mathbb{C})$.

By Theorem 7.1, if the period integral $\mathcal{P}_H(\cdot)$ is nonzero on the space of π, there exists $\phi \in \mathcal{A}_\pi$ such that the Eisenstein series $E(\phi, s)$ has a pole at $s = 1/2$. In this case, the normalizing factor of the intertwining operator is

$$\frac{L(s,\pi,\rho_{X,1})L(2s,\pi,\rho_{X,2})}{L(s+1,\pi,\rho_{X,1})L(2s+1,\pi,\rho_{X,2})}.$$

By Theorem 4.7 of [KK11], the normalized intertwining operator is holomorphic at $s = 1/2$. By Proposition 2.8, $L(3/2,\pi,\rho_{X,1})L(2,\pi,\rho_{X,2}) \neq 0$. It follows that the numerator $L(s,\pi,\rho_{X,1})L(2s,\pi,\rho_{X,2})$ has a pole at $s = 1/2$, which implies that the L-function $L(s,\pi,\rho_{X,1})$ is nonzero at $s = 1/2$ and the L-function $L(s,\pi,\rho_{X,2})$ has a pole at $s = 1$. □

7.2. **The parabolic subgroups.** Let $\underline{B} = \underline{A_0}\underline{N_0}$ be the Borel subgroup of $\underline{G}$ that stabilizes the filtration

$$Span_k\{w_1\} \subset Span_k\{w_1, iw_1\} \subset Span_k\{w_1, w_2, iw_1\} \subset \cdots \subset Span_k\{w_1, \cdots, w_n, iw_1, \cdots, iw_n\},$$

and $\underline{A_0}$ be the maximal torus of $\underline{G}$ that stabilizes the subspaces $W_{j,+}$ and $W_{j,-}$ for $1 \leq j \leq 2n$. Then we have $\underline{B} \subset \underline{P}$ and $\underline{A_0} \subset \underline{M}$. We identify $\underline{A_0}$ with $(\mathrm{GL}_1)^{2n}$ under the natural isomorphism

$$\underline{A_0} \simeq \mathrm{GL}(W_{1,+}) \times \mathrm{GL}(W_{1,-}) \times \cdots \times \mathrm{GL}(W_{n,+}) \times \mathrm{GL}(W_{n,-}). \tag{7.1}$$

As in the $(\mathrm{SO}_{2n+1}, \mathrm{SO}_{n+1} \times \mathrm{SO}_n)$-case, we use $\mathcal{F}(\underline{A_0}, \underline{P})$ to denote the set of semi-standard parabolic subgroups $\underline{Q} \in \mathcal{F}(\underline{A_0})$ of $\underline{G}$ that are conjugated to $\underline{P}$. The following proposition gives a description of the set $\mathcal{F}(\underline{A_0}, \underline{P})$.

Proposition 7.4. *Let S be the set $\{(a_1, \cdots, a_{2n}) |\ a_j = \pm 1\}$. Then there is a natural bijection between S and $\mathcal{F}(\underline{A_0}, \underline{P})$ given as follows. For $\underline{a} = (a_1, \cdots, a_{2n}) \in S$, $\underline{P_{\underline{a}}}$ will be the Siegel parabolic subgroup of $\underline{G}$ that stabilizes the k-subspace*

$$Span_k\{v_{(-1)^{a_j}j} \mid 1 \leq j \leq 2n\}.$$

Here for $1 \leq j \leq 2n$, $v_j = w_{\frac{j}{2}}$ and $v_{-j} = w_{2n+1-\frac{j}{2}}$ if j is even; $v_j = iw_{\frac{j+1}{2}}$ and $v_{-j} = iw_{2n+1-\frac{j+1}{2}}$ if j is odd. In particular $\underline{P}$ corresponds to the element $(0, 0, \cdots, 0)$ in S.

On the mean time, let $\underline{B_H} = \underline{T_H}\underline{N_{0,H}}$ be the Borel subgroup of $\underline{H}$ that stabilizes the filtration

$$Span_{k'}\{w_1\} \subset Span_{k'}\{w_1, w_2\} \subset \cdots \subset Span_{k'}\{w_1, \cdots, w_n\},$$

and $\underline{T_H}$ be the maximal torus of $\underline{G}$ that stabilizes the subspaces W_j for $1 \leq j \leq 2n$. Let $\underline{A_{0,H}} = \underline{T_H} \cap \underline{A_0}$. Then $\underline{A_{0,H}}$ is a maximal split torus of $\underline{H}$. Under the isomorphism (7.1), we have

$$\underline{A_{0,H}} \simeq (\mathrm{GL}(W_{1,+}) \times \mathrm{GL}(W_{1,-}))^{diag} \times \cdots \times (\mathrm{GL}(W_{n,+}) \times \mathrm{GL}(W_{n,-}))^{diag}$$

where $(\mathrm{GL}(W_{j,+}) \times \mathrm{GL}(W_{j,-}))^{diag}$ is the set of elements of $\mathrm{GL}(W_{j,+}) \times \mathrm{GL}(W_{j,-})$ that act by scalar on $W_{j,+} \oplus W_{j,-}$ for $1 \leq j \leq n$.

The following corollary is a direct consequence of the discussions above together with the definition of the set $\mathcal{F}^{\underline{G}}(\underline{B_H}, \underline{P})$ in Section 2.6.

Corollary 7.5. *With the notations above, the set $\mathcal{F}^{\underline{G}}(\underline{B_H}, \underline{P})$ only contains one element $\underline{P}$.*

Remark 7.6. *The corollary above confirms the discussion in Section 3.2.1.*

7.3. **The proof of Theorem 7.1.** In this section, we will prove Theorem 7.1. We use Method 1 introduced in Section 3.2.

With the same notations as in Theorem 7.1 and Section 3.2, we want to study the regularized period integral $\mathcal{P}_{\underline{H},reg}(E(\phi,s))$ for $\phi\in\mathcal{A}_\pi$. First, let's prove statement (1) in Section 3.2 for the current case (there is no need to prove statement (2) of Section 3.2 since in the current case the set $\mathcal{F}^{\underline{G}}(\underline{B}_H,\underline{P})$ only contains one element $\underline{P}$). For (1), since π is generic, together with the argument in Section 3.2, it is enough to show that statement (4) of Section 3.2 holds for the pair $(\underline{G},\underline{H})$. But this just follows from Corollary 3.15(4).

Then we compute the constant $s_0=-c(1-2c_{\underline{P}}^{\underline{H}})$ for the current case. By Remark 2.6, we have

$$c_{\underline{P}}^{\underline{H}}=\frac{\dim(N_{\underline{H}})}{\dim \underline{N}}=\frac{n+1}{2n+1}.$$

By Proposition 2.1, we have $c=\frac{2n+1}{2}$. This implies that

$$s_0=-c(1-2c_{\underline{P}}^{\underline{H}})=1/2.$$

Combining the discussions above, equation (3.2) in Method 1 becomes

$$\begin{aligned}\int_{[\underline{H}]}\Lambda^{T,\underline{H}}Res_{s=1/2}E(h,\phi,s)dh=&\int_{K_{\underline{H}}}\int_{[H]/Z_G(\mathbb{A})}\phi(hk)dhdk\\&-e^{\langle-\varpi_{\underline{P}},T\rangle}\int_{K_{\underline{H}}}\int_{[H]/Z_G(\mathbb{A})}Res_{s=1/2}M(s)\phi(hk)dhdk\end{aligned}$$

for the current case. This finishes the proof of Theorem 7.1.

7.4. **The local result.** Let F be a p-adic field, and E/F be a quadratic extension. As in the previous subsections, we can define the groups $\underline{G},\underline{H},\underline{P}=\underline{M}\underline{N},G,H$ over F. Let π be an irreducible smooth representation of $G(F)=\underline{M}(F)=\mathrm{GL}_{2n}(F)$. We extend π to $\underline{P}(F)$ by making it trivial on $\underline{N}(F)$. As in Section 5.2, for $s\in\mathbb{C}$, we use π_s to denote the representation $\pi\otimes\varpi^s$ and use $I_{\underline{P}}^{\underline{G}}(\pi_s)$ to denote the normalized parabolic induction.

Theorem 7.7. *If π is an irreducible representation of $G(F)$ such that the Hom space*

$$\mathrm{Hom}_{H(F)}(\pi,1)$$

is nonzero, then the representation $I_{\underline{P}}^{\underline{G}}(\pi_{\frac12})$ is $\underline{H}(F)$-distinguished, i.e. $\mathrm{Hom}_{\underline{H}(F)}(I_{\underline{P}}^{\underline{G}}(\pi_{\frac12}),1)\neq\{0\}$.

Theorem 7.7 will follows from the exact same argument as the proof of Theorem 5.6 once we have proved the following lemma which is the analogue 5.10.

Lemma 7.8. *We have the following equality of characters of $M_{\underline{H}}(F)$.*

$$\delta_{\underline{P}}^{-1/2}\delta_{P_{\underline{H}}}=\varpi^{1/2}.$$

Here $\delta_{\underline{P}}$ and ϖ are characters of $\underline{M}(F)$, and we view them as characters of $M_{\underline{H}}(F)$ by restriction.

Proof. By the same argument as in the proof of Lemma 5.10, we have

$$\delta_{\underline{P}}^{-1/2}\delta_{P_{\underline{H}}}=\varpi^{-c+2cc_{\underline{P}}^{\underline{H}}}=\varpi^{\frac{-2n-1}{2}+\frac{2(2n+1)}{2}\cdot\frac{n+1}{2n+1}}=\varpi^{1/2}.$$

This proves the lemma. □

Now we are ready to prove Theorem 1.10. Let π be a tempered representation of $G(F)$ with trivial central character (in particular, π is generic since $G=\mathrm{GL}_{2n}$). Assume that the Hom space $\mathrm{Hom}_{H(F)}(\pi,1)$ is nonzero, we need to show that the local exterior square L-function $L(s,\pi,\rho_{X,2})$ has a pole at $s=0$. By the same argument as in Proposition 5.7, we know that the induced representation $I_{\underline{P}}^{\underline{G}}(\pi_{\frac{1}{2}})$ is reducible. Again by applying Lemma B.2 of [GI16] and the Standard Module Conjecture [HO13], we have that $I_{\underline{P}}^{\underline{G}}(\pi_{\frac{1}{2}})$ is reducible if and only if the local gamma factor

$$\gamma(s,\pi,\rho_{X,1})\gamma(2s,\pi,\rho_{X,2})=\epsilon(s,\pi,\rho_{X,1})\epsilon(2s,\pi,\rho_{X,1})\frac{L(1-s,\pi,\rho_{X,1})L(1-2s,\pi,\rho_{X,2})}{L(s,\pi,\rho_{X,1})L(2s,\pi,\rho_{X,2})}$$

has a pole at $s=\frac{1}{2}$. Since π is tempered, $L(s,\pi,\rho_{X,1})$ and $L(s,\pi,\rho_{X,2})$ are holomorphic and nonzero when $Re(s)>0$ (Theorem 1.1 of [HO13]), which implies that the L-function $L(s,\pi,\rho_{X,2})$ has a pole at $s=0$. This finishes the proof of Theorem 1.10.

8. The model $(\mathrm{GE}_6, A_1\times A_5)$

8.1. **The result.** Fix a quaternion algebra B over the number field k. Denote by $J_B=H_3(B)$ the Hermitian 3×3 matrices over B. Let $\Theta=B\oplus B$ be an octonion algebra over k defined via the Cayley-Dickson construction and denote by $J_\Theta=H_3(\Theta)$ the Hermitian 3×3 matrices over Θ. Then $\dim_k J_B=15$ and $\dim_k J_\Theta=27$; J_Θ is the exceptional cubic norm structure. The Cayley-Dickson construction induces an identification $J_\Theta=J_B\oplus B^3$.

Let $G=GE_6$ be the group preserving the cubic norm on J_Θ up to similitude. Let

$$H=(\mathrm{GL}_1(B)\times\mathrm{GL}_3(B))^0=\{(x,g)\in\mathrm{GL}_1(B)\times\mathrm{GL}_3(B):n_B(x)=N_6(g)\}$$

where n_B, resp. N_6, denotes the reduced norm on B (of degree two), resp. on $M_3(B)$ (of degree six). In this section, we will consider H-periods of cusp forms on G.

Denote by $\underline{G}$ the semisimple, simply-connected group of type E_7 defined in terms of J_Θ. Precisely, $\underline{G}$ is the group preserving Freudenthal's symplectic and quartic form on $W_\Theta=k\oplus J_\Theta\oplus J_\Theta^\vee\oplus k$. Denote by $W_B:=k\oplus J_B\oplus J_B^\vee\oplus k$ the Freudenthal space associated to the cubic norm structure J_B. We write elements of W_B as ordered four-tuples (a,b,c,d), so that $a,d\in k$, $b\in J_B$ and $c\in J_B^\vee$, and similarly for W_Θ. The 32-dimensional space W_B affords one of the half-spin representations of a group of type D_6. The identification $J_\Theta=J_B\oplus B^3$ induces an identification $W_\Theta=W_B\oplus B^6$, which we will use to define $\underline{H}$ and the map $\underline{H}\to\underline{G}$.

In more detail, let $\underline{H}'$ be the subgroup of elements with similitude equal to 1 of the group denoted $\widetilde{G}$ in [Pol17, Appendix A]. The group $\underline{H}'$, by its definition in *loc cit*, comes equipped with maps to D_6^+ and $U_6(B)$. Here D_6^+ is the semisimple half-spin group of type D_6 whose defining representation is W_B and $U_6(B)$ is the subgroup of $\mathrm{GL}_6(B)$ satisfying $g\begin{pmatrix} & 1_3\\ -1_3 & \end{pmatrix}g^*=\begin{pmatrix} & 1_3\\ -1_3 & \end{pmatrix}$ where g^* is the transpose conjugate of g. The group $\underline{H}'$ acts on $W_\Theta=W_B\oplus B^6$ through these two maps, preserving the decomposition. Denote by B^1 the subgroup of $\mathrm{GL}_1(B)$ with reduced norm equal to 1. Let B^1 act on W_Θ by $x(w,v)=(w,xv)$ where $x\in B^1$, $w\in W_B$ and $v\in B^6$. This action commutes with the action of $\underline{H}'$ on W_Θ because $\underline{H}'$ acts on the right of B^6. We set $\underline{H}=B^1\times\underline{H}'$. The action of $\underline{H}$ on W_Θ defines a map $\underline{H}\to\underline{G}$. This map has a diagonal μ_2-kernel; we abuse notation and also let $\underline{H}$ denote the image of this map in $\underline{G}$.

Denote by $\underline{P}$ the subgroup of $\underline{G}$ that stabilizes the line $k(0,0,0,1)$ in W_Θ. The group $\underline{P}$ is a parabolic subgroup of $\underline{G}$ with reductive quotient of type E_6. Denote by $\underline{M}$ the subgroup of $\underline{P}$ that also fixes the line $k(1,0,0,0)$. Then $\underline{M}$ is a Levi subgroup of $\underline{P}$, and one has $\underline{P}=\underline{MN}$ with the unipotent radical $\underline{N}$ of $\underline{P}$ abelian; in fact, $\underline{N}\simeq J_\Theta$. The group $\underline{M}$ is isomorphic to the subgroup $\mathrm{GL}_1\times GE_6$ consisting of pairs (δ,m) with $\delta^2=\nu(m)$, where $\nu:GE_6\to\mathrm{GL}_1$ is the similitude. The map $\underline{M}\to\mathrm{GL}_1$ defined as $(\delta,m)\mapsto\delta$ is the fundamental weight of $\underline{M}$. One has that

$$\underline{M}\cap\underline{H}=\{(x,\delta,g)\in B^1\times\mathrm{GL}_1\times\mathrm{GL}_3(B):\delta^2=N_6(g)\}.$$

Note that the image of $\underline{M}\cap\underline{H}$ in PGE_6 is contained in the image of H in PGE_6.

Let $\pi=\otimes_{v\in|k|}\pi_v$ be a cuspidal automorphic representation of $G(\mathbb{A})$ with trivial central character. It can also be viewed as a cuspidal automorphic representation of $\underline{M}(\mathbb{A})$ with trivial central character. For $\phi \in \mathcal{A}_\pi$ and $s \in \mathbb{C}$, let $E(\phi, s)$ be the Eisenstein series on $\underline{G}(\mathbb{A})$. The goal of this section is to prove the following theorem.

Theorem 8.1. *Assume that there exists a local non-archimedean place $v \in |k|$ such that $G(k_v)$ is split and π_v is a generic representation of $G(k_v)$. If the period integral $\mathcal{P}_H(\cdot)$ is nonzero on the space of π, then there exists $\phi \in \mathcal{A}_\pi$ such that the Eisenstein series $E(\phi, s)$ has a pole at $s = 1$.*

Remark 8.2. *We will prove the theorem using Method 1. As we explained in Section 3.3, it is also possible to prove the theorem by Method 2, but it is more complicated since we need to study all the orbits in the double coset $\underline{P}\backslash\underline{G}/\underline{H}$. However, if the quaternion algebra B is not split, there are two elements in the double coset $\underline{P}\backslash\underline{G}/\underline{H}$, and one can prove the theorem by Method 2 even without the assumption of local genericity.*

Proposition 8.3. *Theorem 8.1 implies Theorem 1.11.*

Proof. Let us recall the statement of Theorem 1.11. Assume that G is split and π is a generic cuspidal automorphic representation of G with trivial central character. Assume that the period integral $\mathcal{P}_H(\cdot)$ is nonzero on the space of π and $L(2, \pi, \rho_X) \neq 0$. We need to show that the L-function $L(s, \pi, \rho_X)$ has a pole at $s = 1$. Here ρ_X is the a fundamental 27-dimensional representation of ${}^L PGE_6 = E_6^{sc}(\mathbb{C})$.

By Theorem 8.1, if the period integral $\mathcal{P}_H(\cdot)$ is nonzero on the space of π, there exists $\phi \in \mathcal{A}_\pi$ such that the Eisenstein series $E(\phi, s)$, and thus the intertwining operator $M(s)$, has a pole at $s = 1$. In this case, the normalizing factor of the intertwining operator is

$$\frac{L(s, \pi, \rho_X)}{L(s+1, \pi, \rho_X)}.$$

By Theorem 4.7 of [KK11], the normalized intertwining operator is holomorphic at $s = 1$. Since we have assumed that $L(2, \pi, \rho_X) \neq 0$, it follows that the numerator $L(s, \pi, \rho_X)$ has a pole at $s = 1$. This proves Theorem 1.11. □

8.2. **The parabolic subgroups.** Let $A_{0,\underline{H}}$ be a maximal split torus of $\underline{H}$, $M_{0,\underline{H}}$ its centralizer in $\underline{H}$, and $P_{0,\underline{H}}$ a minimal parabolic of $\underline{H}$ with $M_{0,\underline{H}}$ as Levi subgroup. We will specify a specific choice of $A_{0,\underline{H}}$ momentarily. Let $\underline{A}_0$ be a maximal split torus of $\underline{G}$ with $A_{0,\underline{H}} \subset \underline{A}_0$. In case $B = M_2(k)$ is split, we will specify $A_{0,\underline{H}}$ below; in this case, $\underline{G}$ is also split and we have $A_{0,\underline{H}} = \underline{A}_0$. When B is a division algebra, the choice of $\underline{A}_0$ will turn out to be irrelevant.

Let $\mathcal{F}(\underline{A}_0, \underline{P})$ be the set of semistandard parabolic subgroups of $\underline{G}$ (i.e. parabolic subgroups that contain $\underline{A}_0$) which are conjugate to $\underline{P}$. Let $\mathcal{F}(M_{0,\underline{H}}, \underline{P})$ be the set parabolic subgroups $\underline{Q} \in \mathcal{F}(\underline{A}_0, \underline{P})$ such that $M_{0,\underline{H}} \subset \underline{Q}$. Finally, denote by $\mathcal{F}'(M_{0,\underline{H}}, \underline{P})$ the set of parabolic subgroups $\underline{Q}$ of $\underline{G}$ conjugate to $\underline{P}$ such that $\underline{Q}$ contains $M_{0,\underline{H}}$. Thus $\mathcal{F}(M_{0,\underline{H}}, \underline{P}) \subseteq \mathcal{F}'(M_{0,\underline{H}}, \underline{P})$. The purpose of this subsection is to make explicit the set $\mathcal{F}'(M_{0,\underline{H}}, \underline{P})$. In case the quaternion algebra B is a division algebra, this set contains 8 elements; in case $B = M_2(k)$ is the split quaternion algebra, this set contains 56 elements.

We have inclusions $\mathrm{GL}_3 \to \mathrm{Sp}_6 \to \underline{H}$ where the first arrow is the Levi of the Siegel parabolic of Sp_6 and the composite arrow $\mathrm{GL}_3 \to \underline{H}$ is defined in [Pol17, Appendix A] in the second paragraph of page 1428. Denote by T_6 the image in $\underline{H}$ of the diagonal maximal torus of Sp_6 or GL_3. In case B is a division algebra, $T_6 = A_{0,\underline{H}}$ is a maximal split torus of $\underline{H}$. The action of T_6 on $W_\Theta = k \oplus J_\Theta \oplus J_\Theta^\vee \oplus k$ has 14 weight spaces. There are 8 one-dimensional spaces of the form

$$(1, 0, 0, 0), (0, e_{ii}, 0, 0), (0, 0, e_{ii}, 0), (0, 0, 0, 1)$$

where $1 \leq i \leq 3$ and e_{ii} is the element of $H_3(\Theta)$ with a 1 in the i^{th}-position on the diagonal and 0's elsewhere. Denote this above set of 8 elements of W_Θ or W_B by X_{long}. The action of T_6 on W_Θ has 6 eight-dimensional weight spaces of the form $(0, v_k(\Theta), 0, 0)$ and $(0, 0, v_k(\Theta), 0)$ for $1 \leq k \leq 3$. Here

$v_1(\Theta)$ is the eight-dimensional subspace of $H_3(\Theta)$ consisting of elements of the form $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & * \\ 0 & * & 0 \end{pmatrix}$ and analogously for $v_2(\Theta)$ and $v_3(\Theta)$. We denote[1] the set of these 6 eight-dimensional spaces by X^{Θ}_{short}.

Denote by M_0^B the centralizer of T_6 in $\underline{H}$. Note that M_0^B contains the B^1-component of $\underline{H} = B^1 \times \underline{H}'$. In case B is a division algebra, $M_0^B = M_{0,\underline{H}}$.

Lemma 8.4. *For $v \in X_{long}$, denote by $\underline{P}_v$ the subgroup of $\underline{G}$ that stabilizes the line kv. If B is a division algebra, then these eight subgroups $\underline{P}_v$ are the elements of $\mathcal{F}'(M_{0,\underline{H}}, \underline{P})$.*

Proof. The subgroups of $\underline{G}$ that are conjugate to $\underline{P}$ are those subgroups that stabilize a rank one line in W_Θ; see, e.g., [Pol18, section 4.3]. It is clear that the eight elements v of X_{long} are rank one, and thus each $\underline{P}_v$ is an element of $\mathcal{F}'(M_{0,\underline{H}}, \underline{P})$.

The lemma now follows immediately from the following claim:

Claim 8.5. *Suppose ℓ is a rank one line in W_Θ stabilized by M_0^B. Then $\ell = kv$ for some element $v \in X_{long}$.*

To prove the claim, we use the action of $B^1 \subseteq M_0^B$. Each of the 6 weight spaces in X^{Θ}_{short} is a copy of the octonion algebra Θ. The action of B^1 breaks Θ into $B \oplus B$, where B^1 acts on the first B by the identity and the second B by left-multiplication. Because B is a division algebra, the reduced norm on B is anisotropic, and thus no element of one of these B's in Θ can be rank one. Therefore, the only rank one lines in W_Θ stabilized by M_0^B are spanned by elements in X_{long}. This proves the claim, and with it, the lemma. □

Remark 8.6. *The subgroups $\underline{P}_v$ of Lemma 8.4 may be described as follows. Recall the maximal diagonal torus $T_6 \subseteq \mathrm{Sp}_6$, which we consider inside of $\underline{G}$. Recall that the Weyl group of Sp_6 is $(\pm 1)^3 \rtimes S_3$. Let $w_1, \ldots, w_8$ be elements of the normalizer of T_6 in Sp_6 that correspond to the elements $(\pm 1)^3$ of the Weyl group. Let $\lambda_i : \mathrm{GL}_1 \to T_6$ be the cocharacters $t \mapsto w_i(\begin{pmatrix} t1_3 & \\ & t^{-1}1_3 \end{pmatrix})$. Abusing notation, also denote by λ_i the cocharacter $\mathrm{GL}_1 \to T_6 \subseteq A_{0,\underline{H}} \subseteq \underline{A}_0$, where the map $\mathrm{GL}_1 \to T_6$ is the one just described. With this notation, the parabolic subgroups of Lemma 8.4 are those parabolic subgroups of $\underline{G}$ that correspond to the cocharacters λ_i. Consequently, $\underline{P}_v$ contains the centralizer of the image of λ_i for some i that depends upon v. In particular, note that each $\underline{P}_v$ contains $\underline{A}_0$ for any choice of maximal split torus $\underline{A}_0$ containing $A_{0,\underline{H}}$. Thus, in the statement of Lemma 8.4, one can replace the set $\mathcal{F}'(M_{0,\underline{H}}, \underline{P})$ with $\mathcal{F}(M_{0,\underline{H}}, \underline{P})$, if one so desires.*

We now suppose that $B = M_2(k)$ is the split quaternion algebra. In this case, the decomposition $W_\Theta = W_B \oplus B^6$ is $W_\Theta = W_B \oplus M_{2,12}(k)$ where $M_{2,12}(k)$ denotes the 2×12 matrices with entries in k, $B^1 = \mathrm{SL}_2$ acts by multiplication on the left and $\underline{H}'$ acts by multiplication on the right of $M_{2,12}$ via its map to $O(12)$.

Consider again the split torus T_6 in $\underline{H}'$. It acts on W_B with again 14 weight spaces, the 8 one-dimensional spaces in X_{long} and 6 four-dimensional spaces X^B_{short} of the form $(0, v_k(B), 0, 0)$ and $(0, 0, v_k(B), 0)$ for $k = 1, 2, 3$. Here $v_1(B)$ is the four-dimensional subspace of $H_3(B)$ consisting of elements of the form $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & * \\ 0 & * & 0 \end{pmatrix}$ and analogously for $v_2(B)$ and $v_3(B)$. We can[2] and do choose a maximal torus T' of $\underline{H}'$ so that

[1]The vector space W_Θ is the abelianization of the unipotent radical of the Heisenberg parabolic of a group of type E_8. This group has a certain relative root system of type F_4; the elements of X_{long} make up the long roots in W_Θ for this system, whereas the spaces in X^{Θ}_{short} make up the short root spaces, hence the "long" and "short" labels.

[2]Note that the half-spin representation of D_6 is miniscule

- T' contains T_6
- the action of T' on W_B breaks up each $v_k(B) \simeq B = M_2(k)$ into four weight spaces, which are spanned by the matrices $\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right)$ of $M_2(k)$.

Finally, if T'' denotes the one-dimensional diagonal torus of $\mathrm{SL}_2 = B^1$, we set $\underline{A}_0 = A_{0,\underline{H}} = T'' \times T'$. This is a maximal split torus of $\underline{H}$ and $\underline{G}$. We also have $M_{0,\underline{H}} = A_{0,\underline{H}}$. The weight spaces for $A_{0,\underline{H}}$ on W_Θ are each one-dimensional, and consist of the 32 weight spaces in $W_B \subseteq W_\Theta$ and the 24 spaces in $M_{2,12}$ spanned by the coordinate entries.

The following lemma computes the elements of $\mathcal{F}(A_{0,\underline{H}}, \underline{P}) = \mathcal{F}(M_{0,\underline{H}}, \underline{P}) = \mathcal{F}'(M_{0,\underline{H}}, \underline{P})$.

Lemma 8.7. *Let w be one of the weight spaces for $A_{0,\underline{H}}$ on W_Θ, and $\underline{P}_w$ the subgroup of $\underline{G}$ stabilizing w. Then the 56 subgroups $\underline{P}_w$ are exactly the elements of $\mathcal{F}(A_{0,\underline{H}}, \underline{P})$.*

Proof. It is immediate to check that each of these weight spaces is a rank one line in W_Θ, and thus each $\underline{P}_w$ is conjugate to $\underline{P}$ and contains $A_{0,\underline{H}}$. Conversely, because the weight spaces for A_0 on W_Θ are each one-dimensional and the parabolics of $\underline{G}$ conjugate to $\underline{P}$ stabilize rank one lines in W_Θ, the $\underline{P}_w$ are all of the elements of $\mathcal{F}(M_{0,\underline{H}}, \underline{P})$. □

8.3. **Proof of Theorem 8.1.** In this subsection, we prove Theorem 8.1 by using Method 1 introduced in Section 3.2. We want to study the regularized period integral $\mathcal{P}_{\underline{H},reg}(E(\phi, s))$.

We first prove statements (1) and (2) in Section 3.2 for the current case. For statement (1), by our assumptions on π together with the argument in Section 3.2, it is enough to show that statement (4) of Section 3.2 holds for the pair $(\underline{G}, \underline{H})$. But this just follows from Corollary 3.15 (7).

To prove statement (2) of Section 3.2, we fix $P_{0,\underline{H}}$ to be any minimal parabolic subgroup of $\underline{H}$ that contains $M_{0,\underline{H}}$ and fixes the line $k(0,0,0,1)$ in $W_B \subseteq W_\Theta$. We want to study the set $\mathcal{F}^{\underline{G}}(P_{0,\underline{H}}, \underline{P})$. By Proposition 2.4, we have $\mathcal{F}^{\underline{G}}(P_{0,\underline{H}}, \underline{P}) \subset \mathcal{F}(M_{0,\underline{H}}, \underline{P}) \subseteq \mathcal{F}'(M_{0,\underline{H}}, \underline{P})$. So it is enough to consider the elements in $\mathcal{F}'(M_{0,\underline{H}}, \underline{P})$.

Suppose first that B is a division algebra. Because the elements v of Lemma 8.4 are fixed by the B^1-factor of $\underline{H}$, one has $\underline{P}_v \cap \underline{H} = B^1 \times (\underline{P}_v \cap \underline{H}')$. Because the subgroups $\underline{P}_v$ of Lemma 8.4 stabilize rank one lines in W_B, the groups $\underline{P}_v \cap \underline{H}'$ are conjugate in $\underline{H}'$ and thus only one of the groups $\underline{P}_v \cap \underline{H}'$ can contain a fixed minimal parabolic of $\underline{H}'$. In fact, the groups $\underline{P}_v \cap \underline{H}'$ give all 8 of the maximal semistandard parabolic subgroups of $\underline{H}'$ of type A_5 in one of the two conjugacy classes with A_5-type Levi. Thus only one can be standard (i.e., the one corresponds to $\underline{P}$). By Proposition 2.4(1), we have that $\mathcal{F}^{\underline{G}}(P_{0,\underline{H}}, \underline{P}) \subseteq \{\underline{P}\}$.

Note that $\underline{P}$ is the parabolic associated to the cocharacter $\mathrm{GL}_1 \to T_6 \subseteq A_{0,\underline{H}} \subseteq \underline{A}_0$, with the map $\mathrm{GL}_1 \to T_6$ given by $t \mapsto \begin{pmatrix} t1_3 & \\ & t^{-1}1_3 \end{pmatrix}$. Thus $\underline{P}$ contains the centralizer of the image of this cocharacter, and in particular, contains $\underline{A}_0$ for any choice of $\underline{A}_0$ containing $A_{0,\underline{H}}$. Consequently, $\mathcal{F}^{\underline{G}}(P_{0,\underline{H}}, \underline{P}) = \{\underline{P}\}$, as desired. Hence statement (2) is trivial in this case.

Now suppose that $B = M_2(k)$ is split. Recall from Lemma 8.7 that there are 56 semistandard parabolic subgroups $\underline{P}_w$ of $\underline{G}$ conjugate to $\underline{P}$. Moreover, these 56 parabolic subgroups are partitioned into two sets, one of size 32 and the other of size 24 depending on whether the rank one line w is contained in W_B or $M_{2,12}(k)$. For the 32 $\underline{P}_w$ with w contained in W_B, we have just as in the division algebra case that $\underline{P}_w \cap \underline{H} = B^1 \times (\underline{P}_w \cap \underline{H}')$ and that the groups $\underline{P}_w \cap \underline{H}'$ give all the maximal semistandard parabolic subgroups of $\underline{H}'$ in one of the two conjugacy classes with Levi of type A_5 (there are 32 of them). Thus only one can be standard (i.e., the one that corresponds to $\underline{P}$).

Similarly, the 24 $\underline{P}_w$ with w contained in $M_{2,12} = V_2 \otimes V_{12}$ all satisfy that $\underline{P}_w \cap \underline{H} = \underline{P}_w \cap (\mathrm{SL}_2 \times \underline{H}')$ are stabilizers of pure tensors $b \otimes v$ with $b \in V_2$ the two-dimensional representation of SL_2 and $v \in V_{12}$ an isotropic vector in the orthogonal 12-dimensional representation of $\underline{H}'$. Thus for any such $\underline{P}_w$, $\underline{P}_w \cap \underline{H} = B \times (\underline{P}_w \cap \underline{H}')$ with B a semistandard Borel subgroup of SL_2 (there are 2 of them) and $\underline{P}_w \cap \underline{H}'$ a maximal semistandard parabolic subgroup of $\underline{H}'$ of type D_5 (there are 12 of them). Thus only one can be standard.

Since $\underline{A}_0 = A_{0,\underline{H}}$ when B is split, by applying Proposition 2.4 again, we have $\mathcal{F}^{\underline{G}}(P_{0,\underline{H}}, \underline{P}) = \{\underline{P}, \underline{P}'\}$ such that $\underline{P}' \cap \underline{H}$ is the parabolic subgroup of type D_5. Then statement (2) follows from Corollary 3.15(6).

Finally, we compute the constant $s_0 = -c(1-2c_{\underline{P}}^{\underline{H}})$ for the current case. We have $c = 9$, and, because the unipotent radicals are abelian, $c_{\underline{P},\underline{H}} = \frac{\dim_k J_B}{\dim_k J_\Theta} = \frac{15}{27}$. Thus $s_0 = 1$. Let $H^0 \subseteq H$ be the image of $\underline{M} \cap \underline{H}$ in GE_6. Combining the discussions above, equation (3.2) in Method 1 becomes

$$\int_{[\underline{H}]} \Lambda^{T,\underline{H}} Res_{s=1} E(h,\phi,s)dh = \int_{K_{\underline{H}}} \int_{[H^0]/Z_G(\mathbb{A})} \phi(hk)dhdk - \frac{e^{\langle -2\varpi_{\underline{P}},T\rangle}}{2} \int_{K_{\underline{H}}} \int_{[H^0]/Z_G(\mathbb{A})} Res_{s=1} M(s)\phi(hk)dhdk$$

for the current case. Because $H^0 \subseteq H$, this finishes the proof of Theorem 8.1.

Remark 8.8. *When $\underline{G}$ and $\underline{H}$ are split, the set $\mathcal{F}^{\underline{G}}(P_{0,\underline{H}}, \underline{P})$ contains two elements $\underline{P}$ and $\underline{P}'$. One can easily show that $c(1-2c_{\underline{P}'}^{\underline{H}}) = 9(1-2\frac{10}{27}) = \frac{5}{3} > s_0 = 1$. This confirms the discussion in Section 3.2.1.*

8.4. **The local result.** Let F be a p-adic field. As in the previous subsections, we can define the groups $\underline{G}, \underline{H}, \underline{P} = \underline{MN}, G, H$ over F. Let π be an irreducible smooth representation of $G(F)$ with trivial central character. We can also view π as a representation of $\underline{M}(F)$ with trivial central character. We then extend π to $\underline{P}(F)$ by making it trivial on $\underline{N}(F)$. As in Section 5.2, for $s \in \mathbb{C}$, we use π_s to denote the representation $\pi \otimes \varpi^s$ and use $I_{\underline{P}}^{\underline{G}}(\pi_s)$ to denote the normalized parabolic induction.

Theorem 8.9. *If π is an irreducible representation of $G(F)$ with trivial central character. Assume that the Hom space $\mathrm{Hom}_{H(F)}(\pi, 1)$ is nonzero, then the representation $I_{\underline{P}}^{\underline{G}}(\pi_1)$ is $\underline{H}(F)$-distinguished.*

Proof. The proof follows from the exact same argument as the proof of Theorem 5.6, we will skip it here. □

Now we are ready to prove Theorem 1.12. Assume that G is split over F. Let π be a tempered generic representation of $G(F)$ with trivial central character. Assume that the Hom space $\mathrm{Hom}_{H(F)}(\pi, 1)$ is nonzero, we need to show that the local L-function $L(s, \pi, \rho_X)$ has a pole at $s = 0$. By the same argument as in Proposition 5.7, we know that the induced representation $I_{\underline{P}}^{\underline{G}}(\pi_1)$ is reducible. Then by applying Lemma B.2 of [GI16] and the Standard Module Conjecture [HO13], we have that $I_{\underline{P}}^{\underline{G}}(\pi_1)$ is reducible if and only if the local gamma factor

$$\gamma(s, \pi, \rho_X) = \epsilon(s, \pi, \rho_X) \frac{L(1-s, \pi, \rho_X)}{L(s, \pi, \rho_X)}$$

has a pole at $s = 1$. Since π is tempered, $L(s, \pi, \rho_X)$ is holomorphic and nonzero when $Re(s) > 0$ (Theorem 1.1 of [HO13]), which implies that the L-function $L(s, \pi, \rho_X)$ has a pole at $s = 0$. This finishes the proof of Theorem 1.12.

9. The model $(\mathrm{GL}_4 \times \mathrm{GL}_2, \mathrm{GL}_2 \times \mathrm{GL}_2)$

The purpose of this section is to prove the local and global results for the pair $(\mathrm{GL}_4 \times \mathrm{GL}_2, \mathrm{GL}_2 \times \mathrm{GL}_2)$. The first several subsections are concerned with the global results, while the final subsection concerns the local results.

9.1. **Overview of argument.** The purpose of this and the following three subsections is to prove Theorem 1.13. We will do this by Method 2. The argument is analogous to that of [PWZ], but the computations are easier. In this subsection, we give an overview of the argument used to prove Theorem 1.13.

Denote by $E = k \times k$ the split quadratic étale extension of k. In fact, almost all of this section is unchanged if E is replaced by a general quadratic étale extension of k, so we frequently write E instead of $k \times k$.[3] Let Θ be a split octonion algebra over k. Define the quadratic space $V = \Theta \oplus E$ with quadratic form $q(x, \lambda) = n_\Theta(x) - n_E(\lambda)$ where $x \in \Theta$, $\lambda \in E$, and n_Θ resp. n_E denote the quadratic norms on Θ resp. E. We define $\underline{G} = \mathrm{GSO}(V)$, which by definition is the subgroup of $\mathrm{GO}(V)$ consisting of those g with $\det(g) = \nu(g)^{\dim(V)/2}$, where $\nu : \mathrm{GO}(V) \to \mathrm{GL}_1$ is the similitude.

In the next subsection, we specify a group $\underline{H}_7$ which is of type GSpin(7) together with its 8-dimensional spin representation on Θ. Set

$$\begin{aligned}\underline{H} = \underline{H}_7 \boxtimes \mathrm{Res}_{E/k}(\mathrm{GL}_1) : &= \{(h, \lambda) \in \underline{H}_7 \times \mathrm{Res}_{E/k}(\mathrm{GL}_1) : \nu(h) = n_E(\lambda)\} \\ &= \{(h, \lambda_1, \lambda_2) \in \underline{H}_7 \times \mathrm{GL}_1 \times \mathrm{GL}_1 : \nu(h) = \lambda_1\lambda_2\}.\end{aligned}$$

Via the representation $t_1 : \mathrm{GSpin}(7) \to \mathrm{GSO}(\Theta)$ specified below, we obtain an inclusion $\underline{H} \to \underline{G}$.

Denote by $\underline{P} = \underline{MN}$ the Heisenberg parabolic of $\underline{G}$, so that the Levi subgroup $\underline{M}$ of $\underline{P}$ is of type $A_1 \times D_3 = A_1 \times A_3$. Suppose that $\pi = \otimes_{v \in |k|} \pi_v$ is a cuspidal automorphic representation of $\underline{M}$ or $G = \mathrm{GL}_2 \times \mathrm{GL}_4$ with trivial central character[4]. Suppose that $\phi \in \mathcal{A}_\pi$, $s \in \mathbb{C}$ and $E(\phi, s)$ denotes the associated Eisenstein series on $\mathcal{A}_\pi(\underline{G}(\mathbb{A}))$.

Let

$$\begin{aligned}H &= (\mathrm{GL}_2 \times \mathrm{GL}_2) \boxtimes \mathrm{Res}_{E/k}(\mathrm{GL}_1) \\ &= \{(g, h, \lambda) \in \mathrm{GL}_2 \times \mathrm{GL}_2 \times \mathrm{Res}_{E/k}(\mathrm{GL}_1) : \det(g)\det(h) = N_{E/k}(\lambda)\}.\end{aligned}$$

Denote by $Z \simeq \mathrm{GL}_1 \times \mathrm{GL}_1$ the subgroup of $(\mathrm{GL}_2 \times \mathrm{GL}_2) \boxtimes \mathrm{Res}_{E/F}(\mathrm{GL}_1)$ consisting of the elements $(\mathrm{diag}(z, z), \mathrm{diag}(w, w), (zw, zw))$. From Lemma 9.4 below we obtain an embedding $H \to \underline{M}$ so that $Z = H \cap Z_{\underline{M}}$ where $Z_{\underline{M}}$ is the center of $\underline{M}$. Note that the map $H \to \mathrm{GL}_2 \times \mathrm{GL}_2$ given by $(g, h, (\lambda_1, \lambda_2)) \mapsto (g, \lambda_1^{-1}\det(g)h)$ induces an isomorphism $H/Z \simeq (\mathrm{GL}_2 \times \mathrm{GL}_2)/\Delta(\mathrm{GL}_1)$, with $\Delta(\mathrm{GL}_1)$ the diagonally embedded central GL_1. For a cuspidal automorphic form φ of $\underline{M}$ with trivial central character, denote by $\mathcal{P}_H$ the period

$$\mathcal{P}_H(\varphi) = \int_{Z(\mathbb{A})H(k)\backslash H(\mathbb{A})} \varphi(h)\, dh.$$

Theorem 9.1. *Suppose that the period $\mathcal{P}_H(\cdot)$ is nonvanishing on the space of π. Then there exists $\phi \in \mathcal{A}_\pi$ such that $E(\phi, s)$ has a pole at $s = 1/2$.*

Note that even though G, H and $\underline{G}$ are classical groups, our proof of Theorem 9.1 proceeds through the non-classical group $\underline{H}_7 \simeq \mathrm{GSpin}(7)$. This is in complete analogy with the triple product period integral considered by Jiang in [Jia98], later generalized by Ginzburg-Jiang-Rallis in [GJR04b], where one considers G_2-periods of certain residual representations on groups of type D.

From Theorem 9.1 we obtain Theorem 1.13 of the introduction.

Proposition 9.2. *Theorem 9.1 implies Theorem 1.13.*

[3]It is essentially for the purpose of proving the absolute convergence of certain integrals below that we choose E split.

[4]Note that the exterior square representation of GL_4 induces a map of algebraic groups $\mathrm{GL}_1 \times \mathrm{GL}_4 \to \mathrm{GSO}(6)$ which is surjective on k-points and has central kernel. Thus this map induces an isomorphism $\mathrm{PGL}_4 \simeq \mathrm{PGSO}_6$, so that a cuspidal automorphic representation on GL_4 with trivial central character may be considered as an automorhpic representation of GSO(6).

Proof. We first recall the statement of Theorem 1.13. Let π be a cuspidal automorphic representation of $\mathrm{GL}_4(\mathbb{A}) \times \mathrm{GL}_2(\mathbb{A})$ with trivial central character. Assume that the $\mathrm{GL}_2 \times \mathrm{GL}_2$-period defined in Section 1.1.6 is nonzero on the space of π. Moreover assume that the L-function $L(s, \pi, \rho_X)$ is nonzero at $s = 3/2$ where $\rho_X = \wedge^2 \otimes std$ is a 12-dimensional representation of ${}^LG$. Then we need to show that the L-function $L(s, \pi, \rho_X)$ is nonzero at $s = 1/2$.

As we explained in the previous page, we can view π as a cuspidal automorphic representation of $\underline{M}(\mathbb{A})$ with trivial central character. Moreover, by the discussion in the end of Section 9.4, we know that the $\mathrm{GL}_2 \times \mathrm{GL}_2$-period integral on π (viewed as a cuspidal automorphic representation of $\mathrm{GL}_4(\mathbb{A}) \times \mathrm{GL}_2(\mathbb{A})$) is just the H-period integral on π (viewed as a cuspidal automorphic representation of $\underline{M}(\mathbb{A})$). As a result, we know that the period $\mathcal{P}_H(\cdot)$ is nonvanishing on the space of π. By Theorem 9.1, there exists $\phi \in \mathcal{A}_\pi$ such that the Eisenstein series $E(\phi, s)$, and thus the intertwining operator $M(s)$, has a pole at $s = 1/2$.

In this case, the normalizing factor of the intertwining operator is

$$\frac{L(s, \pi, \rho_X)\zeta_k(2s)}{L(s+1, \pi, \rho_X)\zeta_k(2s+1)}$$

where $\zeta_k(s)$ is the Dedekind zeta function. By Theorem 4.7 of [KK11], the normalized intertwining operator is holomorphic at $s = 1/2$. By Proposition 2.8, $L(3/2, \pi, \rho_X) \neq 0$. It follows that the numerator $L(s, \pi, \rho_X)\zeta_k(2s)$ has a pole at $s = 1/2$, which implies that $L(\frac{1}{2}, \pi, \rho_X) \neq 0$. This proves Theorem 1.13. $\square$

We now finish this subsection by explaining how Theorem 9.1 is proved and the organization of the next three subsections. Denote by $\Lambda^T E(\phi, s)$ the Arthur-Langlands truncation of $E(\phi, s)$; see subsection 2.5. As explained in subsection 3.3, we will compute an $\underline{H}$-period of $\Lambda^T E(\phi, s)$, $\mathcal{P}_{\underline{H}}(\Lambda^T E(\phi, s))$ and essentially reduce the calculation to a period $\mathcal{P}_H(\phi)$.

More precisely, the proof of Theorem 9.1 proceeds as follows.

(1) First, we prove that the double coset space $\underline{P}(k)\backslash\underline{G}(k)/\underline{H}(k)$ is finite. Denote by $\gamma_1 = 1, \gamma_2, \ldots, \gamma_\ell$ its elements.
(2) Define $\underline{H}_i = \underline{H} \cap (\gamma_i^{-1}\underline{P}\gamma_i)$. Then for each i, we prove that the pair $(\underline{H}, \underline{H}_i)$ is a good pair in the sense of [PWZ, Section 5].
(3) Denote by $I_i(\phi, s)$ and $J_i(\phi, s)$ the integrals specified in Subsection 3.3. Applying Lemma 5.1, Proposition 5.2, and Proposition 5.5 of [PWZ], one obtains that each of the finitely many integrals $I_i(\phi, s)$ and $J_i(\phi, s)$ converges absolutely for $Re(s)$ and T sufficiently large.
(4) Finally, we prove that the integrals $I_i(\phi, s)$ and $J_i(\phi, s)$ vanish if $i > 1$.
(5) Computing the integrals $I_1(\phi, s)$ and $J_1(\phi, s)$, we obtain (3.6) with $s_0 = 1/2$, from which Theorem 9.1 follows immediately.

In the next subsection, we define the group $\underline{H}_7$ precisely, its representation $t_1 : \underline{H}_7 \to \mathrm{GSO}(\Theta)$, and some special subgroups of it. In subsection 9.3 we prove that the double coset $\underline{P}(k)\backslash\underline{G}(k)/\underline{H}(k)$ is finite and that $(\underline{H}, \underline{H}_i)$ is a good pair for all i. In subsection 9.4 we prove that the integrals $I_i(\phi, s)$ and $J_i(\phi, s)$ vanish for each $i > 1$ and deduce Theorem 9.1.

9.2. **Non-classical groups.** In this subsection we define the group $\underline{H}_7$, specify its Lie algebra concretely, and define certain subgroups of it. First, recall from [PWZ] the Zorn model of the octonions Θ. We will use the notation

$$\epsilon_1, e_1, e_2, e_3, e_1^*, e_2^*, e_3^*, \epsilon_2$$

of [PWZ, section 1.1.1] to denote a particular basis of Θ. We write $u_0 = \epsilon_1 - \epsilon_2$.

Define

$$GT(\Theta) = \{(g_1, g_2, g_3) \in \mathrm{GSO}(\Theta) \times \mathrm{GSO}(\Theta) \times \mathrm{SO}(\Theta) : (g_1 x, g_2 y, g_3 z) = (x, y, z) \text{ for all } x, y, z \in \Theta\}.$$

Here $\mathrm{SO}(\Theta)$ is defined via the norm $n_\Theta : \Theta \to F$ and the trilinear form is $(x, y, z) := \mathrm{tr}_\Theta(xyz)$. Denote by $t_1 : GT(\Theta) \to \mathrm{GSO}(\Theta)$ the first projection, and $\nu : GT(\Theta) \to \mathrm{GL}_1$ the map that is t_1 composed with the similitude on $\mathrm{GSO}(\Theta)$. The subgroup of $GT(\Theta)$ with $\nu = 1$ is the group $\mathrm{Spin}(8) = \mathrm{Spin}(\Theta)$. Define $\underline{H}_7$ to be the subgroup of $GT(\Theta)$ consisting of triples (g_1, g_2, g_3) so that $g_3 \cdot 1 = 1$. One can check that the similitude $\nu : \underline{H}_7 \to \mathrm{GL}_1$ is not the trivial map on k-points. We slightly abuse notation and let $t_1 : \underline{H}_7 \to \mathrm{GSO}(\Theta)$ denote the restriction of t_1 from $GT(\Theta)$ to $\underline{H}_7$.

We record facts about the group $\underline{H}_7$ that we will need later. Denote by σ the map $x \mapsto x^*$ on Θ. We begin with a simple lemma, whose proof is an exercise.

Lemma 9.3. *Suppose $(g_1, g_2, g_3) \in \mathrm{Spin}(\Theta)$.*

(1) Then

$$g_1(x)g_2(y) = (\sigma g_3 \sigma)(xy)$$

for all $x, y \in \Theta$.

(2) Suppose $g = (g_1, g_2, g_3) \in GT(\Theta)$, and define $\nu = \nu(g_1)$. If $g_3(1) = 1$, then $g_2 = \nu^{-1}\sigma g_1 \sigma$. Consequently, if $(g_1, g_2, g_3) \in GT(\Theta)$ and $g_3(1) = 1$, then

$$g_1(x)g_1(y)^* = \nu(\sigma g_3 \sigma)(xy^*)$$

for all $x, y \in \Theta$.

(3) Conversely, suppose $(g_1, g_2, g_3) \in \mathrm{Spin}(\Theta)$, and $g_2 = \sigma g_1 \sigma$. Then g_3 stabilizes 1.

Recall that we define the parabolic subgroup $\underline{P}$ of $\underline{G}$ to be the Heisenberg parabolic. This means that $\underline{P}$ stabilizes an isotropic two-dimensional subspace of V and that the flag variety $\underline{P}(k)\backslash\underline{G}(k)$ is the set of isotropic two-dimensional subspaces of V. In order to compute the double coset space $\underline{P}(k)\backslash\underline{G}(k)/\underline{H}(k)$, we will need to use $\underline{H}_7$ to move around various isotropic subspaces of V. To do this, it is helpful to have handy large concrete subgroups of $\underline{H}_7$. We specify such subgroups now.

The subgroups of $\underline{H}_7$ we will use are G_2, $\mathrm{GSpin}(6)$, and $\mathrm{GL}_2 \times \mathrm{GL}_2$. It is clear that $G_2 \subseteq \underline{H}_7$. For $\mathrm{GL}_2 \times \mathrm{GL}_2$, we compute inside the Cayley-Dickson construction (see, e.g., [PWZ, section 1.1]), so that the multiplication is $(x_1, y_1)(x_2, y_2) = (x_1x_2 + \gamma y_2' y_1, y_2x_1 + y_1x_2')$ and the conjugation is $(x, y)^* = (x', -y)$. Here for $h = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M_2(k)$, $h' = \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$ so that $hh' = \det(h)I_2$. Now, suppose $g \in \mathrm{GL}_2$ and $h \in \mathrm{GL}_2$. Define $(g, h) \cdot (x, y) = (gxh', \mu_h y g')$, where $\mu_h = \mathrm{diag}(\det(h), 1)$.

Lemma 9.4. *This action of $\mathrm{GL}_2 \times \mathrm{GL}_2$ on Θ defines a map $\mathrm{GL}_2 \times \mathrm{GL}_2 \to \underline{H}_7$.*

Proof. Indeed, if $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$, then one computes

$$\begin{aligned}((g,h)z_1) \cdot ((g,h)z_2)^* &= (gx_1h', \mu_h y_1 g')(gx_2h', \mu_h y_2 g')^* \\ &= \det(g)\det(h)(g(x_1x_2' - \gamma y_2' y_1)g^{-1}, \mu_h(-y_2x_1 + y_1x_2)h^{-1}). \end{aligned} \tag{9.1}$$

From this, the lemma is clear. □

We now describe the flag variety of Heisenberg parabolic subgroups in $\underline{H}_7$. For a two-dimensional isotropic subspace W of Θ, define $\kappa'(W) = \{xy^* : x, y \in W\}$. Then, because W is two-dimensional and isotropic, $\kappa'(W)$ is contained in $V_7 = \Theta^{\mathrm{tr}=0}$, and is either 0 or a line. If $\kappa'(W) = 0$, we say that W is *null*; otherwise, we say that W is not null. By Lemma 9.3, whether or not W is null is an $\underline{H}_7$-invariant. Moreover, it is clear that being isotropic and null is a closed condition on the Grassmanian of two-spaces in Θ, and thus the set of null isotropic two-spaces is a projective variety.

Lemma 9.5. *One has the following:*

(1) The group $\underline{H}_7$ acts transitively on the set of null-isotropic two-spaces W of Θ, and thus the stabilizer P_W of any such W is a parabolic subgroup of $\underline{H}_7$; these are the Heisenberg parabolics.

(2) Denote by W the null isotropic two-dimensional subspace of Θ that consists of the elements $(0, \left(\begin{smallmatrix} * & * \\ 0 & 0 \end{smallmatrix}\right))$. The map $\mathrm{GL}_2 \times \mathrm{GL}_2 \to \underline{H}_7$ of Lemma 9.4 identifies $\mathrm{GL}_2 \times \mathrm{GL}_2$ with a Levi subgroup of P_W.

Proof. The first statement is easily checked, and in any case, is surely well-known. For the second statement, it is easy to see that this $\mathrm{GL}_2 \times \mathrm{GL}_2$ embeds into $\underline{H}_7$, and that the image preserves W. As the reductive quotient of the parabolic subgroup P_W is exactly $\mathrm{GL}_2 \times \mathrm{GSpin}_3 = \mathrm{GL}_2 \times \mathrm{GL}_2$, the lemma follows. □

We next describe the subgroup GSpin(6) of $\underline{H}_7$ and how it acts on Θ. Recall the elements $\epsilon_1, \epsilon_2 \in \Theta$ with $1 = \epsilon_1 + \epsilon_2$. Define H_6 to be the subgroup of triples (g_1, g_2, g_3) in $GT(\Theta)$ for which $g_3(\epsilon_j) = \epsilon_j$ for $j = 1, 2$.

Lemma 9.6. *The group H_6 fixes the four-dimensional subspaces $U_+ = \left(\begin{smallmatrix} * & * \\ 0 & 0 \end{smallmatrix}\right)$ and $U_- = \left(\begin{smallmatrix} 0 & 0 \\ * & * \end{smallmatrix}\right)$ of Θ under the t_1-representation. Moreover, the image of the map $H_6 \to \mathrm{GL}(U_+)$ includes $\mathrm{SL}(U_+)$.*

Proof. We have the relation $g_3(x)g_1(y) = (\sigma g_2 \sigma)(xy)$ for general triples $(g_1, g_2, g_3) \in \mathrm{Spin}(\Theta)$. Now, U_+ is the subset of $y \in \Theta$ with $\epsilon_2 y = 0$ and $U_- = \{y \in \Theta : \epsilon_1 y = 0\}$. The first part of the lemma follows immediately from this.

For the second part, it is clear that the image contains the $\mathrm{SL}_3 \subseteq G_2$ that stabilizes ϵ_1 and ϵ_2. From (9.1), the subgroup $1 \times \mathrm{SL}_2 \subseteq \mathrm{GL}_2 \times \mathrm{GL}_2$ is in H_6. Under the identification of the Cayley-Dickson and the Zorn model of the octonions, the subspace spanned by e_1, e_2, e_3 in the Zorn model becomes the subspace of elements $(\left(\begin{smallmatrix} 0 & * \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} * & 0 \\ * & 0 \end{smallmatrix}\right))$ in the Cayley-Dickson model. Thus, the subgroup $1 \times \mathrm{SL}_2$ of H_6 is the SL_2 that acts on the span of ϵ_1 and (say) e_1. Because the image of H_6 in $\mathrm{GL}(U_+)$ contains these two subgroups of $\mathrm{SL}(U_+)$, the image must contain all of $\mathrm{SL}(U_+)$, giving the lemma. □

Finally, it will be useful to have a concrete realization of the Lie algebra of $\underline{H}_7$ that makes it easy to compute its action on Θ via the Spin representation t_1 (as opposed to the 7 dimensional orthogonal representation of SO(7).) Denote by $\underline{H}_7'$ the subgroup of $\underline{H}_7$ with $\nu = 1$. We describe the Lie algebra of $\underline{H}_7'$ as a subalgebra of $\mathfrak{so}(\Theta)$ via the representation t_1.

To do this, consider the map $\kappa : \wedge^2\Theta \to V_7 = \Theta^{\mathrm{tr}=0}$ given by $x \wedge y \mapsto Im(xy^*)$. Define K to be the kernel of this map. Note, K is *not* $\wedge^2 V_7$, but it does contain the exceptional Lie algebra $\mathfrak{g}_2$, which is the subspace of $\wedge^2 V_7$ in the kernel of κ.

The following description of the Lie algebra $\underline{H}_7$ is likely well-known. Not knowing a reference, we include a proof.

Lemma 9.7. *The Lie algebra $\mathfrak{h}_7^0$ of $\underline{H}_7'$ is equal to K as subspaces of $\mathfrak{so}(\Theta) = \wedge^2(\Theta)$. In particular, K is a Lie algebra.*

Proof. First we claim that K is fixed under the induced action of $\underline{H}_7'$ on Θ. Indeed, if $x, y \in \Theta$ and $g \in \underline{H}_7'$, then

$$\kappa(g \cdot (x \wedge y)) = \kappa(g(x) \wedge g(y)) = Im(g(x)g(y)^*) = Im(g'(xy^*)) = g' Im(xy^*) = g'\kappa(x \wedge y).$$

Hence κ is equivariant for the action of $\underline{H}_7'$, so K is preserved by $g \in H_7'$.

We have $G_2 \to \underline{H}_7' \to \mathrm{Spin}(8)$. As already mentioned, one has $\mathfrak{g}_2 \subseteq K$. But because K is closed under the action of $\underline{H}_7'$, K contains $\underline{H}_7' \cdot \mathfrak{g}_2 \subseteq \mathfrak{h}_7^0$. But $H_7' \cdot \mathfrak{g}_2$ is all of $\mathfrak{h}_7^0$. Thus $\mathfrak{h}_7^0 \subseteq K$. But both $\mathfrak{h}_7^0$ and K are 21-dimensional, thus $K = \mathfrak{h}_7^0$ as claimed. □

9.3. **Orbits, stabilizers, and convergence.** In this subsection we prove that the double coset $\underline{P}(k)\backslash\underline{G}(k)/\underline{H}(k)$ is finite. We also prove that the integrals $I_i(\phi, s)$ and $J_i(\phi, s)$ associated to these orbits are absolutely convergent.

The variety $\underline{P}(k)\backslash\underline{G}(k)$ is the set of isotropic two-dimensional subspaces W of V. We thus consider the orbits of $\underline{H}(k)$ on these isotropic subspaces. For $W \subseteq V$ isotropic and two-dimensional, we set $\underline{P}_W$ the stabilizer of W inside $\underline{G}$ and $\underline{H}_W = \underline{H} \cap \underline{P}_W$, the stabilizer of W inside $\underline{H}$. In this subsection, we will prove the following two statements:

Claim 9.8 (Proposition 9.12)**.** *There are finitely many $\underline{H}(k)$-orbits of isotropic two-dimensional subspaces in V.*

We will then calculate the stabilizers $\underline{H}_W$ for representatives W of these finitely many orbits, and deduce the following.

Claim 9.9 (Proposition 9.16)**.** *For every isotropic two-dimensional subspace W of V, $(\underline{H}, \underline{H}_W)$ is a good pair in the sense of* [PWZ, Section 5].

As already explained, these two claims imply the equality (3.5) and the absolute convergence of the integrals $I_i(\phi, s)$ and $J_i(\phi, s)$ on the right-hand side of this equality. To compute the orbits, we begin with the following lemma.

Lemma 9.10. *Suppose $v, w \in \Theta$ are nonzero. If $n_\Theta(v) = n_\Theta(w)$, whether 0 or not, then there exists $g \in \underline{H}'_7$ with $gv = w$.*

Proof. Suppose $v \in \Theta$ as above. We claim that we can use the H'_6 inside $\underline{H}'_7$ to move v to $V_7 = \Theta^{\mathrm{tr}=0}$. From this, the lemma follows from the corresponding fact for G_2, by moving both v and w into V_7. To move v into V_7, write $v = v_1 + v_2$, with $v_1 \in U_+$ and $v_2 \in U_-$ in the notation of Lemma 9.6. Thinking about the action of SL_4 on its defining representation and its dual, it is clear that we can simultaneously move v_1 into the three-dimensional subspace $\left(\begin{smallmatrix} 0 & * \\ 0 & 0 \end{smallmatrix}\right)$ of Θ and v_2 into the three-dimensional subspace $\left(\begin{smallmatrix} 0 & 0 \\ * & 0 \end{smallmatrix}\right)$. This proves the lemma. □

Given an isotropic two-dimensional subspace W of V, it falls into one of four (broad) classes. Define $pr_E : V \to E$ the orthogonal projection of V onto E.

(1) $pr_E(W) = E$
(2) $pr_E(W)$ is one-dimensional and anisotropic
(3) $pr_E(W)$ is one-dimensional and isotropic
(4) $pr_E(W)$ is 0.

Clearly, the class of such W is invariant under the action of $\underline{H}$. Moreover, let us record now that there will be two subclasses in the case $pr_E(W) = 0$: those for which $W \subseteq \Theta$ is null, and those for which $W \subseteq \Theta$ is non-null.

Lemma 9.11. *There are two $\underline{H}'_7$ orbits on isotropic two-dimensional subspaces of Θ, consisting of the orbit of null isotropic spaces and of non-null spaces.*

Proof. To see that there are at least two orbits, note that $\mathrm{Span}\{\epsilon_1, e_3\}$ is not null, whereas the span $\mathrm{Span}\{e_1, e_3^*\}$ is null.

To see that there are exactly two orbits, one argues as follows. First, suppose W is two-dimensional isotropic. We claim that there is $g \in \underline{H}'_7$ so that $gW \subseteq V_7$. From this claim, the lemma follows from Lemma 2.5 of [PWZ], by using the action of $G_2 \subseteq H'_7$.

To see that there is $g \in \underline{H}'_7$ with $gW \subseteq V_7$, write x, y for a basis of W. Applying Lemma 9.10, we can assume $x = \epsilon_1$. Because $(x, y) = 0$, we get that $y \in \left(\begin{smallmatrix} * & * \\ * & 0 \end{smallmatrix}\right)$, and thus may assume $y \in \left(\begin{smallmatrix} 0 & * \\ * & 0 \end{smallmatrix}\right) \subseteq V_7$. Acting by H'_6, it is then clear that we can move all of W into V_7, as claimed. This proves the lemma. □

Proposition 9.12. *Each of the four classes of isotropic two-spaces W above makes up finitely many $\underline{H}(k)$-orbits, which are characterized as follows:*

(1) $pr_E(W) = E$;
(2) $pr_E(W)$ is one-dimensional anisotropic;
(3) $pr_E(W) = (k, 0)$ or $(0, k)$ and $pr_\Theta(W)$ is one-dimensional isotropic;
(4) $pr_E(W) = (k, 0)$ or $(0, k)$ and $pr_\Theta(W)$ is two-dimensional isotropic and null;
(5) $pr_E(W) = (k, 0)$ or $(0, k)$ and $pr_\Theta(W)$ is two-dimensional isotropic and non-null;
(6) $pr_E(W) = 0$ and $W \subseteq \Theta$ is non-null;
(7) $pr_E(W) = 0$ and $W \subseteq \Theta$ is null.

Proof. Suppose first that $pr_E(W) = E$. Define $W' = pr_\Theta(W)$; this is a non-degenerate two-space. By Lemma 9.10, we can use $\underline{H}'_7 \subseteq \underline{H}$ to move one element of W' to 1, so we assume without loss of generality that $1 \in W'$. Let $u \in W'$ span the perpendicular space 1; thus, $u \in V_7$. Because W' combines with $pr_E(W) = E$ to make an isotropic two space, $n_\Theta(u) \neq 0$ is determined by E. Because G_2 acts transitively on such elements u, we see that there is one H-orbit in this case, as claimed.

Next suppose that $pr_E(W)$ is one-dimensional and anisotropic. We may assume that $pr_E(W) = k1$, and then that W contains $(1,1) \in \Theta \oplus E$. Let $y \in W \cap \Theta$, so that y is isotropic and perpendicular to 1, i.e., $y \in V_7$. Then, because G_2 acts transitively on the isotropic lines in V_7, we see that there is one orbit of such spaces.

If $pr_E(W) = 0$, then $W \subseteq \Theta$ is two-dimensional isotropic, and thus we have handled these cases by Lemma 9.11. This completes the possible cases when E is anisotropic, i.e., when E is a field.

Thus now assume that $pr_E(W)$ is one-dimensional isotropic. Then $pr_\Theta(W)$ is isotropic, and is either one or two-dimensional. Suppose that $pr_\Theta(W)$ is one-dimensional isotropic. By Lemma 9.10 above, there is one $\underline{H}'_7$-orbit of such lines, thus one orbit in this case. If $pr_\Theta(W)$ is two-dimensional isotropic, then by Lemma 9.11, we have two $\underline{H}'_7$ orbits. This completes the proof of the proposition. □

We next compute the stabilizers of the above two-spaces in $\underline{H}$. These stabilizer computations enable us to apply the results of [PWZ, Section 5] to check the convergence of the integrals $I_i(\phi, s)$ and $J_i(\phi, s)$. In order to prove the *vanishing* of the integrals $I_i(\phi, s)$ and $J_i(\phi, s)$ for $i > 1$, we will need to make a different stabilizer computation, which we do in the next subsection. See Remark 3.12.

To state the result on the various stabilizers, we require the following notation regarding parabolic subgroups. Denote by $Z_{\underline{G}} \simeq \mathrm{GL}_1$ the one-dimensional center of $\underline{G}$. For a nonzero isotropic element $y \in V_7$, denote by $P_{G_2}(y) = M_{G_2}(y)N_{G_2}(y)$ the parabolic subgroup of G_2 stabilizing the line ky and by $P_7(y)$ the maximal (Siegel) parabolic subgroup of $\underline{H}_7$ stabilizing ky. For a null isotropic two-dimensional subspace W of Θ, denote by $P_{7,W}$ the maximal (Heisenberg) parabolic subgroup of $\underline{H}_7$ stabilizing W.

We also require a notation for a certain non-maximal parabolic subgroup of $\underline{H}_7$. For this, denote by $P_{7,G}(e_3^*)$ the non-maximal parabolic subgroup of $\underline{H}_7$ that stabilizes the filtration $ke_3^* \subseteq \mathrm{Span}\{e_3^*, e_1, e_2\}$, so that $P_{7,G}(e_3^*) \supseteq P_{G_2}(e_3^*)$. Let $N_{7,G}(e_3^*)$ be the unipotent radical of $P_{7,G}(e_3^*)$; it is 3-step, $N_{7,G}(e_3^*) \supseteq N_{7,G}(e_3^*)' \supseteq N_{7,G}(e_3^*)''$, with $\dim_k N_{7,G}(e_3^*)/N_{7,G}(e_3^*)' = 4$ and the other two successive quotients of dimension two.

Lemma 9.13. *Except in the case where $pr_\Theta(W)$ is two-dimensional isotropic and non-null, the groups $\underline{H}_W$ are as follows:*

1. *Suppose $pr_E(W) = E$. Then the map $\underline{H} \to \underline{H}_7$ induces an isomorphism of $\underline{H}_W$ with a Levi subgroup of the Siegel parabolic of $\underline{H}_7$. In particular, $\underline{H}_W \simeq \mathrm{GL}_3 \times \mathrm{GL}_1$.*
2. *Suppose $pr_E(W)$ is one-dimensional anisotropic. For concreteness, suppose that W is spanned by $(1,1) \in \Theta \oplus E$ and the isotropic element $e_3^* \in V_7 \subseteq \Theta$. The group $\underline{H}_W$ is $Z_{\underline{G}} \times (M_{G_2}(e_3^*) \ltimes N)$ where N is the six-dimensional subgroup $N_{G_2}(e_3^*)N_{7,G}(e_3^*)'$ of $N_{7,G}(e_3^*)$.*
3. *Suppose that both $pr_E(W)$ and $pr_\Theta(W) = ky$ are one-dimensional isotropic. Then $\underline{H}_W \simeq P_7(y) \times \mathrm{GL}_1$ is the inverse image of $P_7(y) \subset \underline{H}_7$ under the map $\underline{H} \to \underline{H}_7$.*
4. *Suppose that $pr_E(W)$ is one-dimensional isotropic, $Y = pr_\Theta(W)$ is two-dimensional isotropic and null, and $y \in Y$ is nonzero. Denote by $Q_{7,y,Y} \subseteq P_{7,Y}$ a non-maximal parabolic subgroup of $\underline{H}_7$ that stabilizes a flag $ky \subseteq Y$ with Y two-dimensional isotropic and null. Then the map $\underline{H}_W \to \underline{H}_7$ induces an isomorphism of $\underline{H}_W$ with a parabolic subgroup of $\underline{H}_7$ of the form $Q_{7,y,Y}$.*
5. *Suppose that $pr_E(W) = 0$, and $pr_\Theta(W)$ is two-dimensional isotropic and null. Then $\underline{H}_W$ is the inverse image of $P_{7,W} \subset \underline{H}_7$ under the map $\underline{H} \to \underline{H}_7$.*

Proof. For the first item, we may assume $W = \mathrm{Span}\{\epsilon_1 + (1,0), \epsilon_2 + (0,1)\}$. Thus any element of $\underline{H}_W$ must stabilize both the line $k\epsilon_1$ and $k\epsilon_2$. The stabilizers of these two lines in $\underline{H}_7$ is the Levi subgroup of a Siegel parabolic stabilizing, say, $k\epsilon_1$. This proves that the image of $\underline{H}_W$ in $\underline{H}_7$ lands in the Levi subgroup $\mathrm{GL}_3 \times \mathrm{GL}_1$. That this map induces an isomorphism $\underline{H}_W \to \mathrm{GL}_3 \times \mathrm{GL}_1$ is now clear once accounting for the action of $E^\times$ on $(1,0)$ and $(0,1)$.

For the second item, accounting for the action of $Z_{\underline{G}}$, we must compute the subgroup of $P_7(e_3^*)$ that also stabilizes $\mathrm{Span}\{1, e_3^*\}$. By acting by the unipotent elements $\exp(x\epsilon_1 \wedge e_3^*) \in \underline{H}_7'$ for $x \in k$, we can reduce the stabilizer to $P_{G_2}(e_3^*)$. Denote by $N_{G_2}(e_3^*)'$ the three-dimensional commutator subgroup of $N_{G_2}(e_3^*)$. Because the elements $\exp(x\epsilon_1 \wedge e_3^*)$ span the one-dimensional space $N_{7,G}(e_3^*)'/N_{G_2}(e_3^*)'$, this completes the computation of the stabilizer in this case.

The third, fourth and fifth items are handled immediately. □

We now must compute the stabilizers in the case that $pr_\Theta(W)$ is two-dimensional isotropic and non-null. The work is done in the following lemma, which is easily proved once stated.

Lemma 9.14. *Suppose that $W = \mathrm{Span}\{x, y\}$ is a two-dimensional isotropic but non-null subspace of Θ. Set $b = xy^*$ and $U(b) = \{z \in \Theta : bz = 0\}$.*

1. *The space $U(b)$ is a four-dimensional isotropic subspace of Θ, that comes equipped with the symplectic form $\langle z_1, z_2\rangle$ defined by $z_1 z_2^* = \langle z_1, z_2\rangle b$.*
2. *Denote by $Q_{U(b)}$ the subgroup of $\underline{H}_7$ that stabilizes $U(b)$. Then $Q_{U(b)} = L_{U(b)}V_{U(b)}$ is a parabolic subgroup of $\underline{H}_7$ with Levi subgroup $L_{U(b)} = \mathrm{GL}_1 \times \mathrm{GSpin}(5) = \mathrm{GL}_1 \times \mathrm{GSp}_4$. The unipotent radical $V_{U(b)}$ is abelian of dimension 5 and the map $Q_{U(b)} \to \mathrm{GSp}_4$ is induced by the action of $Q_{U(b)}$ on $U(b)$.*
3. *The stabilizer of W inside $\underline{H}_7$ is $L'V_{U(b)}$ with $L' = \mathrm{GL}_1 \times (\mathrm{GL}_2 \times \mathrm{GL}_2)^0 \subseteq \mathrm{GL}_1 \times \mathrm{GSp}_4$. Here $(\mathrm{GL}_2 \times \mathrm{GL}_2)^0$ is the subgroup of pairs $(g_1, g_2) \in \mathrm{GL}_2 \times \mathrm{GL}_2$ with $\det(g_1) = \det(g_2)$.*

Applying Lemma 9.14, one obtains the following for the stabilizers $\underline{H}_W$ in case $pr_\Theta(W)$ is two-dimensional isotropic and non-null.

Lemma 9.15. *Let L' and $V = V_{U(b)}$ be as in Lemma 9.14.*

1. *Suppose that $pr_E(W) = 0$ and $pr_\Theta(W)$ is two-dimensional isotropic and non-null. Then $\underline{H}_W$ is the inverse image of $L' \ltimes V \subset \underline{H}_7$ under the map $\underline{H} \to \underline{H}_7$.*
2. *Suppose that $pr_E(W)$ is one-dimensional isotropic and $pr_\Theta(W)$ is two-dimensional isotropic and non-null. Denote by B a Borel subgroup of GL_2 and set $L'' := \mathrm{GL}_1 \times (B \times \mathrm{GL}_2)^0 \subseteq L'$. Then the map $\underline{H}_W \to \underline{H}_7$ induces an isomorphism $\underline{H}_W \simeq L'' \ltimes V$.*

Proof. This follows immediately from Lemma 9.14. □

Proposition 9.16. *For every isotropic two-space W of V, the pair $(\underline{H}, \underline{H}_W)$ is a good pair in the sense of* [PWZ, Section 5].

Proof. We first consider the cases when $pr_\Theta(W)$ is not two-dimensional isotropic and non-null. By Lemma 9.13, we can find a parabolic subgroup $P_{\underline{H}} = M_{\underline{H}} N_{\underline{H}}$ of $\underline{H}$ and a closed subgroup M' of $M_{\underline{H}}$ ($M' = 1$ for case (3) and (5) in Lemma 9.13; $M' \simeq \mathrm{GL}_1$ for case (1) and (4) in Lemma 9.13; $M' \simeq \mathrm{GL}_1 \times \mathrm{GL}_1$ for case (2) in Lemma 9.13) such that the following two conditions hold.

(1) $\underline{H}_W = (\underline{H}_W \cap M_{\underline{H}}) \ltimes (\underline{H}_W \cap N_{\underline{H}})$.
(2) $M_{\underline{H}} = (\underline{H}_W \cap M_{\underline{H}}) \times M'$.

Then we know that $(\underline{H}, \underline{H}_W)$ is a good pair by Corollary 5.9 of [PWZ].

Now we consider the cases when $pr_\Theta(W)$ is two-dimensional isotropic and non-null. If $pr_E(W)$ is one-dimensional isotropic, by Lemma 9.15, we can still find a parabolic subgroup $P_{\underline{H}} = M_{\underline{H}} N_{\underline{H}}$ of $\underline{H}$ and a closed subgroup M' of $M_{\underline{H}}$ such that condition (1) and (2) above hold. In fact, $P_{\underline{H}}$ is the parabolic subgroup whose Levi part is isomorphic to $\mathrm{GSpin}_3 \times \mathrm{GL}_1 \times \mathrm{GL}_1 \times \mathrm{GL}_1 =$

$\mathrm{GSp}_2 \times \mathrm{GL}_1 \times \mathrm{GL}_1 \times \mathrm{GL}_1$, and $M' \simeq \mathrm{GL}_1$. Then we know that $(\underline{H}, \underline{H}_W)$ is a good pair by Corollary 5.9 of [PWZ].

Hence the only case left is when $pr_E(W) = 0$ and $pr_\Theta(W)$ is two-dimensional isotropic and non-null. By Proposition 5.8(4) of [PWZ] and Lemma 9.15(1) above, in order to show that $(\underline{H}, \underline{H}_W)$ is a good pair, it is enough to prove the following lemma. □

Lemma 9.17. *$(\mathrm{GSp}_4 \ltimes U, (\mathrm{GL}_2 \times \mathrm{GL}_2)^0 \ltimes U')$ is a good pair. Here U is some unipotent group and $U' \subset U$ is a closed subgroup.*

Proof. The proof is very similar to the argument in Section 5.4 of our previous paper [PWZ], we will write it in the Appendix. □

9.4. **Vanishing and reduction of period.** In this subsection we prove the following result, which immediately implies the vanishing of the integrals $I_i(\phi, s)$ and $J_i(\phi, s)$ for $i > 1$.

Proposition 9.18. *Suppose that W is an isotropic two-dimensional subspace of V, but exclude the case that $W \subseteq \Theta$ is isotropic and null. Suppose that $\beta : \underline{P}(k)N(\mathbb{A})\backslash \underline{G}(\mathbb{A}) \to \mathbb{C}$ is a measurable function, with $m \mapsto \beta(mg)$ a cuspidal function on $\underline{M}(\mathbb{A})$ for almost every $g \in \underline{G}(\mathbb{A})$, and that the integral*

$$\mathcal{P}_W(\beta) = \int_{Z_{\underline{G}}(\mathbb{A})\underline{H}_W(k)\backslash \underline{H}(\mathbb{A})} \beta(h)\, dh$$

converges absolutely. Then $\mathcal{P}_W(\beta) = 0$.

Proof. We consider the vanishing of the various orbits one-by-one. For an isotropic two-dimensional subspace W of $V = \Theta \oplus E$, we make and recall the following notations:

- $\underline{P}_W = \underline{M}_W \underline{N}_W \subseteq \mathrm{GSO}(V)$ the Heisenberg parabolic that is the stabilizer of W, with unipotent radical $\underline{N}_W$ and Levi subgroup $\underline{M}_W$.
- $\underline{H}_W \subseteq \underline{H}$ the stabilizer of W inside $\underline{H}$, i.e. $\underline{H}_W = \underline{H} \cap \underline{P}_W$.
- H'_W the image of $\underline{H}_W$ inside the reductive quotient of $\underline{P}_W$.

First consider the case that $pr_E(W) = E$. Then we may assume that $W = \mathrm{Span}\{\epsilon_1 + (1,0), \epsilon_2 + (0,1)\}$. Set $W' = \{\epsilon_1 - (1,0), \epsilon_2 - (0,1)\}$ and denote by V_6 the subspace of Θ perpendicular to $k\epsilon_1 \oplus k\epsilon_2$. Then $V = W' \oplus V_6 \oplus W$. The semisimple part of H'_W in this case is SL_3, acting on V_6 as the direct sum of the standard representation and its dual. The vanishing of this orbit thus follows from the following lemma.

Lemma 9.19. *Suppose α is a cusp form on $\mathrm{SO}(V_6)$. Embed $\mathrm{SL}_3 \subseteq \mathrm{SO}(V_6)$ as the semisimple part of the Levi of a Siegel parabolic. Then the period $\int_{[\mathrm{SL}_3]} \alpha(h)\, dh = 0$.*

Proof. One first Fourier expands α along the abelian unipotent radical of the Siegel parabolic. By standard unfolding arguments, one quickly sees that the integral vanishes by the cuspidality of α along the unipotent radical of the maximal parabolic of $\mathrm{SO}(V_6)$ that stabilizes an isotropic line. □

Next suppose that $pr_E(W)$ is one-dimensional and anisotropic. Then we may assume that $W = \mathrm{Span}\{1_\Theta + 1_E, y\}$ with $y \in V_7$ isotropic. Let $y' \in V_7$ be isotropic with $(y, y') = 1$, and denote by $V_5(y)$ the perpendicular space to $\mathrm{Span}\{y, y'\}$ inside V_7. Set $W' = \mathrm{Span}\{1_\Theta - 1_E, y'\}$ and $V_6^W = V_5(y) \oplus k(1,-1)$. Then $V = W' \oplus V_6^W \oplus W$. Moreover, the stabilizer $\underline{H}_W$ contains the parabolic subgroup $P_{G_2}(y)$. To check the vanishing of this orbit, we must consider the image H'_W inside $\mathrm{SO}(V_6^W) \times \mathrm{GL}(W)$.

To do this, we first consider the image of $P_{G_2}(y)$ inside $\mathrm{SO}(V_5(y))$. We have the following lemma, which is easily checked.

Lemma 9.20. *Let $V_3(y) \supseteq ky$ be the three-dimensional isotropic subspace of V_7 stabilized by $P_{G_2}(y)$, and set P'' the parabolic subgroup of $\mathrm{SO}(V_5(y))$ that stabilizes $V_3(y)/ky$. Then the image of $P_{G_2}(y)$ inside of $\mathrm{SO}(V_5(y))$ is P''.*

Denote by P''' the derived subgroup of P''. Then the image of P''' in $\mathrm{SO}(V_6^W) \times \mathrm{GL}(W)$ is contained in $\mathrm{SO}(V_6^W)$. We are thus left to consider the P''' periods of cusp forms on $\mathrm{SO}(V_6)$, which we do in the following lemma.

Lemma 9.21. *The P''' periods of cusp forms on $\mathrm{SO}(V_6)$ vanish.*

Proof. Denote by N' the unipotent radical of the parabolic subgroup of $\mathrm{SO}(V_6^W)$ that stabilizes the line ke_1 in $V_5(y)$. Then it is simple to show that P'''-period of $\mathrm{SO}(V_6)$-cusp forms vanish by cuspidality along N'. □

Next we suppose that $pr_E(W)$ and $pr_\Theta(W)$ are each one-dimensional isotropic. Then we may assume $W = \mathrm{Span}\{\epsilon_1, (1,0)\}$. Define $W' = \mathrm{Span}\{\epsilon_2, (0,1)\}$ and $V_6 \subseteq \Theta$ the perpendicular space to $\mathrm{Span}\{\epsilon_1, \epsilon_2\}$. Then $V = W' \oplus V_6 \oplus W$. It is clear that H'_W includes the SL_3 acting on V_6, and thus these orbits vanish just as the first case above.

Now suppose that $pr_E(W)$ is one-dimensional isotropic and $pr_\Theta(W)$ is two-dimensional isotropic and null. Then we may assume that $W = \mathrm{Span}\{e_1 + (1,0), e_3^*\}$. Then

$$W^\perp = \mathrm{Span}\{e_1 + (1,0), e_3^*, e_1, \epsilon_1, \epsilon_2, e_2^*, e_2, e_1^* + (0,1)\}.$$

Consider the four elements $e_1 \wedge e_2^*$, $u_0 \wedge e_1 + e_2^* \wedge e_3^*$, $\epsilon_2 \wedge e_1$, $u_0 \wedge e_3^* + e_1 \wedge e_2$ of K. Denote by X the Lie subaglebra of K generated by these elements and by N' the unipotent subgroup of $\underline{H}'_7$ whose Lie algebra is X. We have the following lemma, which implies the vanishing for this orbit.

Lemma 9.22. *The group N' acts as the identity on W, and the image of N' in $\mathrm{SO}(W^\perp/W)$ is the unipotent radical of the parabolic subgroup of $\mathrm{SO}(W^\perp/W)$ that fixes the isotropic line ke_1.*

Proof. One sees easily that X annihilates e_1 and e_3^*. It follows that N' acts as the identity on W, and that the induced action on the six-dimensional space $W^\perp/W$ fixes the isotropic vector e_1. Moreover, one computes immediately $X \cdot (e_1^* + (0,1)) = \mathrm{Span}\{\epsilon_1, \epsilon_2, e_2^*, e_2\}$. From this it follows that the image of N' in $\mathrm{SO}(W^\perp/W)$ is the entire unipotent radical of the parabolic subgroup stabilizing e_1, proving the lemma. □

Now assume that $pr_E(W)$ is one-dimensional isotropic and $pr_\Theta(W)$ is two-dimensional isotropic and non-null. Then we may assume that $W = \mathrm{Span}\{\epsilon_1, e_3 + (1,0)\}$. Then

$$W^\perp = \mathrm{Span}\{e_1, e_2, e_1^*, e_2^*, e_3^* + (0,1), e_3, e_3 + (1,0), \epsilon_1\}.$$

Consider the element $\epsilon_1 \wedge e_3^*$ of K. This element is nilpotent and annihilates $e_1, e_2, e_1^*, e_2^*, e_3^* + (1,0)$, and ϵ_1. Moreover $(\epsilon_1 \wedge e_3^*)(e_3) = -\epsilon_1$. It follows that $\exp(x\epsilon_1 \wedge e_3^*)$ acts as the identity on $W^\perp/W$, and acts on W as the unipotent radical of the a Borel subgroup of $\mathrm{GL}(W)$. Consequently, this orbit vanishes by cuspidality on GL_2.

Finally suppose that $W \subseteq \Theta$ is isotropic and non-null. Then we may assume $W = \mathrm{Span}\{\epsilon_1, e_3\}$, so that $W^\perp = \mathrm{Span}\{(1,0), (0,1), e_1, e_2, e_1^*, e_2^*, \epsilon_1, e_3\}$. Again, consider the elements $\exp(x\epsilon_1 \wedge e_3^*)$ of $\underline{H}'_7$. It is immediate that they act as the identity on $W^\perp/W$ and act on W as the unipotent radical of a Borel subgroup. Consequently, this orbit also vanishes by cuspidality along GL_2. □

Combining Proposition 9.18, Proposition 9.16, and Proposition 9.12, we arrive at the following. Recall that $H = \underline{H} \cap \underline{M}$, where $\underline{P} = \underline{MN}$ is the Heisenberg parabolic subgroup of $\underline{G}$ that stabilizes a two-dimensional isotropic and null subspace of $\Theta \subseteq V$.

$$\mathcal{P}_{\underline{H}}(\Lambda^T E(\phi, s)) = I_1(\phi, s) + J_1(\phi, s)$$

where

$$\begin{aligned} I_1(\phi, s) &= \int_{Z_G(\mathbb{A})H(k)\backslash \underline{H}(\mathbb{A})} (1 - \widehat{\tau_{\underline{P}}}(H_{\underline{P}}(h) - T)) e^{\langle s\omega_{\underline{P}}, H_{\underline{P}}(h)\rangle} \phi(h)\, dh \\ &= \frac{e^{(s-s_0)T}}{s - s_0} \int_{K_{\underline{H}}} \int_{Z(\mathbb{A})H(k)\backslash H(\mathbb{A})} \phi(hk) dh dk, \end{aligned}$$

and similarly

$$J_1(\phi,s)=\int_{Z_G(\mathbb{A})H(k)\backslash \underline{H}(\mathbb{A})}\widehat{\tau_{\underline{P}}}(H_{\underline{P}}(h)-T)e^{\langle -s\varpi_{\underline{P}},H_{\underline{P}}(h)\rangle}M(s)\phi(h)\,dh$$
$$=\frac{e^{(-s-s_0)T}}{-s-s_0}\int_{K_{\underline{H}}}\int_{Z(\mathbb{A})H(k)\backslash H(\mathbb{A})}M(s)\phi(hk)dhdk.$$

Then we compute the constant s_0 in Method 2. Like Method 1, we have $s_0=c(1-2c_{\underline{P}}^{\underline{H}})$. In this case, although the unipotent group $\underline{N}$ is not abelian, it is easy to see that $c_{\underline{P}}^{\underline{H}}=\frac{8}{14}$. On the other hand, by Proposition 2.1(4), we have $c=\frac{7}{2}$. This implies that $s_0=\frac{1}{2}$. As a result, we have proved that

$$\mathcal{P}_{\underline{H}}(\Lambda^T E(\phi,s))=\frac{e^{(s-\frac{1}{2})T}}{s-\frac{1}{2}}\int_{K_{\underline{H}}}\int_{Z(\mathbb{A})H(k)\backslash H(\mathbb{A})}\phi(hk)dhdk+\frac{e^{(-s-\frac{1}{2})T}}{-s-\frac{1}{2}}\int_{K_{\underline{H}}}\int_{Z(\mathbb{A})H(k)\backslash H(\mathbb{A})}M(s)\phi(hk)dhdk.$$

By taking the residue at $s=\frac{1}{2}$, we have

$$\mathcal{P}_{\underline{H}}(\Lambda^T Res_{s=\frac{1}{2}}E(\phi,s))=\int_{K_{\underline{H}}}\int_{Z(\mathbb{A})H(k)\backslash H(\mathbb{A})}\phi(hk)dhdk-e^{-T}\int_{K_{\underline{H}}}\int_{Z(\mathbb{A})H(k)\backslash H(\mathbb{A})}Res_{s=\frac{1}{2}}M(s)\phi(hk)dhdk.$$

This proves Theorem 9.1.

To conclude this subsection, we make explicit the period integral $\mathcal{P}_H$ in terms of the isomorphism $\mathrm{PGL}_4\simeq\mathrm{PGSO}_6$. More precisely, the map $H\to\underline{M}\simeq\mathrm{GL}_2\times\mathrm{GSO}(6)$ induces

$$H/Z\to\mathrm{PGL}_2\times\mathrm{PGSO}(6)\simeq\mathrm{PGL}_2\times\mathrm{PGL}_4.$$

We have already noted that $H/Z\simeq(\mathrm{GL}_2\times\mathrm{GL}_2)/\Delta(\mathrm{GL}_1)$. In the following lemma, we make explicit the induced map

$$(\mathrm{GL}_2\times\mathrm{GL}_2)/\Delta(\mathrm{GL}_1)\to\mathrm{PGL}_2\times\mathrm{PGL}_4. \tag{9.2}$$

Lemma 9.23. *The map* (9.2) *is induced by the map* $\mathrm{GL}_2\times\mathrm{GL}_2\to\mathrm{GL}_4\times\mathrm{GL}_2$ *given by* $(a,b)\mapsto(\left(\begin{smallmatrix}a&\\&b\end{smallmatrix}\right),a)$ *in* 2×2 *block form.*

Proof. From Lemma 9.4 and Lemma 9.5, the map $H\to\underline{M}=\mathrm{GL}_2\times\mathrm{GSO}(6)$ is given by

$$(g,h,(\lambda_1,\lambda_2))\mapsto(g,j(g,h,\lambda))$$

where $j(g,h,\lambda)$ acts on $V_6=M_2(k)\oplus E$ as $(m,\mu)\mapsto(gmh',\lambda\mu)$. Up to the action of $Z=\mathrm{GL}_1\times\mathrm{GL}_1$ which sits inside H as triples $(z,w,(zw,zw))$, we can assume that $(g,h,\lambda)=(g,h,(\det(g),\det(h)))$. Denote by $V_4=V_2\oplus V_2$ the decomposition of the defining representation of GL_4 into two GL_2 representations. Recall that our map $\mathrm{GL}_4\to\mathrm{GSO}(6)$ is induced by the exterior square representation. The element $\left(\begin{smallmatrix}g&\\&h\end{smallmatrix}\right)\in\mathrm{GL}_4$ acts on $V_6=\wedge^2(V_4)$ by $(m,\mu)\mapsto(gmh',(\det(g),\det(h))\mu)$ for an appropriate choice of basis. The lemma follows. □

9.5. **The local result.** Let F be a local field (archimedean or p-adic), and D/F be the unique quaternion algebra if $F\neq\mathbb{C}$. Let

$$G(F)=\mathrm{GL}_4(F)\times\mathrm{GL}_2(F),\ H(F)=\left\{\begin{pmatrix}a&0\\0&b\end{pmatrix}\times\begin{pmatrix}a\end{pmatrix}\mid a,b\in\mathrm{GL}_2(F)\right\}$$

as in the previous subsections. Let π be an irreducible smooth representation of $G(F)$ with trivial central character (we can also consider the nontrivial central character case, but we assume it is trivial here for simplicity), define the multiplicity

$$m(\pi)=\dim(\mathrm{Hom}_{H(F)}(\pi,1)).$$

Similarly, if $F \neq \mathbb{C}$, we can define the quaternion version of the model (G_D, H_D) with $G_D(F) = \mathrm{GL}_2(D) \times \mathrm{GL}_1(D)$ and $H_D(F) \simeq \mathrm{GL}_1(D) \times \mathrm{GL}_1(D)$. We can also define the multiplicity $m(\pi_D)$ for all irreducible smooth representations of $G_D(F)$ with trivial central character.

Assume that F is p-adic. Let $\pi = \pi_1 \otimes \pi_2$ (resp. $\pi_D = \pi_{1,D} \otimes \pi_{2,D}$) be an irreducible tempered representation of $G(F)$ (resp. $G_D(F)$) with trivial central character. We define the geometric multiplicity

$$m_{geom}(\pi) = c_{\pi_1}(1)c_{\pi_2}(1) + \Sigma_{T \in \mathcal{T}_{ell}(\mathrm{GL}_2)} |W(\mathrm{GL}_2, T)|^{-1} \int_{T(F)/Z_{\mathrm{GL}_2}(F)} D^{\mathrm{GL}_2(F)}(t)^2 c_{\pi_1}(\begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}) \theta_{\pi_2}(t) dt.$$

Here $\mathcal{T}_{ell}(\mathrm{GL}_2)$ is the set of all the maximal elliptic tori of $\mathrm{GL}_2(F)$ (up to conjugation), $W(\mathrm{GL}_2, T)$ is the Weyl group, Z_{GL_2} is the center of GL_2, dt is the Haar measure on $T(F)/Z_{\mathrm{GL}_2}(F)$ such that $vol(T(F)/Z_{\mathrm{GL}_2}(F)) = 1$, $D^{\mathrm{GL}_2(F)}(t)$ is the Weyl determinant, θ_{π_i} is the distribution character of π_i, $c_{\pi_1}(1)$ is the regular germ of θ_{π_1} at the identity element, and $c_{\pi_1}(\begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix})$ is the regular germ of θ_{π_1} at $\begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}$. We refer the readers to Section 4.5 of [Beu15] for the definition of regular germs. Similarly, we can also define the quaternion version of the geometric multiplicity

$$m_{geom}(\pi_D) = \Sigma_{T_D \in \mathcal{T}_{ell}(\mathrm{GL}_1(D))} |W(\mathrm{GL}_1(D), T)|^{-1} \int_{T_D(F)/Z_{\mathrm{GL}_1(D)}(F)} D^{\mathrm{GL}_1(D)}(t_D)^2 c_{\pi_{1,D}}(\begin{pmatrix} t_D & 0 \\ 0 & t_D \end{pmatrix}) \theta_{\pi_{2,D}}(t_D) dt_D.$$

Note the for the quaternion case, we don't need to include the regular germ at the identity element because the group is not quasi-split.

The proof of the following theorem follows from a similar (but easier) argument as in the Ginzburg-Rallis model case ([Wana], [Wanb], [Wan17]), we will skip it here. In fact, the argument for the current model is more similar to the argument for the "middle model" (which is a reduced model of the Ginzburg-Rallis model) defined in Appendix B of [Wana]. But it is easier since H is reductive.

Theorem 9.24. *(1) Assume that F is p-adic. Let $\pi = \pi_1 \otimes \pi_2$ (resp. $\pi_D = \pi_{1,D} \otimes \pi_{2,D}$) be an irreducible tempered representation of $G(F)$ (resp. $G_D(F)$) with trivial central character. Then we have a multiplicity formula*

$$m(\pi) = m_{geom}(\pi),\ m(\pi_D) = m_{geom}(\pi_D).$$

(2) Assume that F is p-adic. Let $\pi = \pi_1 \otimes \pi_2$ be an irreducible tempered representation of $G(F)$ with trivial central character, and let π_D be the Jacquet-Langlands correspondence of π from $G(F)$ to $G_D(F)$ if it exists; otherwise let $\pi_D = 0$. Then we have

$$m(\pi) + m(\pi_D) = 1.$$

In other words, the summation of the multiplicities over every tempered local Vogan L-packet is equal to 1.

(3) The statement in (2) also holds when $F = \mathbb{R}$.

(4) When $F = \mathbb{C}$, the multiplicity $m(\pi) = 1$ for all irreducible tempered representation π of $G(F)$ with trivial central character.

Remark 9.25. *As in the Ginzburg-Rallis model case, we can make the epsilon dichotomy conjecture for this model. To be specific, let $\pi = \pi_1 \otimes \pi_2$ be an irreducible tempered representation of $G(F)$ with trivial central character, and let π_D be the Jacquet-Langlands correspondence of π from $G(F)$ to $G_D(F)$ if it exists; otherwise let $\pi_D = 0$. Then the conjecture states that*

$$m(\pi) = 1 \iff \epsilon(1/2, \pi, \rho_X) = 1,\ m(\pi) = 0 \iff \epsilon(1/2, \pi, \rho_X) = -1.$$

Here $\rho_X = \wedge^2 \otimes std$ is the 12-dimensional representation of ${}^L G = \mathrm{GL}_4(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C})$ as in the previous subsections.

By a similar argument as in the Ginzburg-Rallis model case ([Wanb], [Wan17]*), we can prove this conjecture when F is archimedean. And when F is p-adic, we can prove this conjecture when π is not a discrete series.*

Remark 9.26. *In general, we expect the results above hold for all generic representations of $G(F)$.*

Appendix A. The proof of Lemma 9.17

In this appendix, we are going to prove Lemma 9.17. Let $G = G_0 \ltimes U$ where $G_0 = \mathrm{GSp}_4$ and U is some unipotent group. Let $H = H_0 \ltimes U'$ be a subgroup of G with U' being a subgroup of U and $H_0 = (\mathrm{GL}_2 \times \mathrm{GL}_2)^0 \subset \mathrm{GSp}_4 = G_0$. Our goal is to prove the following lemma.

Lemma A.1. *The pair (G, H) is a good pair.*

The proof is very similar to the argument in Section 5.4 of our previous paper [PWZ], we only include it here for completion. We use the same notations as in Section 5 of [PWZ]. We need some preparation. Let $w_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $J_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and $J_4 = \begin{pmatrix} 0 & w_2 \\ -w_2 & 0 \end{pmatrix}$. We define the groups GSp_4 to be

$$\mathrm{GSp}_4 = \{g \in \mathrm{GL}_4 |\ g^t J_4 g = \lambda J_4 \text{ for some } \lambda \in \mathrm{GL}_1\}.$$

The embedding $(\mathrm{GL}_2 \times \mathrm{GL}_2)^0 \to \mathrm{GSp}_4$ is given by

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \times \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \in (\mathrm{GL}_2 \times \mathrm{GL}_2)^0 \mapsto \begin{pmatrix} a_1 & 0 & 0 & b_1 \\ 0 & a_2 & b_2 & 0 \\ 0 & c_2 & d_2 & 0 \\ c_1 & 0 & 0 & d_1 \end{pmatrix} \in \mathrm{GSp}_4.$$

Let $B = TN$ be the upper triangular Borel subgroup of $G_0 = \mathrm{GSp}_4$ with T being the group of diagonal elements in B. Then $B_H = B \cap H_0$ is a Borel subgroup of $H_0 = (\mathrm{GL}_2 \times \mathrm{GL}_2)^0$ with the Levi decomposition $B_H = TN_H$ where $N_H = N \cap H_0$. Let

$$N' = \{ \begin{pmatrix} 1 & a_{12} & a_{13} & a_{14} \\ 0 & 1 & a_{23} & a_{24} \\ 0 & 0 & 1 & a_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix} \in N |\ a_{14} = a_{23} = 0\}$$

be a closed subvariety of N (note that it is not a group). The map

$$N_H \times N' \to N : \ (n, n') \mapsto nn'$$

is an isomorphism of varieties.

Lemma A.2. *For all $h \in H_0(\mathbb{A}_{\bar{k}})$ and $n' \in N'(\mathbb{A})$, we have*

$$||hn'||_G \sim ||h||_G \cdot ||n'||_G.$$

Proof. By the Iwasawa decomposition, it is enough to consider the case when $h = tn$ with $t \in T(\mathbb{A}_{\bar{k}})$ and $n \in N_H(\mathbb{A}_{\bar{k}})$. Since $N = N_H N'$, $B = TN$ is a parabolic subgroup of G_0 and $B_H = TN_H$ is a parabolic subgroup of H_0, we have

$$||hn'||_G = ||tnn'||_G \sim ||t||_G \cdot ||nn'||_G \sim ||t||_G \cdot ||n||_G \cdot ||n'||_G \sim ||tn||_G \cdot ||n'||_G = ||h||_G \cdot ||n'||_G.$$

This proves the lemma. □

Now we are ready to prove Lemma A.1. For $g \in G(\mathbb{A})$, we want to show that

$$\inf_{\gamma \in H(\bar{k})} ||\gamma g||_G \gg \inf_{\gamma \in H(k)} ||\gamma g||_G.$$

By the Iwasawa decomposition, it is enough to consider the case when $g = utnn'$ with $u \in U(\mathbb{A}), t \in T(\mathbb{A}), n \in N_H(\mathbb{A})$ and $n' \in N'(\mathbb{A})$. Since $TN_H \in H_0$, we can write g as uh_0n' with $u \in U(\mathbb{A}), h_0 \in H_0(\mathbb{A})$ and $n' \in H(\mathbb{A})$. In order to prove Lemma 9.17, it is enough to prove the following lemma.

Lemma A.3. *For all $u \in U(\mathbb{A}), h_0 \in H_0(\mathbb{A})$ and $n' \in N'(\mathbb{A})$, we have*

$$\inf_{\gamma \in H_0(\bar{k}), \nu \in U'(\bar{k})} ||\nu\gamma u h_0 n'||_G \gg \inf_{\gamma \in H_0(k), \nu \in U'(k)} ||\nu\gamma u h_0 n'||_G. \tag{A.1}$$

Proof. The proof is exactly the same as the proof of Proposition 5.12 of [PWZ]. All we need to do is replace Lemma 5.11 of loc. cit. by Lemma A.2 above. This finishes the proof of Lemma 9.17. ☐

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