

# FOURIER COEFFICIENTS OF $\times p$ -INVARIANT MEASURES

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ABSTRACT. We consider densities  $D_\Sigma(A)$ ,  $\overline{D}_\Sigma(A)$  and  $\underline{D}_\Sigma(A)$  for a subset  $A$  of  $\mathbb{N}$  with respect to a sequence  $\Sigma$  of finite subsets of  $\mathbb{N}$  and study Fourier coefficients of ergodic, weakly mixing and strongly mixing  $\times p$ -invariant measures on the unit circle  $\mathbb{T}$ . Combining these, we prove the following measure rigidity results: on  $\mathbb{T}$ , the Lebesgue measure is the only non-atomic  $\times p$ -invariant measure satisfying one of the following: (1)  $\mu$  is ergodic and there exist a Følner sequence  $\Sigma$  in  $\mathbb{N}$  and a nonzero integer  $l$  such that  $\mu$  is  $\times(p^j + l)$ -invariant for all  $j$  in a subset  $A$  of  $\mathbb{N}$  with  $D_\Sigma(A) = 1$ ; (2)  $\mu$  is weakly mixing and there exist a Følner sequence  $\Sigma$  in  $\mathbb{N}$  and a nonzero integer  $l$  such that  $\mu$  is  $\times(p^j + l)$ -invariant for all  $j$  in a subset  $A$  of  $\mathbb{N}$  with  $\overline{D}_\Sigma(A) > 0$ ; (3)  $\mu$  is strongly mixing and there exists a nonzero integer  $l$  such that  $\mu$  is  $\times(p^j + l)$ -invariant for infinitely many  $j$ . Moreover, a  $\times p$ -invariant measure satisfying (2) or (3) is either a Dirac measure or the Lebesgue measure.

As an application we prove that for every increasing function  $\tau$  defined on positive integers with  $\lim_{n \rightarrow \infty} \tau(n) = \infty$ , there exists a multiplicative semigroup  $S_\tau$  of  $\mathbb{Z}^+$  containing  $p$  such that  $|S_\tau \cap [1, n]| \leq (\log_p n)^{\tau(n)}$  and the Lebesgue measure is the only non-atomic ergodic  $\times p$ -invariant measure which is  $\times q$ -invariant for all  $q$  in  $S_\tau$ .

## 1. INTRODUCTION

There are two motivations for this paper. Both are related to the celebrated  $\times p, \times q$  conjecture by H. Furstenberg. The first motivation is Lyons' Theorem and Rudolph-Johnson's Theorem, and the second is a theorem due to E. A. Sataev and later independently discovered by M. Einsiedler and A. Fish.

For an integer  $p$ , consider the group homomorphism  $T_p$  (called the  $\times p$  map) on the unit circle  $\mathbb{T} = \mathbb{R}/\mathbb{Z}$  given by  $T_p(x) = px \pmod{\mathbb{Z}}$  for all  $x$  in  $\mathbb{R}/\mathbb{Z}$ .

When  $p$  and  $q$  are positive integers greater than 1 with  $\frac{\log p}{\log q} \notin \mathbb{Q}$ , H. Furstenberg gave a classification of  $\times p, \times q$ -invariant closed subsets in  $\mathbb{T}$  [Fur67, Thm. IV.1].

**Theorem 1.1.** [Furstenberg, 1967]

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A  $\times p, \times q$ -invariant closed subset in  $\mathbb{T}$  is either finite or  $\mathbb{T}$ .

Motivated by this, H. Furstenberg conjectured a classification of  $\times p, \times q$ -invariant measures.

**Conjecture.** *[Furstenberg's  $\times p, \times q$  conjecture]*

For two positive integers  $p, q \geq 2$  with  $\frac{\log p}{\log q} \notin \mathbb{Q}$ , an ergodic  $\times p, \times q$ -invariant measure on  $\mathbb{T}$  is either finitely supported or the Lebesgue measure. That is, the only non-atomic  $\times p, \times q$ -invariant measure on  $\mathbb{T}$  is the Lebesgue measure.

The first progress was made by R. Lyons in 1988 [Lyo88, Thm. 1].

**Theorem 1.2.** *[Lyons' theorem]*

Suppose  $p, q$  are relatively prime. The Lebesgue measure is the only non-atomic  $\times p, \times q$ -invariant measure which is  $T_p$ -exact.

A measure  $\mu$  is  $T_p$ -exact means  $h_\mu(T_p, \xi) > 0$  for any nontrivial finite partition  $\xi$  of  $\mathbb{T}$ , where  $h_\mu(T_p, \xi)$  stands for the measure entropy of  $T_p$  with respect to a finite partition  $\xi$ .

In 1990, D. Rudolph improved Lyons' theorem to the following [Rud90, Thm. 4.9].

**Theorem 1.3.** *[Rudolph's theorem]*

Suppose  $p, q$  are relatively prime. A  $\times p, \times q$ -invariant measure  $\mu$  with  $h_\mu(T_p) = \sup_\xi h_\mu(T_p, \xi) > 0$  must be the Lebesgue measure.

Rudolph's theorem was strengthened by A. S. A. Johnson [Joh82, Thm. A].

**Theorem 1.4.** *[Rudolph-Johnson's theorem]*

Suppose that  $\frac{\log p}{\log q} \notin \mathbb{Q}$ . Then a  $\times p, \times q$ -invariant measure with  $h_\mu(T_p) > 0$  is the Lebesgue measure.

In this paper by assuming that a non-atomic  $\times p$ -invariant measure  $\mu$  satisfies weaker conditions than  $T_p$ -exactness or positive entropy, we prove that if  $\mu$  is invariant under enough many  $\times q$ -maps of special forms, then  $\mu$  is the Lebesgue measure.

**Theorem 5.1.** *The Lebesgue measure is the only non-atomic  $\times p$ -invariant measure on  $\mathbb{T}$  satisfying one of the following:*

- (1) *it is ergodic and there exist a nonzero integer  $l$  and a Følner sequence  $\Sigma = \{F_n\}_{n=1}^\infty$  in  $\mathbb{N}$  such that  $\mu$  is  $(p^j + l)$ -invariant for all  $j$  in some  $A \subseteq \mathbb{N}$  with  $D_\Sigma(A) = 1$ ;*

- (2) *it is weakly mixing and there exist a nonzero integer  $l$  and a Følner sequence  $\Sigma = \{F_n\}_{n=1}^\infty$  in  $\mathbb{N}$  such that  $\mu$  is  $\times(p^j + l)$ -invariant for all  $j$  in some  $A \subseteq \mathbb{N}$  with  $\overline{D}_\Sigma(A) > 0$ ;*
- (3) *it is strongly mixing and there exist a nonzero integer  $l$  and an infinite set  $A \subseteq \mathbb{N}$  such that  $\mu$  is  $\times(p^j + l)$ -invariant for all  $j$  in  $A$ .*

Moreover, a  $\times p$ -invariant measure satisfying (2) or (3) is either a Dirac measure or the Lebesgue measure.

Here  $D_\Sigma(A)$  and  $\overline{D}_\Sigma(A)$  are density and upper density of  $A$  with respect to  $\Sigma$  respectively. See Section 2 for their definitions.

The second motivation is a theorem which is independently discovered by E. A. Sataev in 1975 [Sat75, Thm. 1] and M. Einsiedler and A. Fish in 2010 [EF10, Thm. 1.2].

**Theorem 1.5.** *For a multiplicative semigroup  $S$  of positive integers with*

$$\liminf_{n \rightarrow \infty} \frac{\log |S \cap [1, n]|}{\log n} > 0,$$

*if a Borel probability measure on  $\mathbb{T}$  is an ergodic  $\times p$ -invariant measure for some  $p$  in  $S$  and is  $\times q$ -invariant for every  $q$  in  $S$ , then it is either finitely supported or Lebesgue measure.*

As an application of Theorem 5.1, we prove that there exists a multiplicative semigroup  $S$  of positive integers with  $\lim_{n \rightarrow \infty} \frac{\log |S \cap [1, n]|}{\log n} = 0$  such that Theorem 1.5 still holds (see Theorem 5.3).

The paper is organized as follows.

Firstly we give definitions of density functions of a subset  $A$  of nonnegative integers with respect to a Følner sequence. In Section 3, we lay down some basic facts about Fourier coefficients of a measure on the unit circle. In Section 4, we give the characterizations of ergodic, weakly mixing and strongly mixing  $\times p$ -invariant measures via their Fourier coefficients. In the last section, we prove the main theorem, Theorem 5.1. Applying it, we prove Theorem 5.3 at the end.

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## 2. PRELIMINARIES

Let  $\mathbb{N}$  stand for the set of nonnegative integers and  $\mathbb{Z}^+$  stand for the set of positive integers. Throughout this article, for two integers  $a < b$ , we denote the set  $\{a, \dots, b\}$  by  $[a, b]$ . Denote by  $|F|$  the cardinality of a set  $F$ .

The following definition of Følner sequence in  $\mathbb{N}$  is a special case of Følner sequences in an amenable semigroup [Bow71, p.2].

**Definition 2.1.** *A sequence  $\Sigma = \{F_n\}_{n=1}^\infty$  of finite subsets in  $\mathbb{N}$  is called a **Følner sequence** if*

$$\lim_{n \rightarrow \infty} \frac{|(F_n + m) \Delta F_n|}{|F_n|} = 0$$

for every  $m$  in  $\mathbb{N}$ .

The density  $D(A)$  of a subset  $A$  of  $\mathbb{N}$  is given by  $D(A) = \lim_{n \rightarrow \infty} \frac{|A \cap [0, n-1]|}{n}$ . The upper density of  $A$ ,  $\overline{D}(A) := \limsup_{n \rightarrow \infty} \frac{|A \cap [0, n-1]|}{n}$  and the lower density of  $A$ ,  $\underline{D}(A) := \liminf_{n \rightarrow \infty} \frac{|A \cap [0, n-1]|}{n}$ . These densities are defined via the sequence of finite subsets  $\{[0, n-1]\}_{n=1}^\infty$  in  $\mathbb{N}$ . Generalizing these, one can define densities of  $A$  with respect to every sequence of finite subsets of  $\mathbb{N}$ .

**Definition 2.2.** *Let  $\Sigma = \{F_n\}_{n=1}^\infty$  be a sequence of finite subsets of  $\mathbb{N}$ . The density  $D_\Sigma(A)$  of a subset  $A$  of  $\mathbb{N}$  with respect to  $\Sigma$  is given by*

$$D_\Sigma(A) = \lim_{n \rightarrow \infty} \frac{|A \cap F_n|}{|F_n|}.$$

The upper density  $\overline{D}_\Sigma(A) := \limsup_{n \rightarrow \infty} \frac{|A \cap F_n|}{|F_n|}$ , and the lower density  $\underline{D}_\Sigma(A) := \liminf_{n \rightarrow \infty} \frac{|A \cap F_n|}{|F_n|}$ .

**Remark 2.3.** (1) Denote  $\cup F_n$  by  $F$ . Then  $D_\Sigma(A) = D_\Sigma(A \cap F)$ .

- (2) The density  $D_\Sigma(A)$  depends on choices of  $\Sigma$ . For instance, let  $A = \cup_{n=1}^\infty [2^n, 2^n + n]$ . For the Følner sequence  $\Sigma = \{F_m\}_{m=1}^\infty$  with  $F_m = [1, m]$ , one has  $D_\Sigma(A) = 0$ . On the other hand  $D_{\Sigma'}(A) = 1$  for the Følner sequence  $\Sigma' = \{[2^n, 2^n + n]\}_{n=1}^\infty$ .

Within this paper, a measure on a compact metrizable  $X$  always means a Borel probability measure. A measure  $\mu$  is called **non-atomic** if  $\mu\{x\} = 0$  for every  $x$  in  $X$ .

A topological dynamical system consists of a compact metrizable space  $X$  and a continuous map  $T : X \rightarrow X$ .

A measure  $\mu$  on  $X$  is called  **$T$ -invariant** if  $\mu(B) = \mu(T^{-1}B)$  for any Borel subset  $B$  of  $X$ . A  $T$ -invariant measure  $\mu$  is called **ergodic** if every Borel subset  $B$  with  $T^{-1}B = B$  satisfies that  $\mu(B)^2 = \mu(B)$ , it is called **weakly mixing** if  $\mu \times \mu$  is an ergodic  $T \times T$ -invariant measure on  $X \times X$ , and it is called **strongly mixing** if  $\lim_{j \rightarrow \infty} \mu(T^{-j}A \cap B) = \mu(A)\mu(B)$  for all Borel subsets  $A, B$  in  $X$ .

It's well-known that strongly mixing  $\implies$  weakly mixing  $\implies$  ergodic.

Within this paper, we only consider that  $X = \mathbb{T}$  and  $T = T_p$  is the  $\times_p$  map on  $\mathbb{T} = \mathbb{R}/\mathbb{Z}$  defined by  $T_p(x) = px \pmod{\mathbb{Z}}$  for all  $x$  in  $\mathbb{R}/\mathbb{Z}$  and  $p$  in  $\mathbb{Z}$ .

### 3. SOME BASIC FACTS ABOUT FOURIER COEFFICIENTS

Denote the support of  $\mu$  by  $\text{Supp}(\mu)$ . For  $n$  in  $\mathbb{Z}$ , the **Fourier coefficient**  $\hat{\mu}(n)$  of a measure  $\mu$  on  $\mathbb{T}$  is given by  $\hat{\mu}(n) = \int_{\mathbb{T}} z^n d\mu(z)$  when taking  $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$ .

**Lemma 3.1.** For nonzero  $k$  in  $\mathbb{Z}$  and  $c$  in  $\mathbb{T}$ ,  $\int_{\mathbb{T}} |z^k - c|^2 d\mu(z) = 0$  if and only if  $\text{Supp}(\mu) \subseteq \{z \mid z^k = c\}$ .

*Proof.* Obvious. □

**Proposition 3.2.** For a nonzero integer  $k$ , one has  $|\hat{\mu}(k)| < 1$  if and only if there is no  $c$  in  $\mathbb{T}$  such that  $\text{Supp}(\mu) \subseteq \{z \mid z^k = c\}$ .

*Proof.* Let  $k$  be a nonzero integer. By Lemma 3.1, it suffices to show that  $|\hat{\mu}(k)| < 1$  if and only if  $\int_{\mathbb{T}} |z^k - c|^2 d\mu(z) > 0$  for all  $c \in \mathbb{T}$ .

If  $|\hat{\mu}(k)| < 1$ , then for any  $c \in \mathbb{T}$ , we have

$$\int_{\mathbb{T}} |z^k - c|^2 d\mu(z) = \int_{\mathbb{T}} (z^k - c)(\bar{z}^k - \bar{c}) d\mu(z)$$

$$= 2 - 2\operatorname{Re}(\bar{c}\hat{\mu}(k)) \geq 2 - 2|\hat{\mu}(k)| > 0.$$

Conversely assume that  $\int_{\mathbb{T}} |z^k - c|^2 d\mu(z) > 0$  for all  $c \in \mathbb{T}$ . When choosing  $c \in \mathbb{T}$  such that  $c\hat{\mu}(k) = |\hat{\mu}(k)|$ , we get  $0 < \int_{\mathbb{T}} |z^k - c|^2 d\mu(z) = 2 - 2|\hat{\mu}(k)|$ , which implies that  $|\hat{\mu}(k)| < 1$ .  $\square$

#### 4. FOURIER COEFFICIENTS OF ERGODIC, WEAKLY MIXING OR STRONGLY MIXING $\times p$ -INVARIANT MEASURES

In this section, we give characterizations of ergodic, weakly mixing and strongly mixing  $\times p$ -invariant measures via their Fourier coefficients.

**Theorem 4.1.** *The following are true.*

- (1) *A measure  $\mu$  on  $\mathbb{T}$  is an ergodic  $\times p$ -invariant measure if and only if*

$$\lim_{n \rightarrow \infty} \frac{1}{|F_n|} \sum_{j \in F_n} \hat{\mu}(kp^j + l) = \hat{\mu}(k)\hat{\mu}(l)$$

*for every Følner sequence  $\Sigma = \{F_n\}_{n=1}^{\infty}$  in  $\mathbb{N}$  and all  $k, l$  in  $\mathbb{Z}$ .*

- (2) *A measure  $\mu$  on  $\mathbb{T}$  is a weakly mixing  $\times p$ -invariant measure if and only if*

$$\lim_{n \rightarrow \infty} \frac{1}{|F_n|} \sum_{j \in F_n} |\hat{\mu}(kp^j + l) - \hat{\mu}(k)\hat{\mu}(l)|^2 = 0$$

*for every Følner sequence  $\Sigma = \{F_n\}_{n=1}^{\infty}$  in  $\mathbb{N}$  and all  $k, l$  in  $\mathbb{Z}$ .*

- (3) *A measure  $\mu$  on  $\mathbb{T}$  is a strongly mixing  $\times p$ -invariant measure if and only if*

$$\lim_{j \rightarrow \infty} \hat{\mu}(kp^j + l) = \hat{\mu}(k)\hat{\mu}(l)$$

*for all  $k, l$  in  $\mathbb{Z}$ .*

To prove (1) and (2) of Proposition 4.1, we need a preliminary result, which is a special case of von Neumann's mean ergodic theorem for amenable semigroups proved by Bowley [Bow71, Thm. 1].

**Lemma 4.2.** *For a topological dynamical system  $(X, T)$ , if  $\nu$  is an ergodic  $T$ -invariant measure on  $X$ , then for every Følner sequence  $\{F_n\}_{n=1}^{\infty}$  in  $\mathbb{N}$ , one has*

$$\lim_{n \rightarrow \infty} \frac{1}{|F_n|} \sum_{j \in F_n} f(T^j x) = \int_X f d\nu$$

for every  $f$  in  $L^2(X, \nu)$  (note that the identity holds with respect to  $L^2$ -norm). Consequently

$$\lim_{n \rightarrow \infty} \frac{1}{|F_n|} \sum_{j \in F_n} \int_X f(T^j x) g(x) d\nu(x) = \int_X f d\nu \int_X g d\nu$$

for every  $f, g$  in  $L^2(X, \nu)$

*Proof.* [Proof of Theorem 4.1]

- (1) Suppose  $\mu$  is an ergodic  $\times p$ -invariant measure on  $\mathbb{T}$ . Denote the  $\times p$  map by  $T_p$ . Consider the measurable dynamical system  $(\mathbb{T}, T_p, \mu)$ . Using Lemma 4.2, we get

$$\lim_{n \rightarrow \infty} \frac{1}{|F_n|} \sum_{j \in F_n} \int_{\mathbb{T}} f(T_p^j(x)) g(x) d\mu(x) = \int_{\mathbb{T}} f d\mu \int_{\mathbb{T}} g d\mu$$

for all continuous functions  $f, g$  on  $\mathbb{T}$ . By choosing  $f = z^k$  and  $g = z^l$ , we prove the necessity.

Now assume that  $\lim_{n \rightarrow \infty} \frac{1}{|F_n|} \sum_{j \in F_n} \hat{\mu}(kp^j + l) = \hat{\mu}(k)\hat{\mu}(l)$  for every Følner sequence  $\{F_n\}_{n=1}^{\infty}$  in  $\mathbb{N}$  and all  $k, l$  in  $\mathbb{Z}$ . Let  $l = 0$ , we have

$$\lim_{n \rightarrow \infty} \frac{1}{|F_n|} \sum_{j \in F_n} \hat{\mu}(kp^j) = \hat{\mu}(k)$$

for every  $k$  in  $\mathbb{Z}$ . Replacing  $k$  by  $kp$ , one has

$$\begin{aligned} \hat{\mu}(kp) &= \lim_{n \rightarrow \infty} \frac{1}{|F_n|} \sum_{j \in F_n} \hat{\mu}(kp^{j+1}) \\ &= \lim_{n \rightarrow \infty} \frac{1}{|F_n + 1|} \sum_{j \in F_n + 1} \hat{\mu}(kp^j) \end{aligned}$$

( $\{F_n + 1\}_{n=1}^{\infty}$  is a Følner sequence in  $\mathbb{N}$ .)

$$= \hat{\mu}(k)$$

for every  $k$  in  $\mathbb{N}$ . Hence  $\mu$  is  $\times p$ -invariant.

From  $\lim_{n \rightarrow \infty} \frac{1}{|F_n|} \sum_{j \in F_n} \hat{\mu}(kp^j + l) = \hat{\mu}(k)\hat{\mu}(l)$ , we have  $\lim_{n \rightarrow \infty} \frac{1}{|F_n|} \sum_{j \in F_n} \int_{\mathbb{T}} f(T_p^j x) g(x) d\mu(x) = \int_{\mathbb{T}} f \int_{\mathbb{T}} g$  for all polynomials on  $\mathbb{T}$ . Polynomials are dense in  $L^2(\mathbb{T}, \mu)$ , so  $\lim_{n \rightarrow \infty} \frac{1}{|F_n|} \sum_{j \in F_n} \int_{\mathbb{T}} f(T_p^j x) g(x) d\mu(x) = \int_{\mathbb{T}} f d\mu \int_{\mathbb{T}} g d\mu$  for all  $f, g \in L^2(\mathbb{T}, \mu)$ .

In particular, it is true for  $f = g = 1_A$  for a Borel subset  $A$  with  $T_p^{-1}A = A$ . Hence  $\mu(A) = \mu(A)^2$ . This proves that  $\mu$  is ergodic.

- (2) Suppose  $\mu$  is a weakly mixing  $\times p$ -invariant measure on  $\mathbb{T}$ , which means,  $\mu \times \mu$  is an ergodic  $T_p \times T_p$ -invariant measure on  $\mathbb{T}^2$ . Applying the second identity of Lemma 4.2 to  $X = \mathbb{T}^2$ ,  $\nu = \mu \times \mu$  and letting  $f(z_1, z_2) = z_1^k z_2^{-k}$  and  $g(z_1, z_2) = z_1^l z_2^{-l}$  for any  $k, l$  in  $\mathbb{Z}$ , we have

$$\lim_{n \rightarrow \infty} \frac{1}{|F_n|} \sum_{j \in F_n} |\hat{\mu}(kp^j + l)|^2 = |\hat{\mu}(k)|^2 |\hat{\mu}(l)|^2.$$

Note that

$$\begin{aligned} & |\hat{\mu}(kp^j + l) - \hat{\mu}(k)\hat{\mu}(l)|^2 \\ = & |\hat{\mu}(kp^j + l)|^2 + |\hat{\mu}(k)|^2 |\hat{\mu}(l)|^2 - \hat{\mu}(kp^j + l)\hat{\mu}(-k)\hat{\mu}(-l) - \hat{\mu}(-kp^j - l)\hat{\mu}(k)\hat{\mu}(l). \end{aligned}$$

So we get

$$\begin{aligned} & \lim_{n \rightarrow \infty} \frac{1}{|F_n|} \sum_{j \in F_n} |\hat{\mu}(kp^j + l) - \hat{\mu}(k)\hat{\mu}(l)|^2 \\ = & \lim_{n \rightarrow \infty} \frac{1}{|F_n|} \sum_{j \in F_n} [|\hat{\mu}(kp^j + l)|^2 + |\hat{\mu}(k)|^2 |\hat{\mu}(l)|^2 - \hat{\mu}(kp^j + l)\hat{\mu}(-k)\hat{\mu}(-l) - \hat{\mu}(-kp^j - l)\hat{\mu}(k)\hat{\mu}(l)] \end{aligned}$$

(Use (1) since that  $\mu$  is weakly mixing implies ergodicity of  $\mu$ .)

$$\begin{aligned} = & |\hat{\mu}(k)|^2 |\hat{\mu}(l)|^2 + |\hat{\mu}(k)|^2 |\hat{\mu}(l)|^2 - |\hat{\mu}(k)|^2 |\hat{\mu}(l)|^2 - |\hat{\mu}(k)|^2 |\hat{\mu}(l)|^2 = 0 \\ & \text{for all } k, l \text{ in } \mathbb{Z}. \end{aligned}$$

On the other hand, suppose

$$\lim_{n \rightarrow \infty} \frac{1}{|F_n|} \sum_{j \in F_n} |\hat{\mu}(kp^j + l) - \hat{\mu}(k)\hat{\mu}(l)|^2 = 0$$

for every Følner sequence  $\{F_n\}_{n=1}^\infty$  in  $\mathbb{N}$  and all  $k, l$  in  $\mathbb{Z}$ .

Firstly we have

$$\lim_{n \rightarrow \infty} \frac{1}{|F_n|} \sum_{j \in F_n} \hat{\mu}(kp^j + l) = \hat{\mu}(k)\hat{\mu}(l).$$

So by (1)  $\mu$  is an ergodic  $\times p$ -invariant measure. To prove  $\mu \times \mu$  is an ergodic  $T_p \times T_p$ -invariant measure on  $\mathbb{T}^2$ , it suffices to prove that

$$\lim_{n \rightarrow \infty} \frac{1}{|F_n|} \sum_{j \in F_n} \int_{\mathbb{T}^2} f((T_p \times T_p)^j(z_1, z_2)) g(z_1, z_2) d\mu(z_1) d\mu(z_2) = \int_{\mathbb{T}^2} f d\mu d\mu \int_{\mathbb{T}^2} g d\mu d\mu$$

for all continuous functions  $f$  and  $g$  on  $\mathbb{T}^2$ , which is equivalent to that

$$\lim_{n \rightarrow \infty} \frac{1}{|F_n|} \sum_{j \in F_n} \hat{\mu}(k_1 p^j + l_1) \hat{\mu}(k_2 p^j + l_2) = \hat{\mu}(k_1) \hat{\mu}(k_2) \hat{\mu}(l_1) \hat{\mu}(l_2)$$



for all  $k_1, k_2, l_1, l_2$  in  $\mathbb{Z}$  by letting  $f = z_1^{k_1} z_2^{k_2}$  and  $g = z_1^{l_1} z_2^{l_2}$  whose linear spans are dense in  $C(\mathbb{T}^2)$ .

Note that

$$\begin{aligned} & |\hat{\mu}(k_1 p^j + l_1) \hat{\mu}(k_2 p^j + l_2) - \hat{\mu}(k_1) \hat{\mu}(k_2) \hat{\mu}(l_1) \hat{\mu}(l_2)| \\ & \leq |\hat{\mu}(k_1 p^j + l_1) [\hat{\mu}(k_2 p^j + l_2) - \hat{\mu}(k_2) \hat{\mu}(l_2)]| + |[\hat{\mu}(k_1 p^j + l_1) - \hat{\mu}(k_1) \hat{\mu}(l_1)] \hat{\mu}(k_2) \hat{\mu}(l_2)| \\ & \leq |\hat{\mu}(k_2 p^j + l_2) - \hat{\mu}(k_2) \hat{\mu}(l_2)| + |\hat{\mu}(k_1 p^j + l_1) - \hat{\mu}(k_1) \hat{\mu}(l_1)| \end{aligned}$$

for all  $k_1, k_2, l_1, l_2$  in  $\mathbb{Z}$ .

Hence we have

$$\begin{aligned} & \lim_{n \rightarrow \infty} \frac{1}{|F_n|} \sum_{j \in F_n} |\hat{\mu}(k_1 p^j + l_1) \hat{\mu}(k_2 p^j + l_2) - \hat{\mu}(k_1) \hat{\mu}(k_2) \hat{\mu}(l_1) \hat{\mu}(l_2)|^2 \\ & \leq \lim_{n \rightarrow \infty} \frac{1}{|F_n|} \sum_{j \in F_n} [|\hat{\mu}(k_2 p^j + l_2) - \hat{\mu}(k_2) \hat{\mu}(l_2)| + |\hat{\mu}(k_1 p^j + l_1) - \hat{\mu}(k_1) \hat{\mu}(l_1)|]^2 \end{aligned}$$

(Cauchy-Schwarz inequality)

$$\leq 2 \lim_{n \rightarrow \infty} \frac{1}{|F_n|} \sum_{j \in F_n} [|\hat{\mu}(k_2 p^j + l_2) - \hat{\mu}(k_2) \hat{\mu}(l_2)|^2 + |\hat{\mu}(k_1 p^j + l_1) - \hat{\mu}(k_1) \hat{\mu}(l_1)|^2] = 0.$$

Using the inequality  $(\frac{|x_1| + \dots + |x_n|}{n})^2 \leq \frac{|x_1|^2 + \dots + |x_n|^2}{n}$ , we have

$$\lim_{n \rightarrow \infty} \frac{1}{|F_n|} \sum_{j \in F_n} |\hat{\mu}(k_1 p^j + l_1) \hat{\mu}(k_2 p^j + l_2) - \hat{\mu}(k_1) \hat{\mu}(k_2) \hat{\mu}(l_1) \hat{\mu}(l_2)| = 0.$$

This completes the proof.

- (3) If  $\mu$  is strongly mixing, then  $\lim_{j \rightarrow \infty} \mu(T_p^{-j} A \cap B) = \mu(A) \mu(B)$  for all Borel subsets  $A$  and  $B$  in  $\mathbb{T}$ . This means  $\lim_{j \rightarrow \infty} \int_{\mathbb{T}} 1_A(T_p^j x) 1_B(x) d\mu(x) = \int_{\mathbb{T}} 1_A d\mu \int_{\mathbb{T}} 1_B d\mu$  for all Borel subsets  $A$  and  $B$ , where  $1_A$  stands for the characteristic function of  $A$ .

Note that linear combinations of characteristic functions are dense in  $L^2(\mathbb{T}, \mu)$ , so  $\lim_{j \rightarrow \infty} \int_{\mathbb{T}} f(T_p^j x) g(x) d\mu(x) = \int_{\mathbb{T}} f d\mu \int_{\mathbb{T}} g d\mu$  for all  $f, g$  in  $C(\mathbb{T})$ . In particular, this holds for  $f = z^k$  and  $g = z^l$  for all  $k, l$  in  $\mathbb{Z}$ , which means

$$\lim_{j \rightarrow \infty} \hat{\mu}(k p^j + l) = \hat{\mu}(k) \hat{\mu}(l)$$

for all  $k, l \in \mathbb{Z}$ .

On the other hand, if a measure  $\mu$  satisfies that

$$\lim_{j \rightarrow \infty} \hat{\mu}(kp^j + l) = \hat{\mu}(k)\hat{\mu}(l)$$

for all  $k, l \in \mathbb{Z}$ . Let  $l = 0$  and replace  $k$  by  $kp$ . Then we have

$$\hat{\mu}(kp) = \lim_{j \rightarrow \infty} \hat{\mu}(kp^{j+1}) = \hat{\mu}(k)$$

for all  $k$  in  $\mathbb{Z}$ .

Linear combinations of  $z^k$  and  $z^l$  are polynomials on  $\mathbb{T}$ , which is dense in  $L^2(\mathbb{T}, \mu)$ . Hence

$$\lim_{j \rightarrow \infty} \mu(f(T_p^j)g) = \mu(f)\mu(g)$$

for all  $f, g$  in  $L^2(\mathbb{T}, \mu)$ . In particular, it holds for  $f = 1_A$  and  $g = 1_B$  for any Borel subsets  $A, B$  of  $\mathbb{T}$ , which completes the proof. □

**Remark 4.3.** (1) As shown in [Lyo88], a measure  $\mu$  is  $T_p$ -exact iff

$$\lim_{j \rightarrow \infty} \sup_{k \in \mathbb{Z}} |\hat{\mu}(kp^j + l) - \hat{\mu}(k)\hat{\mu}(l)| = 0$$

for every  $l$  in  $\mathbb{Z}$ . Hence  $T_p$ -exactness is much stronger than being strongly mixing.

(2) So far it is unknown how to characterize that  $h_\mu(T_p) > 0$  via Fourier coefficients of  $\mu$ .

## 5. RIGIDITY OF $\times p$ -INVARIANT MEASURES

With the above preliminaries, we are ready to prove the main theorem.

**Theorem 5.1.** *The Lebesgue measure is the only non-atomic  $\times p$ -invariant measure on  $\mathbb{T}$  satisfying one of the following:*

- (1) *it is ergodic and there exist a nonzero integer  $l$  and a Følner sequence  $\Sigma = \{F_n\}_{n=1}^\infty$  in  $\mathbb{N}$  such that  $\mu$  is  $\times(p^j + l)$ -invariant for all  $j$  in some  $A \subseteq \mathbb{N}$  with  $D_\Sigma(A) = 1$ ;*
- (2) *it is weakly mixing and there exist a nonzero integer  $l$  and a Følner sequence  $\Sigma = \{F_n\}_{n=1}^\infty$  in  $\mathbb{N}$  such that  $\mu$  is  $\times(p^j + l)$ -invariant for all  $j$  in some  $A \subseteq \mathbb{N}$  with  $\overline{D}_\Sigma(A) > 0$ ;*
- (3) *it is strongly mixing and there exist a nonzero integer  $l$  and an infinite set  $A \subseteq \mathbb{N}$  such that  $\mu$  is  $\times(p^j + l)$ -invariant for all  $j$  in  $A$ .*

Moreover, a  $\times p$ -invariant measures satisfying (2) or (3) is either a Dirac measure or the Lebesgue measure.

*Proof.* [Proof of the first part of Theorem 5.1]

- (1) Suppose  $\mu$  is an ergodic  $\times p$ -invariant measure and there exist a nonzero integer  $l$  and a Følner sequence  $\Sigma = \{F_n\}_{n=1}^\infty$  such that  $\mu$  is  $\times(p^j + l)$ -invariant for all  $j$  in some  $A \subseteq \mathbb{N}$  with  $D_\Sigma(A) = 1$ .

If  $\mu$  is not Lebesgue measure, then there exists nonzero  $k$  in  $\mathbb{Z}$  such that  $0\hat{\mu}(k)$  is nonzero.

By Theorem 4.1(1), one has

$$\lim_{n \rightarrow \infty} \frac{1}{|F_n|} \sum_{j \in F_n} \hat{\mu}(kp^j + kl) = \hat{\mu}(k)\hat{\mu}(kl).$$

Note that

$$\begin{aligned} \frac{1}{|F_n|} \sum_{j \in F_n} \hat{\mu}(kp^j + kl) &= \frac{1}{|F_n|} \left[ \sum_{j \in F_n \cap A} + \sum_{j \in F_n \setminus A} \right] \hat{\mu}(kp^j + kl) \\ &= \frac{|F_n \cap A|}{|F_n|} \hat{\mu}(k) + \frac{1}{|F_n|} \sum_{j \in F_n \setminus A} \hat{\mu}(kp^j + kl) \rightarrow \hat{\mu}(k) \end{aligned}$$

as  $n \rightarrow \infty$ . Hence  $\hat{\mu}(k) = \hat{\mu}(k)\hat{\mu}(kl)$ . This implies that  $\hat{\mu}(kl) = 1$  which contradict that  $\mu$  is non-atomic according to Proposition 3.2.

- (2) Suppose  $\mu$  is a weakly mixing  $\times p$ -invariant measure and there exist a nonzero integer  $l$  and a Følner sequence  $\Sigma = \{F_n\}_{n=1}^\infty$  such that  $\mu$  is  $\times(p^j + l)$ -invariant for all  $j$  in some  $A \subseteq \mathbb{N}$  with  $\overline{D}_\Sigma(A) > 0$ .

If  $\mu$  is not Lebesgue measure, then there exists nonzero  $k$  in  $\mathbb{Z}$  such that  $\hat{\mu}(k)$  is nonzero.

By Theorem 4.1(2), one has

$$\lim_{n \rightarrow \infty} \frac{1}{|F_n|} \sum_{j \in F_n} |\hat{\mu}(kp^j + kl) - \hat{\mu}(k)\hat{\mu}(kl)|^2 = 0.$$

It follows that

$$\begin{aligned} 0 &= \limsup_{n \rightarrow \infty} \frac{1}{|F_n|} \sum_{j \in F_n} |\hat{\mu}(kp^j + kl) - \hat{\mu}(k)\hat{\mu}(kl)|^2 \\ &\geq \limsup_{n \rightarrow \infty} \frac{1}{|F_n|} \sum_{j \in F_n \cap A} |\hat{\mu}(kp^j + kl) - \hat{\mu}(k)\hat{\mu}(kl)|^2 \\ &= \limsup_{n \rightarrow \infty} \frac{1}{|F_n|} \sum_{j \in F_n \cap A} |\hat{\mu}(k) - \hat{\mu}(k)\hat{\mu}(kl)|^2 \\ &= |\hat{\mu}(k) - \hat{\mu}(k)\hat{\mu}(kl)|^2 \overline{D}_\Sigma(A). \end{aligned}$$

Hence  $\hat{\mu}(k) - \hat{\mu}(k)\hat{\mu}(kl) = 0$  which implies that  $\hat{\mu}(kl) = 1$ . This again leads to a contradiction.

- (3) Assume that  $\mu$  is a strongly mixing  $\times p$ -invariant measure and there exist a nonzero integer  $l$  and an infinite  $A \subseteq \mathbb{N}$  such that  $\mu$  is  $\times(p^j + l)$ -invariant for all  $j$  in  $A$ .

If  $\mu$  is not Lebesgue measure, then there exists nonzero  $k$  in  $\mathbb{Z}$  such that  $\hat{\mu}(k)$  is nonzero.

By Theorem 4.1(3), we have

$$\lim_{\substack{j \rightarrow \infty \\ j \in A}} \hat{\mu}(kp^j + kl) = \hat{\mu}(k)\hat{\mu}(kl).$$

On the other hand, for all  $j \in A$ , one has  $\hat{\mu}(kp^j + kl) = \hat{\mu}(k)$ . So  $\hat{\mu}(k) = \hat{\mu}(k)\hat{\mu}(kl)$ . Again this leads to a contradiction.

We finish the proof the first part of Theorem 5.1. □

Before proceeding to the proof of the second part of Theorem 5.1, we need a lemma.

An **atom** for a measure  $\mu$  on a compact metrizable space  $X$  is a point  $x$  in  $X$  such that  $\mu\{x\} > 0$ .

**Lemma 5.2.** *Let  $T : X \rightarrow X$  be a continuous map on a compact metrizable space  $X$ . If a  $T$ -invariant measure  $\mu$  has an atom  $x$  with  $\mu\{x\} < 1$ , then  $\mu$  is not weakly mixing.*

*Proof.* Suppose  $\mu$  is weakly mixing and has an atom  $x$  with  $\lambda = \mu\{x\} < 1$ .

Note that  $\mu$  is weakly mixing if and only if

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} |\mu(T^{-j}A \cap B) - \mu(A)\mu(B)|^2 = 0$$

for all Borel subsets  $A, B$  of  $X$  [Wal82, Defn. 1.5(i)&Thm. 1.24].

Choose  $A = X \setminus \{x\}$  and  $B = \{x\}$ . Note that  $\mu(T^{-j}A \cap B)$  can only have two possible values: 0 or  $\lambda$ , hence for all  $j$ , we have

$$|\mu(T^{-j}A \cap B) - \mu(A)\mu(B)| \geq \min\{\lambda(1 - \lambda), \lambda - \lambda(1 - \lambda)\} \geq c$$

for some constant  $c > 0$ . This leads to a contradiction. □

Now we are ready to finish the proof of Theorem 5.1.

*Proof.* [Proof of the second part of Theorem 5.1]

Suppose  $\mu$  is a measure satisfying (2) or (3). By the first part of Theorem 5.1, if  $\mu$  is not a Lebesgue measure, then  $\mu$  has an atom. By Lemma 5.2, we obtain that  $\mu$  is a Dirac measure at some point  $z$  in  $\mathbb{T}$ .  $\square$

Next we prove the following.

**Theorem 5.3.** *Let  $\tau : \mathbb{Z}^+ \rightarrow \mathbb{R}$  be an arbitrary increasing function with  $\lim_{n \rightarrow \infty} \tau(n) = \infty$ . Then there exists a multiplicative semigroup  $S_\tau$  of  $\mathbb{Z}^+$  containing  $p$  and satisfying:*

- (1)  $|S_\tau \cap [1, n]| \leq (\log_p n)^{\tau(n)}$ ;
- (2) *the Lebesgue measure is the only non-atomic ergodic  $\times p$ -invariant measure which is  $\times q$ -invariant for all  $q$  in  $S_\tau$ .*

*In particular, there exists a multiplicative semigroup  $S$  of  $\mathbb{Z}^+$  containing  $p$  and satisfying:*

- (1)  $\lim_{n \rightarrow \infty} \frac{\log |S \cap [1, n]|}{\log n} = 0$ ;
- (2) *the Lebesgue measure is the only non-atomic ergodic  $\times p$ -invariant measure which is  $\times q$ -invariant for all  $q$  in  $S$ .*

*Proof.* Let  $\{l_n\}_{n=1}^\infty$  be a sequence of positive integers such that  $\lim_{n \rightarrow \infty} l_n = \infty$  and define  $f(m) = \sum_{n=1}^m l_n$  for every positive integer  $m$ .

Define  $g(m) = \min\{\log_p N \mid \tau(N) \geq 1 + f(m)\}$  for every positive integer  $m$ .

Define  $F_n = [p^{g(n)}, p^{g(n)} + l_n]$  for every positive integer  $n$ . Then  $\Sigma = \{F_n\}_{n=1}^\infty$  is a Følner sequence in  $\mathbb{N}$ . Denote  $\cup F_n$  by  $A$ .

Let  $S_\tau$  be the multiplicative semigroup generated by  $p$  and  $p^j + 1$  for all  $j \in A$ .

Since  $D_\Sigma(A) = 1$ , by (1) of Theorem 5.1, a non-atomic ergodic  $\times p$ -invariant measure which is  $\times q$ -invariant for all  $q$  in  $S_f$  must be the Lebesgue measure.

The remaining thing is to prove that  $|S_\tau \cap [1, n]| \leq (\log_p n)^{\tau(n)}$ .

Every positive integer  $n$  locates in  $[p^{g(m)}, p^{g(m+1)})$  for some nonnegative integer  $m$ .

Consider  $S_\tau \cap [1, n]$ .

Firstly  $|\{j \in A \mid p^j + 1 \leq n\}| \leq l_1 + l_2 + \cdots + l_m = f(m)$ . This means that  $S_f \cap [1, n]$  has at most  $1 + f(m)$  generators.

Note that  $|\{k | p^k \leq n\}| \leq \log_p n$ . So for each generator, there are at most  $\log_p n$  choices for its powers.

Since  $n$  is in  $[p^{g(m)}, p^{g(m+1)})$ , we have  $g(m) \leq \log_p n$ . Then  $1 + f(m) \leq \tau(n)$  by the definition of  $g$ .

Hence

$$|S_\tau \cap [1, n]| \leq (\log_p n)^{1+f(m)} \leq (\log_p n)^{\tau(n)}.$$

This proves the first half of the theorem.

For the second half, choose  $\tau(n) = \log \log(n+3)$  for every  $n$  in  $\mathbb{Z}^+$ . Then for  $S = S_\tau$ , we obtain that

$$\lim_{n \rightarrow \infty} \frac{\log |S \cap [1, n]|}{\log n} \leq \lim_{n \rightarrow \infty} \frac{[\log \log(n+3)](\log \log_p n)}{\log n} = 0.$$

□

**Remark 5.4.** *Furstenberg's conjecture asks for measure rigidity of a non-lacunary semigroup generated by two positive integers  $p, q$  and this semigroup has asymptotically  $(\log n)^2$  elements in  $[1, n]$ . Sataev, Einsiedler and Fish prove measure rigidity of a semigroup containing asymptotically  $n^\alpha$  elements in  $[1, n]$  for some  $0 < \alpha < 1$ . Theorem 5.3 says that for an arbitrary increasing function  $\tau(n)$  with  $\lim_{n \rightarrow \infty} \tau(n) = \infty$ , there is a semigroup with asymptotically  $(\log n)^{\tau(n)}$  elements in  $[1, n]$  for which measure rigidity still holds. One can choose  $\tau$  such that the semigroup  $S_\tau$  is sparsely scattered in  $\mathbb{Z}^+$ .*

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