ON DEGREES OF BIRATIONAL MAPPINGS

SERGE CANTAT AND JUNYI XIE

ABSTRACT. We prove that the degrees of the iterates $\deg(f^n)$ of a birational map satisfy $\liminf(\deg(f^n)) < +\infty$ if and only if the sequence $\deg(f^n)$ is bounded, and that the growth of $\deg(f^n)$ cannot be arbitrarily slow, unless $\deg(f^n)$ is bounded.

1. DEGREE SEQUENCES

Let $\mathbf{k}$ be a field. Consider a projective variety X, a polarization H of X (given by hyperplane sections of X in some embedding $X \subset \mathbb{P}^N$), and a birational transformation f of X, all defined over the field $\mathbf{k}$. Let k be the dimension of X. The **degree** of f with respect to the polarization H is the integer

$$\deg_H(f) = (f^*H) \cdot H^{k-1} \tag{1.1}$$

where f^*H is the total transform of H, and $(f^*H) \cdot H^{k-1}$ is the intersection product of f^*H with $k-1$ copies of H. The degree is a positive integer, which we shall simply denote by $\deg(f)$, even if it depends on H. When f is a birational transformation of the projective space $\mathbb{P}^k$ and the polarization is given by $O_{\mathbb{P}^k}(1)$ (i.e. by hyperplanes $H \subset \mathbb{P}^k$), then $\deg(f)$ is the degree of the homogeneous polynomial formulas defining f in homogeneous coordinates.

The degrees are submultiplicative, in the following sense:

$$\deg(f \circ g) \leq c_{X,H} \deg(f) \deg(g) \tag{1.2}$$

for some positive constant $c_{X,H}$ and for every pair of birational transformations. Also, if the polarization H is changed into another polarization H', there is a positive constant c which depends on X, H and H' but not on f, such that

$$\deg_H(f) \leq c \deg_{H'}(f) \tag{1.3}$$

We refer to [11, 16, 18] for these fundamental properties.

The **degree sequence** of f is the sequence $(\deg(f^n))_{n\geq 0}$; it plays an important role in the study of the dynamics and the geometry of f. There are

Date: 2018.

infinitely, but only countably many degree sequences (see [4, 19]); unfortunately, not much is known on these sequences when $\dim(X) \geq 3$ (see [3, 10] for $\dim(X) = 2$). In this article, we obtain the following basic results.

- The sequence $(\deg(f^n))_{n\geq 0}$ is bounded if and only if it is bounded along an infinite subsequence (see Theorems A and B in § 2 and § 3).
- If the sequence $(\deg(f^n))_{n\geq 0}$ is unbounded, then its growth can not be arbitrarily slow; for instance, $\max_{0\leq j\leq n} \deg(f^j)$ is asymptotically bounded from below by the inverse of the diagonal Ackermann function when $X = \mathbb{P}^k_{\mathbf{k}}$ (see Theorem C in § 4 for a better result).

We focus on birational transformations because a rational dominant transformation which is not birational has a topological degree $\delta > 1$, and this forces an exponential growth of the degrees: $1 < \delta^{1/k} \leq \lim_n (\deg(f^n)^{1/n})$ where $k = \dim(X)$ (see [11] and [6], pages 120–126).

2. Automorphisms of the affine space

We start with the simpler case of automorphisms of the affine space; the goal of this section is to introduce a p-adic method to study degree sequences.

Theorem A (Urech).– *Let f be an automorphism of the affine space $\mathbb{A}^k_{\mathbf{k}}$. If $\deg(f^n)$ is bounded along an infinite subsequence, then it is bounded.*

2.1. **Urech's proof.** In [19], Urech proves a stronger result. Writing his proof in an intrinsic way, we extend it to affine varieties:

Theorem 2.1. *Let $X = Spec A$ be an irreducible affine variety of dimension k over the field $\mathbf{k}$. Let $f : X \to X$ be an automorphism. If $(\deg(f^n))$ is unbounded there exists $\alpha > 0$ such that $\#\{n \geq 0 |\ \deg(f^n) \leq d\} \leq \alpha d^k$; in particular, $\max_{0\leq j\leq n} \deg(f^j)$ is bounded from below by $(n/\alpha)^{1/k}$.*

Here, the degree of f^n, depends on the choice of a projective compactification Y of X and an ample line bundle L on Y. However, by Equation (1.3), the statement of Theorem 2.1 does not depend on the choice of (Y, L). Since automorphisms of X always lift to its normalization, we may assume that X is normal. To prove this theorem, we shall introduce another equivalent notion of degree.

2.1.1. *Degrees on affine varieties.* Consider X as a subvariety $X \subseteq \mathbb{A}^N \subseteq \mathbb{P}^N$. Let $\bar{X}$ be the Zariski closure of X in $\mathbb{P}^N$ and $H_1 := \mathbb{P}^N \setminus \mathbb{A}^N$ be the hyperplane at infinity. Let $\pi : Y \to \bar{X}$ be its normalization: Y is a normal projective compactification of X. Since $\pi : Y \to \bar{X}$ is finite, there exists $m \geq 1$ such (i) $H := \pi^*(mH_1|_{\bar{X}})$ is very ample on Y and (ii) H is projectively normal on Y i.e. for every $n \geq 0$, the morphism $(H^0(Y,H))^{\otimes n} \to H^0(Y,nH)$ is surjective.

If $P \in A$ is a regular function on X, we extend it as a rational function on Y, we denote by $(P) = (P)_0 - (P)_\infty$ the divisor defined by P on Y, and we define

$$\Delta(P) = \min\{d \geq 0 | \, (P) + dH \geq 0 \text{ on } Y\}, \tag{2.1}$$
$$A_d = \{P \in A | \, \Delta(P) \leq d\}, \quad (\forall d \geq 0). \tag{2.2}$$

Then $A = \cup_{d \geq 0} A_d$. Since $Y \setminus X$ is the support of H, we get an isomorphism $i_n : H^0(Y,nH) \to A_n \subseteq A$ for every $n \geq 0$. Thus, A_1 generates A and the morphism $A_1^{\otimes n} \to A_n$ is surjective. Now we define

$$\deg^H(f) = \min\{m \geq 0 | \, \Delta(f^*P) \leq m \text{ for every } P \in A_1\}. \tag{2.3}$$

For every $P \in A_n$, we can write $P = \sum_{i=1}^{l} g_{1,i} \dots g_{1,n}$ for some $g_{i,j} \in A_1$. We get $f^*P = \sum_{i=1}^{l} f^*g_{1,i} \dots f^*g_{1,n} \in A_{\deg^H(f)n}$ and

$$\Delta(f^*P) \leq \deg^H(f)\Delta(P). \tag{2.4}$$

Since A is generated by A_1, we get an embedding

$$\mathsf{End}\,(A) \subseteq \mathrm{Hom}_{\mathbf{k}}(A_1, A) = \cup_{d \geq 1} \mathrm{Hom}_{\mathbf{k}}(A_1, A_d). \tag{2.5}$$

Set $\mathsf{End}\,(A)_d = \mathsf{End}\,(A) \cap \mathrm{Hom}_{\mathbf{k}}(A_1, A_d)$. For any automorphism $f \colon X \to X$, $\deg^H(f) \leq d$ if and only if $f \in \mathsf{End}\,(A)_d$. By Riemann-Roch theorem, there exists $\gamma > 0$ such that $\dim A_n \leq \gamma n^k$, and this gives the upper bound

$$\dim \mathsf{End}\,(A)_d \leq \mathrm{Hom}_{\mathbf{k}}(A_1, A_d) \leq (\gamma d^k) \dim A_1. \tag{2.6}$$

The following proposition, proved in the Appendix, shows that this new degree $\deg^H(f)$ is equivalent to the degree $\deg_H(f)$ introduced in Section 1.

Proposition 2.2. *For every automorphism $f \in \mathsf{Aut}(X)$ we have*

$$\frac{1}{k}\deg^H(f) \leq \frac{1}{(H^k)}\deg_H(f) \leq \deg^H(f).$$

2.1.2. *Proof of Theorem 2.1.* By Proposition 2.2, the initial notion of degree can be replaced by $\deg^H$. Let γ be as in Equation (2.6). Set $\ell = (\gamma d^k)\dim A_1 + 1$, and assume that $\deg^H(f^{n_i}) \leq d$ for some sequence of positive integers $n_1 < n_2 < \ldots < n_\ell$. Each $(f^*)^{n_i}$ is in $\mathsf{End}\,(A)_d$ and, because $\ell > \dim \mathsf{End}\,(A)_d$, there is a non-trivial linear relation between the $(f^*)^{n_i}$ in the vector space $\mathsf{End}\,(A)_d$:

$$(f^*)^n = \sum_{m=1}^{n-1} a_m (f^*)^m \tag{2.7}$$

for some integer $n \leq n_\ell$ and some coefficients $a_m \in \mathbf{k}$. Then, the subalgebra $\mathbf{k}[f^*] \subseteq \mathsf{End}(A)$ is of finite dimension and $\mathbf{k}[f^*] \subseteq E_B$ for some $B \geq 0$. This shows that the sequence $(\deg^H(f^N))_{N\geq 0}$ is bounded.

Thus, if we set $\alpha = \gamma \dim A_1$, and if the sequence $(\deg^H(f^n))$ is not bounded, we obtain $\#\{n \geq 0 |\ \deg^H(f^n) \leq d\} \leq \alpha d^k$. This proves the first assertion of the theorem; the second follows easily.

2.2. **The p-adic argument.** Let us give another proof of Theorem A when $\mathrm{char}(\mathbf{k}) = 0$, which will be generalized in § 3 for birational transformations.

2.2.1. *Tate diffeomorphisms.* Let p be a prime number. Let K be a field of characteristic 0 which is complete with respect to an absolute value $|\cdot|$ satisfying $|p| = 1/p$; such an absolute value is automatically ultrametric (see [13], Ex. 2 and 3, Chap. I.2). Let $R = \{x \in K;\ |x| \leq 1\}$ be the valuation ring of K; in the vector space K^k, the unit **polydisk** is the subset $\mathsf{U} = R^k$.

Fix a positive integer k, and consider the ring $R[\mathbf{x}] = R[\mathbf{x}_1, ..., \mathbf{x}_k]$ of polynomial functions in k variables with coefficients in R. For f in $R[\mathbf{x}]$, define the norm $\parallel f \parallel$ to be the supremum of the absolute values of the coefficients of f:

$$\parallel f \parallel = \sup_I |a_I| \tag{2.8}$$

where $f = \sum_{I=(i_1,\ldots,i_k)} a_I \mathbf{x}^I$. By definition, the **Tate algebra** $R\langle \mathbf{x} \rangle$ is the completion of $R[\mathbf{x}]$ with respect to this norm. It coincides with the set of formal power series $f = \sum_I a_I \mathbf{x}^I$ converging (absolutely) on the closed polydisk R^k. Moreover, the absolute convergence is equivalent to $|a_I| \to 0$ as $\mathrm{length}(I) \to \infty$. Every element g in $R\langle \mathbf{x} \rangle^k$ determines a **Tate analytic** map $g \colon \mathsf{U} \to \mathsf{U}$.

For f and g in $R\langle \mathbf{x} \rangle$ and c in $\mathbf{R}_+$, the notation $f \in p^c R\langle \mathbf{x} \rangle$ means $\parallel f \parallel \leq |p|^c$ and the notation $f \equiv g \mod (p^c)$ means $\parallel f - g \parallel \leq |p|^c$; we then extend such notations component-wise to $(R\langle \mathbf{x} \rangle)^m$ for all $m \geq 1$.

For indeterminates $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_k)$ and $\mathbf{y} = (\mathbf{y}_1, \ldots, \mathbf{y}_m)$, the composition $R\langle \mathbf{y} \rangle \times R\langle \mathbf{x} \rangle^m \to R\langle \mathbf{x} \rangle$ is well defined, and coordinatewise we obtain

$$R\langle \mathbf{y} \rangle^n \times R\langle \mathbf{x} \rangle^m \to R\langle \mathbf{x} \rangle^n. \tag{2.9}$$

When $m = n = k$, we get a semigroup $R\langle \mathbf{x} \rangle^k$. The group of (Tate) **analytic diffeomorphisms** of U is the group of invertible elements in this semigroup; we denote it by $\mathsf{Diff}^{an}(\mathsf{U})$. Elements of $\mathsf{Diff}^{an}(\mathsf{U})$ are bijective transformations $f \colon \mathsf{U} \to \mathsf{U}$ given by $f(\mathbf{x}) = (f_1, \ldots, f_k)(\mathbf{x})$ where each f_i is in $R\langle \mathbf{x} \rangle$ with an inverse $f^{-1} \colon \mathsf{U} \to \mathsf{U}$ that is also defined by power series in the Tate algebra.

The following result is due to Jason Bell and Bjorn Poonen (see [1, 17]).

Theorem 2.3. *Let f be an element of $R\langle \mathbf{x}\rangle^k$ with $f \equiv \mathrm{id} \mod (p^c)$ for some real number $c > 1/(p-1)$. Then f is a Tate diffeomorphism of $\mathsf{U} = R^k$ and there exists a unique Tate analytic map $\Phi\colon R \times \mathsf{U} \to \mathsf{U}$ such that*

(1) $\Phi(n, \mathbf{x}) = f^n(\mathbf{x})$ *for all* $n \in \mathbf{Z}$*;*
(2) $\Phi(s+t, \mathbf{x}) = \Phi(s, \Phi(t, \mathbf{x}))$ *for all* t, s *in* R.

2.2.2. *Second proof of Theorem A.* Denote by S the finite set of all the coefficients that appear in the polynomial formulas defining f and f^{-1}. Let $R_S \subset \mathbf{k}$ be the ring generated by S over $\mathbf{Z}$, and let K_S be its fraction field:

$$\mathbf{Z} \subset R_S \subset K_S \subset \mathbf{k}. \tag{2.10}$$

Since $\mathrm{char}(\mathbf{k}) = 0$, there exists a prime $p > 2$ such that R_S embeds into $\mathbf{Z}_p$ (see [15], §4 and 5, and [1], Lemma 3.1). We apply this embedding to the coefficients of f and get an automorphism of $\mathbb{A}^k_{\mathbf{Q}_p}$ which is defined by polynomial formulas in $\mathbf{Z}_p[\mathbf{x}_1, \ldots, \mathbf{x}_k]$; for simplicilty, we keep the same notation f for this automorphism (embedding R_S in $\mathbf{Z}_p$ does not change the value of the degrees $\deg(f^n)$). Since f and f^{-1} are polynomial automorphisms with coefficients in $\mathbf{Z}_p$, they determine elements of $\mathsf{Diff}^{an}(\mathsf{U})$, the group of analytic diffeomorphisms of the polydisk $\mathsf{U} = \mathbf{Z}_p^k$.

Reducing the coefficients of f and f^{-1} modulo $p^2\mathbf{Z}_p$, one gets two permutations of the finite set $\mathbb{A}^k(\mathbf{Z}_p/p^2\mathbf{Z})$ (equivalently, f and f^{-1} permute the balls of $\mathsf{U} = \mathbf{Z}_p^k$ of radius p^{-2}, and these balls are parametrized by $\mathbb{A}^k(\mathbf{Z}_p/p^2\mathbf{Z})$; see [7]). Thus, there exists a positive integer m such that $f^m(0) \equiv 0 \mod (p^2)$. Taking some further iterate, we may also assume that the differential Df_0^m satisfies $Df_0^m \equiv \mathsf{Id} \mod (p)$. We fix such an integer m and replace f by f^m. The following lemma follows from the submultiplicativity of degrees (see Equation (1.2) in Section 1). It shows that replacing f by f^m is harmless if one wants to bound the degrees of the iterates of f.

Lemma 2.4. *If the sequence* $\deg(f^{mn})$ *is bounded for some* $m > 0$*, then the sequence* $\deg(f^n)$ *is bounded too.*

Denote by $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_k)$ the coordinate system of $\mathbb{A}^k$, and by m_p the multiplication by p: $m_p(\mathbf{x}) = p\mathbf{x}$. Change f into $g := m_p^{-1} \circ f \circ m_p$; then $g \equiv \mathsf{Id} \mod (p)$ in the sense of Section 2.2.1. Since $p \geq 3$, Theorem 2.3 gives a Tate analytic flow $\Phi\colon \mathbf{Z}_p \times \mathbb{A}^k(\mathbf{Z}_p) \to \mathbb{A}^k(\mathbf{Z}_p)$ which extends the action of g: $\Phi(n, \mathbf{x}) = g^n(\mathbf{x})$ for every integer $n \in \mathbf{Z}$. Since Φ is analytic, one can write

$$\Phi(\mathbf{t}, \mathbf{x}) = \sum_J A_J(\mathbf{t})\mathbf{x}^J \tag{2.11}$$

where J runs over all multi-indices $(j_1,\ldots,j_k)\in(\mathbf{Z}_{\geq 0})^k$ and each A_J defines a p-adic analytic curve $\mathbf{Z}_p\to\mathbb{A}^k(\mathbf{Q}_p)$. By submultiplicativity of the degrees, there is a constant $C>0$ such that $\deg(g^{n_i})\leq CB^m$. Thus, we obtain $A_J(n_i)=0$ for all indices i and all multi-indices J of length $|J|>CB^m$. The A_J being analytic functions of $t\in\mathbf{Z}_p$, the principle of isolated zeros implies that

$$A_J=0 \text{ in } \mathbf{Z}_p\langle t\rangle,\ \forall J \text{ with } |J|>CB^m. \tag{2.12}$$

Thus, $\Phi(t,\mathbf{x})$ is a polynomial automorphism of degree $\leq CB^m$ for all $t\in\mathbf{Z}_p$, and $g^n(\mathbf{x})=\Phi(n,\mathbf{x})$ has degree at most CB^m for all n. By Lemma 2.4, this proves that $\deg(f^n)$ is a bounded sequence.

3. Birational transformations

Theorem B.– *Let* $\mathbf{k}$ *be a field of characteristic* 0*. Let* X *be a projective variety and* $f\colon X\dashrightarrow X$ *be a birational transformation of* X*, both defined over* $\mathbf{k}$*. If the sequence* $(\deg(f^n))_{n\geq 0}$ *is not bounded, then it goes to* $+\infty$ *with* n*:*

$$\liminf_{n\to+\infty}\deg(f^n)=+\infty.$$

This extends Theorem A to birational transformations. With a theorem of Weil, we get: *if* f *is a birational transformation of the projective variety* X*, over an algebraically closed field of characteristic* 0*, and if the degrees of its iterates are bounded along an infinite subsequence* f^{n_i}*, then there exist a birational map* $\psi\colon Y\dashrightarrow X$ *and an integer* $m>0$ *such that* $f_Y:=\psi^{-1}\circ f\circ\psi$ *is in* $\mathsf{Aut}(Y)$*, and* f_Y^m *is in the connected component* $\mathsf{Aut}(Y)^0$ (see [5] and references therein).

Urech's argument does not apply to this context; the basic obstruction is that rational transformations of $\mathbb{A}^k_{\mathbf{k}}$ of degree $\leq B$ generate an infinite dimensional $\mathbf{k}$-vector space for every $B\geq 1$ (the maps $z\in\mathbb{A}^1_{\mathbf{k}}\mapsto(z-a)^{-1}$ with $a\in\mathbf{k}$ are linearly independent); looking back at the proof in Section 2.1.2, the problem is that the field of rational functions on an affine variety X is not finitely generated as a $\mathbf{k}$-algebra. We shall adapt the p-adic method described in Section 2.2.2. In what follows, f and X are as in Theorem B; we assume, without loss of generality, that $\mathbf{k}=\mathbf{C}$ and X is smooth. We suppose that there is an infinite sequence of integers $n_1<\ldots<n_j<\ldots$ and a number B such that $\deg(f^{n_j})\leq B$ for all j. We fix a finite subset $S\subset\mathbf{C}$ such that X, f and f^{-1} are defined by equations and formulas with coefficients in S, and we embed the ring $R_S\subset\mathbf{C}$ generated by S in some $\mathbf{Z}_p$, for some prime number $p>2$. According to [7, Section 3], we may assume that X and f have good reduction modulo p.

3.1. **The Hrushovski's theorem and p-adic polydisks.** According to a theorem of Hrushovski (see [12]), there is a periodic point z_0 of f in $X(\mathbf{F})$ for some finite field extension $\mathbf{F}$ of the residue field $\mathbf{F}_p$, the orbit of which does not intersect the indeterminacy points of f and f^{-1}. If ℓ is the period of z_0, then $f^\ell(z_0) = z_0$ and $Df^\ell_{z_0}$ is an element of the finite group $\mathsf{GL}\left((TX_{\mathbf{F}_q})_{z_0}\right) \simeq \mathsf{GL}\left(k, \mathbf{F}_q\right)$. Thus, there is an integer $m > 0$ such that $f^m(z_0) = z_0$ and $Df^m_{z_0} = \mathsf{Id}$.

Replace f by its iterate $g = f^m$. Then, g fixes z_0 in $X(\mathbf{F})$, g is an isomorphism in a neighborhood of z_0, and $Dg_{z_0} = \mathsf{Id}$. According to [2] and [7, Section 3], this implies that there is

- a finite extension K of $\mathbf{Q}_p$, with valuation ring $R \subset K$;
- a point z in $X(K)$ and a polydisk $\mathsf{V}_z \simeq R^k \subset X(K)$ which is g-invariant and such that $g|_{V_z} \equiv \mathsf{Id} \mod (p)$ (in the coordinate system $(\mathbf{x}_1, \ldots, \mathbf{x}_k)$ of the polydisk).

When the point z_0 is in $X(\mathbf{F}_p)$ and is the reduction of a point $z \in X(\mathbf{Z}_p)$, the polydisk V_z is the set of points $w \in X(\mathbf{Z}_p)$ with $|z - w| < 1$; one identifies this polydisk to $\mathsf{U} = (\mathbf{Z}_p)^k$ via some p-adic analytic diffeomorphism $\varphi\colon \mathsf{U} \to \mathsf{V}_z$; changing φ into $\varphi \circ m_p$ if necessary, we obtain $g_{V_z} \equiv \mathsf{Id} \mod (p)$ (see Section 2.2.2 and [7], Section 3.2.1). In full generality, a finite extension K of $\mathbf{Q}_p$ is needed because z_0 is a point in $X(\mathbf{F})$ for some extension $\mathbf{F}$ of $\mathbf{F}_p$.

3.2. **Controling the degrees.** As in Section 2.2.1, denote by U the polydisk $R^k \simeq \mathsf{V}_z$; thus, U is viewed as the polydisk R^k and also as a subset of $X(K)$. Applying Theorem 2.3 to g, we obtain a p-adic analytic flow

$$\Phi\colon R \times \mathsf{U} \to \mathsf{U}, \quad (t, \mathbf{x}) \mapsto \Phi(t, \mathbf{x}) \tag{3.1}$$

such that $\Phi(n, \mathbf{x}) = g^n(\mathbf{x})$ for every integer n. In other words, the action of g on U extends to an analytic action of the additive compact group $(R, +)$.

Let $\pi_1 : X \times X \to X$ denote the projection onto the first factor. Denote by $\mathsf{Bir}_D(X)$ the set of birational transformations of X of degree D; once birational transformations are identified to their graphs, this set becomes naturally a finite union of irreducible, locally closed algebraic subsets in the Hilbert scheme of $X \times X$ (see [5], Section 2.2, and references therein). Taking a subsequence, there is a positive integer D, an irreducible component B_D of $\mathsf{Bir}_D(X)$, and a strictly increasing, infinite sequence of integers (n_j) such that

$$g^{n_j} \in B_D \tag{3.2}$$

for all j. Denote by $\overline{B_D}$ the Zariski closure of B_D in the Hilbert scheme of $X \times X$. To every element $h \in \overline{B_D}$ corresponds a unique algebraic subset $\mathcal{G}_h$ of

$X \times X$ (the graph of h, when h is in B_D). Our goal is to show that, for every $t \in R$, the graph of $\Phi(t,\cdot)$ is the intersection $\mathcal{G}_{h_t} \cap \mathsf{U}^2$ for some element $h_t \in \overline{B_D}$; this will conclude the proof because $g^n(\mathbf{x}) = \Phi(n,\mathbf{x})$ for all $n \geq 0$.

We start with a simple remark, which we encapsulate in the following lemma.

Lemma 3.1. *There is a finite subset $E \subset \mathsf{U} \subset X(K)$ with the following property. Given any subset $\tilde{E}$ of $\mathsf{U} \times \mathsf{U}$ with $\pi_1(\tilde{E}) = E$, there is at most one element $h \in \overline{B_D}$ such that $\tilde{E} \subset \mathcal{G}_h$.*

Fix such a set E, and order it to get a finite list $E = (x_1, \ldots, x_{\ell_0})$ of elements of U. Let $E' = (x_1, \ldots, x_{\ell_0}, x_{\ell_0+1}, \ldots, x_\ell)$ be any list of elements of U which extends E. For every element h in $\overline{B_D}$, the variety $\mathcal{G}_h$ determines a correspondance $\mathcal{G}_h \subset X \times X$. The subset of elements $(h, (x_i, y_i)_{1 \leq i \leq \ell})$ in $\overline{B_D} \times (X \times X)^\ell$ defined by the incidence relation

$$(x_i, y_i) \in \mathcal{G}_h \tag{3.3}$$

for every $1 \leq i \leq \ell$ is an algebraic subset of $\overline{B_D} \times (X \times X)^\ell$. Add one constraint, namely that the first projection $(x_i)_{1 \leq i \leq \ell}$ coincides with E', and project the resulting subset on $(X \times X)^\ell$: we get a subset $G(E')$ of $(X \times X)^\ell$. Then, define a p-adic analytic curve $\Lambda \colon R \to (X \times X)^\ell$ by

$$\Lambda(t) = (x_i, \Phi(t, x_i))_{1 \leq i \leq \ell}. \tag{3.4}$$

If $t = n_j$, g^{n_j} is an element of B_D and $\Lambda(n_j)$ is contained in the graph of g^{n_j}; hence, $\Lambda(n_j)$ is an element of $G(E')$. By the principle of isolated zeros, the analytic curve $t \mapsto \Lambda(t) \subset (X \times X)^\ell$ is contained in $G(E')$ for all $t \in R$. Thus, for every t there is an element $h_t \in \overline{B_D}$ such that $\Lambda(t)$ is contained in the subset $\mathcal{G}_{h_t}^\ell$ of $(X \times X)^\ell$. From the choice of E and the inclusion $E \subset E'$, we know that h_t does not depend on E'. Thus, the graph of $\Phi(t,\cdot)$ coincides with the intersection of $\mathcal{G}_{h_t}$ with $\mathsf{U} \times \mathsf{U}$. This implies that the graph of $g^n(\cdot) = \Phi(n,\cdot)$ coincides with $\mathcal{G}_{h_n}$, and that the degree of g^n is at most D for all values of n.

4. Lower bounds on degree growth

We now prove that the growth of $(\deg(f^n))$ can not be arbitrarily slow unless $(\deg(f^n))$ is bounded. For simplicity, we focus on birational transformations of the projective space; there is no restriction on the characteristic of $\mathbf{k}$.

4.1. **A family of integer sequences.** Fix two positive integers k and d; k will be the dimension of $\mathbb{P}^k_{\mathbf{k}}$, and d will be the degree of $f \colon \mathbb{P}^k \dashrightarrow \mathbb{P}^k$. Set

$$m = (d-1)(k+1). \tag{4.1}$$

Then, consider an auxiliary integer $D \geq 1$, which will play the role of the degree of an effective divisor in the next paragraphs, and define

$$q = (dD+1)^m. \tag{4.2}$$

Thus, q depends on k, d and D because m depends on k and d. Then, set

$$a_0 = \left(\begin{array}{c} k+D \\ k \end{array} \right) - 1, \quad b_0 = 1, \quad c_0 = D+1. \tag{4.3}$$

Starting from the triple (a_0, b_0, c_0), we define a sequence $((a_j, b_j, c_j))_{j \geq 0}$ inductively by

$$(a_{j+1}, b_{j+1}, c_{j+1}) = (a_j, b_j - 1, qc_j^2) \tag{4.4}$$

if $b_j \geq 2$, and by

$$(a_{j+1}, b_{j+1}, c_{j+1}) = (a_j - 1, qc_j^2, qc_j^2) = (a_j - 1, c_{j+1}, c_{j+1}) \tag{4.5}$$

if $b_j = 1$. By construction, $(a_1, b_1, c_1) = (a_0 - 1, qc_0^2, qc_0^2)$.

Define $\Phi \colon \mathbf{Z}^+ \to \mathbf{Z}^+$ by

$$\Phi(c) = qc^2. \tag{4.6}$$

Lemma 4.1. *Define the sequence of integers $(F_i)_{i \geq 1}$ recursively by $F_1 = q(D+1)^2$ and $F_{i+1} = \Phi^{F_i}(F_i)$ for $i \geq 1$ (where Φ^{F_i} is the F_i-iterate of Φ). Then*

$$(a_{1+F_1+\cdots+F_i},\, b_{1+F_1+\cdots+F_i},\, c_{1+F_1+\cdots+F_i}) = (a_0 - i - 1,\, F_{i+1},\, F_{i+1}).$$

The proof is straightforward. Now, define $S \colon \mathbf{Z}^+ \to \mathbf{Z}^+$ as the sum

$$S(j) = 1 + F_1 + F_2 + \cdots + F_j \tag{4.7}$$

for all $j \geq 1$; it is increasing and goes to $+\infty$ extremely fast with j. Then, set

$$\chi_{d,k}(n) = \max \left\{ D \geq 0 \mid S(\left(\begin{array}{c} k+D \\ k \end{array} \right) - 2) < n \right\}. \tag{4.8}$$

Lemma 4.2. *The function $\chi_{d,k} \colon \mathbf{Z}^+ \to \mathbf{Z}^+$ is non-decreasing and goes to $+\infty$ with n.*

Remark 4.3. *The function S is primitive recursive* (see [9], Chapters 3 and 13). In other words, S is obtained from the basic functions (the zero function, the successor $s(x) = x+1$, and the projections $(x_i)_{1 \leq i \leq m} \to x_i$) by a finite sequence of compositions and recursions. Equivalently, there is a program computing S, all of whose instructions are limited to (1) the zero initialization $V \leftarrow 0$, (2) the increment $V \leftarrow V+1$, (3) the assignement $V \leftarrow V'$, and (4) loops of definite length. Writing such a program is an easy exercise. Now, consider the diagonal Ackermann function $A(n)$ (see [9], Section 13.3). It grows asymptotically faster than any primitive recursive function; hence, the inverse of the Ackermann diagonal function $\alpha(n) = \max\{D \geq 0 \mid \mathrm{Ack}(D) \leq n\}$ is, asymptotically, a lower bound for $\chi_{d,k}(n)$. Showing that $\chi_{d,k}$ is in the $\mathcal{L}_6$ hierarchy of [9], Chapter 13, one gets an asymptotic lower bound by the inverse of the function f_7 of [9], independent of the values of d and k.

4.2. **Statement of the lower bound.** We can now state the result that will be proved in the next paragraphs.

Theorem C.– *Let f be a birational transformation of the complex projective space $\mathbb{P}^k_{\mathbf{k}}$ of degree d. If the sequence $(\max_{0\leq j\leq n}(\deg(f^j)))_{n\geq 0}$ is unbounded, then it is bounded from below by the sequence of integers $(\chi_{d,k}(n))_{n\geq 0}$.*

Remark 4.4. There are infinitely, but only countably many sequences of degrees $(\deg(f^n))_{n\geq 0}$ (see [4, 19]). Consider the countably many sequences

$$\left(\max_{0\leq j\leq n}(\deg(f^j))\right)_{n\geq 0} \tag{4.9}$$

restricted to the family of birational maps for which $(\deg(f^n))$ is unbounded. We get a countable family of *non-decreasing, unbounded sequences of integers*. Let $(u_i)_{i\in\mathbf{Z}_{\geq 0}}$ be any countable family of such sequences of integers $(u_i(n))$. Define $w(n)$ as follows. First, set $v_j = \min\{u_0, u_1, \ldots, u_j\}$; this defines a new family of sequences, with the same limit $+\infty$, but now $v_j(n) \geq v_{j+1}(n)$ for every pair (j,n). Then, set $m_0 = 0$, and define m_{n+1} recursively to be the first positive integer such that $v_{n+1}(m_{n+1}) \geq v_n(m_n) + 1$. We have $m_{n+1} \geq m_n + 1$ for all $n \in \mathbf{Z}_{\geq 0}$. Set $w(n) := v_{r_n}(m_{r_n})$ where r_n is the unique non-negative integer satisfying $m_{r_n} \leq n \leq m_{r_n+1} - 1$. By construction, $w(n)$ goes to $+\infty$ with n and $u_i(n)$ is *asymptotically* bounded from below by $w(n)$.

In Theorem C, the result is more explicit. Firstly, the lower bound is explicitely given by the sequence $(\chi_{d,k}(n))_{n\geq 0}$. Secondly, the lower bound is not asymptotic: it works for every value of n. In particular, if $\deg(f^j) < \chi_{d,k}(n)$ for $0 \leq j \leq n$ and $\deg(f) = d$, then the sequence $(\deg(f^n))$ is bounded.

4.3. **Divisors and strict transforms.** To prove Theorem C, we consider the action of f by strict transform on effective divisors. As above, $d = \deg(f)$ and $m = (d-1)(k+1)$ (see Section 4.1).

4.3.1. *Exceptional locus.* Let X be a smooth projective variety and π_1 and $\pi_2\colon X\to\mathbb{P}^k$ be two birational morphisms such that $f=\pi_2\circ\pi_1^{-1}$; then, consider the exceptional locus $\mathrm{Exc}(\pi_2)\subset X$, project it by π_1 into $\mathbb{P}^k$, and list its irreducible components of codimension 1: we obtain a finite number

$$E_1,\ \ldots,\ E_{m(f)} \tag{4.10}$$

of irreducible hypersurfaces, contained in the zero locus of the jacobian determinant of f. Since this critical locus has degree m, we obtain:

$$m(f)\le m, \quad \text{and} \;\; \deg(E_i)\le m \quad (\forall i\ge 1). \tag{4.11}$$

4.3.2. *Effective divisors.* Denote by M the semigroup of effective divisors of $\mathbb{P}^k_{\mathbf{k}}$. There is a partial ordering $\le$ on M, which is defined by $E\le E'$ if and only if the divisor $E'-E$ is effective.

We denote by $\deg\colon M\to\mathbf{Z}_{\ge 0}$ the degree function. For every degree $D\ge 0$, we denote by M_D the set $\mathbb{P}(H^0(\mathbb{P}^k_{\mathbf{k}},\mathcal{O}_{\mathbb{P}^k_{\mathbf{k}}}(D)))$ of effective divisors of degree D; thus, M is the disjoint union of all the M_D, and each of these components will be endowed with the Zariski topology of $\mathbb{P}(H^0(\mathbb{P}^k_{\mathbf{k}},\mathcal{O}_{\mathbb{P}^k_{\mathbf{k}}}(D)))$. The dimension of M_D is equal to the integer $a_0=a_0(D,k)$ from Section 4.1:

$$\dim(M_D)=\begin{pmatrix} k+D \\ k \end{pmatrix}-1. \tag{4.12}$$

Let $G\subset M$ be the semigroup generated by the E_i:

$$G=\bigoplus_{i=1}^{m(f)}\mathbf{Z}_{\ge 0}E_i. \tag{4.13}$$

The elements of G are the effective divisors which are supported by the exceptional locus of f. For every $E\in G$, there is a translation operator $T_E\colon M\to M$, defined by $T_E\colon E'\mapsto E+E'$; it restricts to a linear projective embedding of the projective space M_D into the projective space $M_{D+\deg(E)}$. We define

$$M_D^\circ=M_D\setminus\bigcup_{E\in G\setminus\{0\},\deg(E)\le D} T_E(M_{D-\deg(E)}). \tag{4.14}$$

Thus, M_D° is the complement in M_D of finitely many proper linear projective subspaces. Also, $M_0^\circ=M_0$ is a point and M_1° is obtained from $M_1=(\mathbb{P}^k_{\mathbf{k}})^\vee$ by removing finitely many points, corresponding to the E_i of degree 1 (the hyperplanes contracted by f). Set $M^\circ=\bigcup_{D\ge 0}M_D^\circ$. This is the set of effective divisors without any component in the exceptional locus of f. The inclusion of M° in M will be denoted by $\iota\colon M^\circ\to M$. There is a natural projection $\pi_G\colon M\to G$; namely, $\pi_G(E)$ is the maximal element such that $E-\pi_G(E)$ is effective.

We denote by $\pi_\circ\colon M\to M^\circ$ the projection $\pi_\circ = \mathsf{Id}-\pi_G$; this homomorphism removes the part of an effective divisor E which is supported on the exceptional locus of f.

Remark 4.5. The restriction of the map $\pi_\circ$ to the projective space M_D is piecewise linear, in the following sense. Consider the subsets $U_{E,D}$ of M_D which are defined for every $E\in G$ with $\deg(E)\le D$ by

$$U_{E,D} = T_E(M_{D-\deg(E)}) \setminus \bigcup_{E'>E,\, E'\in G,\, \deg(E')\le D} T_{E'}(M_{D-\deg(E')}).$$

They define a stratification of M_D by (open subsets of) linear subspaces, and $\pi_\circ$ coincides with the linear map inverse of T_E on each $U_{E,D}$. Moreover, $\pi_\circ(Z)$ is closed for any closed subset $Z\subseteq M_D$.

We say that a scheme theoritic point $x\in M$ (resp. M°) is **irreducible** if the divisor of $\mathbb{P}^k$ corresponding to x is irreducible. In other words, x is irreducible, if a general closed point $y\in\overline{\{x\}}\subseteq M$ is irreducible.

4.3.3. *Strict transform.* First, we consider the total transform $f^*\colon M\to M$, which is defined by $f^*(E)=(\pi_1)_*\pi_2^*(E)$ for every divisor $E\in M$. This is a homomorphism of semigroups; it is injective on non-closed irreducible points. Let $[x_0,\dots,x_k]$ be homogeneous coordinates on $\mathbb{P}^k$. If E is defined by the homogeneous equation $P=0$, then $f^*(E)$ is defined by $P\circ f=0$; thus, f^* induces a linear projective embedding of M_D into M_{dD} for every D.

Then, we denote by $f^\circ\colon M^\circ\to M^\circ$ the strict transform. It is defined by

$$f^\circ(E) = (\pi_\circ\circ f^*\circ\iota)(E). \tag{4.15}$$

This is a homomorphism of semigroups. If $x\in M$ is an irreducible point, its total transform $f^*(x)$ is not necessarily irreducible, but $f^\circ(x)$ is irreducible.

In general, $(f^\circ)^n\neq(f^n)^\circ$, but for non-closed irreducible point $x\in M$, we have $(f^\circ)^n(x)=(f^n)^\circ(x)$ for $n\ge 0$. Indeed, a non-closed irreducible point $x\in M$ can be viewed as an irreducible hypersurface on X which is defined over some transcendental extension of $\mathbf{k}$, but not over $\mathbf{k}$. Then $f^\circ(x)$ is the unique irreducible component E of $f^*(x)$, on which $f|_E$ is birational to its image. (Note that when $\mathbf{k}$ is uncountable, one can also work with very general points of M_D for every $D\ge 1$, instead of irreducible, non-closed points).

4.4. **Proof of Theorem C.** Let η be the generic point of M_1° (η corresponds to a generic hyperplane of $\mathbb{P}^k_{\mathbf{k}}$). Note that η is non-closed and irreducible. The degree of $f^*(\eta)$ is equal to the degree of f, and since η is generic, $f^*(\eta)$ coincides with $f^\circ(\eta)$. Thus, $\deg(f) = \deg(f^\circ(\eta))$ and more generally

$$\deg(f^n) = \deg((f^\circ)^n \eta) \quad (\forall n \geq 1). \tag{4.16}$$

Fix an integer $D \geq 0$. Write $M^\circ_{\leq D}$ for the disjoint union of the $M^\circ_{D'}$ with $D' \leq D$, and define recursively $Z_D(0) = M^\circ_{\leq D}$ and

$$Z_D(i+1) = \{E \in Z_D(i) \mid f^\circ(E) \in Z_D(i)\} \tag{4.17}$$

for $i \geq 0$. A divisor $E \in M^\circ_{\leq D}$ is in $Z_D(i)$ if its strict transform $f^\circ(E)$ is of degree $\leq D$, and $f^\circ(f^\circ(E))$ is also of degree $\leq D$, up to $(f^\circ)^i(E)$ which is also of degree at most D.

Let us describe $Z_D(i+1)$ more precisely. For each i, and each $E \in G$ of degree $\deg(E) \leq dD$ consider the subset $T_E(\overline{\iota(Z_D(i))}) \cap M_{dD}$; this is a subset of M_{dD} which is made of divisors W such that $\pi_\circ(W)$ is contained in $Z_D(i)$, and the union of all these subsets when E varies is exactly the set of points W in M_{dD} with a projection $\pi_\circ(W)$ in $Z_D(i)$. Thus, we consider

$$(f^*)^{-1}(T_E(\overline{\iota(Z_D(i))})) = \{V \in M_{\leq D} \mid f^*(V) \in T_E(\overline{\iota(Z_D(i))})\}. \tag{4.18}$$

These sets are closed subsets of $M_{\leq D}$, and

$$Z_D(i+1) = Z_D(i) \bigcap \bigcup_{E \in G, \deg(E) \leq dD} \pi_\circ\left((f^*)^{-1}(T_E(\overline{\iota(Z_D(i))})\right). \tag{4.19}$$

Since $Z_D(0)$ is closed in $M^\circ_{\leq D}$ and $\pi_\circ$ is closed on $M_{\leq D}$, by induction, $Z_D(i)$ is closed for all $i \geq 0$. The subsets $Z_D(i)$ form a decreasing sequence of Zariski closed subsets (in the disjoint union $M^\circ_{\leq D}$ of the $M^\circ_{D'}$, $D' \leq D$). The strict transform f° maps $Z_D(i+1)$ into $Z_D(i)$. By Noetherianity, there exists a minimal integer $\ell(D) \geq 0$ such that

$$Z_D(\ell(D)) = \bigcap_{i \geq 0} Z_D(i); \tag{4.20}$$

we denote this subset by $Z_D(\infty) = Z_D(\ell(D))$. By construction, $Z_D(\infty)$ is stable under the operator f°; more precisely, $f^\circ(Z_D(\infty)) = Z_D(\infty) = (f^\circ)^{-1}(Z_D(\infty))$.

Let $\tau\colon \mathbf{Z}_{\geq 0} \to \mathbf{Z}_{\geq 0}$ be a lower bound for the inverse function of ℓ:

$$\ell(\tau(n)) \leq n \quad (\forall n \geq 0). \tag{4.21}$$

Assume that $\max\{\deg(f^m) \mid 0 \leq m \leq n_0\} \leq \tau(n_0)$ for some $n_0 \geq 1$. Then $\deg((f^\circ)^i(\eta)) \leq \tau(n_0)$ for every integer i between 0 and n_0; this implies that η is in the set $Z_{\tau(n_0)}(\ell(\tau(n_0))) = Z_{\tau(n_0)}(\infty)$, so that the degree of $(f^\circ)^m(\eta)$ is bounded from above by $\tau(n_0)$ for all $m \geq 0$. From Equation (4.16) we deduce that the sequence $(\deg(f^m))_{m\geq 0}$ is bounded. This proves the following lemma.

Lemma 4.6. *Let τ be a lower bound for the inverse function of ℓ. If*

$$\max\{\deg(f^m) \mid 0 \leq m \leq n_0\} \leq \tau(n_0)$$

for some $n_0 \geq 1$, then the sequence $(\deg(f^n))_{n\geq 0}$ is bounded by $\tau(n_0)$.

So, to conclude, we need to compare $\ell \colon \mathbf{Z}_{\geq 0} \to \mathbf{Z}^+$ to the function $S \colon \mathbf{Z}_{\geq 0} \to \mathbf{Z}^+$ of paragraph 4.1 (recall that S depends on the parameters $k = \dim(\mathbb{P}^k_{\mathbf{k}})$ and $d = \deg(f)$ and that ℓ depends on f). Now, write $Z'_D(i) = Z_D(i) \setminus Z_D(\infty)$, and note that it is a strictly decreasing sequence of open subsets of $Z_D(i)$ with $Z'_D(j) = \emptyset$ for all $j \geq \ell(D)$. We shall say that a closed subset of $M^\circ_{\leq D} \setminus Z_D(\infty)$ for the Zariski topology is **piecewise linear** if all its irreducible components are equal to the intersection of $M^\circ_{\leq D} \setminus Z_D(\infty)$ with a linear projective subspace of some $M_{D'}$, $D' \leq D$. We note that the intersection of two irreducible linear projective subspaces is still an irreducible linear projective subspace.

Let $\mathrm{Lin}(a,b,c)$ be the family of closed piecewise linear subsets of $M^\circ_{\leq D} \setminus Z_D(\infty)$ of dimension a, with at most c irreducible components, and at most b irreducible components of maximal dimension a. Then,

(1) $Z'_D(i+1) = \{F \in Z'_D(i) \mid f^\circ(F) \in Z'_D(i)\} = \pi_\circ(f^* Z'_D(i) \bigcap \cup_E T_E(Z'_D(i)))$, where E runs over the elements of G of degree $\deg(E) \leq dD$;
(2) in this union, each irreducible component of $T_E(Z'_D(i))$ is piecewise linear.

Recall that $q = (dD+1)^m$ (see Section 4.1). If Z is any closed piecewise linear subset of $M^\circ_{\leq D} \setminus Z_D(\infty)$ that contains exactly c irreducible components, the set

$$\begin{aligned}\pi_\circ(f^*Z \bigcap \bigcup_{E\in G,\, \deg(E)\leq dD} T_E(Z)) &= \bigcup_{E\in G,\, \deg(E)\leq dD} \pi_\circ(f^*Z \bigcap T_E(Z)) \\ &= \bigcup_{E\in G,\, \deg(E)\leq dD} T_E^{-1}|_{T_E(Z)}(f^*Z \bigcap T_E(Z))\end{aligned}$$

has at most $qc^2 = (dD+1)^m c^2$ irreducible components (this is a crude estimate: $f^*Z \bigcap T_E(Z)$ has at most c^2 irreducible components, $T_E^{-1}|_{T_E(Z)}$ is injective and the factor $(dD+1)^m$ comes from the fact that G contains at most $(dD+1)^m$ elements of degree $\leq dD$). Let us now use that the sequence $Z'_D(i)$ decreases strictly as i varies from 0 to $\ell(D)$, with $Z'_D(\ell(D)) = \emptyset$. If $0 \leq i \leq \ell(D) - 1$, and if $Z'_D(i)$ is contained in $\mathrm{Lin}(a,b,c)$, we obtain

(1) if $b \geq 2$, then $Z'_D(i+1)$ is contained in $\mathrm{Lin}(a, b-1, qc^2)$;

(2) if $b=1$, then $Z'_D(i+1)$ is contained in $\mathrm{Lin}(a-1,qc^2,qc^2)$.

This shows that

$$\ell(D)\leq S(\begin{pmatrix} k+D \\ k \end{pmatrix}-2)+1 \tag{4.22}$$

where S is the function introduced in the Equation (4.7) of Section 4.1. Since $\chi_{d,k}$ satisfies $\ell(\chi_{d,k}(n))\leq n$ for every $n\geq 1$, the conclusion follows.

5. Appendix: Proof of Proposition 2.2

We keep the notation from Section 2.1.1. Let f be an automorphism of X. There exist a normal projective irreducible variety Z and two birational morphisms $\pi_1\colon Z\to Y$ and $\pi_2\colon Z\to Y$ such that π_1 and π_2 are isomorphisms over X, and $f=\pi_2\circ\pi_1^{-1}$.

Lemma 5.1. *We have $\Delta(f^*P)\leq k(H^k)^{-1}\Delta(P)\deg_H(f)$ for every $P\in A$.*

Proof of Lemma 5.1. By Siu's inequality (see [14] Theorem 2.2.15, and [8] Theorem 1), we get

$$\pi_2^*H\leq\frac{k(\pi_2^*H\cdot(\pi_1^*H)^{k-1})}{((\pi_1^*H)^k)}\pi_1^*H=\frac{k\deg_H(f)}{(H^k)}\pi_1^*H. \tag{5.1}$$

Since $(P)+\Delta(P)H\geq 0$ we have $(\pi_2^*P)+\Delta(P)\pi_2^*H\geq 0$. It follows that

$$(\pi_1^*f^*P)+\frac{\Delta(P)k\deg_H(f)}{(H^k)}\pi_1^*H=(\pi_2^*P)+\frac{\Delta(P)k\deg_H(f)}{(H^k)}\pi_1^*H\geq 0. \tag{5.2}$$

Since $(\pi_1)_*\circ(\pi_1)^*=\mathsf{Id}$ we obtain $(f^*P)+(k\Delta(P)(H^k)^{-1}\deg_H(f))H\geq 0$. This implies $\Delta(f^*P)\leq k(H^k)^{-1}\Delta(P)\deg_H(f)$. □

Lemma 5.1 shows that $\deg^H(f)\leq k(H^k)^{-1}\deg_H(f)$. We now prove the reverse direction: $\deg_H(f)\leq(H^k)\deg^H(f)$.

Since H is very ample, Bertini's theorem gives an irreducible divisor $D\in|H|$ such that $\pi_2(E)\not\subseteq D$ for every prime divisor E of Z in $Z\setminus\pi_2^*(X)$; hence, π_2^*D is equal to the strict transform $\pi_2^\circ D$. By definition, $D=(P)+H$ for some $P\in A_1$. Thus, $(\pi_1)_*\pi_2^*H$ is linearly equivalent to $(\pi_1)_*\pi_2^*D=(\pi_1)_*\pi_2^\circ D$, and this irreducible divisor $(\pi_1)_*\pi_2^\circ D$ is the closure D_{f^*P} of $\{f^*P=0\}\subseteq X$ in Y. Writing $(f^*P)=D_{f^*P}-F$ where F is supported on $Y\setminus X$ we also get that $(\pi_1)_*\pi_2^*H$ is linearly equivalent to F. Since $\Delta(f^*P)\leq\deg^H(f)\Delta(P)=\deg^H(f)$, the definition of Δ gives

$$D_{f^*P}-F+\deg^H(f)H=(f^*P)+\deg^H(f)H\geq 0. \tag{5.3}$$

Thus, $F\leq\deg^H(f)H$ because D_{f^*P} is irreducible and is not supported on $Y\setminus X$. Altogether, this gives $\deg_H(f)=((\pi_1)_*\pi_2^*H\cdot H^{k-1})=(F\cdot H^{k-1})\leq\deg^H(f)(H^k)$.

REFERENCES

[1] Jason P. Bell. A generalised Skolem-Mahler-Lech theorem for affine varieties. *J. London Math. Soc. (2)*, 73(2):367–379, 2006.

[2] Jason P. Bell, Dragos Ghioca, and Thomas. J. Tucker. The dynamical Mordell-Lang problem for étale maps. *Amer. J. Math.*, 132(6):1655–1675, 2010.

[3] Jérémy Blanc and Serge Cantat. Dynamical degrees of birational transformations of projective surfaces. *J. Amer. Math. Soc.*, 29(2):415–471, 2016.

[4] Araceli M. Bonifant and John Erik Fornæ ss. Growth of degree for iterates of rational maps in several variables. *Indiana Univ. Math. J.*, 49(2):751–778, 2000.

[5] Serge Cantat. Morphisms between Cremona groups, and characterization of rational varieties. *Compos. Math.*, 150(7):1107–1124, 2014.

[6] Serge Cantat, Antoine Chambert-Loir, and Vincent Guedj. *Quelques aspects des systèmes dynamiques polynomiaux*, volume 30 of *Panoramas et Synthèses [Panoramas and Syntheses]*. Société Mathématique de France, Paris, 2010.

[7] Serge Cantat and Junyi Xie. Algebraic actions of discrete groups: the p-adic method. *Acta Math.*, 220(2):239–295, 2018.

[8] Steven Dale Cutkosky. Teissier's problem on inequalities of nef divisors. *J. Algebra Appl.*, 14(9):1540002, 37, 2015.

[9] Martin D. Davis and Elaine J. Weyuker. *Computability, complexity, and languages*. Computer Science and Applied Mathematics. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York, 1983. Fundamentals of theoretical computer science.

[10] Jeffrey Diller and Charles Favre. Dynamics of bimeromorphic maps of surfaces. *Amer. J. Math.*, 123(6):1135–1169, 2001.

[11] Tien-Cuong Dinh and Nessim Sibony. Une borne supérieure pour l'entropie topologique d'une application rationnelle. *Ann. of Math. (2)*, 161(3):1637–1644, 2005.

[12] Ehud Hrushovski. The elementary theory of the Frobenius automorphism. *http://arxiv.org/pdf/math/0406514v1*, pages 1–135, 2004.

[13] Neal Koblitz. *p-adic numbers, p-adic analysis, and zeta-functions*, volume 58 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1984.

[14] Robert Lazarsfeld. *Positivity in algebraic geometry. I*, volume 48 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2004. Classical setting: line bundles and linear series.

[15] Christer Lech. A note on recurring series. *Ark. Mat.*, 2:417–421, 1953.

[16] Bac-Dang Nguyen. Degrees of iterates of rational transformations of projective varieties. *arXiv*, arXiv:1701.07760:1–46, 2017.

[17] Bjorn Poonen. p-adic interpolation of iterates. *Bull. Lond. Math. Soc.*, 46(3):525–527, 2014.

[18] Tuyen Trung Truong. Relative dynamical degrees of correspondances over fields of arbitrary characteristic. *J. Reine Angew. Math.*, to appear:1–44, 2018.

[19] Christian Urech. Remarks on the degree growth of birational transformations. *Math. Res. Lett.*, 25(1):291–308, 2018.

UNIV RENNES, CNRS, IRMAR - UMR 6625, F-35000 RENNES, FRANCE
E-mail address: serge.cantat@univ-rennes1.fr, junyi.xie@univ-rennes1.fr