The decay law can have an irregular character

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(Dated: October 16, 2018)

Within a well-known decay model describing a particle confined initially within a spherical δ potential shell, we consider the situation when the undecayed state has an unusual energy distribution decaying slowly as $k \to \infty$; the simplest example corresponds to a wave function constant within the shell. We show that the non-decay probability as a function of time behaves then in a highly irregular, most likely fractal way.

PACS numbers: 03.65.Xp

The decay of an unstable quantum systems is one of the effects frequently discussed and various aspects of such processes were considered. To name just a few, recall the long-time deviation from the exponential decay law[1], the short-time behavior related to the Zeno and anti-Zeno effects [3, 4, 5, 6], revival effects such the classical one in the kaon-antikaon system, etc. In all the existing literature [7], however, the decay law is treated as a smooth function, either explicitly or implicitly, e.g. by dealing with its derivatives. The aim of this letter is to show there are situation when it is not the case.

A hint why it could be so comes from the behavior of Schrödinger wave functions during the time evolution. While in most cases the evolution causes smoothing [8], it may not be true for for a particle confined in a potential well and the initial state does not belong to the domain of the Hamiltonian. A simple and striking example was found by M. Berry [9] for a rectangular hard-wall box, and independently by B. Thaller [10] for a one-dimensional infinite potential well. It appears that if the initial wave function is constant, it evolves into a steplike-shaped $\psi(x,t)$ for times which are rational multiples of the period, $t = qT$ with $q = N/M$, and the number of steps increases with growing M, while for an irrational q the function $\psi(x,t)$ is fractal w.r.t. the variable x.

One can naturally ask what will happen if the hard wall is replaced by a semitransparent barrier through which the particle can tunnel into the outside space. In a broad sense this is one of the most classical decay model which can be traced back to [11]. We will deal with its particular case when the barrier is given by a spherical δ potential which is sometimes called *Winter model* being introduced for the first time, to our knowledge, in [12]; see also [13]. The described behavior of the wave function in the absence of tunneling suggests that in the decaying system the irregular time dependence could also be visible, both in the wave function and in various quantities derived from it [14], at least in the weak coupling case. The aim of the present letter is to demonstrate that this conjecture is indeed valid.

To be concrete, we will study a spinless nonrelativistic quantum particle described by the Hamiltonian

$$H_\alpha = -\triangle + \alpha\delta(|\vec{r}| - R)\,, \quad \alpha > 0\,, \tag{1}$$

with a fixed $R > 0$; we use rational units, $\hbar = 2m = 1$. For simplicity we restrict our attention to the s-wave part of the problem, writing thus the wave functions as $\psi(\vec{r},t) = \frac{1}{\sqrt{4\pi}} r^{-1}\phi(r,\,t)$ with the associated Hamiltonian

$$\mathsf{h}_\alpha = -\frac{\mathsf{d}^2}{\mathsf{d}r^2} + \alpha\delta(r-R) \tag{2}$$

in the lowest partial wave. We are interested in the time evolution determined by the Hamiltonian (1), $\psi(\vec{r},t) = e^{-iH_\alpha t}\psi(\vec{r},0)$ for a fixed initial condition $\psi(\vec{r},0)$ with the support inside the ball of radius R; the advantage of the used model is that the propagator can be computed explicitly. Of a particular interest is the decay law,

$$P(t) = \int_0^R |\phi(r,\,t)|^2\,\mathsf{d}r\,, \tag{3}$$

i.e. the probability that the system localized initially within the shell will be still found there at the measurement performed at an instant t. We are going to derive an exact formula for the decay law which we will then allow us to evaluate the function (3) numerically for a given initial state.

It is straightforward to check [13] that the Hamiltonian (1) has no bound states. On the other hand, it has infinitely many resonances with the widths increasing logarithmically w.r.t. the resonance index. A natural and well known idea [15, 16] is to employ them as a tool to expand the quantities of interest.

First of all, we have to find Green's function $g(k,\,r,\,r')$, i.e. the integral kernel of $(\mathsf{h}_\alpha - k^2)^{-1}$ which determines the time evolution in the standard way [17],

$$e^{-i\mathsf{h}_\alpha t} = \frac{1}{\pi}\lim_{\varepsilon\downarrow 0}\int_0^\infty e^{-i\lambda t}\,\mathrm{Im}\,\frac{1}{\mathsf{h}_\alpha - \lambda - i\varepsilon}\,\mathsf{d}\lambda\,, \tag{4}$$

recall that $\sigma(\mathsf{h}_\alpha) = [0,\infty)$ for $\alpha > 0$; we are going to perform a resonance expansion of the integral (4). The Green function for a system with singular potential is obtained from Krein's formula,

$$\frac{1}{\mathsf{h}_\alpha - k^2} = \frac{1}{\mathsf{h}_0 - k^2} + \lambda(k)(\Phi_k,\cdot)\Phi_k(r)\,,$$

where $\Phi_k(r)$ is the free Green function, in particular, $\Phi_k(r) = \frac{1}{k}\sin(kr)\,e^{ikR}$ holds for $r < R$, and $\lambda(k)$ is determined by boundary conditions at the singular point R; by a direct calculation [13] one finds

$$\lambda(k) = -\frac{\alpha}{1+\frac{i\alpha}{2k}(1-e^{2ikR})}\,. \tag{5}$$

This allows us to write the sought reduced wave function at time t as $\phi(r,\,t) = \int_0^\infty u(t,\,r,\,r')\phi(r',\,0)\,\mathsf{d}r'$ with

$$u(t,\,r,\,r') = \frac{1}{\pi}\lim_{\varepsilon\downarrow 0}\int_0^\infty e^{-ik^2t}\,\mathrm{Im}\,g(k+i\varepsilon,\,r,\,r')\,2k\,\mathsf{d}k$$

Since h_α has no eigenvalues we may pass in the last formula to the limit $\varepsilon \to 0$ obtaining

$$u(t,\,r,\,r') = \int_0^\infty p(k,\,r,\,r')\exp(-ik^2t)\,2k\,\mathsf{d}k, \tag{6}$$

where $p(k,\,r,\,r') = \frac{1}{\pi}\mathrm{Im}\,g(k,\,r,\,r')$. This can be using equation (5) written explicitly as

$$p(k,\,r,\,r') = \frac{2k\sin(kr)\sin(kr')}{\pi(2k^2+2\alpha^2\sin^2 kR+2k\alpha\sin 2kR)}\,. \tag{7}$$

The resonances of the problem are identified with the poles of $g(\cdot,\,r,\,r')$ continued analytically to the lower momentum halfplane. They exist in pairs, those in the fourth quadrant, denoted as k_n in the increasing order of their real parts, and $-\bar{k}_n$. The set of singularities of the kernel (7) then include also the mirrored points, being

$$S = \{k_n,\,-k_n,\,\bar{k}_n,\,-\bar{k}_n\,:\;n\in\mathbb{N}\}. \tag{8}$$

In the vicinity of the singular point k_n the function $p(\cdot,\,r,\,r')$ can be written as

$$p(k,\,r,\,r') = \frac{i}{2\pi}\frac{v_n(r)v_n(r')}{k^2-k_n^2} + \chi(k,\,r,\,r'), \tag{9}$$

where $v_n(r)$ solves the differential equation $\mathsf{h}_\alpha v_n(r) = k_n^2 v_n(r)$ and the function χ is locally analytic.

The factor $\frac{i}{2\pi}$ is chosen to get the conventional normalization of the resonant state $v_n(r)$ [15]. The last named paper also demonstrates that for a finite-range potential barrier and $r,\,r' < R$ the function $p(\cdot,\,r,\,r')$ decreases in every direction of the k-plane; in the present case it is not difficult to verify this claim directly. Then one can express $p(k,\,r,\,r')$ as the sum over the pole singularities

$$p(k,\,r,\,r') = \sum_{\tilde{k}\in S}\frac{1}{k-\tilde{k}}\,\mathrm{Res}_{\tilde{k}}\,p(k,\,r,\,r') \tag{10}$$

and derive from the residue theorem the following useful formula

$$\sum_{\tilde{k}\in S}\mathrm{Res}_{\tilde{k}}\,p(k,\,r,\,r') = 0\,. \tag{11}$$

The last two equations can be rewritten in view of eq. (9) and the symmetry of the set S in the form

$$p(k,\,r,\,r') = \sum_{n\in\mathbb{Z}}\frac{i}{2\pi}\frac{1}{k^2-k_n^2}\frac{k}{k_n}\,v_n(r)v_n(r')\,, \tag{12}$$

$$\sum_{n\in\mathbb{Z}}\frac{1}{k_n}\,v_n(r)v_n(r') = 0\,, \tag{13}$$

where we denote $k_{-n} := -\bar{k}_n$ and v_{-n} is the associated solution of the equation $\mathsf{H}_\alpha v_{-n}(r) = k_{-n}^2 v_{-n}(r)$.

Next we substitute from eq. (12) into (6) and using the identity (13) we arrive at the formula

$$\begin{aligned} u(t,\,r,\,r') &= \frac{i}{2\pi}\int_0^\infty\sum_{n\in\mathbb{Z}}\frac{\exp(-ik^2t)}{k^2-k_n^2}\frac{2k^2}{k_n}v_n(r)v_n(r')\mathsf{d}k \\ &= \sum_{n\in\mathbb{Z}} M(k_n,\,t)v_n(r)v_n(r') \end{aligned} \tag{14}$$

with $M(k_n,\,t) = \frac{1}{2}\,e^{u_n^2}\,\mathrm{erfc}(u_n)$ and $u_n := -e^{-i\pi/4}k_n\sqrt{t}$. Indeed, using $2k^2 = 2(k^2-k_n^2)+2k_n^2$ we write $u(t,\,r,\,r')$ as a sum of two terms the first of which vanishes in view of (13). The second one decomposes again into a sum of two integrals containing $k_n \pm k$ in their denominators, which gives the right-hand side of (14) with

$$M(k_n,\,t) = \frac{i}{2\pi}\int_{-\infty}^\infty\frac{e^{-ik^2t}}{k-k_n}\,\mathsf{d}k\,, \tag{15}$$

in other words, the above expression.

Now a straightforward calculation using (3) and (14) allows us to express the decay law in the form

$$P(t) = \sum_{n,\,l} C_n\bar{C}_l I_{nl} M(k_n,\,t)\overline{M(k_l,\,t)}\,, \tag{16}$$

with the coefficients

$$C_n := \int_0^R \phi(r,\,0)v_n(r)\,\mathsf{d}r\,,\;\; I_{nl} := \int_0^R v_n(r)\bar{v}_l(r)\,\mathsf{d}r\,. \tag{17}$$

This expression holds generally, see [16]; in our particular case of the Hamiltonian (2) we can specify $v_n(r) = \sqrt{2}Q_n\sin(k_n r)$ with the coefficient Q_n equal to

$$\left(\frac{-2ik_n^2}{2k_n+\alpha^2R\sin 2k_nR+\alpha\sin 2k_nR+2k_n\alpha R\cos 2k_nR}\right)^{1/2}$$

Now we are ready to compute the decay law for a given initial state. Without loss of generality we may put $R = 1$, we choose the value $\alpha = 500$ for numerical evaluation and replace the infinite series by a cut-off one with $|n| \le 1000$. As the first example we consider initial wave function constant within the well, i.e.

$$\phi(r,\,0) = R^{-3/2}\sqrt{3}r\,, \quad r < R\,. \tag{18}$$

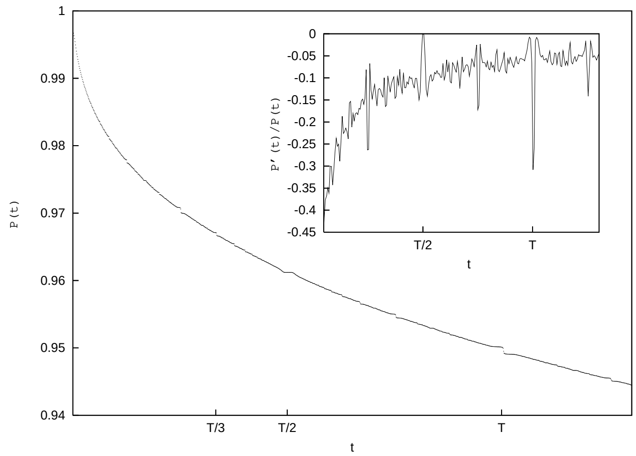

FIG. 1: Decay law for the initial state $\phi(r,\,0) = R^{-3/2}\sqrt{3}r$. In the inset we plot the logarithmic derivative averaged over intervals of the length approximately $T/200$.

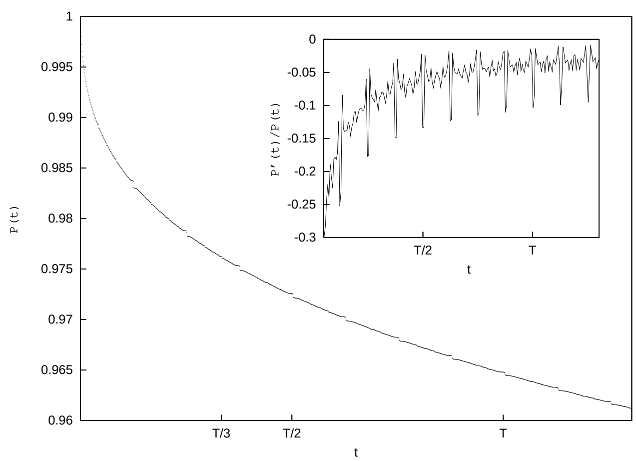

FIG. 2: Decay law for initial state $\phi(r,\,0) = R^{-1/2}$ and its logarithmic derivative, locally smeared, in the inset.

The corresponding decay law is plotted in Figure 1. It is irregular having numerous steps [18], the most pronounced at the period $T = 2R^2/\pi$ and its simple rational multiples. To make them more visible we plot in the inset the logarithmic derivative of the function $P(t)$; it is locally smeared, otherwise the picture would be a fuzzy band. The irregular structure is expected to be fractal; it persists at higher time but its amplitude decreases relatively w.r.t. the smooth background.

In the next example we choose the initial state having constant reduced wave function, $\phi(r,\,0) = R^{-1/2}$ for $r < R$, so $\psi(r,\,0)$ has a (square integrable) singularity at the origin; the advantage is that the reduced problem offers a straightforward comparison to the one-dimensional example treated in [10] including the shapes of the wave functions. The decay law for this case is plotted in Figure 2; it again exhibits derivative jumps around simple rational multiples of the period. The corresponding function $|\phi(r,\,t)|^2$ for three such values is plotted in Figure 3.

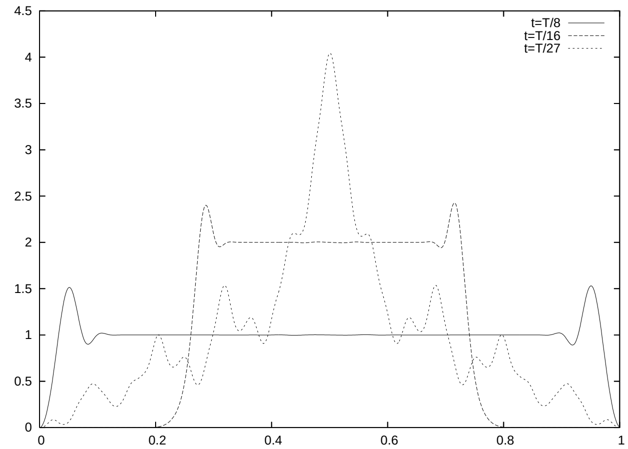

FIG. 3: The probability density inside the sphere with $R = 1$ multiplied by r^2 at the instants $t = T/8$, $T/16$, and $T/27$.

To compare with [10] notice that the one-dimensional well has length $R = 1$ so the revival period of this state is $T/8$. In the absence of decay the function for $t = T/8$ is just constant, the other two are simple step functions. We see that the tunneling through δ-barrier modifies the shape of the function mostly in the vicinity of the origin, the barrier, and the jump points.

To support the conjecture about the fractal character of $P(t)$ let us look how its derivative behaves in the limit $\alpha \to \infty$ when $\phi(r,\,t)$ expands asymptotically as

$$\phi(r,\,t) \approx \sum_n C_n \exp(-ik_n^2 t) v_n(r)\,. \tag{19}$$

It is easy to see that for a fixed n and $\alpha \to \infty$ the resonance position expands around $k_{n,0} := n\pi/R$ as

$$k_n \approx k_{n,0} - \frac{k_{n,0}}{\alpha R} + \frac{k_{n,0}}{(\alpha R)^2} - i\frac{k_{n,0}^2}{\alpha^2 R} + \cdots \tag{20}$$

which shows that in the leading order we have $v_n(r) \approx \sqrt{\frac{2}{R}}\sin(k_n r)$, and furthermore, that the substantial contribution to the sum in (19) comes from terms with $n \lesssim [\alpha^{1-\varepsilon}\frac{R}{\pi}]$ for some $0 < \varepsilon < 1/3$.

The derivative $\dot{P}(t)$ can be computed from the probability current conservation $j(r) = -\frac{\mathsf{d}}{\mathsf{d}t}|\phi(r,\,t)|^2$; integrating it over the interval $(0, R)$ and using $j(0) = 0$ we get

$$\dot{P}(t) = -2\mathrm{Im}\,(\phi'(R,\,t)\bar{\phi}(R,\,t))\,. \tag{21}$$

We plug the above expansion into (19) obtaining

$$\phi(R,\,t) \;\approx\; \sqrt{\frac{2}{R}}\sum_{n=1}^{\infty}(-1)^n C_n \exp\left[-ik_{n,0}^2 t\left(1-\frac{2}{\alpha R}\right)\right] \exp\left(-\frac{2k_{n,0}^3}{\alpha^2 R}t\right)\left(-\frac{k_{n,0}}{\alpha} - i\frac{k_{n,0}^2}{\alpha^2}\right) \tag{22}$$

and a similar expansion for $\phi'(R,\,t)$ with the last bracket replaced by $k_{n,0}$. We observe that for $j>-1$ we have

$$\sum_{n=1}^{\infty}\exp\left(-2\frac{k_{n,0}^3}{\alpha^2 R}t\right)k_{n,0}^j\approx\frac{R}{\pi}\left(\frac{R}{2t}\right)^{(j+1)/3}\alpha^{2(j+1)/3}I_j\,,$$

where we have denoted $I_j:=\int_0^\infty e^{-x^3}x^j\,\mathsf{d}x=\frac{1}{3}\Gamma\left(\frac{j+1}{3}\right)$.

Let us now we assume that the coefficients in (19) satisfy $C_n\sim k_{n,0}^{-p}$ as $n\to\infty$. Suppose first that the decay is fast enough, $p>1$; notice that this is certainly true in the finite-energy case with $p>3/2$. The term $-k_{n,0}/\alpha$ in (22) obviously does not contribute to the imaginary part, hence we find that $|\dot P(t)|\le\text{const}\,\alpha^{4/3-4/3p}\to 0$ holds as $\alpha\to\infty$ uniformly in the time variable.

The situation is different if the decay is slow, $p\le 1$. As an illustration take $C_n=(-1)^{n+1}\frac{\sqrt{6}}{Rk_n}$, which corresponds to the first one of the above numerical examples. Since the real part of the resonances changes with α, cf. (20), it is natural to study the limit of $\dot P(t_\alpha)$ as $\alpha\to\infty$ at the moving time value $t_\alpha:=t(1+2/\alpha R)$. Up to higher order terms the appropriate value $\phi(R,\,t_\alpha)$ is obtained by removing the bracket $(1-2/\alpha R)$ at the right-hand side of (22) and $\phi'(R,\,t_\alpha)$ is obtained similarly.

Consider first irrational multiples of T. We use the observation made in [19] that the modulus of $\sum_{n=1}^{L}e^{i\pi n^2 t}$ is for an irrational t bound by $C\,L^{1-\varepsilon}$ where C,ε depend on t only. In combination with a Cauchy-like estimate, $\sum_{n=1}^{\infty}a_nb_n\le\sum_{n=1}^{\infty}|\sum_{j=1}^{n}a_j|\,|b_n-b_{n+1}|$ which yields

$$\sum_{n=1}^{\infty}\exp(-ik_{n,0}^2t)\exp\left(-\frac{2k_{n,0}^2}{\alpha^2R}t\right)k_{n,0}^j\lesssim\text{const}\,\alpha^{2/3(j+1-\varepsilon)}$$

and consequently, $\dot P(t_\alpha)\to 0$ as $\alpha\to\infty$ similarly as in the case of fast decaying coefficients.

Let us assume next rational times, $t=\frac{p}{q}T$. If pq is odd then $S_L(t):=\sum_{n=1}^{L}e^{i\pi n^2 t}$ repeatedly retraces by [19] the same pattern, hence $\dot P(t_\alpha)\to 0$ – cf. Fig. 1 at the half period. On the other hand, for pq even $|S_L(t)|$ grows linearly with L, and consequently, $\lim_{\alpha\to\infty}\dot P(t_\alpha)>0$. As a example let us compute this limit for the period T, i.e. $p=q=1$. Using (21) we find

$$\lim_{\alpha\to\infty}\dot P(T_\alpha)=-\frac{24}{R^2}\lim_{\alpha\to\infty}\text{Im}\left(\sum_{n=1}^{\infty}\exp\left(-\frac{2k_{n,0}^3}{\alpha^2R}T\right)\right.$$
$$\left.\sum_{n=1}^{\infty}\exp\left(-\frac{2k_{n,0}^3}{\alpha^2R}T\right)\left(-\frac{1}{\alpha}+i\frac{k_{n,0}}{\alpha^2}\right)\right)$$
$$=-\frac{24}{R^2}\left(\frac{R}{\pi}\right)^2\frac{1}{2T/R}\,I_1I_0=-\frac{4}{3\sqrt{3}}\approx -0.77\,;$$

notice that without a local smearing $\dot P(T)$ in the inset of Fig. 1 would take approximately this value.

In conclusion, we have reexamined time decay in Winter model and found indications that the decay law is a highly irregular function if the energy distribution decays slowly as $k\to\infty$.

The research was partially supported by GAAS and MEYS of the Czech Republic under projects A100480501 and LC06002.

[1] L.A. Khalfin, *Zh. Eksp. Teor. Fiz.* **33**, 1371 (1957); for a more complete bibliography see Ref. [2]

[2] P. Exner: *Open Quantum Systems and Feynman Integrals*, D. Reidel, Dordrecht 1985.

[3] J. Beskow, J. Nilsson, *Arkiv Fys.* **34**, 561 (1967).

[4] B. Misra, E.C.G. Sudarshan, *J. Math. Phys.* **18**, 756 (1977).

[5] A.U. Schmidt, in *Mathematical Physics Research on Leading Edge* (Ch. Benton, ed.), Nova Sci, Hauppauge NY, 2004; pp. 113-143.

[6] P. Exner, *J. Phys.* **A38**, L449 (2005).

[7] The bibliography concerning unstable systems is vast. A large part can be derived from sources like Refs. 2 and 5; one can also mention, e.g., L. Fonda, G.C. Ghirardi, A. Rimini, *Rep. Progr. Phys.* **41**, 587 (1978).

[8] K. Yajima, in *Functional-analytic methods for partial differential equations*, Springer Lecture Notes in Math., vol. 1450, Berlin 1990; p. 20

[9] M.V. Berry, *J. Phys.* **A29**, 6617 (1996).

[10] B. Thaller: *Visual Quantum Mechanics*, Springer, Heidelberg 2000.

[11] G. Gamow, *Z. Phys.* **51**, 204 (1928).

[12] R.G. Winter, *Phys. Rev.* **123**, 1503 (1961).

[13] A thorough analysis of Winter's model can be found in J.-P. Antoine, F. Gesztesy, J. Shabani, *J. Phys.* **A20**, 3687 (1987). There are various generalizations of the model; we refer to the S. Albeverio, F. Gesztesy, R. Høegh-Krohn, H. Holden: *Solvable Models in Quantum mechanics*, 2nd edition, AMS Chelsea 2005, for a bibliography.

[14] This conjecture is also supported by the study of revivals in an infinite square well with a δ barrier, see G.A. Vugalter, A.K. Das, V.A. Sorokin, *Phys. Rev.* **A66**, 012104 (2002).

[15] G. García-Calderón, R. Peierls, *Nucl. Phys.* **A265**, 443 (1976).

[16] G. García-Calderón, J.L. Mateos, M .Moshinsky, *Phys. Rev. Lett.* **74**, 337 (1995).

[17] M. Reed, B. Simon: *Methods of Modern Mathematical Physics I. Functional Analysis*, Academic Press, New York 1972.

[18] Another striking deviation from the exponential decay law is that the decay rate explodes as $t\to 0$, which is due to the fact that the energy distribution of the state ψ decays too slowly at high energies – cf. P. Exner, J. Phys. **A38**, L449 (2005).

[19] M. V. Berry, J. Goldberg, *Nonlinearity* **1**, 1 (1988).