THE EXISTENCE OF ZARISKI DENSE ORBITS FOR POLYNOMIAL ENDOMORPHISMS OF THE AFFINE PLANE

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Abstract. In this paper we prove the following theorem. Let f be a dominant polynomial endomorphism of the affine plane defined over an algebraically closed field of characteristic 0. If there is no nonconstant invariant rational function under f, then there exists a closed point in the plane whose orbit under f is Zariski dense.

This result gives us a positive answer to a conjecture proposed by Medvedev and Scanlon, by Amerik, Bogomolov and Rovinsky, and by Zhang, for polynomial endomorphisms of the affine plane.

1. Introduction

Denote by k an algebraically closed field of characteristic 0.

The aim of this paper is to prove

Theorem 1.1. *Let $f : \mathbb{A}^2_k \to \mathbb{A}^2_k$ be a dominant polynomial endomorphism. If there are no nonconstant rational functions g satisfying $g \circ f = g$, there exists a point $p \in \mathbb{A}^2(k)$ such that the orbit $\{f^n(p)|\ n \geq 0\}$ of p is Zariski dense in $\mathbb{A}^2_k$.*

We cannot ask g in Theorem 1.1 to be a polynomial. Indeed, let $P(x, y)$ be a polynomial which is neither zero nor a root of unity. Let $f : \mathbb{A}^2_k \to \mathbb{A}^2_k$ be the endomorphism defined by $(x, y) \mapsto (P(x, y)x, P(x, y)y)$. It is easy to see that $g \circ f = g$ if $g = y/x$, but there does not exist any polynomial h satisfying $h \circ f = h$.

The following conjecture was proposed by Medvedev and Scanlon [14, Conjecture 5.10] and also by Amerik, Bogomolov and Rovinsky [3]

Conjecture 1.2. Let X be a quasi-projective variety over k and $f : X \to X$ be a dominant endomorphism for which there exists no nonconstant rational function g satisfying $g \circ f = g$. Then there exists a point $p \in X(k)$ whose orbit is Zariski dense in X.

Conjecture 1.2 strengthens the following conjecture of Zhang [20].

Conjecture 1.3. Let X be a projective variety and $f : X \to X$ be an endomorphism defined over k. If there exists an ample line bundle L on X satisfying $f^*L = L^{\otimes d}$ for some integer $d > 1$, then there exists a point $p \in X(k)$ whose orbit $\{f^n(p)|\ n \geq 0\}$ is Zariski dense in $X(k)$.

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Theorem 1.1 setlles Conjecture 1.2 for polynomial endomorphisms of $\mathbb{A}^2_k$.

When k is uncountable, Conjecture 1.2 was proved by Amerik and Campana [4]. In [9], Fakhruddin proved Conjecture 1.2 for *generic*[1] endomorphisms on projective spaces over arbitrary algebraically closed fields k of characteristic zero. In [19], the author proved Conjecture 1.2 for birational surface endomorphisms with dynamical degree great than 1. Recently in [15], Medvedev and Scanlon proved Conjecture 1.2 when $f := (f_1(x_1), \cdots, f_N(x_N))$ is an endomorphism of $\mathbb{A}^N_k$ where the f_i's are one-variable polynomials defined over k.

We mention that in [2], Amerik proved that there exists a nonpreperiodic algebraic point when f is of infinite order. In [6], Bell, Ghioca and Tucker proved that if f is an automorphism, then there exists a subvariety of codimension 2 whose orbit under f is Zariski dense.

We note that Conjecture 1.2 is not true in the case when k is the algebraic closure of a finite field, since in this case all orbits of k-points are finite.

Our proof of Theorem 1.1 is based on the valuative techniques developed in [10, 11, 12, 17]. Here is an outline of the proof.

For simplicity, suppose that f is a dominant polynomial map $f := (F(x,y), G(x,y))$ defined over $\mathbb{Z}$.

When f is birational, our Theorem 1.1 is essentially proved in [19]. So we may suppose that f is not birational.

By [12], there exists a projective compactification X of $\mathbb{A}^2$ for which the induced map by f at infinity is *algebraically stable* i.e. it does not contract any curve to a point of indeterminacy. Moreover, we can construct an "attracting locus" at infinity, in the sense that

(i) either there exists a superattracting fixed point $q \in X \setminus \mathbb{A}^2$ such that there is no branch of curve at q which is periodic under f;
(ii) or there exists an irreducible component $E \in X \setminus \mathbb{A}^2$ such that $f^*E = dE + F$ where $d \geq 2$ and F is an effective divisor supported by $X \setminus \mathbb{A}^2$.

In Case (i), we can find a point $p \in \mathbb{A}^2(\bar{\mathbb{Q}})$, near q w.r.t. the euclidean topology. Then we have $\lim_{n\to\infty} f^n(p) = q$. It is easy to show that the orbit of p is Zariski dense in $\mathbb{A}^2$.

In Case (ii), E is defined over $\mathbb{Q}$. There exists a prime number $\mathfrak{p} \geq 3$, such that $f_{\mathfrak{p}}|_{E_{\mathfrak{p}}}$ is dominant where $f_{\mathfrak{p}} := f \bmod \mathfrak{p}$ and $E_{\mathfrak{p}} := E \bmod \mathfrak{p}$.

We first treat the case $f^n|_E \neq \mathrm{id}$ for all $n \geq 1$. After replacing f by a suitable iterate, we may find a fixed point $x \in E_{\mathfrak{p}}$ such that $df_{\mathfrak{p}}(x) = 1$ in $\bar{\mathbb{F}}_{\mathfrak{p}}$. Denote by U the $\mathfrak{p}$-adic open set of $X(\mathbb{Q}_{\mathfrak{p}})$ consisting the points y such that $y \bmod \mathfrak{p} = \mathrm{x}$. Then U is fixed by f. By [16, Theorem 1], all the preperiodic points in $U \cap E$ are fixed. Moreover $\bigcap_{n\geq 0} f^n(U) = U \cap E$. Denote by S the set of fixed points in $U \cap E$. Then S is finite. If S is empty, pick a point $p \in \mathbb{A}^2(\bar{\mathbb{Q}} \cap \mathbb{Q}_p) \cap U$, it is easy to see that the orbit of p is Zariski dense in $\mathbb{A}^2$. If S is not empty, by [1,

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[1] An endomorphism $f : \mathbb{P}^N_k \to \mathbb{P}^N_k$ satisfying $f^*O_{\mathbb{P}^N_k}(1) = O_{\mathbb{P}^N_k}(d)$ is said to be generic if it conjugates by a suitable linear automorphism on $\mathbb{P}^N_k$ to an endomorphism $[x_0 : \cdots : x_N] \mapsto [\sum_{|I|=d} a_{0,I}x^I : \cdots : \sum_{|I|=d} a_{N,I}x^I]$ where the set $\{a_{i,I}\}_{0\leq i\leq N, |I|=d}$ is algebraically independent over $\bar{\mathbb{Q}}$.

Theorem 3.1.4], at each point $q_i \in S$, there exists at most one algebraic curve C_i passing through q_i which is preperiodic. Set $C_i = \emptyset$ if no such curve does exists. We have that C_i is fixed. Pick a point $p \in \mathbb{A}^2(\bar{\mathbb{Q}} \cap \mathbb{Q}_p) \setminus C_1$ very closed to q_1. We can show that the orbit of p is Zariski dense in $\mathbb{A}^2$.

Next, we treat the case $f|_E = \mathrm{id}$. By [1, Theorem 3.1.4], at each point $q \in E$, there exists at most one algebraic curve C_q passing through q which is preperiodic. Set $C_q = \emptyset$, if such curve does not exists. We have that C_q is fixed and transverse to E. If $C_q = \emptyset$ for all but finitely many $q \in E$, there exists $q \in E$ and a $\mathfrak{p}$-adic neighborhood U of q such that for any point $y \in U \cap E$, $C_y = \emptyset$, $f(U) \subseteq U$ and $\bigcap_\infty f^n(U) = U \cap E$. Then for any point $p \in \mathbb{A}^2(\bar{\mathbb{Q}} \cap \mathbb{Q}_p) \cap U$, the orbit of p is Zariski dense in $\mathbb{A}^2$. Otherwise there exists a sequence of points $q_i \in E$, such that $C_i := C_{q_i}$ is an irreducible curve. Since $f|_{C_i}$ is an endomorphism of $C_i \cap \mathbb{A}^2$ of degree at least two, C_i has at most two branches at infinity. Since C_i is transverse to E at q_i, we can bound the intersection number $(E \cdot C_i)$. By the technique developed in [17], we can also bound the intersection of C_i with the other irreducible components of $X \setminus \mathbb{A}^2$. Then we bound the degree of C_i which allows us to construct a nonconstant invariant rational function.

The article is organized in 2 parts.

In Part 1, we gather some results on the geometry and dynamics at infinity and metrics on projective varieties defined over a valued field. We first introduce the valuative tree at infinity in Section 2, and then we recall the main properties of the action of a polynomial map on the valuation space in Section 3. Next we introduce the Green function on the valuative tree for a polynomial endomorphism in Section 4. Finally we give background information on metrics on projective varieties defined over a valued field in Section 5.

In Part 2, we prove Theorem 1.1. We first prove it in some special cases in Section 6. In most of these cases, we find a Zariski dense orbit in some attracting locus. Then we study totally invariant curves in Section 7 and prove Theorem 1.1 when there are infinitely many such curves. Finally we finish the proof of Theorem 1.1 in Section 8.

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Part 1. Preliminaries

In this part, we denote by k an algebraically closed field of characteristic zero. We also fix affine coordinates on $\mathbb{A}^2_k = \operatorname{Spec} k[x, y]$.

2. The valuative tree at infinity

We refer to [13] for details, see also [10, 11, 12].

2.1. **The valuative tree at infinity.** In this article by a valuation on a unitary k-algebra R we shall understand a function $v : R \to \mathbb{R} \cup \{+\infty\}$ such that the restriction of v to $k^* = k - \{0\}$ is constant equal to 0, and v satisfies $v(fg) = v(f) + v(g)$ and $v(f+g) \geq \min\{v(f), v(g)\}$. It is usually referred to as a semivaluation in the literature, see [10]. We will however make a slight abuse of notation and call it a valuation.

Denote by V_∞ the space of all normalized valuations centered at infinity i.e. the set of valuations $v : k[x,y] \to \mathbb{R} \cup \{+\infty\}$ satisfying $\min\{v(x), v(y)\} = -1$. The topology on V_∞ is defined to be the weakest topology making the map $v \mapsto v(P)$ continuous for every $P \in k[x,y]$.

The set V_∞ is equipped with a *partial ordering* defined by $v \leq w$ if and only if $v(P) \leq w(P)$ for all $P \in k[x,y]$. Then $-\deg : P \mapsto -\deg(P)$ is the unique minimal element.

Given any valuation $v \in V_\infty \setminus \{-\deg\}$, the set $\{w \in V_\infty, | -\deg \leq w \leq v\}$ is isomorphic as a poset to the real segment $[0,1]$ endowed with the standard ordering. In other words, $(V_\infty, \leq)$ is a rooted tree in the sense of [10, 13].

Given any two valuations $v_1, v_2 \in V_\infty$, there is a unique valuation in V_∞ which is maximal in the set $\{v \in V_\infty |\ v \leq v_1 \text{ and } v \leq v_2\}$. We denote it by $v_1 \wedge v_2$.

The segment $[v_1, v_2]$ is by definition the union of $\{w \,|\, v_1 \wedge v_2 \leq w \leq v_1\}$ and $\{w \,|\, v_1 \wedge v_2 \leq w \leq v_2\}$.

Pick any valuation $v \in V_\infty$. We say that two points v_1, v_2 lie in the same direction at v if the segment $[v_1, v_2]$ does not contain v. A *direction* (or a tangent vector) at v is an equivalence class for this relation. We write $\mathrm{Tan_v}$ for the set of directions at v.

When $\mathrm{Tan_v}$ is a singleton, then v is called an endpoint. In V_∞, the set of endpoints is exactly the set of all maximal valuations. When $\mathrm{Tan_v}$ contains exactly two directions, then v is said to be regular. When $\mathrm{Tan_v}$ has more than three directions, then v is a branch point.

Pick any $v \in V_\infty$. For any tangent vector $\vec{v} \in \mathrm{Tan_v}$, denote by $U(\vec{v})$ the subset of those elements in V_∞ that determine $\vec{v}$. This is an open set whose boundary is reduced to the singleton $\{v\}$. If $v \neq -\deg$, the complement of $\{w \in V_\infty, |w \geq v\}$ is equal to $U(\vec{v}_0)$ where $\vec{v}_0$ is the tangent vector determined by $-\deg$.

It is a fact that finite intersections of open sets of the form $U(\vec{v})$ form a basis for the topology of V_∞.

2.2. **Compactifications of $\mathbb{A}^2_k$.** A *compactification* of $\mathbb{A}^2_k$ is the data of a projective surface X together with an open immersion $\mathbb{A}^2_k \to X$ with dense image.

A compactification X dominates another one X' if the canonical birational map $X \dashrightarrow X'$ induced by the inclusion of $\mathbb{A}^2_k$ in both surfaces is in fact a regular map.

The category $\mathcal{C}$ of all compactifications of $\mathbb{A}^2_k$ forms an inductive system for the relation of domination.

Recall that we have fixed affine coordinates on $\mathbb{A}^2_k = \operatorname{Spec} k[x,y]$. We let $\mathbb{P}^2_k$ be the standard compactification of $\mathbb{A}^2_k$ and denote by $l_\infty := \mathbb{P}^2_k \setminus \mathbb{A}^2_k$ the line at infinity in the projective plane.

An *admissible compactification* of $\mathbb{A}^2_k$ is by definition a smooth projective surface X endowed with a birational morphism $\pi_X : X \to \mathbb{P}^2_k$ such that π_X is an isomorphism over $\mathbb{A}^2_k$ with the embedding $\pi^{-1}|_{\mathbb{A}^2_k} : \mathbb{A}^2_k \to X$. Recall that π_X can then be decomposed as a finite sequence of point blow-ups.

We shall denote by $\mathcal{C}_0$ the category of all admissible compactifications. It is also an inductive system for the relation of domination. Moreover $\mathcal{C}_0$ is a subcategory of $\mathcal{C}$ and for any compactification $X \in \mathcal{C}$, there exists that $X' \in \mathcal{C}_0$ dominates X.

2.3. **Divisorial valuations.** Let $X \in \mathcal{C}$ be a compactification of $\mathbb{A}^2_k = \operatorname{Spec} k[x,y]$ and E be an irreducible component of $X \setminus \mathbb{A}^2$. Set $b_E := -\min\{\operatorname{ord}_E(x), \operatorname{ord}_E(y)\}$ and $v_E := b_E^{-1}\operatorname{ord}_E$. Then we have $v_E \in V_\infty$.

By Poincaré Duality there exists a unique *dual divisor* $\check{E}$ of E defined as the unique divisor supported on $X \setminus \mathbb{A}^2$ such that $(\check{E} \cdot F) = \delta_{E,F}$ for all irreducible components F of $X \setminus \mathbb{A}^2$.

Remark 2.1. Recall that l_∞ is the line at infinity of $\mathbb{P}^2$. Let s be a formal curve centered at some point $q \in l_\infty$. Suppose that the strict transform of s in X intersects E transversally at a point in E which is smooth in $X \setminus \mathbb{A}^2$. Then we have $(s \cdot l_\infty) = b_E$.

2.4. **Classification of valuations.** There are four kinds of valuations in V_∞. The first one corresponds to the *divisorial valuations* which we have defined above. We now describe the three remaining types of valuations.

Irrational valuations. Consider any two irreducible components E and E' of $X \setminus \mathbb{A}^2_k$ for some compactification $X \in \mathcal{C}$ of $\mathbb{A}^2_k$ intersecting at a point p. There exists local coordinates (z,w) at p such that $E = \{z = 0\}$ and $E' = \{w = 0\}$. To any pair $(s,t) \in (\mathbb{R}^+)^2$ satisfying $sb_E + tb_{E'} = 1$, we attach the valuation v defined on the ring O_p of germs at p by $v(\sum a_{ij} z^i w^j) = \min\{si + tj |\ a_{ij} \neq 0\}$. Observe that it does not depend on the choice of coordinates. By first extending v to the common fraction field $k(x,y)$ of O_p and $k[x,y]$, then restricting it to $k[x,y]$, we obtain a valuation in V_∞, called *quasimonomial.* It is divisorial if and only if either $t = 0$ or the ratio s/t is a rational number. Any divisorial valuation is quasimonomial. An *irrational valuation* is by definition a nondivisorial quasimonomial valuation.

Curve valuations. Recall that l_∞ is the line at infinity of $\mathbb{P}^2_k$. For any formal curve s centered at some point $q \in l_\infty$, denote by v_s the valuation defined by $P \mapsto (s \cdot l_\infty)^{-1}\operatorname{ord}_\infty(P|_s)$. Then we have $v_s \in V_\infty$ and call it a *curve valuation.*

Let C be an irreducible curve in $\mathbb{P}^2_k$. For any point $q \in C \cap l_\infty$, denote by O_q the local ring at q, m_q the maximal ideal of O_q and I_C the ideal of height 1 in O_q defined by C. Denote by $\widehat{O}_q$ the completion of O_q w.r.t. m_q, $\widehat{m}_q$ the completion of m_q and $\widehat{I}_C$ the completion of I_C. For any prime ideal $\widehat{p}$ of height 1 containing $\widehat{I}_C$, the morphism $\operatorname{Spec} \widehat{O}_q/\widehat{p} \to \operatorname{Spec} \widehat{O}_q$ defines a formal curve centered at q. Such a formal curve is called a *branch of C at infinity.*

Infinitely singular valuations. Let h be a formal series of the form $h(z) = \sum_{k=0}^{\infty} a_k z^{\beta_k}$ with $a_k \in k^*$ and $\{\beta\}_k$ an increasing sequence of rational numbers with unbounded denominators. Then $P \mapsto -\min\{\mathrm{ord}_\infty(x), \mathrm{ord}_\infty(h(x^{-1}))\}^{-1}\mathrm{ord}_\infty P(x, h(x^{-1}))$ defines a valuation in V_∞ called an infinitely singular valuation.

A valuation $v \in V_\infty$ is a branch point in V_∞ if and only if it is diviorial, it is a regular point in V_∞ if and only if it is an irrational valuation, and it is an endpoint in V_∞ if and only if it is a curve valuation or an infinitely singular valuation. Moreover, given any smooth projective compactification X in which $v = v_E$, one proves that the map sending an element V_∞ to its center in X induces a map $\mathrm{Tan_v} \to \mathrm{E}$ that is a bijection.

2.5. **Parameterizations.** The *skewness* function $\alpha : V_\infty \to [-\infty, 1]$ is the unique function on V_∞ that is continuous on segments, and satisfies

$$\alpha(v_E) = \frac{1}{b_E^2}(\check{E} \cdot \check{E})$$

where E is any irreducible component of $X \setminus \mathbb{A}_k^2$ of any compactification X of $\mathbb{A}_k^2$ and $\check{E}$ is the dual divisor of E as defined above.

The skewness function is strictly decreasing, and upper semicontinuous. In an analogous way, one defines the *thinness* function $A : V_\infty \to [-2, \infty]$ as the unique, increasing, lower semicontinuous function on V_∞ such that for any irreducible exceptional divisor E in some compactification $X \in \mathcal{C}$, we have

$$A(v_E) = \frac{1}{b_E}\left(1 + \mathrm{ord}_E(dx \wedge dy)\right) .$$

Here we extend the differential form $dx \wedge dy$ to a rational differential form on X.

2.6. **Computation of local intersection numbers of curves at infinity.** Let s_1, s_2 be two different formal curves at infinity. We denote by $(s_1 \cdot s_2)$ the intersection number of these two formal curves in $\mathbb{P}^2$. This intersection number is always nonnegative, and it is positive if and only if s_1 and s_2 are centered at the same point.

Denote by l_∞ the line at infinity in $\mathbb{P}_k^2$. Denote by v_{s_1}, v_{s_2} the curve valuations associated to s_1 and s_2.

By [17, Proposition 2.2], we have

$$(s_1 \cdot s_2) = (s_1 \cdot l_\infty)(s_2 \cdot l_\infty)(1 - \alpha(v_{s_1} \wedge v_{s_2})).$$

3. Background on dynamics of polynomial maps

Recall that the affine coordinates have been fixed, $\mathbb{A}_k^2 = \mathrm{Spec}\, k[x, y]$.

3.1. **Dynamical invariants of polynomial mappings.** The (algebraic) degree of a dominant polynomial endomorphism $f = (F(x, y), G(x, y))$ defined on $\mathbb{A}_k^2$ is defined by

$$\deg(f) := \max\{\deg(F), \deg(G)\} .$$

It is not difficult to show that the sequence $\deg(f^n)$ is sub-multiplicative, so that the limit $\lambda_1(f) := \lim_{n\to\infty}(\deg(f^n))^{\frac{1}{n}}$ exists. It is referred to as the *dynamical* *degree* of f, and it is a theorem of Favre and Jonsson that $\lambda_1(f)$ is always a quadratic integer, see [12].

The (topological) degree $\lambda_2(f)$ of f is defined to be the number of preimages of a general closed point in $\mathbb{A}^2(k)$; one has $\lambda_2(fg) = \lambda_2(f)\lambda_2(g)$.

It follows from Bézout's theorem that $\lambda_2(f) \leq \deg(f)^2$ hence

$$\lambda_1(f)^2 \geq \lambda_2(f) \ . \tag{3.1}$$

The resonant case $\lambda_1(f)^2 = \lambda_2(f)$ is quite special and the following structure theorem for these maps is proven in [12].

Theorem 3.1. *Any polynomial endomorphism f of $\mathbb{A}^2_k$ such that $\lambda_1(f)^2 = \lambda_2(f)$ is proper, and we are in exactly one of the following two exclusive cases.*

1. $\deg(f^n) \asymp \lambda_1(f)^n$*; there exists a compactification X of $\mathbb{A}^2_k$ to which f extends as a regular map $f : X \to X$.*
2. $\deg(f^n) \asymp n\lambda_1(f)^n$*; there exist affine coordinates x, y in which f takes the form*

$$f(x,y) = (x^l + a_1 x^{l-1} + \ldots + a_l, A_0(x)y^l + \ldots + A_l(x))$$

where $a_i \in k$ and $A_i \in k[x]$ with $\deg A_0 \geq 1$, and $l = \lambda_1(f)$.

Remark 3.2. Regular endomorphisms as in (1) have been classified in [12].

3.2. **Valuative dynamics.** Any dominant polynomial endomorphism f as in the previous section induces a natural map on the space of valuations at infinity in the following way.

For any $v \in V_\infty$ we set

$$d(f,v) := -\min\{v(F), v(G), 0\} \geq 0 \ .$$

In this way, we get a non-negative continuous decreasing function on V_∞. Observe that $d(f, -\deg) = \deg(f)$. It is a fact that f is proper if and only if $d(f,v) > 0$ for all $v \in V_\infty$.

We now set

- $f_* v := 0$ if $d(f,v) = 0$;
- $f_* v(P) = v(f^* P)$ if $d(f,v) > 0$.

In this way one obtains a valuation on $k[x,y]$ (that may be trivial); we then get a continuous map

$$f_\bullet : \{v \in V_\infty |\ d(f,v) > 0\} \to V_\infty$$

defined by

$$f_\bullet(v) := d(f,v)^{-1} f_* v \ .$$

This map extends to a continuous map $f_\bullet : \overline{\{v \in V_\infty |\ d(f,v) > 0\}} \to V_\infty$. The image of any $v \in \partial\{v \in V_\infty |\ d(f,v) > 0\}$ is a curve valuation defined by a rational curve with one place at infinity.

We now recall the following key result, [12, Proposition 2.3 ,Theorem 2.4, Proposition 5.3.].

We say a polynomial endomorphism f of $\mathbb{A}^2_k$ is proper if it is a proper morphism between schemes. When $k = \mathbb{C}$, it means that the preimage of any compact set of $\mathbb{C}^2$ is compact.

Theorem 3.3. *There exists a valuation v_* such that $\alpha(v_*) \geq 0 \geq A(v_*)$, and $f_* v_* = \lambda_1 v_*$.*

If $\lambda_1(f)^2 > \lambda_2(f)$, this valuation is unique .

If $\lambda_1(f)^2 = \lambda_2(f)$, the set of such valuations is a closed segment in V_∞.

This valuation v_* is called the *eigenvaluation* of f when $\lambda_1(f)^2 > \lambda_2(f)$.

4. The Green function of f

4.1. **Subharmonic functions on V_∞.** We refer to [18, Section 3] for details.

To any $v \in V_\infty$ we attach its Green function

$$Z_v(w) := \alpha(v \wedge w) \ .$$

This is a decreasing continuous function taking values in $[-\infty, 1]$, satisfying $g_v(-\deg) = 1$.

Given any positive Radon measure ρ on V_∞ we define

$$Z_\rho(w) := \int_{V_\infty} Z_v(w)\, d\rho(v) \ .$$

Observe that $g_v(w)$ is always well-defined as an element in $[-\infty, 1]$ since $g_v \leq 1$ for all v.

Then we recall the following result.

Theorem 4.1 ([18])**.** *The map $\rho \mapsto Z_\rho$ is injective.*

One can thus make the following definition.

Definition 4.2. A function $\phi : V_\infty \to \mathbb{R} \cup \{-\infty\}$ is said to be *subharmonic* if there exists a positive Radon measure ρ such that $\phi = Z_\rho$. In this case, we write $\rho = \Delta\phi$ and call it the *Laplacian* of ϕ.

4.2. **Basic properties of the Green function of f.** We refer to [17, Section 12] for details.

Let f be a dominant polynomial endomorphism on $\mathbb{A}^2_k$ with $\lambda_1(f)^2 > \lambda_2(f)$.

By [17, Section 12], there exists a unique subharmonic function θ^* on V_∞ such that

(i) $f^*\theta^* = \lambda_1\theta^*$;
(ii) $\theta^*(v) \geq 0$ for all $v \in V_\infty$;
(iii) $\theta^*(-\deg) = 1$;
(iv) for all $v \in V_\infty$ satisfying $\alpha(v) > -\infty$, we have $\theta^*(v) > 0$ if and only if $d(f^n, v) > 0$ for all $n \geq 0$ and

$$\lim_{n\to\infty} f^n_\bullet(v) = v_*.$$

5. Metrics on projective varieties defined over a valued field

A field with an absolute value is called a valued field.

Definition 5.1. Let $(K, |\cdot|_v)$ be a valued field. For any integer $n \geq 1$, we define a metric d_v on the projective space $\mathbb{P}^n(K)$ by

$$d_v([x_0 : \cdots : x_n], [y_0 : \cdots : y_n]) = \frac{\max_{0\leq i,j\leq n} |x_i y_j - x_j y_i|_v}{\max_{0\leq i\leq n} |x_i|_v \max_{0\leq j\leq n} |y_j|_v}$$

for any two points $[x_0 : \cdots : x_n], [y_0 : \cdots : y_n] \in \mathbb{P}^n(K)$.

Observe that when $|\cdot|_v$ is archimedean, then the metric d_v is not induced by a smooth riemannian metric. However it is equivalent to the restriction of the Fubini-Study metric on $\mathbb{P}^n(\mathbb{C})$ or $\mathbb{P}^n(\mathbb{R})$ to $\mathbb{P}^n(K)$ induced by any embedding $\sigma_v : K \hookrightarrow \mathbb{R}$ or $\mathbb{C}$.

More generally, for a projective variety X defined over K, if we fix an embedding $\iota : X \hookrightarrow \mathbb{P}^n$, we may restrict the metric d_v on $\mathbb{P}^n(K)$ to a metric $d_{v,\iota}$ on $X(K)$. This metric depends on the choice of embedding ι in general, but for different embeddings ι_1 and ι_2, the metrics d_{v,ι_1} and d_{v,ι_2} are equivalent. Since we are mostly intersecting in the topology induced by these metrics, we shall usually write d_v instead of $d_{v,\iota}$ for simplicity.

Part 2. The existence of Zariski dense orbit

The aim of this part is to prove Theorem 1.1.

6. The attracting case

In this section, we prove Theorem 1.1 in some special cases. In most of these cases, we find a Zariski dense orbit in some attracting locus. We also prove Theorem 1.1 when $\lambda_1^2 = \lambda_2 > 1$.

Denote by k an algebraically closed field of characteristic 0. Let $f : \mathbb{A}^2 \to \mathbb{A}^2$ be a dominant polynomial endomorphism defined over k. We have the following result.

Lemma 6.1. *If $\lambda_2^2(f) > \lambda_1(f)$ and the eigenvaluation v_* is not divisorial, then there exists a point $p \in \mathbb{A}^2(k)$ whose the orbit is Zariski dense in $\mathbb{A}^2$.*

Proof of Lemma 6.1. After replacing k by an algebraically closed subfield of k containing all the coefficients of f, we may suppose that the transcendence degree of k over $\bar{\mathbb{Q}}$ is finite.

By [12, Theorem 3.1], there exists a compactification X of $\mathbb{A}^2$ defined over k and a superattacting point $q \in X \setminus \mathbb{A}^2_k$ such that for any valuation $v \in V_\infty$ whose center in X is q, we have $f_\bullet^n v \to v_*$ as $n \to \infty$.

By embedding k in $\mathbb{C}$, we endow X with the usual Euclidean topology. There exists a neighborhood U of q in X such that

(i) $I(f) \cap U = \emptyset$;
(ii) $f(U) \subseteq U$;
(iii) for any point $p \in U$, $f^n(p) \to q$ as $n \to \infty$.

Since $\mathbb{A}^2(k)$ is dense in $\mathbb{A}^2(\mathbb{C})$, there exists a point $p \in \mathbb{A}^2(k) \cap U$. If the orbit of p is not Zariski dense, then its Zariski closure Z is a union of finitely many curves. Since $f^n(p) \to q$, p is not preperiodic. It follows that all the one dimensional irreducible components of Z are periodic under f. Let C be a one dimensional irreducible component of Z. Since there exists an infinite sequence $\{n_0 < n_1 < \cdots\}$ such that $f^{n_i}(p) \in C$ for $i \geq 0$, we have C contains $q = \lim_{i\to\infty} f^{n_i}(p)$. Then there exists a branch C_1 of C at infinity satisfying $q \in C_1$. Thus v_{C_1} is periodic which is a contradiction. It follows that $O(p)$ is Zariski dense. □

In many cases, for example $\lambda_1(f)^2 > \lambda_2(f)$ and v_* is divisorial, there exists a projective compactification X of $\mathbb{A}^2$ and an irreducible component E of $X \setminus \mathbb{A}^2$ satisfying $f_\bullet(v_E) = v_E$. The following result proves Theorem 1.1 when $f|_E$ is of infinite order.

Lemma 6.2. *Let X be a projective compactification of $\mathbb{A}^2$ defined over k. Then f extends to a rational selfmap on X. Let E be an irreducible component of $X \setminus \mathbb{A}^2$ satisfying $f_\bullet(v_E) = v_E$, $d(f, v_E) \geq 2$ and $f^n|_E \neq \mathrm{id}$ for all $n \geq 1$. Then there exists a point $p \in \mathbb{A}^2(k)$ whose orbit is Zariski dense in $\mathbb{A}^2$.*

Proof of Lemma 6.2. There exists a finitely generated $\mathbb{Z}$-subalgebra R of k such that X, E and f are defined over the fraction field K of R.

By [5, Lemma 3.1], there exists a prime $\mathfrak{p} \geq 3$, an embedding of K into $\mathbb{Q}_\mathfrak{p}$, and a $\mathbb{Z}_\mathfrak{p}$-scheme $\mathcal{X}_{Z_\mathfrak{p}}$ such that the generic fiber is X, the specialization $E_\mathfrak{p}$ of E is isomorphic to $\mathbb{P}^1_{\mathbb{F}_\mathfrak{p}}$, the specialization $f_\mathfrak{p} : X_\mathfrak{p} \dashrightarrow X_\mathfrak{p}$ of f at the prime ideal $\mathfrak{p}$ of $Z_\mathfrak{p}$ is dominant, and $\deg f_\mathfrak{p}|_{E_\mathfrak{p}} = \deg f_E$.

Since there are only finitely many points in the orbits of $I(f_\mathfrak{p})$ and the orbits of ramified points of $f_\mathfrak{p}$, by [8, Proposition 5.5], there exists a closed point $x \in E_\mathfrak{p}$ such that x is periodic, $E_\mathfrak{p}$ is the unique irreducible component of $X_\mathfrak{p} \setminus \mathbb{A}^2_\mathfrak{p}$ containing x, $x \notin I(f_\mathfrak{p}^n)$ for all $n \geq 0$ and $f_\mathfrak{p}|_{E_\mathfrak{p}}$ is not ramified at any point on the orbit of x. After replacing $\mathbb{Q}_\mathfrak{p}$ by a finite extension $K_\mathfrak{p}$ we may suppose that x is defined over $O_{K_\mathfrak{p}}/\mathfrak{p}$. After replacing f by a positive iterate, we may suppose that x is fixed by $f_\mathfrak{p}$.

The fixed point x of $f_\mathfrak{p}$ defines an open and closed polydisc U in $X(K_\mathfrak{p})$ with respect to the $\mathfrak{p}$-adic norm $|\cdot|_\mathfrak{p}$. We have $f(U) \subseteq U$. Observe that $f^*E = d(f,v)E$ in U and $d(f,v) \geq 2$. So for all point $q \in U \cap \mathbb{A}^2(K_\mathfrak{p})$, we have $d_\mathfrak{p}(f^n(q), E) \to 0$ as $n \to \infty$.

Since $f_\mathfrak{p}|_{E_\mathfrak{p}}$ is not ramified at x, after replacing f by some positive iterate, we may suppose that $df_\mathfrak{p}|_{E_\mathfrak{p}}(x) = 1$. By [16, Theorem 1], we have that for any point $q \in U \cap E$, there exists a $\mathfrak{p}$-adic analytic map $\Psi : O_{K_\mathfrak{p}} \to U \cap E$, such that for any $n \geq 0$, we have $f^n(q) = \Psi(n)$.

If there exists a preperiodic point q in $U \cap E$, the there exists $m \geq 0$ such that $f^m(q)$ is periodic. Then there are infinitely many $n \in \mathbb{Z}^+ \subseteq O_{K_\mathfrak{p}}$ such that $\Psi(n) = f^m(q)$. The fact that $O_{K_\mathfrak{p}}$ is compact shows that Ψ is constant. It follows that q is fixed. Thus all preperiodic points in $U \cap E$ are fixed by $f|_E$.

Let S be the set of all fixed points in $U \cap E$. Since $f^n|_E \neq \mathrm{id}$ for all $n \geq 1$, S is finite.

We first treat the case $S = \emptyset$. Pick a point $p \in U \cap \mathbb{A}^2(\bar{\mathbb{Q}})$. Then p is not preperiodic. If $O(p)$ is not Zariski dense, we denote by Z its Zariski closure. Pick a one dimensional irreducible component C of Z. We have $C \cap E \cap U \neq \emptyset$, and for all points $q \in C \cap E \cap U$, q is preperiodic under $f|_E$. This contradicts our assumption, so $O(p)$ is Zariski dense.

Next we treat the case $S \neq \emptyset$. Since $|df|_E(q_i)|_\mathfrak{p} = 1$ for all $i = 1, \cdots, m$, we have $df|_E(q_i) \neq 0$. By embedding K in $\mathbb{C}$, X we endow X with the usual Euclidean topology. By [1, Theorem 3.1.4], for any $i = 1, \cdots, m$, there exists a unique complex analytic manifold W not contained in E such that $f(W) = W$. It follows that there are at most one irreducible algebraic curve $C_i \neq E$ in X such that $q_i \in C_i$ and $f(C_i) \subseteq C_i$. For convenience, if such an algebraic curve does not exists, we define C_i to be $\emptyset$.

For any $n \geq 1$, by applying [1, Theorem 3.1.4] for f^n, if C is a curve satisfying $q_i \in C$ and $f(C) \subseteq C$, then $C = C_i$. Moreover if C' is an irreducible component of $f^{-1}(C_i)$ such that $q \in C'$, then for any point $y \in C'$ near q w.r.t. the Euclidean topology, we have $f(p) \in C$. Then by [1, (iv) of Theorem 3.1.4], we have $p \in C$. It follows that $C' = C$. Thus there exists a small open and closed neighborhood U_i of q_i w.r.t. to the norm $|\cdot|_\mathfrak{p}$ such that for all $j \neq i$, $q_j \notin U_i$, $f(U_i) \subseteq U_i$ and $f^{-1}(C_i \cap U_i) \cap U_i = C_i \cap U_i$.

Observe that $\mathbb{A}^2(\bar{\mathbb{Q}} \cap K_\mathfrak{p})$ is dense in $\mathbb{A}^2(K_\mathfrak{p})$ w.r.t. $|\cdot|_\mathfrak{p}$.

There exists a $\bar{\mathbb{Q}}$-point p in $U_1 \setminus C_1(K_{\mathfrak{p}})$. If the orbit $O(p)$ of p is not Zariski dense, denote by Z its Zariski closure. Since $d_{\mathfrak{p}}(f^n(p), E) \to 0$ as $n \to \infty$, p is not preperiodic. It follows that there exists a one dimensional irreducible component C of Z which is periodic. There exists $a, b > 0$ such that $f^{an+b}(p) \in C$ for all $n \geq 0$. Since $d_{\mathfrak{p}}(f^n(p), E) \to 0$ as $n \to \infty$ and U_1 is closed, there exists a point $q \in C \cap E \cap U_1$. It follows that q is periodic under $f|_E$.Then q is fixed and $q = q_1$. This implies $C = C_1$. Since $f^b(p) \in C_1 \cap V$ and $f^{-1}(C_1) \cap V = C_1$, we have $p \in C_1$ which is a contradiction. It follows that $O(p)$ is Zariski dense. □

Proposition 6.3. *If $\lambda_2^2(f) = \lambda_1(f) > 1$ then Theorem 1.1 holds.*

Proof of Proposition 6.3. By [12, Proposition 5.1] and [12, Proposition 5.3] there exists a divisorial valuation $v_* \in V_\infty$, satisfying $f_\bullet(v_*) = v_*$ and $d(f, v_*) = \lambda_1 \geq 2$. Moreover, there exists a compactification X of $\mathbb{A}^2$ and an irreducible component E in $X \setminus \mathbb{A}^2$ satisfying $v_* = v_E$ and $\deg(f|_E) = \lambda_1 \geq 2$. We conclude our theorem by invoking Lemma 6.2. □

7. Totally invariant curves

Let $f : \mathbb{A}^2 \to \mathbb{A}^2$ be a dominant polynomial endomorphism defined over an algebraically closed field k of of characteristic 0. Let X be a compactification of $\mathbb{A}^2_k$. Then f extends to a rational selfmap on X.

As in [7], a curve C in X is said to be *totally invariant* if the strict transform $f^\# C$ equals C.

If there are infinitely many irreducible totally invariant curves in $\mathbb{A}^2_k$, then [7, Theorem B] shows that f preserves a nontrivial fibration:

Proposition 7.1. *If there are infinitely many irreducible curves in $\mathbb{A}^2_k$ that are totally invariant under f, then there is a nonconstant rational function g satisfying $g \circ f = g$.*

In this section, we give a direct proof of this result.

Proof of Proposition 7.1. Let $\{C_i\}_{i \geq 1}$ be an infinite sequence of distinct irreducible totally invariant curves in $\mathbb{A}^2_k$. Since the ramification locus of f is of dimension at most one, after replacing $\{C_i\}_{i \geq 1}$ by an infinite subsequence, we may suppose that C_i is not contained in the ramification locus of f for any $i \geq 0$. Then we have $\mathrm{ord}_{C_i} f^* C_i = 1$ for all $i \geq 0$.

Let $E_1, \cdots, E_s$ be the set of irreducible curves in $\mathbb{A}^2$ contracted by f. Let V be the $\mathbb{Q}$-subspace in $\mathrm{Div}(\mathbb{A}^2) \otimes \mathbb{Q}$ spanned by E_i, $i = 1, \cdots s$. Then we have $f^* C_i = C_i + F_i$ where $F_i \in V$ for all $i \geq 1$. Set $W_i := \bigcap_{j \geq i} (\sum_{t \geq j} \mathbb{Q} F_t) \subseteq V$. We have $W_{i+1} \subseteq W_i$ for $i \geq 1$. Since V is of finite dimension, there exists $n_0 \geq 1$, such that $W_i = W_{n_0}$ for all $i \geq n_0$. We may suppose that $n_0 = 1$ and set $W := W_{n_0}$. Moreover we may suppose that W is generated by $F_1, \cdots, F_l$ where $l = \dim W$.

For all $i \geq l+1$, we have $F_i = \sum_{j=1}^l a_j^i F_j$ where $a_j^i \in \mathbb{Q}$. Since $F_i = f^* C_i - r C_i$, we have $f^*(C_i - \sum_{j=1}^l a_j^i C_j) = r(C_i - \sum_{j=1}^l a_j^i C_j)$. There exists $n_i \in \mathbb{Z}^+$ such that $n_i a_j^i \in \mathbb{Z}$ for all $j = 1, \cdots, l$. Then we have $f^*(n_i C_i - \sum_{j=1}^l n_i a_j^i C_j) = r(n_i C_i - \sum_{j=1}^l n_i a_j^i C_j)$. Up to multiplication by a nonzero constant, there exists a unique $g_i \in k(x,y) \setminus \{0\}$ such that $\mathrm{Div}(g_i) = n_iC_i - \sum_{j=1}^{l} n_i a_j^i C_j$. It follows that $f^*g_i = A_i g_i$ where $A_i \in k \setminus \{0\}$. Since $n_iC_i - \sum_{j=1}^{l} n_i a_j^i C_j \neq 0$ for $i \geq l+1$, g_i is non constant for $i \geq l+1$. This concludes the proof. □

Corollary 7.2. *If f is birational, then either there is a nonconstant rational function g satisfying $g \circ f = g$, or there exists a point $p \in \mathbb{A}^2(k)$ with Zariski dense orbit.*

Proof of Corollary 7.2. There exists a finite generated $\mathbb{Q}$-subalgebra R of k such that f is defined over R. Denote by K the fraction field of R.

By [5, Lemma 3.1], there exists a prime $\mathfrak{p} \geq 3$, and an embedding of R into $\mathbb{Z}_{\mathfrak{p}}$ such that all coefficients of f are of $\mathfrak{p}$-adic norm 1. Denote by $\mathbb{F}$ the algebraic closure of $\mathbb{F}_{\mathfrak{p}}$. Then the degree of the specialization $f_{\mathfrak{p}} : \mathbb{A}^2_{\mathbb{F}} \dashrightarrow \mathbb{A}^2_{\mathbb{F}}$ of f equals $\deg f$.

By [19, Proposition 6.2], there exists a noncritical periodic point $x \in \mathbb{A}^2(\mathbb{F})$. After replacing $\mathbb{Q}_p$ by a finite extension $K_{\mathfrak{p}}$, we may suppose that x is defined over $O_{K_{\mathfrak{p}}}/\mathfrak{p}$. Replacing f by a suitable iterate, we may suppose that x is fixed. Since x is noncritical, $df_{\mathfrak{p}}(x)$ is invertible. After replacing f by a suitable iterate, we may suppose that $df_{\mathfrak{p}}(x) = \mathrm{id}$.

The fixed point x defines an open and closed neighborhood U in $\mathbb{A}^2(K_{\mathfrak{p}})$ with respect to $d_{\mathfrak{p}}$ such that $f(U) \subseteq U$. By applying [16, Theorem 1], we have that for any point $q \in U$, there exists a $\mathfrak{p}$-adic analytic map $\Psi : O_{K_{\mathfrak{p}}} \to U$ such that for any $n \geq 0$, we have $f^n(q) = \Psi(n)$.

Arguing by contradiction, we suppose that the orbit $O(q)$ of q is not Zariski dense for any $q \in \mathbb{A}^2(k \cap K_{\mathfrak{p}})$. As in the proof of Lemma 6.2, if q is preperiodic, then q is fixed. Suppose that q is non-preperiodic. There exists an irreducible curve C such that $f^n(q) \in C$ for infinitely many $n \geq 0$. Let P be a polynomial such that C is defined by $P = 0$. Then $P \circ \Psi$ is an analytic function on $O_{K_{\mathfrak{p}}}$ having infinitely many zeros. It follows that $P \circ \Psi \equiv 0$ and then $f^n(q) \in C$ for all $n \geq 0$. Then we have $f(C) = C$. Since f is birational, a curve C is totally invariant by f if and only if $f(C) = C$. We may suppose that $f \neq \mathrm{id}$. Since $\overline{\mathbb{Q}} \cap K_{\mathfrak{p}} \subseteq k$ and the $\overline{\mathbb{Q}} \cap K_{\mathfrak{p}}$- points in U are Zariski dense in $\mathbb{A}^2_{K_{\mathfrak{p}}}$, there are infinitely many irreducible totally invariant curves in $\mathbb{A}^2$. We conclude our Corollary by invoking Proposition 7.1. □

8. Proof of Theorem 1.1

Let $f : \mathbb{A}^2 \to \mathbb{A}^2$ be a dominant polynomial endomorphism defined over an algebraically closed field k of characteristic 0.

After replacing k by an algebraically closed subfield which contains all the coefficients of f, we may suppose that the transcendence degree of k over $\bar{\mathbb{Q}}$ is finite.

By Lemma 6.1, Proposition 6.3 and Corollary 7.2, we may suppose that $\lambda_1^2 > \lambda_2 > 1$ and v_* is divisorial. Suppose that $v_* = v_E$ for some irreducible exceptional divisor E in some compactification X. If $f^n|_E \neq \mathrm{id}$ for all $n \geq 1$, Lemma 6.2 concludes the proof. So after replacing f by a suitable iterate, we may suppose that $f|_E = \mathrm{id}$.

By choosing a suitable compactification $X \in \mathcal{C}_0$, we may suppose that $E \cap I(f) = \emptyset$. There exists a subfield K of k which is finite generated over $\mathbb{Q}$ such that $X, f, E, I(f)$ are defined over K. Moreover we may suppose that $E \simeq \mathbb{P}^1$ over K.

By [5, Lemma 3.1], there exists a prime $\mathfrak{p} \geq 3$, such that we can embed K into $\mathbb{Q}_{\mathfrak{p}}$. Further there exists an open and closed set U of $X(\mathbb{Q}_{\mathfrak{p}})$ w.r.t. the norm $|\cdot|_{\mathfrak{p}}$ containing E and satisfying $U \cap I(f) = \emptyset$, $f(U) \subseteq U$ and $d_{\mathfrak{p}}(f^n(p), E) \to 0$ as $n \to \infty$ for any $p \in U$.

By [1, Theorem 3.1.4], for any point $q \in E(\bar{K} \cap Q_{\mathfrak{p}})$, there is at most one irreducible algebraic curve $C_q \neq E$ in X such that $q \in C_q$ and $f(C_q) \subseteq C_q$. For convenience, if such an algebraic curve does not exists, set $C_q := \emptyset$. Further if $C_q \neq \emptyset$, C_q is smooth at q and intersects E transitively.

If there are only finitely many points $q \in E(\bar{K} \cap \mathbb{Q}_{\mathfrak{p}})$ such that C_q is an algebraic curve, there exists a point $q \in E$ and an open and closed set V w.r.t. the norm $|\cdot|_{\mathfrak{p}}$ containing q such that $f(V) \subseteq V$ and $C_x = \emptyset$ for all $x \in V \cap E(\bar{K} \cap \mathbb{Q}_{\mathfrak{p}})$. Pick a point $p \in V \cap \mathbb{A}^2(\bar{K} \cap Q_{\mathfrak{p}})$, then p is not preperiodic. If $O(p)$ is not Zariski dense, we denote by Z its Zariski closure. There exists a one dimensional irreducible component C of Z which is periodic under f. Since $C \cap O(p)$ is infinite, C is definied over $\bar{K} \cap Q_{\mathfrak{p}}$. Since $d_v(f^n(p), E) \to 0$ as $n \to \infty$, we have $C \cap E(\bar{K} \cap Q_{\mathfrak{p}}) \cap V \neq \emptyset$. Pick $q \in C \cap E(\bar{K} \cap Q_{\mathfrak{p}}) \cap V$, then we have that $C_q = C$ is an algebraic curve which contradicts our assumption. Thus $O(p)$ is Zariski dense.

Otherwise there exists an infinite sequence of points q_i, $i \geq 1$ such that $C_i := C_{q_i}$ is an algebraic curve. Since $f|_{C_i}$ is an endomorphism of C_i of degree $\lambda_1 > 1$, every C_i is rational and has at most two branches at infinity. We may suppose that E is the unique irreducible component of $X \setminus \mathbb{A}^2$ containing q_i for all $i \geq 1$ and $C_i \neq C_j$ for $i \neq j$. We need the following result, which is proved below.

Lemma 8.1. *After replacing $\{C_i\}_{i \geq 1}$ by an infinite subsequence, we have that either* $\deg(C_i)$ *is bounded or Theorem 1.1 holds.*

Suppose that $\deg(C_i)$ is bounded. Pick an ample line bundle L on X. Then there exists $M > 0$ such that $(C_i \cdot L) \leq M$ for all $i \geq 1$.

There exist a smooth projective surface Γ, a birational morphism $\pi_1 : \Gamma \to X$ and morphism $\pi_2 : \Gamma \to X$ satisfying $f = \pi_2 \circ \pi_1$. We denote by f_* the map $\pi_{2*} \circ \pi_1^* : \mathrm{Div}X \to \mathrm{Div}X$. Let E_{π_1} be the union of exceptional irreducible divisors of π_1 and $\mathfrak{E}$ be the set of effective divisors in X supported by $\pi_2(E_{\pi_1})$. It follows that for any curve C in X, there exists $D \in \mathfrak{E}$ such that $f_*C = \deg(f|_C) f(C) + D$.

For any effective line bundle $M \in \mathrm{Pic}(X)$, the projective space $H_M := \mathbb{P}(H^0(M))$ parameterizes the curves C in the linear system $|M|$. Since $\mathrm{Pic}^0(X) = 0$, for any $l \geq 0$, there are only finitely many effective line bundles satisfying $(M \cdot L) \leq l$.

Then $H^l := \coprod_{(M \cdot L) \leq l} H_M$ is a finite union of projective spaces and it parameterizes the curves C in X satisfying $(C \cdot L) \leq l$.

There exists $d \geq 1$ such that $dL - f^*L$ is nef. Then, for any curve C in X, we have $(f_*C \cdot L) = (C \cdot f^*L) \leq d(C \cdot L)$. It follows that f_* induces a morphism $F : H^l \to H^{dl}$ by $C \to f_*C$ for all $l \geq 1$. For all $l \geq 1$, $a \in \mathbb{Z}^+$ and $D \in \mathfrak{E}$, there exists an embedding $i_{a,D} : H_l \to H_{al+(D \cdot L)}$ by $C \mapsto aC + D$. Let $Z_1, \cdots, Z_m$ be all irreducible components of the Zariski closure of $\{C^j\}_{j \leq -1}$ in H^M of maximal dimension. For any $i \in \{1, \cdots, m\}$, there exists $l \leq M$ such that $(C \cdot L) = l$ for all $C \in Z_i$. Let S be the finite set of pairs (a, D) where $a \in \mathbb{Z}^+$, $D \in \mathfrak{E}$ satisfying $al + (D \cdot L) \leq dM$. Then we have $F(Z_i) \subseteq \bigcup_{j=1,\cdots,m} \bigcup_{(a,D)\in S} i_{a,D}(Z_j)$. It follows that there exists a unique $j_i \in \{1, \cdots, m\}$, and a unique $(a, D) \in S$ such that $F(Z_i) = i_{a,D}(Z_{j_i})$. Observe that, the map $i \mapsto j_i$ is an one to one map of $\{1, \cdots, m\}$. After replacing f by a positive iterate, we may suppose that $j_i = i$ and $F(Z_i) \subseteq i_{a_{Z_i}, D_{Z_i}} Z_i$ for all $i = 1, \cdots, m$. Set $Z := Z_1$, $a = a_{Z_1}$, $D = D_{Z_1}$. We may suppose that $C_i \in Z$ for all $i \geq 1$. Since $f(C_i) = C_i$, for all $i \geq 1$, we have $i^{-1}_{a_{Z_1}, D_{Z_1}} \circ F|_Y = \mathrm{id}$.

For any point $t \in Z$, denote by C^t the curve parameterized by t. Then $f(C^t) = C^t$ for all $t \in Z$. Since C_i intersects E transversely at at most two points, there exists $s \in \{1, 2\}$ such that C^t intersects E transversely at s points for a general $t \in Z$. It follows that, for a general point $t \in Z$, there are exactly s points $q_t^1, \cdots, q_t^s \in E$ such that $C^t = C_{q_t^j}$ for $j = 1, \cdots, s$.

Set $Y := \{(p, t) \in X \times Z |\ p \in C^t\}$. Denote by $\pi_1 : Y \to X$ the projection to the first coordinate and by $\pi_2 : Y \to Z$ the projection to the second coordinate. Since $C^t \cap E$ is not empty for general $t \in Z$, the map $\pi_2|_{\pi_1^* E}$ is dominant. We see that f induces a map $T : Y \to Y$ defined by $(p, t) \to (f(p), t)$. Since there are infinitely many points in E contained in $\pi_1(\pi_1^* E)$, so $\pi_1|_{\pi_1^* E} : \pi_1^* E \to E$ is dominant. For a general point $t \in Z$, there are exactly s points in $\pi_1^* E$. It follows that the map $\pi_2|_{\pi_1^* E}$ is generically finite of degree s. For a general point $q \in E$, there exists only one point $(q, C_q) \in \pi_1^* E$. Hence $\pi_1(\pi_1^* E) : \pi_1^* E \to E$ is birational. It follows that Z is a rational curve and Y is a surface. Thus the morphism $\pi_1 : Y \to X$ is generically finite. Let p be a general point in $\mathbb{A}^2$. If $\#\pi_1^{-1}(p) \geq 2$, then there are $t_1 \neq t_2 \in Z$ such that $p \in C^{t_1} \cap C^{t_2}$. Since there exists $M' > 0$ such that $\deg C^t \leq M'$ for all $t \in Z$, we have $\#(C_{t_1} \cap C_{t_2}) \leq M'^2$. If follows that there exist $a < b \in \{0, \cdots, M'^2\}$ such that $f^a(p) = f^b(p)$. This contradicts the assumption that p is general. It follows that the morphism $\pi_1 : Y \to X$ is birational. Identify Z with $\mathbb{P}^1$. Set $g := \pi_2 \circ \pi_1^{-1}$. Then we have $g \circ f = g$, which completes the proof.

Proof of Lemma 8.1. If C_i has only one place at infinity for all $i \geq 1$, then $\deg C_i = b_E$. So we may suppose that C_i has two places at infinity for all $i \geq 1$.

We first suppose that $\# C_i \cap E = 2$ for all $i \geq 1$. We may suppose that $C_i \cap E \cap \mathrm{Sing}(X \setminus \mathbb{A}^2) = \emptyset$ for all $i \geq 1$. Then for all $i \geq 1$, we have $\deg C_i = 2b_E$.

Then we may suppose that $C_i \cap E = \{q_i\}$ for all $i \geq 1$. Let c_i be the unique branch of C at infinity centered at q_i and w_i the unique branch of C at infinity not centred at q_i. Since $f(C_i) = C_i$, we have $f_\bullet(c_i) = c_i$ and $f_\bullet(w_i) = w_i$.

We first treat the case $\theta^* = Z_{v^*}$ where $v^* \in V_\infty$ is divisorial. It follows that $\lambda_2/\lambda_1 \in \mathbb{Z}^+$. Observe that $\deg(f|_{C_i}) = \lambda_1$ for i large enough.

If $\lambda_1 = \lambda_2$, then the strict transform $f^\#(C_i)$ equals to C_i for i large enough, and the lemma follows from Proposition 7.1.

Otherwise we have $\lambda_2/\lambda_1 \geq 2$. Set $v^* := v_{E'}$. Then we have $d(f, v^*) = \lambda_2/\lambda_1 \geq 2$ and $\deg(f|_{E'}) = \lambda_1 > 1$. By Lemma 6.2, Theorem 1.1 holds.

Next we treat the case $\theta^* = Z_{v^*}$ where $v^* \in V_\infty$ is not divisorial. Since $\alpha(v^*) = 0$, v^* can not be irrational. Thus v^* is infinitely singular, and hence an end in V_∞. Then f is proper.

By [17, Proposition 15.2], there exists $v_1 < v^*$ such that for any valuation $v \neq v^*$ in $U := \{w \in V_\infty |\ w > v_1\}$, there exists $N \geq 1$ such that $f_\bullet^n(v) \notin U$ for all $n \geq N$. It follows that there is no curve valuation in U which is periodic under $f_\bullet$. Let U_E be the open set in V_∞ consisting of all valuations whose center in X is contained E. If follows that $w_i \notin U \cup U_E$ for all $i \geq 1$. Hence $w_i \notin U \cup f_\bullet^{-N}(U_E)$ for all $N \geq 0$. Set $W_{-4} := \{v \in V_\infty |\ \alpha(v) \geq -4\}$. By [17, Proposition 11.6] and the fact that $W_{-4} \backslash U$ is compact, there exists $N \geq 0$ such that $f_\bullet^N(W_{-4} \backslash U) \subseteq U_E$. Then we have $W_{-4} \subseteq U \cup f_\bullet^{-N}(U_E)$.

Since the boundary $\partial(V_\infty \setminus (U \cup f_\bullet^{-N}(U_E)))$ of $V_\infty \setminus (U \cup f_\bullet^{-N}(U_E))$ is finite and $w_i \in V_\infty \setminus (U \cup f_\bullet^{-N}(U_E))$ for all $i \geq 1$, we may suppose that there exists $w \in \partial(V_\infty \setminus (U \cup f_\bullet^{-N}(U_E)))$ satisfying $w_i > w$ for all $i \geq 1$.

If, for all $i \geq 1$, we have $(w_i \cdot l_\infty) \leq 1/2 \deg(C_i)$, then $\deg C_i = (w_i \cdot l_\infty) + (c_i \cdot l_\infty) \leq 1/2 \deg(C_i) + b_E$. It follows that $\deg(C_i) \leq 2b_E$.

Thus we may suppose that $(w_i \cdot l_\infty) \geq 1/2 \deg(C_i)$ for all $i \geq 1$. For any $i \neq j$, the intersection number $(C_i \cdot C_j)$ is the sum of the local intersection numbers at all points in $C_i \cap C_j$. Since all the local intersection numbers are positive, we have

$$\deg(C_i) \deg(C_j) \geq (c_i \cdot c_j) + (w_i \cdot w_j).$$

By the calculation in Section 2.6, we have

$$\begin{aligned} &(c_i \cdot c_j) + (w_i \cdot w_j) \\ \geq\ & b_E^2(1 - \alpha(v_E)) + (w_i \cdot l_\infty)(w_j \cdot l_\infty)(1 - \alpha(w)) \\ \geq\ & b_E^2(1 - \alpha(v_E)) + 5/4 \deg(C_i) \deg(C_j). \end{aligned}$$

Thus $\deg(C_i) \deg(C_j) \leq -4b_E^2(1 - \alpha(v_E)) < 0$, which is a contradiction.

Finally, we treat the case when $\#\mathrm{Supp}\Delta\theta^* \geq 2$. By [12, Theorem 2.4], we have $\theta^* > 0$ on the set $W_0 := \{v \in V_\infty |\ \alpha(v) \geq 0\}$. Set $W_{-1} := \{v \in V_\infty |\ \alpha(v) \geq -1\}$ and $Y := \{v \in W_{-1} |\ \theta^*(v) = 0\}$. By [17, Proposition 11.2], Y is compact. For any point $y \in Y$, there exists $w_y < y$ satisfying $\alpha(w_y) \in (-1, 0)$. Set $U_y := \{v \in V_\infty |\ v > w_y\}$. There are finitely many points $y_1, \cdots, y_l$ such that $Y \subseteq \bigcup_{i=1}^l U_{y_i}$. Pick $r := 1/2 \min\{-\alpha(w_{y_i})\}_{i=1,\cdots,l}$. Then $r \in (0, 1)$, and $W_{-r} \cap (\bigcup_{i=1}^l U_{y_i}) = \emptyset$. It follows that there exists $t > 0$ such that $\theta^* \geq t$ on W_{-r}. By [17, Proposition 11.6], there exists $N \geq 0$ such that $f_\bullet^N(W_{-r}) \subseteq U_E$. Then we have $W_{-r} \subseteq f_\bullet^{-N}(U_E)$.

Since the boundary $\partial(V_\infty \setminus (U \cup f_\bullet^{-N}(U_E)))$ of $V_\infty \setminus (U \cup f_\bullet^{-N}(U_E))$ is finite and $w_i \in V_\infty \setminus (U \cup f_\bullet^{-N}(U_E))$ for all $i \geq 1$, we may suppose that there exists $w \in \partial(V_\infty \setminus (U \cup f_\bullet^{-N}(U_E)))$ satisfying $w_i > w$ for all $i \geq 1$.

Pick $\delta \in (0, \frac{r}{2(1+r)})$. If, for all $i \geq 1$, we have $(w_i \cdot l_\infty) \leq (1 - \delta) \deg(C_i)$, then $\deg C_i = (w_i \cdot l_\infty) + (c_i \cdot l_\infty) \leq (1 - \delta) \deg(C_i) + b_E$. It follows that $\deg(C_i) \leq b_E/\delta$.

Thus we may suppose that $(w_i \cdot l_\infty) \geq (1 - \delta) \deg(C_i)$ for all $i \geq 1$. For any $i \neq j$, we have

$$\deg(C_i) \deg(C_j) \geq (c_i \cdot c_j) + (w_i \cdot w_j)$$

$$\geq b_E^2(1-\alpha(v_E)) + (w_i \cdot l_\infty)(w_j \cdot l_\infty)(1-\alpha(w))$$
$$\geq b_E^2(1-\alpha(v_E)) + (1-\delta)^2(1+r)\deg(C_i)\deg(C_j).$$

Set $t := (1-\delta)^2(1+r)-1$, then we have

$$t > (1-2\delta)(1+r)-1 > (1-r/(1+r))(1+r)-1 = 0,$$

and hence $\deg(C_i)\deg(C_j) \leq -t^{-1}b_E^2(1-\alpha(v_E)) < 0$ which is a contradiction. This concludes the proof. □

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