THE EXISTENCE OF ZARISKI DENSE ORBITS FOR ENDOMORPHISMS OF PROJECTIVE SURFACES (WITH AN APPENDIX IN COLLABORATION WITH THOMAS TUCKER)

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Abstract. In this paper we prove the following theorem. Let f be a dominant endomorphism of a smooth projective surface over an algebraically closed field of characteristic 0. If there is no nonconstant invariant rational function under f, then there exists a closed point whose orbit under f is Zariski dense. This result gives us a positive answer to the Zariski dense orbit conjecture proposed by Medvedev and Scanlon, by Amerik, Bogomolov and Rovinsky, and by Zhang, for endomorphisms of smooth projective surfaces.

Moreover, we define a new canonical topology on varieties over an algebraically closed field which has finite transcendence degree over $\mathbb{Q}$. We call it the adelic topology. The adelic topology is stronger than the Zariski topology and an irreducible variety is still irreducible in this topology. Using the adelic topology, we propose an adelic verison of the Zariski dense orbit conjecture. This version is stronger then the original one and it quantifies how many such orbits there are. We also proved this adelic version for endomorphisms of smooth projective surfaces. Moreover, we proved the adelic verison of the Zariski dense orbit conjecture for endomorphisms of abelian varieties and split polynomial maps. This yields new proofs for the original version in this two cases.

In Appendix A, we study the endomorphisms on the k-affinoid spaces. We show that for certain endomorphism f on a k-affinoid space X, the attractor Y of f is a Zariski closed subset and the dynamics of f semi-conjugates to its restriction on Y. A special case of this result is used in the proof of the main theorem.

In Appendix B, written in collaboration with Thomas Tucker, we prove the Zariski dense orbit conjecture for endomorphisms of $(\mathbb{P}^1)^N$.

Contents

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1. Introduction

Denote by $\mathbf{k}$ an algebraically closed field of characteristic 0.

1.1. Zariski dense orbit conjecture.

1.1. Zariski dense orbit conjecture. One of the aim of this paper is to prove the following result.

Theorem 1.1. *Let X be an irreducible smooth projective surface over $\mathbf{k}$. Let $f : X \to X$ be a dominant endomorphism. If there are no nonconstant rational functions H satisfying $H \circ f = H$, then there exists a closed point whose orbit under f is Zariski dense in X.*

This theorem setlles the Zariski dense orbit conjecture for endomorphisms of smooth projective surfaces.

Conjecture 1.2. Let X be an irreducible projective variety over $\mathbf{k}$ and $f : X \dashrightarrow X$ be a dominant rational endomorphism for which there exists no nonconstant rational function H satisfying $H \circ f = H$. Then there exists a point $p \in X(\mathbf{k})$ whose orbit is well defined and is Zariski dense in X.

This conjecture was proposed by Medvedev and Scanlon [33, Conjecture 5.10] and also by Amerik, Bogomolov and Rovinsky [2], which strengthens the following conjecture of Zhang [48].

Conjecture 1.3. Let X be an irreducible projective variety and $f : X \to X$ be an endomorphism defined over k. If f is polarized[1], then there exists a point $p \in X(\mathbf{k})$ whose orbit $\{f^n(p) |\ n \geq 0\}$ is Zariski dense in X.

We note that the presentation form of Conjecture 1.2 is slightly unbalanced. We note that the nonexistence of nonconstant invariant rational function is birational invariant, but it is not obvious that the existence of Zariski dense orbits is birational invariant without assuming the dynamical Mordell-Lang Conjecture. To eliminate this imbalance, we propose to reformulate Conjecture 1.2 in the following strong form[2].

Conjecture 1.4. Let X be an irreducible projective variety over $\mathbf{k}$ and $f : X \dashrightarrow X$ be a dominant rational endomorphism for which there exists no nonconstant rational function H satisfying $H \circ f = H$. Then for every Zariski dense open subset U of X, there exist a point $p \in X(\mathbf{k})$ whose orbit $O_f(p)$ under f is well defined, contained in U and Zariski dense in X.

Conjecture 1.4 is equivalent to Conjecture 1.2. But it is stronger than the original one for every pair (X, f). Indeed it is easy to see that the strong form holds for (X, f) if and only if the original form holds for every birational model of (X, f). See Section 2 for more discussion on the strong form of the Zariski dense orbit conjecture.

In this paper, we also prove Conjecture 1.4 for endomorphism of surfaces.

Theorem 1.5. *Let X be an irreducible smooth projective surface over $\mathbf{k}$. Let $f : X \to X$ be a dominant endomorphism. If there are no nonconstant rational functions H satisfying $H \circ f = H$, then for every Zariski dense open set U of X, there exist a closed point whose orbit is contained in U and is Zariski dense in X.*

This theorem implies Theorem 1.1.

1.2. **Historical note.** When $\mathbf{k}$ is uncountable, Conjecture 1.2 was proved by Amerik and Campana [3]. If $\mathbf{k}$ is countable, Conjecture 1.1 has only been proved in a few special cases. In [34], Medvedev and Scanlon proved Conjecture 1.2 when $f := (f_1(x_1), \cdots, f_N(x_N))$ is an endomorphism of $\mathbb{A}^N_{\mathbf{k}}$ where the f_i's are one-variable polynomials defined over $\mathbf{k}$. In [8], Bell, Ghoica and Tucker proved

[1] A dominant endomorphism f on a projective variety X is said to be *polarized* if there exists an ample line bundle L on X satisfying $f^*L = L^{\otimes d}$ for some integer $d > 1$

[2] This strong form is inspired by [39, Conjecture 7.2] in an earlier paper of Benedetto, Ingram, Jones, Manes, Silverman and Tucker. In [39, Conjecture 7.2], they also proposed a strong form of Conjecture 1.2. However, their strong and the original form in [39, Conjecture 7.2] are equivalent for every pair (X, f).

Conjecture 1.2 when $f := (f_1(x_1), \cdots, f_N(x_N))$ is an endomorphism of $\mathbb{P}^N_{\overline{\mathbb{Q}}}$ where the f_i's are endomorphisms of $\mathbb{P}^1_{\overline{\mathbb{Q}}}$ which are not post critically finite. In [2], Amerik, Bogomolov and Rovinsky proved Conjecture 1.2 under the assumption that $\mathbf{k} = \overline{\mathbb{Q}}$ and f has a fixed point o which is smooth and such that the eigenvalues of $df|_o$ are nonzero and multiplicatively independent. The author proved Conjecture 1.2 for surface birational self-maps with dynamical degree great than 1 in [44], and for all polynomial endomorphisms f of $\mathbb{A}^2$ in [45].

In [24], Ghioca and the author shows that if Conjecture 1.5 holds for a rational self-map $g : X \dashrightarrow X$ of a projective variety then Conjecture1.4 holds for a skew-linear rational self-map $f : X \times \mathbb{A}^N \dashrightarrow X \times \mathbb{A}^N$ takes form $(x, y) \mapsto (f(x), A(x)y)$ where $A(x) \in M_{N\times N}(\mathbf{k}(X))$.

We mention that in [1], Amerik proved that there exists a nonpreperiodic algebraic point when f is of infinite order. In [9], Bell, Ghioca and Tucker proved that if f is an automorphism without nonconstant invariant rational function, then there exists a subvariety of codimension 2 whose orbit under f is Zariski dense. See [2, 17, 4, 6, 22, 24, 19, 21] for more previous results.

1.3. **Adelic topology.** The Zariski dense orbit conjecture indicates the existence of Zariski dense orbit for a dynamical system, but it says very few about how many such orbits there are. In order to quantify this problem, we define a canonical topology and call it the adelic topology.

Assume that the transcendence degree of $\mathbf{k}$ over $\mathbb{Q}$ is finite. Let X be a variety over $\mathbf{k}$. The *adelic topology* is a topology on $X(\mathbf{k})$, which is defined by considering all embeddings of $\mathbf{k}$ in $\mathbb{C}$ and $\mathbb{C}_p$ where p is some prime. See Section 3 for the precise definition. The adelic topology has the following basic properties.

(i) It is stronger then the Zariski topology.

(ii) It is T_1 i.e. for every distinct points $x, y \in X(\mathbf{k})$ there are adelic open subsets U, V of $X(\mathbf{k})$ such that $x \in U, y \notin U$ and $y \in V, x \notin V$.

(iii) The morphisms between algebraic varieties over $\mathbf{k}$ are continuous under the adelic topology.

(iv) Étale morphisms are open w.r.t. the adelic topology.

(v) The irreducible components of $X(\mathbf{k})$ in the Zariski topology are the irreducible components of $X(\mathbf{k})$ in the adelic topology.

(vi) Let K be any subfield of $\mathbf{k}$ which is finitely generated over $\mathbb{Q}$ such that X is defined over K. Then the action
$$\mathrm{Gal}(\mathbf{k}/K) \times X(\mathbf{k}) \to X(\mathbf{k})$$
is continuous w.r.t. the adelic topology.

In particular, when X is irreducible, the intersection of finitely many nonempty adelic open subsets of $X(\mathbf{k})$ is nonempty. In general, the adelic topology is strictly stronger than the Zariski topology. For example, on $\mathbb{P}^1(\mathbf{k})$ there exists an adelic open set U such that both U and $\mathbb{P}^1(\mathbf{k}) \setminus U$ are infinite.

Remark 1.6. In [47, Section 2.1], Yuan and Zhang introduced an analytic space X^{an}, which is the union of the original Berkovich spaces over all places. The same as the adelic topology, the space X^{an} is canonical and is defined by considering all valuations. On the other hand, the topology of X^{an} is Hausdorff, but the adelic topology is not Hausdorff in general.

Then we propose an adelic veriosn of the Zariski dense orbit conjecture.

Conjecture 1.7. Assume that the transcendence degree of $\mathbf{k}$ over $\mathbb{Q}$ is finite. Let X be an irreducible projective variety over $\mathbf{k}$ and $f : X \dashrightarrow X$ be a dominant rational endomorphism for which there exists no nonconstant rational function H satisfying $H \circ f = H$. Then there exists a nonempty adelic open subset $U \subseteq X(\mathbf{k})$ such that for every point $x \in U$, its orbit $O_f(x)$ is well defined and is Zariski dense in X.

In Section 3, we will show that this conjecture has good behaviors under the changing of birational models and it is stronger than the original version of the Zariski dense orbit conjecture.

In this paper, we prove Conjecture 1.7 for endomorphism of surfaces.

Theorem 1.8. *Assume that the transcendence degree of* $\mathbf{k}$ *over* $\mathbb{Q}$ *is finite. Let* X *be an irreducible smooth projective surface over* $\mathbf{k}$*. Let* $f : X \to X$ *be a dominant endomorphism. If there are no nonconstant rational functions* H *satisfying* $H \circ f = H$*, then there exists a nonempty adelic open subset* U *of* $X(\mathbf{k})$ *such that for every point* $x \in X(\mathbf{k})$*, the orbit of* x *is Zariski dense in* X*.*

By Corollary 3.24, this implies Theorem 1.1 and Theorem 1.5.

Most of the known results of the Zariski dense orbit conjecture can be generalized to an adelic version. In Section 3, we do this generalization for some results we need in the proof of Theorem 1.8.

As an application of the adelic topology, we give fast proofs of Conjecture 1.7 for split polynomial endomorphisms on $(\mathbb{A}^1)^N$ and for endomorphisms of abelian varieties.

Theorem 1.9. *Let* $f : \mathbb{A}^N \to \mathbb{A}^N, N \geq 1$ *be a dominant endomorphism over* $\mathbf{k}$ *taking form* $(x_1, \ldots, x_N) \mapsto (f_1(x_1), \ldots, f_N(x_N))$*. If there are no nonconstant rational functions* H *satisfying* $H \circ f = H$*, then there exists a nonempty adelic open subset* U *of* $\mathbb{A}^N(\mathbf{k})$ *such that for every point* $x \in \mathbb{A}^N(\mathbf{k})$*, the orbit of* x *is Zariski dense in* X*.*

Moreover, if $\deg f_i \geq 2, i = 1, \ldots, N$*, then there exists a nonempty adelic open subset* U *of* $\mathbb{A}^N(\mathbf{k})$ *such that for every point* $x \in \mathbb{A}^N(\mathbf{k})$*, the orbit of* x *is Zariski dense in* X*.*

Theorem 1.10. *Let* A *be an abelian variety over* $\mathbf{k}$*. Let* $f : A \to A$ *be a dominant endomorphism. If there are no nonconstant rational functions* H *satisfying* $H \circ f = H$*, then there exists a nonempty adelic open subset* U *of* $A(\mathbf{k})$ *such that for every point* $x \in A(\mathbf{k})$*, the orbit of* x *is Zariski dense in* X*.*

These two theorem generalizes [34, Theorem 7.16] and [22, Theorem 1.2], and they yields new proofs of [34, Theorem 7.16] and [22, Theorem 1.2].

In the original proofs of [34, Theorem 7.16], Medvedev and Scanlon used their deep result on the classification of all invariant subvarieties of split polynomial endomorphisms on $(\mathbb{A}^1)^N$, which is proved using Model theory and polynomial decomposition theory. The original proofs of [22, Theorem 1.2] relies on the Mordell-Lang conjecture, due to Faltings. Our new proofs do not rely on the classification or the Mordell-Lang conjecture.

1.4. **Endomorphisms of $(\mathbb{P}^1)^N$.** In Appendix B, we prove Conjecture 1.7 for endomorphism of $(\mathbb{P}^1)^N$ which generalizes Theorem 1.9. As a consequence, this setlles the Zariski dense orbit conjecture for endomorphisms of $(\mathbb{P}^1)^N$.

Theorem 1.11. *Let $f : (\mathbb{P}^1)^N \to (\mathbb{P}^1)^N, N \geq 1$ be a dominant endomorphism over $\mathbf{k}$. If there are no nonconstant rational functions H satisfying $H \circ f = H$, then there exists a closed point whose orbit under f is Zariski dense.*

This result generalizes [34, Theorem 7.16] and [8, Theorem 14.3.4.2].

In [8, Theorem 14.3.4.2], Bell, Ghoica and Tucker proved Theorem 1.11 when $\mathbf{k} = \overline{\mathbb{Q}}$ and f takes form $f := (f_1(x_1), \cdots, f_N(x_N))$ where the f_i's are not post critically finite. In their proof, after replacing f by a positive iterate, the assumption of not post critically finite guarantee a fixed point o, such that the eigenvalues of $df|_o$ are multiplicatively independent. This derives the existence of a Zariski dense orbit by [2].

In [34, Theorem 7.16], Medvedev and Scanlon proved Conjecture 1.2 when $f := (f_1(x_1), \cdots, f_N(x_N))$ is an endomorphism of $\mathbb{A}^N_{\mathbf{k}}$ where the f_i's are one-variable polynomials defined over $\mathbf{k}$. Their proof based on their classification of all invariant subvarieties of split polynomial endomorphisms on $(\mathbb{A}^1)^N$. More precisely, using model theory, they shows that when all factors f_i are not of some special type, all invariant subvarieties come from some invariant curves of $(f_i, f_j) : \mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1 \times \mathbb{A}^1$ for some $i \neq j$. Basically, this reduces the problem to the case $N = 2$. When $N = 2$, they classifies all invariant curves using polynomial decomposition theory. Then we may construct a point which avoids all such invariant curves of very concrete form. This theorem is generalized by our Theorem 1.9 by different method. But the strategy of its proof has been inherited by our proof of Theorem 1.11.

Now we explain the strategy of the proof of Theorem 1.11. It is easy to show that after replacing f by a positive iterate, we may assume that f takes form $f := (f_1(x_1), \cdots, f_N(x_N))$. As in the proof of [34, Theorem 7.16], we first need a description of invariant subvarieties of endomorphisms of $(\mathbb{P}^1)^N$ (see Proposition 9.2). Basically this description shows that when all factors f_i are not of some special type, all invariant subvarieties come from some invariant curves of $(f_i, f_j) : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$ for some $i \neq j$. Such a description was already obtained by Medvedev and Scanlon in [34] using Model theory. Here we give a new and elementary proof. Using this description, we reduce the problem to the case $N = 2$. Then we may conclude the proof by Theorem 1.8.

1.5. **Strategy of the proof.** In this paper, Theorem 1.1 is implied by its adelic version Theorem 1.8. Here, for the simplicity, we explain a strategy of a direct proof of Theorem 1.1. However, except the systematic use of the adelic topology and some technical difficulties, Theorem 1.8 follows the same idea.

We first explain this strategy for an endomorphism f of $\mathbb{P}^2$ of degree at least 2. For the simplicity, we assume that $\mathbf{k} = \overline{\mathbb{Q}}$. In this case, there is no nonconstant rational function invariant under f. So we only need to show that there exits a closed point which has a Zariski dense orbit. The idea of the proof is to combine the p-adic local dynamic near a certain periodic point with a constraint on the definition field of an invariant curve which is obtain by some global information.

By [2], if there exists a fixed point o of f^m, $m \geq 1$ such that the two eigenvalues λ_1, λ_2 of $df^m|_o$ are multiplicatively independent, then there exits a closed point which has a Zariski dense orbit. So we may assume that such point does not exist.

At first, we study the invariant curves of f. Assume that f and all fixed points of f are defined over a number field K. We show that there exists a positive integer N depend on f, such that for every irreducible invariant curve C of f, C is defined over a field K_C such that $[K_C : K] | N^n$ for some $n \geq 0$. Moreover, we show that the number of invariant branches of C at a fixed point is bounded from above by some integer $B > N$.

Next we want to find a fixed point o of f^m, $m \geq 1$ and a field embedding $\tau : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$ such that

(i) $df^m|_o$ is invertible;
(ii) $|\tau(\lambda_1)|, |\tau(\lambda_2)| \leq 1$ where λ_1, λ_2 are the two eigenvalues of $df^m|_o$;
(iii) $|\tau(\lambda_1)||\tau(\lambda_2)| < 1$.

By studying of multipliers of endomorphisms on curves and assuming that there is no Zariksi dense orbits of closed points, we show that the existence of such point is ensured by the existence of repelling periodic point. The later is ensured by [26, Theorem 3.4, iv)]. After replacing f by f^m, we may assume that o is a fixed point of f. Using τ, we may view $\overline{\mathbb{Q}}$ as a subfield of $\mathbb{C}_p$. We may assume that λ_1, λ_2 are contained in K. Denote by K_p the closure of K in $\mathbb{C}_p$.

If $|\lambda_1| = 1$ and $|\lambda_2| < 1$, we show that there exists a p-adic neighborhood U of o in $X(K_p)$ which is isomorphic to a polydisc $(K_p^\circ)^2$ and invariant by f. We show that after shrinking U, there exist an analytic curve $Y \subseteq U$ which is invariant by f, and an analytic morphism $\psi : U \to Y$ such that $\psi|_Y = \mathrm{id}$ and $\psi \circ f = f|_Y \circ \psi$. Moreover, we have $\cap_{n \geq 0} f^n(U) = Y$. Indeed, in Appendix A, we proved a more general result for endomorphisms of affinoid spaces. For endomorphism of $\mathbb{P}^2$, the periodic points are isolated. It shows that $f|_Y$ is not of finite order. In this case, we can show that there is a point in $X(\overline{\mathbb{Q}}) \cap U$ which has Zariski dense orbit.

If both $|\lambda_1|$ and $|\lambda_2|$ are strictly less than 1, since λ_1, λ_2 are not multiplicatively independent, there exists $m_1, m_2 \geq 1$ such that $\lambda_1^{m_1} = \lambda_2^{m_2}$. After replacing f by a suitable iterate, we may assume that $(m_1, m_2) = 1$. We show that there exist a birational morphism $\pi : X' \to X$ which is a composition of blowups of K-points, an irreducible component E in $\pi^{-1}(o)$ such that the induced rational endomorphism f' on X' is regular along E and fix E, and a fixed point $o' \in E(K)$ such that the two eigenvalues of $df'|_{o'}$ is $1, \mu$ where $|\mu| < 1$ and the eigenvectors for 1 is in the tangent space of E. Let M be a finite field extension of K such that $[M : K]$ is prime to $B!$. Denote by M_p the closure of M in $\mathbb{C}_p$. The argument in the previous paragraph shows that there exists a p-adic neighborhood U of o

in $X'(M_p)$ which is isomorphic to a polydisc $(M_p^\circ)^2$ and invariant by f' satisfying $\cap_{n\geq 0}(f')^n(U) = Y := U \cap E$ and an analytic morphism $\psi : U \to Y$ such that $\psi|_Y = \mathrm{id}$ and $\psi \circ f' = f'|_Y \circ \psi$. Moreover the construction of U and ψ shows that they are defined over K_p. If $f'|_Y \neq \mathrm{id}$, we may conclude the proof by the argument in the previous paragraph. If $f'|_Y = \mathrm{id}$, such argument is not sufficient. Here we need the constraint on definition fields of invariant curves. We show that for every irreducible periodic curve C passing through U are indeed invariant by f'. So it is defined over a field K_C such that $[K_C : K]|N^n$ for some $n \geq 0$. It follows that $C \cap U$ is defined over $(K_C)_p$ which is the closure of K_C in $\mathbb{C}_p$. We show that $C \cap U$ is a disjoint union of $\psi^{-1}(x_i), i = 1, \ldots, s$ where $s \leq B$ and $x_i \in Y = U \cap E$. Then there exists a finite field extension H_p over K_p satisfying $[H_p : (K_C)_p]|B!$ such that all x_i are defined over H_p. It follows that there exists $n \geq 0$ such that $[H_p : K_p]|(B!)^n$. Since $[M_p : K_p]$ is prime to $B!$, $M_p \cap H_p = K_p$. Then $x_i \in X'(K_p) \cap Y, i = 1, \ldots, s$. Observe that $X(K_p) \cap Y$ is not dense in Y. We can show that there exists a point $x \in X'(\overline{\mathbb{Q}}) \cap \psi^{-1}(Y \setminus X'(K_p))$ which has a Zariski dense orbit for f'. Then $\pi(x)$ has a Zariski dense orbit for f.

In the general case, by [9, Theorem 1.3], we may assume that f is not an automorphism. Using the classification of surface and the works of Fujimoto, Nakayama, Matsuzawa, Sano and Shibata, we may reduce to a case either can be treated by the same argument for $\mathbb{P}^2$ or preserve a fibration to a curve. In the later case, we can conclude the proof using this fibration.

1.6. **Organization of the paper.** The article is organized in 6 Sections and two appendixes.

In Section 2, we discuss some some basic facts on the Zariski dense orbit conjecture. We show that Conjecture 1.4 implies Conjecture 1.2. In particular, Theorem 1.5 implies Theorem 1.1. We also discuss the relation between the Zariski dense orbit conjecture and the dynamical Mordell-Lang conjecture.

In Section 3, we introduce the adelic topology and prove some basic facts of this topology. Using this topology, we propose the adelic version of the Zariski dense orbit conjecture. We show that Conjecture 1.7 implies Conjecture 1.4. In particular, Theorem 1.8 implies Theorem 1.5. We generalize some former results on the Zariski dense orbit conjecture to an adelic version.

In Section 4, we gives some applications of the adelic topology. It yields the proofs of Theorem 1.9 and Theorem 1.10.

In Section 5, we prove some general facts of endomorphisms of projective surfaces. In particular, we prove a constraint on definition field of an invariant curve.

In Section 6, we first study the multipliers of periodic points and the dynamics near a fixed point. Then we focus on the amplified endomorphism. In particular, we prove Theorem 1.8 for endomorphisms of $\mathbb{P}^2$.

In Section 7 we prove Theorem 1.8 in the general case.

In Appendix A, we study the endomorphisms on the k-affinoid spaces. We show that for certain endomorphism f on a k-affinoid space X, the attractor Y of f is a Zariski closed subset and the dynamics of f is semi-conjugates to the its restriction on Y.

In Appendix B, written in collaboration with Thomas Tucker, we prove Theorem 1.11.

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2. The Zariski density orbit conjecture

Let X be an irreducible projective variety over $\mathbf{k}$ and $f : X \dashrightarrow X$ be a dominant rational endomorphism.

Definition 2.1. We say that a pair (X, f) satisfies the *ZD-property*, if either there exists a nonconstant rational function $H \in \mathbf{k}(X) \setminus \mathbf{k}$ satisfying $H \circ f = H$ or there exist a point $p \in X(\mathbf{k})$ whose orbit $O_f(p)$ under f is well defined and Zariski dense in X.

Definition 2.2. We say that a pair (X, f) satisfies the *strong ZD-property*, if either there exists a nonconstant rational function $H \in \mathbf{k}(X)\setminus\mathbf{k}$ satisfying $H \circ f = H$ or for every Zariski dense open subset U of X, there exist a point $p \in X(\mathbf{k})$ whose orbit $O_f(p)$ under f is well defined, contained in U and Zariski dense in X.

We note that a pair (X, f) satisfies the ZD-property (resp. strong ZD-property) if and only if Conjecture 1.2 (resp. Conjecture 1.4) holds for it.

Remark 2.3. It is obvious that the strong ZD-property implies the ZD-property.

The following lemma shows that the strong ZD-property is invariant under birational conjugation and iterations.

Proposition 2.4. *The following statements are equivalents:*

(i) *(X, f) satisfies the strong ZD-property;*
(ii) *there exists $m \geq 1$, such that (X, f^m) satisfies the strong ZD-property;*
(iii) *there exists a pair (Y, g) which is birational to the pair (X, f), and (Y, f) satisfies the strong ZD-property.*

Remark 2.5. Lemma 2.6 implies also that the ZD-property is invariant under iterations. However, a priori it is not clear that whether the ZD-property is invariant under birational conjugation.

Proof of Proposition 2.4. It is clear that (i) implies (ii), (i) implies (iii) and (iii) implies (i).

We only need to prove that (ii) implies (i).

The following Lemma is a special case of [22, Lemma 4.1].

Lemma 2.6. *If there exists $m \geq 1$, and a nonconstant rational function H on X, such that $(f^m)^*H = H$, then there exists a nonconstant rational function G on X, such that $f^*G = G$.*

Now assume that (X, f^m) satisfies the strong ZD-property. By Lemma 2.6, we may assume that exists no nonconstant rational function $H \in \mathbf{k}(X) \setminus \mathbf{k}$ satisfying $H \circ f^m = H$. Let U be a Zariski dense open subset of X. We only need to show that there exist a point $p \in X(\mathbf{k})$ whose orbit $O_f(p)$ under f is contained in U and is Zariski dense in X. We may assume that $U \cap (I(f) \cup I(f^2) \cdots \cup I(f^{m-1})) = \emptyset$. Set

$$V := U \cap (\cap_{i=1}^{m-1} f|_U^{-i}(U)).$$

We note that for every point $p \in V$, the sequence $p, f(p), \ldots, f^{m-1}(p)$ are well defined and are contained in U.

Since the pair (X, f^m) satisfies the strong ZD-property, there exist a point $p \in X(\mathbf{k})$ whose orbit $O_{f^m}(p)$ under f^m is contained in V and is Zariski dense in X. It follows that the orbit $O_f(p)$ under f is contained in U and is Zariski dense in X, which implies concludes the proof. $\square$

Proof of Lemma 2.6. Let $P_1 = \sum_{i=0}^m (f^{i-1})^*H, \ldots, S_m = \prod_{i=0}^m (f^{i-1})^*H$ be the elementary symmetric polynomials of $H, \ldots, (f^{m-1})^*H$. For every $i = 1, \ldots, m$, $f^*P_i = P_i$. We only need to show that there exists $i = 1, \ldots, m$ such that P_i is not constant on X. We have

$$H^m + \sum_{i=1} (-1)^i P_i H^i = 0.$$

If all P_i are constant on X, then H is also constant on X, which is a contradiction. Then we concludes the proof. $\square$

The dynamical Mordell-Lang conjecture was proposed by Ghioca and Tucker [23]. The following is a slight generalization of the dynamical Mordell-Lang conjecture for rational endomorphisms for rational endomorphisms.

Conjecture 2.7. Let X be a projective variety defined over $\mathbf{k}$, let $f : X \dashrightarrow X$ be an endomorphism, and V be any subvariety of X. Then for every subvariety V of X and every point $p \in X(\mathbf{k})$ whose orbit is well defined, the set $\{n \geq 0 |\ f^n(x) \in V\}$ is a finite union of arithmetic progressions[3].

Inspired by this conjecture, we introduce the following definition.

Definition 2.8. We say that a pair (X, f) satisfies the *DML-property*, if for every subvariety V of X and every point $p \in X(\mathbf{k})$ whose orbit is well defined, the set $\{n \geq 0 |\ f^n(x) \in V\}$ is a finite union of arithmetic progressions.

This definition was introduced in [43] when X is a surface.

Proposition 2.9. *Assume that (X, f) has both the ZD-property and the DML-property, then it has the strong ZD-property.*

[3]An arithmetic progression is a set of the form $\{an + b |\ n \in \mathbb{N}\}$ with $a, b \in \mathbb{N}$ possibly with $a = 0$.

Proof of Proposition 2.9. We may assume that there exists no nonconstant f-invariant rational function on X. Then there exists $p \in X(\mathbf{k})$ whose orbit is well defined and Zariski dense in X.

Let U be any Zariski dense open subset. Set $Z := X \setminus U$. Since (X, f) satisfies the DML-property, the set $\{n \geq 0 |\ f^n(p) \in Z\}$ is a finite union of arithmetic progressions. If it is infinite, there exist $a, b \in \mathbb{N}, a \neq 0$ such that

$$\{an + b |\ n \in \mathbb{N}\} \subseteq \{n \geq 0 |\ f^n(x) \in Z\}.$$

It follows that

$$\{f^n(p) |\ n \geq b\} \subseteq Z \cup \cdots \cup f^{a-1}(Z).$$

Then the orbit $O_f(p)$ is not Zariski dense, which contradicts our assumption. So there exists $N > 0$ such that $f^n(p) \notin Z$ for every $n \geq N$. Then the orbit of $f^N(p)$ is well defined, contained in U and Zariski dense in X. □

3. The adelic topology

Assume that the transcendence degree of $\mathbf{k}$ over $\mathbb{Q}$ is finite. Let X be a variety defined over $\mathbf{k}$. Let K_0 be a finitely generated field extension over $\mathbb{Q}$ such that $\overline{K_0} = \mathbf{k}$, and X is defined over K_0 i.e. there exists a variety X_{K_0} defined over K_0 such that $X = X_{K_0} \times_{\operatorname{Spec} K_0} \operatorname{Spec} \mathbf{k}$. In this section, we will define the adelic topology on $X(\mathbf{k})$.

For every algebraic extension K over K_0, we define $X_K := X \times_{\operatorname{Spec} K_0} \operatorname{Spec} K$. We may canonically identify $X_K(\mathbf{k})$ with $X(\mathbf{k})$.

Define $\mathbb{C}_\infty := \mathbb{C}$. Let K be any finite field extension of K_0. Denote by $\mathcal{I}_K$ the set of embeddings for fields $\tau : K \hookrightarrow \mathbb{C}_{p_\tau}$ for some p_τ prime or ∞. Denote by $\mathcal{I}_K^f$ the set of $\tau \in \mathcal{I}_K$ for which p_τ is a prime. We say two embeddings for fields $\tau, \tau' \in \mathcal{I}_K$ are equivalent if the absolute values $|\tau(\cdot)|, |\tau'(\cdot)|$ on K are the same. Denote by $\mathcal{M}_K$ (resp. $\mathcal{M}_K^f$) the set of equivalent classes in $\mathcal{I}_K^f$.

For every $\tau \in \mathcal{I}_K$, denote by $\mathcal{I}_\tau$ the set of embeddings for fields $\overline{\tau} : \mathbf{k} = \overline{K} \hookrightarrow \mathbb{C}_{p_\tau}$ satisfying $\overline{\tau}|_K = \tau$. Every $\overline{\tau} \in \mathcal{I}_\tau$, induces an embedding $\phi_{\overline{\tau}} : X(\mathbf{k}) \hookrightarrow X_K(\mathbb{C}_{p_\tau})$. On $X_K(\mathbb{C}_{p_\tau})$, we have the natural p_τ-adic topology when p_τ is a prime and the complex topology when $p_\tau = \infty$. We note that for every $\overline{\tau} \in \mathcal{I}_\tau$, the image $\phi_{\overline{\tau}}(X(\mathbf{k})) \subseteq X_K(\mathbb{C}_{p_\tau})$ are the same.

Let $\tau_i : K \hookrightarrow \mathbb{C}_{p_i}, i = 1, \ldots, m$ be m (not necessarily different) elements in $\mathcal{I}_K$. For every $i = 1, \ldots, m$, let U_i be a nonempty p_i-adic open subset of $X_K(\mathbb{C}_{p_i})$.

We define

$$X_K((\tau_i, U_i), i = 1, \ldots, m) := \cap_{i=1}^m (\cup_{\overline{\tau_i} \in \mathcal{I}_{\tau_i}} \phi_{\overline{\tau_i}}^{-1}(U_i)) \subseteq X(\mathbf{k}).$$

Definition 3.1. A subset S of $X(\mathbf{k})$ is called an *adelic subset over* K for some finite extension K over K_0, if it is takes form $X_K((\tau_i, U_i), i = 1, \ldots, m)$ where $\tau_i \in \mathcal{I}_K, i = 1, \ldots, m$ and $U_i, i = 1, \ldots m$ are open subsets of $X_K(\mathbb{C}_{p_i})$.

We say that S is an *adelic subset*, if it is adelic over K for some finite extension K over K_0.

Remark 3.2. For every Zariski open subset U of X defined over K, the subset $U(\mathbf{k}) \subseteq X(\mathbf{k})$ is an adelique subset over K and we have

$$X_K((\tau_i, U_i), i = 1, \ldots, m) \cap U = X_K((\tau_i, U_i \cap U), i = 1, \ldots, m).$$

Remark 3.3. Let K' be a finite extension of K. Then every adelic subsets over K is a finite union of adelic subsets over K'.

Let $\tau_i' : K' \hookrightarrow \mathbb{C}_{p_i}$ be an extension of of τ_i for $i = 1, \dots, m$. Then we have

$$X_{K'}((\tau_i', U_i), i = 1, \dots, m) \subseteq X_K((\tau_i, U_i), i = 1, \dots, m).$$

Remark 3.4. Let Y be a variety over K. Let $\pi : Y \to X$ be a morphism over K. Then we have

$$\pi^{-1}(X_K((\tau_i, U_i), i = 1, \dots, m)) = Y_K((\tau_i, \pi^{-1}(U_i)), i = 1, \dots, m).$$

Moreover, if π is étale, then we have $\pi(Y_K(i, U)) = X_K(i, \pi(U))$ which is an adelic subset.

Proposition 3.5. *If X is irreducible and all $U_i, i = 1, \dots, m$ are not empty, then the adelic subset $X_K((\tau_i, U_i), i = 1, \dots, m)$ is not empty.*

Remark 3.6. Observe that

$$\begin{aligned} &X_K((\tau_i, U_i), i = 1, \dots, m) \cap X_K((\tau_i', U_i'), i = 1, \dots, m') \\ &= X_K((\tau_i, U_i), i = 1, \dots, m, (\tau_j', U_j'), j = 1, \dots, m'). \end{aligned}$$

When X is irreducible, by Remark 3.3 and Proposition 3.5, the intersection of finitely many nonempty adelic subsets is a nonempty adelic subset.

Definition 3.7. We say that S is an *adelic open subset*, if it is a union of adelic subsets. Remark 3.3 and 3.6 shows that the adelic open subsets forms a topology on X. We call it the *adelic topology* on $X(\mathbf{k})$.

We note that, this adelic topology does not depend on the choice of base field K_0.

Example 3.8. Assume that $\mathbf{k} = \overline{\mathbb{Q}}$, $X := \mathbb{A}^1$. Then it is defined over $\mathbb{Q}$. Let $\tau : \mathbb{Q} \hookrightarrow \mathbb{C}$ be the unique embedding. Let $U_1 := \{x \in \mathbb{C} = \mathbb{A}^1(\mathbb{C}) |\ |x| < 1\}, U_2 := \{x \in \mathbb{C} = \mathbb{A}^1(\mathbb{C}) |\ |x| > 1\}$ be two disjoint open set in $\mathbb{A}^1(\mathbb{C})$. Then we have

$$\{1/2, 1/3, \dots\} \subseteq \mathbb{A}^1_{\mathbb{Q}}(\tau, U_1) \setminus \mathbb{A}^1_{\mathbb{Q}}(\tau, U_2);$$

$$\{2, 3, \dots\} \subseteq \mathbb{A}^1_{\mathbb{Q}}(\tau, U_2) \setminus \mathbb{A}^1_{\mathbb{Q}}(\tau, U_1)$$

and

$$\{n \pm \sqrt{n^2 - 1}, n = 2, 3, \dots\} \subseteq \mathbb{A}^1_{\mathbb{Q}}(\tau, U_1) \cap \mathbb{A}^1_{\mathbb{Q}}(\tau, U_2) = \mathbb{A}^1_{\mathbb{Q}}((\tau, U_1), (\tau, U_2)).$$

We note that $\mathbb{A}^1_{\mathbb{Q}}(\tau, U_1)$ is an adelic open subset of $\mathbb{A}^1(\overline{\mathbb{Q}})$. Both $\mathbb{A}^1_{\mathbb{Q}}(\tau, U_1)$ and $\mathbb{A}^1(\overline{\mathbb{Q}}) \setminus (\mathbb{A}^1_{\mathbb{Q}}(\tau, U_1))$ are infinite. This shows that the adelic topology on $\mathbb{A}^1(\overline{\mathbb{Q}})$ is strictly stronger than the Zariski topology.

We note that $\mathbb{A}^1_{\mathbb{Q}}(\tau, U_1) \cap \mathbb{A}^1_{\mathbb{Q}}(\tau, U_2) \neq \emptyset$, even when $U_1 \cap U_2 = \emptyset$.

Proposition 3.9. *The adelic topology has the following basic properties.*

(i) *It is stronger then the Zariski topology.*
(ii) *It is T_1 i.e. for every distinct points $x, y \in X(\mathbf{k})$ there are adelic open subsets U, V of $X(\mathbf{k})$ such that $x \in U, y \notin U$ and $y \in V, x \notin V$.*
(iii) *The morphisms between algebraic varieties over $\mathbf{k}$ are continuous under the adelic topology.*

(iv) *Étale morphisms are open w.r.t. the adelic topology.*
(v) *The irreducible components of $X(\mathbf{k})$ in the Zariski topology are the irreducible components of $X(\mathbf{k})$ in the adelic topology.*
(vi) *Let K be any subfield of $\mathbf{k}$ which is finitely generated over $\mathbb{Q}$ such that X is defined over K. Then the action*

$$\rho : \mathrm{Gal}(\mathbf{k}/K) \times X(\mathbf{k}) \to X(\mathbf{k})$$

sending (σ, x) to $\sigma(x)$ is continuous w.r.t. the adelic topology.

In particular, when X is irreducible, the intersection of finitely many nonempty adelic open subsets of $X(\mathbf{k})$ is nonempty. This also shows that, in general, the adelic topology is not Hausdorff.

Proof of Proposition 3.9. The properties (i)-(v) easily follows from Remarks 3.2, Remark 3.3, Remark 3.4 and Proposition 3.5.

We only need to prove (vi). Let U be an adelic subset of $X(\mathbf{k})$ over a finite Galois extension L over K. We only need to show that $\rho^{-1}(U)$ is open in $\mathrm{Gal}(\mathbf{k}/K) \times X(\mathbf{k})$. Let (σ, x) be a point in $\rho^{-1}(U)$. Observe that $\sigma^{-1}(U)$ is an adelic subset of $X(\mathbf{k})$. We have $x \in \sigma^{-1}(U)$. Moreover, since L is Galois over K, $\sigma(U)$ is still over L. It follows that for every $\tau \in \mathrm{Gal}(\mathbf{k}/L)$, $\tau(\sigma(U)) = \sigma(U)$. Then we have $(\sigma, x) \in (\mathrm{Gal}(\mathbf{k}/L)\sigma) \times (\sigma^{-1}(U))$ and $(\mathrm{Gal}(\mathbf{k}/L)\sigma) \times (\sigma^{-1}(U)) \subseteq \rho^{-1}(U)$. This concludes the proof. $\square$

Remark 3.10. Here we define the adelic topology by using all valuations of $\mathbf{k}$ from some emdeddings in $\mathbb{C}_p$ where p is a prime or ∞. In fact, this is not necessary. If we shrink or enlarge the range of valuations we consider, for example, only the archimedean ones or the nonarchimedean ones, or all possible valuations etc., we may get some topology which satisfies similar properties. However, the current range is sufficient for this paper.

Remark 3.11. In this paper, we usually use the adelic topology in the following way. Assume that X is irreducible. Let $A_1, \dots, A_m$ be finitely many algebraic objects defined over $\mathbf{k}$. For example they can be a rational endomorphism of X, a subvariety of X, a number in $\mathbf{k}$ etc. Let $P_1, \dots, P_n$ be finitely many algebraic properties for points in X, which involve only $A_1, \dots, A_m$. There exists a subfield K of $\mathbf{k}$ which is finitely generated over $\mathbb{Q}$ such that $\overline{K} = \mathbf{k}$ and $A_1, \dots, A_m, X$ are defined over K. Once we showed that, for every $i = 1, \dots, n$, there exists an embedding $\tau_i : \mathbf{k} \hookrightarrow \mathbb{C}_p$ and a nonempty open subset U_i of $X(\mathbb{C}_p)$ such that the property P_i is satisfied for all points in U_i, we get automatically that all points in the nonempty adelic open subset $X_K((\tau_1|_K, U_1), \dots, (\tau_n|_K, U_n))$ satisfy all the properties $A_1, \dots, A_m$.

Proof of proposition 3.5. There exists a finite extension K' of K, such that there are extensions $\tau_i' : K' \hookrightarrow \mathbb{C}_{p_i}$ of $\tau_i, i = 1, \dots, m$ such that the absolute values $|\tau_i'(\cdot)|, i = 1, \dots, m$ on K' are distinct. After replacing K by K' and τ_i by $\tau_i', i = 1, \dots, m$, we may assume that $|\cdot|_i := |\tau_i(\cdot)|, i = 1, \dots, m$ on K are distinct.

Denote by K_{p_i} the closure of $\tau_i(K)$ in $\mathbb{C}_{p_i}$.

Lemma 3.12. [30, Page 35, Theorem 1] *The image of the diagonal embedding*

$$x \mapsto (\tau_1(x), \ldots, \tau_m(x)) : K \hookrightarrow \prod_{i=1}^m K_{p_i}$$

is dense.

We may assume that X is smooth and affine. Set $d := \dim X$. There exists a finite morphism $\pi : X_K \to \mathbb{A}^d_K$. We still denote by π the induced morphism $X \to \mathbb{A}^d_{\mathbf{k}}$.

There exists a Zariski dense open subset V of $\mathbb{A}^d_K$ such that $\pi|_{\pi^{-1}(V)}$ is an étale covering.

Let $\psi_i : \mathbb{A}^d_K(K) \hookrightarrow \mathbb{A}^d_K(K_{p_i})$ be the morphism

$$\psi_i : (x_1, \ldots, x_d) \mapsto (\tau_i(x_1), \ldots, \tau_i(x_d)).$$

Let $\psi : \mathbb{A}^d_K(K) \hookrightarrow \prod_{i=1}^m \mathbb{A}^d_K(K_{p_i})$ the diagonal embedding

$$\psi : x \mapsto (\psi_1(x), \ldots, \psi_m(x)).$$

Lemma 3.12 shows that the image of ψ is dense. It follows that

$$\psi : V(K) \hookrightarrow \prod_{i=1}^m V(K_{p_i})$$

is dense. Indeed Lemma 3.12 showed that the image of the diagonal map

$$\phi_\infty : V(K) \hookrightarrow \prod_{\tau \in \mathcal{M}_K} V(K_{p_\tau})$$

is dense.

Now we replace X_K by $\pi^{-1}(V)$. By Remark 3.4, we may replace X_K by an Galois étale cover over V which dominant X_K. Then we may assume that π is Galois. By [41, Proposition 3.3.1], there exists a thin set $A \subseteq V(K)$ such that for every point $x \in V(K) \setminus A$, the fiber $\pi^{-1}(x)$ is integral.

Set $W_i := \pi(U_i) \subseteq V(\mathbb{C}_{p_i}), i = 1, \ldots, m$. They are open subsets. After replacing K by a finite extension, we may assume that there exists $x_i \in V(K_{p_i}) \cap W_i$.

Lemma 3.13. *The image $\psi(V(K) \setminus A)$ is dense in $\prod_{i=1}^m V(K_{p_i})$.*

By Lemma 3.13, there exists a point $x \in V(K) \setminus A$ such that for $i = 1, \ldots, m$, $\phi_i(x) \in V(K_{p_i}) \cap W_i$. Since $\pi^{-1}(x)$ is integral, we have $\pi^{-1}(x) = \mathrm{Spec}\,(L)$ for some finite field extension L over K. The inclusion $L \subseteq \mathbf{k} = \overline{K}$ gives a morphism $\mathrm{Spec}\,(\mathbf{k}) \to \pi^{-1}(x) \subseteq X$. This defines a point $z \in X(\mathbf{k})$.

For $i = 1, \ldots, m$, there exists $y \in \pi^{-1}(x)(\mathbb{C}_{p_i}) \cap U_i$. It gives a morphism $\tau_y : L \hookrightarrow \mathbb{C}_{p_i}$ which extends τ_i. We extend τ_y to a morphism $\overline{\tau_i} : \overline{K} \hookrightarrow \mathbb{C}_{p_i}$. Then we have $\phi_i(z) \in U_i, i = 1, \ldots, m$. We conclude the proof. $\square$

Proof of Lemma 3.13. When K is a number field, the following lemma is [41, Theorem 3.5.3].

Lemma 3.14. *Let V be a geometrically irreducible variety over K and let A be a thin subset of $V(K)$. For every finite subset $S_0 \subseteq \mathcal{M}_K^f$, there exists a finite set $S \subseteq \mathcal{M}_K^f \setminus S_0$ such that the image of A in $\prod_{\tau\in S} V(K_{p_\tau})$ under the diagonal map ϕ_S is not dense.*

It follows that the image of the diagonal map $\phi_f : A \hookrightarrow \prod_{\tau\in\mathcal{M}_K^f} V(K_{p_\tau})$ is nowhere dense. Then the image of $\phi_\infty : A \hookrightarrow \prod_{\tau\in\mathcal{M}_K} V(K_{p_\tau})$ is nowhere dense. It follows that $\phi_\infty(V(K) \setminus A)$ is dense in $\prod_{\tau\in\mathcal{M}_K} V(K_{p_\tau})$, this concludes the proof. □

Proof of Lemma 3.14. Let L be a finite Galois extension of K. There exists a subring R of K which is finitely generated over $\mathbb{Z}$ such that $K = \operatorname{Frac} R$. Set $W_K := \operatorname{Spec} R$ and let W_L be the normalization of W_K in L. After shrinking W_K, we may assume that W_K is regular.

Lemma 3.15. *For every $N \geq 0$, and every $g \in R \setminus \{0\}$, there exists a prime $p \geq N$ and an embedding $\tau : K \hookrightarrow \mathbb{C}_p$ in $\mathcal{I}_K^f$ such that the absolute value $|\tau(\cdot)|$ on K is complete splitting in L, and $\tau(R_g) \subseteq \mathbb{C}_p^\circ$.*

In the proof of [41, Theorem 3.5.3], we used the fact that when K is a number field, there are infinitely many places of K which is completely splitting in L. Once we replace this fact by Lemma 3.15 in the proof of [41, Theorem 3.5.3], the same proof still works without assuming K to be a number field. □

Proof of Lemma 3.15. After replacing R by R_g, we may assume that $g = 1$. Pick any $y \in W_K(\overline{\mathbb{Q}})$, such that the morphism $W_L \to W_R$ is étale at y. Denote by o the image of x. Then the residue field $\kappa(o)$ is a number field. Then o induces an embedding $\iota : \operatorname{Spec} O_{\kappa(o),S} \hookrightarrow W_K$, where S is a finite set of places of $\kappa(o)$ containing all infinite places.

Denote by $q_1, \ldots, q_s$ the pre-image of o is W_L. Since the extension L over K is Galois, the extensions $\kappa(q_i)$ over $\kappa(o), i = 1, \ldots, s$ are isomorphic to each other.

For every $N > 0$, there exists a closed point $x \in \operatorname{Spec} O_{\kappa(o),S}$ which is completely splitting in the extension $\kappa(q_1)$ over $\kappa(o)$ and whose residue field $\kappa(x)$ is of characteristic $p > N$. Then it is completely splitting in the extension $\kappa(q_i)$ over $\kappa(o)$ for every $i = 1, \ldots, s$. Let m be the maximal ideal of R corresponding to $\iota(x)$. The pre-image of $\iota(x)$ in W_L has exactly $[L : K]$ points. Now we only need to show that there exists an embedding $\tau : R \hookrightarrow \mathbb{C}_p^\circ$, such that $m = \tau^{-1}(\mathbb{C}_p^{\circ\circ})$.

For every $P \in R \setminus \{0\}$, denote by Z_P the set $\{z \in W_K(\mathbb{C}_p) | \ P(z) = 0\}$. It is a nonwhere dense closed subset of $W_K(\mathbb{C}_p)$. Observe that the topology of $W_K(\mathbb{C}_p)$ can be defined by a complete metric. Since $R\setminus\{0\}$ is countable, by Baire category theorem, $W_K(\mathbb{C}_p) \setminus (\cup_{R\setminus\{0\}} Z_P)$ is dense in $W_K(\mathbb{C}_p)$. Let $P_1, \ldots, P_n$ be a set of generator of m. Then $B := \{z \in W_K(\mathbb{C}_p) | \ |P_i(z)| < 1\}$ is an open subset of $W_K(\mathbb{C}_p)$. Pick any inclusion $\overline{\mathbb{Q}} \subseteq \mathbb{C}_p$. Using this inclusion, we may view y as a point in $B \subseteq W_K(\mathbb{C}_p)$. So B is not empty. Pick any point $z \in B \setminus ((\cup_{R\setminus\{0\}} Z_P))$. Then the inclusion $\tau_z : R \hookrightarrow \mathbb{C}_p$ defined by z is what we need. □

3.1. **Invariant polydisk and the dynamical Mordell-Lang conjecture.** Assume that the transcendence degree of $\mathbf{k}$ over $\mathbb{Q}$ is finite. Let X be an irreducible variety defined over $\mathbf{k}$ of dimension d. Let K be a finitely generated field extension over $\mathbb{Q}$ such that $\overline{K} = \mathbf{k}$, and X is defined over K. Let $f : X \dashrightarrow X$ be a dominant rational endomorphism of X defined over K. There exists a projective variety $X_K \to \mathrm{Spec}\,(K)$ such that $X = X_K \times_{\mathrm{Spec}\,(K)} \mathrm{Spec}\,(\mathbf{k})$ and an endomorphism $f_K : X_K \dashrightarrow X_K$ such that $f = f_K \times_{\mathrm{Spec}\,(K)} \mathrm{id}$.

As we showed in Proposition 2.9, under the assumption of the dynamical Mordell-Lang conjecture, the ZD-property implies the strong ZD-property. Unfortunately, the dynamical Mordell-Lang conjecture is still open in general. In this section, we use the strategy in [1] and [7] to show that the the dynamical Mordell-Lang conjecture holds for the points in a nonempty adic open subsets of $X(\overline{K})$.

We first need the following result.

Proposition 3.16. *There exists $m \geq 1$, a prime $p \geq 3$ and an embedding $i : K \hookrightarrow \mathbb{C}_p$, such that there exists an open subset $V \simeq (\mathbb{C}_p^{\circ})^d$ of $X_K(\mathbb{C}_p)$, which is invariant by f^m, the orbit of the points in V are well defined and*

$$f^m|_V = \mathrm{id} \mod p.$$

Moreover, there exists an analytic action $\Phi : \mathbb{C}_p^{\circ} \times V \to V$ of $(\mathbb{C}_p^{\circ}, +)$ on V such that for every $n \in \mathbb{Z}_{\geq 0}$, $\Phi(n, \cdot) = f^{mn}|_V(\cdot)$.

In particular, for every $x \in V$, the Zariski closure of the orbit $O_{f^m}(x)$ in X is irreducible.

Proof of Proposition 3.16. There exists a subring R of K, which is finitely generated over $\mathbb{Z}$, such that $\overline{K} = \mathbf{k}$.

Pick a model $\pi : X_R \dashrightarrow \mathrm{Spec}\,(R)$ which is projective over $\mathrm{Spec}\,(R)$ and whose generic fiber is X_K. Then f extends to a rational self-map $f_R : X_R \dashrightarrow X_R$. Denote by B_R the union of indeterminacy locus of f_R, the nonétale locus of f_R, and the singular locus of X_R.

Lemma 3.17. *There exists a nonempty, affine, open subset U of* $\mathrm{Spec}\,(R)$ *such that*

1. *U is of finite type over* $\mathrm{Spec}\,(\mathbb{Z})$*;*
2. *for every point $y \in U$, the fiber X_y is absolutely irreducible and* $\dim_{K(y)} X_y = \dim_K X_K$*, where $K(y)$ is the residue field at y;*
3. *for every $y \in U$, the fiber X_y is not contained in B_R.*

Proof of Lemma 3.17. To prove the lemma, we shall use the following fact: For any integral affine scheme $\mathrm{Spec}\,(A)$ of finite type over $\mathrm{Spec}\,(\mathbb{Z})$ and any nonempty open subset V_1 of $\mathrm{Spec}\,(A)$, there exists an affine open subset V_2 of V_1 which is of finite type over $\mathrm{Spec}\,(\mathbb{Z})$. Indeed, we may pick any nonzero element $g \in I$ where I is the ideal of A that defines the closed subset $\mathrm{Spec}\,(A) \setminus V$ and set $U := \mathrm{Spec}\,(A) \setminus \{g = 0\}$. Then $U = \mathrm{Spec}\,(A[1/g])$ is of finite type over $\mathrm{Spec}\,(\mathbb{Z})$.

Since X_K is absolutely irreducible, Proposition 9.7.8 of [25] gives an affine open subset V of $\mathrm{Spec}\,(R)$ such that X_y is absolutely irreducible for every $y \in V$. We may suppose that V is of finite type over $\mathrm{Spec}\,(\mathbb{Z})$. By generic flatness (see

[25], Thm. 6.9.1), we may change V in a smaller subset and suppose that the restriction of π to $\pi^{-1}(V)$ is flat. Then, the fiber X_y is absolutely irreducible and of dimension $\dim_{K(y)} X_y = \dim_K X_K$ for every point $y \in V$.

Denote by B_K the union of the indeterminacy locus, the nonétale locus of f in X_K, the singular locus of X_K and Z_K. Observe that B_K is exactly the generic fiber of $\pi_{|B_R} \colon B_R \to \mathrm{Spec}\,(R)$. By generic flatness, there exists a nonempty, affine, open subset U of V such that the restriction of π to every irreducible component of $\pi^{-1}_{|B_R}(U)$ is flat. Then for $y \in U$, the fiber X_y is not contained in B_R. Then, we shrink U to suppose that U is of finite type over $\mathrm{Spec}\,(\mathbb{Z})$. Since

$$\dim_{K(y)}(B_R \cap X_y) = \dim_K(B_K) < \dim_K X_K = \dim_{K(y)} X_y$$

for every $y \in V$, the fiber X_y is not contained in B_R. $\square$

By Lemma 3.17, we may replace $\mathrm{Spec}\,(R)$ by U and assume that

- for every $y \in \mathrm{Spec}\,(R)$, the fiber X_y is absolutely irreducible;
- for every $s \in S$ and $y \in \mathrm{Spec}\,(R)$, the fiber X_y is not contained in $B_{R,s}$.

Recall the following Lemma.

Lemma 3.18 (see [31, 5])**.** *Let L be a finitely generated extension of $\mathbb{Q}$ and B be a finite subset of L. The set of primes p for which there exists an embedding of L into $\mathbb{Q}_p$ that maps B into $\mathbb{Z}_p$ has positive density[4] among the set of all primes.*

Since R is integral and finitely generated over $\mathbb{Z}$, by Lemma 3.18 there exists infinitely many primes $p \geq 3$ such that R can be embedded into $\mathbb{Z}_p \subseteq \mathbb{C}_p^\circ$. This induces an embedding $\mathrm{Spec}\,(\mathbb{Z}_p) \to \mathrm{Spec}\,(R)$. Set $X_{\mathbb{C}_p} := X_R \times_{\mathrm{Spec}\,(R)} \mathrm{Spec}\,(\mathbb{C}_p)$, and $f_{\mathbb{C}_p} := f_R \times_{\mathrm{Spec}\,(R)} \mathrm{id}$. All fibers X_y, for $y \in \mathrm{Spec}\,(R)$, are absolutely irreducible and of dimension d; hence, the special fiber $X_{\overline{\mathbb{F}_p}}$ of $X_{\mathbb{C}_p^\circ} \to \mathrm{Spec}\,(\mathbb{C}_p^\circ)$ is absolutely irreducible. Denote by $B_{\mathbb{C}_p}$ the union of indeterminacy locus, the nonétale of $f_{\mathbb{C}_p}$, the singular locus of $X_{\mathbb{C}_p}$. Since $B_{\mathbb{C}_p} \subset B_R \cap X_{\overline{\mathbb{F}_p}}$, the fiber $X_{\overline{\mathbb{F}_p}}$ is not contained in $B_{\mathbb{C}_p}$. We note that $X_{\overline{\mathbb{F}_p}}$ and $f|_{X_{\overline{\mathbb{F}_p}} \setminus B_{\mathbb{C}_p}}$ are indeed defined over $\mathbb{F}_p$.

Apply [1, Corollary 2] to the rational map $f|_{X_{\overline{\mathbb{F}_p}} \setminus B_{\mathbb{C}_p}} : X_{\overline{\mathbb{F}_p}} \setminus B_{\mathbb{C}_p} \dashrightarrow X_{\overline{\mathbb{F}_p}} \setminus B_{\mathbb{C}_p}$ there exists a periodic point $x \in X_{\overline{\mathbb{F}_p}}(\overline{\mathbb{F}_p}) \setminus B_{\mathbb{C}_p}$ whose orbit under $f_{\mathbb{C}_p}$ is contained in $X_{\overline{\mathbb{F}_p}}(\overline{\mathbb{F}_p}) \setminus B_{\mathbb{C}_p}$. Observe that x is a regular closed point in $X_{\mathbb{C}_p^\circ}$. There exists $m \geq 1$ such that $f|^m_{X_{\overline{\mathbb{F}_p}}}(x) = x$ and $(df|^m_{X_{\overline{\mathbb{F}_p}}})_x = \mathrm{id}$. Let U be the open subset of $X(\mathbb{C}_p)$ consisting of points whose specialization is y. Then we have $U \simeq (\mathbb{C}_p^{\circ\circ})^d$. Then we have $f^m(U) = U$ and the orbit of the points in U are well defined. The restriction of f^m on U is an analytic automorphism taking form

$$f^m|_U : (x_1, \ldots, x_d) \mapsto (F_1, \ldots, F_d)$$

where $F_n = \sum_I a_I^n x^I, n = 1, \ldots d$ are analytic functions on U with $a_I^n \in \mathbb{C}_p^{\circ\circ}$. Since $(df|^m_{X_{\overline{\mathbb{F}_p}}})_x = \mathrm{id}$, we have

$$f^m|_U = \mathrm{id} \mod \mathbb{C}_p^{\circ\circ}.$$

[4]By **positive density**, we mean that the proportion of primes p among the first N primes that satisfy the statement is bounded from below by a positive number if N is large enough.

There exists $l \in \mathbb{Q}^+$, such that

$$f^m|_U = \mathrm{id} \mod p^{-2l}.$$

Set $V := \{(x_1, \dots, x_d) \in U| \ |x_i| \leq p^{-l}\} \simeq (\mathbb{C}_p^\circ)^d$. Then V is invariant by f^m and

$$f^m|_V = \mathrm{id} \mod p^{-l}.$$

After replacing m by some multiple of m, we get

$$f^m|_V = \mathrm{id} \mod p.$$

By [37, Theorem 1], there exists an analytic action $\Phi : \mathbb{C}_p^\circ \times V \to V$ of $(\mathbb{C}_p^\circ, +)$ on V such that for every $n \geq \mathbb{Z}_{\geq 0}$, $\Phi(n, \cdot) = f^{mn}|_V(\cdot)$.

For every $x \in V$, denote by $\Phi_x : \mathbb{C}_p^\circ \to V$ the analytic morphism $t \to \Phi(t, x)$. Denote Z the Zariski closure of $O_{f^m}(x)$ in X. Let $Z_1, \dots, Z_s$ be the irreducible component of Z. We have $\mathbb{C}_p^\circ = \cap_{i=1}^s \Phi_x^{-1}(Z_i)$. Since $\mathbb{C}_p^\circ$ is irreducible in the Zariski topology, we have $\Phi_x^{-1}(Z_i) = \mathbb{C}_p$ for some $i = 1, \dots, s$. It follows that $O_{f^m}(x) \subseteq \Phi_x(\mathbb{C}_p^\circ) \subseteq Z_i$. It follows that $Z = Z_i$ is irreducible. We concludes the proof. □

The following result says that the dynamical Mordell-Lang conjecture holds for the points in some dense adelic open subset of $X(\mathbf{k})$.

Proposition 3.19. *There exists a prime $p \geq 3$, an embedding $i : K \hookrightarrow \mathbb{C}_p$, and an open subset $V \simeq (\mathbb{C}_p^\circ)^d$, such that for every proper subvariety Z of X and point $x \in X_K(i, V)$, the orbit of x is well defined and the set $\{n \geq 0| \ f^n(x) \in Z\}$ is a finite union of arithmetic progressions. In particular, if the orbit of x is Zariski dense in X, then $\{n \geq 0| \ f^n(x) \in Z\}$ is finite.*

Proof of Proposition 3.19. By Proposition 3.16, there exists $m \geq 1$, a prime $p \geq 3$ and an embedding $i : K \hookrightarrow \mathbb{C}_p$, such that on $X_K(\mathbb{C}_p)$ there exists an open subset $V \simeq (\mathbb{C}_p^\circ)^d$, such that V is invariant by f^m, the orbit of the points in V are well defined and there exists an analytic action $\Phi : \mathbb{C}_p^\circ \times V \to V$ of $(\mathbb{C}_p^\circ, +)$ on V such that for every $n \in \mathbb{Z}_{\geq 0}$, $\Phi(n, \cdot) = f^{mn}|_V(\cdot)$.

Let x be a point in $X_K(i, V)$, there exists $\bar{i} \in \mathcal{I}_i$ such that $\phi_{\bar{i}}(x) \in V$. Using i to view $\mathbf{k}$ as a subfield of $\mathbb{C}_p$ and identify x with $\phi_{\bar{i}}(x) \in V$. Set $Z_j := (f|_V^j)^{-1}(Z)$, $j = 0, \dots, m-1$. Set $g := f^m|_V$, we only need to show that for every $j = 0, \dots, m-1$, the set $S_j := \{n \geq 0| \ g^n(x) \in Z_j\}$ is a finite union of arithmetic progressions. Observe that $S_j := \{n \geq 0| \ g^n(x) \in Z_j\} \subseteq T_j := \{t \in \mathbb{C}_p^\circ| \ \Phi(t, x) \in Z_j\}, j = 0, \dots, m-1$. Observe that T_j is a Zariski closed subset of the disk $\mathbb{C}_p^\circ$. If S_j is infinite, then T_j is Zariski dense in $\mathbb{C}_p^\circ$. It follows that $S_j = \mathbb{Z}_{\geq 0}$. So $S_j, j = 0, \dots, m-1$ is is either finite or $\mathbb{Z}_{\geq 0}$.

Now assume that the orbit $O_f(x)$ of x is Zariski dense in X. If $\{n \geq 0| \ f^n(x) \in Z\}$ is not finite, there exists $a \geq 0, b \geq 1$ such that $f^{a+bn}(x) \in Z$ for $n \geq 0$. It follows that

$$O_f(x) \subseteq \{x, \dots, f^{a-1}(x)\} \cup Z \cup \dots \cup f^{b-1}(Z)$$

which is not Zariski dense. This contradicts to our assumption, which concludes the proof. □

3.2. **The adelic version of the Zariski density orbit conjecture.** Assume that the transcendence degree of $\mathbf{k}$ over $\mathbb{Q}$ is finite. Let X be an irreducible variety defined over $\mathbf{k}$. and $f : X \dashrightarrow X$ be a dominant rational endomorphism.

Definition 3.20. We say that a pair (X, f) satisfies the *adelic ZD-property*, if either there exists a nonconstant rational function $H \in \mathbf{k}(X) \setminus \mathbf{k}$ satisfying $H \circ f = H$ or there exist a nonempty adelic open subset A of $X(\mathbf{k})$ such that for every a point $x \in X(\mathbf{k})$ whose orbit $O_f(x)$ under f is well defined and Zariski dense in X.

We say that a pair (X, f) satisfies the *adelic ZD2-property*, if there exist a nonempty adelic open subset A of $X(\mathbf{k})$ such that for every a point $x \in X(\mathbf{k})$ whose orbit $O_f(x)$ under f is well defined and Zariski dense in X.

Then adelic ZD2-property implies adelic ZD-property.

Remark 3.21. Proposition 3.16 shows that when X is a curve, (X, f) satisfies the adelic ZD2-property if and only if f is not of finite order. In particular, (X, f) satisfies the adelic ZD-property

Proposition 3.22. *The following statements are equivalents:*

(i) (X, f) *satisfies the adelic ZD-property (resp. adelic ZD2-property);*
(ii) *there exists* $m \geq 1$, *such that* (X, f^m) *satisfies the adelic ZD-property (resp. adelic ZD2-property);*
(iii) *there exists a pair* (Y, g) *which is birational to the pair* (X, f), *and* (Y, f) *satisfies the adelic ZD-property (resp. adelic ZD2-property).*

Proof of Proposition 3.22. We only prove it for adelic ZD-property. For adelic ZD2-property, the proof is similar.

It is clear that (i) implies (ii) and (iii). Lemma 2.6 shows that (ii) implies (i).

We only need to show that (iii) implies (i). Let $\pi : Y \dashrightarrow X$ be a birational morphism satisfying $\pi \circ g = f \circ \pi$. If there exists a nonconstant rational function $H \in \mathbf{k}(Y) \setminus \mathbf{k}$ satisfying $H \circ g = H$, there is nothing to proof.

Now assume that exist a nonempty adelic open subset A of $X(\mathbf{k})$ such that for every a point $x \in X(\mathbf{k})$ whose orbit $O_f(x)$ under f is well defined and Zariski dense in X. Let U be a Zariski subset of Y such that $\pi|_U : U \to X$ is an isomorphism to its image. Apply Proposition 3.19 to $g|_U : U \dashrightarrow U$, there exists a nonempty adelic open subset A_1 of $U(\mathbf{k})$ such that for every point $x \in A$, its orbit under $g|_U$ is well defined. Then for every $x \in A \cap A_1$, the orbit of x under g is well defined, contained in U and is Zariski dense in Y. By Remark 3.6, $A \cap A_1$ is a nonempty adelic open subset of $Y(\mathbf{k})$. Then $\pi(A \cap A_1)$ a nonempty adelic open subset of $X(\mathbf{k})$. Then for every $x \in \pi(A \cap A_1)$, the orbit of x under f is well defined, contained in $\pi(U)$ and is Zariski dense in X, which concludes the proof. $\square$

Lemma 3.23. *Let* X' *be an irreducible variety over* $\mathbf{k}$. *Let* $f' : X' \dashrightarrow X'$ *be a rational dominant endomorphism. Let* $\pi : X' \dashrightarrow X$ *be a generically finite morphism such that* $\pi \circ f' = f \circ \pi$. *Then* (X', f') *satisfies the adelic ZD-property (resp. adelic ZD2-property) if and only if* (X, f) *satisfies the adelic ZD-property (resp. adelic ZD2-property).*

Proof of Lemma 3.23. We only prove it for adelic ZD-property. For adelic ZD2-property, the proof is similar.

After shrinking X' we may assume that π is well defined, locally finite and étale.

We first assume that (X', f') satisfies the adelic ZD-property.

Assume that there exists a nonconstant rational function H' on X' such that $(f')^*H' = H'$. We have $H' \in \mathbf{k}(X') \subseteq \overline{\mathbf{k}(X)}$. Set $m := [\mathbf{k}(X') : \mathbf{k}(X)]$. Then $\mathbf{k}(X')$ is a m dimensional $\mathbf{k}(X)$ vector space. Denote by

$$T^m + \sum_{i=1}^{m}(-1)^i P_i T^{m-i}$$

the characteristic polynomial of the $\mathbf{k}(X)$-linear operator

$$\mathbf{k}(X') \to \mathbf{k}(X') : g \mapsto H'g.$$

We have $P_i \in \mathbf{k}(X)$ and $f^*(P_i) = P_i, i = 1, \dots, m$. If $P_i \in \mathbf{k}$ for $i = 1, \dots, m$, then $H' \in \mathbf{k}$, which is a contradiction. It follows that there exists $i = 1, \dots, m$, such that P_i is a nonconstant rational function on X.

Now we may assume that there exists a nonempty adelic open subset A of $X'(\mathbf{k})$ such that for every point $p \in A$, the orbit of p is well defined, contained in $\pi^{-1}(X \setminus I(f))$ and Zariski dense in X'. Then for every p in the nonempty adelic open subset $\pi(A) \subseteq X'(\mathbf{k})$, the orbit $O_f(p)$ is Zariski dense in X.

Next we assume that (X, f) satisfies the adelic ZD-property. If there exists a nonconstant rational function H on X such that $(f)^*H = H$, then $H' := H \circ \pi$ is a nonconstant rational function on X' such that $(f')^*H' = H'$.

Now we may assume that there exists a nonempty adelic open subset A of $X(\mathbf{k})$ such that for every point $p \in A$, the orbit of p is well defined, contained in $\pi(X' \setminus I(f')) \cap (X \setminus I(f))$ and Zariski dense in X. Then for every p in the nonempty adelic open subset $\pi^{-1}(A) \subseteq X(\mathbf{k})$, the orbit $O_{f'}(p)$ is Zariski dense in X', which concludes the proof. □

The following result shows that the adelic veriosn of the Zariski densiy conjecture implies the strong Zariski densiy conjecture.

Corollary 3.24. *Let $\mathbf{k}'$ be an algebraically closed field extension over $\mathbf{k}$. Set $X_{\mathbf{k}'} := X \times_{\operatorname{Spec} \mathbf{k}} \operatorname{Spec} \mathbf{k}'$ and $f_{\mathbf{k}'} := f \times_{\operatorname{Spec} \mathbf{k}} \mathrm{id}$. If the pair (X, f) satisfies the adelic ZD-property, then $(X_{\mathbf{k}'}, f_{\mathbf{k}'})$ satisfies the strong ZD-property.*

Proof of Corollary 3.24. Let U' be a nonempty Zariski open of $X_{\mathbf{k}'}$. Set $V := \cup_{\sigma \in \mathrm{Gal}(\mathbf{k}'/\mathbf{k})} \sigma(U')$. Then there exists a nonempty Zariski open U of X such that $U' = U \times_{\operatorname{Spec} \mathbf{k}} \operatorname{Spec} \mathbf{k}'$. Denote by $\phi : X(\mathbf{k}) \hookrightarrow X_{\mathbf{k}'}(\mathbf{k}')$ the natural embedding. Observe that for every $y \in X(\mathbf{k})$, $\phi(y)$ is invariant under the action of $\mathrm{Gal}(\mathbf{k}'/\mathbf{k})$.

By Proposition 3.22, the pair $(U, f|_U)$ is satisfies the adelic ZD-property. Then there exists a nonempty adelic open subset A of $U(\mathbf{k})$ for every $x \in A$, the orbit of x under f is well defined, contained in U and is Zariski dense in X. Then we have $\phi(x) \in V$, its orbit under $f_{\mathbf{k}'}$ is well defined, contained in V and is

Zariski dense in $X_{\mathbf{k}'}$. For every $n \geq 0$, there exists $\sigma \in \mathrm{Gal}(\mathbf{k}'/\mathbf{k})$ such that $\phi(f_{\mathbf{k}}^n(x)) = f_{\mathbf{k}'}^n(\phi(x)) \in \sigma(U)$. It follows that

$$\phi(f_{\mathbf{k}}^n(x)) = \sigma^{-1}(\phi(f_{\mathbf{k}}^n(x))) \in U,$$

which concludes the proof. □

3.3. **Invariant curves.** In this section, we assume further that X is surface. The aim of this section is to prove the following result.

Proposition 3.25. *If the pair (X, f) does not satisfy the adelic ZD2-property, then there exists $m \geq 1$, such that there exist infinitely many irreducible curves C of X satisfying $f^m(C) \subseteq C$.*

Proof of Proposition 3.25. Let K be a subfield of $\mathbf{k}$ which is finitely generated over $\mathbb{Q}$, $\overline{K} = \mathbf{k}$ such that X and f are defined over K.

By Proposition 3.16, there exists $m \geq 1$, a prime $p \geq 3$ and an embedding $i : K \hookrightarrow \mathbb{C}_p$, such that on $X_K(\mathbb{C}_p)$ there exists an open subset $V \simeq (\mathbb{C}_p^{\circ})^d$, such that V is invariant by f^m, the orbit of the points in V are well defined and there exists an analytic action $\Phi : \mathbb{C}_p^{\circ} \times V \to V$ of $(\mathbb{C}_p^{\circ}, +)$ on V such that for every $n \in \mathbb{Z}_{\geq 0}$, $\Phi(n, \cdot) = f^{mn}|_V(\cdot)$. Observe that for every point $x \in V \setminus \mathrm{Fix}(f^m)$, the orbit of x is infinite. Since the pair (X, f) does not satisfy the adelic Zariski dense property, $V \setminus \mathrm{Fix}(f^m) \neq \emptyset$.

Observe that for every point x in $X_K(i, V \setminus \mathrm{Fix}(f^m))$, the orbit of x is well defined and is infinite. Let B be any proper Zariski close subset of X containing $\mathrm{Fix}(f)$. Since the pair (X, f) does not satisfy the adelic Zariski dense property, there exists $z \in X_K(i, V \setminus B)$, whose orbit is not Zariski dense. Denote by Z_z the Zariski closure of $O_{f^m}(z)$ in X. There exists $\bar{i} \in \mathcal{I}_i$ such that $\phi_{\bar{i}}(z) \in V$. By Proposition 3.16, Z_z is irreducible. Then we have $f^m(Z_z) = Z_z$ and $Z_z \not\subseteq B$. Since z is not preperiodic, $\dim Z_z = 1$. It follows that for every proper Zariski close subset of X, there exists an irreducible and f^m-invariant curve C of X which is not contained in B. This concludes the proof. □

The following result generalizes [44, Theorem 1.3] and [9, Theorem 1.3.] in the adelic setting.

Corollary 3.26. *Assume that f is a birational morphism on the surface X, then the pair (X, f) satisfies the adelic ZD-property.*

Proof of Corollary 3.26. If the pair (X, f) does not satisfy the adelic ZD2-property, then there exists $m \geq 1$, such that there exist infinitely many irreducible curves C of X satisfying $f^m(C) \subseteq C$.

By [15, Theorem B], f^m preserves a nonconstant rational function. Then (X, f) satisfies the adelic ZD-property, which concludes the proof. □

3.4. **Skew-linear self-maps.** In [24, Theorem 1.4], Ghioca and the author proved the following theorem.

Theorem 3.27. [5] *Let $g : X \dashrightarrow X$ be a dominant rational map defined over an algebraically closed field K of characteristic zero, let $N \geq 1$, and let $f : X \times \mathbb{A}^N \dashrightarrow X \times \mathbb{A}^N$ be a dominant rational map defined by $(x, y) \mapsto (g(x), A(x)y)$ where $A \in \mathrm{GL}_N(k(X))$. If the pair (X, g) satisfies the strong ZD-property, then the pair $(X \times \mathbb{A}^N, f)$ satisfies the ZD-property.*

In this section, we will prove the following adelic version of it.

Theorem 3.28. *Let $g : X \dashrightarrow X$ be a dominant rational map defined over $\mathbf{k}$ and $N \geq 1$. Let $f : X \times \mathbb{A}^N \dashrightarrow X \times \mathbb{A}^N$ be a dominant rational map defined by $(x, y) \mapsto (g(x), A(x)y)$ where $A \in \mathrm{GL}_N(\mathbf{k}(X))$ and $h : X \times \mathbb{P}^{N-1} \dashrightarrow X \times \mathbb{P}^{N-1}$ be a dominant rational map defined by $(x, y) \mapsto (g(x), C(x)y)$ where $C \in \mathrm{PGL}_N(\mathbf{k}(X))$. If the pair (X, g) satisfies the adelic ZD-property, then the pairs $(X \times \mathbb{A}^N, f)$ and $(X \times \mathbb{P}^N, h)$ also satisfy the adelic ZD-property.*

Proof of Theorem 3.28. Assume that the pair (X, g) satisfies the adelic ZD-property.

We first prove that $(X \times \mathbb{A}^N, f)$ satisfies the adelic ZD-property. Denote by $\pi : X \times \mathbb{A}^N \to X$ the projection to the first coordinate. Let $\mathcal{B}$ be the set of points $x \in X$ such that f is not a locally isomorphism on the fiber $\pi^{-1}(x)$. Then $\mathcal{B}$ is a proper closed subset of X.

If there exists a nonconstant rational function H on X invariant under g, then the nonconstant rational function $H \circ \pi$ on $X \times \mathbb{A}^N$ is invariant under f. So Theorem 3.28 holds.

Now we may assume that there is no nonconstant rational function on X invariant under g. Then there exists a nonempty adelic open subset A of $(X \setminus \mathcal{B})(\mathbf{k})$ such that for all point $x \in A$, the orbit of x under $g|_{X \setminus \mathcal{B}}$ is well defined and is Zariski dense in X.

Let I be the set of all invariant subvarieties in $X \times \mathbb{A}^N$ for which every irreducible component of V dominates X under the projection map $\pi : X \times \mathbb{A}^N \to X$. Then [24, Theorem 2.1] yields that (perhaps, at the expense of replacing f by a suitable iterate) there exists an irreducible variety Y endowed with a dominant rational self-map

$$g' : Y \dashrightarrow Y$$

and a generically finite map $\tau : Y \dashrightarrow X$ satisfying $\tau \circ g' = g \circ \tau$ such that there exists a birational map h on $Y \times \mathbb{A}^N = Y \times_X X \times \mathbb{A}^N$ of the form $(x, y) \mapsto (x, T(x)y)$ where $T(x) \in \mathrm{GL}_N(\mathbf{k}(Y))$ such that for any subvariety $V \in I$, we have

$$h^{-1}((\tau \times_X \mathrm{id})^{\#}(V)) = Y \times \alpha(V) \subseteq Y \times \mathbb{A}^N,$$

where $\alpha(V)$ is a subvariety of $\mathbb{A}^N$ and $(\tau \times_X \mathrm{id})^{\#}(V)$ is the strict transform of V by the rational map $\tau \times_X \mathrm{id}$. Let $f' : Y \times \mathbb{A}^N \to Y \times \mathbb{A}^N$ be the rational map defined by

$$g' \times_{(X,g)} f : (x, y) \mapsto (g'(x), A(\tau(x))y).$$

[5] In [24], we said (X, g) is a *good pair* if is satisfies the strong ZD-property. The original statement of [24, Theorem 1.4] said that if (X, g) satisfies the strong ZD-property, then the pair $(X \times \mathbb{A}^N, f)$ satisfies the strong ZD-property. But its proof only showed the slightly weaker conclusion as we stated here.

We have $(\tau \times \mathrm{id}) \circ f' = f \circ (\tau \times \mathrm{id})$. Let

$$F := h^{-1} \circ f' \circ h : Y \times \mathbb{A}^N \to Y \times \mathbb{A}^N.$$

Then F is the map $(x, y) \mapsto (g'(x), B(x)y)$ where $B(x) := T^{-1}(g'(x))A(\tau(x))T(x)$. Let $\rho := (\tau \times \mathrm{id}) \circ h$. Then we have $\rho \circ F = f \circ \rho$. For any $V \in I$, we see that $\rho^{\#}(V)$ is invariant by F and it has the form $Y \times \alpha(V)$. Denote by G^V the subgroup of $\mathrm{GL}_N(\mathbf{k})$ consisting for $g \in \mathrm{GL}_N(\mathbf{k})$ satisfying $g\alpha(V) = \alpha(V)$. Set $G := \cap_{V \in I} G^V$.

After replacing Y by some smaller open subset, we may assume that τ, ρ are regular morphism. Furthermore, we may assume that τ, ρ are locally finite and étale. Then $\tau^{-1}(A)$ is a nonempty adelic open subset of Y. Let

$$p : Y \times \mathbb{A}^N \to Y$$

be the projection to the first coordinate. Let $\mathcal{B}'$ be the set of points $x \in Y$ such that F is not locally an isomorphism on the fiber $p^{-1}(x)$. Then $\mathcal{B}'$ is a proper closed subset of Y. Since $Y \times \alpha(V)$ is invariant by F for every $V \in I$, we get $B(x) \in G$ for $x \in Y \setminus \mathcal{B}'$. By Lemma 3.23, we only need to show that $(Y \times \mathbb{A}^N, F)$ satisfies the adelic ZD-property.

By Proposition 3.19, there exists a nonempty adelic open subset A_1 of $Y \setminus \mathcal{B}'$ such that for every point $x \in A_1$, its orbit under $F|_{Y \setminus \mathcal{B}'}$ is well defined. Since τ is flat, $\tau(A_1)$ is a nonempty adelic open subset of $X(\mathbf{k})$. For every point $x \in \tau^{-1}(A) \cap A_1$, the orbit of x is well defined, contained in $Y \setminus \mathcal{B}'$ and Zariski dense in Y.

By [40, Theorem 2], either there exists a nonconstant rational function $\phi \in \mathbf{k}(\mathbb{A}^N)$ such that $\phi \circ g' = \phi$ for all $g' \in G$ or there exists a nonempty G-invariant Zariski open subset $U_G \subseteq \mathbb{A}^N$ such that for every $y \in U_G$, $G \cdot y$ is Zariski dense in $\mathbb{A}^N$.

First assume that the later holds. Then $(A_1 \cap \rho^{-1}(A)) \times U_G(\mathbf{k})$ is a nonempty adelic open subset of $Y \times \mathbb{A}^N$. For every $q \in (A_1 \cap \rho^{-1}(A)) \times U_G(\mathbf{k})$, the orbit of q is well defined and contained in $p^{-1}(Y \setminus B_1) \cap \rho^{-1}(X \setminus \mathcal{B})$. We need to show that that the orbit $O_F(q)$ is Zariski dense in $Y \times \mathbb{A}^N$. Denote by Z the Zariski closure of $O_F(q)$. Since $O_{g'}(p(q))$ is Zariski dense in Y, then Z has at least one irreducible component which dominates X. Let W be the union of all irreducible components of Z which dominate Y; then $\overline{\rho(W)} \in I$ and $\overline{\rho(W)} \neq Y \times \mathbb{A}^N$. There exists $m \geq 0$ such that $F^m(q) \in \rho^{\#}(\overline{\rho(W)}) = Y \times \alpha(\overline{\rho(W)})$ and so,

$$F^n(q) \in Y \times \alpha(\overline{\rho(W)})$$

for all $n \geq m$. Write $q = (x, y) \in (A_1 \cap \rho^{-1}(A)) \times U_G(\mathbf{k})$, we have

$$F^n(q) = (g'^m(x), B(g'^{m-1}(x)) \ldots B(x)y).$$

It follows that $B(g'^{m-1}(x)) \ldots B(x)y \in \alpha(\overline{\rho(W)})$. Since $B(z) \in G \subseteq G^{\overline{\rho(W)}}$ for $z \in Y \setminus \mathcal{B}'$, we get $y \in \alpha(\overline{\rho(W)})$. It follows that $G \cdot y \subseteq \alpha(\overline{\rho(W)})$, which is not Zariski dense in $\mathbb{A}^N$. Then we get a contradiction.

Now we assume that there exists a nonconstant rational function $\phi \in \mathbf{k}(\mathbb{A}^N)$ such that $\phi \circ g' = \phi$ for all $g' \in G_\alpha$. Let χ be the rational function on $Y \times \mathbb{A}^N$ defined by $(x, y) \mapsto \phi(y)$. Then χ is nonconstant and is invariant by F.

Now we prove that $(X \times \mathbb{P}^{N-1}, h)$ satisfies the adelic ZD-property. Denote by $\pi : X \times \mathbb{P}^{N-1} \to X$ the projection to the first coordinate. Let $\mathcal{B}$ be the set of points $x \in X$ such that h is not a locally isomorphism on the fiber $\pi^{-1}(x)$. Then $\mathcal{B}$ is a proper closed subset of X.

If there exists a nonconstant rational function H on X invariant under g, then the nonconstant rational function $H \circ \pi$ on $X \times \mathbb{P}^{N-1}$ is invariant under h. So Theorem 3.28 holds.

Now we may assume that there is no nonconstant rational function on X invariant under g. Then there exists a nonempty adelic open subset A of $(X \setminus \mathcal{B})(\mathbf{k})$ such that for all point $x \in A$, the orbit of x under $g|_{X\setminus\mathcal{B}}$ is well defined and is Zariski dense in X.

Let J be the set of all invariant subvarieties in $X \times \mathbb{P}^{N-1}$ for which every irreducible component of V dominates X under the projection map $\pi : X \times \mathbb{P}^{N-1} \to X$.

The following result is an analogue of [24, Theorem 2.1] in this setting.

Lemma 3.29. *At the expense of replacing h by a suitable iterate, there exists an irreducible variety Y endowed with a dominant rational self-map*

$$g' : Y \dashrightarrow Y$$

and a generically finite map $\tau : Y \dashrightarrow X$ satisfying $\tau \circ g' = g \circ \tau$ such that there exists a birational map β on $Y \times \mathbb{P}^{N-1} = Y \times_X X \times \mathbb{P}^{N-1}$ of the form $(x, y) \mapsto (x, T(x)y)$ where $T(x) \in \mathrm{PGL}_N(\mathbf{k}(Y))$ such that for any subvariety $V \in J$, we have

$$\beta^{-1}((\tau \times_X \mathrm{id})^{\#}(V)) = Y \times \gamma(V) \subseteq Y \times \mathbb{P}^{N-1},$$

where $\gamma(V)$ is a subvariety of $\mathbb{P}^{N-1}$.

After replacing [24, Theorem 2.1] by Lemma 3.29, the proof above for the pair $(X \times \mathbb{A}^N, f)$ yields directly the proof for the pair $(X \times \mathbb{P}^{N-1}, g)$. We concludes the proof. □

Proof of Lemma 3.29. Since $H^1_{ét}(\mathbf{k}(X), \mathbb{G}_m) = 0$, there exists $D(x) \in \mathrm{GL}_N(\mathbf{k}(X))$ whose image in $\mathrm{PGL}_N(\mathbf{k}(X))$ is $C(x)$. Consider the rational morphism $f : X \times (\mathbb{A}^N \setminus \{0\}) \dashrightarrow X \times (\mathbb{A}^N \setminus \{0\})$ sending (x, y) to $(x, D(x)y)$. Denote $\theta : (\mathbb{A}^N \setminus \{0\}) \to \mathbb{P}^{N-1}$ the morphism $(x_1, \dots, x_N) \mapsto [x_1 : \dots, : x_N]$. Set

$$\phi := \mathrm{id} \times \theta : X \times (\mathbb{A}^N \setminus \{0\}) \to X \times \mathbb{P}^{N-1}.$$

Then we have $\phi \circ f = h \circ \phi$. Denote by I the set of all f-invariant subvarieties in $X \times \mathbb{A}^N$ for which every irreducible component of V dominates X under the projection map $\pi_1 : X \times \mathbb{A}^N \to X$. For every $W \in J$, define $\hat{W} := \overline{\phi^{-1}(W)}$. We have $\hat{W} \in I$.

By [24, Theorem 2.1], at the expense of replacing h, f by a suitable iterate, there exists an irreducible variety Y endowed with a dominant rational self-map

$$g' : Y \dashrightarrow Y$$

and a generically finite map $\tau : Y \dashrightarrow X$ satisfying $\tau \circ g' = g \circ \tau$ such that there exists a birational map $\hat{\beta}$ on $Y \times \mathbb{A}^N = Y \times_X X \times \mathbb{A}^N$ of the form $(x, y) \mapsto$

$(x, \hat{T}(x)y)$ where $\hat{T}(x) \in \mathrm{GL}_N(\mathbf{k}(Y))$ such that for any subvariety $V \in I$, we have

$$\hat{\beta}^{-1}((\tau \times_X \mathrm{id})^{\#}(V)) = Y \times \alpha(V) \subseteq Y \times \mathbb{A}^N,$$

where $\alpha(V)$ is a subvariety of $\mathbb{A}^N$ and $(\tau \times_X \mathrm{id})^{\#}(V)$ is the strict transform of V by the rational map $\tau \times_X \mathrm{id}$. Let $T(x)$ be the image of $\hat{T}(x)$ in $\mathrm{PGL}_N(\mathbf{k}(X))$. Let $\beta : Y \times \mathbb{P}^{N-1} \to Y \times \mathbb{P}^{N-1}$ be the morphism $(x, y) \mapsto (x, T(x)y)$. For every $V \in J$, define $\gamma(V) := \theta(\alpha(\hat{V}))$. Then we get

$$\beta^{-1}((\tau \times_X \mathrm{id})^{\#}(V)) = Y \times \gamma(V) \subseteq Y \times \mathbb{P}^{N-1},$$

which concludes the proof. □

3.5. **Diophantine condition.** We say that $\lambda_1, \lambda_2 \in \mathbb{C}_p \setminus \{0\}$ satisfy the *Diophantine condition* [27], if $|\lambda_1| = |\lambda_2| = 1$ and there exists $C, \beta > 0$ such that for every $n_1, n_2 \in \mathbb{Z}_{\geq 0}, n_1 + n_2 \geq 2$ and $i = 1, 2$, we have

$$|\lambda_1^{n_1}\lambda_2^{n_2} - \lambda_i| \geq C|n_1 + n_2|_{\mathbb{R}}^{-\beta},$$

here we denote by $|\cdot|_{\mathbb{R}}$ the absolute value on $\mathbb{R}$. By [46], if $\lambda_1, \lambda_2 \in \mathbb{C}_p \setminus \{0\}$ are algebraic numbers, then this condition is always satisfied.

Proposition 3.30. *Let $\lambda_1, \lambda_2 \in \mathbf{k} \setminus \{0\}$ be two multiplicatively independent*[6] *elements. Then there exists a prime p, a positive integer $m \in \mathbb{Z}_{>0}$ and an embedding $\tau : \mathbb{Q}(\lambda_1, \lambda_2) \hookrightarrow \mathbb{C}_p$ such that $\tau(\lambda_1)^m, \tau(\lambda_2)^m$ satisfy the Diophantine condition.*

Proof of Proposition 3.30. If $\lambda_1, \lambda_2 \in \overline{\mathbb{Q}}$, we conclude the proof by [46].

If the transcendence degree of $\mathbb{Q}(\lambda_1, \lambda_2)$ over $\mathbb{Q}$ is two, then we only need to show that there are elements $\mu_1, \mu_2 \in 1 + p\mathbb{Z}_p$ which are algebraically independent over $\mathbb{Q}$ and satisfy the Diophantine condition. This is clear by [27, Proposition 3].

Now we may assume that the transcendence degree of $\mathbb{Q}(\lambda_1, \lambda_2)$ over $\mathbb{Q}$ is one. We may assume that $\lambda_1 \notin \overline{\mathbb{Q}}$ and $\lambda_2 \in \overline{\mathbb{Q}(\lambda_1)}$. Assume that the minimal polynomial of λ_2 over $\overline{\mathbb{Q}}(\lambda_1)$ is as follows:

$$P(t, x) = x^d + a_{d-1}x^{d-1} + \cdots + a_0 = 0$$

where $a_i \in \overline{\mathbb{Q}}(\lambda_1), i = 1, \ldots, d$. Observe that $a_0 \neq 0$.

There exists a finite set $S \subseteq \overline{\mathbb{Q}}$ such that $a_i \in \mathcal{O}(\mathbb{A}^1 \setminus S)$. Denote by Y the curve $\{(t, y) \in (\mathbb{A}^1 \setminus S) \times \mathbb{A}^1 \setminus \{0\} |\ y^d + a_{d-1}(t)y^{d-1} + \cdots + a_0(t) = 0\}$ and $\pi : Y \to \mathbb{A}^1 \setminus S$ the projection to the first coordinate. After enlarge S, we may assume that π is étale. Pick a root of unity $\mu \in \overline{\mathbb{Q}} \setminus S$ and a point $(\mu, c) \in \pi^{-1}(\mu)$. Observe that $c \in \overline{\mathbb{Q}}$. Let K be a number field who contains μ, c and all coefficients of $a_i, i = 0, \ldots, d-1$. By Lemma 3.18, there exists a prime $p > 2$ which does not

[6] We say that λ_1, λ_2 are *multiplicatively independent* if for every $(m_1, m_2) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$,

$$\lambda_1^{m_1}\lambda_2^{m_2} \neq 1.$$

divide the order of μ, an embedding $\tau_1 : K \to \mathbb{Q}_p$ such that $a_i(\mu) \in \mathbb{Z}_p$ and $(d\pi)|_{(\mu,c)}$ is invertible modulo p. Then there exists

$$\phi(t) = \sum_{i\geq 1} c_i t^i,$$

where $c_i \in \mathbb{Z}_p, c_i \to 0$ as $i \to \infty$ and $P(\mu+t, c+\phi(t)) = 0$. There exists $m \in \mathbb{Z}_{>0}$, such that $p \not| m$, $\mu^m = 1$ and $c^m = 1 \mod p$. Set

$$\alpha(t) := (\mu + t)^m - 1.$$

We have $\alpha(0) = 0$ and $\alpha|_{p\mathbb{Z}_p} : p\mathbb{C}_p \to p\mathbb{C}_p$ is an analytic automorphism. Set $\beta(t) := (c + \phi(\alpha^{-1}(t)))^m - 1$. Then β converges on $p\mathbb{C}_p$ and $\beta(p\mathbb{C}_p) \subseteq p\mathbb{C}_p$. Observe that the coefficients of β are contained in $\mathbb{Z}_p \cap \overline{\mathbb{Q}}$. For every $u \in p\mathbb{C}_p \setminus \overline{\mathbb{Q}}$, there exists an embedding

$$\tau_u : \mathbb{Q}(\lambda_1, \lambda_2) \hookrightarrow \mathbb{C}_p$$

sending λ_1 to $1 + u$ and λ_2 to $1 + \beta(u)$. Then we only need to show that there exists $u \in p\mathbb{C}_p \setminus \overline{\mathbb{Q}}$, such that $1+u$ and $1+\beta(u)$ satisfy the Diophantine condition.

We note that $\exp(x)$ and $\log(x)$ are well defined analytic function on $p\mathbb{C}_p$ and satisfy:

$$|\log(1 + x)| = |\exp(x) - 1| = |x|, x \in p\mathbb{C}_p.$$

Set $\delta(t) := \log(1 + \beta(\exp(t)))$, which converges on $p\mathbb{C}_p$. Write $\delta(t) = \sum_{i\geq 0} b_i t^i$.

Lemma 3.31. *There exists $r \in \mathbb{Q}_{>0} \cap [1, +\infty)$, and $C, \beta > 0$ such that for every $v \in \mathbb{C}_p$ with norm $|v| = p^{-r}$, for every $m, n \in \mathbb{Z}, m + n \geq 1$ and $i = 1, 2$, we have*

$$|mv + n\delta(v)| \geq C|m + n + 1|_{\mathbb{R}}^{-\beta}.$$

Pick r as in Lemma 3.31. Pick $u \in p\mathbb{C}_p \setminus \overline{\mathbb{Q}}$ with norm $|u| = p^{-r}$. Then $v := \log(1 + u)$ has norm $|v| = p^{-r}$. We conclude the proof by Lemma 3.31. $\square$

Proof of Lemma 3.31. We first have the following observations:

(i) for $m, n \in \mathbb{Z}, m + n \geq 1$, we have

$$\max\{|m|_{\mathbb{R}}, |n|_{\mathbb{R}}\} \leq |m + n + 1|_{\mathbb{R}};$$

(ii) for $n \in \mathbb{Z} \setminus \{0\}$, we have $|n| \geq |n|_{\mathbb{R}}^{-1}$.

Write $\delta(t) = \sum_{i\geq 0} b_i t^i$. We note that $b_i \in \overline{\mathbb{Q}}, i \geq 0$.

We first treat the case where $b_i = 0$ for all $i \neq 1$. Then we have $\delta(t) = b_1 t$. Since $\exp(t)$ and $\exp(\delta(t))$ is multiplicatively independent, we have $b_1 \notin \mathbb{Q}$. Pick any $r = 1$ and $\beta := 3$. By the p-adic Thue-Siegel-Roth theorem[38], there exists $C_1 > 0$ such that for $m, n \in \mathbb{Z}, m + n \geq 1$, we have

$$|m + nb_1| \geq C_1 \max\{|m|_{\mathbb{R}}, |n|_{\mathbb{R}}\}^{-3}.$$

Set $C := C_1 p^{-1}$. Then for every $v \in p\mathbb{C}_p$ with norm $|v| = p^{-1}$, $m, n \in \mathbb{Z}, m+n \geq 1$, we have

$$|mv + n\delta(v)| = |m + nb_1|p^{-1} \geq C_1 p^{-1} \max\{|m|_{\mathbb{R}}, |n|_{\mathbb{R}}\}^{-3} \geq C|m + n + 1|_{\mathbb{R}}^{-3}.$$

This concludes the proof.

Now, we may assume that the set $\{i \in \mathbb{Z}_{\geq 0} \setminus \{1\} | \, b_i \neq 0\}$ is nonempty and let s be the smallest integer in this set.

There exists $l \in \mathbb{Z}_{>0}$ such that $|b_l p^l| > |b_i p^i|$ for all $i \neq 1, l$. Pick $r := l + 1/(s+2)$. Let v be any element with norm $|v| = p^{-r}$. If $n = 0, m \geq 1$, we get

$$|mv + n\delta(v)| = |mv| \geq p^{-r}|m|_{\mathbb{R}}^{-1} = p^{-r}|m+n+1|_{\mathbb{R}}^{-1}.$$

For every $n \in \mathbb{Z} \setminus \{0\}$, We have

- $|nb_s v^s| > |n(\sum_{i \geq 0, i \neq 1, s} b_i v^i)|$;
- $|nb_s v^s| \neq |(m + nb_1)v|$.

It follows that for $m, n \in \mathbb{Z}, m+n \geq 1, n \neq 0$, we have

$$|mv + n\delta(v)| = |(m+nb_1)v + nb_s v^s + n(\sum_{i \geq 0, i \neq 1, s} b_i v^i)|$$

$$= \max\{|(m+nb_1)v|, |nb_s v^s|\} \geq |nb_s| p^{-sr}$$
$$\geq |b_s| p^{-sr} |n|_{\mathbb{R}}^{-1} \geq |b_s| p^{-sr} |n+m+1|_{\mathbb{R}}^{-1}$$

We conclude the proof by setting $\beta := 1$ and $C := \min\{|b_0|p^{-sr}, p^{-r}\}$. □

The proof of [2, Proposition 2.3], [2, Lemma 2.6] and [2, Corollary 2.7] shows that

Proposition 3.32. *Let p be a prime. Let $X_{\mathbb{C}_p}$ be an irreducible surface defined over $\mathbb{C}_p$. and $f : X_{\mathbb{C}_p} \dashrightarrow X_{\mathbb{C}_p}$ be a dominant rational endomorphism. Let o be a smooth point in $X_{\mathbb{C}_p}(\mathbb{C}_p) \setminus I(f)$ satisfying $f(x) = x$. Let λ_1, λ_2 be the two eigenvalues of $df|_o$. Assume that λ_1, λ_2 are nonzero and satisfy the Diophantine condition. Then for every p-adic neighborhood V of o, there exists a nonempty p-adic open set $U \subseteq V$ such that for every point $y \in U$, the orbit of y is well defined and Zariski dense in $X_{\mathbb{C}_p}$.*

Combine Proposition 3.30 with Proposition 3.32, we get the following result.

Corollary 3.33. *Let X be an irreducible surface defined over $\mathbf{k}$. and $f : X \dashrightarrow X$ be a dominant rational endomorphism. Let o be a smooth point in $X(\mathbf{k}) \setminus I(f)$ satisfying $f(x) = x$. Let λ_1, λ_2 be the two eigenvalues of $df|_o$. Assume that λ_1, λ_2 are nonzero and multiplicatively independent. Then the pair (X, f) satisfies the adelic ZD2-property.*

4. Some applications of the adelic topology

In this section, assume that the transcendence degree of $\mathbf{k}$ over $\mathbb{Q}$ is finite.

4.1. **Product by some endomorphism of $\mathbb{P}^1$.** Let X be an irreducible projective variety over $\mathbf{k}$. Let $g : X \dashrightarrow X$ be a dominant rational endomorphism. Let $h : \mathbb{P}^1 \to \mathbb{P}^1$ be a dominant endomorphism. Denote by $f : X \times \mathbb{P}^1 \to X \times \mathbb{P}^1$ the rational endomorphism defined by $(x, y) \mapsto (g(x), h(y))$.

Theorem 4.1. *Assume that (X, g) satisfies the adelic ZD-property (resp. adelic ZD2-property). If h has a superattracting fixed point, then $(X \times \mathbb{P}^1, f)$ satisfies the adelic ZD-property (resp. adelic ZD2-property).*

Remark 4.2. Theorem 4.1 can be generalized in two direction.

1). Combine the proof of Theorem 4.1 and Lemma [8, Lemma 14.3.4.1] (see Lemma 6.8 also), we can replacing the assumption that h has a superattracting fixed point by assuming h is not a post critially finite.

2). A slight modification of the proof of Theorem 4.1, show that the following result:

Let $f : Y \to Y$ be an endomorphism of a normal projective variety and Z is an invariant hyperplane of Y. If the generic point of Z is contained in the critical locus of f, and the pair $(Z, f|_Z)$ satisfies the adelic ZD2-property, then the pair (Y, f) satisfies the adelic ZD2-property.

Combine this result with Theorem 1.8, we can show that for every endomorphism $f : \mathbb{A}^3 \to \mathbb{A}^3$ which extends to an endomorphism of $\mathbb{P}^3$, the pair $(\mathbb{A}^3, f)$ satisfies the adelic ZD-property.

Proof of Theorem 4.1. Denote by $\pi : X \times \mathbb{P}^1 \to X$ the projection to the first coordinate. If g has a nonconstant invariant rational function H, then $H \circ \pi$ is a nonconstant invariant rational function of f.

So we only need to do the proof for adelic ZD2-property.

Now we may assume that there exists a nonempty adelic open subset A of X, such that for every $x \in A$, the orbit of x is well defined and Zariski dense in X.

Let o be a superattracting fixed point of h. Let K be a subfield of $\mathbf{k}$ which is finitely generated over $\mathbb{Q}$, such that $\overline{K} = \mathbf{k}$ and f, X, h, o are defined over K.

By Proposition 3.16 , after replacing f by a positive iterate, we may assume that there exists a nonempty adelic open subset B of $(X \times \mathbb{P}^1)(\mathbf{k})$ such that for every point $z \in B$, the orbit of z is well defined and its closure is irreducible.

By Proposition 3.16 again, after replacing f by a positive iterate, we may assume that there exists a prime $p \geq 3$ and an embedding $i : K \hookrightarrow \mathbb{C}_p$, such that there exists an open subset $V \simeq (\mathbb{C}_p^\circ)^d$ of $X_K(\mathbb{C}_p)$, which is invariant by f, the orbit of the points in V are well defined and

$$f|_V = \mathrm{id} \mod p.$$

Moreover, there exists an analytic action $\Phi : \mathbb{C}_p^\circ \times V \to V$ of $(\mathbb{C}_p^\circ, +)$ on V such that for every $n \in \mathbb{Z}_{\geq 0}$, $\Phi(n, \cdot) = f^{mn}|_V(\cdot)$.

Let U be an invariant neighborhood of o in $\mathbb{P}^1_K(\mathbb{C}_p)$ such that for every $y \in U$, $h^n(y) \to o$ when $n \to \infty$.

For every $z := (x, y) \in V \times U$, we have $f^{p^n}(z) \to (x, o) = (\pi(z), o)$ when $n \to \infty$. Denote by Z_z the Zariski closure of the orbit of z. Then we have $(\pi(z), o) \in Z_z$. It follows that $Z_{(\pi(z),o)} \subseteq Z_z$.

Then for every $z \in (X \times \mathbb{P}^1)_K(i, (V \times U) \setminus X \times \{o\}) \cap \pi^{-1}(A) \cap B$, we have

(i) the orbits of z and $\pi(z)$ are well defined;
(ii) the Zariski closure $Z_{\pi(z)}$ of the orbit of $\pi(z)$ is X;
(iii) the Zariski closure Z_z of the orbit of z is irreducible;
(iv) $Z_{\pi(z)} \times \{o\} \subseteq Z_z$;
(v) $z \in Z_z \setminus Z_{\pi(z)} \times \{o\}$.

It follows that $Z_z = X \times \mathbb{P}^1$, which concludes the proof. $\square$

By induction, this theorem easily implies the adelic veriosn of the Zariski densiy conjecture for splitting polynomial endomorphisms on $(\mathbb{A}^1)^N$.

Proof of Theorem 1.9. Extend f to an endomorphism on $(\mathbb{P}^1)^N$. By Theorem 3.28, we may assume that $\deg(f_i) \geq 2$ for all $i = 1, \ldots, N$. Using Theorem 4.1, we concludes the proof by induction on the number of factors $N \geq 1$. $\square$

4.2. **Endomorphisms of abelian varieties.** Let A be an abelian variety defined over $\mathbf{k}$. Let $f : A \to A$ be a dominant endomorphism.

The aim is to prove Theorem 1.10. In particular, this gives a new proof of [22, Theorem 1.2].

For every subvariety V, define

$$S_V := \{a \in A |\ a + V = V\}.$$

Then S_V is a group subvariety of A. Denote by S_V^0 the identity component of S_V. Then S_V^0 is an abelian subvariety.

Proposition 4.3. [28, Theorem 1.2] *Assume that V is irreducible and invariant under f. Then there is an irreducible subvariety $W \subseteq V$ with $\kappa(W) = \dim(W) = \kappa(V)$, and some iterate f^m, such that $V = W + S_V^0$ and $f^m(S_V^0 + w) = S_V^0 + w$ for every $w \in W$.*

Set $B := A/S_V^0$ and denote by $\pi : A \to B$ the quotient morphism. There exists a unique endomorphism $f_B : B \to B$ such that $f_B \circ \pi = \pi \circ f$. Since f is dominant, f_B is dominant. Set $W_B := \pi(W) = \pi(W + S_V^0) = \pi(V)$. We have $(f_B^m)|_{W_B} = \mathrm{id}$. Observe that if $\dim(W_B) = 0$, V takes form $a + S_V^0$ where $a \in V$.

Lemma 4.4. *Assume that* $\dim(W_B) \geq 1$*, then there exists a nonconstant rational function H of A satisfying $f^*H = H$.*

Proof of Lemma 4.4. By Lemma 2.6, after replacing f by f^m, we may assume that $(f_B)|_{W_B} = \mathrm{id}$. Since π is surjective, we only need to show that there exists a nonconstant rational function G of B satisfying $f_B^*G = G$. We may assume that $0 \in W_B$. Then f_B is an isogeny. We have $W_B \subseteq \ker(f - \mathrm{id})$. So $\dim \ker(f - \mathrm{id}) \geq 1$. Write the minimal polynomial of f_B as $(1-t)^r P(t)$ where $P(1) \neq 0$. We have $r \geq 1$. Set $N := (\mathrm{id} - f_B)^{r-1} P(f_B)$. Then $\dim N(A) \geq 1$ and $N(A) \subseteq \ker(f - \mathrm{id})$. Pick a nonconstant rational function F on $N(A)$. Set $G := F \circ N$, which is a nonconstant rational function on A. We have

$$f_B^*G = F \circ N \circ f_B = F \circ f_B \circ N = F \circ \mathrm{id} \circ N = G,$$

which concludes the proof. $\square$

Then we showed the following result.

Lemma 4.5. *Assume that there exists no nonconstant rational function H of A satisfying $f^*H = H$. Then every irreducible f-invariant subvariety takes form $a + A_0$ where $a \in A$ and A_0 is a subabelian variety of A.*

Proof of Theorem 1.10. Let K be a subfield of $\mathbf{k}$ which is finitely generated over $\mathbb{Q}$ such that $\mathbf{k} = \overline{K}$ and A, f are defined over K. There exists an abelian variety A_K over K and an endomorphism $f_K : A_K \to A_K$ such that $A = A_K \times_{\mathrm{Spec}\, K} \mathrm{Spec}\, \mathbf{k}$ and $f = f_K \times_{\mathrm{Spec}\, K} \mathrm{id}$.

By Proposition 3.16, there exists $m \geq 1$, a prime $p \geq 3$, $m \geq 1$ and an embedding $i : K \hookrightarrow \mathbb{C}_p$, such that on $A_K(\mathbb{C}_p)$ there exists a nonempty open subset V, such that for every $x \in V$, the Zariski closure of the orbit $O_{f^m}(x)$ in A is irreducible. After replacing f by f^m, we may assume that for every $x \in A_K(i, V)$, the Zariski closure Z_x of the orbit $O_f(x)$ in A is irreducible.

We note that for every $x \in A_K(i, V)$, we have $f(Z_x) = Z_z$. By Lemma 4.5, Z_z is a translation of subabelian variety of A.

Denote by $A[2]$ the finite subgroup of the 2-torsion points in A. We have $|A[2]| = 2^{2\dim A}$. Moreover, for every abelian subvariety B of A, we have

$$|A[2] \cap B| = |B[2]| = 2^{2\dim B}.$$

For every $l \in \mathbb{Z}$, denote by $[l] : A \to A$ the morphism $x \to lx$.

Pick an embedding $\tau : K \hookrightarrow \mathbb{C}_3$. We note that $0 \in A_K(\mathbb{C}_3)$ is an attracting fixed point for $[3]$. There exists a open neighborhood $U \subseteq A_K(\mathbb{C}_3)$ of 0 such that for every $x \in U$,

$$\lim_{n\to\infty} [3^n]x = 0.$$

Set $C := A_K(i, V) \cap (\cap_{y\in A[2]} A_K(\tau, y + U))$, which is a nonempty adelic subset of $A(\mathbf{k})$.

We only need to show that for every $x \in C$, $Z_x = A$. For every $j \in \mathcal{I}_\tau$, we denote by $\phi_j : A(\mathbf{k}) \hookrightarrow A_K(\mathbb{C}_3)$ the embedding induced by the embedding $j : \mathbf{k} \hookrightarrow \mathbb{C}_3$. For every $y \in A[2]$, there exists $j_y \in \mathcal{I}_\tau$ such that $a_y := \phi_{j_y}(x) \in U + y$. We note also that $a_y = \phi_{j_y}(x) \in Z_x$ for every $y \in A[2]$. Set $B := Z_x - a_0$, which is a subabelian variety of A. It follows that for every $n \geq 0, y \in A[2]$, we have

$$y + [3^n](a_y - y) - [3^n]a_0 = [3^n](a_y - a_0) \in B.$$

Since $a_0, a_y - y \in U$, let $n \to \infty$, we get $y \in B$. It follows that $A[2] \subseteq B$. Then $B = A$. It follows that $Z_x = A$, which concludes the proof. □

5. General facts of endomorphisms of projective surfaces

Let X be an irreducible projective surface over $\mathbf{k}$ and $f : X \dashrightarrow X$ be a dominant rational endomorphism. We mainly interest in the case when f is an endomorphism. When f is an endomorphism, by [16, Lemma 5.6], f is finite. Denote by

$$d_f := [\mathbf{k}(X) : f^*(\mathbf{k}(X))]$$

the topological degree of f.

5.1. **Amplified endomorphisms.** Assume that f is an endomorphism. Recall that $f : X \to X$ is said to be *amplified* [29], if there exists a line bundle L on X such that $f^*L \otimes L^{-1}$ is ample. In particular, a polarized endomorphism is amplified.

Lemma 5.1. *Let n be a positive integer. Then f is amplified if and only if f^n is amplified.*

Proof of Lemma 5.1. If f is amplified, then there exists a line bundle L on X such that $H := f^*L \otimes L^{-1}$ is ample. Since f is finite, for every $i \geq 0$, $(f^i)^*H = (f^{i+1})^*L \otimes (f^i)^*L^{-1}$ is ample. It follows that

$$(f^n)^*L \otimes L^{-1} = \otimes_{i=0}^{n-1} H$$

is ample. Then f^n is amplified.

If f^n is amplified, then there exists a line bundle L on X such that $(f^n)^*L \otimes L^{-1}$ is ample. Set $M := \otimes_{i=0}^{n-1}(f^i)^*L$. Then we have

$$f^*M \otimes M^{-1} = (f^n)^*L \otimes L^{-1}$$

is ample. Then f is amplified. □

Denote by $\mathrm{Fix}(f)$ the set of fixed points of f. The proof of [16, Theorem 5.1] shows that when f is amplified, the set of periodic points of f is Zariski dense and for all $n \geq 1$, $\mathrm{Fix}(f^n)$ is finite.

Lemma 5.2. *Assume that f is amplified. Let C be an irreducible curve in X satisfying $f(C) = C$. Then the degree of $f|_C$ is at least 2 and at most d_f. In particular the normalization of C is either $\mathbb{P}^1$ or an elliptic curve curve.*

Proof of Lemma 5.2. Since f is amplified, there exists a line bundle L on X such that $H := f^*L \otimes L^{-1}$ is ample. Since f is finite, $\deg(f|_C) \geq 1$. If $\deg(f|_C) = 1$, then $(f|_C)^*L|_C$ is numerically equivalent to $L|_C$. It follows that

$$f^*L \otimes L^{-1}|_C = (f|_C)^*L|_C \otimes L|_C^{-1}$$

is both ample and numerically trivial, which is a contradiction. Then we get $\deg(f|_C) \geq 2$. So the normalization of C is either $\mathbb{P}^1$ or an elliptic curve.

Let x be a general point in $C(\mathbf{k})$, we have

$$\deg(f|_C) = |f|_C^{-1}(x)| \leq |f^{-1}(x)| \leq d_f,$$

which concludes the proof. □

For an irreducible curve C in X satisfying $f(C) = C$, denote by $\pi_C : \overline{C} \to C$ the normalization of C and $f_{\overline{C}} : \overline{C} \to \overline{C}$ the endomorphism induced by $f|_C$. For a point $o \in \mathrm{Fix}(f)$ and an irreducibe curve C of X, denote by $m_C(o)$ the number of branches of C centered at o which is invariant by f. We claim that if f is amplified, then we have

$$m_C(o) \leq [d_f + 2d_f^{1/2} + 1] + 1. \tag{5.1}$$

Indeed, if C is not invariant by f, we have $m_C(o) = 0$ for every $o \in \mathrm{Fix}(f)$. If C is invariant by f, $\overline{C}$ is either $\mathbb{P}^1$ or an elliptic curve. When $\overline{C} \simeq \mathbb{P}^1$, we have

$$|\mathrm{Fix}(f_{\overline{C}})| \leq \deg(f_{\overline{C}}) + 1 \leq d_f + 1 \leq [d_f + 2d_f^{1/2} + 1] + 1.$$

When $\overline{C}$ is an elliptic curve, we have

$$|\mathrm{Fix}(f_{\overline{C}})| = |(f_{\overline{C}} - \mathrm{id})^{-1}(0)| = |\alpha - 1|^2$$

where α is some complex number satisfying $|\alpha|^2 = \deg(f_{\overline{C}}) \geq 2$. It follows that

$$|\mathrm{Fix}(f_{\overline{C}})| = |\alpha|^2 - 2\mathrm{Re}(\alpha) + 1 \leq \deg(f_{\overline{C}}) + 2\deg(f_{\overline{C}})^{1/2} + 1$$
$$\leq d_f + 2d_f^{1/2} + 1 \leq [d_f + 2d_f^{1/2} + 1] + 1.$$

Since every invariant branch of C corresponds to a fixed point of $f_{\overline{C}}$ in $\overline{C}$, we get

$$m_C(o) \leq |\mathrm{Fix}(f_{\overline{C}})| \leq [d_f + 2d_f^{1/2} + 1] + 1.$$

Lemma 5.3. *Assume that f is amplified. Let C be an irreducible curve in X satisfying $f(C) = C$. Then there exists a sequence of distinct points $o_i \in C(\mathbf{k}), i \geq 0$ such that*

(i) $o_0 \in \mathrm{Fix}(f) \cap C$;
(ii) $f(o_i) = o_{i-1}$ *for* $i \geq 1$.

Proof of Lemma 5.3. By Lemma 5.2, we have $\deg(f_{\overline{C}}) \geq 2$. Denote by $\mathrm{Exc}(f_{\overline{C}})$ the set of exceptional points i.e. the points $x \in \overline{C}$ whose inverse orbit $\cup_{i\geq 0} f_{\overline{C}}^{-i}(x)$ is finite. We claim that

$$\mathrm{Fix}(f_{\overline{C}}) \setminus \mathrm{Exc(f)} \neq \emptyset.$$

Recall that $\overline{C}$ is a either $\mathbb{P}^1$ or an elliptic curve.

When $\overline{C} \simeq \mathbb{P}^1$, it is well know that $|\mathrm{Exc}(f_{\overline{C}})| \leq 2$. If there exists $x \in \mathrm{Fix}(f_{\overline{C}})$ with multiplicity at least 2, then $x \notin \mathrm{Exc}(f_{\overline{C}})$. Otherwise if all fixed points of $f_{\overline{C}}$ are of multiplicity 1, then we have

$$|\mathrm{Fix}(f_{\overline{C}})| = \deg(f|_{\overline{C}}) + 1 \geq 3 > |\mathrm{Exc}(f_{\overline{C}})|.$$

We concludes the claim.

When $\overline{C}$ is an elliptic curve, $f_{\overline{C}}$ is étale. So we have $\mathrm{Exc}(f_{\overline{C}}) = \emptyset$. On the other hand

$$|\mathrm{Fix}(f_{\overline{C}})| = |(f_{\overline{C}} - \mathrm{id})^{-1}(0)| = |\alpha - 1|^2$$

where α is some complex number satisfying $|\alpha|^2 = \deg(f_{\overline{C}}) \geq 2$. Since $\alpha \neq 1$, we get $|\mathrm{Fix}(f_{\overline{C}})| > 0$, which concludes the claim.

Pick $q_0 \in \mathrm{Fix}(f_{\overline{C}}) \setminus \mathrm{Exc(f)}$. There exists a sequence $q_i \in \overline{C}, i \geq 1$ such that $f_{\overline{C}}(q_i) = q_{i-1}$. Then $q_i, i \geq 0$ are distinct. Set $o_i := \pi_C(q_i) \in C(\mathbf{k}), i \geq 0$. Since π_C is finite, the sequence $o_i, i \geq 0$ is infinite. We have $f(o_i) = o_{i+1}, i \geq 1$ and $o_0 \in \mathrm{Fix}(f)$. It follows that $o_i, i \geq 0$ are distinct, which concludes the proof. $\square$

5.2. **Definition field of a subvariety.** Let K be a subfield of $\mathbf{k}$ such that X, f are defined over K.

Remark 5.4. There exists always such field K which is finitely generated over $\mathbb{Q}$.

Set $G := \mathrm{Gal}(\mathbf{k}/K)$. It naturally acts on $X(\mathbf{k})$. For every $x \in X(\mathbf{k})$, we denote by G_x the stabilizer of x under this action. For every sub-extension K'/K of $\mathbf{k}/K$, we write $X(K')$ for the set of points in $X(\mathbf{k})$ defined over K'. We particularly interest the case $K' = \overline{K}$.

For a subvariety S of X, define

$$G_S := \{g \in G |\ G(S) = S\}$$

which is a closed subgroup of G. Define $K_S := \mathbf{k}^{G_S}$, which is the smallest field extension of K, over which S is defined. In particular, if S is G-invariant, then we have $K_S = K$.

Define

$$G^S := \cap_{x\in S(\mathbf{k})} G_x$$

which is a closed subgroup of G_S. Define $K^S := \mathbf{k}^{G^S}$ which is the the smallest field extension of K such that all points in $S(\mathbf{k})$ are defined over K. Observe that K^S is a Galois extension of K_S whose Galois group G_S/G^S is the image of G in the permutation group of S. It follows that, when S is finite, $[K^S : K_S]$ divides $|S|!$.

Lemma 5.5. *Assume that f is an endomorphism. Let $p_0, \dots, p_n$ be a sequence of points in $X(\mathbf{k})$ satisfying $f(p_i) = p_{i-1}, i = 1, \dots, n$. Then we have*

$$[K^{\{p_0,\dots,p_n\}} : K^{\{p_0\}}] |\ (d_f!)^n.$$

Proof of Lemma 5.5. We have a filtration of fields

$$K^{\{p_0\}} \subseteq K^{\{p_0,p_1\}} \subseteq \cdots \subseteq K^{\{p_0,\dots,p_n\}}.$$

We only need to show that

$$[K^{\{p_0,\dots,p_{i+1}\}} : K^{\{p_0,\dots,p_i\}}] |\ d_f!, i = 0, \dots, n-1.$$

After replacing K by $K^{\{p_0,\dots,p_i\}}$, we only need to prove this lemma in the case $n = 1$ and $K = K^{\{p_0\}}$.

Now assume that $n = 1$ and $K = K^{\{p_0\}}$. Since $f^{-1}(p_0)$ is G-invariant, we have $K_{f^{-1}(p_0)} = K$. Then we have

$$K = K_{f^{-1}(p_0)} \subseteq K^{\{p_0,p_1\}} \subseteq K^{f^{-1}(p_0)}.$$

It follows that

$$[K^{\{p_0,p_1\}} : K] |\ [K^{f^{-1}(p_0)} : K] |\ |f^{-1}(p_0)|! \mid d_f!,$$

which concludes the proof. □

Then we get the following constraint on definition fields of invariant curves.

Corollary 5.6. *Assume that f is an amplified endomorphism. Let C be an irreducible curve in X satisfying $f(C) = C$. Then $K^{\mathrm{Fix}(f)}$ is a finite field extension of K and there exists $n \geq 1$ such that*

$$[K_C : K^{\mathrm{Fix}(f)}] |\ (d_f!)^n.$$

Proof of Corollary 5.6. Since f is amplified, $\mathrm{Fix}(f)$ is finite. Then all points in $\mathrm{Fix}(f)$ are defined over $\overline{K}$. It follows that $K^{\mathrm{Fix}(f)}$ is a finite field extension of K.

By Lemma 5.3, there exists a sequence of distinct points $o_i \in C(\mathbf{k}), i \geq 0$ such that

(i) $o_0 \in \mathrm{Fix}(f) \cap C$;
(ii) $f(o_i) = o_{i-1}$ for $i \geq 1$.

Let M be an ample line bundle on X defined over K. Denote by Y the space of curves D in X satisfying $M\cdot D\leq M\cdot C$, which is a quasi-projective variety over $\mathbf{k}$. Moreover, it is defined over K. So G naturally acts on Y.

For every $i\geq 0$, denote by H_i the closed subset of Y consisting of curves $D\in Y$ satisfying $o_i\in D, i=0,\dots,i$. Then $H_i, i\geq 0$ is decreasing and $\cap_{i\geq 0}H_i=\{C\}$. There exists $n\geq 1$ such that $\cap_{i=0}^n H_i=\{C\}$. For every $g\in G^{\{o_0,\dots,o_n\}}$, we have $g(C)\in Y$ and $o_i\in g(C)$ for $i=0,\dots,n$. It follows that $g(C)\in\cap_{i=0}^n H_i=\{C\}$. Then we have $G^{\{o_0,\dots,o_n\}}<G_C$. It follows that

$$K_C\subseteq K^{\{o_0,\dots,o_n\}}.$$

By Lemma 5.5, we have

$$[K^{\{o_0,\dots,o_n\}}:K^{\{o_0\}}]|\ (d_f!)^n.$$

Since $K^{\{o_0\}}\subseteq K^{\mathrm{Fix}(f)}$, Then we get $[K_C:K^{\mathrm{Fix}(f)}]|\ (d_f!)^n$. $\square$

6. Local dynamics

Assume that the transcendence degree of $\mathbf{k}$ is finite. Let X be a smooth irreducible projective surface over $\mathbf{k}$ and $f:X\dashrightarrow X$ be a dominant rational endomorphism.

6.1. **Fixed points.** Let o be a fixed point of f. Let λ_1,λ_2 be the eigenvalues of the tangent map $df|_o:T_{X,o}\to T_{X,o}$.

If we blow up o, we get an new surface X_1. Denote by E the exceptional curve. Then f induces a rational endomorphism f_1 on X_1. Assume that $df|_o$ is invertible. Then f_1 is regular along E.

If $\lambda_1\neq\lambda_2$, then there are exact two fixed points o_1,o_2 of f_1 in E. At $o_i, i=1,2$, $df|_{o_i}$ is semi-simple and the tangent vectors in E is an eigenvector of $df_1|_{o_1}$. We may assume that the eigenvalue for this vector at o_1 is λ_2/λ_1 and the other eigenvalue is λ_1. Then the eigenvalues of $df|_{o_2}$ are $\lambda_1/\lambda_2,\lambda_2$.

If $\lambda_1=\lambda_2$ and $df|_o$ is semi-simple, then every point in E is fixed by f_1. At a point q in E, $df|_q$ is semi-simple and the eigenvalues of $df|_q$ are $1,\lambda_1=\lambda_2$. If $\lambda_1=\lambda_2$ and $df|_o$ is not semi-simple, then there exists a unique point q in E fixed by f_1. The eigenvalues of $df|_q$ are $1,\lambda_1=\lambda_2$.

If C is a branch of curve centered at o and invariant under f. Then the strict transform of C in X' is a branch of curve passing through a fixed point in E and it is invariant by f'. After a finite sequence of blowups at the center of the strict transform of C, we may get a strict transform $\overline{C}$ of C where the composition $\pi_C:\overline{C}\to C$ of these blowups is the normalization of C. The induces morphism $f_{\overline{C}}:\overline{C}\to\overline{C}$ from the blows coincides the one induced by the normalization. Denote by $\overline{o}$ the center of $\overline{C}$. The above computation shows that

$$df|_{\overline{o}}=\lambda_1^s\lambda_2^t \tag{6.1}$$

for some $s,t\in\mathbb{Z}$.

Lemma 6.1. *Assume that $df|_o$ is invertible and semi-simple. Assume that $\lambda_1 = \mu^{m_1}$ and $\lambda_2 = \mu^{m_2}$, where $\mu \in \mathbf{k}$ and $m_1, m_2 \in \mathbb{Z}_{>0}$ satisfying $(m_1, m_2) = 1$. Then there exists a sequence of birational maps $\pi_i : X_i \to X_{i-1}, i = 1, \dots, l$ with a point $o_i \in X_i, i = 0, \dots, l$ such that*

(i) $X_0 = X, o_0 = o$;
(ii) π_i *is the blowup at* o_{i-1};
(iii) o_i *is a fixed point of the rational map* $f_i : X_i \dashrightarrow X_i$ *induced by* f;
(iv) o_i *is in the exceptional curve* E_i *of* π_i;
(v) *the eigenvalues of* $df_i|_{o_i}, i = 0, \dots, l-1$ *take form* $\mu^s, s \geq 1$;
(vi) *the two eigenvalues of* $df_{l-1}|_{o_{l-1}}$ *are* μ, μ;
(vii) $f_l|_{E_l} = \mathrm{id}$.

Moreover, if K is a subfield of $\mathbf{k}$ such that X, f, o and μ are defined over K, then we may ask that o_i are defined over K for $i = 0, \dots, l$.

Proof of Lemma 6.1. We prove the lemma by induction on $\max\{m_1, m_2\}$.

When $\max\{m_1, m_2\} = 1$, we have $m_1 = m_2 = 1$. Define $\pi_1 : X_1 \to X_1$ the blowup of o. Then $f_l|_{E_l} = \mathrm{id}$. Let o_1 be any point in E_1 (if $\mu \in K$, then pick $o_1 \in E_1(K)$), we conclude the proof.

Now assume that we have proved the lemma for $\max\{m_1, m_2\} \leq N$ where $N \geq 1$. Assume that $\max\{m_1, m_2\} = N + 1 \geq 2$. Since $(m_1, m_2) = 1$, we have $m_1 \neq m_2$. Assume that $m_1 < m_2$. Define $\pi_1 : X_1 \to X_1$ the blowup of o. If $\mu \in K$, the two fixed points in E_1 are defined over K. In E_1, there exists a fixed point o_1 of f_1 such that the eigenvalues of $df_1|_{o_1}$ is $\mu^{m_1}, \mu^{m_2-m_1}$.

Since $m_2 - m_1 \geq 1$, $(m_1, m_2 - m_1) = 1$ and $\max\{m_1, m_2 - m_1\} \leq m_2 - 1 \leq N$, we may apply the induction hypothesis the (f_1, X_1, o_1) to conclude the proof. ☐

Definition 6.2. The fixed point $o \in X(\mathbf{k})$ is said to be *good* if $df|_o$ is invertible and one of the following holds:

(i) λ_1, λ_2 are multiplicatively independent;
(ii) there exists a prime p and an embedding $\tau : \mathbf{k} \hookrightarrow \mathbb{C}_p$ such that

$$|\tau(\lambda_1) + \tau(\lambda_2)| \leq 1 \text{ and } |\tau(\lambda_1)||\tau(\lambda_2)| < 1$$

where $|\cdot|$ is the p-adic norm on $\mathbb{C}_p$.

Remark 6.3. We note the that condition (ii) just means that both $|\tau(\lambda_1)|$ and $|\tau(\lambda_1)|$ are at most one and there exists $i = 1, 2$ much that $|\tau(\lambda_i)| < 1$.

Definition 6.4. We say that f has R-property if there exists a fixed point o of f and an embedding $\sigma : \mathbf{k} \hookrightarrow \mathbb{C}$ such that both $|\sigma(\lambda_1)|$ and $|\sigma(\lambda_2)|$ strictly great then 1, where λ_1, λ_2 are the eigenvalues of the tangent map $df_o : T_{X,o} \to T_{X,o}$.

6.2. **The existence of good fixed points.** In this section, assume that f is an amplified endomorphism on X. Let L be a line bundle on X such that $f^*L \otimes L^{-1}$ is ample.

The aim of this section is to prove the following result.

Lemma 6.5. *Assume that f is an amplified endomorphism which has R-property. Then either (X, f) satisfied the adelic ZD2-property or there exists $n \geq 1$, such that f^n has a good fixed point.*

Let R be a finitely generated $\overline{\mathbb{Q}}$ sub-algebra of $\mathbf{k}$, such that $\mathbf{k}$ is the algebraically closure of $\operatorname{Frac} R$ and X, f, L are defined over $\operatorname{Frac} R$. There exists a variety $X_{\operatorname{Frac} R}$ over $\operatorname{Frac} R$ and an endomorphism $f_{\operatorname{Frac} R} : X_{\operatorname{Frac} R} \to X_{\operatorname{Frac} R}$, such that $X = X_{\operatorname{Frac} R} \times_{\operatorname{Spec} \operatorname{Frac} R} \operatorname{Spec} \mathbf{k}$ and and $f = f_{\operatorname{Frac} R} \times_{\operatorname{Spec} \operatorname{Frac} R} \mathrm{id}$.

After shrink $W := \operatorname{Spec} R$, we may assume that W is smooth, there exists a smooth projective R-scheme $\pi : X_R \to W$ whose generic fiber is $X_{\operatorname{Frac} R}$, $f_{\operatorname{Frac} R}$ extends to a finite endomorphism f_R on $X_{\operatorname{Frac} R}$ and there exists a line bundle L_R on X_R such that $f^* L_R \otimes L_R^{-1}$ is ample over π. For every point $t \in W(\overline{\mathbb{Q}})$, denote by X_t the special fiber $X_R \times_W \operatorname{Spec} \overline{\mathbb{Q}}$. Let L_t, f_t be the restriction of L_R, f_R on X_t.

Lemma 6.6. *Assume that there exists $t \in W(\overline{\mathbb{Q}})$ such that f_t has a good fixed point in X_t. Then f has a good fixed point in X.*

Proof of Lemma 6.6. Denote by $\operatorname{Fix}(f_R)$ the subscheme of X_R of the fixed points of f_R. It is isomorphic to the intersection of the graph of f_R and the diagonal of $X_R \times_W X_R$. Let o be a good fixed point of $f_t \in X_t \subseteq X_R$. We have $o \in \operatorname{Fix}(f_R)$. Since o is smooth, every irreducible component of $\operatorname{Fix}(f_R)$ passing through o has absolute dimension at least $2(\dim W + 2) - (\dim W + 2 + 2) = \dim W$. Pick S an irreducible component of $\operatorname{Fix}(f_R)$ passing through o. For every $s \in W(\overline{\mathbb{Q}})$, $X_s \cap S \subseteq \operatorname{Fix}(f_s)$, which is finite. It follows that $\pi|_S : S \to R$ is finite and surjective.

Let λ_1, λ_2 be the eigenvalues of the tangent map $d(f_t)_o : T_{X_t,o} \to T_{X_t,o}$. Since o is a good fixed point, then if $d(f_t)_o$ is invertible and one of the following holds:

(i) λ_1, λ_2 are multiplicatively independent;

(ii) there exists a prime p and an embedding $\tau : \mathbf{k} \hookrightarrow \mathbb{C}_p$ such that
$$|\tau(\lambda_1)|, |\tau(\lambda_2)| \leq 1 \text{ and } |\tau(\lambda_1)||\tau(\lambda_2)| < 1$$
where $|\cdot|$ is the p-adic norm on $\mathbb{C}_p$.

If we identify $X_{\mathbf{k}}$ as the geometric generic fiber of π. Then there exists a point $o_{\mathbf{k}}$ of $X_{\mathbf{k}}$, whose Zariski closure in X_R is S. Denote by $(\lambda_1)_{\mathbf{k}}, (\lambda_2)_{\mathbf{k}}$ be the eigenvalues of the tangent map $d(f)_{o_{\mathbf{k}}} : T_{X,o_{\mathbf{k}}} \to T_{X,o_{\mathbf{k}}}$. Observe that λ_1, λ_2 are the specializations of $(\lambda_1)_{\mathbf{k}}, (\lambda_2)_{\mathbf{k}}$ (up to some permutation). If λ_1, λ_2 are multiplicatively independent, then $(\lambda_1)_{\mathbf{k}}, (\lambda_2)_{\mathbf{k}}$ are multiplicatively independent. Now we may assume that there exists a prime p and an embedding $\tau : \mathbf{k} \hookrightarrow \mathbb{C}_p$ such that
$$|\tau(\lambda_1) + \tau(\lambda_2)| \leq 1 \text{ and } |\tau(\lambda_1)||\tau(\lambda_2)| < 1$$
where $|\cdot|$ is the p-adic norm on $\mathbb{C}_p$.

The embedding τ induces embedings $W(\overline{\mathbb{Q}}) \hookrightarrow W(\mathbb{C}_p)$ and $X_R(\overline{\mathbb{Q}}) \hookrightarrow X_R(\mathbb{C}_p)$. For every $x \in S(\mathbb{C}_p)$, we denote by $\lambda_1(x), \lambda_2(x)$ the eigenvalues of the tangent map $d(f|_{X_{\pi(x)}})_x : T_{X_{\pi(x)},x} \to T_{X_{\pi(x)},x}$.

Since $\lambda_1(x) + \lambda_2(x)$ and $\lambda_1(x)\lambda_2(x)$ are continuous functions on $S(\mathbb{C}_p)$, there exists a neighbourhood $U \subseteq S(\mathbb{C}_p)$ of o such that for every $x \in U$, $|\lambda_1(x) +$

$\lambda_2(x)| \leq 1$ and $0 < |\lambda_1(x)\lambda_2(x)| < 1$. We note that $\pi(U)$ is a nonempty open subset of $W(\mathbb{C}_p)$.

For every $P \in R \setminus \{0\}$, denote by Z_P the set $\{z \in W(\mathbb{C}_p)|\ P(z) = 0\}$. It is a nonwhere dense closed subset of $W(\mathbb{C}_p)$. Observe that the topology of $W(\mathbb{C}_p)$ can be defined by a complete metric. Since $R \setminus \{0\}$ is countable, by Baire category theorem, $W(\mathbb{C}_p)\setminus(\cup_{R\setminus\{0\}} Z_P)$ is dense in $W(\mathbb{C}_p)$. It follows that $\pi(U)\setminus(\cup_{R\setminus\{0\}} Z_P)$ is not empty. Pick any point $z \in \pi(U) \setminus (\cup_{R\setminus\{0\}} Z_P)$. Then z induces an inclusion $\tau_z : R \hookrightarrow \mathbb{C}_p$. It extends to a inclusion $\tau_z : \mathrm{Frac}\,(R) \hookrightarrow \mathbb{C}_p$. Pick $x \in U \cap \pi^{-1}(z)$, we have $|\lambda_1(x) + \lambda_2(x)| \leq 1$ and $0 < |\lambda_1(x)\lambda_2(x)| < 1$. Then x induces an extension $\sigma := \overline{\tau_z} : \mathbf{k} = \mathrm{Frac}\,(R) \hookrightarrow \mathbb{C}_p$, such that $|\sigma((\lambda_1)_{\mathbf{k}} + (\lambda_2)_{\mathbf{k}})| \leq 1$ and $|\sigma((\lambda_1)_{\mathbf{k}}(\lambda_2)_{\mathbf{k}})| < 1$. Then f has a good fixed point in X, which concludes the proof. $\square$

Lemma 6.7. *Let o be a fixed point of X. Let C be an irreducible curve in X passing through o. Assume that $f(C) = C$, and every branch of C at o is invariant under f. Denote by $\pi_C : \overline{C} \to C$ the normalization of C and $f_{\overline{C}} : \overline{C} \to \overline{C}$ the endomorphism induced by $f|_C$. Let $q \in \pi_C^{-1}(o)$ and set $\mu := df_{\overline{C}}|_q \in \overline{K}$. Assume that there exists an embedding $\alpha : \mathbf{k} \hookrightarrow \mathbb{C}$ such that $0 < |\alpha(\mu)| < 1$. Then there exists $n \geq 0$ such that f^n has a good fixed point in X.*

Proof of Lemma 6.7. After enlarge K, we may assume that o, C, q are defined over K. In this case, we have $\mu \in R$. After shrinking W, we may assume that there exists an irreducible subscheme C_R of X_R whose generic fiber is C and a section $o_R \in X_R(R)$ whose generic fiber is o. For every point $t \in W$, denote by C_t and o_t the specializations of C_R and o_R. After shrinking W, we may assume that C_t are irreducible for $t \in W$. There exists a projective morphism $\pi_{C_R} : \overline{C}_R \to C_R$ over R whose generic fiber is π_C and a point $q_R \in \overline{C}_R(R)$, whose generic fiber is q. After shrinking W, we may assume that for all $t \in W$, the specialization $\pi_{C_t} : \overline{C}_t \to C_t$ of π_{C_R} is the normalization of C_t.

The embedding $\alpha : R \subseteq \mathbf{k} \hookrightarrow \mathbb{C}$ defined a point $\eta \subseteq W(\mathbb{C})$. We view μ as a function on $W(\mathbb{C})$. We have $|\mu(\eta)| = |\alpha(\mu)| \in (0, 1)$. There exists an euclidean open neighborhood U of η, such that $|\mu(\cdot)| < 1$ on U. Pick $t \in U \cap W(\overline{\mathbb{Q}})$, we have $|\mu(t)| < 1$. By Lemma 6.6, we only need to prove that there exists $n \geq 0$ such that f_t^n has a good fixed point in X_t. Then we reduce to the case where $\mathbf{k} = \overline{\mathbb{Q}}$.

Now we may assume that $\mathbf{k} = \overline{\mathbb{Q}}$. Assume that X, f are defined over a number field K. There exists a variety X_K over K and an endomorphism $f_K : X_K \to X_K$, such that $X = X_K \times_{\mathrm{Spec}\,K} \mathrm{Spec}\,\mathbf{k}$ and and $f = f_K \times_{\mathrm{Spec}\,K} \mathrm{id}$.

Let O_K be the ring of integers of K. There exists a projective O_K-scheme X_{O_K} which is flat over $\mathrm{Spec}\,O_K$ whose generic fiber is X_{O_K}. Denote by $\pi_{O_K} : X_{O_K} \to \mathrm{Spec}\,O_K$ the structure morphism. The endomorphism f_K on the generic fiber extends to a rational endomorphism f_{O_K} on X_{O_K}.

Denote by $\pi_{\mathbb{Z}}^{O_K} : \mathrm{Spec}\,O_K \to \mathrm{Spec}\,\mathbb{Z}$ the morphism induced by the inclusion $\mathbb{Z} \hookrightarrow O_K$. Let $X_{\mathbb{Z}}$ be the $\mathbb{Z}$-scheme which is the same as X_{O_K} as an absolute scheme with the structure morphism $\pi_{\mathbb{Z}} := \pi_{\mathbb{Z}}^{O_K} \circ \pi_{O_K} : X_{O_K} \to \mathrm{Spec}\,\mathbb{Z}$. Then $X_{\mathbb{Z}}$

is a projective $\mathbb{Z}$-scheme. Denote by $f_{\mathbb{Z}} : X_{\mathbb{Z}} \dashrightarrow X_{\mathbb{Z}}$ the rational endomorphism induced by f_{O_K}.

Since the generic fiber of $X_{\mathbb{Z}}$ is smooth and $f_{\mathbb{Z}}$ is regular above the generic fiber, there exists a finite set $B(f,\mathbb{Z})$ of primes such that $\pi_{\mathbb{Z}}^{-1}(\mathrm{Spec}\,\mathbb{Z}\setminus B)$ is smooth and $f_{\mathbb{Z}}$ is regular on $\pi_{\mathbb{Z}}^{-1}(\mathrm{Spec}\,\mathbb{Z}\setminus B)$. Set $B(f,O_K) := (\pi_{\mathbb{Z}}^{O_K})^{-1}(B(f,\mathbb{Z}))$, which is a finite subset of $\mathrm{Spec}\,(O_K)$. Then $\pi_{O_K}^{-1}(\mathrm{Spec}\,(O_K)\setminus B(f,O_K))$ is smooth and f_{O_K} is regular on $\pi_{O_K}^{-1}(\mathrm{Spec}\,(O_K)\setminus B(f,O_K))$. We note that for every prime $p\notin B(f,\mathbb{Z})$, and every embedding $\tau : \overline{K}\hookrightarrow \mathbb{C}_p$, we have $|\tau(\lambda_1)|, |\tau(\lambda_2)|\leq 1$.

By Lemma 5.2, $\deg(f_{\overline{C}})\geq 2$. Observe that $\overline{C}$ is either $\mathbb{P}^1$ or an elliptic curve. Since on a complex elliptic curve, an endomorphism of degree at least 2 is everywhere repelling, $\overline{C}$ could not be an elliptic curve. Then we have $\overline{C}\simeq\mathbb{P}^1$. Since $0<|\alpha(\mu)|<1$, by [35, Corollary 11.6], $f_{\overline{C}}$ is not post-critically finite.

We need the following lemma, which is almost the same as [8, Lemma 14.3.4.1].

Lemma 6.8. *Let $g:\mathbb{P}^1\to\mathbb{P}^1$ be an endomorphism over $\overline{\mathbb{Q}}$ of degree at least 2 which is not post-critically finite. Then for every $N\geq 0$ and a finite subset Z of $\mathbb{P}^1$, there exists a prime $p>N$, a point $x\in\mathbb{P}^1(\overline{\mathbb{Q}})$, $l\geq 1$, and an embedding $\tau(\overline{\mathbb{Q}})\hookrightarrow\mathbb{C}_p$ such that*

(i) $x\notin Z$;
(ii) $g^l(x)=x$;
(iii) *and* $|\tau(d(g^l)|_x)|<1$.

Denote by $J(f)$ the critical locus of f. Since $o\notin J(f)$ and $o\in C$, we have $C\not\subseteq J(f)$. Then $C\cap J(f)$ is finite. Let $P(f,C)$ be the union of the orbits of periodic points in $C\cap J(f)$. Then $P(f,C)$ is finite. Observe the for every $n\geq 1$, $P(f^n,C)=P(f,C)$.

By Lemma 6.8, after replacing f by a suitable positive iterate, there exists a prime $p\notin B(f,\mathbb{Z})$, an embedding $\tau(\overline{\mathbb{Q}})\hookrightarrow\mathbb{C}_p$ and $x\in\mathrm{Fix}(f|_{\overline{C}})\setminus\pi_C^{-1}(P(f,C))$ such that

(i) C is smooth at $\pi_C(x)$;
(ii) and $|\tau(d(f|_{\overline{C}})|_x)|<1$.

Set $q:=\pi_C(x)$. Since $q\notin P(f,C)$, $df|_q$ is invertible. Since $d(f|_{\overline{C}})|_x$ is an eigenvalue of $df|_q$, q is a good fixed point of f, which concludes the proof. □

Proof of Lemma 6.8. Denote by $J(g)$ the set of critical points of g. Since g is not post critically finite, there exists $b\in J(g)$ such that the orbit $O_g(b)$ of b is infinite. There exists $b_1\in\mathbb{P}^1(\overline{\mathbb{Q}})$ such that $g(b_1)=b$. We have $b\neq b_1$. Let W be the union of all orbits of periodic points in $J(g)\cup Z$. Then W is finite.

After a base change, we may assume that g,b,b_1, all points in Z and all points of W are defined over a number field K. Set $T:=\{b,b_1\}\cup Z\cup W$.

Then g defines a rational map $g_{O_K}:\mathbb{P}^1_{O_K}\dashrightarrow\mathbb{P}^1_{O_K}$ over O_K. There exists a finite subset $B\subseteq\mathrm{Spec}\,O_K$ such that

(i) g_{O_K} is regular over $\mathrm{Spec}\,O_K\setminus B$;
(ii) for every $v\in\mathrm{Spec}\,O_K\setminus B$, the characteristic of the residue field at v is strictly great then N;

(iii) for every $v \in \operatorname{Spec} O_K \setminus B$, the specialization of points of T are distinct.

For every $v \in \operatorname{Spec} O_K \setminus B$, denote by $\mathbb{P}^1_v$ the special fiber at v, $f : \mathbb{P}^1_v \to \mathbb{P}^1_v$ the specialization of f at v and for every $x \in \mathbb{P}^1(K)$, $r_v(x)$ the specialization of x in $\mathbb{P}^1_v$. By [10, Lemma 4.1], there are infinitely may $v \in \operatorname{Spec} O_K \setminus B$, such that there exists $n \geq 1$ such that $f_v^n(r_v(b)) = r_v(b_1)$. It follows that $f_v^{n+1}(r_v(b)) = r_v(b)$. Denote by p the characteristic of the residue field at v. We have $p > N$. Then $r_v(b)$ is a critical periodic point of f_v. Denote by K_v the completion of K by v and fix an embedding $K \hookrightarrow K_v \subseteq \mathbb{C}_p$.

Then there exists a point in $y \in \mathbb{P}^1(K_v \cap \overline{K})$ whose reduction is $r_v(b)$ and satisfying $f^{n+1}(y) = y$. Since $b \notin W$, $r_v(b) \notin r_v(W)$. It follows that $y \notin W$. Since y is periodic, $y \notin Z$. Since the reduction of $df^{n+1}|_y$ is $df_v^{n+1}|_{r_v(b)} = 0$, we have $|df^{n+1}|_y| < 1$. Extend the inclusion $K \subseteq \mathbb{C}_p$ to an embedding $\tau : \overline{K} \hookrightarrow \mathbb{C}_p$, we concludes the proof. $\square$

Proof of Lemma 6.5. Since f has R-property, there exists a fixed point o of f at which X is smooth, and an embedding $\sigma : \mathbf{k} \hookrightarrow \mathbb{C}$ such that both $|\sigma(\lambda_1)|$ and $|\sigma(\lambda_2)|$ are strictly great then 1, where λ_1, λ_2 are the eigenvalues of the tangent map $df_o : T_{X,o} \to T_{X,o}$. It follows that $df|_o$ is invertible.

If λ_1, λ_2 are multiplicatively independent, then o is a good fixed point of f.

Now we may assume that λ_1, λ_2 are not multiplicatively independent. There exists $(m_1, m_2) \in \mathbb{Z}^2 \setminus \{(0,0)\}$ such that

$$\lambda_1^{m_1} \lambda_2^{m_2} = 1.$$

Since $|\sigma(\lambda_1)|, |\sigma(\lambda_2)| > 1$, we have $m_1 m_2 < 0$. We may assume that $m_1 > 0$ and $m_2 < 0$.

If for every embedding $\alpha : \mathbf{k} \hookrightarrow \mathbb{C}$ we have $|\alpha(\lambda_1)| \geq 1$, then $\lambda_1 \in \overline{\mathbb{Q}}$. Then by product formula, there exists a prime p and an embedding $\tau : \mathbf{k} \hookrightarrow \mathbb{C}_p$ such that $|\tau(\lambda_1)| < 1$. Since

$$|\tau(\lambda_1)|^{m_1} = |\tau(\lambda_2)|^{-m_2},$$

we have $0 < |\tau(\lambda_1)|, |\tau(\lambda_2)| < 1$. Then o is good for f.

Now we may assume that there exists an embedding $\alpha : \mathbf{k} \hookrightarrow \mathbb{C}$ such that $|\alpha(\lambda_1)| < 1$. Then we have $|\alpha(\lambda_2)| < 1$. View $X(\mathbb{C})$ as a complex surface using the inclusion $\alpha : \mathbf{k} \hookrightarrow \mathbb{C}$. Let ϕ_α be the natural morphism $\phi_\alpha : X(\mathbf{k}) \hookrightarrow X(\mathbb{C})$ induced by α. We note that $X(\mathbf{k})$ is dense in $X(\mathbb{C})$ in this topology. Then o is an attracting fixed point of f in $X(\mathbb{C})$. There exists an euclidean open set U of $X(\mathbb{C})$ containing o such that $f(\overline{U}) \subseteq U$ and

$$\lim_{n\to\infty} f^n(x) = o$$

for every $x \in U$.

Lemma 6.9. *If (X, f) does not satisfies the adelic ZD2-property, then there exists an irreducible curve C of X over $\mathbf{k}$ passing through o and $m \geq 1$ such that $f^m(C) = C$.*

Now assume that (X, f) does not satisfies the adelic ZD2-property. After replacing f by a suitable positive iterate, we may assume that there exists an irreducible curve C of X passing through o such that $f(C) = C$. Denote by $\pi_C : \overline{C} \to C$ the normalization of C and $f|_{\overline{C}} : \overline{C} \to \overline{C}$ the endomorphism induced by $f|_C$. After replacing f by a suitable positive iterate, we may assume that every branch of C at o is invariant under f. Pick $q \in \pi_C^{-1}(o)$. It is a fixed point of $f|_{\overline{C}}$. Set $\mu := df_{\overline{C}}|_q \in \overline{K}$. By Equation 6.1, there exists $l_1, l_2 \in \mathbb{Z}$ such that $\mu = \lambda_1^{l_1}\lambda_2^{l_2}$. Since f is attracting at $o \in X(\mathbb{C})$, we have $|\alpha(\mu)| < 1$. Since $\lambda_1^{m_1} = \lambda_2^{-m_2}$, there exists $m \in \mathbb{Z}_{>0}, s \in \mathbb{Z}_{>0}$ such that

$$\lambda_1^{sm_1} = \lambda_2^{-sm_2} = \mu^m.$$

In particular, we have $0 < |\alpha(\mu)| < 1$. We conclude the proof by Lemma 6.7. $\square$

Proof of Lemma 6.9. Let K be a subfield of $\mathbf{k}$ which is finitely generated over $\mathbb{Q}$, such that $\overline{K} = \mathbf{k}$ and X, f, o are defined over K. By Proposition 3.16, there exists $m \geq 1$ and a nonempty adelic open subset B of $X(\mathbf{k})$ such that for every $x \in A$, the Zariski closure of the orbit $O_{f^m}(x)$ in X is irreducible. After replacing f by f^m, we may assume that for every $x \in B$, the Zariski closure Z_x of the orbit $O_f(x)$ in X is irreducible.

Since f is finite, there exists an open neighborhood V of o in U such that $f^{-1}(o) \cap V = \{o\}$. There exists $l \geq 1$ such that $f^l(\overline{U}) \subseteq V$. It follows that

$$S := \cup_{i\geq 0} f^{-i}(o) \cap U = \cup_{i=0}^{l} f^{-i}(o) \cap U$$

is finite. For every $x \in U \setminus S$, $O_f(x)$ is infinite.

Then for every $x \in X_K(\alpha|_K, U \setminus S) \cap A$, Z_x is irreducible and positive dimensional. Since (X, f) does not satisfy the adelic ZD2-property, there exists $x \in X_K(\alpha|_K, U \setminus S) \cap A$, such that $\dim Z_x = 1$. Since $f^n(\phi_\alpha(x)) \to o$ in $X_K(\mathbb{C})$ for $n \to \infty$, we have $o \in Z_x$, which concludes the proof. $\square$

6.3. **Invariant neighborhood.** Let o be a fixed point of f and let λ_1, λ_2 be the two eigenvalues of $df|_o$. Let K be a subfield of $\mathbf{k}$ which is finitely generated over $\overline{\mathbb{Q}}$, such that $\overline{K} = \mathbf{k}$ and $X, f, o, \lambda_1, \lambda_2$ are defined over K. Let $\tau : K \hookrightarrow \overline{\mathbb{Q}_p} \subseteq \mathbb{C}_p$ be an embedding for some prime p. Assume that

$$|\tau(\lambda_1)|, |\tau(\lambda_2)| \leq 1.$$

Let K_p be the closure of $\tau(K)$ in $\mathbb{C}_p$ which is a finite extension of $\mathbb{Q}_p$.

Let W be an affine chart of X containing o. Assume that W is defined over K. Since o is smooth, we may assume that W is a complete intersection. Then W can be viewed as a closed subvariety of $\mathbb{A}^N$ which is defined by the ideal $(F_1, \ldots, F_{N-2})$ where $F_i, i = 1, \ldots, N-2$ are contained in $K_p[x_1, \ldots, x_N]$. We may assume that o is the origin in $\mathbb{A}^N$. Since X is smooth at o, the matrix $(\partial_{x_j} F_i(0))_{1\leq i\leq N-2, 1\leq j\leq N}$ has rank $N-2$. Observe that the tangent plan of W at o in $\mathbb{A}^N$ is defined over K_p. After a K_p linear transform, we may assume that tangent plan of W at o in $\mathbb{A}^N$ is spanned by $\partial_{x_1}(0)$ and $\partial_{x_2}(0)$ and moreover the matrix of $df|_o$ under the base $\partial_{x_1}(0)$, $\partial_{x_2}(0)$ is a Jordan block

$$\begin{pmatrix} \lambda_1 & \epsilon \\ 0 & \lambda_2 \end{pmatrix}$$

where $\epsilon = 0$ or 1. Then the matrix $(\partial_{x_j} F_i(0))_{1\leq i\leq N-2, 3\leq j\leq N}$ is invertible. Denote by $\pi : W \to \mathbb{A}^2$ the projection $(x_1, \dots, x_N) \mapsto (x_1, x_2)$. For every $l \geq 0$, denote by $U_l := \{(x, y) \in \mathbb{A}^2(\mathbb{C}_p) |\ x, y \in p^l C_p^\circ\}$ which is a p-adic neighborhood of $(0, 0)$ in $\mathbb{A}^2(\mathbb{C}_p)$. By implicit function theorem, there exists a $l \in \mathbb{Z}_{>0}$ and an analytic morphism $\phi_l : U_l \to W(\mathbb{C}_p) \subseteq \mathbb{A}^N(\mathbb{C}_p)$ such that

$$\pi \circ \phi_l = \mathrm{id} \text{ and } \phi_l \circ \pi|_{\pi^{-1}(U_l)} = \mathrm{id}.$$

Moreover, ϕ_l is defined over K_p.

For every $n \geq l$, define $V_n := \phi_l(U_n) = \pi^{-1}(U_n)$ which is a p-adic neighborhood of o in $X_K(\mathbb{C}_p)$. Then there exists $m \geq l$ such that $f(V_m) \subseteq V_l$. Then f induces an analytic morphism $F : U_m \to U_l$. Observe that $(0, 0)$ is fixed by F and

$$dF|_{(0,0)} = \begin{pmatrix} \lambda_1 & \epsilon \\ 0 & \lambda_2 \end{pmatrix}.$$

We may write F as

$$F : (x_1, x_2) \mapsto (\lambda_1 x_1 + \epsilon x_2 + \sum_{i,j\geq 0, i+j\geq 2} a_{i,j} x_1^i x_2^j, \lambda_2 x_2 + \sum_{i,j\geq 0, i+j\geq 2} b_{i,j} x_1^i x_2^j)$$

where $a_{i,j}, b_{i,j} \in K_p$. There exists $r \in \mathbb{Z}_{>0}$ such that

$$\max\{|a_{i,j}|, |b_{i,j}||\ i, j \geq 0, i + j \geq 2\} \leq |p|^{-r+1}.$$

Then we have $F(U_r) \subseteq U_r$. There exists an isomorphism $U := (\mathbb{C}_p^\circ)^2 \to U_r$ sending (z_1, z_2) to $(p^r z_1, p^r z_2)$. Then F induces a morphism $G : U \to U$ taking form

$$G : (z_1, z_2) \mapsto (\lambda_1 z_1 + \epsilon z_2 + \sum_{i,j\geq 0, i+j\geq 2} p^{(i+j-1)r} a_{i,j} z_1^i z_2^j, \lambda_2 z_2 + \sum_{i,j\geq 0, i+j\geq 2} p^{(i+j-1)r} b_{i,j} z_1^i z_2^j).$$

Observe that

$$|p^{(i+j-1)r} a_{i,j}|, p^{(i+j-1)r} b_{i,j} \leq |p|$$

for $i, j \geq 0, i + j \geq 2$. The reduction $\widetilde{G} : \widetilde{U} = \widetilde{M_p}^2 \to \widetilde{U}$ of G takes form

$$(z_1, z_2) \mapsto (\widetilde{\lambda_1} z_1 + \widetilde{\epsilon} z_2, \widetilde{\lambda_2} z_2).$$

Summarizing the above, we get the following result.

Proposition 6.10. *Assume that $|\lambda_1|, |\lambda_2| < 1$. Then there exists an analytic diffeomorphism ϕ from the unit polydisk $U := (\mathbb{C}_p^\circ)^2$ to the open subset V of $X_K(\mathbb{C}_p)$ which is defined over K_p such that,*

(i) *$\phi((0, 0)) = o$;*
(ii) *the set V is f-invariant;*
(iii) *the action of f on V is conjugate, via ϕ, to an analytic endomropshim on $U = (\mathbb{C}_p^\circ)^2$ taking form*

$$G : (z_1, z_2) \mapsto (\lambda_1 z_1 + \epsilon z_2 + \sum_{i,j\geq 0, i+j\geq 2} c_{i,j} z_1^i z_2^j, \lambda_2 z_2 + \sum_{i,j\geq 0, i+j\geq 2} d_{i,j} z_1^i z_2^j).$$

where $c_{i,j}, d_{i,j} \in pK_p^\circ$, $\epsilon = 0$ if $df|_o$ is semi-simple and $\epsilon = 1$ if $df|_o$ is not semi-simple.

In particular, G is defined over K_p and the reduction of G takes form

$$\widetilde{G} : (z_1, z_2) \mapsto (\widetilde{\lambda_1} z_1 + \widetilde{\epsilon} z_2, \widetilde{\lambda_2} z_2).$$

Lemma 6.11. *Assume that $|\lambda_1| < 1$, $|\lambda_2| = 1$ and f is amplified. Then there exists a nonempty open subset U of $X_K(\mathbb{C}_p)$, such that for every point $x \in U$, the orbit $O_f(x)$ is well defined and Zariski dense in X.*

Proof of Lemma 6.11. Denote by q a uniformizer of K_p. Since $|\lambda_1| < 1$ and $|\lambda_2| = 1$, $df|_o$ is semi-simple. Then the reduction of $\widetilde{F}$ takes form

$$\widetilde{G} : (z_1, z_2) \mapsto (\widetilde{\lambda_1} z_1, \widetilde{\lambda_2} z_2).$$

By Section 8.1, there exists $g \in K_p\{z_1, z_2\}$ taking form $g = z_2 + h$ where $h \in qK_p^\circ\{z_1, z_2\}$ such that $Y := \{g = 0\}$ is invariant by f, $f|_Y$ is an isomorphism, $Y \simeq \mathbb{C}_p^\circ$ and $\cap_{n\geq 0} f^n(U) = Y$. There exists a morphism $\psi : U \to Y$ satisfying $\psi|_Y = \mathrm{id}$ and

$$f|_Y \circ \psi = \psi \circ f.$$

Since f is finite, $G(U) \not\subseteq Y$. Since f is amplified, then all periodic points are isolated. It follows that $G|_Y$ is not of finite order. Then we concludes the proof by Proposition 8.12 and Remark 8.13. □

Lemma 6.12. *Assume that f is an amplified endomorphism, every point in* $\mathrm{Fix}(f)$ *is defined over K. Assume that $df|_o$ is invertible, $|\lambda_1|, |\lambda_2| < 1$ and λ_1, λ_2 are not multiplicatively independent. Then there exists a nonempty open subset U of $X_K(\mathbb{C}_p)$, such that for every point $x \in U$, the orbit $O_f(x)$ is well defined and Zariski dense in X.*

Proof of Lemma 6.12. Since λ_1, λ_2 are not multiplicatively independent, there exists $(l_1, l_2) \in \mathbb{Z}^2 \setminus \{(0,0)\}$ such that

$$\lambda_1^{l_1} \lambda_2^{-l_2} = 1.$$

Since $|\lambda_1|, |\lambda_2| < 1$, we may assume that $l_1, l_2 > 0$. After replacing f by $f^{(l_1, l_2)}$ we may assume that $(l_1, l_2) = 1$. There exists $s, t \in \mathbb{Z}$ such that $sl_1 + tl_2 = 1$. Set $\mu := \lambda_1^t \lambda_2^s$. Since $\lambda_1, \lambda_2 \in L$,we have $\mu \in L$. Observe that

$$\mu_2^l = \lambda_1^{tl_2} \lambda_2^{sl_2} = \lambda_1^{1-sl_1} \lambda_2^{sl_2} = \lambda_1 (\lambda_1^{l_1} \lambda_2^{-l_2})^{-s} = \lambda_1.$$

The same, we have $\mu_1^l = \lambda_2$.

We first treat the case where $df|_o$ is not semi-simple. In this case, $\lambda_1 = \lambda_2 = \mu$. Denote by $\pi_1 : X_1 \to X$ the blowup of o. Denote by E_1 the exceptional curve and f_1 the rational self-map of X_1 induced by f. Observe that f_1 is regular along E_1. In E_1, there exists a unique fixed point o_1 of f_1 which is defined over K. The two eigenvalues of $df_1|_{o_1}$ is $1, \mu$. By Proposition 6.10, there exists an analytic diffeomorphism ϕ from the unit polydisk $U := (\mathbb{C}_p^\circ)^2$ to the open subset V of $X_{1,K}(\mathbb{C}_p)$ which is defined over K_p such that,

(i) $\phi((0,0)) = o$;
(ii) the set V is f_1-invariant;

(iii) the action of f_1 on V is conjugate, via ϕ, to an analytic endomropshim on $U=(\mathbb{C}_p^\circ)^2$ taking form

$$G:(z_1,z_2)\mapsto (z_1+\sum_{i,j\geq 0, i+j\geq 2} c_{i,j}z_1^iz_2^j, \mu z_2+\sum_{i,j\geq 0, i+j\geq 2} d_{i,j}z_1^iz_2^j).$$

where $c_{i,j}, d_{i,j}\in pK_p^\circ$.

In particular, the reduction of G takes form

$$\widetilde{G}:(z_1,z_2)\mapsto (z_1,0).$$

Since E_1 is fixed by f_1, we have $d_{i,0}=0$ for $i\geq 2$. By Section 8.1 and Remark 8.10, $Y:=\{z_2=0\}$ is invariant by G, $G|_Y$ is an isomorphism, $Y\simeq \mathbb{C}_p^\circ$ and $\cap_{n\geq 0}f^n(U)=Y$. There exists a morphism $\psi: U\to Y$ satisfying $\psi|_Y=\mathrm{id}$ and

$$f|_Y\circ\psi=\psi\circ f.$$

Observe that $Y=\phi^{-1}(V\cap E_1)$. Since f is finite, $G(U)\not\subseteq Y$. Since $f_1|_{E_1}$ is not of finite order, $G|_Y$ is not of finite order. Then we concludes the proof by Proposition 8.12 and Remark 8.13.

Now we may assume that $df|_o$ is semi-simple. By Lemma 6.1, there exists a sequence of birational map $\pi: X'\to X$ defined over K, an irreducible component E of $\pi^{-1}(o)$ defined over L, and a point $o'\in E(K)$ such that

(i) π is an isomorphism above $X\setminus\{o\}$;
(ii) $\pi^{-1}(o)$ is a smooth point at o';
(iii) the induced rational map $f': X'\to X'$ is regular along $\pi^{-1}(o)$;
(iv) the eigenvalues of $df'|_{o'}$ are $1,\mu$;
(v) $f'|_E=\mathrm{id}$.

By Proposition 6.10 and the fact that $f'|_E=\mathrm{id}$, there exists an analytic diffeomorphism ϕ from the unit polydisk $U:=(\mathbb{C}_p^\circ)^2$ to the open subset V of $X'_K(\mathbb{C}_p)$ which is defined over K_p such that,

(i) $\phi((0,0))=o$;
(ii) the set V is f'-invariant;
(iii) the action of f' on V is conjugate, via ϕ, to an analytic endomropshim on $U=(\mathbb{C}_p^\circ)^2$ taking form

$$G:(z_1,z_2)\mapsto (z_1+z_2(\sum_{i,j\geq 0, i+j\geq 1} c_{i,j}z_1^iz_2^j), \mu z_2+z_2(\sum_{i,j\geq 0, i+j\geq 1} d_{i,j}z_1^iz_2^j)).$$

where $c_{i,j}, d_{i,j}\in p\mathbb{C}_p^\circ$.

In particular, the reduction of G takes form

$$\widetilde{G}:(z_1,z_2)\mapsto (z_1,0).$$

We have that $Y:=\{z_2=0\}=\phi^{-1}(E)$ and the morphism $\beta: U\setminus Y\to X_K(\mathbb{C}_p)$ is an homeomorphism on to an open subset of $X_K(\mathbb{C}_p)$. Observe that $Y\simeq\mathbb{C}_p^\circ$, and $G|_Y=\mathrm{id}$. Let q be a uniformizer of K_p. Let r be a positive integer which is prime to $([d_f+2d_f^{1/2}+1]+1)!$. Set $W_Y:=\{(z_1,0)\in Y|\ |z_1|=|q|^{1/r}\}$. Then W_Y is a no empty open subset of Y.

Lemma 6.13. *Let L be any finite extension of K satisfying*

$$[L:K]|((([d_f+2d_f^{1/2}+1]+1)!)^n$$

for some $n\geq 1$. Let $\tau_L: L\hookrightarrow \mathbb{C}_p$ be any extension of τ. Denote by $\phi_{\tau_L}: (X_K(L))\hookrightarrow X_K(\mathbb{C}_p)$ the induced inclusion. Then $W_Y\cap\phi^{-1}(\phi_{\tau_L}(E_K(L)))=\emptyset$.

By Section 8.1 and Remark 8.10, there exists a morphism $\psi: U\to Y$ such that $\psi|_Y=\mathrm{id}$, $\psi=\psi\circ f$ and for every point $x\in U$, $f^n(x)$ tends to $\psi(x)$. Since $\pi|_{U\setminus E}$ is a homeomorphism to its image and f is injective in a neighborhood of o, after shrinking U, we may assume that $f(U\setminus E)\subseteq U\setminus E$.

Set $W:=\psi^{-1}(W_Y)\setminus Y$, which is a nonempty open in U. Then $\beta(W)$ is a nonempty open subset of $X_K(\mathbb{C}_p)$. Then we conclude the proof by the following lemma.

Lemma 6.14. *For every $y\in\beta(W)$, the orbit $O_f(y)$ is Zariski dense in X.*

□

Proof of Lemma 6.13. Denote by L_p the closure of $\tau_L(L)$ in $\mathbb{C}_p$. Then we have

$$[L_p:K_p]|[L:K]|((([d_f+2d_f^{1/2}+1]+1)!)^n.$$

Let t be a uniformizer of K_p, we have $|t|^e=|q|$ for some positive integer $e|[L_p:K_p]$. We have $|L_p|=\{0\}\cup|q|^{e^{-1}\mathbb{Z}}$. We note that $|q|^{1/l}\not\in|L_p|$.

Since ϕ is defined over $K_p\subseteq L_p$, we have $\phi^{-1}(E(L_p))\subseteq Y(L_p):=\{(z_1,0)|z_1\in L_p^\circ\}\subseteq Y$. In particular, for every $x=(z_1,0)\in\phi^{-1}(E(L_p))$, we have $|z_1|\neq|q|^{1/r}$. It follows that $x\not\in W_Y$. We conclude the proof. □

Proof of Lemma 6.14. Extension τ to an embedding $\mathbf{k}\hookrightarrow\mathbb{C}_p$. Using this embedding, we may view $X(\mathbf{k})$ as a subset of $X_K(\mathbb{C}_p)$.

Assume that the orbit $O_f(y)$ is not Zariski dense in X. Denote by C the Zariski closure of $O_f(y)$ in X. We have $\dim C=1$.

Set $x:=\beta^{-1}(y)$ which is contained in W. Set $c:=W_Y$. By Example 8.11, $D_c:=\psi^{-1}(c)\simeq\mathbb{C}_p^\circ$ and it contains the orbit of x. It follows that $\beta(D_c)$ is an irreducible analytic curve in $X_K(\mathbb{C}_p)$ which contains $O_f(y)$. Then C is exactly the Zariski closure of $\beta(D_c)$ in X. It follows that C is irreducible curve. Moreover, since $f(\beta(D_c))\subseteq\beta(D_c)$, we have $f(C)=C$. By Corollary 5.6, there exists a finite field extension H over K satisfying $[H:K]|(d_f!)^l$ for some $l\geq 0$ such that C is defined over U.

Denote by C' the strict transform of C in X'. Then $\phi^{-1}(C')$ is a Zariski closed subset of U. Observe that $Y\not\subseteq\phi^{-1}(C')$. Then $\phi^{-1}(C')$ takes form $\sqcup_{i=1}^s D_{c_i}$ where $c_i\in Y$ and $D_{c_i}:=\psi^{-1}(c_i)\simeq\mathbb{C}_p^\circ$. We may assume that $c_0=c$. Then D_c is the unique irreducible component of $\phi^{-1}(C')$ which meets c. It follows that C' has only one branch passing through $\phi(c)$.

Denote by $\pi_C:\overline{C}\to C$ the normalization of C and $f_{\overline{C}}$ the endomorphism induced by $f|_C$. Set $F_o:=\mathrm{Fix}(f_{\overline{C}})\cap\pi^{-1}(o)$. We have $|F_o|\leq m_C(o)\leq[d_f+2d_f^{1/2}+1]+1$. Since C and o are defined over H, F_o is defined over H. Then there exists a finite field extension I over H satisfying $[I : H] | ([d_f + 2d_f^{1/2} + 1] + 1)!$ such that every point in F_o is defined over I. We note that

$$[I : L] = [I : H][H : L] | ([d_f + 2d_f^{1/2} + 1] + 1)!(d_f)^l.$$

The rational map $\pi^{-1} \circ \pi_C : \overline{C} \dashrightarrow C'$ extends to a morphism $\pi_{C'} : \overline{C} \to C'$. It is the normalization of C'. The morphism $\pi_{C'}$ is defined over H. It follows that the image of every point of F_o in X' is defined over I. Then we have $\phi(c) \in \pi_{C'}(F_o) \subseteq E_K(I)$. Then we have $c \in W_Y \cap \phi^{-1}(E_K(I))$. Since $[I : L]$ divides some power of $([d_f + 2d_f^{1/2} + 1] + 1)!$, this contradicts Lemma 6.13. Then we concludes the proof. □

6.4. **Amplified endomorphisms of smooth surfaces.**

Proposition 6.15. *Let X be a smooth projective variety over $\mathbf{k}$. Let $f : X \to X$ be an amplified endomorphism. Assume that f satisfies the R-property. Then the pair (X, f) satisfies the adelic ZD2-property.*

Proof of Proposition 6.15. By Lemma 6.5, after replacing f by a positive iterate, we may assume that f has a good fixed point $o \in X(\mathbf{k})$. Let λ_1, λ_2 be the two eigenvalue of $df|_o$. By Corollary 3.33, we may assume that λ_1, λ_2 are not multiplicatively independent and there exists an embedding $\tau : \mathbf{k} \hookrightarrow \mathbb{C}_p$ for some prime p such that $|\tau(\lambda_1)|, |\tau(\lambda_2)| \leq 1$ and $0 < |\tau(\lambda_1)||\tau(\lambda_2)| < 1$. Let K be a subfield of $\mathbf{k}$ which is finitely generated over $\mathbb{Q}$ such that $\overline{K} = \mathbf{k}$ and $X, f, o, \lambda_1, \lambda_2$ are defined over K. Lemma 6.11 and 6.12 show that there exists a nonempty open subset $U \subseteq X_K(\mathbb{C}_p)$ such that for every $x \in X_K(\tau|_K, U)$, the orbit of x is Zariski dense in X. This concludes the proof □

Let X be a smooth projective variety over $\mathbf{k}$. Let $f : X \to X$ be a dominant endomorphism. We denote by $\lambda_1(f)$ the first dynamical degree i.e.

$$\lambda_1(f) := \lim_{n \to \infty} ((f^*)^n L \cdot L)^{1/n}$$

where L is an ample line bundle on X. The limit always exists and does not depend on the choice of the ample line bundle L.

Pick any embedding $\sigma : \mathbf{k} \hookrightarrow \mathbb{C}$. We view $X_K(\mathbb{C})$ as a complex surface induced by σ.

By [26, Theorem 3.4, iv)], if $d_f > \lambda_1(f)$, then there exists a repelling periodic point of f. It implies that f^n has R-Property for some $n \geq 1$. Then Proposition 6.15 implies the following result.

Corollary 6.16. *Let X be a smooth projective variety over $\mathbf{k}$. Let $f : X \to X$ be an amplified endomorphism. Assume that $d_f > \lambda_1(f)$. Then the pair (X, f) satisfies the adelic ZD2-property.*

In particular, when $X = \mathbb{P}^2$ and f is an endomorphism of $\mathbb{P}^2$ of degree at least 2, the pair (X, f) satisfies the adelic ZD2-property.

7. Proof of Theorem 1.8

Proof. Assume that the transcendence degree of $\mathbf{k}$ over $\mathbb{Q}$ is finite. Let X be a smooth projective surface over $\mathbf{k}$. Let $f : X \to X$ be a dominant endomorphism.

If X is an automorphism, then we conclude the proof by Corollary 3.26. Now we may assume that $d_f \geq 2$.

If the Kodaira dimension of X equals to 2, by [18, Proposition 2.6], f is an automorphism, which concludes the proof.

Recall the following result [18, Lemma 2.3 and Proposition 3.1].

Lemma 7.1. *If the Kodaira dimension of X is nonnegative and f is not an automorphism, then X is minimal and f is étale.*

If the Kodaira dimension of X equals to 1, by [32, Section 8], there exists a projective curve B, surjective morphism $\pi : X \to B$ and $m \geq 1$ such that

$$\pi \circ f^m = \pi.$$

Pick a nonconstant rational function h on B. Then $H := h \circ \pi$ is a nonconstant rational function on X. We have $(f^m)^* H = H$. Then we concludes the proof by Lemma 2.6.

Now we assume that the Kodaira dimension of X equals to 0.

So X is either an abelian surface, a hyperelliptic surface, a K3 surface, or an Enriques surface. Since f is étale, by [18, Corollary 2.3], we have

$$\chi(X, \mathcal{O}_X) = d_f \chi(X, \mathcal{O}_X).$$

Since $d_f \geq 2$, we have $\chi(X, \mathcal{O}_X) = 0$. Then X is either an abelian surface or a hyperelliptic surface, because K3 surfaces and Enriques surfaces have nonzero Euler characteristics. When X is an abelian surface, we concludes the proof by Theorem 1.10. Now we may assume that X is a hyperelliptic surface.

Let $\pi : X \to E$ be the Albanese map of X. Then E is a genus one curve, π is a surjective morphism with connected fibers. There exists a morphism $g : E \to E$ satisfying $g \circ \pi = \pi \circ f$. Moreover there is an étale cover $\phi : E' \to E$ such that $X' := X \times_E E' = F \times E'$,where F is a genus one curve. Denote by $\pi_1 : X' \to X$ the projection to the first factor, which is a finite étale morphism.

By [32, Lemma 6.3], after a further étale base change, we may assume that there exists an endomorphism $g' : E' \to E'$ such that $\phi \circ g' = g \circ \phi$. Define $f' := f \times_E g' : X' \to X'$ the induced endomorphism on X'. Then we have $\pi_1 \circ f' = f \circ \pi_1$. Since X' is an abelian surface, Theorem 1.8 holds for f' by Theorem 1.10. Then we conclude the proof by Lemma 3.23.

Now we assume that the Kodaira dimension of X equals to $-\infty$.

Recall the following result [36, Proposition 10].

Lemma 7.2. *Assume that Kodaira dimension of X equals to $-\infty$ and f is not an automorphism. Then there is a positive integer m such that for every irreducible curve E on X with negative self-intersection, we have $f^m(E) = E$.*

By Lemma 7.2, after replacing f by f^m, we may assume that f fixes all (-1)-curves. If we contract a (-1)-curve of X to get a new surface X', f induces an endomorphism f' on X'. By Lemma 3.23, we only need to show Theorem 1.5 for X'. Continue this process until there is no (-1)-curve, we may assume that X is minimal. Then X is either $\mathbb{P}^2$ or a $\mathbb{P}^1$-bundle over a smooth projective curve B.

If $X = \mathbb{P}^2$, then there exists $d \geq 2$ such that $f^*O(1) = O(d)$. Then f is amplified and $\lambda_1(f) = d < d^2 = d_f$. Then we conclude the proof by Corollary 6.16.

Now we may assume that X is a $\mathbb{P}^1$-bundle $\pi : X \to B$ over a smooth projective curve B. By [32, Lemma 5.4], after replacing f by f^2, we may assume that there exists an endomorphism $f_B : B \to B$ such that $\pi \circ f = f_B \circ \pi$. Denote by d_B the degree of f_B. For $b \in B$, set $F_b := \pi^{-1}(b)$. Denote by d_F the degree of the morphism $f|_{F_b} : F_b \to F_{f_B(b)}$. We have

$$d_F \times d_B = d_f.$$

Since $d_f \geq 2$, either $d_B \geq 2$ or $d_F \geq 2$.

Denote by $N^1(X)$ the $\mathbb{R}$-Neron Serveri group. We have $\dim N^1(X) = 2$. Denote by A the nef cone of X in $N^1(X)$. Denote by F the class of a fiber of π in $N^1(X)$. There exists $E \in N^1(X)$ such that the boundary of A is the union of $\mathbb{R}_{\geq 0}F$ and $\mathbb{R}_{\geq 0}E$. We note that for every $s, t > 0$, $sE + tF$ is ample. Since f_*, f^* preserve the nef cone and $f_*f^* = d_f\mathrm{id}$, we have $f^*(A) = A$. Since f preserves π, we have $f^*(F) = d_BF$ and $f^*(E) = d_FE$.

If $d_F, d_B \geq 2$, then

$$f^*(F + E) - (F + E) = (d_B - 1)F + (d_F - 1)E$$

is ample. It follows that f is amplified. Observe that

$$\lambda_1(f) = \max\{d_B, d_F\} < d_B \times d_F = d_f.$$

We conclude the proof by Corollary 6.16.

Now we may assume that there is exactly one of d_B, d_F equals to 1. In particular, $d_E \neq d_F$. Then we have

$$E \cdot E = d_F^{-1}(f^*E \cdot E) = d_F^{-1}(E \cdot f_*E) = d_B/d_F(E \cdot E).$$

It follows that $E \cdot E = 0$.

If f_B is of finite order, then there exists $m \geq 1$ such that

$$\pi \circ f^m = \pi.$$

Pick a nonconstant rational function h on B. Then $H := h \circ \pi$ is a nonconstant rational function on X. We have $(f^m)^*H = H$. Then we concludes the proof by Lemma 2.6. Now we may assume that $f|_B$ is not of finite order.

For a curve C in X, we denote by $[C]$ its class in $N^1(X)$. Write $C = aF + bE$. For every $m \geq 0$, we have

$$f_*^mC = af_*^mF + bf_*^mE = ad_F^mF + bd_B^mE.$$

Then if C is an irreducible periodic curve, we have $[C] \in \mathbb{R}_+E \cup \mathbb{R}_+F$. Moreover, if $[C] \in \mathbb{R}_+F$, we have $[C] \cdot F = 0$. It follows that C is a fiber of π.

By Proposition 3.25, after replacing f by a suitable iterate, we may assume that there are infinitely many distinguished irreducible curves $C_i, i \geq 1$ of f. Since $f_B \neq \mathrm{id}$, there are at most finite many fixed point of f_B. Then there are at most finitely many C_i are fibers of π. After replacing C_i by a subsequence, we may assume that $[C_i] \in \mathbb{R}_+ E, i \geq 1$.

By Lemma 3.23, we may replace (X, f) by $(X \times_B C_1 \times_B C_2 \times_B C_3, f \times_B f|_{C_1} \times_B f|_{C_2} \times_B f|_{C_3})$ and assume that C_1, C_2, C_3 are sections of π. Since $C_i \cdot C_j = 0, i, j \geq 0$, $C_i \cap C_j = \emptyset, i \neq j$. For every $b \in B(\mathbf{k})$, denote by $C_{i,b} \in F_b(\mathbf{k})$ the fiber of $C_i, i = 1, 2, 3$. Fix 3 distinct points $o_i, i = 1, 2, 3$ in $\mathbb{P}^1(\mathbf{k})$. There exists a unique morphism $\psi_b : \mathbb{P}^1 \to F_b$ sending o_i to $C_{i,b}$. The morphism $\psi : B \times \mathbb{P}^1 \to X$ sending (b, x) to $\psi_b(x) \in F_b \subseteq X$ is an isomorphism. Then we may identify X with $B \times \mathbb{P}^1$. Denote by $\pi' : X = B \times \mathbb{P}^1 \to \mathbb{P}^1$ the projection to the second factor. Then we may assume that E is the class of a fiber of π'. Since

$$f_* E \cdot E = d_B E \cdot E = 0,$$

f preserves π'. Then $f : B \times \mathbb{P}^1 \to B \times \mathbb{P}^1$ takes form $(x, y) \to (f_B(x), g(y))$ where g is an endomorphism of $\mathbb{P}^1$ of degree d_F. If g is of finite order, we conclude by Lemma 2.6.

Now we may assume that both f_B and g are of infinite order. By Remark 3.21, there are nonempty adelic open subsets V_1, V_2 of $B(\mathbf{k})$ and $\mathbb{P}^1(\mathbf{k})$ respectively, such that the for every $x \in V_1$ (resp. $x \in V_2$), the orbit of x under f_B (resp. g) is Zariski dense in $B(\mathbf{k})$ (resp. $\mathbb{P}^1(\mathbf{k})$). Set $U_1 := \pi^{-1}(V_1)$ and $U_2 := (\pi')^{-1}(V_2)$. By Proposition 3.16, there exists $m \geq 1$ and a nonempty adelic open subsets U of $X(\mathbf{k})$, such that for every $x \in U$, the Zariski closure of $O_{f^m}(x)$ is irreducible. After replacing f by f^m, we may assume that $m = 1$. We only need to show that for every $p = (a, b) \in U \cap U_1 \cap U_2$, the orbit of x is Zariski dense.

Assume that $O_f(p)$ is not Zariski dense. Since p is not preperiodic, the Zariski closure Z of $O_f(p)$ is of dimension 1. Then Z is irreducible and invariant by f. Then Z is either a fiber of π or a fiber of π'. If Z is a fiber of π, then $\pi(Z) = \pi(p) = b$ is a fixed point which contradicts our assumption. The same, if Z is a fiber of π', then $\pi'(Z) = \pi'(p) = a$ is a fixed point which contradicts our assumption. Then we concludes the proof. $\square$

8. Appendix A: Endomorphisms on the k-affinoid spaces

In this appendix, we use the terminology of Berkovich space. See [11, 12] for the general theory of Berkovich spaces. Our aim is to show that for certain endomorphism f on a k-affinoid space X, the attractor Y of f is a Zariski closed subset and the dynamics of f is semi-conjugates to the its restriction on Y. A special case of this result is used in the proof of the main theorem. In the sequels to the papers [42], we will generate this result to the global setting.

Denote by $\mathbf{k}$ a complete valued field with a nontrivial nonArchimedean norm $|\cdot|$. Denote by $\mathbf{k}^\circ := \{f \in \mathbf{k} | \, |f| \leq 1\}$ the valuation ring and $\mathbf{k}^{\circ\circ} := \{f \in \mathbf{k} | \, |f| < 1\}$ its maximal ideal. Denote by $\widetilde{\mathbf{k}} := \mathbf{k}^\circ / \mathbf{k}^{\circ\circ}$ the residue field.

Let A be a strict and reduced $\mathbf{k}$-affinoid space. Let $\rho(\cdot)$ be the spectral norm on A. Set $X := \mathcal{M}(A)$. Denote by $\tilde{X} := \mathrm{Spec}\,(A^\circ / A^{\circ\circ})$ the reduction of X and

$\pi : X \to \widetilde{X}$ the reduction map. Let $f : X \to X$ be an endomorphism. Denote by $\widetilde{f} : \widetilde{X} \to \widetilde{X}$ the reduction of f.

For every $h \in A$, the sequence $\rho((f^n)^*h), n \geq 0$ is decreasing, so the limit

$$\rho_f(h) := \lim_{n\to\infty} \rho((f^n)^*h)$$

exists. It is easy to see that $\rho_f(\cdot) : A \to [0, +\infty)$ is a power multiplicative seminorm on A which is bounded by ρ. Define $\mathcal{J}^f$ to be the ideal of A consisting of the $h \in A$ satisfying $\rho_f(h) = 0$. The following result shows that for $h \in J^f$, $(f^n)^*h$ converges to 0 uniformly.

Proposition 8.1. *There exists $b \in (0, 1)$ and $m \geq 1$ such that for all $g \in J^f$, $\rho((f^*)^m(g)) \leq b\rho(g)$.*

Proof of Proposition 8.1. Write $J^f = (g_1, \ldots, g_s)$ where $\rho(g_i) = 1, i = 1, \ldots, s$. There exists $C > 0$ such that for every $g \in J^f$, we may write

$$g = \sum_{i=1}^{s} g_i h_i$$

where $\rho(h_i) \leq C\rho(g)$. There exists $m \geq 1$ such that

$$\rho((f^*)^m(g_i)) < (1 + C)^{-1}, i = 1, \ldots .s$$

We have

$$(f^*)^m(g) = \sum_{i=1}^{s} (f^*)^m(g_i)(f^*)^m(h_i).$$

For $i = 1, \ldots, s$, we have

$$\begin{aligned}\rho((f^*)^m(g_i)(f^*)^m(h_i)) &\leq \rho((f^*)^m(g_i))\rho((f^*)^m(h_i))\\ &< (1 + C)^{-1}C\rho(g).\end{aligned}$$

It follows that

$$\rho((f^*)^m(g)) \leq (1 + C)^{-1}C\rho(g)$$

for all $g \in J^f$. Set $b := (1 + C)^{-1}C$. We conclude the proof. □

The main result of Appendix A is the following theorem.

Theorem 8.2. *Assume that X is distinguished. Assume that there exists a subvariety $Z \subseteq \widetilde{X}$ such that $\widetilde{f}(\widetilde{X}) = Z$ and $\widetilde{f}|_Z$ is an automorphism of Z. Denote by $\widetilde{I}$ the ideal of $\widetilde{A}$ defined by Z. Let Y be the Zariski closed subset of X defined by J^f.*

Then we have

1) *$\widetilde{J^f} = \widetilde{I}$ where $\widetilde{J^f} := (J^f \cap A^\circ)/(J^f \cap A^{\circ\circ})$;*
2) *the residue norm on A/J^f w.r.t. the spectral norm of A and the spectral norm on A/J^f are equal to the norm on Y induced by $\rho_f(\cdot)$;*
3) *$\widetilde{Y} \simeq \pi(Y) = Z$ where the first isomorphism is induced by the inclusion of Y in X;*
4) *$f(Y) = Y$;*

5) $f|_Y$ is an automorphism of Y.

There exists a unique morphism $\psi : X \to Y$ satisfying $\psi|_Y = \mathrm{id}$ and

$$f|_Y \circ \psi = \psi \circ f.$$

Moreover there exists $C > 0, \beta \in (0,1)$ such that for every $h \in A$, $x \in X$ and $n \geq 0$, we have

$$||h(f^n(x))| - |h(f^n(\psi(x)))|| \leq C\beta^n \rho(h).$$

Remark 8.3. Since $\widetilde{f}(\widetilde{X}) \subseteq \widetilde{Z}$ and $\psi|_Y = \mathrm{id}$, we have $\widetilde{\psi} = \widetilde{f}|_Z^{-1} \circ \widetilde{f}$.

Remark 8.4. By [13, Theorem 6.4.3/1], when $\mathbf{k}$ is stable, X is always distinguished.

Remark 8.5. Under the assumption of Theorem 8.2, we have $Y = \cap_{n\geq 0} f^n(X)$.

Proof of Theorem 8.2. There exists a distinguished epimorphism

$$\phi^* : T := \mathbf{k}\{T_1, \ldots, T_r\} \twoheadrightarrow A.$$

There exists a morphism $F^* : \mathbf{k}\{T_1, \ldots, T_r\} \to \mathbf{k}\{T_1, \ldots, T_r\}$ such that

$$\phi^* \circ F^* = f^* \circ \phi^*.$$

Denote by K the kernel of ϕ^*. Set $K^\circ := K \cap T^\circ$ and $\widetilde{K} := K^\circ/(K^\circ \cap T^{\circ\circ})$. Since ϕ^* is distinguished, $\widetilde{K}$ is exactly the kernel of $\widetilde{\phi}$. Denote by $\widetilde{I}$ the ideal of $\widetilde{A}$ defined by Z. Set $\widetilde{I}_1 := \widetilde{\phi}^{-1}(\widetilde{I})$. Since $\widetilde{f}(\widetilde{X}) = Z$, we have

$$\widetilde{F}^*\widetilde{I}_1 \subseteq \widetilde{K}.$$

Moreover, since $\widetilde{f}_Z$ is an automorphism of Z, for every $h \in \widetilde{T}, j \geq 1$, there exists $h' \in \widetilde{T}$ such that $h - (\widetilde{F}^*)^j(h') \in \widetilde{I}_1$. In other words, we have

$$\widetilde{T} = \widetilde{I}_1 + (\widetilde{F}^*)^j(\widetilde{T}).$$

Write $\widetilde{I}_1 = (\widetilde{G}_1, \ldots, \widetilde{G}_s)$. Set $\widetilde{g}_i = \phi(\widetilde{G_i}), i = 1, \ldots, s$, then we have $\widetilde{I} = (\widetilde{g}_1, \ldots, \widetilde{g}_s)$.

There are $G_i \in T^\circ, i = 1, \ldots, s$ such that $\widetilde{G}_i, i = 1, \ldots, s$ is the reduction of G_i. Since $\widetilde{F}^*(\widetilde{G}_i) \in \widetilde{K}, i = 1, \ldots, s$, there exists $c \in (0,1)$ such that for all $i = 1, \ldots, s$,

$$\rho(f^*(\phi(G_i))) \leq c.$$

By [14, Corollary 7], we may write $K = (K_1, \ldots, K_m)$ where $\rho(K)_i = 1, i = 1, \ldots, m$ such that $\widetilde{K} = (\widetilde{K}_1, \ldots, \widetilde{K}_m)$ and $K^\circ = \sum_{i=1}^m K_i T^\circ$.

Lemma 8.6. *There exist three disjoint sets S_1, S_2, S_3 and elements $E_i, i \in S_1 \sqcup S_2$ and $E_i^j, i \in S_3, j \geq 1$ of T such that*

- *for every $j \geq 1$, $E_i, i \in S := S_1 \sqcup S_2, E_i^j, i \in \sqcup S_3$ is an orthonormal basis of T;*
- *$E_i, i \in S_1$ is an orthonormal basis of K;*
- *$\widetilde{E}_i, i \in S_1 \sqcup S_2$ is a base of $\widetilde{I}_1$;*
- *for every $i \in S_2$, E_i takes form $G_j T^I$ for some $j \in \{1, \ldots, s\}$ and some multi-index I;*

- *for every $i \in S_3, j \geq 1$, E_i^j takes form $(F^*)^j(P_i^j)$ where $P_i^j = T^I$ for some multi-index I.*

For every $i \in S_2$, we have $\rho(F^*(E_i)) \leq c$. Set $W := \oplus_{i\in S_2}\mathbf{k}E_i$. We note that $\widetilde{W}$ is generated by $\widetilde{E_i}, i \in S_2$. So we have

$$\widetilde{\phi}(\widetilde{W}) = \widetilde{I}.$$

For every $H \in W$, we may write $H = \sum_{i\in S_2} a_iE_i$ with $\rho(H) = \max_{i\in S_2} a_i$. It follows that

$$\rho(F^*(H)) \leq c\rho(H)$$

for all $H \in W$.

For the convenience, we set $E_i^j := E_i$ for $i \in S_1 \sqcup S_2, j \geq 1$.

For every $H \in T, j \geq 1$, we may wrtie $H = \sum_{i\in S} a_i^jE_i^j, a_i^j \in \mathbf{k}$ where $\rho(G) = \max_{i\in S} a_i^j$. Write

$$K^j(H) := \sum_{i\in S_1} a_i^jE_i \in K,$$

$$W^j(H) := \sum_{i\in S_2} a_i^jE_i \in W^j,$$

and

$$Q^j(H) := \sum_{i\in S_3} a_i^jP_i^j.$$

Then we have

$$H = K^j(H) + W^j(H) + (F^*)^j(Q^j(H))$$

and

$$\rho(H) = \max\{\rho(K^j(H)), \rho(W^j(H)), \rho(Q^j(H))\}.$$

For every $H \in (K \oplus W) \cap T^\circ$, we will define sequences $H_i \in T^\circ, i \geq 0, a_i \in \mathbf{k}, i \geq 1$ such that $|a_i| \leq c$,

- $(F^*)^i(H_i) \in (W \oplus K) \cap T^\circ$;
- $\rho(W^i((F^*)^i(H_i))) \leq \prod_{j=1}^i |a_j|$;
- $\rho(H_j - H_i) \leq c^{i+1}$ for $j > i \geq 0$.

In particular, we have $\widetilde{H_i} = \widetilde{H}$ for $i \geq 0$ and the sequence H_i converges when $i \to \infty$.

Now we do the construction by recurrence. Set $H_0 := H$. For $i \geq 0$, we have

$$(F^*)^i(H_i) = K^i((F^*)^i(H_i)) + W^i((F^*)^i(H_i)).$$

Pick $V_i \in W \cap T^\circ$ such that $W^i((F^*)^i(H_i)) = (\prod_{j=1}^i a_j)V_i$. It follows that

$$(F^*)^{i+1}(H_i) = (\prod_{j=1}^i a_j)F^*(V_i) \mod K.$$

Since $V_i \in T^\circ \cap W$, we have $\rho(\phi(F^*(V_i))) \leq c$. Pick $a_{i+1} \in \mathbf{k}$ with $|a_{i+1}| = \rho(\phi(F^*V_i))$. Since ϕ is distinguished, we may have

$$F^*(V_i) = a_{i+1}U_i \mod K$$

where $U_i\in T^\circ$. We have
$$U_i=W^{i+1}(U_i)+(F^*)^{i+1}(Q^{i+1}(U_i))\mod K.$$
It follows that
$$(F^*)^{i+1}(H_i)=(\prod_{j=1}^{i+1}a_j)W^{i+1}(U_i)+(\prod_{j=1}^{i+1}a_j)(F^*)^{i+1}(Q^{i+1}(U_i))\mod K,$$
thus
$$(F^*)^{i+1}(H_i-(\prod_{j=1}^{i+1}a_j)Q^{i+1}(U_i))=(\prod_{j=1}^{i+1}a_j)W^{i+1}(U_i)\mod K.$$
We set
$$H_{i+1}:=H_i-(\prod_{j=1}^{i+1}a_j)Q^{i+1}(U_i).$$
We note that $\rho((\prod_{j=1}^{i+1}a_j)Q^{i+1}(U_i))\leq c^{i+1}$. The sequences $H_i\in T^\circ, i\geq 0, a_i\in \mathbf{k}, i\geq 1$ are what we need.

We claim that for every $\widetilde{g}\in\widetilde{I}$, there exists $g\in J^f\cap A^\circ$ such that $\widetilde{g}$ is the reduction of g. Now we prove the claim. For $\widetilde{g}\in\widetilde{I}$, there exists $\widetilde{G}\in\widetilde{I}_1$ such that $\widetilde{g}=\widetilde{\psi}(\widetilde{G})$. Write
$$\widetilde{G}=\sum_{i\in S_1\sqcup S_2}\widetilde{a_i}\widetilde{E_i}$$
where $a_i\in\mathbf{k}^\circ$. Set $H:=\sum_{i\in S_1\sqcup S_2}a_iE_i\in(K\oplus W)\cap T^\circ$. The reduction of H equal to $\widetilde{G}$. By the construction in the previous paragraph, we have a sequence $H_i, i\geq 0$.

Set $H_\infty:=\lim\limits_{i\to\infty}H_i$ and $g:=\phi(H_\infty)$. We have
$$\rho(H_\infty-H_i)\leq c^{i+1}$$
for $i\geq 0$. In particular, we have $\widetilde{H}_\infty=\widetilde{H}$. It follows that the reduction of g is $\widetilde{g}$. For every $i\geq 1$, we have
$$\rho((F^*)^i(H_\infty)-(F^*)^i(H_i))\leq c^{i+1}$$
and
$$(F^*)^i(H_i)\in K+(\prod_{j=1}^{i}a_i)W^\circ,$$
we have
$$\rho((f^*)^i(g))=\rho((F^*)^i(\phi(H_\infty)))\leq\max\{c^i,c^{i+1}\}=c^i.$$
Then we get $g\in J^f$.

The above argument shows that $\widetilde{I}\subseteq\widetilde{J^f}$. Since $\widetilde{f}_Z$ is an automorphism of Z, we have $\widetilde{I}=\ker(\widetilde{f^*}^i)$ for all $i\geq 1$. For every $g\in J^f\cap A^\circ$ there exists $n\geq 1$ such that $(f^*)^n(g)\in A^{\circ\circ}$. Then we have $\widetilde{g}\in\ker(\widetilde{f^*}^n)=\widetilde{I}$. Then we get
$$\widetilde{I}=\widetilde{J^f},$$

which proves 1).

Define Y to be the Zariski closed subset of X defined by the ideal J^f. We have $Y = \mathcal{M}(A/J^f)$. Denote by $\|\cdot\|_Y$ the residue norm on A/J^f. Denote by $\rho_Y(\cdot)$ the spectral norm on A/J^f. We still denote by $\rho_f(\cdot)$ the norm on A/J^f induced by $\rho_f(\cdot)$ on A. Since $\rho_f(\cdot) \leq \rho(\cdot)$ on A and it is power-multiplicative, we have

$$\|\cdot\|_Y \geq \rho_Y(\cdot) \geq \rho_f(\cdot)$$

on A/J^f. To prove 2), we only need to show that for every $\overline{g} \in A/J^f$, we have

$$\rho_f(\overline{g}) \geq \|\overline{g}\|_Y.$$

Lemma 8.7. *Let A be a distinguished* $\mathbf{k}$*-affinoid algebra. Let I be a reduced ideal of A. Denote by $\pi : A \twoheadrightarrow B := A/I$ the quotient map. Denote by $\|\cdot\|$ the residue norm on B w.r.t. the spectral norm on A. Then for every $g \in B$, there exists $G \in \pi^{-1}(g)$ such that $\rho(G) = \|g\|$.*

By Lemma 8.7, there exists $g \in A$ whose image in A/J^f is $\overline{g}$ such that

$$\rho(g) = \|\overline{g}\|_Y.$$

We may assume that $\rho(g) = \|\overline{g}\|_Y = 1$ and we only need to show that $\rho_f(g) = 1$. Otherwise $\rho_f(g) < 1$, then there exists $n \geq 1$ such that $\rho((f^*)^n(g)) < 1$. In other words, $\widetilde{g} \in \ker(\widetilde{f^*})^n = \widetilde{I}$. Since $\widetilde{I} = \widetilde{J^f}$, there exists $w \in J^f$ and $h \in A^{\circ\circ}$ such that $g = w + h$. It follows that $\|\overline{g}\|_Y \leq \rho(h) < 1$, which is a contradiction. Then we proved 2).

The inclusion of Y in X induces a morphism $\widetilde{Y} \to \widetilde{X}$. By 2), this morphism is a closed embedding given by the morphism

$$\widetilde{A} \twoheadrightarrow \widetilde{A}/\widetilde{J^f} = \widetilde{A/J^f}.$$

By 1), we have $\widetilde{I} = \widetilde{J^f}$. It implies that the image of this inclusion is exactly Z. This proves 3). Since $f^*(J^f) \subseteq J^f$, we have $f(Y) \subseteq Y$. Since $\widetilde{f}|_Z$ is an automorphism of Z, $\widetilde{f|_Y}$ is an automorphism of $\widetilde{Y}$. By 2) and the assumption that X is distinguished, Y is distinguished.

Proposition 8.8. *Let A, B be two distinguished* $\mathbf{k}$*-affinoid algebra. Let $g : A \to B$ be a morphism. If the reduction $\widetilde{g} : \widetilde{A} \twoheadrightarrow \widetilde{B}$ is surjective. Then the morphism $g : A \to B$ is surjective.*

By Proposition 8.8, $f|_Y^*$ is surjective. By [13, 6.3.1 Theorem 6], $f|_Y^*$ is injective. Then $f|_Y^*$ is an isomorphism of Y. Then we get 4) and 5).

We now construct the morphism $\psi : X \to Y$. Denote by τ the quotient morphism $\tau : A \to A/J^f$. Pick a bounded $\mathbf{k}$-linear map $\chi : A/J^f \to A$ satisfying

$$\tau \circ \chi = \mathrm{id}.$$

There exists $C > 0$ such that

$$\rho(\chi(h)) \leq C\rho_Y(h).$$

By Proposition 8.1, there exists $b \in (0, 1)$ and $m \geq 1$ such that for all $g \in J^f$,

$$\rho((f^*)^m(g)) \leq b\rho(g).$$

It follows that for every $n\geq m, g\in J^f$ we have
$$\rho((f^*)^n(g))\leq \rho((f^*)^{[n/m]m}(g))\leq b^{n/m}\rho(g)\leq b^{n/2m}\rho(g).$$
For $n\geq 0$, define bounded $\mathbf{k}$-linear maps
$$\psi_n:=(f^*)^n\circ\chi\circ(f|_Y^*)^{-n}:A/J^f\to A.$$
Denote by $\|\cdot\|$ the operator norm. We have $\|\psi_n\|\leq C$.

For every $h\in A/J^f, j\geq i\geq m$, we have
$$\begin{aligned}\psi_j(h)-\psi_i(h)&=(f^*)^j\circ\chi\circ(f|_Y^*)^{-j}(h)-(f^*)^i\circ\chi\circ(f|_Y^*)^{-i}(h)\\&=(f^*)^i\circ((f^*)^{(j-i)}\circ\chi\circ(f|_Y^*)^{-j}(h)-\chi\circ(f|_Y^*)^{-i}(h)).\end{aligned}$$
Observe that
$$\begin{aligned}&\tau((f^*)^{(j-i)}\circ\chi\circ(f|_Y^*)^{-j}(h)-\chi\circ(f|_Y^*)^{-i}(h))\\&=\tau\circ(f^*)^{(j-i)}\circ\chi\circ(f|_Y^*)^{-j}(h)-\tau\circ\chi\circ(f|_Y^*)^{-i}(h)\\&=(f|_Y^*)^{(j-i)}\circ(\tau\circ\chi)\circ(f|_Y^*)^{-j}(h)-(\tau\circ\chi)\circ(f|_Y^*)^{-i}(h)\\&=0.\end{aligned}$$
We have $(f^*)^{(j-i)}\circ\chi\circ(f|_Y^*)^{-j}(h)-\chi\circ(f|_Y^*)^{-i}(h)\in J^f$ and
$$\rho((f^*)^{(j-i)}\circ\chi\circ(f|_Y^*)^{-j}(h)-\chi\circ(f|_Y^*)^{-i}(h))\leq\|\chi\|\rho(h).$$
It follows that
$$\rho(\psi_j(h)-\psi_i(h))=\rho((f^*)^i((f^*)^{(j-i)}\circ\chi\circ(f|_Y^*)^{-j}(h)-\chi\circ(f|_Y^*)^{-i}(h)))\leq b^{i/2m}\rho(h).$$
Then the sequence of operators $\psi_i, i\geq 0$ converges to a bounded $\mathbf{k}$-linear map
$$\psi^*:A/J^f\to A$$
with $\|\psi^*\|\leq C$.

For $g,h\in A/J^f, n\geq m$, we have
$$\psi_n(gh)-\psi_n(g)\psi_n(h)=(f^*)^n\circ(\chi\circ(f|_Y^*)^{-n}(gh)-\chi\circ(f|_Y^*)^{-n}(g)\chi\circ(f|_Y^*)^{-n}(h))$$
Observe that
$$\begin{aligned}&\tau(\chi\circ(f|_Y^*)^{-n}(gh)-\chi\circ(f|_Y^*)^{-n}(g)\chi\circ(f|_Y^*)^{-n}(h))\\&=(f|_Y^*)^{-n}(gh)-(f|_Y^*)^{-n}(g)(f|_Y^*)^{-n}(h)=0.\end{aligned}$$
We have $\chi\circ(f|_Y^*)^{-n}(gh)-\chi\circ(f|_Y^*)^{-n}(g)\chi\circ(f|_Y^*)^{-n}(h)\in J_f$ of of norm at most $C\rho_Y(g)\rho_Y(h)$. So we have
$$\begin{aligned}\rho(\psi_n(gh)-\psi_n(g)\psi_n(h))&=\rho((f^*)^n\circ(\chi\circ(f|_Y^*)^{-n}(gh)-\chi\circ(f|_Y^*)^{-n}(g)\chi\circ(f|_Y^*)^{-n}(h)))\\&\leq Cb^{n/2m}\rho_Y(g)\rho_Y(h).\end{aligned}$$
Let $n\to\infty$, we get $\psi^*(gh)=\psi^*(g)\psi^*(h)$. Then ψ^* is indeed a morphism of $\mathbf{k}$-algebra. It defines a morphism $\psi:X\to Y$. Observe that
$$\tau\circ\psi_n=\tau\circ(f^*)^n\circ\chi\circ(f|_Y^*)^{-n}=\tau\circ(f|_Y^*)^n\circ(\tau\circ\chi)\circ(f|_Y^*)^{-n}=\mathrm{id}$$
for all $n\geq 0$. Then we have $\tau\circ\psi^*=\mathrm{id}$. This shows that $\psi|_Y=\mathrm{id}$. We have
$$\psi_n\circ f|_Y^*=f^*\circ\psi_{n-1}.$$
Let $n\to\infty$, we get $\psi^*\circ f|_Y^*=f^*\circ\psi^*$. It follows that $\psi\circ f=f|_Y\circ\psi$.

For every $x \in X$, $h \in A^{\circ}, n \geq m$ we have

$$\rho((f^*)^n \circ \psi^* \circ \tau(h) - (f^*)^n \circ \psi_n^* \circ \tau(h)) \leq \|\psi^* - \psi_n\| \rho(h) \leq C b^{n/2m}.$$

We note that $\tau(\psi_n^* \circ \tau(h) - h) = \tau(h) - \tau(h) = 0$. Then we have $\psi_n^* \circ \tau(h) - h \in J^f$ and its norm is at most C. It follows that

$$\rho((f^*)^n \circ \psi_n^* \circ \tau(h) - (f^*)^n(h)) \leq b^{n/2m} C.$$

Then we have

$$\begin{aligned} ||h(\psi(f^n(x)))| - |h(\psi(f^n(x)))|| &\leq |h(\psi(f^n(x))) - h(\psi(f^n(x)))| \\ &\leq \rho((f^*)^n \circ \psi^* \circ \tau(h) - (f^*)^n(h)) \\ \leq \max\{\rho((f^*)^n \circ \psi_n^* \circ \tau(h) - (f^*)^n(h)), &\rho((f^*)^n \circ \psi_n^* \circ \tau(h) - (f^*)^n(h))\} \\ &\leq b^{n/2m} C. \end{aligned}$$

Now we only need to prove the uniqueness of ψ. If we have another morphism $\psi_1 : X \to Y$ satisfying $\psi_1|_Y = \mathrm{id}$ and $f|_Y \circ \psi_1 = \psi_1 \circ f$, we want to show $\psi = \psi_1$. Since $f^* \circ \psi^* = \psi^* \circ f|_Y^*$, for every $n \geq 0$, we have $(f^n)^* \circ \psi^* = \psi^* \circ (f|_Y^n)^*$. Then we have

$$(f^n)^* \circ \psi^* \circ (f|_Y^{-n})^* = \psi^*.$$

The same, we get

$$(f^n)^* \circ \psi_1^* \circ (f|_Y^{-n})^* = \psi_1^*.$$

For every $h \in A/J^f, n \geq 0$, we have

$$(\psi^* \circ (f|_Y^{-n})^*(h) - \psi_1^* \circ (f|_Y^{-n})^*(h))|_Y = 0.$$

Then we have $\psi^* \circ (f|_Y^{-n})^*(h) - \psi_1^* \circ (f|_Y^{-n})^*(h) \in J_f$. Then for every $n \geq m$, we have

$$\begin{aligned} \rho(\psi^*(h) - \psi_1^*(h)) &= \rho((f^*)^n(\psi^* \circ (f|_Y^{-n})^*(h) - \psi_1^* \circ (f|_Y^{-n})^*(h))) \\ &\leq b^{n/2m} \rho(\psi^* \circ (f|_Y^{-n})^*(h) - \psi_1^* \circ (f|_Y^{-n})^*(h)) \leq b^{n/2m} \rho(h). \end{aligned}$$

Let $n \to \infty$, we get $\psi^*(h) - \psi_1^*(h) = 0$, which implies that $\psi = \psi_1$. □

Proof of Lemma 8.6. By [14, Proposition 3], there exists a Bald subring R of $\mathbf{k}^{\circ}$ such that all coefficients of $F, G_i, i = 1, \ldots, s$ and $K_i, i = 1, \ldots, m$ are contained in R. After localizing R by all elements of norm 1, we may assume that R is a B-ring. Moreover, after taking completion, we may assume that R is complete. Then $\widetilde{R} = R^{\circ}/R^{\circ\circ}$ is a subfield of $\widetilde{\mathbf{k}}$.

We have $\widetilde{K} \subseteq \widetilde{I_1}$ and $\widetilde{T} = \widetilde{I_1} + (\widetilde{F}^*)^j(\widetilde{T}), j \geq 1$. We have a base $\widetilde{E}_i, i \in S_1$ of $\widetilde{K}$ such that for all $i \in S_1$, $\widetilde{E}_i$ takes form $\widetilde{K}_{j_i}\widetilde{T}^{I_i}$ for some $j_i \in \{1, \ldots, m\}$ and some multi-index I_i. Since $\widetilde{I_1}$ is spanned by $\widetilde{G_j}\widetilde{T}^I, j = 1, \ldots s, I \in \mathbb{Z}_{\geq 0}^s$, there exist $\widetilde{E_i}, i \in S_2$ such that $\widetilde{E}_i, i \in S_1 \sqcup S_2$ is a base of $\widetilde{I_1}$ and for all $i \in S_2$, $\widetilde{E}_i$ takes form $\widetilde{G}_{j_i}\widetilde{T}^{I_i}$ for some $j_i \in \{1, \ldots, s\}$ and some multi-index I_i.

For every $j \geq 1$, since $\widetilde{T} = \widetilde{I_1} + (\widetilde{F}^*)^j(\widetilde{T})$, and $(\widetilde{F}^*)^j(\widetilde{T}^I), I \in \mathbb{Z}_{\geq 2}^r$ spans $(\widetilde{F}^*)^j(\widetilde{T})$, there exist $\widetilde{E_i^j}, i \in S_3$ such that $\widetilde{E}_i, i \in S_1 \sqcup S_2, \widetilde{E}_i^j \in S_3$ is a base of $\widetilde{T}$ and for all $i \in S_3$, $\widetilde{E}_i^j$ takes form $(\widetilde{F}^*)^j(\widetilde{T}^{I_i^j})$ for some multi-index I_i^j.

Define $E_i := K_{j_i}T^{I_i}$ for $i \in S_1$, $E_i := G_{j_i}T^{I_i}$ for $i \in S_2$, $E_i^j := (F^*)^j(T^{I_i^j})$ for $i \in S_3, j \geq 1$ and $S := S_1 \sqcup S_2 \sqcup S_3$. We note that $E_i, E_n^j \in R\{T_1, \dots, T_r\}$ for $i \in S_1 \sqcup S_2, n \in S_3, j \geq 1$. Now [14, Theorem 6] implies that for every $j \geq 1$, $E_i, i \in S := S_1 \sqcup S_2, E_i^j, i \in S_3$ forms an orthonormal basis of T, which concludes the proof. □

Proof of Lemma 8.7. Pick a distinguished epimorphism

$$\phi : T := \mathbf{k}\{T_1, \dots, T_r\} \twoheadrightarrow A.$$

The spectral norm on A is the residue norm w.r.t. the spectral norm on T. It implies that the norm $\|\cdot\|$ on B is the residue norm w.r.t. the spectral norm on T.

So we may assume that $A = T$. The we conclude the proof by [14, Corollary 7]. □

Proof of Proposition 8.8. Let $\phi_A : T_A := \mathbf{k}\{T_1, \dots, T_r\} \twoheadrightarrow A$ be a distinguished epimorphism. Let $\phi_B : T_B := \mathbf{k}\{U_1, \dots, U_s\} \twoheadrightarrow B$ be a distinguished epimorphism. There exists a morphism $F : T_A \to T_B$ such that

$$g \circ \phi_A = \phi_B \circ F.$$

Denote by K the kernel of ϕ_B. By [14, Corollary 7], we may write $K = (K_1, \dots, K_m)$ where $\rho(K)_i = 1, i = 1, \dots, m$ such that $\widetilde{K} = (\widetilde{K}_1, \dots, \widetilde{K}_m)$ and $K^\circ = \sum_{i=1}^m K_i T_B^\circ$.

By [14, Proposition 3], there exists a Bald subring R of $\mathbf{k}^\circ$ such that all coefficients of F and $K_i, i = 1, \dots, m$ are contained in R. After localizing R by all elements of norm 1, we may assume that R is a B-ring. Moreover, after taking completion, we may assume that R is complete. Then $\widetilde{R} = R^\circ/R^{\circ\circ}$ is a subfield of $\widetilde{\mathbf{k}}$.

We have a base $\widetilde{E}_i, i \in S_1$ of $\widetilde{K}$ such that for all $i \in S_1$, $\widetilde{E}_i$ takes form $\widetilde{K}_{j_i}\widetilde{U}^{I_i}$ for some $j_i \in \{1, \dots, m\}$ and some multi-index I_i.

Since $\widetilde{f}$ and $\widetilde{\phi}_A$ are surjective, $\widetilde{T}_B = \widetilde{K} + \widetilde{F}(\widetilde{T}_A)$. We note that $\widetilde{F}(\widetilde{T}^I), I \in \mathbb{Z}_{\geq 2}^r$ spans $\widetilde{F}(\widetilde{T})$. There exist $\widetilde{E_i^j}, i \in S_2$ such that $\widetilde{E}_i, i \in S_1 \sqcup S_2$ is a base of $\widetilde{T}$ and for all $i \in S_2$, $\widetilde{E}_i^j$ takes form $\widetilde{F}(\widetilde{T}^{I_i})$ for some multi-index I_i.

Define $E_i := K_{j_i}U^{I_i}$ for $i \in S_1$, $E_i := F^*(T^{I_i})$ for $i \in S_2$. We note that $E_i \in R\{U_1, \dots, U_s\}$ for $i \in S_1 \sqcup S_2$. Now [14, Theorem 6] implies that $E_i, i \in S := S_1 \sqcup S_2$ forms an orthonormal basis of T_B. Since $E_i \in K$ for $i \in S_1$ and $E_i \in F^*(T)$ for $i \in S_2$, we get

$$T_B = K + F^*(T).$$

Then $B = F^*(T)/(K \cap F^*(T))$, which implies that g is surjective. □

Proposition 8.9. *Assume that A is distinguished. Let J be a reduced ideal of A such that the residue norm on A/J w.r.t. the spectral norm of A equals to the spectral norm on A/J. Let $g_1, \dots, g_m$ be elements in $A^\circ \cap J$ such that their reductions $\widetilde{g_1}, \dots, \widetilde{g_m}$ generate $\widetilde{J} := (J \cap A^\circ)/(J \cap A^{\circ\circ})$. Then $g_1, \dots, g_m$ generate J.*

Proof of Proposition 8.9. Since the spectral norm of A equals to the spectral norm on A/J, we have

$$\widetilde{A/J} = \widetilde{A}/\widetilde{J}.$$

Pick a distinguished epimorphism

$$\phi : T := \mathbf{k}\{T_1, \ldots, T_r\} \twoheadrightarrow A.$$

Denote by $\|\cdot\|$ the spectral norm on T. The spectral norm on A is the residue norm w.r.t. the spectral norm on T. Set $I := \ker(\phi)$ and $\widetilde{I} := (I \cap T^\circ)/(I \cap T^{\circ\circ})$. Pick $F_1, \ldots, F_s$ in $I \cap T^\circ$ such that their reductions $\widetilde{F_1}, \ldots, \widetilde{F_s}$ generate $\widetilde{I}$. Since ϕ is distinguished, by [14, Corollary 7], for every $i = 1, \ldots, m$, there exists $G_1, \ldots, G_m \in T$ such that

$$\|G_i\| = \rho(g_i) \leq 1, i = 1, \ldots, m.$$

We have $F_i, i = 1, \ldots, F_s, G_1, \ldots, G_m \in \phi^{-1}(J)$. We only need to show that $F_1, \ldots, F_s, G_1, \ldots, G_m$ generate $\phi^{-1}(J)$.

Denote by $\psi : T \twoheadrightarrow A/J = T/\phi^{-1}(J)$ the composition of ϕ and the quotient morphism $A \to A/J$. Since ϕ is distinguished and the residue norm on A/J w.r.t. the spectral norm of A equals to the spectral norm on A/J, ψ is distinguished. It follows that

$$\widetilde{A}/\widetilde{J} = \widetilde{A/J} = \widetilde{T}/\widetilde{\phi^{-1}(J)}$$

where

$$\widetilde{\phi^{-1}(J)} := (\phi^{-1}(J) \cap A^\circ)/(\phi^{-1}(J) \cap A^{\circ\circ}).$$

Since ϕ is distinguished, we have $\widetilde{A} = \widetilde{T/I} = \widetilde{T}/\widetilde{I}$. Then we get

$$\widetilde{T}/\widetilde{\phi^{-1}(J)} = \widetilde{T}/\widetilde{\phi}^{-1}(\widetilde{J}),$$

which implies that

$$\widetilde{\phi^{-1}(J)} = \widetilde{\phi}^{-1}(\widetilde{J}).$$

Observe that $\widetilde{F}_1, \ldots, \widetilde{F}_s, \widetilde{G}_1, \ldots, \widetilde{G}_m$ generate $\widetilde{\phi}^{-1}(\widetilde{J})$. Then the proof of [14, Corollary 7], shows that $F_1, \ldots, F_s, G_1, \ldots, G_m$ generate $\phi^{-1}(J)$, which concludes the proof. □

8.1. **The Zariski density of orbits.** In this section, we assume that $\mathbf{k} = \mathbb{C}_p$. Let $K_p \subseteq \mathbb{C}_p$ be a finite field extension of $\mathbb{Q}_p$.

Let $f : \mathbb{D}^2 \to \mathbb{D}^2$ be an endomorphism defined over K_p whose reduction $\widetilde{f} : \mathbb{A}^2_{\widetilde{\mathbf{k}}} \to \mathbb{A}^2_{\widetilde{\mathbf{k}}}$ takes form

$$\widetilde{f} : (x, y) \mapsto (\widetilde{a}x + \widetilde{b}, 0)$$

where $\widetilde{a} \in \widetilde{\mathbf{k}} \setminus \{0\}, \widetilde{b} \in \widetilde{\mathbf{k}}$.

By Theorem 8.2 and Proposition 8.9, there exists $g \in K_p\{x, y\}$ taking form $g = y + h$ where $h \in K_p\{x, y\}^{\circ\circ}$ such that $J^f = (g)$. Set $Y := \mathcal{M}(\mathbf{k}\{x, y\})/(g)$. We have $Y \simeq \mathbb{D}^1$. There exists a unique morphism $\psi : \mathbb{D}^2 \to Y$ satisfying $\psi|_Y = \mathrm{id}$ and

$$f|_Y \circ \psi = \psi \circ f.$$

There exists $C>0,\beta\in(0,1)$ such that for every $F\in A$, $x\in X$ and $n\geq 0$, we have

$$||F(f^n(x))|-|F(f^n(\psi(x)))||\leq C\beta^n\rho(F).$$

Remark 8.10. Assume that f takes form $(x,y)\mapsto(ax+b+P,yQ)$ where $|a|=1,P,Q\in\mathbf{k}\{x,y\},\rho(P),\rho(Q)<1$. Then for $n\geq 1$, we have

$$(f^*)^n(y)=yQf^*(Q)\cdots(f^*)^nQ.$$

It follows that $\rho((f^*)^n(y))\leq\rho(Q)^n$. In particular, we have $y\in J^f$. By Proposition 8.9, we have $J^f=(y)$.

Example 8.11. Assume that f takes form $(x,y)\mapsto(x+yP,yQ)$ where $P,Q\in\mathbf{k}\{x,y\},\rho(P),\rho(Q)<1$. By Proposition 8.9, we have $J^f=(y)$. So we have $Y:=\{y=0\}$. In this case we may compute the morphism $\psi:\mathbb{D}^2\to Y$ explicitly. Follows the proof Theorem 8.2, ψ equal to $\lim\limits_{n\to\infty}f^n$, which is defined by

$$(x,y)\mapsto(\lim_{n\to\infty}(f^*)^n(x),\lim_{n\to\infty}(f^*)^n(y))=(x+\sum_{i\geq 1}(f^*)^i(y)(f^*)^i(P),0).$$

We note that $\rho((f^*)^i(y)(f^*)^i(P))\leq\rho((f^*)^i(y))\leq\rho(Q)^i$. In particular, for every $c\in\mathbf{k}^\circ$, $\psi^{-1}((c,0))=\{x+\sum_{i\geq 1}(f^*)^i(y)(f^*)^i(P)=c\}$. By implicit function theorem, $\psi^{-1}((c,0))\simeq\mathbb{D}$.

Proposition 8.12. *Assume that $f|_Y$ is not torsion and $f^{-1}(Y)\neq\mathbb{D}^2$. Then there exists a nonempty strict affinoid subdomain V of X such that for every $q\in V(\mathbf{k})$, the orbit $O_f(o)$ of is Zariski dense in $\mathbb{D}^2$.*

Remark 8.13. Assume that X is an projective surface over $\mathbf{k}$. Denote by X^{an} the analytification of X. Then we have a natural morphism $\pi_X:X^{\mathrm{an}}\to X$. We note that π_X gives a bijection between $X^{\mathrm{an}}(\mathbf{k})$ and $X(\mathbf{k})$.

Assume that there exists a strict affinoid subdomain U of X^{an}. Then the Zariski topology of U is finer than the pullback by $\pi_X|_U$ of the Zariski topology of X. So if X is irreducible and a set S of $U(\mathbf{k})$ is Zariski dense in $U(\mathbf{k})$, $\pi_X(U(\mathbf{k}))$ is Zariski dense in X.

Proof of Proposition 8.12. Fix an identification $\mathbb{D}^1=\mathcal{M}(\mathbf{k}\{T\})\simeq Y$. The reduction of $f|_Y$ takes form $\widetilde{f|_Y}:T\mapsto\widetilde{a}T+\widetilde{b}$. There exists $m\geq 0$ such that $\widetilde{a}^m=1$. After replacing f by f^{mp}, we may assume that $\widetilde{a}=1,\widetilde{b}=0$. Then we have $\widetilde{f|_Y}=\mathrm{id}$.

Denote by $\Delta_{f|_Y}:=f|_Y^*-\mathrm{id}:\mathbf{k}\{T\}\to\mathbf{k}\{T\}$ the difference operator which is a bounded linear operator on the Banach space $\mathbf{k}\{T\}$. Denote by $\|\Delta_{f|_Y}\|$ th operator norm of $\Delta_{f|_Y}$. Since $\widetilde{f|_Y}=\mathrm{id}$, and $Y,f|_Y$ are defined over a discrete valuation field K_p, we have $\|\Delta_{f|_Y}\|<1$. By [37, Remark 4], there exists $r\geq 1$ such that $\|\Delta_{f|_Y{}^r}\|<p^{-2}$. After replacing f by f^r, we may assume that $\|\Delta_{f|_Y}\|<p^{-2}$. Then [37, Theorem 1], shows that the set of preperiodic points of $f|_Y$ in $\mathbb{D}^1(\mathbf{k})$ is the set of fixed points $\mathrm{Fix}(f|_Y)$ of $f|_Y$ in $\mathbb{D}^1(\mathbf{k})$. Since $f|_Y$ is not torsion, $\mathrm{Fix}(f|_Y)$ is finite. Since $f^{-1}(Y)\neq\mathbb{D}^2$, $f^{-1}(Y)$ is a union of finitely many irreducible curves. Let Y_1 to be the union of all irreducible components of $f^{-1}(Y)$ except Y. Then $Y\cap Y_1$ is a finite union of closed points.

Pick $a \in Y(\mathbf{k}) \setminus (\mathrm{Fix}(f|_Y) \cup (Y \cap Y_1))$. There exists $s \geq 1$ such that the ball $B := \{t \in Y = \mathbb{D}^1 |\ |(T-a)(t)| \leq p^{-s}\}$ does not meet $\mathrm{Fix}(f|_Y) \cup (Y \cap Y_1)$. Observe that B is a strict affinoid subdomain of Y. By [37, Remark 4], after replacing f by some positive iterate, we may assume that $\|\Delta_{f|_Y}\| < p^{-s}$. It follows that the ball $B := \{t \in \mathbb{D}^1 |\ |(T-a)(t)| \leq p^{-s}\}$ is invariant under $f|_Y$.

We note that $Y \cap \psi^{-1}(B) \cap Y_1 = B \cap Y_1 = \emptyset$. There exists $l \geq 1$ such that

$$Y^l \cap \psi^{-1}(B) \cap Y_1 = \emptyset$$

where Y^l is the affinoid subdomain $\{t \in \mathbb{D}^2 |\ |g(x)| \leq p^l\}$. Observe that $f(Y^l) \subseteq Y^l$. It follows that $Y^l \cap \psi^{-1}(B)$ is an analytic subdomain of $\mathbb{D}^2$ which is invariant by f. Moreover $(Y^l \cap \psi^{-1}(B)) \setminus Y$ is also invariant by f. Since $W := (Y^l \cap \psi^{-1}(B)) \setminus Y$ contains a strict affinoid subdomain of X, we only need to show that for every $o \in W(\mathbf{k})$, the orbit of o is Zariski dense. Otherwise, denote by Z the Zariski closure of $O_f(o)$. We have $\dim(Z) \leq 1$. Since $f^n(o) \notin Y, n \geq 0$ and tends to Y, we have $\dim(Z) = 1$. After replacing f by a positive iterate, we may assume that Z is an irreducible curve. The intersection $Z \cap Y$ is a finite set of closed points. Since $f|_Y$ is an automorphism and $f(Z) \subseteq Z$, every point in $Z \cap Y$ is periodic. It follows that $(Z \cap Y)(\mathbf{k}) \subseteq \mathrm{Fix}(f|_Y)$. By [37, Remark 4], we have $f^{p^n}(\psi(o)) \to \psi(o)$ when $n \to \infty$. Assume that Z is defined by $g_1, \ldots, g_r$. For every $i = 1, \ldots, r$, $n \geq 0$ we have

$$|g_i(f^{p^n}(\psi(o)))| = ||g_i(f^{p^n}(\psi(o)))| - |g_i(f^{p^n}(o))|| \leq C\beta^{p^n}\rho(g_i).$$

Let $n \to \infty$, we have $g_i(\psi(o)) = 0$. It follows that $\psi(o) \in (Z \cap Y)(\mathbf{k}) \subseteq \mathrm{Fix}(f|_Y)$, which is a contradiction. Then we concludes the proof. $\square$

9. Appendix B (Joint work with Thomas Tucker): The Zariski dense orbit conjecture for endomorphisms of $(\mathbb{P}^1)^N$

Let $\mathbf{k}$ be an algebraically closed field of characteristic zero.

The aim of this appendix is to prove Theorem 1.11. By Corollary 3.24, we only need to show the following adelic version of it.

Theorem 9.1. *Assume that the transcendence degree of $\mathbf{k}$ over $\mathbb{Q}$ is finite. Let $f : (\mathbb{P}^1)^N \to (\mathbb{P}^1)^N$ be a dominant endomorphism of $(\mathbb{P}^1)^N, N \geq 1$. Then the pair $((\mathbb{P}^1)^N, f)$ satisfies the adelic ZD-propety.*

Now, we assume further that the transcendence degree of $\mathbf{k}$ over $\mathbb{Q}$ is finite.

For $i = 1, \ldots, N$, denote by $\pi_i : (\mathbb{P}^1)^N \to \mathbb{P}^1$ the projection to the i-th coordinate. Denote by $H_i, i = 1, \ldots, N$ the class in $N^1((\mathbb{P}^1)^N)_{\mathbb{R}}$ represented by $\pi_i^*(\mathcal{O}_{\mathbb{P}^1}(1))$. The Nef cone $\mathcal{C}$ of $(\mathbb{P}^1)^N$ in $N^1((\mathbb{P}^1)^N)_{\mathbb{R}}$ is the convex cone spanned by $H_1, \ldots, H_N$. Since f^*, f_* preserve the Nef cone, after replacing f by some positive iteration, we may assume that f^*H_i is some multiple to H_i. It follows that f preserves all $\pi_i, i = 1, \ldots, N$. Now we may assume that f takes form $(x_1, \ldots, x_N) \mapsto (f_1(x_1), \ldots, f_N(x_N))$. Where f_i is an endomorphism of $\mathbb{P}^1$ of degree at least 1.

9.1. **Exceptional endomorphisms of curves.** Let C be an irreducible projective curve. Let g be an endomorphism $g : \mathbb{P}^1 \to \mathbb{P}^1$ of $\deg g \geq 2$. Then C is either $\mathbb{P}^1$ or an elliptic curve.

We say that g is *of Lattés type*, if it semi-conjugates to an endomorphism of an elliptic curve i.e. there exists an endomorphism of an elliptic curve $h : E \to E$ and a finite morphism $\pi : E \to C$ such that $f \circ \pi = \pi \circ h$.

We say that g is *of monomial type*, if it semi-conjugates to an endomorphism of a monomial map i.e. there exists a monomial endomorphism $h : \mathbb{P}^1 \to \mathbb{P}^1$ taking form $x \mapsto x^d, d \geq 2$ and a finite morphism $\pi : \mathbb{P}^1 \to C$ such that $f \circ \pi = \pi \circ h$. We note that in this case $C \simeq \mathbb{P}^1$.

We say that g is *exceptional* if it is of Lattés type or monomial type. Otherwise, it is said to be *nonexceptional*.

For every endomorphism $g : \mathbb{P}^1 \to \mathbb{P}^1$ of $\deg g \geq 2$, it has exactly one type in Lattés, monomial and nonexception. Moreover, the types of $g^n, n \geq 1$ are the same.

The following facts are well known.

(i) If two endomorphisms of curves are semi-conjugacy, then they have the same type.
(ii) If there is a nonzero rational differential form ω, such that $g^*\omega = \mu\omega$ for some $\mu \in \mathbf{k}^*$, then g is exceptional.
(iii) If g has an exceptional point i.e. a point in C whose inverse orbits is finite, then g^2 is polynomial. In particular, when g is of monomial type, g^2 is polynomial.

9.2. **Invariant subvarieties.** For every $l = 1, \ldots, N$, denote by S_l the set of subsets of $\{1, \ldots, N\}$ of l elements. For every $I \subseteq \{1, \ldots, N\}$, the ordering in I is induced by the ordering in $\{1, \ldots, N\}$. For every subset I of $\{1, \ldots, N\}$, we denote by $\pi_I : (\mathbb{P}^1)^N \to (\mathbb{P}^1)^{|I|}$ the projection $(x_i)_{i=1,\ldots,N} \mapsto (x_i)_{i\in I}$. Denote by $f_I : (\mathbb{P}^1)^{|I|} \to (\mathbb{P}^1)^{|I|}$ the endomorphism $(x_i)_{i\in I} \mapsto (f_i(x_i))_{i\in I}$. Then we have $\pi_I \circ f = f_I \circ \pi_I$.

The following results on the invariant subvarieties was obtained in [34] using model theory. When $\mathbf{k} = \overline{\mathbb{Q}}$, it was also obtained by Ghioca, Nguyen and Ye in [20, Theorem 1.2], as a consequence of their solution of the Dynamical Manin-Mumford Conjecture in this case. Here we give a new proof which is purely geometric.

Proposition 9.2. *Assume that $N \geq 2$, $\deg f_i \geq 2, i = 1, \ldots, N$ and all $f_i, i = 1, \ldots, N$ are nonexceptional. Let V be a proper irreducible subvariety of $(\mathbb{P}^1)^N$ which is invariant under f. Then there exists $I \in S_2$ such that $V \subseteq \pi_I^{-1}(C)$ where C is a f_I-invariant curve in $(\mathbb{P}^1)^2$.*

To prove this, we need the following Lemmas.

Lemma 9.3. *Let V be an irreducible hypersurface of $(\mathbb{P}^1)^N$. For $J \in S_l, l \leq N-2$, for a general point $z \in (\mathbb{P}^1)^{|J|}$, $(\pi_J|_V)^{-1}(z)$ is irreducible.*

Proof of Lemma 9.3. We may assume that $J = \{1, \ldots, l\}$. Let $\mu : Y \to V$ be a desingularization of V. Set $\mu_i := \pi_i \circ \mu, i = 1, \ldots, N$.

By Theorem of Bertini, for general $a_1 \in \mathbb{P}^1$, $\mu_1^{-1}(a_1)$ is irreducible and smooth of dimension $N-2$; for general $a_2 \in \mathbb{P}^1$, $\mu_1^{-1}(a_1) \cap \mu_2^{-1}(a_2)$ is irreducible and smooth of dimension $N-3$;...; for general $a_l \in \mathbb{P}^1$, $\mu_1^{-1}(a_1) \cap \cdots \cap \mu_l^{-1}(a_l)$ is irreducible and smooth of dimension $N-l-1$. It follows that the geometric generic fiber of $\pi_J \circ \mu$ is irreducible and smooth. Then for a general point $z \in (\mathbb{P}^1)^{|J|}$, $(\pi_J \circ \mu)^{-1}(z)$ is irreducible and smooth. Then, $(\pi_J|_V)^{-1}(z) = \mu((\pi_J \circ \mu)^{-1}(z))$ is irreducible. We concludes the proof. $\square$

Lemma 9.4. *Assume that $N = 3$, $\deg f_i \geq 2, i = 1,2,3$ and f_1 is nonexceptional. Let V be a proper irreducible hypersurface of $(\mathbb{P}^1)^3$ which is invariant under f. Assume that $\pi_{\{1,3\}}(V) = (\mathbb{P}^1)^2, \pi_{\{2,3\}}(V) = (\mathbb{P}^1)^2$, then $\pi_{\{1,2\}}(V) \neq (\mathbb{P}^1)^2$.*

Proof of Lemma 9.4. Assume that $\pi_{\{1,2\}}(V) = (\mathbb{P}^1)^2$.

Set $g := f|_V$. Set $P_i := \pi_i|_V, i = 1,2,3$. Then we have three nonzero rational differential forms $dP_i, i = 1,2,3$ on V. We have

$$g^* dP_i = d(P_i \circ g) = d(f_i \circ P_i) = P_i^* f_i' dP_i.$$

Since $\pi_{\{1,2\}}(V) = (\mathbb{P}^1)^2$, dP_1, dP_2 are linearly independent at a general point in V. So there are rational functions $G_1, G_2 \in \mathbf{k}(V)$ such that $dP_3 = G_1 dP_1 + G_2 dP_2$. Since $\pi_{\{1,3\}}(V) = (\mathbb{P}^1)^2, \pi_{\{2,3\}}(V) = (\mathbb{P}^1)^2$, G_1, G_2 are nonzero.

For every $a \in \mathbb{P}^1$, define $V_a := \pi_1^{-1}(a) \cap V$. By Lemma 9.3, there exists a nonempty Zariski open subset U of $\mathbb{P}^1$, such that for every $a \in U$, V_a is irreducible. After shrinking U, we may assume that for every $a \in U$, $G_1|_{V_a}$ and $G_2|_{V_a}$ are nonzero.

We note that $\mathrm{Per}\,(f_1)$ is Zariski dense in $\mathbb{P}^1$. For every $a \in U \cap \mathrm{Per}\,(f_1)$, there exists $s \geq 1$, such that $f_1^s(a) = a$. Then V_a is g^s invariant. The set of critical periodic points of f_1 is finite. After shrinking U, we may assume that for all $a \in U \cap \mathrm{Per}\,(f_1)$, $(f_1^n)'(a) \neq 0$ for $n \geq 0$.

We have

$$(P_3|_{V_a}^*(f_3^s)')dP_3|_{V_a} = (g^s|_{V_a})^* dP_3|_{V_a} = (g^s|_{V_a})^*(G_1|_{V_a} dP_1|_{V_a} + G_2|_{V_a} dP_2|_{V_a})$$

$$= ((g^s|_{V_a})^* G_1|_{V_a})((f_1^s)'(a)) dP_1|_{V_a} + ((g|_{V_a}^s)^* G_2|_{V_a})(P_2|_{V_a}^*(f_2^s)') dP_2|_{V_a}.$$

It follows that

$$\frac{((g^s|_{V_a})^* G_1|_{V_a})((f_1^s)'(a))}{((g|_{V_a}^s)^* G_2|_{V_a})(P_2|_{V_a}^*(f_2^s)')} = \frac{G_1|_{V_a}}{G_2|_{V_a}}.$$

It follows that

$$(g^s|_{V_a})^*(\frac{G_2|_{V_a}}{G_1|_{V_a}} dP_2|_{V_a}) = (g^s|_{V_a})^*(\frac{G_2|_{V_a}}{G_1|_{V_a}})(P_2|_{V_a}^*(f_2^s)') dP_2|_{V_a}$$

$$= ((f_1^s)'(a)) \frac{G_2|_{V_a}}{G_1|_{V_a}} dP_2|_{V_a}.$$

Then $g^s|_{V_a}$ is exceptional. Since g^s semi-conjugates to f_1^s, f_1 is exceptional, which is a contradiction. Then we concludes the proof. $\square$

Proof of Proposition 9.2. We do the proof by induction on $N \geq 2$. When $N = 2$, there is nothing to prove.

Assume that Proposition 9.2 is known for $N = 2, \ldots, l$. We need to show it when $N = l + 1 \geq 3$.

We first assume that there exists $J \in S_l$ such that $\pi_J(V) \neq (\mathbb{P}^1)^{|J|}$. Then it is a proper irreducible subvariety of $(\mathbb{P}^1)^{|J|}$ which is invariant under f^J. We may conclude by the induction hypothesis.

Now we may assume that for every $J \in S_l$, we have $\pi_J(V) = (\mathbb{P}^1)^{|J|}$. Then we have $\dim V = l$, and for every $J \in S_t, 1 \leq t \leq l$, $\pi_J(V) = (\mathbb{P}^1)^{|J|}$.

When $l = 2, N = 3$, we conclude the proof by Lemma 9.3. Now we may assume that $l \geq 3$. Set $K := \{1, \ldots, N\}$,$J := \{4, \ldots, N\}$ and $I := \{1, 2, 3\}$. By Lemma 9.3, there exists a nonempty Zariski open subset U of $(\mathbb{P}^1)^{N-3}$ such that for every $a \in U$, $V_a := \pi_J^{-1}(a) \cap V$ is an irreducible surface.

For every $i \in J$, $\mathrm{Per}\,(f_i)$ is Zariski dense in $\mathbb{P}^1$. Since the set C_i of critical f_i-periodic points of is finite, $P_i := \mathrm{Per}\,(f_i) \setminus C_i$ is Zariski dense in $\mathbb{P}^1$. Then $(\prod_{i \in J} P_i) \cap U$ is Zariski dense in $(\mathbb{P}^1)^{|J|}$. Pick $a \in (\prod_{i \in J} P_i) \cap U$, there exists $s \geq 1$ such that $f_J^s(a) = a$. Then V_a is invariant under f_I^s. By Lemma 9.4, there exists $i \in I$ such that $\pi_{I \setminus \{i\}}(V_a) \neq (\mathbb{P}^1)^2$. Pick $o \in (\mathbb{P}^1)^2 \setminus \pi_{I \setminus \{i\}}(V_a)$. Then we have $(o, a) \subseteq (\mathbb{P}^1)^{N-1} \setminus \pi_{K \setminus i}(V)$. It follows that $\pi_{K \setminus i}(V) \neq (\mathbb{P}^1)^{N-1}$, which contradicts our assumption. We conclude the proof. □

9.3. **Proof of Theorem 9.1.**

Proof. By Theorem 3.28, we may assume that $\deg f_i \geq 2, i = 1, \ldots, N$. By Theorem 4.1, we may assume that $f_i^2, i = 1, \ldots, N$ are not polynomial. In particular, none of $f_i, i = 1, \ldots, N$ are of monomial type.

By Proposition 3.16, after replacing f by some positive iterate, there exists a nonempty adelic open subset A of $(\mathbb{P}^1)^N(\mathbf{k})$ such that for every $x \in A$, the Zariski closure Z_x of the orbit $O_f(x)$ is irreducible.

We may assume that there exists $0 \leq s \leq N$ such that f_i is nonexceptional for $i \leq s$ and it is of type Lattés for $i \geq s + 1$. Define $l(f) := \min\{s, N - s\} \geq 0$.

We first treat the case $l(f) = 0$.

If $s = 0$, then all f_i are of type Lattés. Then there exists an abelian variety A, a dominant endomorphism $g : A \to A$ and a finite morphism $\pi : A \to (\mathbb{P}^1)^N$ such that $f \circ \pi = \pi \circ g$. By Theorem 1.10, the pair (A, g) satisfies the adelic ZD-property. Then we concludes the proof by Lemma 3.23.

If $s = N$, then all f_i are nonexceptional. For every $J \in S_2$, f_J is an amplified endomorphism on $(\mathbb{P}^1)^2$, whose topological degree is strictly larger than $\lambda_1(f_J)$. By Corollary 6.16, there exists a nonempty adelic open subset A_J of $(\mathbb{P}^1)^2$ such that the f_J-orbits of every point in A_J are Zariski dense in $(\mathbb{P}^1)^2$.

Then for every point x in the nonempty adelic open subset $A \cap (\cap_{J \in S_2} \pi_J^{-1}(A_J))$, the Zariski closure Z_x of the orbit of x is irreducible, invariant by f and $\pi_J(Z_x) = (\mathbb{P}^1)^2$ for $J \in S_2$. Proposition 9.2 shows that $Z_x = (\mathbb{P}^1)^N$ which concludes the proof.

Now we do the proof by induction on $l(f)N \geq 0$. Assume that Theorem 9.1 holds when $0 \leq l(f)N \leq m$. We need to prove it when $l(f)N = m+1 \geq 1$.

By the induction hypothesis, for every $J \subseteq S_{N-1}$, there exists a nonempty adelic open subset A_J of $(\mathbb{P}^1)^{N-1}(\mathbf{k})$, such that the f_J-orbit of every $x \in A_J$ is Zariski dense in $(\mathbb{P}^1)^{N-1}(\mathbf{k})$.

Set $B := A \cap (\cap_{J \in S_{N-1}} \pi_J^{-1}(A_J))$. For every $x \in B$, the Zariski closure Z_x of the orbit $O_f(x)$ is irreducible, invariant by f and for every $J \in S_{N-1}$, we have $\pi_J(Z_x) = (\mathbb{P}^1)^{N-1}$. Then we have $\dim Z_x \geq N-1$, and for every $J \in S_t, 1 \leq t \leq N-1$, $\pi_J(V) = (\mathbb{P}^1)^{|J|}$.

Assume that $\dim Z_x = N-1$. We note that f_1 is nonexceptional and f_N is of Lattés type. Set $I := \{2, \ldots, N-1\}$. Lemma 9.3 shows that there exists a nonempty Zariski open subset $U \subseteq (\mathbb{P}^1)^{N-2}$, such that for $a \in U$, $(\pi_I|_{Z_x})^{-1}(a)$ is an irreducible curve. Denote by $\mathrm{Per}\,(f_I)$ the set of periodic points of f_I. It is Zariski dense in $(\mathbb{P}^1)^{N-2}$. Pick $a \in \mathrm{Per}\,(f_I) \cap U$. Then $C_a := (\pi_I|_{Z_x})^{-1}(a)$ is an irreducible curve in $(\mathbb{P}^1)^2$, which is invariant under $f^s_{\{1,N\}}$ for some $s \geq 1$. Then $f^s_{\{1,N\}}|_{C_a} : C_a \to C_a$ is an endomorphism of degree at least 2. Denote by $p_i, i = 1, N : (\mathbb{P}^1)^2 \to \mathbb{P}^1$ the projection $(x_1, x_N) \mapsto x_i$. If $p_i(C_a) = \mathbb{P}^1$ for $i = 1, 2$, then $f^s_{\{1,N\}}|_{C_a}$ semiconjugates to both f_1^s and f_N^s. In particular, f_1 and f_N has the same type, which is a contradiction. It follows that C_a is a fiber of p_i, $i = 1$ or N. This implies that for all $b \in (\mathbb{P}^1)^{N-2}$, $C_b := (\pi_I|_{Z_x})^{-1}(b)$ is a fiber of p_i. Then, for $J := \{1, \ldots, N\} \setminus \{N+1-i\}$, we have $\pi_J(Z_x) \neq (\mathbb{P}^1)^{N-1}$, which contradicts our assumption. It follows that $\dim Z_x = N$, which concludes the proof. $\square$

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