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Structure Constants from Modularity in Warped CFT

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Abstract: We derive a universal formula for the asymptotic growth of the mean value of three-point coefficient for Warped Conformal Field Theories (WCFTs), and provide a holographic calculation in BTZ black holes. WCFTs are two dimensional quantum field theories featuring a chiral Virasoro and U(1) Kac-Moody algebra, and are conjectured to be holographically dual to quantum gravity on asymptotically AdS_3 spacetime with Compère-Song-Strominger boundary conditions. The WCFT calculation amounts to the calculation of one-point functions on torus, whose high temperature limit can be approximated by using modular covariance of WCFT, similar to the derivation of Cardy formula. The bulk process is given by a tadpole diagram, with a massive spinning particle propagates from the infinity to the horizon, and splits into particle and antiparticle which annihilate after going around the horizon of BTZ black holes. The agreement between the bulk and WCFT calculations indicates that the black hole geometries in asymptotically AdS_3 spacetimes can emerge upon coarse-graining over microstates in WCFTs, similar to the results of Kraus and Maloney in the context of AdS/CFT [1].

Contents

1 Introduction

Field theories with $SL(2,R)\times U(1)$ global symmetry are of great interest from many perspectives. One motivation comes from holography for a large class of geometries with $SL(2,R)\times U(1)$ isometry, including the near horizon of extremal Kerr(NHEK) [2, 3] and the warped AdS_3 (WAdS) [4] spacetime. Such geometries are not asymptotically locally AdS space, and thus the conjectured Kerr/CFT [5], WAdS/CFT [4], WAdS/WCFT [6], AdS_3/WCFT [7] correspondence explores properties of holographic dualilty beyond the standard AdS/CFT correspondence. Both the asymptotic analysis [7–10] in the bulk and the field theoretical analysis [11] indicates that the field theory with $SL(2,R)\times U(1)$ global symmetry have infinite dimensional local symmetries. One minimal possibility is the warped conformal field theory (WCFT) [6] featuring Virasoro-Kac-Moody symmetries, and the other is to have both left and right Virasoro symmetries. Further discussions of the two possibilities can be found in [10, 12–14], and in particular the recent developments on $J\bar{T}$ deformations of CFTs [15–18].

In this paper, we focus on the WCFTs, which are two dimensional, nonrelativistic quantum field theory characterized by a Virasoro algebra and one U(1) Kac-Moody algebra [6, 11]. Specific models of WCFT include chiral Liouville gravity [19], free Weyl fermions [20], free scalars [21], and also the Sachdev-Ye-Kitaev models with complex fermions [22] as a symmetry-broken phase [23]. WCFTs have many nice universal features like their CFT cousins. In particular, WCFTs are covariant under modular transformations [6, 24]. As a result, a Cardy-like asymptotic formula of density of states [6] has been derived, which successfully matches the Bekenstein-Hawking entropy of black holes in the bulk. Entanglement entropy [24–27], partition functions [28], quantum chaos [29] of WCFTs have also been worked out.

Similar to CFTs, data of WCFTs are the spectrum of operators and the three-point function coefficients or the structure constants. Using warped conformal symmetry and Operator Product Expansion (OPE), an arbitrary correlation function of a WCFT can be constructed in terms of these data. Similar to the conformal bootstrap [30–32], a warped conformal bootstrap program has been initiated in [33] using crossing symmetry. Relatedly, via the modular bootstrap [34], it is furthermore known that WCFTs with a negative U(1) level require states with purely imaginary charge, in agreement with bulk studies [7, 35].

In this paper we further explore universal features of WCFT and its holographic dual by considering modular covariance of torus one-point function, along the lines of [1] in the context of $\mathrm{AdS}_3/\mathrm{CFT}_2$. In [1], asymptotic formula of the average three-point coefficient in the heavy-heavy-light limit for scalar operators was derived. Using the $\mathrm{AdS}_3/\mathrm{CFT}_2$ dictionary, the aforementioned quantity can be reproduced by a bulk tadpole diagram calculation on BTZ black hole background. In particular, the result can be approximated using the geodesic of massive point particles.

In the following, we will first generalize the discussion of [1] to operators with arbitrary spins in $\mathrm{AdS}_3/\mathrm{CFT}_2$, and find that the bulk approximation requires massive spinning particles. Then we explore how the story works in the context of AdS_3/WCFT. Similarly, we apply modular transformation to one-point functions on the torus, and find an asymptotic formula of the average three-point coefficient for WCFT. We will show that this average three-point coefficient has a gravity interpretation in the bulk AdS_3 with CSS boundary conditions. The dual objects of generic operators in the WCFT are massive spinning particles, and the on-shell worldline actions of the massive spinning particles can help us to calculate an amplitude of a tadpole diagram in the bulk AdS_3. This amplitude can be compared to the average heavy-heavy-light three-point coefficient for the WCFT. The results here we present hint that the black hole geometries in asymptotically AdS_3 spacetimes can emerge from coarse-graining of microstates in WCFTs.

This paper is organized as follows. In section 2 we compute a tadpole diagram in BTZ black holes. In section 3 we calculate the average three-point coefficient in CFT_2, and match it with the bulk amplitude in the context of $\mathrm{AdS}_3/\mathrm{CFT}_2$. In section 5 we derive the average three-point coefficient in WCFT, and match it with the bulk amplitude in the context of AdS_3/WCFT.

2 A tadpole diagram on BTZ black holes

In the quest for quantum gravity, many lessons have been learned from the study of three dimensional gravity. Under the Brown-Henneaux boundary conditions, Einstein gravity on AdS_3 with a negative cosmological constant is holographically dual to a two dimensional

CFT. Under the CSS boundary conditions [7], the holographic dual is conjectured to be a WCFT, which is relevant for extremal black holes in higher dimensions. Recently a new entry of holographic dictionary in $\mathrm{AdS}_3/\mathrm{CFT}_2$ has been found [1], that is, the asymptotic growth of average three-point coefficient for scalar operators match a tadpole diagram in BTZ black holes in some parameter regions. In the semiclassical limit, the bulk process can be approximated by point massive particles and hence the amplitude can be obtained from the geodesics. In this section we revisit the same bulk process for generic massive spinning particles, generalizing the result of [1] for scalars. In the following two sections, we will reproduce this bulk amplitude from either CFT or WCFT, respectively.

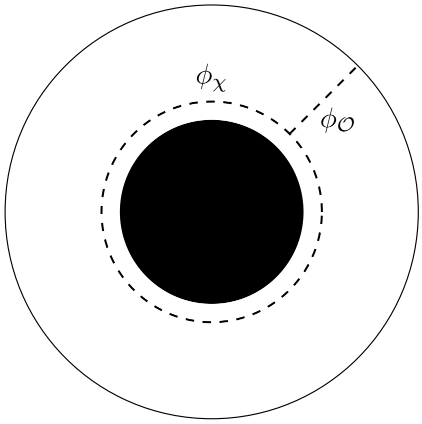

Figure 1. A massive spinning particle $\phi_{\mathcal{O}}$ propagates from infinity to horizon, where it splits into a pair of particles ϕ_χ and $\phi_\chi^\dagger$ which wrap the horizon.

In the semiclassical limit, the bulk process [1] is as depicted in Figure 1: a massive spinning particle $\phi_{\mathcal{O}}$ propagates from infinity to horizon, where it splits into a pair of massive spinning particles ϕ_χ and $\phi_\chi^\dagger$ which wrap the horizon. In the bulk theory, the two bulk field $\phi_{\mathcal{O}}$ and ϕ_χ have a cubic interaction $\phi_{\mathcal{O}}\phi_\chi^\dagger\phi_\chi$ proportional to the three point coefficient $\langle\chi|\mathcal{O}|\chi\rangle$ in the dual field theory. This amplitude $\mathcal{C}^{bk}$ is the product of the cubic-interaction vertex, the propagator of $\phi_{\mathcal{O}}$ from the boundary to the bulk, and the propagator of ϕ_χ around the horizon. Schematically, we have $\mathcal{C}^{bk} \sim \langle\chi|\mathcal{O}|\chi\rangle\langle\phi_{\mathcal{O}}\phi_{\mathcal{O}}\rangle\langle\phi_\chi^\dagger\phi_\chi\rangle$. In the following we will calculate the two propagators.

2.1 Massive spinning particles in BTZ black holes

In this subsection we describe the propagation of a massive spinning particles in BTZ spacetime. The metric of the BTZ black hole can be written in a form with light-like

[1] As was discussed in [36], to reproduce the contributions from all the descendent states of $|\chi\rangle$, a more careful geodesic Witten diagram [37] should be considered. In that case, ϕ_χ will be allowed to propagate in the entire BTZ spacetime, and the interaction point will be determined by minimizing the total worldline action. As a consequence, the one-point block in the bulk will be deformed. However, in this note, we will only focus on the contribution from the primary field and check how the picture works in the context of $\mathrm{AdS}_3/\mathrm{WCFT}$, as well as for arbitrary spins in AdS/CFT.

coordinates,

$$ds^2 = \ell^2 \left(T_u^2 du^2 + 2\rho du dv + T_v^2 dv^2 + \frac{d\rho^2}{4(\rho^2 - T_u^2 T_v^2)} \right) , \tag{2.1}$$

and identifications

$$u \sim u + 2\pi , \qquad v \sim v + 2\pi . \tag{2.2}$$

T_u and T_v are variable constants. The local isometry for the BTZ black hole is $SL(2,R) \times SL(2,R)$, while only the $U(1) \times U(1)$ part is globally well defined due to the spatial circle. It is useful to label the coordinates $u = \varphi + t$ and $v = \varphi - t$ where t is the time and φ is the space.

In the semiclassical approximation, a bulk two-point function for a field with mass m and spin s is given by e^{-S}, where S is the on-shell worldline action of a spinning particle [38],

$$S = \int d\tau \left(m\sqrt{g_{\mu\nu}\dot{X}^\mu \dot{X}^\nu} + s\tilde{n} \cdot \nabla n \right) + S_{constraints} . \tag{2.3}$$

Here τ is the length parameter. $S_{constraints}$ contains Lagrange multipliers which require that the two normalized vector fields n and $\tilde{n}$ should be mutually orthogonal and perpendicular to the worldline, namely

$$n^2 = -1, \qquad \tilde{n}^2 = 1, \qquad n \cdot \tilde{n} = 0, \qquad n \cdot \dot{X} = \tilde{n} \cdot \dot{X} = 0 . \tag{2.4}$$

The symbol ∇ with no subscript indicates a covariant derivative alone the worldline:

$$\nabla V^\mu := \frac{dX^\nu}{d\tau} \nabla_\nu V^\mu . \tag{2.5}$$

The equations of motion with respect to $X^\mu(\tau)$ are known as the Mathisson-Papapetrou-Dixon (MPD) equations,

$$\nabla [m\dot{X}^\mu + \dot{X}^\nu \nabla s^\mu{}_\nu] = -\frac{1}{2}\dot{X}^\nu s^{\rho\sigma} R^\mu{}_{\nu\rho\sigma} , \tag{2.6}$$

where $s^{\mu\nu}$ is the spin tensor,

$$s^{\mu\nu} = s(n^\mu \tilde{n}^\nu - \tilde{n}^\mu n^\nu) . \tag{2.7}$$

As was noticed in [38], in locally AdS spacetimes, the contraction of the Riemann tensor with $s^{\mu\nu}\dot{X}^\rho$ vanishes. The MPD equations reduce to

$$\nabla [m\dot{X}^\mu - s^\mu{}_\nu \nabla \dot{X}^\nu] = 0 . \tag{2.8}$$

One obvious solution to the MPD equation above is a geodesic

$$\nabla \dot{X}^\mu = 0 . \tag{2.9}$$

While other solutions to (2.8) also exist, we will focus on the geodesic solution in this paper.

2.2 A propagator from boundary to bulk

For the particle $\phi_{\mathcal{O}}$ propagating from boundary to horizon along a radial geodesic, the on-shell worldline action (2.3) has two parts. The first part is the length of the worldline times its mass. By using the argument above, a radial geodesic from infinity to the horizon can be viewed as the trajectory followed by $\phi_{\mathcal{O}}$, and its length with a cutoff Λ can be evaluated through the metric (2.1),

$$L_{\mathcal{O}} = \ell \int_{T_u T_v}^{\Lambda} \frac{d\rho}{2\sqrt{\rho^2 - T_u^2 T_v^2}} \approx \frac{\ell}{2}(\log 2\Lambda - \log T_u T_v) + \cdots . \tag{2.10}$$

The second part in (2.3) comes from the spin contribution, which we denote as S_{spin}. As discussed in [38], it can be shown that this term only depends on the boundary data of the normal vectors $n(\tau)$ or $\tilde{n}(\tau)$. Explicitly,

$$S_{\text{spin}} = s_{\mathcal{O}} \log \left(\frac{q(\tau_f) \cdot n_f - \tilde{q}(\tau_f) \cdot n_f}{q(\tau_i) \cdot n_i - \tilde{q}(\tau_i) \cdot n_i} \right) , \tag{2.11}$$

where two vectors q^μ and $\tilde{q}^\mu$ are mutually orthogonal, perpendicular to the geodesic, and furthermore are parallel transported along and the geodesic, i.e.,

$$q^2 = -1, \quad \tilde{q}^2 = 1, \quad q \cdot \tilde{q} = 0, \quad q \cdot \dot{X} = \tilde{q} \cdot \dot{X} = 0, \quad \nabla q = \nabla \tilde{q} = 0. \tag{2.12}$$

The two sets of vectors $(n(\tau), \tilde{n}(\tau))$ and $(q(\tau), \tilde{q}(\tau))$ can be related via a Lorentz boost. In fact, we can expand $n(\tau)$ and $\tilde{n}(\tau)$ in terms of $q(\tau)$ and $\tilde{q}(\tau)$,

$$n(\tau) = \cosh(\eta(\tau)) q(\tau) + \sinh(\eta(\tau)) \tilde{q}(\tau) , \tag{2.13}$$

$$\tilde{n}(\tau) = \sinh(\eta(\tau)) q(\tau) + \cosh(\eta(\tau)) \tilde{q}(\tau) , \tag{2.14}$$

where $\eta(\tau)$ is the rapidity of this Lorentz boost. S_{spin} measures the total change of this boost and can also be expressed in terms of $\tilde{n}$, q, and $\tilde{q}$,

$$S_{\text{spin}} = s_{\mathcal{O}} \log \left(\frac{\tilde{q}(\tau_f) \cdot \tilde{n}_f - q(\tau_f) \cdot \tilde{n}_f}{\tilde{q}(\tau_i) \cdot \tilde{n}_i - q(\tau_i) \cdot \tilde{n}_i} \right) . \tag{2.15}$$

Using the metric (2.1), it is straightforward to find the tangent vector and a parallel transport normal frame of the radial geodesic,

$$\dot{X}^\mu \partial_\mu = \frac{2}{\ell} \sqrt{\rho^2 - T_u^2 T_v^2}\, \partial_\rho, \tag{2.16}$$

$$\tilde{q}^\mu \partial_\mu = \frac{T_v}{\ell \sqrt{2 T_u T_v (\rho + T_u T_v)}} \partial_u + \frac{T_u}{\ell \sqrt{2 T_u T_v (\rho + T_u T_v)}} \partial_v ,$$

$$q^\mu \partial_\mu = \frac{T_v}{\ell \sqrt{2 T_u T_v (\rho - T_u T_v)}} \partial_u - \frac{T_u}{\ell \sqrt{2 T_u T_v (\rho - T_u T_v)}} \partial_v . \tag{2.17}$$

Setting the initial point at infinity and final point at the horizon, it is easy to find [2],

$$\tilde{n}_i^\mu \partial_\mu = \lim_{\rho\to\infty} \frac{1}{\ell\sqrt{2\rho}}(\partial_u + \partial_v), \qquad \tilde{n}_f^\mu \partial_\mu = \lim_{\rho\to T_u T_v} \frac{1}{\ell(\rho + T_u T_v)}(T_v \partial_u + T_u \partial_v)\,. \tag{2.18}$$

Substituting the above formulas into (2.15), we find,

$$S_{\text{spin}} = s_{\mathcal{O}} \log \sqrt{\frac{T_v}{T_u}}\,. \tag{2.19}$$

Putting the two parts together, we can explicitly write down the regularized radial boundary to bulk propagator,

$$e^{-S_{\mathcal{O}}} = T_u^{\frac{\ell m_{\mathcal{O}} + s_{\mathcal{O}}}{2}} T_v^{\frac{\ell m_{\mathcal{O}} - s_{\mathcal{O}}}{2}}\,. \tag{2.20}$$

2.3 A propagator at the horizon

In this subsection, we consider the propagator of a massive spinning particle ϕ_χ going around the horizon as shown in the Figure 1. Similar to the discussion for the particle $\phi_{\mathcal{O}}$, the amplitude for ϕ_χ wrapping the horizon can be evaluated as e^{-S_χ} in the semi-classical limit. Again we will take the geodesic solution to the MPD equation. Then the total on-shell worldline action of ϕ_χ, analogous to (2.3), will become

$$\begin{aligned} S_\chi &= m_\chi L_\chi + S_\chi^{\text{spin}} \\ &= 2\pi \ell m_\chi (T_u + T_v) + s_\chi \int d\tau \tilde{n} \cdot \nabla n \end{aligned} \tag{2.21}$$

where the first part is the length of the trajectory at the horizon times the mass, and the second part is the spin contribution, m_χ and s_χ are the mass and spin of ϕ_χ respectively, n^μ and $\tilde{n}^\mu$ are mutually orthogonal normal vectors as defined in (2.4). The trajectory of ϕ_χ is the horizon at constant time t and is also a geodesic satisfying (2.9).

The horizon is a degenerate hypersurface, and direct solution for n and $\tilde{n}$ will be singular. Instead, we take the orbit of the particle ϕ_χ to be a circular orbit with constant radius slightly greater than the horizon's. It can be shown that the constant radius orbits outside the horizon are no longer geodesics, but with accelerations. Then we evaluate the action of ϕ_χ on that orbit, this is not on-shell since (2.8) is not satisfied. Finally, we take the radius of that orbit tends to the horizon. On the horizon, the worldline action become on-shell, and we will get the result. By using the metric (2.1), we can find out the tangent

[2] Here we choose the time-like co-vector field $n_\mu dx^\mu$ to be proportional to dt along the worldline, and the spatial normal co-vector $\tilde{n}_\mu dx^\mu$ which is perpendicular to n always takes a form,

$$\tilde{n}_\mu dx^\mu \propto d\varphi + \frac{T_u^2 - T_v^2}{2\rho + T_u^2 + T_v^2} dt\,.$$

The worldline action (2.3) is covariant, thus insensitive to the specific choices of n and $\tilde{n}$.

and normal vectors of the horizon at constant time followed by ϕ_χ,

$$\dot{X}^\mu \partial_\mu = \lim_{\rho\to TuTv} \frac{1}{\ell\sqrt{2\rho+T_u^2+T_v^2}}\partial_u + \frac{1}{\ell\sqrt{2\rho+T_u^2+T_v^2}}\partial_v, \tag{2.22}$$

$$\tilde{n}^\mu \partial_\mu = \lim_{\rho\to TuTv} \frac{2\sqrt{\rho^2-T_u^2T_v^2}}{\ell}\partial_\rho\,, \tag{2.23}$$

$$n^\mu \partial_\mu = \lim_{\rho\to TuTv} \frac{\rho+T_v^2}{\ell\sqrt{(\rho^2-T_u^2T_v^2)(2\rho+T_u^2+T_v^2)}}\partial_u - \frac{\rho+T_u^2}{\ell\sqrt{(\rho^2-T_u^2T_v^2)(2\rho+T_u^2+T_v^2)}}\partial_v\,. \tag{2.24}$$

Substituting the above expressions in to the action (2.21), we learn that the limit $\rho \to T_uT_v$ is finite and furthermore the integrand of S_χ^{spin} is independent of the affine parameter. Therefore we can directly perform the integral at the horizon, and get $S_\chi^{\text{spin}} = s_\chi 2\pi(T_u - T_v)$. Alternatively, the same result can be obtained by the formula (2.15), similar to the discussion for $\mathcal{O}$. More details for the alternative derivation can be found in Appendix A.

Putting the two contributions together, the total on-shell action of the massive spinning particle ϕ_χ going round the horizon once is given by

$$S_\chi = 2\pi(\ell m_\chi + s_\chi)T_u + 2\pi(\ell m_\chi - s_\chi)T_v\,. \tag{2.25}$$

Note that (2.25) is independent of boundary conditions or the holographic dictionaries, and will be used in the story of $\text{AdS}_3/\text{CFT}_2$ and AdS_3/WCFT in next sections.

Backreactions

If the mass of the particle is comparable to $\frac{1}{G}$, backreactions to the background geometry will be important if we try to match the local mass and spin in the bulk to quantum numbers in the dual field theory. Solving the Einstein equation with a local source, the backreaction of a particle with mass m_χ and s_χ geometry will be a rotational conical defect [39, 40],

$$ds^2 = \ell^2\left[-(1+r^2)\left(dt - \frac{s_\chi}{\frac{\ell}{4G}\left(1-\frac{\delta\varphi}{2\pi}\right)}d\varphi\right)^2 + \frac{dr^2}{1+r^2} + r^2 d\varphi^2\right], \tag{2.26}$$

$$\varphi \sim \varphi + 2\pi - \delta\varphi\,. \tag{2.27}$$

Here the deficit angle $\delta\varphi$ is related to the mass of the ϕ_χ through

$$m_\chi = \frac{\delta\varphi}{8\pi G}\,. \tag{2.28}$$

More details of the solution can be found in Appendix B, where we show that the mass and spin of the local source can be obtained from the geometry by the quasi-local energy.

In order to match results in the boundary, we need to put the conical defect solution (2.26) into the form of (2.1) with standard identification (2.2), from which we can read the asymptotic charges easily. One can verify that the coordinate transformation is

$$u = \frac{\varphi}{1-4Gm_\chi} + \frac{t}{1-4G(m_\chi + s_\chi/\ell)}\,, \tag{2.29}$$

$$v = \frac{\varphi}{1-4Gm_\chi} - \frac{t}{1-4G(m_\chi - s_\chi/\ell)}\,, \tag{2.30}$$

$$\rho = \frac{1}{2}\left[(1-4Gm_\chi)^2 - 16G^2 s_\chi^2/\ell^2\right](\frac{1}{2}+r^2)\,. \tag{2.31}$$

The resulting metric has fixed boundary metric and identifications (2.2), and takes the standard form,

$$\begin{aligned} ds^2 &= \ell^2\left[T_{\chi u}^2 du^2 + 2\rho du dv + T_{\chi v}^2 dv^2 + \frac{d\rho^2}{4(\rho^2 - T_{\chi u}^2 T_{\chi v}^2)}\right]\,, \\ (u,v) &\sim (u+2\pi,\, v+2\pi) \end{aligned} \tag{2.32}$$

where

$$T_{\chi u}^2 = -\frac{(1-4G(m_\chi + s_\chi/\ell))^2}{4}, \qquad T_{\chi v}^2 = -\frac{(1-4G(m_\chi - s_\chi/\ell))^2}{4}\,. \tag{2.33}$$

Note that (2.32) is in both the Brown-Henneaux phase space and the CSS phase space. From (2.32), we can easily read off the asymptotic charges, and are related to quantum numbers in the field theory according to the holographic dictionary.

2.4 A tadpole diagram in the bulk

Combing the two contributions from $\phi_\mathcal{O}$ (2.20) and ϕ_χ(2.25), we get the total amplitude for the process given by Figure 1,

$$\mathcal{C}_\mathcal{O}^{bk}(E_L, E_R) = \underbrace{\langle\chi|\mathcal{O}|\chi\rangle}_{\text{vertex}}\, \underbrace{T_u^{\frac{\ell m_\mathcal{O}+s_\mathcal{O}}{2}}\, T_v^{\frac{\ell m_\mathcal{O}-s_\mathcal{O}}{2}}}_{\langle\phi_\mathcal{O}\phi_\mathcal{O}\rangle}\, \underbrace{e^{-\ell m_\chi 2\pi(T_v+T_u) - s_\chi 2\pi(T_u - T_v)}}_{\langle\phi_\chi^\dagger\phi_\chi\rangle}, \tag{2.34}$$

which describes a massive spinning particle $\phi_\mathcal{O}$ propagates from infinity, and decays into a loop of ϕ_χ around the black hole horizon.

3 Structure constant in $\mathrm{AdS}_3/\mathrm{CFT}_2$ with spinning operators

In this section, we consider the mean value of three-point coefficients in the context of $\mathrm{AdS}_3/\mathrm{CFT}_2$. Our results generalize those of [1] to operators with arbitrary spins. On the CFT_2 side, we derive the typical expectation value of an operator $\mathcal{O}$ with conformal weights $(h_\mathcal{O}, \bar{h}_\mathcal{O})$ on states with weights $(E_L + \frac{c}{24}, E_R + \frac{c}{24})$. We follow the strategy of [1] where the same quantity was calculated for scalar operator $\mathcal{O}$. By changing the ensemble, the typical value of three-point function can be calculated from the torus one-point function. Similar to the derivation of Cardy's formula, modular invariance of two dimensional CFTs allows an estimation of torus one-point function at high temperature, or equivalently at high energy. On the gravity side, we consider Einstein gravity in asymptotic AdS_3 spacetime with the Brown-Henneaux boundary conditions. The gravity picture is then a tadpole diagram on BTZ background as depicted in Figure 1. Due to the spin, the propagator in the bulk will be given by the worldline action of a spinning particle instead of a spinless particle as discussed in [1]. In the following we will show the CFT_2 calculation and bulk calculation respectively.

3.1 Asymptotic structure constant in CFT_2 with arbitrary spins

In this subsection, we derive the mean structure constant of an operator $\mathcal{O}$ with conformal weights $(h_{\mathcal{O}}, \bar{h}_{\mathcal{O}})$ and two operators with conformal weights $(E_L + \frac{c}{24}, E_R + \frac{c}{24})$ in the limit of $E_L, E_R \gg 1$. In the following, we will carry out a detailed calculation in CFT_2, for the purpose of easy comparison with that of WCFT in the next section.

In a two dimensional CFT, the one-point function of a primary operator $\mathcal{O}$ of conformal weights $(h_{\mathcal{O}}, \bar{h}_{\mathcal{O}})$ on a torus with complex modular parameter $\tau = i\tau_1 + \tau_2$ can be written in terms of three-point coefficients as follows,

$$\langle \mathcal{O} \rangle_{\tau,\bar{\tau}} = \mathrm{Tr}\mathcal{O} e^{2\pi i \tau L_0 - 2\pi i \bar{\tau} \bar{L}_0} = \int dE_L dE_R T_{\mathcal{O}}(E_L, E_R) e^{2\pi i \tau E_L - 2\pi i \bar{\tau} E_R}\,, \tag{3.1}$$

where

$$T_{\mathcal{O}}(E_L, E_R) = \sum_i \langle i|\mathcal{O}|i\rangle \delta(E_L - E_{Li})\delta(E_R - E_{Ri})\,. \tag{3.2}$$

Up to normalizations, $\langle i|\mathcal{O}|i\rangle$ is the OPE coefficient $c_{ii\mathcal{O}}$, and hence $T_{\mathcal{O}}(E_L, E_R)$ is the total three-point coefficient from all operators with (E_{Li}, E_{Ri}). L_0 and $\bar{L}_0$ are the zero modes of the standard Virasoro algebra. E_L and E_R are the eigenvalues of L_0 and $\bar{L}_0$ on the cylinder, respectively, with relative shifts $E_L(E_R) = h(\bar{h}) - \frac{c}{24}$ to the conformal weights $(h, \bar{h})$ defined on the plane. The imaginary part of τ is proportional to the inverse temperature and the real part is the angular potential. Here we keep them arbitrary, and treat τ and $\bar{\tau}$ as independent complex variables.

We can invert the relation (3.1) between the torus one-point function and $T_{\mathcal{O}}(E_L, E_R)$ by an inverse Laplace transformation. From this perspective, the total three-point coefficient $T_{\mathcal{O}}(E_L, E_R)$ is also the density of states weighted by the one-point function,

$$T_{\mathcal{O}}(E_L, E_R) = \int d\tau d\bar{\tau} e^{-2\pi i \tau E_L} e^{2\pi i \bar{\tau} E_R} \langle \mathcal{O} \rangle_{\tau,\bar{\tau}}\,. \tag{3.3}$$

Two dimensional CFTs are invariant under the large conformal transformations on the torus, which act on τ as modular transformations,

$$\tau \to \gamma\tau \equiv \frac{a\tau + b}{c\tau + d}\,, \tag{3.4}$$

where $ad - cd = 1$. The partition function of theory is invariant under such modular transformation, and the primary operator $\mathcal{O}$ transforms with the modular weights $(h_{\mathcal{O}}, \bar{h}_{\mathcal{O}})$,

$$\langle \mathcal{O} \rangle_{\gamma\tau,\gamma\bar{\tau}} = (c\tau + d)^{h_{\mathcal{O}}} (c\bar{\tau} + d)^{\bar{h}_{\mathcal{O}}} \langle \mathcal{O} \rangle_{\tau,\bar{\tau}}\,. \tag{3.5}$$

In particular, under the S transformation $\tau \to -1/\tau$, the one-point function transforms as

$$\langle \mathcal{O} \rangle_{\tau,\bar{\tau}} = \tau^{-h_{\mathcal{O}}} \bar{\tau}^{-\bar{h}_{\mathcal{O}}} \langle \mathcal{O} \rangle_{-1/\tau,-1/\bar{\tau}}\,. \tag{3.6}$$

This formula is very useful as it relates the high temperature behaviour of the theory to the behaviour at low temperature. Taking the limit $\tau \to i0^+$, $\bar{\tau} \to -i0^+$, and assuming the eigenvalues of L_0 and $\bar{L}_0$ are bounded from below, we can project the right hand side of (3.6) onto a lightest state $|\chi\rangle$ with non-vanishing three-point coefficient $\langle\chi|\mathcal{O}|\chi\rangle \neq 0$ [3],

$$\langle\mathcal{O}\rangle_{\tau,\bar{\tau}} = \langle\chi|\mathcal{O}|\chi\rangle \tau^{-h_{\mathcal{O}}} \bar{\tau}^{-\bar{h}_{\mathcal{O}}} e^{-2\pi i \frac{1}{\tau} E_{L\chi}} e^{2\pi i \frac{1}{\bar{\tau}} E_{R\chi}}\,, \tag{3.7}$$

where $(E_{L\chi}, E_{R\chi})$ are the conformal dimensions of the state $|\chi\rangle$. Substituting (3.7) back into (3.3), we can write down the total three-point coefficient in terms of $E_{L\chi}$ and $E_{R\chi}$,

$$T_{\mathcal{O}}(E_L, E_R) = \langle\chi|\mathcal{O}|\chi\rangle \int d\tau d\bar{\tau} \tau^{-h_{\mathcal{O}}} \bar{\tau}^{-\bar{h}_{\mathcal{O}}} e^{-2\pi i \tau E_L} e^{2\pi i \bar{\tau} E_R} e^{-2\pi i \frac{1}{\tau} E_{L\chi}} e^{2\pi i \frac{1}{\bar{\tau}} E_{R\chi}}\,. \tag{3.8}$$

At large E_L and E_R, the above integral is dominated by a saddle point with

$$\tau = i\sqrt{-\frac{E_{L\chi}}{E_L}}, \qquad \bar{\tau} = -i\sqrt{-\frac{E_{R\chi}}{E_R}}\,. \tag{3.9}$$

Further assuming $E_{L\chi}$, $E_{R\chi} < 0$, the saddle points are pure imaginary. Under the saddle point approximation, the integral in (3.8) can be performed and we get,

$$T_{\mathcal{O}}(E_L, E_R) = i^{\bar{h}_{\mathcal{O}} - h_{\mathcal{O}}} \langle\chi|\mathcal{O}|\chi\rangle \left(-\frac{E_{L\chi}}{E_L}\right)^{-\frac{h_{\mathcal{O}}}{2}} \left(-\frac{E_{R\chi}}{E_R}\right)^{-\frac{\bar{h}_{\mathcal{O}}}{2}} e^{4\pi\sqrt{-E_L E_{L\chi}} + 4\pi\sqrt{-E_R E_{R\chi}}}\,. \tag{3.10}$$

$T_{\mathcal{O}}(E_L, E_R)$ characterizes the total contribution of different degenerate states to the three-point function coefficient of the underling CFT at given large E_L and E_R. However, it is useful to define the typical value of the three-point coefficient by dividing $T_{\mathcal{O}}(E_L, E_R)$ by the density of states. Choosing the operator $\mathcal{O}$ being the identity and setting the state $|\chi\rangle$ to the vacuum in (3.10), we can write down the density of states $\rho(E_L, E_R)$ at large E_L and E_R,

$$\rho(E_L, E_R) = e^{4\pi\sqrt{-E_L E_{Lvac}} + 4\pi\sqrt{-E_R E_{Rvac}}}\,. \tag{3.11}$$

Taking $E_{Lvac} = E_{Rvac} = -\frac{c}{24}$, the above equation is nothing but the Cardy's formula in two dimensional CFT. The mean structure constant for the operator $\mathcal{O}$ and two operators with same large conformal dimension E_L and E_R can be defined,

$$\begin{aligned} \mathcal{C}_{\mathcal{O}}(E_L, E_R) &\equiv \frac{T_{\mathcal{O}}(E_L, E_R)}{\rho(E_L, E_R)} \\ &= i^{\bar{h}_{\mathcal{O}} - h_{\mathcal{O}}} \langle\chi|\mathcal{O}|\chi\rangle \left(-\frac{E_{L\chi}}{E_L}\right)^{-\frac{h_{\mathcal{O}}}{2}} \left(-\frac{E_{R\chi}}{E_R}\right)^{-\frac{\bar{h}_{\mathcal{O}}}{2}} e^{4\pi\sqrt{E_L}(\sqrt{-E_{L\chi}} - \sqrt{-E_{Lvac}}) + 4\pi\sqrt{E_R}(\sqrt{-E_{R\chi}} - \sqrt{-E_{Rvac}})}\,. \end{aligned} \tag{3.12}$$

This is the asymptotic formula of the average value of three-point coefficient for a general CFT_2, assuming $E_L, E_R \gg 1$, and the existence of an operator χ with $E_{L\chi}$, $E_{R\chi} < 0$ and

[3] Throughout this paper, we focus on the contribution from the primary field χ, without considering the possible degeneracy. When χ has degeneracies, we need to replace $\langle\chi|\mathcal{O}|\chi\rangle$ by a sum over all degenerate states. Subtleties may appear when the sum is zero. Related discussions can be found in [1]. Here we will only consider the most general theories where such coincidence doesn't show up. See [36] for the contributions from all the descendants.

$\langle\chi|\mathcal{O}|\chi\rangle \neq 0$. When $\mathcal{O}$ is a scalar operator with $h_{\mathcal{O}} = \bar{h}_{\mathcal{O}}$, (3.12) reduces to the CFT results of [1]. In the following subsection, we will match the bulk calculation (2.34) to the CFT result (3.12).

3.2 Matching with $\mathrm{AdS}_3/\mathrm{CFT}_2$

Under Brown-Henneaux boundary conditions, asymptotically AdS_3 spacetimes are dual to CFT_2s. In the context of $\mathrm{AdS}_3/\mathrm{CFT}_2$, [1] proposed a bulk dual for the asymptotic formula of mean three-point coefficient for scalar operators in the heavy-heavy-light limit. In this subsection, we check the picture for the generic spins by matching the bulk tadpole diagram with the CFT quantity $\mathcal{C}_{\mathcal{O}}(E_L, E_R)$ (3.12). In the following, we will first use the $\mathrm{AdS}_3/\mathrm{CFT}_2$ dictionary to rewrite the result (3.12) in terms of BTZ parameters, and then consider the contributions from $\phi_{\mathcal{O}}$ and ϕ_χ, and finally match the bulk result (2.34) to (3.12).

We consider the average value of the three-point coefficient $\mathcal{C}_{\mathcal{O}}(E_L, E_R)$ (3.12), in the heavy-heavy-light limit with $1 \ll h_{\mathcal{O}}, \bar{h}_{\mathcal{O}} \ll \frac{c}{24}$ and E_L, $E_R \geq \frac{c}{24}$, $c \gg 1$. At large E_L and E_R, a typical state $|E_L, E_R\rangle$ is well described by the black hole geometry in AdS_3, which emerges after coarse-graining over many states at fixed E_L and E_R. The difference from [1] is that for $E_L \neq E_R$, we will need to consider rotating BTZ black holes. More explicitly, the quantum numbers E_L and E_R in CFT_2 are the conserved charges associate with the Killing vectors ∂_u and $-\partial_v$ evaluated on the BTZ metric (2.1),

$$E_L = Q[\partial_u] = \frac{\ell T_u^2}{4G}, \qquad E_R = Q[-\partial_v] = \frac{\ell T_v^2}{4G}\,. \tag{3.13}$$

The dual geometry for the CFT vacuum is the global AdS_3, which corresponds to $T_{u,v} = -\frac{i}{2}$, with eigenvalues

$$E_{Lvac} = E_{Rvac} = -\frac{\ell}{16G}\,. \tag{3.14}$$

For a light operator $\mathcal{O}$ with $1 \ll h_{\mathcal{O}}, \bar{h}_{\mathcal{O}} \ll \frac{c}{24}$, its dual field $\phi_{\mathcal{O}}$ can be approximated by a massive spinning particle, instead of a spinless particle when $h_{\mathcal{O}} = \bar{h}_{\mathcal{O}}$ as considered in [1]. Under the massive particle approximation, the conformal weights for $\mathcal{O}$ are related to the mass and spin by

$$h_{\mathcal{O}} = \frac{1}{2}(\ell m_{\mathcal{O}} + s_{\mathcal{O}}), \qquad \bar{h}_{\mathcal{O}} = \frac{1}{2}(\ell m_{\mathcal{O}} - s_{\mathcal{O}})\,. \tag{3.15}$$

Therefore the propagator for $\phi_{\mathcal{O}}$ (2.20) can be also written as

$$e^{-S_{\mathcal{O}}} = T_u^{h_{\mathcal{O}}} T_v^{\bar{h}_{\mathcal{O}}}\,. \tag{3.16}$$

We will further take the lightest state $|\chi\rangle$ coupled to $\mathcal{O}$ to be a heavy operator in the region $1 \ll h_\chi, \bar{h}_\chi < \frac{c}{24}$. Then the dual bulk field, denoted by ϕ_χ, will correspond to a non-perturbative field which backreacts on the bulk AdS_3 geometry but still below the black hole threshold. This means that ϕ_χ will create a spinning conical defect for generic $h_\chi - \bar{h}_\chi$. Now we need to relate the local bulk mass m_χ and spin s_χ to conformal weights h_χ, $\bar{h}_\chi$ in the dual CFT_2. As described in section 2.3, the spinning conical defect (2.26) can be brought into a standard form (2.32) in the Brown-Henneax phase space. Then the conformal weights can be read from the asymptotic charges evaluated on (2.32) with the parameters (2.33),

$$E_{L\chi} = Q_\chi[\partial_u] = \frac{\ell T_{\chi u}^2}{4G}, \qquad E_{R\chi} = Q_\chi[-\partial_v] = \frac{\ell T_{\chi v}^2}{4G} \tag{3.17}$$

Using the above map and Eqs. (2.33), one can rewrite the mass and spin of ϕ_χ in terms of its conformal dimensions,

$$\ell m_\chi = \frac{\ell}{4G} - \sqrt{-\frac{\ell E_{L\chi}}{4G}} - \sqrt{-\frac{\ell E_{R\chi}}{4G}}, \qquad s_\chi = -\sqrt{-\frac{\ell E_{L\chi}}{4G}} + \sqrt{-\frac{\ell E_{R\chi}}{4G}}\,. \tag{3.18}$$

Substituting the mass and spin formula above into (2.25), the amplitude for ϕ_χ wrapping the horizon can be recast as

$$e^{-S_\chi} = e^{4\pi\left[T_u\left(\sqrt{-\frac{\ell E_{L\chi}}{4G}}-\frac{\ell}{8G}\right)+T_v\left(\sqrt{-\frac{\ell E_{R\chi}}{4G}}-\frac{\ell}{8G}\right)\right]}\,. \tag{3.19}$$

Putting the two parts (3.16) and (3.19) together, the bulk amplitude (2.34) for the process given by Figure 1 under Brown-Henneaux boundary conditions can be written as

$$\mathcal{C}^{bk}_{\mathcal{O}}(E_L, E_R) = \underbrace{<\chi|\mathcal{O}|\chi>}_{\text{vertex}}\ \underbrace{T_u^{h_\mathcal{O}} T_v^{\bar{h}_\mathcal{O}}}_{\langle\phi_\mathcal{O}\phi_\mathcal{O}\rangle}\ \underbrace{e^{4\pi\left[T_u\left(\sqrt{-\frac{\ell E_{L\chi}}{4G}}-\frac{\ell}{8G}\right)+T_v\left(\sqrt{-\frac{\ell E_{R\chi}}{4G}}-\frac{\ell}{8G}\right)\right]}}_{\langle\phi_\chi\phi_\chi\rangle} \tag{3.20}$$

Up to an overall normalization that does not depend on E_L and E_R, the above bulk amplitude agrees with the average value of three-point coefficient (3.12) rewritten in terms of temperatures.

4 The AdS$_3$/WCFT correspondence

4.1 Warped CFT and modular invariance

In this subsection, we briefly review some basic features of WCFT. A warped conformal field theory is characterized by the warped conformal symmetry. The global symmetry is $SL(2,R)\times U(1)$, while the local symmetry algebra is a Virasoro algebra plus a $U(1)$ Kac-Moody algebra [6, 11]. In position space, a general warped conformal symmetry transformation can be written as

$$x' = f(x), \qquad y' = y + g(x)\,, \tag{4.1}$$

where x and y are $SL(2,R)$ and $U(1)$ local coordinates, and $f(x)$ and $g(x)$ are two arbitrary functions. Denote $T(x)$ and $P(x)$ as the Noether currents associated with the translation along x and y axis, respectively. The commutation relations for the Noether charges form a canonical warped conformal algebra consists of one Virasoro algebra and a Kac-Moody algebra,

$$\begin{aligned}
[L_n, L_m] &= (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n,-m},\\
[L_n, P_m] &= -mP_{n+m},\\
[P_n, P_m] &= n\frac{k}{2}\delta_{n,-m}\,,
\end{aligned} \tag{4.2}$$

where c is the central charge and k is the Kac-Moody level. One can also construct the spectral flow invariant Virasoro generators,

$$L_n^{inv} = L_n - \frac{1}{k}\Big(\sum_{m\leq -1} P_m P_{n-m} + \sum_{m\geq 0} P_{n-m} P_m\Big)\,. \tag{4.3}$$

One can check that L_n^{inv} commutate with the Kac-Moody generators, and form a Virasoro algebra with central charge $c-1$.

Now consider a WCFT with coordinates (x, y) on a torus with two circles

$$\textit{spatial circle}: \qquad (x, y) \sim (x + 2\pi, y)\,, \tag{4.4}$$

$$\textit{thermal circle}: \qquad (x, y) \sim (x + i\beta, y - i\bar{\beta})\,, \tag{4.5}$$

where β and $\bar{\beta}$ are the inverse temperatures along x and y, respectively. The torus partition function can be written as

$$Z(\beta, \bar{\beta}) = \mathrm{Tr}(e^{-\beta L_0 + \bar{\beta} P_0})\,, \tag{4.6}$$

where L_0 and P_0 are the zero modes of the generators, which are the conserved charges associated with the coordinates x and y respectively,

$$L_0 = Q[\partial_x], \qquad P_0 = Q[\partial_y]\,, \tag{4.7}$$

The partition function (4.6) transforms covariantly under modular transformation, as first obtained in [6], and further explained in [24]. First note that the partition function is invariant under swapping the circles to

$$\textit{spatial circle}: \qquad (x, y) \sim (x + i\beta, y - i\bar{\beta})\,, \tag{4.8}$$

$$\textit{thermal circle}: \qquad (x, y) \sim (x - 2\pi, y)\,. \tag{4.9}$$

Then the following warped conformal transformation

$$x' = -i\frac{2\pi}{\beta}x, \qquad y' = y + \frac{\bar{\beta}}{\beta}x\,, \tag{4.10}$$

leads to a torus with new circles,

$$\textit{spatial circle}: \qquad (x', y') \sim (x' + 2\pi, y')\,, \tag{4.11}$$

$$\textit{thermal circle}: \qquad (x', y') \sim (x' + i\beta', y' - i\bar{\beta}')\,, \tag{4.12}$$

where

$$\beta' = \frac{4\pi^2}{\beta}, \qquad \bar{\beta}' = -\frac{2\pi i\bar{\beta}}{\beta}\,. \tag{4.13}$$

The partition function transforms according to the following equation with anomaly [6, 24],

$$Z(\beta, \bar{\beta}) = e^{k\frac{\bar{\beta}^2}{4\beta}} Z\left(\frac{4\pi^2}{\beta}, -\frac{2\pi i\bar{\beta}}{\beta}\right)\,. \tag{4.14}$$

4.2 The AdS_3/WCFT correspondence

The AdS_3/WCFT setup— version I

Under CSS boundary conditions [7], asymptotically AdS_3 spacetimes are conjectured to be dual to WCFTs. To be more precise, there are two different versions of the boundary conditions, demonstrated in the text and appendix of [7], respectively. Let us first setup the holographic dictionary for WCFT in AdS_3 using the version in the text. In the light-like coordinate system, the BTZ black hole metric (2.1) with identification (2.2) satisfies the CSS boundary conditions,

$$g^{(0)}_{uv} = 1, \quad g^{(0)}_{vv} = 0, \quad \partial_v g^{(0)}_{uu} = 0, \quad g^{(2)}_{vv} = T_v^2 \,, \tag{4.15}$$

with T_v fixed. The asymptotic Killing vectors obeying the boundary condition (4.15) are,

$$\xi_n = e^{inu}\left(\partial_u - \frac{1}{2} in\partial_\rho\right), \quad \eta_n = -e^{inu}\partial_v \,, \tag{4.16}$$

The asymptotic symmetry algebra under above boundary conditions is the Virasoro-Kac-Moody algebra, which can be written as

$$\begin{aligned}
[\tilde{L}_n, \tilde{L}_m] &= (n-m)\tilde{L}_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n,-m}, \\
[\tilde{L}_n, \tilde{P}_m] &= -m\tilde{P}_{n+m} + m\tilde{P}_0\delta_{n,-m}, \\
[\tilde{P}_n, \tilde{P}_m] &= \frac{\tilde{k}}{2} n\delta_{n,-m} \,,
\end{aligned} \tag{4.17}$$

where

$$c = \frac{3\ell}{2G}, \quad \tilde{k} = -\frac{\ell T_v^2}{G} \,. \tag{4.18}$$

$\tilde{L}_0$ and $\tilde{P}_0$ are the conserved charges associate with left and right moving Killing vectors ∂_u and ∂_v respectively, and are related to the bulk mass and energy momentum by $\tilde{L}_0 = \frac{1}{2}(E+J)$, $\tilde{P}_0 = -\frac{1}{2}(E-J)$. The asymptotic charges $\tilde{L}_n, \tilde{P}_n$ are both finite and integrable with fixed T_v. The above algebra (4.17) is not the canonical WCFT algebra (4.2). One way to relate them is through the following charge redefinition [6] [4],

$$\tilde{L}_n = L_n - \frac{2P_0P_n}{k} + \frac{P_0^2\delta_{n,0}}{k}, \quad \tilde{P}_n = \frac{2P_0P_n}{k} - \frac{P_0^2\delta_{n,0}}{k} \,. \tag{4.19}$$

In this paper, we are interested in states with $\langle P_n \rangle = 0$, $\forall n \neq 0$, and this amounts to a nonlocal reparameterization of the theory,

$$u = x, \quad v = \frac{ky}{2P_0} + x \,. \tag{4.20}$$

[4] Alternatively, we can choose the field dependent asymptotic Killing vectors following Appendix A of [34], which is a modification of the Appendix B of [7]. Then the bulk charges will automatically match the Virasoro-Kac-Moody generators in WCFT. In particular, for states with $\langle P_n \rangle = 0$, $\forall n \neq 0$, it will be obvious that $\langle L_0 \rangle$ is the angular momentum.

On such states we further have the expectation values

$$
\begin{aligned}
E_L \equiv \langle L_0 \rangle = \langle \tilde{L}_0 \rangle + \langle \tilde{P}_0 \rangle = J, \quad \frac{Q^2}{k} = \langle \tilde{P}_0 \rangle = \frac{1}{2}(J - E) \\
E_L^{inv} \equiv \langle L_0^{inv} \rangle = \tilde{L}_0 = \frac{1}{2}(E + J),
\end{aligned}
\tag{4.21}
$$

Note that the WCFT weight E_L corresponds to angular momentum in the bulk, while the spectral flow invariant generator L_0^{inv} plays the same role of the left moving energy L_0^{CFT} in a CFT_2. In particular, if a state has a vanishing charge, the bulk energy equals to its angular momentum which is furthermore given by the eigenvalue of L_0,

$$
E = J = E_L, \quad if \ Q = 0 \tag{4.22}
$$

For the BTZ black holes (2.1), the expectation values of the zero modes of the canonical WCFT algebra (4.2) can now be read from the background geometry,

$$
E_L^{inv} = \langle L_0^{inv} \rangle = \frac{\ell}{4G} T_u^2, \qquad Q = \langle P_0 \rangle = -\frac{T_v}{2} \sqrt{-\frac{\ell k}{G}} \,. \tag{4.23}
$$

The WCFT vacuum corresponds to the global AdS_3 with $T_u = T_v = -\frac{i}{2}$, and thus the vacuum has the following quantum numbers

$$
E_L^{inv}{}_{vac} = -\frac{\ell}{16G}, \qquad Q_{vac} = \frac{i}{4} \sqrt{-\frac{\ell k}{G}} \,. \tag{4.24}
$$

Because of the map (4.19), the AdS_3/WCFT correspondence has the following features:

- From (4.23), the phase space of fixed T_v is mapped to a sector with fixed charge Q. On the other hand, WCFT contains sectors with different charges, as is required by modular covariance [6]. This suggests that the phase space of WCFT consists of the union of the bulk phase spaces with different T_v. This interpretation was proposed by [25], and is necessary for reproducing the Bekenstein-Hawking entropy using the DHH formula. The calculations of holographic entanglement entropy [25] and one-loop partition function [41] both support this interpretation.

- It is convenient to choose a negative level k, which makes the charge of black holes real whereas the charge of global AdS_3 pure imaginary, as can be seen in eq (4.23). Alternatively, one could choose a positive k. Then global AdS_3 will has positive charge, and the vacuum sector will remain unitary.

- Relatedly, a modular bootstrap with $k < 0$ can be performed assuming that all the Virasoro-Kac-Moody primary states have positive norms [34]. The negative level k leads to descendent states with negative norms, whose contribution to the partition function can be estimated and is much smaller than the primaries. Furthermore, modular covariance requires that states with pure imaginary charges have to exist [34], consistent with the fact that global AdS_3 has a pure imaginary charge.

Version II

There is an alternatively version of CSS boundary conditions as in Appendix A of [34], which is a modification of Appendix B of [7]. The boundary conditions agree with (4.15), with $g^{(2)}_{vv}$ still coordinate independent but allowed to vary. The resulting asymptotic Killing vectors are state-dependent,

$$\xi'_n = \xi_n + \eta_n, \qquad \eta'_n = \frac{\eta_n}{T_v}\,, \tag{4.25}$$

from which the corresponding charges L_n and P_n automatically satisfy the canonical WCFT algebra (4.2) with central charge $c = 3\ell/2G$ and a negative level k. Both L_n and P_n charges are integrable for arbitrary variations of T_v. These charges are related to the bulk charges (4.17) in version I by the non-local map eq (4.19). Then it is straightforward to show that the zero mode of the Virasoro-Kac-Moody is just the angular momentum

$$E_L \equiv \langle L_0 \rangle = J, \quad \frac{Q^2}{k} = \frac{\langle P_0 \rangle^2}{k} = \frac{1}{2}(J - E) \tag{4.26}$$

In particular, the BTZ black holes has the charges are $L_0 = \frac{\ell}{4G}(T_u^2 - T_v^2), P_0 = -\frac{T_v}{2}\sqrt{-\frac{\ell k}{G}}$. In this version of the boundary conditions, T_v remains constant, but is allowed to vary. The map between the bulk and boundary is straightforward, though at the cost of field-dependent asymptotic Killing vectors. Black hole entropy, entanglement entropy, modular bootstrap can be discussed in this version, and the results remain the same.

5 Structure constant in AdS$_3$/WCFT

In this section, we will first derive the mean structure constant in WCFT by using the modular properties of warped conformal symmetry, and then provide a bulk calculation on the background of BTZ black holes. We found that the gravity picture of the mean structure constant in WCFT will still be given by Figure 1. However, a crucial difference from the previous discussion is that we will use the holographic dictionary of AdS$_3$/WCFT, instead of AdS$_3$/CFT.

5.1 Torus one-point functions

Due to the warped conformal algebra (4.2), the operator spectrum in WCFT can be labelled by conformal weights and charges, i.e. the eigenvalues of L_0 and charge under P_0. Due to charge conservation, only chargeless operators will have non-vanishing one-point function. For a primary operator $\mathcal{O}$ with conformal weight $h_{\mathcal{O}}$ and zero charge, the one-point function on a torus (4.4) (4.5) is defined by

$$\langle \mathcal{O} \rangle_{\beta,\bar{\beta}} = \mathrm{Tr}(\mathcal{O} e^{-\beta L_0 + \bar{\beta} P_0})\,. \tag{5.1}$$

Similar to the story in CFT, the warped conformal transformation (4.10) which swap the spatial and thermal circle on the torus plays the role as modular transformation in WCFT. Under the warped conformal transformation the one-point function transforms as [33]

$$\langle \mathcal{O} \rangle_{\beta,\bar{\beta}} = e^{k\frac{\bar{\beta}^2}{4\beta}} \left(\frac{\partial x'}{\partial x}\right)^{h_{\mathcal{O}}} \langle \mathcal{O} \rangle_{\frac{4\pi^2}{\beta}, -\frac{2\pi i \bar{\beta}}{\beta}}\,. \tag{5.2}$$

Here the form of the finite transformation of $\mathcal{O}$ is determined by the chiral scaling symmetry of the theory. It behaves like an $h_{\mathcal{O}}$-form under x-direction diffeomorphism and like a scalar under $U(1)$ transformation.

Now take limit $\beta \to 0^+$, suppose the eigenvalues of L_0 are bounded from below, we have

$$\langle \mathcal{O} \rangle_{\beta,\bar{\beta}} = \langle \chi|\mathcal{O}|\chi\rangle \left(-i\frac{2\pi}{\beta}\right)^{h_{\mathcal{O}}} e^{-\frac{4\pi^2}{\beta}E_{L\chi}-\frac{2\pi i\bar{\beta}}{\beta}Q_{\chi}+k\frac{\bar{\beta}^2}{4\beta}} \tag{5.3}$$

where $|\chi\rangle$ is the lightest state with non-vanishing three-point coefficient $\langle\chi|\mathcal{O}|\chi\rangle \neq 0$. The charge conservation for the three-point function requires $\mathcal{O}$ to be chargeless [33]. Here we use $E_{L\chi}$ for the eigenvalue of L_0 on the cylinder acting on $|\chi\rangle$ and Q_χ is the eigenvalue of P_0 acting on $|\chi\rangle$ which is known as the charge of $|\chi\rangle$.

On the other hand, the one-point function of the primary operator $\mathcal{O}$ can be rewritten as

$$\langle \mathcal{O} \rangle_{\beta,\bar{\beta}} = \int\int dE_L dQ T_{\mathcal{O}}(E_L,Q)e^{-\beta E_L+\bar{\beta}Q}\,, \tag{5.4}$$

where

$$T_{\mathcal{O}}(E_L,Q) = \sum_i \langle i|\mathcal{O}|i\rangle \delta(Q-Q_i)\delta(E_L-E_{Li})\,, \tag{5.5}$$

is the density of states weighted by the one-point function. The integral above can be inverted to find $T_{\mathcal{O}}(E_L,Q)$,

$$\begin{aligned} T_{\mathcal{O}}(E_L,Q) &= \int \frac{d\beta}{2\pi}\frac{d\bar{\beta}}{2\pi}\langle \mathcal{O}\rangle_{\beta,\bar{\beta}}e^{\beta E_L-\bar{\beta}Q} \\ &= \int \frac{d\beta}{2\pi}\frac{d\bar{\beta}}{2\pi}\langle\chi|\mathcal{O}|\chi\rangle \left(-i\frac{2\pi}{\beta}\right)^{h_{\mathcal{O}}} e^{-\frac{4\pi^2}{\beta}E_{L\chi}-\frac{2\pi i\bar{\beta}}{\beta}Q_\chi+k\frac{\bar{\beta}^2}{4\beta}+\beta E_L-\bar{\beta}Q} \end{aligned} \tag{5.6}$$

At large E_L and $-Q$, this integral is dominated by a saddle point with

$$\beta = 2\pi\sqrt{-\frac{E^{inv}_{L\ \chi}}{E^{inv}_L}}, \qquad \bar{\beta} = \frac{4\pi}{k}\left(iQ_\chi + Q\sqrt{-\frac{E^{inv}_{L\ \chi}}{E^{inv}_L}}\right)\,. \tag{5.7}$$

where $E^{inv}_L = E_L - \frac{Q^2}{k}$, $E^{inv}_{L\ \chi} = E_{L\chi} - \frac{Q_\chi^2}{k}$. When k is negative, the condition for the validity of the saddle point method is $E^{inv}_L \gg 1$. It will become clear that negative k is responsible for the $U(1)$ charge for excited states to be real when considering its gravity dual. It is easy to check that these shifted conformal dimensions E^{inv}_L and $E^{inv}_{L\ \chi}$ are the eigenvalues of the zero modes of spectral flow invariant generators defined in (4.3). To have the real saddle points, it is assumed that $E^{inv}_{L\ \chi} < 0$ and Q_χ is pure imaginary. Under saddle point approximation, the expression of the $T_{\mathcal{O}}(E_L,Q)$ can be written as

$$T_{\mathcal{O}}(E_L,Q) = \frac{(-i)^{h_{\mathcal{O}}}\langle\chi|\mathcal{O}|\chi\rangle}{\sqrt{-kE^{inv}_{L\ \chi}}}\left(\sqrt{-\frac{E^{inv}_{L\ \chi}}{E^{inv}_L}}\right)^{2-h_{\mathcal{O}}} e^{4\pi\sqrt{-E^{inv}_{L\ \chi}E^{inv}_L}-\frac{4\pi i}{k}Q_\chi Q}\,. \tag{5.8}$$

We can also carry out the density of states at large E_L and $-Q$ by choosing the operator $\mathcal{O}$ being the identity and setting ${E_L^{inv}}_\chi \to {E_L^{inv}}_{vac}$ in $T_{\mathcal{O}}(E_L, Q)$,

$$\rho(E_L, Q) = \frac{1}{\sqrt{-k{E_L^{inv}}_{vac}}}\left(\sqrt{-\frac{{E_L^{inv}}_{vac}}{E_L^{inv}}}\right)^2 e^S\,, \tag{5.9}$$

$$\tag{5.10}$$

where S is the entropy formula derived in [6],

$$S = 4\pi\sqrt{-{E_L^{inv}}_{vac}E_L^{inv}} - \frac{4\pi i}{k}Q_{vac}Q\,. \tag{5.11}$$

As discussed in CFT, one can define the typical value of the three-point coefficient $\mathcal{C}_{\mathcal{O}}(E_L, Q)$ by

$$\mathcal{C}_{\mathcal{O}}(E_L, Q) \equiv \frac{T_{\mathcal{O}}(E_L, Q)}{\rho(E_L, Q)}\,. \tag{5.12}$$

At large E_L^{inv}, $\mathcal{C}_{\mathcal{O}}(E_L, Q)$ can be approximated by

$$\mathcal{C}_{\mathcal{O}}(E_L, Q) \sim \tag{5.13}$$

$$(-i)^{h_{\mathcal{O}}}\langle\chi|\mathcal{O}|\chi\rangle\sqrt{\frac{{E_L^{inv}}_\chi}{{E_L^{inv}}_{vac}}}\sqrt{-\frac{E_L^{inv}}{{E_L^{inv}}_\chi}}^{h_{\mathcal{O}}} e^{4\pi\left(\sqrt{\frac{-{E_L^{inv}}_\chi}{-{E_L^{inv}}_{vac}}}-1\right)\sqrt{-{E_L^{inv}}_{vac}E_L^{inv}}-\frac{4\pi i}{k}\left(\frac{Q_\chi}{Q_{vac}}-1\right)Q_{vac}Q}.$$

The above formula (5.13) characterizes the asymptotic growth of the mean three point coefficient in WCFT. This is similar to the result of [42] in the context of CFTs with an additional U(1) symmetry. Although the expression (5.13) looks complicated, we will see that it can also be interpreted as the tadpole diagram (2.34) in the bulk gravity.

5.2 Matching in AdS$_3$/WCFT

Now we consider the bulk interpretation of the average three-point coefficient $\mathcal{C}_{\mathcal{O}}(E_L, Q)$ (5.13) in the context of AdS$_3$/WCFT. Under CSS boundary conditions [7], Einstein gravity on asymptotically AdS$_3$ spacetimes are dual to WCFTs. Similar to the previous section, in this section we also consider the large c limit of the heavy-heavy-light type correlators with $\mathcal{O}$ light and $|E_L, Q\rangle$ heavy. For the light operator $\mathcal{O}$ with $1 \ll h_{\mathcal{O}} \ll \frac{c}{24}$, its dual field can be viewed as perturbative bulk field lives in AdS$_3$. The heavy state with $E_L^{inv}, -\frac{Q^2}{k} \geq \frac{c}{24}$ is well described by the BTZ geometry (2.1) with the identification (2.2).

We propose that the bulk dual of the expectation value for the torus one-point function is still given by the tadpole diagram as depicted in Figure 1. Denote the bulk duals of $\mathcal{O}$ by $\phi_{\mathcal{O}}$, and χ by ϕ_χ. Both $\phi_{\mathcal{O}}$ and ϕ_χ carry mass and spin. The amplitude contains a propagator of $\phi_{\mathcal{O}}$ emanating from infinity, a propagator of ϕ_χ wrapping the black hole, and a cubic coupling $\langle\chi|\mathcal{O}|\chi\rangle$. In the semiclassical limit, the propagator can be approximated by the on-shell action of massive spinning particles

$$\begin{aligned}\mathcal{C}^{bk}_{\mathcal{O}}(E_L, Q) &= \langle\chi|\mathcal{O}|\chi\rangle e^{-S_{\mathcal{O}}}e^{-S_\chi} \\ e^{-S_{\mathcal{O}}} &= T_u^{\frac{\ell m_{\mathcal{O}}+s_{\mathcal{O}}}{2}}T_v^{\frac{\ell m_{\mathcal{O}}-s_{\mathcal{O}}}{2}}\,, \quad e^{-S_\chi} = e^{-\ell m_\chi 2\pi(T_v+T_u)-s_\chi 2\pi(T_u-T_v)}\end{aligned} \tag{5.14}$$

where in the second line we have used the results (2.20) and (2.25). Note that the bulk result written in terms of the bulk variables is the same as the story of AdS_3/CFT_2 proposed in [1] and generalized in the previous section. In the following we will use the holographic dictionary of AdS_3/WCFT to show that the bulk amplitude (5.14) will match the average three-point coefficient in WCFT (5.13) up to some normalization factors. This indicates that holographic dualities for WCFT has another universal property, that is, the agreement between the tadpole diagram in the bulk and the mean three-point coefficient on the boundary.

The bulk dual of $\mathcal{O}$

As discussed in the story of AdS_3/CFT_2, we expect to reproduce the contribution from the operator $\mathcal{O}$ by the boundary to bulk propagator of a bulk field $\phi_{\mathcal{O}}$ from infinity to the horizon. Under the WKB approximation, the propagator of a massive particle from infinity to the horizon of the BTZ metric (2.1) is given by (2.20). Now we need to use the AdS_3/WCFT dictionary to match the quantum numbers.

The bulk-boundary map of AdS_3/WCFT (4.21) relates the conformal dimension and charge to the dimensionless energy and angular momentum. For a perturbative field under the WKB approximation, it increases the dimensionless energy and angular momentum by its mass and spin, $\delta E = \ell m$, $\delta J = s$. From (4.22), we learn that the bulk dual of the chargeless operator $\mathcal{O}$ will have equal mass and spin, which is further given by its conformal weight,

$$\ell m_{\mathcal{O}} = s_{\mathcal{O}} = E_{L\mathcal{O}} + \frac{c}{24} = h_{\mathcal{O}}\,. \tag{5.15}$$

where the shift $\frac{c}{24}$ comes from the background. The dual particle $\phi_{\mathcal{O}}$ is thus a massive spinning particle with equal mass and spin. Its regularized amplitude propagating from infinity to the horizon is given by (2.20). Plugging the above relation into the general formula (2.20) for massive spinning particles, the radial boundary to bulk propagator can be written as

$$e^{-S_{\mathcal{O}}} = T_u^{h_{\mathcal{O}}}. \tag{5.16}$$

The bulk dual of χ

For the lightest state $|\chi\rangle$ coupled to $\mathcal{O}$, we consider the region $1 \ll E_L^{inv}{}_\chi + \frac{c}{24} < \frac{c}{24}$, $1 \ll -\frac{Q_\chi^2}{k} + \frac{c}{24} < \frac{c}{24}$. The bulk dual is again a massive spinning particle going around the horizon, whose on-shell action in terms of mass and spin was given in (2.25). In the following, we use the AdS_3/WCFT dictionary to further relate mass and spin to quantum numbers in WCFT.

When ϕ_χ is in the perturbative region, the discussion is similar to the operator $\mathcal{O}$, but with arbitrary charge Q_χ turned on. One can relate the mass and spin of ϕ_χ to the conformal dimension and charge of $|\chi\rangle$ through (4.21). Comparing to CFTs, L_0^{inv} and $-\frac{P_0^2}{k}$ play the role as left and right moving energies, respectively. So we have the following relation,

$$E_L^{inv}{}_\chi + \frac{c}{24} = \frac{\ell m_\chi + s_\chi}{2}, \qquad -\frac{Q_\chi^2}{k} + \frac{c}{24} = \frac{\ell m_\chi - s_\chi}{2}\,. \tag{5.17}$$

In the non-perturbative region, the relation between the mass and spin in the bulk and conformal dimension and charge in WCFT can be determined through the backreacted geometry, similar to the discussion in section 2.3. For a spinning particle with mass m_χ and spin s_χ, the backreacted geometry is still given by (2.26), which can be rewritten in the standard form (2.32) with the standard spatial identification (2.2), and the "temperatures" $T_{\chi u}$, $T_{\chi v}$ determined by the mass and spin (2.33). However, the crucial difference now is that we should use the AdS$_3$/WCFT dictionary (4.21) to further relate the bulk quantities $T_{\chi u}$, $T_{\chi v}$ to WCFT quantum numbers. To be more explicit, we have

$$E^{inv}_{L\ \ \chi} = \frac{\ell}{4G} T_{\chi u}^2, \qquad Q_\chi = -\frac{T_{\chi v}}{2}\sqrt{-\frac{\ell k}{G}} \tag{5.18}$$

Plugging the above relation to (2.33) and using (4.24), one can rewrite the mass and spin of ϕ_χ in terms of its spectral flow invariant dimension and charge,

$$m_\chi = \frac{1}{4G} - \frac{1}{8G}\left(\sqrt{\frac{-E^{inv}_{L\ \ \chi}}{-E^{inv}_{L\ \ vac}}} + \frac{Q_\chi}{Q_{vac}}\right), \qquad s_\chi = -\frac{\ell}{8G}\left(\sqrt{\frac{-E^{inv}_{L\ \ \chi}}{-E^{inv}_{L\ \ vac}}} - \frac{Q_\chi}{Q_{vac}}\right). \tag{5.19}$$

Note that the relation in the perturbative region (5.17) can be obtained as the leading order expansion in the limit ℓm_χ, $s_\chi \ll c$. We will henceforth focus on the relation (5.19). Substituting the mass and spin formula above into (2.25), we find the contribution to amplitude from ϕ_χ can be recast as

$$e^{-S_\chi} = e^{\frac{\pi\ell}{2G}\left(\sqrt{\frac{-E^{inv}_{L\ \ \chi}}{-E^{inv}_{L\ \ vac}}}-1\right)T_u + \frac{\pi\ell}{2G}\left(\frac{Q_\chi}{Q_{vac}}-1\right)T_v}. \tag{5.20}$$

The bulk dual for the structure constant

Putting the two parts (5.16) and (5.20) together, we can rewrite the total amplitude (5.14) for the process given by Figure 1 under CSS boundary conditions,

$$\mathcal{C}^{bk}_{\mathcal{O}}(E_L, Q) = \underbrace{\langle\chi|\mathcal{O}|\chi\rangle}_{\text{vertex}}\ \underbrace{T_u^{h_\mathcal{O}}}_{\langle\phi_\mathcal{O}\phi_\mathcal{O}\rangle}\ \underbrace{e^{\frac{\pi\ell}{2G}\left(\sqrt{\frac{-E^{inv}_{L\ \ \chi}}{-E^{inv}_{L\ \ vac}}}-1\right)T_u + \frac{\pi\ell}{2G}\left(\frac{Q_\chi}{Q_{vac}}-1\right)T_v}}_{\langle\phi^\dagger_\chi\phi_\chi\rangle}, \tag{5.21}$$

Up to an overall normalization that does not depend on the E_L and Q, this amplitude recover the WCFT result (5.13) rewritten in terms of temperatures.

Note that the bulk process has the same structure as (3.20) in AdS/CFT, consisting of the vertex, the propagator $\langle\phi_\mathcal{O}\phi_\mathcal{O}\rangle$ from boundary to the horizon, and the propagator $\langle\phi_\chi\phi_\chi\rangle$ around the horizon. However, here we would like to point out some features of the results in AdS/WCFT. In the highest weight representation of Virasora-Kac-Moody algebra, all states are eigenstates of the $U(1)$ generator P_0. Charge eigenstate together with charge conservation impose strong constraints in the bulk [33]. For instance, charge conservation requires that the operator $\mathcal{O}$ to be chargeless. In the semiclassical bulk picture, the geodesic is along the radial direction, with no momentum along v, and the massive particle has the equal mass and spin (5.15). Thus there is no T_v dependence for $\langle \phi_{\mathcal{O}} \phi_{\mathcal{O}} \rangle$ in WCFT, as opposed to the CFT result (3.20). The exponential part in the above formula (5.21) is the contribution from ϕ_χ. The $|\chi\rangle$ state in the dual WCFT is a state with pure imaginary charge which is predicted by the modular bootstrap of WCFT [34], so Q_χ and Q_{vac} are both pure imaginary numbers and the amplitude is real.

To summarize, we derived a universal formula for the asymptotic growth of the average three-point coefficient, and performed a bulk calculation in the context of AdS/WCFT correspondence. In addition to the fact that the thermal entropy of a BTZ black hole is captured by the DHH formula in WCFT [6], the matching of (2.34) and (5.13) gives further evidence of the black hole geometries in asymptotically AdS_3 spacetimes can emerge upon coarse-graining over microstates in WCFTs.

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Appendix A: an alternative derivation of the on-shell action at the horizon

In this appendix, we will use the non-rotating frame to the calculate the on-shell action for particle ϕ_χ.

Since the horizon is a degenerate hypersurface, here we use the trick to calculate the on-shell action. We take the orbit of the particle ϕ_χ to be a circular orbit with constant radius greater than the horizon's. It can be shown that the constant radius orbits outside the horizon are no longer geodesics, but with accelerations. Then we evaluate the action of ϕ_χ on that orbit, this is not on-shell since (2.8) is not satisfied. Finally, we take the radius of that orbit tends to horizon. On the horizon, the worldline action become on-shell, and we will get the desired result.

By using the metric (2.1), we can find out the tangent and two normal vectors of the constant radius obit outside the horizon at constant time,

$$\dot{X}^\mu\partial_\mu = \frac{1}{\ell\sqrt{2\rho+T_u^2+T_v^2}}\partial_u + \frac{1}{\ell\sqrt{2\rho+T_u^2+T_v^2}}\partial_v, \tag{5.22}$$

$$\tilde{n}^\mu\partial_\mu = \frac{2\sqrt{\rho^2-T_u^2T_v^2}}{\ell}\partial_\rho\,, \tag{5.23}$$

$$n^\mu\partial_\mu = \frac{\rho+T_v^2}{\ell\sqrt{(\rho^2-T_u^2T_v^2)(2\rho+T_u^2+T_v^2)}}\partial_u - \frac{\rho+T_u^2}{\ell\sqrt{(\rho^2-T_u^2T_v^2)(2\rho+T_u^2+T_v^2)}}\partial_v\,. \tag{5.24}$$

Similar to the discussion of particle $\phi_\mathcal{O}$, the spin contribution can be written as

$$S_\chi^{\text{spin}} = s_\chi \log\left(\frac{\tilde{q}(\tau_f)\cdot\tilde{n}_f - q(\tau_f)\cdot\tilde{n}_f}{\tilde{q}(\tau_i)\cdot\tilde{n}_i - q(\tau_i)\cdot\tilde{n}_i}\right)\,, \tag{5.25}$$

Before taking the limit, the orbit has acceleration, the notion of parallel transport should be replaced by Fermi-Walker transport. Here the vectors q^μ and $\tilde{q}^\mu$ satisfying

$$\begin{aligned} &q^2=-1, \quad \tilde{q}^2=1, \quad q\cdot\tilde{q}=0, \quad q\cdot\dot{X}=\tilde{q}\cdot\dot{X}=0, \\ &\nabla q^\mu = -\dot{X}^\mu(q\cdot A), \quad \nabla\tilde{q}^\mu = -\dot{X}^\mu(\tilde{q}\cdot A)\,, \end{aligned} \tag{5.26}$$

where $A^\mu = \nabla\dot{X}$ is the acceleration corresponds to the tangent vector $\dot{X}$. The Fermi-Walker transport was imposed to ensure that the frame $(q(\tau),\tilde{q}(\tau))$ is non-rotated along the orbit, and compare to another frame $(n(\tau),\tilde{n}(\tau))$, we can obtain the relative change of the Lorentz boost. The explicit expressions of q and $\tilde{q}$ for constant radius orbit outside the horizon can be written down by solving (5.26),

$$\begin{aligned} q^\mu\partial_\mu =& \frac{(\rho+T_v^2)\cosh\left(\frac{(T_u^2-T_v^2)\varphi}{\sqrt{2\rho+T_u^2+T_v^2}}\right)}{\ell\sqrt{(\rho^2-T_u^2T_v^2)(2\rho+T_u^2+T_v^2)}}\partial_u - \frac{(\rho+T_u^2)\cosh\left(\frac{(T_u^2-T_v^2)\varphi}{\sqrt{2\rho+T_u^2+T_v^2}}\right)}{\ell\sqrt{(\rho^2-T_u^2T_v^2)(2\rho+T_u^2+T_v^2)}}\partial_v \\ &- \frac{2\sqrt{\rho^2-T_u^2T_v^2}\sinh\left(\frac{(T_u^2-T_v^2)\varphi}{\sqrt{2\rho+T_u^2+T_v^2}}\right)}{\ell}\partial_\rho\,, \end{aligned} \tag{5.27}$$

$$\begin{aligned} \tilde{q}^\mu\partial_\mu =& -\frac{(\rho+T_v^2)\sinh\left(\frac{(T_u^2-T_v^2)\varphi}{\sqrt{2\rho+T_u^2+T_v^2}}\right)}{\ell\sqrt{(\rho^2-T_u^2T_v^2)(2\rho+T_u^2+T_v^2)}}\partial_u + \frac{(\rho+T_u^2)\sinh\left(\frac{(T_u^2-T_v^2)\varphi}{\sqrt{2\rho+T_u^2+T_v^2}}\right)}{\ell\sqrt{(\rho^2-T_u^2T_v^2)(2\rho+T_u^2+T_v^2)}}\partial_v \\ &+ \frac{2\sqrt{\rho^2-T_u^2T_v^2}\cosh\left(\frac{(T_u^2-T_v^2)\varphi}{\sqrt{2\rho+T_u^2+T_v^2}}\right)}{\ell}\partial_\rho\,. \end{aligned} \tag{5.28}$$

The metric (2.1) has identification as (2.2), so let us set the initial point with $\varphi=0$ and final point with $\varphi=2\pi$, the difference in q and $\tilde{q}$ between initial and final point are the spin effects. Substituting the formulas (5.27), (5.28), and (5.23) into (5.25), we can get,

$$S_{\text{spin}} = s_\chi \frac{2\pi(T_u^2-T_v^2)}{\sqrt{2\rho+T_u^2+T_v^2}}\,. \tag{5.29}$$

Now we can take limit $\rho\to T_uT_v$ to get the spin contribution on-shelly and combine with the first part of (2.21), we can write down the on-shell action of particle ϕ_χ.

$$S_\chi = \ell m_\chi 2\pi(T_v+T_u) + s_\chi 2\pi(T_u-T_v)\,. \tag{5.30}$$

Appendix B: conical defects with spin

In this appendix, we will check that (2.26) is indeed a solution in three dimensional spacetime with negative cosmological constant, sourced by a point particle with mass m and spin s at the origin,

$$\mathcal{G}_{\mu\nu} = 8\pi G T_{\mu\nu}\,, \tag{5.31}$$

$$T^{00} = m\delta^2(x), \qquad x^i T^{0j} - x^j T^{0i} = s\epsilon^{ij}\delta^2(x)\,, \tag{5.32}$$

where $\mathcal{G}_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} - \frac{1}{\ell^2}g_{\mu\nu}$. The indices $(0, i, j)$ labels the local Cartesian coordinates (x^0, x^1, x^2). One can directly check that (2.26) indeed satisfies the above Einstein equation (5.31) following the lines of arguments of [39, 40]. Alternatively, by integrating both sides of (5.31) over a Cauchy surface, (5.32) indicates that the mass and spin can be calculated from the quasi-local gravitational charges. In the covariant formalism, charges associated to exact Killing vectors can be calculated anywhere, and the difference between the usual ADM charges and quasi-local charges comes from the choice of Killing vectors [43]. In general charge integrability imposes strong constraints on the normalizations. More explicitly, the parameters of the local source can be calculated from the covariant charges

$$m = Q[\partial_0], \quad sC = Q[\partial_\theta], \tag{5.33}$$

where $x^1 = r\cos C\theta$, $x^2 = r\sin C\theta$, $x^0 = t - A\theta$ with $\theta \sim \theta + 2\pi$ are the locally polar coordinates. Explicit expression of the charges can be found in [43].

Now let us check the local source of the conical solution (2.26) using the prescription above. Consider the a conical defect solution in global AdS with "jump"

$$\frac{ds^2}{\ell^2} = \frac{dr^2}{1+r^2} + r^2 d\varphi^2 - (1+r^2)(dt - \frac{A}{C}d\varphi)^2\,. \tag{5.34}$$

$$\varphi \sim \varphi + 2\pi C, \quad C = 1 - \frac{\delta\varphi}{2\pi}, \tag{5.35}$$

which can be brought to the standard form (2.32) where we can read the asymptotic charges by,

$$u = \frac{\varphi}{C} + \frac{t}{C-A}, \quad v = \frac{\varphi}{C} - \frac{t}{C+A}\,. \tag{5.36}$$

The total gravitational energy and angular momentum are measured using the asymptotic Killing vectors $E_L = Q[\partial_u] = -\frac{\ell}{16G}(C-A)^2$, $\quad E_R = Q[-\partial_v] = -\frac{\ell}{16G}(C+A)^2$. On the other hand, the local source can be measured by the Killing vectors normalized properly near the origin, using the locally Cartesian coordinates $x^0 = \ell(t - \frac{A}{C}\varphi)$, $x^1 = r\cos\varphi$, $x^2 = r\sin\varphi$. Therefore the particle mass can be calculated by considering the infinitesimal charge

$$\begin{aligned} \ell\delta m &= \delta Q[\partial_t] \\ &= \frac{1}{C-A}\delta Q[\partial_u] + \frac{1}{C+A}\delta Q[\partial_v] \\ &= -\ell\frac{\delta C}{4G}\,. \end{aligned} \tag{5.37}$$

By requiring that smoothness when $m = 0$, we can integrate the above charge and get the particle mass

$$m = \frac{1-C}{4G} = \frac{\delta\varphi}{8\pi G}\,. \tag{5.38}$$

Similarly, we can get the spin of the particle by

$$s = \frac{1}{C}Q[C\partial_\varphi] = \frac{1}{C}(Q[\partial_u] + Q[\partial_v]) = \frac{A\ell}{4G}\,. \tag{5.39}$$

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