Resonance asymptotics in the generalized Winter model

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We consider a modification of the Winter model describing a quantum particle in presence of a spherical barrier given by a fixed generalized point interaction. It is shown that the three classes of such interactions correspond to three different types of asymptotic behaviour of resonances of the model at high energies.

Models with contact interactions are popular because they allow us to study properties of quantum systems in a framework which makes explicit solutions possible. It was found in the beginning of the eighties that the usual δ interaction on the line has a counterpart, named not quite fortunately δ', and a little later the complete four-parameter class of the generalized point interactions (GPI's) was introduced [1, 2]. Properties of these interactions are now well understood – see [3] for a rather complete bibliography.

The GPI's fall into different classes according to their behaviour at low and high energies. The most simple manifestation can be found in scattering. While a δ-type barrier behaves as a "usual" regular potential becoming transparent at high energies, the δ'-like one on the contrary decouples asymptotically in the same limit. In addition, there is an *intermediate class* for which both the reflection and transmission amplitudes have nonzero limits as the energy tends to zero or infinity [4]. Furthermore, in Kronig-Penney-type models describing periodic arrays of such interactions, the indicated classes differ by the gap behaviour at high energies; this has important consequences for spectral nature of the corresponding Wannier-Stark systems [5, 6].

In this letter we discuss the GPI's in the context of the generalized *Winter model* describing a quantum particle in presence of a rotationally symmetric surface interaction supported by a sphere; we suppose that the interaction is repulsive corresponding to a barrier which gives rise to tunneling decay and scattering resonances. Our aim is to show that the said GPI classes correspond to different high-energy asymptotics of the resonances, specifically, we are going to show that for the δ-type interaction the resonance lifetime tends to zero with the increasing energy, for δ' interaction it increases to infinity asymptotically linearly in the resonance index, and that there is an intermediate case for which the resonance lifetime still tends to zero but in a slower way than it the δ case, $\emptyset(n^{-1})$ vs. $\emptyset((n\ln n)^{-1})$.

Recall that the Winter model was introduced in [7, 8], originally for the δ barrier in which case the Hamiltonian can be formally written as

$$H = -\triangle + \alpha\delta(|x| - R)\,,$$

where R is the radius of the sphere S_R and α is the coupling constant. A thorough analysis including the δ' extension can be found in [9]; an extension to multiple spheres is given in [10] and other related results are reviewed in [3]. From the mathematical point of view such models are described by means of spherically symmetric self-adjoint extensions of the symmetric operator $\dot{H} : \dot{H}\psi = -\triangle\psi$, defined on $D(\dot{H}) = \{f \in W^{2,2}(\mathbb{R}^3 \setminus S_R),\ f(x) = \nabla f(x) = 0 \quad \text{for} \quad x \in S_R\}$, i.e. the restriction of the free Hamiltonian to functions vanishing together with their derivatives on the sphere. These extensions form a four-parameter family characterized by the conditions (1) below.

The spherical symmetry allows us to reduced the analysis to a family of halfline problems by partial wave decomposition [10]). Using the isometry $U : L^2((0,\infty), r^2 dr) \to L^2(0,\infty)$ defined by $Uf(r) = rf(r)$ we write

$$L^2(R^3) = \bigoplus_l U^{-1} L^2(0,\infty) \otimes S_1^{(l)}\,, \quad \dot{H} = \bigoplus_l U^{-1}\,\dot{h}_l\,U \otimes I_l$$

where I_l is the identity operator on $S_1^{(l)}$ and $\dot{h}_l = -\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2}$ with the domain $D(\dot{h}_l) = \{f \in D(\mathsf{h}_l^*);\ f(R) = f'(R) = 0\}$; here $D(\mathsf{h}_l^*) = W^{2,2}((0,\infty) \setminus R)$ if $l \neq 0$ while for $l = 0$ the requirement $f(0+) = 0$ is added. The equation

$$\dot{h}_l^* f = k^2 f, \quad \mathrm{Im}\, k > 0\,,$$

has two linearly independent solutions [9, 10], namely

$$\Phi_{k,l}^{(1)}(r) \;=\; \begin{cases} \frac{i\pi}{2}\sqrt{r}J_\nu(kr)H_\nu^{(1)}(kR)\sqrt{R} \\ \frac{i\pi}{2}\sqrt{R}J_\nu(kR)H_\nu^{(1)}(kr)\sqrt{r} \end{cases}$$

$$\Phi^{(2)}_{k,l}(r) \;=\; \begin{cases} i\frac{\pi}{2}J_\nu(kr)\sqrt{r}(H_\nu^{'(1)}(kR)\sqrt{R}k + H_\nu^{(1)}(kR)\frac{1}{2\sqrt{R}}) \\ i\frac{\pi}{2}(J_\nu'(kR)k\sqrt{R} + J_\nu(kR)\frac{1}{2\sqrt{R}})H_\nu^{(1)}(kr)\sqrt{r} \end{cases}$$

for $0 < r \le R$ and $r > R$, respectively, where $\nu = l+1/2$; thus we have a four-parameter family of extensions which can be parameterized via boundary conditions at the singular point R.

There are many different, mutually equivalent forms of such boundary conditions. We will use those introduced in [4]: a (rotationally invariant) surface interaction Hamiltonian is characterized by

$$\begin{aligned} f'(R_+)-f'(R_-) &= \tfrac{\alpha}{2}(f(R_+)+f(R_-)) + \tfrac{\gamma}{2}(f'(R_+)+f'(R_-)) \\ f(R_+)-f(R_-) &= -\tfrac{\bar\gamma}{2}(f(R_+)+f(R_-)) + \tfrac{\beta}{2}(f'(R_+)+f'(R_-)) \end{aligned} \tag{1}$$

for α, $\beta \in \mathbb{R}$ and $\gamma \in \mathbb{C}$; we will call such an operator $H_{\alpha,\beta,\gamma}$. The three GPI classes mentioned in the introduction are the following,

- *δ-type:* $\mathrm{Re}\,\gamma = \beta = 0$,
- *intermediate type:* $\mathrm{Re}\,\gamma \ne 0$, $\beta = 0$,
- *δ'-type:* $\beta \ne 0$.

Notice that the intermediate class contains the "scale-invariant" interaction considered recently by Hejčík and Cheon [11]; it corresponds to $\alpha = \beta = 0$ and

$$\gamma = \frac{h - h^{-1} + 2i\sin\phi}{h + h^{-1} + 2\cos\phi}\,,$$

where h is their α. Notice also that this interaction is not as exotic as it might look; it appears in description of a free quantum motion on a regular metric tree [12] where $\phi = 0$ and $h = \sqrt{N}$ where N is the branching number.

Of the other forms of the boundary conditions let us mention those of [13]. Denote by F the column vector of $f(R+)$ and $f(R-)$, with F' similarly corresponding to the one-sided derivatives at $r = R$. Then the self-adjoint extensions are parametrized by

$$(U-I)F + i(U+I)F' = 0\,, \tag{2}$$

where

$$U = e^{i\xi}\begin{pmatrix} u_1 & u_2 \\ -\bar u_2 & \bar u_1 \end{pmatrix}$$

is a 2×2 unitary matrix, $u_1,\, u_2 \in \mathbb{C}$ with $|u_1|^2 + |u_2|^2 = 1$ and $\xi \in [0,\, \pi)$. The boundary conditions (2) are related to (1) by

$$\begin{aligned} u_1 &= \frac{-2(\alpha+\beta) + 4i\mathrm{Re}\,\gamma}{\sqrt{(\alpha\beta + |\gamma|^2)^2 + 4\alpha^2 + 4\beta^2 + 8|\gamma|^2 + 16}} \\ u_2 &= \frac{1}{2i}\frac{(\alpha\beta + |\gamma|^2 - 4) - 4i\mathrm{Im}\,\gamma}{\sqrt{(\alpha\beta + |\gamma|^2)^2 + 4\alpha^2 + 4\beta^2 + 8|\gamma|^2 + 16}} \\ \tan\xi &= \frac{\alpha\beta + |\gamma|^2 + 4}{2(\alpha - \beta)}\,, \end{aligned}$$

in particular, the case $u_2 = 0$, or

$$\alpha\beta + |\gamma|^2 = 4 \quad \text{and} \quad \mathrm{Im}\,\gamma = 0$$

corresponds to the separated motion inside and outside the barrier; in such a case there is infinitely many embedded eigenvalues on the positive real axis. The three distinct classes in the question can also be described through the matrix U. For instance, the δ'-type requires $\mathrm{Re}\, u_1 + \cos\xi \neq 0$, while for the former two types this quantity vanishes, the δ-type corresponding to $\mathrm{Im}\, u_1 = 0$. It is easy to cast these conditions into a more elegant form,

- *δ-type:* $\det(U + I) = 0$, $\sigma_1 U^T \sigma_1 = U$,
- *intermediate type:* $\det(U + I) = 0$, $\sigma_1 U^T \sigma_1 \neq U$,
- *δ'-type:* $|\det(U + I)| > 0$.

In particular, the δ and intermediate type are the family of point interactions for which -1 is an eigenvalue of corresponding matrix U, with the δ-type being the subfamily invariant under the $\mathcal{PT}$-transformation, induced by the map $U \to \sigma_1 U^T \sigma_1$, where σ_1 is the first Pauli matrix [14].

Another equivalent form of the boundary conditions which can be traced back to [1, 2] is

$$\begin{pmatrix} f(R+) \\ f'(R+) \end{pmatrix} = \Lambda \begin{pmatrix} f(R-) \\ f'(R-) \end{pmatrix},$$

where Λ is a matrix of the form

$$\Lambda = e^{i\chi} \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \chi \in [0, \pi), \quad a,\, b,\, c,\, d \in \mathbb{R}, \quad ad - bc = 1\,;$$

in terms of parameters used above Λ can be written as

$$\frac{1}{\alpha\beta+|\gamma|^2-4+4i\mathrm{Im}\,\gamma}\left(\begin{array}{cc} \alpha\beta+|\gamma|^2+4-4\mathrm{Re}\,\gamma & 4\beta \\ 4\alpha & \alpha\beta+|\gamma|^2+4+4\mathrm{Re}\,\gamma \end{array}\right).$$

Using the transformations

$$f \to \left\{ \begin{array}{ll} e^{i\phi}f & r<R \\ e^{i\theta}f & r\geq R \end{array}\right.$$

it is easy to see that extension corresponding to U and U' are unitarily equivalent if $|u_2|=|u_2'|$. Moreover, to any extension described by $(\alpha',\,\beta',\,\gamma')$ there is a unitarily equivalent one with $(\alpha,\,\beta,\,\gamma)$ such that $\mathrm{Im}\,\gamma=0$. Indeed, the above transformation with $\theta=0$ means the replacement

$$\Lambda \to \tilde{\Lambda} = e^{i(\chi-\varphi)}\left(\begin{array}{cc} a & b \\ c & d \end{array}\right),$$

so in any class of unitarily equivalent extensions there is one with a real transfer matrix; this occurs if $\mathrm{Im}\,\gamma=0$.

After these preliminaries let us turn to our main subject. If $u_2\neq 0$ the Hamiltonian has no embedded eigenvalues. The singularities do not disappear, however, instead we have an infinite family of resonances. To find them, we have derive an explicit expression for the resolvent which can be achieved by means of Krein's formula [3, App. A]: we have

$$(H_{(\alpha,\,\beta,\,\gamma),\,l}-k^2)^{-1} = (H_{0,\,l}-k^2)^{-1} + \sum_{m,n=1}^{2} \lambda_{mn}(\alpha,\,\beta,\,\gamma)(\Phi^{(n)}_{-\bar{k},\,l},\,\cdot)\Phi^{(m)}_{k,\,l}\,,$$

where $\Phi^{(m)}_{k,\,l}$ are the solutions to the deficiency equations given above. The coefficients $\lambda_{i,\,j}$ are found from the fact that the resolvent maps into the domain of $H_{(\alpha,\beta,\gamma)}$ which means, in particular, that $(H_{(\alpha,\,\beta,\,\gamma),\,l}-k^2)^{-1}g$ must for any $g\in L^2((0,\infty))$ satisfy the boundary conditions (1). A straightforward computation then gives

$$\begin{aligned} \lambda_{11} &= \frac{\alpha-\Phi_k^{'(2)}(R)(\alpha\beta+|\gamma|^2)}{\det\lambda}\,, \\ \lambda_{12} &= \frac{\gamma+\Phi_k^{(2)}(\bar{R})(\alpha\beta+|\gamma|^2)}{\det\lambda}\,, \end{aligned}$$

$$\lambda_{21} = \frac{\bar\gamma + \Phi_k^{(2)}(\bar R)(\alpha\beta + |\gamma|^2)}{\det\lambda},$$

$$\lambda_{22} = \frac{-\beta - \Phi_k^{(1)}(R)(\alpha\beta + |\gamma|^2)}{\det\lambda},$$

where the denominator is given explicitly by

$$\det\lambda = -1 - \alpha\Phi_k^{(1)}(R) + \beta\Phi_k^{'(2)}(R) - (\gamma+\bar\gamma)\Phi_k^{(2)}(\bar R) - \frac{1}{4}(\alpha\beta + |\gamma|^2) \quad (3)$$

and by $\Phi_{k,l}^{(2)}(\bar R) := \frac{1}{2}\Big(\Phi_{k,l}^{(2)}(R+) + \Phi_{k,l}^{(2)}(R-)\Big)$.

As usual the resonances are identified with the poles of the resolvent at the unphysical sheet, $\mathrm{Im}\, k < 0$, their distance from the real axis being inversely proportional to the lifetime of the resonance state. We will discuss the asymptotic in momentum k-plane, more specifically, the behaviour of $\mathrm{Im}\, k_n$ corresponding to the nth resonance, given through roots of the equation

$$\det\lambda(k,\, \alpha,\, \beta,\, \gamma) = 0 \quad (4)$$

in the open lower halfplane. We are going to demonstrate that

- for the *δ-type* the quantity $\mathrm{Im}\, k_n$ increases logarithmically w.r.t. the resonance index as $n \to \infty$,
- for the *intermediate type* $\mathrm{Im}\, k_n$ is asymptotically constant,
- finally, for the *δ'-type* with $\beta > 0$ the quantity $\mathrm{Im}\, k_n$ behaves like $\mathcal{O}(n^{-2})$ as $n \to \infty$.

As the poles occur in pairs in the momentum plane we restrict ourselves to those in the fourth quadrant. To prove the claims made we are going to compute explicitly the functions $\Phi_{k,l}^{(i)}$ and to expand then the equation (4) in terms of k^{-1}. The computation is tedious but straightforward; it yields

$$\begin{aligned}\det\lambda &= e^{2ikR}\left\{2kR\, L_{-1} + L_0 + \frac{L_1}{2kR} + \frac{L_2}{(2kR)^2} + \O(k^{-3})\right\} \\ &\quad + 2kR\,\tilde L_{-1} + \tilde L_0 + \frac{\tilde L_1}{2kR} + \O(k^{-2})\,, \end{aligned} \quad (5)$$

where

$$L_{-1} = -\beta\frac{i}{4R}$$

$$
\begin{aligned}
L_0 &= \beta \frac{1}{R} B_{-1} - 2\mathrm{Re}\,\gamma C_{-1} \\
L_1 &= \beta \frac{i}{R} B_0 - 2i\mathrm{Re}\,\gamma C_0 - \alpha i D_1 \\
L_2 &= \beta \frac{1}{R} B_1 - 2\mathrm{Re}\,\gamma C_1 - \alpha D_2 \\
\tilde{L}_{-1} &= \beta \frac{i}{4R} \\
\tilde{L}_0 &= -1 - \frac{1}{4}(\alpha\beta + |\gamma|^2) \\
\tilde{L}_1 &= \beta \frac{i}{R} \tilde{B}_0 - \alpha i \tilde{D}_1
\end{aligned}
$$

and the real constants B, C, D , which we do not present explicitly, depends only on the partial-wave index l. Three distinct cases naturally arise:

δ-type interaction: If $\beta = \gamma = 0$ (general δ-type interaction with $\mathrm{Im}\,\gamma \neq 0$, is unitarily equivalent to this case with $\alpha' = \frac{4\alpha}{\mathrm{Im}^2\gamma + 4}$) it holds $L_{-1} = L_0 = 0$ and the resolvent-pole equation (4) takes the form

$$
1 + \frac{i\alpha}{2k} = \frac{i\alpha}{2k}(-1)^l \, e^{2ikR} \left(1 + i\frac{l(l+1)}{kR} \right) + \emptyset(k^{-2})
$$

In the leading order we have $1 = \frac{i\alpha}{2k}(-1)^l \, e^{2ikR}$. Taking the absolute value we see that $\mathrm{Im}\, k = o(k)$ as $|k| \to \infty$ so we obtain $\frac{2\mathrm{Re}\, k}{|\alpha|} = e^{-2R\,\mathrm{Im}\, k}$ up to higher order terms; substituting back to the first equation we get the asymptotic formulae

$$
k_n = \begin{cases} \frac{1}{2R}\left(2n\pi + l\pi + \frac{3\pi}{2}\right) + O(n^{-1}\ln n) & \text{for} \quad \alpha > 0 \\ \frac{1}{2R}\left(2n\pi + l\pi + \frac{\pi}{2}\right) + O(n^{-1}\ln n) & \text{for} \quad \alpha < 0 \end{cases}
$$

and

$$
\mathrm{Im}\, k_n = -\frac{1}{2R} \ln \frac{2|\mathrm{Re}\, k_n|}{|\alpha|} \left(1 + \emptyset(n^{-1})\right) .
$$

Intermediate-type interaction: If $\beta = 0$ and $\mathrm{Re}\,\gamma \neq 0$ the resolvent-pole equation (4) takes the form

$$
e^{2ikR} \left(L_0 + \frac{L_1}{2kR} + \frac{L_2}{(2kR)^2} \right) + \tilde{L}_0 + \frac{\tilde{L}_1}{2kR} = \emptyset(k^{-2}) \, ,
$$

so in the leading order of k^{-1} we have

$$
L_0 e^{2ikR} + \tilde{L}_0 = 0
$$

which shows that $\operatorname{Im} k$ is asymptotically constant because both coefficients are nonzero. Using the explicit forms of L_i for $\beta = 0$ we get

$$k_n = \begin{cases} -\frac{i}{2R}\ln\left(\frac{1+1/4|\gamma|^2}{\operatorname{Re}\gamma}\right) + \frac{1}{R}\left(\pi n + \frac{\pi l}{2} + \frac{\pi}{2}\right) + \mathcal{O}(n^{-1}) \\ -\frac{i}{2R}\ln\left(\frac{1+1/4|\gamma|^2}{-\operatorname{Re}\gamma}\right) + \frac{1}{R}\left(\pi n + \frac{\pi l}{2} + \frac{3\pi}{2}\right) + \mathcal{O}(n^{-1}) \end{cases}$$

where the upper expression holds for $\operatorname{Re}\gamma > 0$ and the lower one for $\operatorname{Re}\gamma < 0$.

δ'-type interaction: In the most general case when $\beta \neq 0$ the same is true for L_{-1}, and therefore in the leading order of k^{-1} we get the equation

$$2kR\, L_{-1}e^{2ikR} + 2kR\, \tilde{L}_{-1} = 0.$$

Its solutions,

$$k_n^{(0)} = \frac{\pi n}{R} + \frac{\pi(l+1)}{2R}\,,$$

are real which implies that $\operatorname{Im} k_n \to 0$ as $n \to \infty$. To obtain the convergence rate, we have to expand the expression (5) in the vicinity of $k_n^{(0)}$. This shows that the first nontrivial contribution to the imaginary part of resonance is of the second order in $(k_n^{(0)})^{-1}$, hence we deal with a quadratic equation which yields the asymptotic formula

$$\begin{aligned} k_n \;&=\; k_n^{(0)} - \frac{1}{k_n^{(0)}}\left(\frac{1}{R^2}\frac{l^2+l}{2} + \frac{1}{\beta R}\left(\operatorname{Re}\gamma - 1 - \frac{1}{4}(\alpha\beta + |\gamma|^2)\right)\right) \\ &\quad - \frac{i}{(\beta R k_n^{(0)})^2}\left(1 + \frac{1}{2}|\gamma|^2 - (\operatorname{Re}\gamma)^2 - \frac{\alpha\beta}{2} + \frac{1}{16}(\alpha\beta + |\gamma|^2)^2\right) + \mathcal{O}(n^{-3}) \end{aligned}$$

This finishes the proof of the above claims.

In order to suppress additional indices we restrict ourselves to $\mathbb{R}^3$, but the above formulae holds for general n with simple substitution $l \to \nu - 1/2$.

To illustrate this result, we present in Fig. 1 the pole behaviour for a sphere of radius $R = 1$ in the three cases. To show them on the single chart we choose the parameters $\alpha = 50,\ \gamma = \beta = 0,$ for the δ-interaction, $\alpha = \beta = 0,\ \gamma = 1 + i,$ for the intermediate interaction and $\alpha = \gamma = 0,\ \beta = 0.01,$ for the δ'-interaction.

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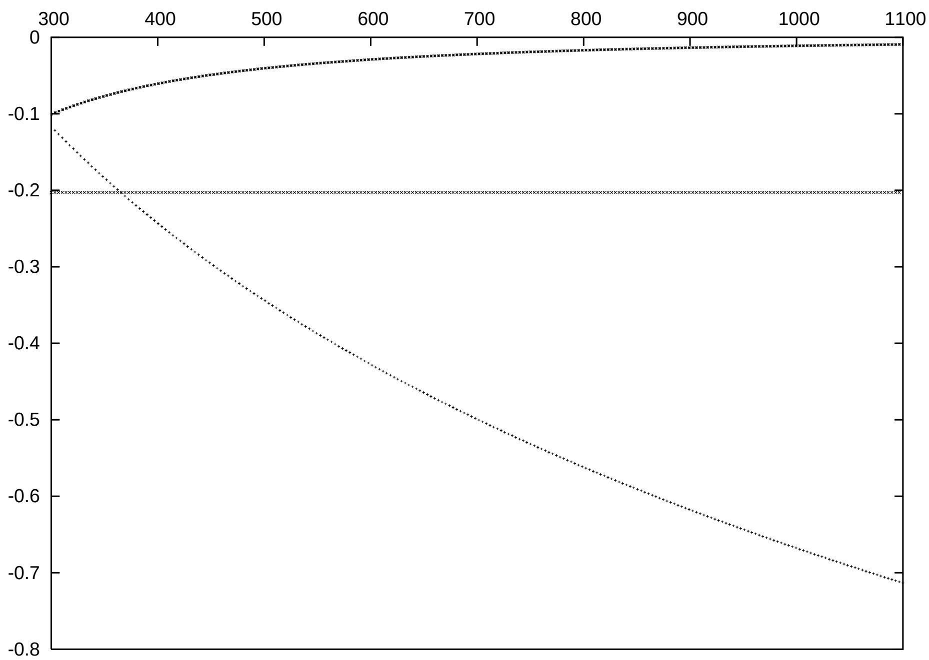

Figure 1: Asymptotics of the resonances in the momentum plane. Here the δ-type poles are marked by $+$, the intermediate ones by $\times$ and the δ'-type ones by $*$.

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