INVARIANT SUBSETS UNDER COMPACT QUANTUM GROUP ACTIONS

HUICHI HUANG

Abstract. We investigate compact quantum group actions on unital C^*-algebras by analyzing invariant subsets and invariant states. In particular, we come up with the concept of compact quantum group orbits and use it to show that countable compact metrizable spaces with infinitely many points are not quantum homogeneous spaces.

1. Introduction

A compact quantum group is a unital C^*-algebra A together with a unital $*$-homomorphism $\Delta : A \to A \otimes A$ satisfying the coassociativity

$$(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta$$

and the cancellation laws that both $\Delta(A)(1 \otimes A)$ and $\Delta(A)(A \otimes 1)$ are dense in $A \otimes A$. If A is a commutative C^*-algebra, then $A = C(G)$ for some compact group G. From the viewpoint of noncommutative topology $A = C(\mathcal{G})$ for some compact quantum space $\mathcal{G}$. So compact quantum groups are generalizations of compact groups. There are lots of similarities and differences between $C(G)$ and $C(\mathcal{G})$. For instance, firstly both of them have the unique bi-invariant state called the Haar state. But unlike the Haar state of $C(G)$, the Haar state of $C(\mathcal{G})$ need to be neither faithful nor tracial. Secondly, although there is a linear functional called the counit which plays the same role in $C(\mathcal{G})$ as the unit in G, the counit is only densely defined and not necessarily bounded.

An action of a compact quantum group $\mathcal{G}$ on a unital C^*-algebra B is a unital $*$-homomorphism $\alpha : B \to B \otimes A$ satisfying that

(1) $(\alpha \otimes id)\alpha = (id \otimes \Delta)\alpha$;
(2) $\alpha(B)(1 \otimes A)$ is dense in $B \otimes A$.

Date: April 29, 2016.

2010 *Mathematics Subject Classification.* Primary: 46L65; Secondary:16W22.

Key words and phrases. Compact quantum group, action, invariant state, invariant subset, compact Hausdorff space, orbit.

If $A = C(G)$ for some compact group G and $B = C(X)$ for some compact Hausdorff space X, then the action α is just the action of G on X as homeomorphisms. Therefore actions of compact quantum groups on unital C^*-algebras are generalizations of compact groups on compact Hausdorff spaces. Moreover when a group acts on a space, the group elements are symmetries on the space. So when a compact quantum group $\mathcal{G}$ acts on a unital C^*-algebra B, then $\mathcal{G}$ can be understood as a set of quantum symmetries of the compact quantum space B.

A compact quantum group action α of $\mathcal{G}$ on B is called ergodic if $\{b \in B | \alpha(b) = b \otimes 1\} = \mathbb{C}$. If $\mathcal{G}$ is a compact group and $B = C(X)$ for a compact Hausdorff space X, then α is ergodic just means that the action is transitive. In this case X is called homogeneous. Generalizing the classical homogeneous space, we call a unital C^*-algebra B a homogeneous space if B admits an ergodic compact group action or a quantum homogeneous space if B admits an ergodic compact quantum group action. Note that there are different definitions of quantum homogeneous spaces (see [?] for example) we adopt the one given by P. Podleś in [? , Definition 1.8].

A compact group is a compact quantum group, hence a homogeneous space is a quantum homogeneous space. However, a quantum homogeneous space is not necessarily a homogeneous space.

It was shown by Høegh-Krohn, Landstad and Størmer that a homogeneous space has a finite trace [?]. But the class of quantum homogeneous spaces includes operator algebras of some other types. For instance, S. Wang showed that some type III factors and Cuntz algebras are quantum homogeneous spaces [?]. So there exists compact quantum spaces which are quantum homogeneous space, but not homogeneous. Thus on some compact quantum spaces, namely Cuntz algebra, although there are not enough symmetries to make these spaces to be homogeneous spaces, there are enough quantum symmetries such that these spaces are quantum homogeneous spaces.

But when one considers compact quantum group actions on classical compact spaces, the situation is quite different. So far, all classical quantum homogeneous spaces are homogeneous spaces [? ? ?]. This means that on a classical compact space, if there are not enough symmetries, then there are not enough quantum symmetries. This interesting phenomena leads us to conjecture that a compact Hausdorff space is a quantum homogeneous space if and only if it is a homogeneous space. Our main result in the paper is to confirm this conjecture in the case of compact Hausdorff spaces with countably infinitely many points.

Theorem 1.1. Any compact Hausdorff space with countably infinitely many points is not a quantum homogeneous space.

To prove the main theorem, we use invariant subsets and invariant states, formulate the concept of compact quantum group orbits and adopt them to study ergodic actions on compact spaces.

The paper is organized as follows. In section 2 we collect some facts about compact quantum groups and their actions on unital C^*-algebras. In section 3, we derive some results about invariant subsets and invariant states which will be used later. Especially, we show that a compact quantum group action is ergodic iff there is a unique invariant state (Theorem **??**). Next we show that the "support" of an invariant state is an invariant subset (Theorem **??**) and show that as long as all invariant states are tracial or there exists a faithful tracial invariant state, the compact quantum group is a Kac algebra (Theorem **??**). Section 4 is about compact quantum group actions on classical compact spaces. We formulate the concept of orbits. Then we prove that an orbit is an invariant subset (Theorem **??**) and that an action is ergodic iff there exists a unique orbit (Theorem **??**). In section 4.4 we prove Theorem **??** which says the invariant measure on a quantum homogeneous compact Hausdorff space with infinitely many points is non-atomic and the main theorem, Theorem **??** follows immediately.

Acknowledgements

This paper is one part of my PhD thesis. I am grateful to my PhD supervisor Hanfeng Li for his long term support and encouragement. His advice is indispensable for the writing of this paper. The paper goes to the final stage when I was a postdoctoral from June 2013 to January 2016 supported by ERC Advanced Grant No. 267079. I express my gratitude to my mentor Joachim Cuntz. Part of the paper was carried out during my staying in Chongqing University in the summer of 2012. I thank Dechao Zheng for his hospitality. I also thank Shuzhou Wang for many helpful discussions about compact quantum group theory and his pointing out an error in Remark **??**. I also thank Piotr Sołtan for his comments. Lastly I thank Masoud Khalkhali and the anonymous referee for their detailed comments which improve readability of the paper.

2. Preliminaries

In this section, we recall some definitions and basic properties of compact quantum groups and their actions. We refer to [**? ? ?**] for basics of compact quantum groups and [**?** **?** **?**] for some background of compact quantum group actions.

Throughout this paper, for two unital C*-algebras A and B, the notations $A \otimes B$ and $A \odot B$ stand for the minimal and the algebraic tensor product of A and B respectively.

For a $*$-homomorphism $\beta : B \to B \otimes A$, use $\beta(B)(1 \otimes A)$ and $\beta(B)(B \otimes 1)$ to denote the linear span of the set $\{\beta(b)(1_B \otimes a) | b \in B, a \in A\}$ and the linear span of the set $\{\beta(b_1)(b_2 \otimes 1_A) | b_1, b_2 \in B\}$ respectively.

For a C*-algebra B, we use $S(B)$ to denote the state space of B. For $\mu \in S(B)$, we denote $\{b \in B | \mu(b^*b) = 0\}$ by N_μ. If $N_\mu = \{0\}$, then μ is called faithful. If $\mu(ab) = \mu(ba)$ for all $a, b \in B$, then μ is called tracial.

Let's first recall the definition of compact quantum group, which, briefly speaking, is the C^*-algebra of continuous functions on some compact quantum space with a group-like structure.

Definition 2.1. [**?** , Definition 1.1]

A **compact quantum group** is a pair (A, Δ) consisting of a unital C*-algebra A and a unital $*$-homomorphism $\Delta : A \to A \otimes A$ such that

(1) $(id \otimes \Delta)\Delta = (\Delta \otimes id)\Delta$.
(2) $\Delta(A)(1 \otimes A)$ and $\Delta(A)(A \otimes 1)$ are dense in $A \otimes A$.

The $*$-homomorphism Δ is called the **coproduct** or **comultiplication** of $\mathcal{G}$. The first condition in the definition of compact quantum groups just means that the coproduct is associative, and the second condition says that the left cancellation law and the right cancellation law hold. Note that a compact semigroup in which cancellation laws hold is a group. Hence compact quantum groups are the quantum analogue of compact groups.

Furthermore, one can think of A as $C(\mathcal{G})$, i.e., the C*-algebra of continuous functions on some quantum space $\mathcal{G}$ and in the rest of the paper we write a compact quantum group (A, Δ) as $\mathcal{G}$.

There exists a unique state h on A such that

$$(h \otimes id)\Delta(a) = (id \otimes h)\Delta(a) = h(a)1_A$$

for all a in A. The state h is called the **Haar state** of $\mathcal{G}$ or the **Haar state** on A. Throughout this paper, we use h to denote the Haar state of $\mathcal{G}$.

Example 2.2. [Examples of compact quantum groups]

(1) For every non-singular $n \times n$ complex matrix Q ($n > 1$), the universal compact quantum group $(A_u(Q), \Delta_Q)$ [**?** , Theorem 1.3] is generated by u_{ij} ($i, j = 1, \cdots, n$) with defining relations

(with $u = (u_{ij})$):

$u^* u = I_n = u u^*, \quad u^t Q \bar{u} Q^{-1} = I_n = Q \bar{u} Q^{-1} u^t;$

and the coproduct Δ_Q given by $\Delta_Q(u_{ij}) = \sum_{k=1}^n u_{ik} \otimes u_{kj}$ for $1 \le i, j \le n$. In particular, when Q is the identity matrix, we denote $(A_u(Q), \Delta_Q)$ by $A_u(n)$.

(2) The **quantum permutation group** $(A_s(n), \Delta_n)$ [**?** , Theorem 3.1] is the universal C^*-algebra generated by a_{ij} for $1 \le i, j \le n$ under the relations

$$a_{ij}^* = a_{ij} = a_{ij}^2, \quad \sum_{i=1}^n a_{ij} = \sum_{j=1}^n a_{ij} = 1.$$

The coproduct $\Delta_n : A_s(n) \to A_s(n) \otimes A_s(n)$ is the $*$-homomorphism satisfying that

$$\Delta_n(a_{ij}) = \sum_{k=1}^n a_{ik} \otimes a_{kj}.$$

Definition 2.3. Let A be an associative $*$-algebra over $\mathbb{C}$ with an identity. Assume that Δ is a unital $*$-homomorphism from A to $A \odot A$ such that $(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta$. Also assume that there are linear maps $\varepsilon : A \to \mathbb{C}$ and $\kappa : A \to A$ such that

$$(\varepsilon \otimes id)\Delta(a) = (id \otimes \varepsilon)\Delta(a) = a$$

$$m(\kappa \otimes id)\Delta(a) = m(id \otimes \kappa)\Delta(a) = \varepsilon(a)1$$

for all $a \in A$, where $m : A \odot A \to A$ is the multiplication map. Then (A, Δ) is called a **Hopf $*$-algebra** [**?** , Definition 2.3].

A nondegenerate (unitary) **representation** U of a compact quantum group $\mathcal{G}$ is an invertible (unitary) element in $M(K(H) \otimes A)$ for some Hilbert space H satisfying that $U_{12}U_{13} = (id \otimes \Delta)U$. Here $K(H)$ is the C^*-algebra of compact operators on H and $M(K(H) \otimes A)$ is the multiplier C*-algebra of $K(H) \otimes A$. We write U_{12} and U_{13} respectively for the images of U by two maps from $M(K(H) \otimes A)$ to $M(K(H) \otimes A \otimes A)$ where the first one is obtained by extending the map $x \mapsto x \otimes 1$ from $K(H) \otimes A$ to $K(H) \otimes A \otimes A$, and the second one is obtained by composing this map with the flip on the last two factors. The Hilbert space H is called the **carrier Hilbert space** of U. From now on, we always assume representations are nondegenerate. If the carrier Hilbert space H is of finite dimension, then U is called a finite dimensional representation of $\mathcal{G}$.

For two representations U_1 and U_2 with the carrier Hilbert spaces H_1 and H_2 respectively, the set of **intertwiners** between U_1 and U_2, $Mor(U_1, U_2)$, is defined as

$$Mor(U_1, U_2) = \{T \in B(H_1, H_2) | (T \otimes 1)U_1 = U_2(T \otimes 1)\}.$$

Two representations U_1 and U_2 are equivalent if there exists an invertible element T in $Mor(U_1, U_2)$. A representation U is called **irreducible** if $Mor(U, U) \cong \mathbb{C}$.

Moreover, we have the following well-established facts about representations of compact quantum groups:

(1) Every finite dimensional representation is equivalent to a unitary representation.
(2) Every irreducible representation is finite dimensional.

Let $\widehat{\mathcal{G}}$ be the set of equivalence classes of irreducible representations of $\mathcal{G}$. For every $\gamma \in \widehat{\mathcal{G}}$, let $U^\gamma \in \gamma$ be unitary and H_γ be its carrier Hilbert space with dimension d_γ. After fixing an orthonormal basis of H_γ, we can write U^γ as $(u_{ij}^\gamma)_{1 \leq i,j \leq d_\gamma}$ with $u_{ij}^\gamma \in A$. The matrix $\overline{U^\gamma}$ is still an irreducible representation (not necessarily unitary) with the carrier Hilbert space $\overline{H_\gamma}$. It is called the **contragradient** representation of U^γ and the equivalence class of $\overline{U^\gamma}$ is denoted by γ^c. There is a unique positive invertible element F^γ in $Mor(U^\gamma, U^{\gamma^{cc}})$ such that $tr(F^\gamma) = tr(F^\gamma)^{-1}$. Denote $tr(F^\gamma)$ by M_γ and M_γ is called the **quantum dimension** of γ. Note that $F^\gamma > 0$ is in $B(H_\gamma)$ and can be expressed as a $d_\gamma \times d_\gamma$ matrix under the same orthonormal basis of H_γ adopted by U^γ.

The linear space $\mathscr{A}$ spanned by $\{u_{ij}^\gamma\}_{\gamma \in \widehat{\mathcal{G}},\, 1 \leq i,j \leq d_\gamma}$ is a Hopf $*$-algebra [**?** **?**] such that

$$\Delta|_{\mathscr{A}} : \mathscr{A} \to \mathscr{A} \odot \mathscr{A}, \qquad \Delta(u_{ij}^\gamma) = \sum_{m=1}^{d_\gamma} u_{im}^\gamma \otimes u_{mj}^\gamma.$$

Moreover, the following are true.

(1) The Haar state h is **faithful** on $\mathscr{A}$, that is, if $h(a^*a) = 0$ for an $a \in \mathscr{A}$, then $a = 0$.
(2) There exist uniquely a linear multiplicative functional $\varepsilon : \mathscr{A} \to \mathbb{C}$ and a linear antimultiplicative map $\kappa : \mathscr{A} \to \mathscr{A}$ such that

$$\varepsilon(u_{ij}^\gamma) = \delta_{ij}, \qquad \kappa(u_{ij}^\gamma) = (u_{ji}^\gamma)^*.$$

The two maps ε and κ are called the **counit** and the **antipodle** of $\mathcal{G}$ respectively.

For $\gamma_1, \gamma_2 \in \widehat{\mathcal{G}}$, $1 \leq m, k \leq d_{\gamma_1}$ and $1 \leq n, l \leq d_{\gamma_2}$, we have

$$h(u^{\gamma_1}_{mk} u^{\gamma_2 *}_{nl}) = \frac{\delta_{\gamma_1 \gamma_2} \delta_{mn} F^{\gamma_1}_{lk}}{M_{\gamma_1}}, \tag{1}$$

and

$$h(u^{\gamma_1 *}_{km} u^{\gamma_2}_{ln}) = \frac{\delta_{\gamma_1 \gamma_2} \delta_{mn} (F^{\gamma_1})^{-1}_{lk}}{M_{\gamma_1}}. \tag{2}$$

A compact quantum group (A', Δ') is called a **quantum subgroup** of $\mathcal{G}$ if there exists a surjective $*$-homomorphism $\pi : A \to A'$ such that

$$(\pi \otimes \pi)\Delta = \Delta' \pi.$$

We can identify A' with a quotient C^*-algebra of A, i.e., $A' \cong A/I$ for some ideal of A. We call the ideal I a **Woronowicz C^*-ideal** of A. If we write A' as $C(\mathcal{H})$ for some quantum space $\mathcal{H}$, we also call $\mathcal{H}$ a quantum subgroup of $\mathcal{G}$ [**?** , Definition 2.13].

Definition 2.4. [**?** , Definition 1.4]

An **action** of a compact quantum group $\mathcal{G}$ on a unital C*-algebra B is a unital $*$-homomorphism $\alpha : B \to B \otimes A$ satisfying that

(1) $(\alpha \otimes id)\alpha = (id \otimes \Delta)\alpha$;
(2) $\alpha(B)(1 \otimes A)$ is dense in $B \otimes A$.

An action α of a compact quantum group $\mathcal{G}$ on B is called **ergodic** if the fixed point algebra $B^\alpha = \{b \in B | \alpha(b) = b \otimes 1\}$ equals $\mathbb{C}1_B$.

Consider an action of $\mathcal{G}$ on B. For every $\gamma \in \widehat{\mathcal{G}}$, there is a linear subspace B_γ of B with a basis $\mathscr{S}_\gamma = \{e_{\gamma k i} | k \in J_\gamma, 1 \leq i \leq d_\gamma\}$ such that α maps B_γ into $B_\gamma \odot \mathscr{A}$ and $\alpha(e_{\gamma k i}) = \sum_{j=1}^{d_\gamma} e_{\gamma k j} \otimes u^\gamma_{ji}$. Moreover B_γ contains any other subspace of B satisfying these two conditions. The **quantum multiplicity** $\mathrm{mul}(B, \gamma)$ of γ is defined as cardinality of J_γ, which does not depend on the choice of J_γ [**?** , Theorem 1.5]. Moreover, $B^*_\gamma = B_{\gamma^c}$ [**?** , Lemma 11]. Hence $\mathrm{mul}(B, \gamma) > 0$ implies $\mathrm{mul}(B, \gamma^c) > 0$.

Take $\mathscr{B} = \bigoplus_{\gamma \in \widehat{\mathcal{G}}} B_\gamma$. It is known from [**?** , Theorem 1.5] that $\mathscr{B}$ is a dense $*$-subalgebra of B, which is called the **Podlés algebra** of B. Also

$$\alpha|_{\mathscr{B}} : \mathscr{B} \to \mathscr{B} \odot \mathscr{A}, \qquad (id \otimes \varepsilon)\alpha|_{\mathscr{B}} = id_{\mathscr{B}}.$$

We say a bounded linear functional μ on B is **α-invariant** or briefly **invariant** if $(\mu \otimes id)\alpha(b) = \mu(b)1_A$ for all $b \in B$. Denote by Inv_α the set of α-invariant states on B. It is known that

$$Inv_\alpha = \{(\psi \otimes h)\alpha | \psi \in S(B)\}.$$

Denote by $C(X)$ the C^*-algebra of complex-valued continuous functions on a compact Hausdorff space X. If a compact quantum group $\mathcal{G}$ acts on $B = C(X)$, then briefly we say that $\mathcal{G}$ acts on X.

Definition 2.5. [**?** , Definition 1.8]

A unital C^*-algebra B is called a **quantum homogeneous space** if B admits an ergodic compact quantum group action.

Briefly speaking, the investigation of actions of compact quantum groups on unital C^*-algebras is to study how compact quantum groups behave as symmetries of compact quantum spaces. Certainly there are many interesting examples of compact quantum group actions. Below we list some of them for later use, in particular, we give two examples of compact quantum group actions on compact Hausdorff spaces.

Example 2.6. [Examples of compact quantum group actions]

(1) Every compact quantum group $\mathcal{G}$ acts on A by the coproduct Δ, and $\mathscr{A}$ is the Podlés algebra of A.

(2) The adjoint action Ad_u of $(A_u(Q), \Delta_Q)$ on $M_n(\mathbb{C})$ is given by

$$Ad_u(b) = u(b \otimes 1)u^*,$$

for every $b \in M_n(\mathbb{C})$.

(3) Recall that the Cuntz algebra $\mathcal{O}_n$ [**?**] is the universal C^*-algebra generated by n(≥ 2) isometries $S_1, S_2, ..., S_n$ such that

$$\sum_{i=1}^{n} S_i S_i^* = 1.$$

The compact quantum group $(A_u(Q), \Delta_Q)$ acts on $\mathcal{O}_n$ by

$$\alpha(S_i) = \sum_{j=1}^{n} S_j \otimes u_{ji},$$

for $1 \leq i \leq n$ [**?** , Equation 5.2].

(4) The quantum permutation group $A_s(n)$ acts on $\{x_1, x_2, \cdots, x_n\}$ [**?** , Theorem 3.1] by

$$\alpha(e_i) = \sum_{j=1}^{n} e_j \otimes a_{ji},$$

where e_i is the characteristic function of $\{x_i\}$ for $1 \leq i \leq n$.

(5) Let Y be a connected compact Hausdorff space and Y_1 is a closed subset of Y. Define an equivalence relation in $X_n \times Y$ as the following: $(x_i, y) \sim (x_j, y)$ if $(x_i, y) = (x_j, y)$ or $y \in Y_1$. Then $A_s(n)$ acts on the connected compact space $X_n \times Y/\sim$

faithfully and the action α is given by

$$\alpha(\sum_{i=1}^{n} e_i \otimes f_i) = \sum_{i=1}^{n}\sum_{j=1}^{n} e_j \otimes f_i \otimes a_{ji}$$

for all $\sum_{i=1}^{n} e_i \otimes f_i \in C(X_n \times Y/\sim)$ [**?**].

3. Actions on compact quantum spaces

3.1. **Faithful actions.**

In this section, we give some equivalent conditions of faithful compact quantum group actions for future use. This is well known for experts, but for completeness and convenience, we give a proof here. Part of these results can be found in [**?** , Lemma 2.4].

We first recall some definitions.

Definition 3.1. [**?** , Definition 2.9]

For a compact quantum group $\mathcal{G}$, a unital C*-subalgebra Q of A is called a **compact quantum quotient group** of $\mathcal{G}$ if $\Delta(Q) \subseteq Q \otimes Q$, and $\Delta(Q)(1 \otimes Q)$ and $\Delta(Q)(Q \otimes 1)$ are dense in $Q \otimes Q$. That is, $(Q, \Delta|_Q)$ is a compact quantum group. If $Q \neq A$, we call Q a **proper compact quantum quotient group**.

We say that a compact quantum group action α on B is **faithful** if there is no proper compact quantum quotient group Q of $\mathcal{G}$ such that α induces an action α_q of $(Q, \Delta|_Q)$ on B satisfying $\alpha(b) = \alpha_q(b)$ for all b in B [**?** , Definition 2.4].

There are several equivalent descriptions of faithful actions.

Proposition 3.2. Consider a compact quantum group action α of $\mathcal{G}$ on B. The following are equivalent:

(1) The action α is faithful.
(2) The $*$-subalgebra of A generated by $(\omega \otimes id)\alpha(B)$ for all bounded linear functionals ω on B is dense in A.
(3) The $*$-subalgebra $\mathscr{A}_1$ of $\mathscr{A}$ generated by $(\omega \otimes id)\alpha(\mathscr{B})$ for all bounded linear functionals ω on B is dense in A.
(4) The $*$-subalgebra $\mathscr{A}_2$ of $\mathscr{A}$ generated by u_{ij}^{γ} for all $\gamma \in \widehat{\mathcal{G}}$ and $1 \leq i, j \leq d_\gamma$ such that $\mathrm{mul}(B, \gamma) > 0$ is dense in A.
(5) $\mathscr{A}_2 = \mathscr{A}$.

Proof. (2) $\Rightarrow$ (1). Suppose that the action α of $\mathcal{G}$ on B induces an action α_q of a quotient group Q of $\mathcal{G}$ on B such that $\alpha(b) = \alpha_q(b)$ for all b in B. The $*$-subalgebra generated by $(\omega \otimes id)\alpha(B)$ for all bounded linear functional ω on B is a subalgebra of Q. Hence $Q = A$ and α is faithful.

(1) $\Rightarrow$ (4). Let A_2 be the closure of $\mathscr{A}_2$ in A. We want to show that $(A_2, \Delta|_{A_2})$ is a quotient group of $\mathcal{G}$. First, since $\Delta(\mathscr{A}_2) \subseteq \mathscr{A}_2 \odot \mathscr{A}_2$, we have that $\Delta(A_2) \subseteq A_2 \otimes A_2$.

We next show that $\Delta(A_2)(1 \otimes A_2)$ is dense in $A_2 \otimes A_2$. Since u^γ is unitary for all $\gamma \in \widehat{\mathcal{G}}$ with $\mathrm{mul}(B, \gamma) > 0$, we first have

$$\sum_{t=1}^{d_\gamma} \Delta(u^\gamma_{it})(1 \otimes u^{\gamma *}_{jt}) = u^\gamma_{ij} \otimes 1,$$

for all $1 \leq i, j \leq d_\gamma$. Note that $\sum_{t=1}^{d_\gamma} \Delta(u^\gamma_{it})(1 \otimes u^{\gamma *}_{jt})$ belongs to $\Delta(A_2)(1 \otimes A_2)$, so does $u^\gamma_{ij} \otimes 1$ for all $1 \leq i, j \leq d_\gamma$. It follows that

$$u^{\gamma_1}_{ij} u^{\gamma_2}_{kl} \otimes 1 \in \Delta(A_2)(1 \otimes A_2)(u^{\gamma_2}_{kl} \otimes 1) = \Delta(A_2)(u^{\gamma_2}_{kl} \otimes 1)(1 \otimes A_2) \subseteq \Delta(A_2)(1 \otimes A_2)$$

for all $\gamma_1, \gamma_2 \in \widehat{\mathcal{G}}$ with positive multiplicity in B and all $1 \leq i, j \leq d_{\gamma_1}$ and $1 \leq k, l \leq d_{\gamma_2}$. Inductively $u^{\gamma_1}_{i_1 j_1} \cdots u^{\gamma_s}_{i_s j_s} \otimes 1 \in \Delta(A_2)(1 \otimes A_2)$ for all $\gamma_1, \cdots, \gamma_s \in \widehat{\mathcal{G}}$ with positive multiplicity in B and all $1 \leq i_t, j_t \leq d_{\gamma_t}$ with $1 \leq t \leq s$.

Note that $\mathscr{A}_2$ is the $*$-subalgebra of $\mathscr{A}$ generated by the matrix elements of u^γ for all $\gamma \in \widehat{\mathcal{G}}$ with $\mathrm{mul}(B, \gamma) > 0$. Also the adjoint of the matrix elements of u^γ are the matrix elements of u^{γ^c}, the contragradient representation of γ. Hence $\mathscr{A}_2$ is the subalgebra of $\mathscr{A}$ generated by the matrix elements of u^γ for all $\gamma \in \widehat{\mathcal{G}}$ with positive multiplicity in B. So $A_2 \otimes 1$ is in the closure of $\Delta(A_2)(1 \otimes A_2)$. Then for any $a, b \in A_2$, we have $a \otimes b = (a \otimes 1)(1 \otimes b)$ is in the closure of $\Delta(A_2)(1 \otimes A_2)$ since $\Delta(A_2)(1 \otimes A_2)(1 \otimes b) \subseteq \Delta(A_2)(1 \otimes A_2)$. Hence $\Delta(A_2)(1 \otimes A_2)$ is dense in $A_2 \otimes A_2$.

Similarly, we can prove that $1 \otimes u^{\gamma *}_{ij} \in \Delta(A_2)(A_2 \otimes 1)$ for all $\gamma \in \widehat{\mathcal{G}}$ with $\mathrm{mul}(B, \gamma) > 0$ and all $1 \leq i, j \leq d_\gamma$, and that $\Delta(A_2)(A_2 \otimes 1)$ is dense in $A_2 \otimes A_2$. Therefore, A_2 is a compact quantum quotient group of A. Next we show that α is an action of $(A_2, \Delta|_{A_2})$ on B.

Obviously $\alpha(B) \subseteq B \otimes A_2$. To show that $\alpha(B)(1 \otimes A_2)$ is dense in $B \otimes A_2$, it is enough to prove that $e_{\gamma ki} \otimes 1 \in \alpha(B)(1 \otimes A_2)$ for all $\gamma \in \widehat{\mathcal{G}}$ such that $\mathrm{mul}(B, \gamma) > 0$ and all $1 \leq i \leq d_\gamma$ and $1 \leq k \leq \mathrm{mul}(B, \gamma)$. This follows from the following identity:

$$\sum_{t=1}^{d_\gamma} \alpha(e_{\gamma kt})(1 \otimes u^{\gamma *}_{it}) = e_{\gamma ki} \otimes 1.$$

Hence α is also an action of A_2 on B. By the faithfulness of α, we have that $A_2 = A$.

(3) ⇔ (4). To prove the equivalence of (3) and (4), it suffices to show that $\mathscr{A}_1 = \mathscr{A}_2$. Obviously $\mathscr{A}_1 \subseteq \mathscr{A}_2$. For $\gamma \in \widehat{\mathcal{G}}$ such that $\mathrm{mul}(B,\gamma) > 0$, we have that $\alpha(e_{\gamma ki}) = \sum_{j=1}^{d_\gamma} e_{\gamma kj} \otimes u_{ji}^{\gamma}$ for $1 \leq i \leq d_\gamma$ and $1 \leq k \leq \mathrm{mul}(B,\gamma)$. Note that $e_{\gamma ki}$'s are linearly independent. For every $1 \leq s \leq d_\gamma$ and every $1 \leq l \leq \mathrm{mul}(B,\gamma)$, by the Hahn-Banach Theorem, there exists a bounded linear functional ω_{ls}^{γ} on B such that $\omega_{ls}^{\gamma}(e_{\gamma ki}) = \delta_{kl}\delta_{si}$ for $1 \leq i \leq d_\gamma$ and $1 \leq k \leq \mathrm{mul}(B,\gamma)$. Therefore $(\omega_{ks}^{\gamma} \otimes id)\alpha(e_{\gamma ki}) = u_{si}^{\gamma} \in \mathscr{A}_1$ for all $\gamma \in \widehat{\mathcal{G}}$ such that $\mathrm{mul}(B,\gamma) > 0$, and for every $1 \leq i \leq d_\gamma$ and every $1 \leq s \leq \mathrm{mul}(B,\gamma)$. This implies that $\mathscr{A}_2 \subseteq \mathscr{A}_1$, which proves the equivalence of (3) and (4).

(2) ⇔ (3). The equivalence of (2) and (3) is immediate from the density of $\mathscr{B}$ in B and the continuity of $(\omega \otimes id)\alpha$ for every bounded linear functional ω on B.

(4) ⇔ (5). It is obvious that (5) implies (4). Now suppose that (4) is true. The $*$-subalgebra $\mathscr{A}_2$ is a Hopf $*$-subalgebra of A. A compact quantum group has a unique dense Hopf $*$-subalgebra [**?** , Theorem A.1], so (5) follows. □

3.2. **Invariant states.** In this subsection, we prove Theorem **??** and Theorem **??**.

First, for a compact quantum group, there is a reduced version of it in which the Haar state is faithful [**?** , Theorem 2.1].

For a compact quantum group $\mathcal{G}$ with the Haar state h and the counit ε, let $N_h = \{a \in A | h(a^*a) = 0\}$ and $\pi_r : A \to A/N_h$ be the quotient map. Then N_h is a two-sided ideal of A [**?** , Proposition 7.9]. Furthermore, the following is true.

Theorem. [**?** , Theorem 2.1]

For a compact quantum group$\mathcal{G}$, the C^*-algebra $A_r = A/N_h$ is a compact quantum subgroup of $\mathcal{G}$ with coproduct Δ_r determined by $\Delta_r(\pi_r(a)) = (\pi_r \otimes \pi_r)\Delta(a)$, for all $a \in A$. The Haar state h_r of (A_r, Δ_r) is given by $h = h_r\pi_r$ and h_r is faithful. Also, the quotient map π_r is injective on $\mathscr{A}$ and the Hopf $*$-algebra of (A_r, Δ_r) is $\pi_r(\mathscr{A})$, with the counit ε_r and the antipodle κ_r determined by $\varepsilon = \varepsilon_r\pi_r$ and $\pi_r\kappa = \kappa_r\pi_r$, respectively.

Definition 3.3. The compact quantum group (A_r, Δ_r) is called the **reduced** compact quantum group of $\mathcal{G}$, and we write it as $\mathcal{G}_r$.

From the theorem above, it is easy to check that any compact quantum group action of $\mathcal{G}$ on B induces an action α_r of (A_r, Δ_r) on B defined by

$$\alpha_r = (id \otimes \pi_r)\alpha.$$

Let α be an action of compact quantum group $\mathcal{G}$ on a unital C*-algebra B. Let $B^\alpha = \{b \in B | \alpha(b) = b \otimes 1_A\}$. It is known that

$$B^\alpha = (id \otimes h)\alpha(B).$$

Next we show that the space of invariant linear bounded functionals on B is isometrically isomorphic to the dual space of B^α.

Let $(B^\alpha)'$ be the dual space of B^α and $\mathrm{Inv}(B)$ be the space of α-invariant bounded linear functionals on B. Define $T : \mathrm{Inv}(B) \to (B^\alpha)'$ as $T(\psi) = \psi|_{B^\alpha}$. Then

Theorem 3.4. The linear map T is a bijective isometry.

Proof. Obviously $\|T\| \leq 1$, so T is bounded. Define the map $S : (B^\alpha)' \to B'$ by $S(\varphi) = \widetilde{\varphi}$ for every φ in $(B^\alpha)'$ where $\widetilde{\varphi}$ is the linear functional on B defined by

$$\widetilde{\varphi}(b) = \varphi((id \otimes h)\alpha(b))$$

for every $b \in B$. Next we show that S is the inverse of T.

First we show that $\widetilde{\varphi}$ is α-invariant. From $(\alpha \otimes id)\alpha = (id \otimes \Delta)\alpha$ and $(h \otimes id)\Delta = h(\cdot)1_A$, we have

$$\begin{aligned}(\widetilde{\varphi} \otimes id)\alpha &= ((\varphi \otimes h)\alpha \otimes id)\alpha = (\varphi \otimes h \otimes id)(\alpha \otimes id)\alpha \\ &= (\varphi \otimes h \otimes id)(id \otimes \Delta)\alpha = (\varphi \otimes ((h \otimes id)\Delta))\alpha \\ &= (\varphi \otimes (h(\cdot)1_A))\alpha = \varphi((id \otimes h)\alpha(\cdot))1_A = \widetilde{\varphi}(\cdot)1_A.\end{aligned}$$

Hence S maps $(B^\alpha)'$ into $\mathrm{Inv}(B)$. Moreover $\alpha(b) = b \otimes 1_A$ for any $b \in B^\alpha$. Hence $\widetilde{\varphi}(b) = \varphi(b)$ for any $b \in B^\alpha$. So φ is the restriction of $\widetilde{\varphi}$ on B^α. Therefore $\widetilde{\varphi}$ is α-invariant and $TS(\varphi) = T(\widetilde{\varphi}) = \varphi$. This shows the surjectivity of T.

Secondly for all $\phi \in \mathrm{Inv}(B)$ and all $b \in B$, we have $(\phi \otimes id)\alpha(b) = \phi(b)1_A$. Applying h on both sides of the above equation, we get $(\phi \otimes h)\alpha(b) = \phi(b)$. So

$$\widetilde{T(\phi)}(b) = T(\phi)((id \otimes h)\alpha(b)) = \phi((id \otimes h)\alpha(b)) = (\phi \otimes h)\alpha(b) = \phi(b)$$

for all $b \in B$. That is to say that $ST(\phi) = \widetilde{T(\phi)} = \phi$ for all $\phi \in \mathrm{Inv}(B)$. Therefore S is the inverse of T and T is bijective.

Moreover for every $\phi \in \mathrm{Inv}(B)$, we see that $\|T(\phi)\| \leq \|\phi\|$ and $\phi(b) = (\phi \otimes h)\alpha(b) = \phi((id \otimes h)\alpha(b))$ for each $b \in B$. If $b \in B$ and $\|b\| \leq 1$, then $(id \otimes h)\alpha(b) \in B^\alpha$ and $\|(id \otimes h)\alpha(b)\| \leq 1$. So

$$\begin{aligned}\|\phi\| &= \sup_{\|b\| \leq 1} |\phi(b)| = \sup_{\|b\| \leq 1} |\phi((id \otimes h)\alpha(b))| \\ &= \sup_{\|b\| \leq 1} |T(\phi)((id \otimes h)\alpha(b))| \leq \|T(\phi)\|.\end{aligned}$$

Therefore $\|T(\phi)\| = \|\phi\|$ for every $\phi \in \mathrm{Inv}(B)$ and T is an isometry from $\mathrm{Inv}(B)$ onto $(B^\alpha)'$. $\square$

The following theorem follows from Theorem **??** immediately.

Theorem 3.5. A compact quantum group action α of $\mathcal{G}$ on B is ergodic if and only if there is a unique α-invariant state on B.

Proof. The "only if" part is well-known [**?** , Lemma 4], and we just prove the "if" part.

Assume that there is a unique α-invariant state on B. By Theorem **??**, we have that $\mathrm{Inv}(B) \cong (B^\alpha)'$. So there is a unique state on $(B^\alpha)'$. Every bounded linear functional on B^α is a linear combination of states on B^α, so $(B^\alpha)' = \mathbb{C}$. Hence $B^\alpha \subseteq (B^\alpha)'' = \mathbb{C}$. Therefore $B^\alpha = \mathbb{C}$ and α is ergodic. $\square$

For a compact quantum group action α of $\mathcal{G}$ on B, recall that the reduced action α_r of $\mathcal{G}_r$ on B is defined by

$$\alpha_r = (id \otimes \pi_r)\alpha.$$

A state μ on B is α-invariant if and only if μ is α_r-invariant since $(\mu \otimes h)\alpha = (\mu \otimes h_r)\alpha_r$. So by Theorem **??**, the following is true.

Corollary 3.6. A compact quantum group action α of $\mathcal{G}$ on B is ergodic if and only if the reduced action α_r of $\mathcal{G}_r$ on B is ergodic.

3.3. **Invariant subsets.** From now on, an ideal I of a unital C^*-algebra B always means a closed two-sided ideal, and we denote the quotient map from B onto B/I by π_I.

Definition 3.7. Suppose a compact quantum group $\mathcal{G}$ acts on B by α. An ideal I of B is called **α-invariant** if for all $b \in I$,

$$(\pi_I \otimes id)\alpha(b) = 0.$$

A proper I is called **maximal** if any proper α-invariant ideal $J \supseteq I$ of B satisfies that $I = J$.

Remark 3.8. If an ideal I of B is α-invariant, then α induces an action α_I of $\mathcal{G}$ on B/I given by

$$\alpha_I(b + I) = (\pi \otimes id)\alpha(b)$$

for all $b \in B$.

If $B = C(X)$ for a compact Hausdorff space X, then there is a one-one correspondence between closed subsets of X and ideals of B. To say that an ideal is invariant under a compact group action is equivalent to say that the corresponding closed subset of X is invariant. An ideal is maximal just means that the corresponding closed subset is a minimal invariant subset of X.

Take an α-invariant state μ on B. Let $\Phi_\mu : B \to B(H_\mu)$ be the GNS representation of B with respect to μ and denote $\ker \Phi_\mu$ by I_μ. If B is commutative, then

$$I_\mu = N_\mu = \{f \in B | \mu(f^* f) = 0\} = \{f \in B| \ f|_{support\ of\ \mu} = 0\}.$$

For a compact group action on a commutative C^*-algebra $B = C(X)$, the ideal I_μ is invariant is equivalent to that the support of μ is an invariant subset of X. The following theorem says that this is also true in the quantum case.

Theorem 3.9. Suppose that $\mathcal{G}$ acts on B by α and μ is an α-invariant state on B. The ideal I_μ of B is α-invariant, and the induced action on B/I_μ, denoted by α_μ, is injective.

To prove Theorem **??**, we need the following lemma:

Lemma 3.10. There exists an injective $*$-homomorphism $\beta : B(H_\mu) \to L(H_\mu \otimes A)$ such that

$$\beta\Phi_\mu = (\Phi_\mu \otimes id)\alpha,$$

where $H_\mu \otimes A$ is the right Hilbert A-module with the inner product $\langle .,. \rangle$ given by $\langle b_1 \otimes a_1, b_2 \otimes a_2\rangle = \mu(b_1^* b_2)a_1^* a_2$ for $a_i \in A$ and $b_i \in B$, and $L(H_\mu \otimes A)$ is the set of adjointable maps on $H_\mu \otimes A$

Proof. We can define a bounded linear map $U : H_\mu \otimes A \to H_\mu \otimes A$ by

$$U(b \otimes a) = \alpha(b)(1 \otimes a),$$

for all $b \in B$ and $a \in A$.

Using the argument in [**?** , Lemma 5], we get that U is a unitary representation of $\mathcal{G}$ with the carrier Hilbert space H_μ.

Let $\beta(T) = U(T \otimes 1)U^*$ for $T \in B(H_\mu)$. It is easy to see that β is an injective $*$-homomorphism from $B(H_\mu)$ into $L(H_\mu \otimes A)$. To prove $\beta\Phi_\mu = (\Phi_\mu \otimes id)\alpha$, it is enough to show that

$$\beta\Phi_\mu(b)(\alpha(b_1)(1 \otimes a_1)) = (\Phi_\mu \otimes id)\alpha(b)(\alpha(b_1)(1 \otimes a_1))$$

for all $a_1 \in A$ and $b, b_1 \in B$, since $\alpha(B)(1 \otimes A)$ is dense in $B \otimes A$. From the definitions of U and β and that U is unitary,

$$\begin{aligned}\beta\Phi_\mu(b)(\alpha(b_1)(1 \otimes a_1)) &= u(b \otimes 1)u^*(\alpha(b_1)(1 \otimes a_1)) \\ &= u(bb_1 \otimes a_1) = \alpha(bb_1)(1 \otimes a_1).\end{aligned}$$

On the other hand, we have that

$$(\Phi_\mu \otimes id)\alpha(b)(\alpha(b_1)(1 \otimes a_1)) = \alpha(bb_1)(1 \otimes a_1).$$

This completes the proof. □

Now we are ready to prove Theorem **??**.

Proof. By Lemma **??**, we have that $\beta\Phi_\mu = (\Phi_\mu \otimes id)\alpha$. Hence $(\Phi_\mu \otimes id)\alpha(b) = \beta\Phi_\mu(b) = 0$ for any $b \in I_\mu$. Let π_μ be the quotient map from B onto B/I_μ and $\widehat{\Phi_\mu}$ be the injective $*$-homomorphism from B/I_μ into $B(H_\mu)$ induced by Φ_μ, then

$$\Phi_\mu = \widehat{\Phi_\mu}\pi_\mu.$$

The injectivity of $\widehat{\Phi_\mu}$ gives us the injectivity of $\widehat{\Phi_\mu} \otimes id$. So for $b \in I_\mu$, the identities

$$0 = \beta\Phi_\mu(b) = (\Phi_\mu \otimes id)\alpha(b) = (\widehat{\Phi_\mu} \otimes id)(\pi_\mu \otimes id)\alpha(b)$$

implies that $(\pi_\mu \otimes id)\alpha(b) = 0$, which proves the invariance of I_μ.

If $\alpha_\mu(b + I_\mu) = 0$ for some $b \in B$, then $(\pi_\mu \otimes id)\alpha(b) = 0$. Hence $(\widehat{\Phi_\mu} \otimes id)(\pi_\mu \otimes id)\alpha(b) = 0$. Then it follows from $\Phi_\mu = \widehat{\Phi_\mu}\pi_\mu$ that $(\Phi_\mu \otimes id)\alpha(b) = 0$. Since $\beta\Phi_\mu = (\Phi_\mu \otimes id)\alpha$, we have that $\beta\Phi_\mu(b) = 0$. That is to say $\beta\widehat{\Phi_\mu}\pi_\mu(b) = 0$. Since β and $\widehat{\Phi_\mu}$ are both injective, we have that $\pi_\mu(b) = 0$, which proves the injectivity of α_μ. □

Example 3.11. [Examples of invariant ideals]

(1) Consider the action of a compact quantum group $\mathcal{G}$ on A given by Δ. The Haar state h is the unique Δ-invariant state on A. Since N_h is an ideal [**?** , Proposition 7.9], we have that $I_h = N_h$. Hence N_h is an invariant ideal of A.

(2) If B is commutative, then $N_\mu = I_\mu$ for every α-invariant state μ on B and N_μ is an α-invariant ideal of B by Theorem **??**.

3.4. **Kac algebra and tracial invariant states.**

Definition 3.12. A compact quantum group $\mathcal{G}$ is called a **Kac algebra** if one of the following equivalent conditions holds [**?** , Theorem 1.5] [**?** , Example 1.1] [**?** , Definition 8.1]:

(1) The Haar state h of $\mathcal{G}$ is tracial.

(2) The antipode κ of $\mathcal{G}$ satisfies that $\kappa^2 = id$ on $\mathscr{A}$.

(3) $F^\gamma = id$ for all $\gamma \in \widehat{\mathcal{G}}$.

For an ergodic action α of a compact quantum group $\mathcal{G}$ on B, in general, the unique α-invariant state μ on B is not necessarily tracial (See Remark 3.34 below). In [**?**], Goswami showed that if $\mathcal{G}$ acts on a unital C^*-algebra B ergodically and faithfully, and the unique α-invariant state μ on B is tracial, then $\mathcal{G}$ is a Kac algebra [**?** , Corollary 2.3]. Actually Goswami proved this result with the assumption that B is commutative, but his proof works in the noncommutative case with the assumption of the traciality of μ.

Using a different method, we generalize this result to faithful (not necessarily ergodic) actions, and show that traciality of h depends on traciality of invariant states (see Theorem **??** below).

Lemma 3.13. *Suppose that $\mathcal{G}$ acts on B by α. Take $\gamma \in \widehat{\mathcal{G}}$ such that $\mathrm{mul}(B, \gamma) > 0$. If there exists a state φ on B satisfying that*

$$\varphi(\sum_{1 \le s \le d_\gamma} e_{\gamma k s} e^*_{\gamma k s}) > 0$$

and $(\varphi \otimes h)\alpha(e_{\gamma kj} e^*_{\gamma ki}) = (\varphi \otimes h)\alpha(e^*_{\gamma ki} e_{\gamma kj})$ for some $1 \le k \le \mathrm{mul}(B, \gamma)$ and all $1 \le i, j \le d_\gamma$, then $F^\gamma = id$.

Proof. For convenience, in the proof we denote F^γ by F for $\gamma \in \widehat{\mathcal{G}}$. Recall that for $\gamma_1, \gamma_2 \in \widehat{\mathcal{G}}$, $1 \le m, k \le d_{\gamma_1}$ and $1 \le n, l \le d_{\gamma_2}$, we have that

$$h(u^{\gamma_1}_{mk} u^{\gamma_2 *}_{nl}) = \frac{\delta_{\gamma_1 \gamma_2} \delta_{mn} F_{lk}}{M_{\gamma_1}},$$

and

$$h(u^{\gamma_1 *}_{km} u^{\gamma_2}_{ln}) = \frac{\delta_{\gamma_1 \gamma_2} \delta_{mn} (F^{-1})_{lk}}{M_{\gamma_1}}.$$

Hence

$$\begin{aligned}
&(\varphi \otimes h)\alpha(e_{\gamma kj} e^*_{\gamma ki}) \\
&= \sum_{1 \le s,t \le d_\gamma} \varphi(e_{\gamma ks} e^*_{\gamma kt}) h(u^\gamma_{sj} (u^\gamma_{ti})^*) \\
&= \sum_{1 \le s,t \le d_\gamma} \varphi(e_{\gamma ks} e^*_{\gamma kt}) \delta_{st} \frac{F_{ij}}{M_\gamma} \\
&= \sum_{1 \le s \le d_\gamma} \varphi(e_{\gamma ks} e^*_{\gamma ks}) \frac{F_{ij}}{M_\gamma},
\end{aligned}$$

and

$$\begin{aligned}
&(\varphi \otimes h)\alpha(e^*_{\gamma ki} e_{\gamma kj}) \\
&= \sum_{1 \le s,t \le d_\gamma} \varphi(e^*_{\gamma ks} e_{\gamma kt}) h((u^\gamma_{si})^* u^\gamma_{tj}) \\
&= \sum_{1 \le s,t \le d_\gamma} \varphi(e^*_{\gamma ks} e_{\gamma kt}) (F^{-1})_{ts} \frac{\delta_{ij}}{M_\gamma}.
\end{aligned}$$

From $(\varphi \otimes h)\alpha(e_{\gamma kj} e^*_{\gamma ki}) = (\varphi \otimes h)\alpha(e^*_{\gamma ki} e_{\gamma kj})$ and $\sum_{1 \le s \le d_\gamma} \varphi(e_{\gamma ks} e^*_{\gamma ks}) > 0$, we have that

$$F_{ij} = \frac{\sum_{1 \le s,t \le d_\gamma} \varphi(e^*_{\gamma ks} e_{\gamma kt}) (F^{-1})_{ts} \delta_{ij}}{\sum_{1 \le s \le d_\gamma} \varphi(e_{\gamma ks} e^*_{\gamma ks})},$$

which implies that F is a scalar matrix under a fixed orthonormal basis of H_γ. Note that $tr(F) = tr(F^{-1})$, hence we get $F = I$ under a fixed orthonormal basis of H_γ, which means that $F = id$. $\square$

Proposition 3.14. Suppose that a compact quantum group $\mathcal{G}$ acts on B by α. If one of the following two conditions is true:

(1) every invariant state on B is tracial,
(2) there exists a faithful tracial invariant state,

then for all $\gamma \in \widehat{\mathcal{G}}$ such that $\mathrm{mul}(B, \gamma) > 0$, we have that $F^\gamma = id$.

Proof. Suppose that every invariant state on B is tracial. Note that $(\varphi \otimes h)\alpha$ is an α-invariant state for any $\varphi \in S(B)$. By assumption $(\varphi \otimes h)\alpha$ is tracial. For any $\gamma \in \widehat{\mathcal{G}}$ with $\mathrm{mul}(B, \gamma) > 0$, since for any $1 \leq k \leq \mathrm{mul}(B, \gamma)$, $\sum_{1 \leq s \leq d_\gamma} e_{\gamma ks} e^*_{\gamma ks} > 0$, there exists a $\varphi_\gamma \in S(B)$ satisfying that $\sum_{1 \leq s \leq d_\gamma} \varphi_\gamma(e_{\gamma ks} e^*_{\gamma ks}) > 0$. Hence by Lemma **??** we have that $F^\gamma = id$.

On the other hand, if there exists a faithful tracial invariant state on B, say ψ, then $(\psi \otimes h)\alpha = \psi$ and ψ satisfies the conditions of Lemma **??**. Hence $F^\gamma = id$ for all $\gamma \in \widehat{\mathcal{G}}$ with positive $\mathrm{mul}(B, \gamma)$. $\square$

Remark 3.15. A special case of Proposition **??** is the following:

If α is ergodic and the unique α-invariant state μ is tracial, then for all $\gamma \in \widehat{\mathcal{G}}$ such that $\mathrm{mul}(B, \gamma) > 0$, we have that $F^\gamma = id$.

A slightly different version of this result appears in [**?** , Theorem 3.1] where a necessary and sufficient condition of traciality of the unique invariant state of an ergodic action is given.

Theorem 3.16. Suppose that a compact quantum group $\mathcal{G}$ acts on B by α faithfully. If one of the following two conditions is true:

(1) every invariant state on B is tracial,
(2) there exists a faithful tracial invariant state on B,

then $\mathcal{G}$ is a Kac algebra.

Proof. Note that for all $\gamma \in \widehat{\mathcal{G}}$ and a unitary $u^\gamma \in \gamma$, it follows from [**?** , Theorem 5.4] that $(id \otimes \kappa^2)u^\gamma = F^\gamma u^\gamma (F^\gamma)^{-1}$. By Proposition **??**, we see that $F^\gamma = id$ for all $\gamma \in \widehat{\mathcal{G}}$ such that $\mathrm{mul}(B, \gamma) > 0$. So

$$\kappa^2(u^\gamma_{ij}) = u^\gamma_{ij}$$

for all $\gamma \in \widehat{\mathcal{G}}$ such that $\mathrm{mul}(B, \gamma) > 0$ and $1 \leq i, j \leq d_\gamma$. Note that κ^2 is a linear multiplicative map on $\mathscr{A}$. Hence κ^2 is the identity map when restricted on the algebra $\mathscr{A}'_2$ generated by u^γ_{ij}'s for all $\gamma \in \widehat{\mathcal{G}}$ such that $\mathrm{mul}(B, \gamma) > 0$ and $1 \leq i, j \leq d_\gamma$. If $\mathrm{mul}(B, \gamma) > 0$, then $\mathrm{mul}(B, \gamma^c) > 0$. Note that $\overline{u^\gamma} = (u^{\gamma*}_{ij})_{1 \leq i, j \leq d_\gamma} \in \gamma^c$ for all $\gamma \in \widehat{\mathcal{G}}$. So $u^{\gamma*}_{ij} \in \mathscr{A}'_2$ for all $\gamma \in \widehat{\mathcal{G}}$ such that $\text{mul}(B, \gamma) > 0$ and $1 \le i, j \le d_\gamma$, and $\mathscr{A}_2'$ is a $*$-algebra. Thus $\mathscr{A}_2' = \mathscr{A}_2$ where $\mathscr{A}_2$ is defined in Proposition **??** and is the $*$-algebra generated by u_{ij}^γ for all $\gamma \in \widehat{\mathcal{G}}$ such that $\text{mul}(B, \gamma) > 0$ and $1 \le i, j \le d_\gamma$.

Note that α is faithful, hence $\mathscr{A}_2 = \mathscr{A}$ by Proposition **??**. So $\kappa^2 = id$ on $\mathscr{A}$. This completes the proof. □

Remark 3.17. Theorem **??** includes Theorem 2.10 (i) in [**?**] as special cases.

However, the converse of Theorem **??** is not true.

By [**?** , Theorem 5.1], there exists an ergodic and faithful action α of $A_u(n)$ on the Cuntz algebra $\mathcal{O}_n$ by

$$\alpha(S_j) = \sum_{i=1}^{n} S_i \otimes u_{ij}.$$

Although $A_u(n)$ is a Kac algebra, there is no tracial state on $\mathcal{O}_n$.

4. Actions on compact Hausdorff spaces

In this section, we consider a compact quantum group $\mathcal{G}$ acts on a compact Hausdorff space X by α and denote $C(X)$ by B. Let ev_x be the evaluation functional on B at $x \in X$, i.e., $ev_x(f) = f(x)$ for all $f \in B$.

4.1. **Compact quantum group orbit.**

We define compact quantum group orbits and derive some basic properties.

Definition 4.1. Let $\mathcal{G}$ act on X by α. For $x \in X$. We call the subset

$$\{x' \in X | (ev_x \otimes h)\alpha = (ev_{x'} \otimes h)\alpha\}$$

of X the **orbit** of x, and denote it by Orb_x.

For a closed subset Y of X, let $J_Y = \{f \in B | \, f = 0 \,\text{on}\, Y\}$ and π_Y be the quotient map from B onto B/J_Y. Suppose that a compact quantum group $\mathcal{G}$ acts on X by α. We say that Y is an α-**invariant subset** of X if J_Y is an α-invariant ideal of B.

Define the induced action α_Y of $\mathcal{G}$ on Y by $\alpha_Y(f + J_Y) = (\pi_Y \otimes id)\alpha(f)$ for $f \in B$. For a state μ on B, since B is commutative, $N_\mu = \{f \in B | \mu(f^* f) = 0\}$ is a two-sided ideal of B. Let $X_\alpha = \{x \in X | f(x) = 0 \text{ for all}\, f \in \ker \alpha\}$.

We now give another characterization of invariant subsets. First we need the following lemma.

Lemma 4.2. For a closed subset Y of X and $f \in B$, $(\pi_Y \otimes id)\alpha(f) = 0$ if and only if $(ev_x \otimes id)\alpha(f) = \alpha(f)(x) = 0$ for all x in Y.

Proof. Suppose that $(\pi_Y \otimes id)\alpha(f) = 0$. For any x in Y, we define a linear functional $\widetilde{ev_x}$ on B/J_Y by $\widetilde{ev_x}(f + J_Y) = f(x)$ for all $f \in B$. If $f \in J_Y$, then $f(x) = 0$ for all x in Y. Hence $\widetilde{ev_x}$ is well-defined. Furthermore, $\widetilde{ev_x}\pi_Y = ev_x$. Applying $\widetilde{ev_x} \otimes id$ to both sides of $(\pi_Y \otimes id)\alpha(f) = 0$, we get $(ev_x \otimes id)\alpha(f) = 0$ for all x in Y.

On the other hand, for all x in Y and some $f \in B$, if $(ev_x \otimes id)\alpha(f) = 0$, then $(\widetilde{ev_x}\pi_Y \otimes id)\alpha(f) = 0$. Note that $(\pi_Y \otimes id)\alpha(f) \in (B/J_Y) \otimes A \cong C(Y) \otimes A \cong C(Y, A)$. Hence for all $x \in Y$, if $(\widetilde{ev_x} \otimes id)(\pi_Y \otimes id)\alpha(f) = 0$, then $(\pi_Y \otimes id)\alpha(f) = 0$. □

Using Lemma **??**, we have the following.

Proposition 4.3. A closed subset Y of X is α-invariant if and only if $(ev_x \otimes id)\alpha(f) = 0$ for all x in Y and f in J_Y.

Next, we show that every orbit is an invariant subset.

Recall that $B^\alpha \cong C(Y_\alpha)$ and we denote the canonical quotient map from X onto Y_α by π. Then we have the following,

Lemma 4.4. For every $y \in Y_\alpha$, two points x_1 and x_2 are in $\pi^{-1}(y)$ if and only if x_1 and x_2 are in the same orbit.

Proof. Note that $B^\alpha = (id \otimes h)\alpha(B)$. We have that $x_1, x_2 \in \pi^{-1}(y)$ for $y \in Y_\alpha$ if and only if

$$(ev_{x_1} \otimes h)\alpha(g) = (ev_y \otimes h)\alpha(g) = (ev_{x_2} \otimes h)\alpha(g)$$

for every $g \in B$. That is to say, x_1 and x_2 are in the same orbit. □

Theorem 4.5. For every $x \in X$, the orbit Orb_x is an α-invariant subset of X.

Proof. By Proposition **??**, it suffices to show that for any $f \in C(X)$, if $f|_{\mathrm{Orb}_x} = 0$, then $(ev_{x'} \otimes id)\alpha(f) = 0$ for every $x' \in \mathrm{Orb}_x$.

By Lemma **??**, there exists $y \in Y_\alpha$ such that $\pi^{-1}(y) = \mathrm{Orb}_x$.

Let $f \in B$ such that $f|_{\mathrm{Orb}_x} = 0$. For arbitrary $\varepsilon > 0$, denote the closed subset $\{x \in X \,|\, |f(x)| \geq \varepsilon\}$ by E_ε. Both X and Y_α are compact Hausdorff spaces,

hence $\pi(E_\varepsilon)$, denoted by K_ε, is also compact and Hausdorff. Since $y \notin K_\varepsilon$, by Urysohn's Lemma, there exists a $g_\varepsilon \in B^\alpha$, such that $0 \leq g_\varepsilon \leq 1$, $g_\varepsilon(y) = 0$ and $g_\varepsilon|_{K_\varepsilon} = 1$. Since B^α is a C*-subalgebra of B, the function g_ε is also in B and satisfies that $0 \leq g_\varepsilon \leq 1$, $g_\varepsilon|_{\mathrm{Orb}_x} = 0$ and $g_\varepsilon|_{E_\varepsilon} = 1$.

Now consider $f - fg_\varepsilon$. Then $|f(x) - g_\varepsilon(x)f(x)| = 0$ for every x in E_ϵ, and $|f(x) - g_\varepsilon(x)f(x)| < \varepsilon$ for all $x \in X \setminus E_\varepsilon$ since $|f(x)| < \varepsilon$ and

$0 \leq g_\varepsilon \leq 1$. Therefore $||f - fg_\varepsilon|| < \varepsilon$ which implies

$$||(ev_{x'} \otimes id)\alpha(f) - (ev_{x'} \otimes id)\alpha(fg_\varepsilon)|| < \varepsilon$$

for every $x' \in X$.

Note that $g_\varepsilon \in B^\alpha$ and $g_\varepsilon|_{\mathrm{Orb}_x} = 0$. For every $x' \in \mathrm{Orb}_x$, we have that

$$(ev_{x'} \otimes id)\alpha(fg_\varepsilon) = (ev_{x'} \otimes id)(\alpha(f)(g_\varepsilon \otimes 1)) = (ev_{x'} \otimes id)\alpha(f)g_\varepsilon(x') = 0.$$

Consequently, $||(ev_{x'} \otimes id)\alpha(f)|| < \varepsilon$ for all $x' \in \mathrm{Orb}_x$. Note that ε is arbitrary. So $(ev_{x'} \otimes id)\alpha(f) = 0$ for every $x' \in \mathrm{Orb}_x$. This ends the proof. □

Theorem 4.6. The action α is ergodic iff $\mathrm{Orb}_x = X$ for some $x \in X$.

Proof. Suppose that α is ergodic. Then $(id \otimes h)\alpha(f)$ is a constant function on X for every $f \in B$. Therefore, $(ev_x \otimes h)\alpha(f) = (ev_{x'} \otimes h)\alpha(f)$ for all x and x' in X. Consequently $\mathrm{Orb}_x = X$.

If there exists $x \in X$ such that $\mathrm{Orb}_x = X$. We have that $(ev_y \otimes h)\alpha(f) = (ev_x \otimes h)\alpha(f)$ for every $f \in B$ and $y \in X$. So $(id \otimes h)\alpha(f)$ is a constant function on X for every $f \in B$. Therefore α is ergodic. □

4.2. **Non-atomic invariant measures.**

We first prove the following result.

Theorem 4.7. If a compact quantum group $\mathcal{G}$ acts ergodically by α on a compact metrizable space X with infinitely many points, then the unique α-invariant measure μ of X is non-atomic. That is, every point of X has zero μ-measure.

Denote $C(X)$ by B. For $y \in X$, denote by e_y the characteristic function of $\{y\}$. For a compact quantum group action $\alpha : B \to B \otimes A$, we use ev_x to denote the evaluation functional on B at a point $x \in X$.

Take a regular Borel probability measure μ on X. Denote $\mu(\{x\})$ by μ_x and define a linear functional ν_x on B by $\nu_x(f) = f(x)\mu_x$ for all $f \in B$. With abuse of notation, we also use μ to denote the corresponding linear functional on B such that $\mu(f) = \int_X f \, d\mu$ for $f \in B$. For a subset U of X, if an $f \in B$ satisfies that $0 \leq f \leq 1$ and $f|_U = 1$, then we write it as $U \prec f$. If f satisfies that $0 \leq f \leq 1$ and supportof $f \subseteq U$, then we denote it by $f \prec U$.

Before proceeding to the main theorem, we prove two preliminary lemmas.

Lemma 4.8. Suppose that a compact quantum group $\mathcal{G}$ acts on a compact Hausdorff space X by α. Take an α-invariant measure μ on X. If $\mu_x > \mu_y$ for two points x and y in X, then there exists an open neighborhood V of y satisfying that $(ev_x \otimes id)\alpha(g) = 0$ for all $g \in B$ with $g \prec V$.

Proof. Note that μ is a state on B and X is a compact Hausdorff space. Hence μ is a regular Borel measure on X by the Riesz representation theorem. Since $\mu_x > \mu_y$, there exists an open neighborhood U of y such that $\mu_x > \mu(U)$. We claim that

$$\|(ev_x \otimes id)\alpha(f)\| < 1$$

for all $f \in B$ with $f \prec U$. Since $0 \leq f \leq 1$, we have that $\|(ev_x \otimes id)\alpha(f)\| \leq 1$. If $\|(ev_x \otimes id)\alpha(f)\| = 1$, then there exists a state ϕ on A such that $\phi((ev_x \otimes id)\alpha(f)) = \|(ev_x \otimes id)\alpha(f)\| = 1$ since $(ev_x \otimes id)\alpha(f) \geq 0$. Moreover,

$$\begin{aligned}(\mu \otimes \phi)\alpha(f) = \phi((\mu \otimes id)\alpha(f)) &= \phi(\int_X (ev_x \otimes id)\alpha(f)\, d\mu) \\ &\geq \phi((ev_x \otimes id)\alpha(f))\mu_x = \mu_x.\end{aligned}$$

Since μ is α-invariant, on the other hand

$$(\mu \otimes \phi)\alpha(f) = \phi((\mu \otimes id)\alpha(f)) = \phi(\mu(f)1_A) = \mu(f).$$

Therefore combining these, we get that $\mu(f) \geq \mu_x$. Since $f \prec U$, we also have that $\mu_x > \mu(U) \geq \mu(f)$. This leads to a contradiction. Hence $\|(ev_x \otimes id)\alpha(f)\| < 1$ for all $f \in B$ with $f \prec U$.

Since X is a compact Hausdorff space, there exist an open subset V and a compact subset K of X such that $y \in V \subseteq K \subseteq U$.

By Urysohn's lemma, there is an $f \in B$ such that $K \prec f \prec U$. For any $g \in B$ with $g \prec V$, we see that $0 \leq g \leq f^n$ for every positive integer n. Thus

$$\|(ev_x \otimes id)\alpha(g)\| \leq \|(ev_x \otimes id)\alpha(f^n)\| = \|(ev_x \otimes id)\alpha(f)\|^n \to 0$$

as $n \to \infty$. Therefore $(ev_x \otimes id)\alpha(g) = 0$. ☐

Lemma 4.9. Suppose that a compact quantum group $\mathcal{G}$ has the faithful Haar state and acts ergodically by α on a compact Hausdorff space X with infinitely many points. Denote the unique α-invariant measure on X by μ. Assume that there exists some $x \in X$ such that $\mu_x > 0$. Let $E_1 = \{y \in X | \mu_y = \max\{\mu_x | x \in X\}\}$. For any $f \in B$, if $f|_{E_1} = 0$, we have $\alpha(f) = 0$.

Proof. First E_1 is a finite subset of X since μ is a finite measure on X. Let $E_1 = \{x_1, ..., x_n\}$ and $ev_i = ev_{x_i}$ for $1 \leq i \leq n$. For any $\epsilon > 0$, there exists an open neighborhood V_i of x_i for each $x_i \in E_1$ such that $|f(x)| < \epsilon$ for all $x \in \bigcup_{i=1}^n V_i$. For any $y \notin E_1$, by Lemma **??**, there exists an open neighborhood V_y of y such that $V_y \bigcap E_1 = \emptyset$ and $(ev_i \otimes id)\alpha(g) = 0$ for all $g \in B$ with $g \prec V_y$ and all $1 \leq i \leq n$. Then $\mathcal{V} = \{V_y\}_{y \notin E_1} \bigcup \{V_i\}_{i=1}^n$ is an open cover of X. Since X is a compact

Hausdorff space, there exists a finite subcover $\mathcal{V}'$ of $\mathcal{V}$. Let $\{g_V\}_{V\in\mathcal{V}'}$ be a partition of unity of X subordinate to $\mathcal{V}'$. Then $f = \sum_{V\in\mathcal{V}'} fg_V$.

Now let $i = 1$ for convenience. By Lemma **??**, we have that $(ev_1 \otimes id)\alpha(g_V) = 0$ for all $V \in \mathcal{V}' \setminus \{V_i\}_{i=1}^n$. Hence

$$\begin{aligned}(ev_1 \otimes id)\alpha(f) &= (ev_1 \otimes id)\alpha(\sum_{V\in\mathcal{V}'} fg_V) \\ &= \sum_{V\in\mathcal{V}' \bigcap \{V_i\}_{i=1}^n} (ev_1 \otimes id)\alpha(fg_V) + \sum_{V\in\mathcal{V}'\setminus\{V_i\}_{i=1}^n} (ev_1 \otimes id)\alpha(fg_V) \\ &= \sum_{V\in\mathcal{V}' \bigcap \{V_i\}_{i=1}^n} (ev_1 \otimes id)\alpha(fg_V).\end{aligned}$$

Take any $x \in X$. If $x \in \bigcup_{i=1}^n V_i$, then $|\sum_{V\in\mathcal{V}' \bigcap\{V_i\}_{i=1}^n} f(x)g_V(x)| \leq |f(x)| < \epsilon$. If $x \notin \bigcup_{i=1}^n V_i$, then $\sum_{V\in\mathcal{V}' \bigcap\{V_i\}_{i=1}^n} f(x)g_V(x) = 0$. Therefore $\| \sum_{V\in\mathcal{V}' \bigcap\{V_i\}_{i=1}^n} fg_V\| \leq \epsilon$.

Thus

$$\|(ev_1 \otimes id)\alpha(f)\| = \|(ev_1 \otimes id)\alpha(\sum_{V\in\mathcal{V}' \bigcap\{V_i\}_{i=1}^n} fg_V)\| \leq \epsilon.$$

Since ϵ is arbitrary, we have that $(ev_1 \otimes id)\alpha(f) = 0$. Note that $(ev_1 \otimes id)\alpha$ is a $*$-homomorphism, so $(ev_1 \otimes id)\alpha(f^*f) = 0$. The action α is ergodic, hence $(ev_x \otimes h)\alpha(f^*f) = (ev_1 \otimes h)\alpha(f^*f) = 0$ for any $x \in X$. The Haar state h is faithful and $(ev_x \otimes id)\alpha(f^*f) \geq 0$, therefore $(ev_x \otimes id)\alpha(f^*f) = 0$ for all $x \in X$ which means $\alpha(f) = 0$. □

Now we are ready to prove the main theorem in this subsection.

Proof. [Proof of Theorem **??**]

We can assume the Haar state h of $\mathcal{G}$ is faithful otherwise we replace α by the reduced compact quantum group action α_r of $\mathcal{G}_r$ which has the faithful Haar state. The action α_r is also ergodic by Corollary **??**. Moreover, a state on B is α-invariant if and only if it is α_r-invariant (see the argument preceding Corollary **??**).

Suppose that $\mu(\{x\}) > 0$ for some $x \in X$. Define $E_1 = \{x_1, ..., x_n\}$ as in Lemma **??**. Let $\mathscr{B}$ be the Podlés algebra of $B = C(X)$. Define a linear map T from $\alpha(\mathscr{B})$ into $\mathbb{C}^n$ by

$$T(\alpha(f)) = (f(x_1), f(x_2), ..., f(x_n))$$

for all $f \in \mathscr{B}$. Note that α is injective on $\mathscr{B}$. So T is well-defined. Also T is linear. By Lemma **??**, T is injective. The space X contains infinitely many points, hence B is infinite dimensional. Since $\mathscr{B}$ is a dense subspace of B, we have that $\mathscr{B}$ is also infinite dimensional. This leads to a contradiction to that $\mathbb{C}^n$ is finite dimensional and that T is injective. □

Now we consider compact quantum group actions on a compact Hausdorff space X_∞ with countably infinitely many points. We complete the paper with the following main result by using Theorem **??**.

Theorem 4.10. X_∞ is not a quantum homogeneous space.

Proof. For every Borel probability measure μ on X_∞, there exists an $x \in X_\infty$ such that $\mu(\{x\}) > 0$. So by Theorem **??**, the space X_∞ cannot admit an ergodic compact quantum group action. □

References

[1] T. Banica. Fusion rules for representations of compact quantum groups. *Exposition. Math.* **17** (1999), no. 4, 313–337.

[2] T. Banica. Representations of compact quantum groups and subfactors. *J. Reine Angew. Math.* **509** (1999), 167–198.

[3] T. Banica and A. Skalski. Quantum symmetry groups of C^*-algebras equipped with orthogonal filtrations. *Proc. Lond. Math. Soc.* (3) **106** (2013), 980-1004.

[4] J. Bhowmick, D. Goswami and A. Skalski. Quantum isometry groups of 0-dimensional manifolds. *Trans. Amer. Math. Soc.* **363** (2011), no. 2, 901–921.

[5] E. Bédos, G. J. Murphy and L. Tuset. Co-amenability of compact quantum groups. *J. Geom. Phys.* **40** (2001), no. 2, 130–153.

[6] F. P. Boca. Ergodic actions of compact matrix pseudogroups on C^*-algebras. *Recent advances in operator algebras,* Orlèans, 1992, Astérisque **232** (1995), 93–109.

[7] J. Cuntz. Simple C^*-algebras generated by isometries. *Comm. Math. Phys.* **57** (1977), no. 2, 173–185.

[8] D. Goswami. Quantum symmetries and quantum isometries of compact metric spaces. arXiv: 0811.0095 v7.

[9] R. Høegh-Krohn, M.B. Landstad and E. Størmer. Compact ergodic groups of automorphisms. *Ann. of Math.* **114** (1981), no.1, 75–86.

[10] H. Huang. Faithful compact quantum group actions on connected compact metrizable spaces. *J. Geom. Phys.* **70** (2013), 232-236.

[11] H. Li. Compact quantum metric spaces and ergodic actions of compact quantum groups. *J. Funct. Anal.* **256** (2009), no. 10, 3368–3408.

[12] M. Maes and A. Van Daele. Notes on compact quantum groups. *Nieuw Arch. Wisk.* (4) **16** (1998), no. 1-2, 73–112.

[13] C. Pinzari. Growth rates of dimensional invariants of compact quantum groups and a theorem of Høegh-Krohn, Landstad and Størmer. *Proc. Amer. Math. Soc.* **141** (2013), no. 3, 895-907.

[14] P. Podleś. Symmetries of quantum spaces. Subgroups and quotient spaces of quantum SU(2) and SO(3) groups. *Comm. Math. Phys.* **170** (1995), no.1, 1–20.

[15] S. Vaes. A new approach to induction and imprimitivity results. *J. Funct. Anal.* **229** (2005), no. 2, 317-374.

[16] A. Van Daele and S. Wang. Universal quantum groups. *International J. Math.* **7** (1996), no. 2, 255–264.

[17] S. Wang. Free products of compact quantum groups. *Comm. Math. Phys.* **167** (1995), no. 3, 671–692.

[18] S. Wang. Quantum symmetry groups of finite spaces. *Comm. Math. Phys.* **915** (1998), no. 1, 195–211.

[19] S. Wang. Ergodic actions of universal quantum groups on operator algebras. *Comm. Math. Phys.* **203** (1999), no. 2, 481–498.

[20] S. L. Woronowicz. Compact matrix pseudogroups. *Comm. Math. Phys.* **111** (1987), no. 4, 613–665.

[21] S. L. Woronowicz. Compact quantum groups. In: *Symétries Quantiques (Les Houches, 1995)*, pp. 845–884, North-Holland, Amsterdam, 1998.

Huichi Huang, College of Mathematics and Statistics, Chongqing University, Chongqing, 401331, PR China

E-mail address: huichi-huang@hotmail.com