

A BOUND FOR THE EIGENVALUE COUNTING FUNCTION FOR HIGHER-ORDER KREIN LAPLACIANS ON OPEN SETS

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ABSTRACT. For an arbitrary nonempty, open bounded set $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, we consider the minimally defined higher-order Laplacian $(-\Delta)^m|_{C_0^\infty(\Omega)}$, $m \in \mathbb{N}$, and its Krein–von Neumann extension $A_{K,\Omega,m}$ in $L^2(\Omega)$. With $N(\lambda, A_{K,\Omega,m})$, $\lambda > 0$, denoting the eigenvalue counting function corresponding to the strictly positive eigenvalues of $A_{K,\Omega,m}$, we derive the bound

$$N(\lambda, A_{K,\Omega,m}) \leq (2\pi)^{-n} |\Omega| \lambda^{n/(2m)} \min_{\alpha > 0} \left(\alpha^{-1} \int_{\mathbb{R}^n} [\alpha - |\xi|^{4m} + |\xi|^{2m}]_+ d^n \xi \right).$$

This bound remains valid for unbounded domains $\Omega \subset \mathbb{R}^n$ of finite volume, finite width, and such that $\dot{W}^{2m}(\Omega)$ embeds compactly into $L^2(\Omega)$.

The proof relies on variational considerations and exploits the fundamental link between the Krein–von Neumann extension and an underlying (abstract) buckling problem.

1. INTRODUCTION

To set the stage, suppose that S is a densely defined, symmetric, closed operator with nonzero deficiency indices in a separable complex Hilbert space \mathcal{H} that satisfies

$$S \geq \varepsilon I_{\mathcal{H}} \text{ for some } \varepsilon > 0. \quad (1.1)$$

Then, according to M. Krein’s celebrated 1947 paper [34], among all nonnegative self-adjoint extensions of S , there exist two distinguished ones, S_F , the Friedrichs extension of S and S_K , the Krein–von Neumann extension of S , which are, respectively, the largest and smallest such extension (in the sense of quadratic forms). In particular, a nonnegative self-adjoint operator \tilde{S} is a self-adjoint extension of S if and only if \tilde{S} satisfies

$$S_K \leq \tilde{S} \leq S_F \quad (1.2)$$

(again, in the sense of quadratic forms).

An abstract version of [25, Proposition 1], presented in [6], describing the following intimate connection between the nonzero eigenvalues of S_K , and a suitable abstract buckling problem, can be summarized as follows:

$$\text{There exists } 0 \neq v_\lambda \in \text{dom}(S_K) \text{ satisfying } S_K v_\lambda = \lambda v_\lambda, \quad \lambda \neq 0, \quad (1.3)$$

Date: March 18, 2014.

2010 *Mathematics Subject Classification.* Primary 35J25, 35J40, 35P15; Secondary 35P05, 46E35, 47A10, 47F05.

Key words and phrases. Krein Laplacian, eigenvalues, spectral analysis, Weyl asymptotics, buckling problem.

Work of M. M. was partially supported by the Simons Foundation Grant # 281566 and by a University of Missouri Research Leave grant.

Mathematical Results in Quantum Mechanics, QMath12 Proceedings, P. Exner, W. König, and H. Neidhardt (eds), World Scientific, Singapore, to appear.

if and only if

$$\text{there exists a } 0 \neq u_\lambda \in \text{dom}(S^*S) \text{ such that } S^*Su_\lambda = \lambda Su_\lambda, \quad (1.4)$$

and the solutions v_λ of (1.3) are in one-to-one correspondence with the solutions u_λ of (1.4) given by the pair of formulas

$$u_\lambda = (S_F)^{-1}S_K v_\lambda, \quad v_\lambda = \lambda^{-1}Su_\lambda. \quad (1.5)$$

As briefly recalled in Section 2, (1.4) represents an abstract buckling problem. The latter has been the key in all attempts to date in proving Weyl-type asymptotics for eigenvalues of S_K when S represents an elliptic partial differential operator in $L^2(\Omega)$. In fact, it is convenient to go one step further and replace the abstract buckling eigenvalue problem (1.4) by the variational formulation,

$$\begin{aligned} &\text{there exists } u_\lambda \in \text{dom}(S) \setminus \{0\} \text{ such that} \\ &\mathfrak{a}(w, u_\lambda) = \lambda \mathfrak{b}(w, u_\lambda) \text{ for all } w \in \text{dom}(S), \end{aligned} \quad (1.6)$$

where the symmetric forms \mathfrak{a} and \mathfrak{b} in \mathcal{H} are defined by

$$\mathfrak{a}(f, g) := (Sf, Sg)_{\mathcal{H}}, \quad f, g \in \text{dom}(\mathfrak{a}) := \text{dom}(S), \quad (1.7)$$

$$\mathfrak{b}(f, g) := (f, Sg)_{\mathcal{H}}, \quad f, g \in \text{dom}(\mathfrak{b}) := \text{dom}(S). \quad (1.8)$$

In the present context of higher-order Krein Laplacians, the role of S will be played by the closure of the minimally defined operator in $L^2(\Omega)$,

$$A_{min, \Omega, m} := (-\Delta)^m, \quad \text{dom}(A_{min, \Omega, m}) := C_0^\infty(\Omega). \quad (1.9)$$

Under the assumption that $\emptyset \neq \Omega \subset \mathbb{R}^n$ has finite width, this closure, $\overline{A_{min, \Omega, m}}$, is denoted by $A_{\Omega, m}$ and explicitly given by

$$A_{\Omega, m} = (-\Delta)^m, \quad \text{dom}(A_{\Omega, m}) = \mathring{W}^{2m}(\Omega). \quad (1.10)$$

The Krein–von Neumann and Friedrichs extension of $A_{\Omega, m}$ will then be denoted by $A_{K, \Omega, m}$ and $A_{F, \Omega, m}$, respectively.

If $A_{K, \Omega, m}$ has purely discrete spectrum in $(0, \infty)$, let $\{\lambda_{K, \Omega, j}\}_{j \in \mathbb{N}} \subset (0, \infty)$ be the strictly positive eigenvalues of $A_{K, \Omega, m}$ enumerated in nondecreasing order, counting multiplicity, and let

$$N(\lambda, A_{K, \Omega, m}) := \#\{j \in \mathbb{N} \mid 0 < \lambda_{K, \Omega, j} < \lambda\}, \quad \lambda > 0, \quad (1.11)$$

be the (strictly positive) eigenvalue distribution function for $A_{K, \Omega, m}$. The function $N(\cdot, A_{K, \Omega, m})$ is the principal object of this note. Similarly, $N(\lambda, A_{F, \Omega, m})$, $\lambda > 0$, denotes the eigenvalue counting function for $A_{F, \Omega, m}$.

In Section 2 we recall the basic abstract facts on the Friedrichs extension, S_F and the Krein–von Neumann extension S_K of a strictly positive, closed, symmetric operator S in a complex, separable Hilbert space \mathcal{H} and describe the intimate link between the Krein–von Neumann extension and an underlying abstract buckling problem. Section 3 then focuses on the concrete case of higher-order Laplacians $(-\Delta)^m$, $m \in \mathbb{N}$, on appropriate open, finite volume subsets $\Omega \subset \mathbb{R}^n$ (without imposing any constraints on Ω in the case where Ω is bounded) and derives the bound

$$\begin{aligned} N(\lambda, A_{K, \Omega, m}) &\leq (2\pi)^{-n} |\Omega| \lambda^{n/(2m)} \min_{\alpha > 0} \left(\alpha^{-1} \int_{\mathbb{R}^n} [\alpha - |\xi|^{4m} + |\xi|^{2m}]_+ d^n \xi \right), \\ &\lambda > 0. \end{aligned} \quad (1.12)$$

We remark that the power law behavior $\lambda^{n/(2m)}$ coincides with the one in the known Weyl asymptotic behavior. This in itself is not surprising as it is *a priori* known that

$$N(\lambda, A_{K,\Omega,m}) \leq N(\lambda, A_{F,\Omega,m}), \quad \lambda > 0, \quad (1.13)$$

and $N(\lambda, A_{F,\Omega,m})$ is known to have the power law behavior $\lambda^{n/(2m)}$ (cf. (4.3), due to [35], which in turn extends the corresponding result in [37] in the case $m = 1$). Rather than using known estimates for $N(\cdot, A_{F,\Omega,m})$ (cf., e.g., [10], [11], [12], [13], [14], [15], [19], [20], [28], [29], [35], [38], [37], [41], [44], [45], [46], [47], [48], [52]), we will use the one-to-one correspondence of nonzero eigenvalues of $A_{K,\Omega,m}$ with the eigenvalues of its underlying buckling problem (cf. (1.3)–(1.5)) and estimate the eigenvalue counting function for the latter in Section 3. Numerical results in Section 4 suggest the superiority of the buckling problem based bound (1.12) over the known estimates for $N(\cdot, A_{F,\Omega,m})$ (cf. (4.3) and Table 4.1). Elementary analytical considerations confirm this also in the cases $m = 1$ and $1 \leq n \leq 4$ (cf. (4.11)–(4.16)).

Since Weyl asymptotics for $N(\cdot, A_{K,\Omega,m})$ and $N(\cdot, A_{F,\Omega,m})$ are not considered in this paper we just refer to the monographs [36] and [49], but note that very detailed bibliographies on this subject appeared in [5] and [7]. At any rate, the best known result on Weyl asymptotics for $N(\cdot, A_{K,\Omega,m})$ to date is proven for bounded Lipschitz domains [8], whereas the estimate (1.12) assumes no regularity of Ω at all.

We conclude this introduction by summarizing the notation used in this paper. Throughout this paper, the symbol \mathcal{H} is reserved to denote a separable complex Hilbert space with $(\cdot, \cdot)_{\mathcal{H}}$ the scalar product in \mathcal{H} (linear in the second argument), and $I_{\mathcal{H}}$ the identity operator in \mathcal{H} . Next, let T be a linear operator mapping (a subspace of) a Banach space into another, with $\text{dom}(T)$ and $\text{ran}(T)$ denoting the domain and range of T . The closure of a closable operator S is denoted by \bar{S} . The kernel (null space) of T is denoted by $\ker(T)$. The spectrum, point spectrum (i.e., the set of eigenvalues), discrete spectrum, essential spectrum, and resolvent set of a closed linear operator in \mathcal{H} will be denoted by $\sigma(\cdot)$, $\sigma_p(\cdot)$, $\sigma_d(\cdot)$, $\sigma_{ess}(\cdot)$, and $\rho(\cdot)$, respectively. The symbol *s-lim* abbreviates the limit in the strong (i.e., pointwise) operator topology (we also use this symbol to describe strong limits in \mathcal{H}).

The Banach spaces of bounded and compact linear operators on \mathcal{H} are denoted by $\mathcal{B}(\mathcal{H})$ and $\mathcal{B}_{\infty}(\mathcal{H})$, respectively. Similarly, the Schatten–von Neumann (trace) ideals will subsequently be denoted by $\mathcal{B}_p(\mathcal{H})$, $p \in (0, \infty)$. In addition, $U_1 \dot{+} U_2$ denotes the direct sum of the subspaces U_1 and U_2 of a Banach space \mathcal{X} .

The symbol $L^2(\Omega)$, with $\Omega \subseteq \mathbb{R}^n$ open, $n \in \mathbb{N} \setminus \{1\}$, is a shortcut for $L^2(\Omega, d^n x)$, whenever the n -dimensional Lebesgue measure is understood. For brevity, the identity operator in $L^2(\Omega)$ will typically be denoted by I_{Ω} . The symbol $\mathcal{D}(\Omega)$ is reserved for the set of test functions $C_0^{\infty}(\Omega)$ on Ω , equipped with the standard inductive limit topology, and $\mathcal{D}'(\Omega)$ represents its dual space, the set of distributions in Ω . In addition, $\#(M)$ abbreviates the cardinality of the set M . In addition, we define $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, so that \mathbb{N}_0^n becomes the collection of all multi-indices with n components. As is customary, for each $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$ we denote by $|\alpha| := \alpha_1 + \dots + \alpha_n$ the length of α , and set $\alpha! := \alpha_1! \cdots \alpha_n!$.

Moreover, $A \approx B$ signifies the existence of a finite constant $C \geq 1$, independent of the main parameters entering the quantities A, B , such that $C^{-1}A \leq B \leq CA$.

Finally, a notational comment: For obvious reasons, which have their roots in quantum mechanical applications, we will, with a slight abuse of notation, dub the expression $-\Delta = -\sum_{j=1}^n \partial_j^2$ (rather than Δ) as the “Laplacian” in this paper.

2. BASIC FACTS ON THE KREIN–VON NEUMANN EXTENSION AND THE ASSOCIATED ABSTRACT BUCKLING PROBLEM

In this preparatory section we recall the basic facts on the Krein–von Neumann extension of a strictly positive operator S in a complex, separable Hilbert space \mathcal{H} and its associated abstract buckling problem as discussed in [5, 6]. For an extensive survey of this circle of ideas and an exhaustive list of references as well as pertinent historical comments we refer to [7].

To set the stage, we denote by S a linear, densely defined, symmetric (i.e., $S \subseteq S^*$), and closed operator in \mathcal{H} throughout this section. We recall that S is called *nonnegative* provided $(f, Sf)_{\mathcal{H}} \geq 0$ for all $f \in \text{dom}(S)$. The operator S is called *strictly positive*, if for some $\varepsilon > 0$ one has $(f, Sf)_{\mathcal{H}} \geq \varepsilon \|f\|_{\mathcal{H}}^2$ for all $f \in \text{dom}(S)$; one then writes $S \geq \varepsilon I_{\mathcal{H}}$. Next, we recall that two nonnegative, self-adjoint operators A, B in \mathcal{H} satisfy $A \leq B$ (in the sense of forms) if

$$\text{dom}(B^{1/2}) \subset \text{dom}(A^{1/2}) \quad (2.1)$$

and

$$(A^{1/2}f, A^{1/2}f)_{\mathcal{H}} \leq (B^{1/2}f, B^{1/2}f)_{\mathcal{H}}, \quad f \in \text{dom}(B^{1/2}). \quad (2.2)$$

We also recall ([18, Section I.6], [30, Theorem VI.2.21]) that for A and B both self-adjoint and nonnegative in \mathcal{H} one has

$$0 \leq A \leq B \text{ if and only if } (B + aI_{\mathcal{H}})^{-1} \leq (A + aI_{\mathcal{H}})^{-1} \text{ for all } a > 0. \quad (2.3)$$

Moreover, we note the useful fact that $\ker(A) = \ker(A^{1/2})$.

The following is a fundamental result to be found in M. Krein’s celebrated 1947 paper [34] (cf. also Theorems 2 and 5–7 in the English summary on page 492):

Theorem 2.1. *Assume that S is a densely defined, closed, nonnegative operator in \mathcal{H} . Then, among all nonnegative self-adjoint extensions of S , there exist two distinguished ones, S_K and S_F , which are, respectively, the smallest and largest such extension (in the sense of (2.1)–(2.2)). Furthermore, a nonnegative self-adjoint operator \tilde{S} is a self-adjoint extension of S if and only if \tilde{S} satisfies*

$$S_K \leq \tilde{S} \leq S_F. \quad (2.4)$$

In particular, (2.4) determines S_K and S_F uniquely. In addition, if $S \geq \varepsilon I_{\mathcal{H}}$ for some $\varepsilon > 0$, one has $S_F \geq \varepsilon I_{\mathcal{H}}$, and

$$\text{dom}(S_F) = \text{dom}(S) \dot{+} (S_F)^{-1} \ker(S^*), \quad (2.5)$$

$$\text{dom}(S_K) = \text{dom}(S) \dot{+} \ker(S^*), \quad (2.6)$$

$$\begin{aligned} \text{dom}(S^*) &= \text{dom}(S) \dot{+} (S_F)^{-1} \ker(S^*) \dot{+} \ker(S^*) \\ &= \text{dom}(S_F) \dot{+} \ker(S^*), \end{aligned} \quad (2.7)$$

and

$$\ker(S_K) = \ker((S_K)^{1/2}) = \ker(S^*) = \text{ran}(S)^\perp. \quad (2.8)$$

One calls S_K the *Krein–von Neumann extension* of S and S_F the *Friedrichs extension* of S . We also recall that

$$S_F = S^*|_{\text{dom}(S^*) \cap \text{dom}((S_F)^{1/2})}. \quad (2.9)$$

Furthermore, if $S \geq \varepsilon I_{\mathcal{H}}$ then (2.6) implies

$$\ker(S_K) = \ker((S_K)^{1/2}) = \ker(S^*) = \text{ran}(S)^\perp. \quad (2.10)$$

For abstract results regarding the parametrization of all nonnegative self-adjoint extensions of a given strictly positive, densely defined, symmetric operator we refer the reader to Krein [34], Višik [51], Birman [9], Grubb [23, 24], subsequent expositions due to Alonso and Simon [4], Faris [18, Sect. 15], and [26, Sect. 13.2], [50, Ch. 13], and Derkach and Malamud [16, 39], see also [22, Theorem 9.2].

Let us collect a basic assumption which will be imposed in the rest of this section.

Hypothesis 2.2. *Suppose that S is a densely defined, symmetric, closed operator with nonzero deficiency indices in \mathcal{H} that satisfies $S \geq \varepsilon I_{\mathcal{H}}$ for some $\varepsilon > 0$.*

For subsequent purposes we note that under Hypothesis 2.2, one has

$$\dim(\ker(S^* - zI_{\mathcal{H}})) = \dim(\ker(S^*)), \quad z \in \mathbb{C} \setminus [\varepsilon, \infty). \quad (2.11)$$

We recall that two self-adjoint extensions S_1 and S_2 of S are called *relatively prime* (or *disjoint*) if $\text{dom}(S_1) \cap \text{dom}(S_2) = \text{dom}(S)$. The following result will play a role later on (cf., e.g., [5, Lemma 2.8] for an elementary proof):

Lemma 2.3. *Suppose Hypothesis 2.2. Then the Friedrichs extension S_F and the Krein–von Neumann extension S_K of S are relatively prime, that is,*

$$\text{dom}(S_F) \cap \text{dom}(S_K) = \text{dom}(S). \quad (2.12)$$

Next, we consider a self-adjoint operator T in \mathcal{H} which is bounded from below, that is, $T \geq \alpha I_{\mathcal{H}}$ for some $\alpha \in \mathbb{R}$. We denote by $\{E_T(\lambda)\}_{\lambda \in \mathbb{R}}$ the family of strongly right-continuous spectral projections of T , and introduce for $-\infty \leq a < b$, as usual,

$$E_T((a, b)) = E_T(b_-) - E_T(a) \quad \text{and} \quad E_T(b_-) = \text{s-lim}_{\varepsilon \downarrow 0} E_T(b - \varepsilon). \quad (2.13)$$

In addition, we set

$$\mu_{T,j} := \inf \{ \lambda \in \mathbb{R} \mid \dim(\text{ran}(E_T((-\infty, \lambda)))) \geq j \}, \quad j \in \mathbb{N}. \quad (2.14)$$

Then, for fixed $k \in \mathbb{N}$, either:

(i) $\mu_{T,k}$ is the k th eigenvalue of T counting multiplicity below the bottom of the essential spectrum, $\sigma_{\text{ess}}(T)$, of T ,

or,

(ii) $\mu_{T,k}$ is the bottom of the essential spectrum of T ,

$$\mu_{T,k} = \inf \{ \lambda \in \mathbb{R} \mid \lambda \in \sigma_{\text{ess}}(T) \}, \quad (2.15)$$

and in that case $\mu_{T,k+\ell} = \mu_{T,k}$, $\ell \in \mathbb{N}$, and there are at most $k - 1$ eigenvalues (counting multiplicity) of T below $\mu_{T,k}$.

We now record a basic result of M. Krein [34] with an extension due to Alonso and Simon [4] and some additional results recently derived in [6]. For this purpose we introduce the *reduced Krein–von Neumann operator* \widehat{S}_K in the Hilbert space

$$\widehat{\mathcal{H}} := (\ker(S^*))^\perp = (\ker(S_K))^\perp \quad (2.16)$$

by

$$\widehat{S}_K := P_{(\ker(S_K))^\perp} S_K|_{(\ker(S_K))^\perp}, \quad \text{dom}(\widehat{S}_K) = \text{dom } S_K \cap \widehat{\mathcal{H}}, \quad (2.17)$$

where $P_{(\ker(S_K))^\perp}$ denotes the orthogonal projection onto $(\ker(S_K))^\perp$. One then obtains

$$(\widehat{S}_K)^{-1} = P_{(\ker(S_K))^\perp} (S_F)^{-1}|_{(\ker(S_K))^\perp}, \quad (2.18)$$

a relation due to Krein [34, Theorem 26] (see also [39, Corollary 5]).

Theorem 2.4. *Suppose Hypothesis 2.2. Then*

$$\varepsilon \leq \mu_{S_F, j} \leq \mu_{\widehat{S}_K, j}, \quad j \in \mathbb{N}. \quad (2.19)$$

In particular, if the Friedrichs extension S_F of S has purely discrete spectrum, then, except possibly for $\lambda = 0$, the Krein–von Neumann extension S_K of S also has purely discrete spectrum in $(0, \infty)$, that is,

$$\sigma_{\text{ess}}(S_F) = \emptyset \text{ implies } \sigma_{\text{ess}}(S_K) \subseteq \{0\}. \quad (2.20)$$

In addition, if $p \in (0, \infty]$, then $(S_F - z_0 I_{\mathcal{H}})^{-1} \in \mathcal{B}_p(\mathcal{H})$ for some $z_0 \in \mathbb{C} \setminus [\varepsilon, \infty)$ implies

$$(S_K - z I_{\mathcal{H}})^{-1}|_{(\ker(S_K))^\perp} \in \mathcal{B}_p(\widehat{\mathcal{H}}) \text{ for all } z \in \mathbb{C} \setminus [\varepsilon, \infty). \quad (2.21)$$

In fact, the $\ell^p(\mathbb{N})$ -based trace ideal $\mathcal{B}_p(\mathcal{H})$ (resp., $\mathcal{B}_p(\widehat{\mathcal{H}})$) of $\mathcal{B}(\mathcal{H})$ (resp., $\mathcal{B}(\widehat{\mathcal{H}})$) can be replaced by any two-sided symmetrically normed ideal of $\mathcal{B}(\mathcal{H})$ (resp., $\mathcal{B}(\widehat{\mathcal{H}})$).

We note that (2.20) is a classical result of Krein [34]. Apparently, (2.19) in the context of infinite deficiency indices was first proven by Alonso and Simon [4] by a somewhat different method. Relation (2.21) was proved in [6].

Assuming that S_F has purely discrete spectra, let $\{\lambda_{K, j}\}_{j \in \mathbb{N}} \subset (0, \infty)$ be the strictly positive eigenvalues of S_K enumerated in nondecreasing order, counting multiplicity, and let

$$N(\lambda, S_K) := \#\{j \in \mathbb{N} \mid 0 < \lambda_{K, j} < \lambda\}, \quad \lambda > 0, \quad (2.22)$$

be the eigenvalue distribution function for S_K . Similarly, let $\{\lambda_{F, j}\}_{j \in \mathbb{N}} \subset (0, \infty)$ denote the eigenvalues of S_F , again enumerated in nondecreasing order, counting multiplicity, and by

$$N(\lambda, S_F) := \#\{j \in \mathbb{N} \mid \lambda_{F, j} < \lambda\}, \quad \lambda > 0, \quad (2.23)$$

the corresponding eigenvalue counting function for S_F . Then inequality (2.19) implies

$$N(\lambda, S_K) \leq N(\lambda, S_F), \quad \lambda > 0. \quad (2.24)$$

In particular, any estimate for the eigenvalue counting function for the Friedrichs extension S_F , in turn, yields one for the Krein–von Neumann extension S_K (focusing on strictly positive eigenvalues of S_K according to (2.22)). While this is a viable approach to estimate the eigenvalue counting function (2.22) for S_K , we will proceed along a different route in Section 3 and directly exploit the one-to-one correspondence between strictly positive eigenvalues of S_K and the eigenvalues of its underlying abstract buckling problem to be described next.

To describe the abstract buckling problem naturally associated with the Krein–von Neumann extension as described in [6], we start by introducing an abstract version of [25, Proposition 1] (see [6] for a proof):

Lemma 2.5. *Assume Hypothesis 2.2 and let $\lambda \in \mathbb{C} \setminus \{0\}$. Then there exists some $f \in \text{dom}(S_K) \setminus \{0\}$ with*

$$S_K f = \lambda f \quad (2.25)$$

*if and only if there exists $w \in \text{dom}(S^*S) \setminus \{0\}$ such that*

$$S^*S w = \lambda S w. \quad (2.26)$$

In fact, the solutions f of (2.25) are in one-to-one correspondence with the solutions w of (2.26) as evidenced by the formulas

$$w = (S_F)^{-1} S_K f, \quad (2.27)$$

$$f = \lambda^{-1} S w. \quad (2.28)$$

Of course, since $S_K \geq 0$ is self-adjoint, any $\lambda \in \mathbb{C} \setminus \{0\}$ in (2.25) and (2.26) necessarily satisfies $\lambda \in (0, \infty)$.

It is the linear pencil eigenvalue problem $S^*S w = \lambda S w$ in (2.26) that we call the *abstract buckling problem* associated with the Krein–von Neumann extension S_K of S .

Next, we turn to a variational formulation of the correspondence between the inverse of the reduced Krein–von Neumann extension \widehat{S}_K and the abstract buckling problem in terms of appropriate sesquilinear forms by following [31]–[33] in the elliptic PDE context. This will then lead to an even stronger connection between the Krein–von Neumann extension S_K of S and the associated abstract buckling eigenvalue problem (2.26), culminating in the unitary equivalence result in Theorem 2.6 below.

Given the operator S , we introduce the following symmetric forms in \mathcal{H} ,

$$\mathbf{a}(f, g) := (Sf, Sg)_{\mathcal{H}}, \quad f, g \in \text{dom}(\mathbf{a}) := \text{dom}(S), \quad (2.29)$$

$$\mathbf{b}(f, g) := (f, Sg)_{\mathcal{H}}, \quad f, g \in \text{dom}(\mathbf{b}) := \text{dom}(S). \quad (2.30)$$

Then S being densely defined and closed implies that the sesquilinear form \mathbf{a} shares these properties, while $S \geq \varepsilon I_{\mathcal{H}}$ from Hypothesis 2.2 implies that \mathbf{a} is bounded from below, that is,

$$\mathbf{a}(f, f) \geq \varepsilon^2 \|f\|_{\mathcal{H}}^2, \quad f \in \text{dom}(S). \quad (2.31)$$

(The inequality (2.31) follows based on the assumption $S \geq \varepsilon I_{\mathcal{H}}$ by estimating $(Sf, Sg)_{\mathcal{H}} = [(S - \varepsilon I_{\mathcal{H}}) + \varepsilon I_{\mathcal{H}}]f, [(S - \varepsilon I_{\mathcal{H}}) + \varepsilon I_{\mathcal{H}}]g)_{\mathcal{H}}$ from below.)

Thus, one can introduce the Hilbert space

$$\mathcal{W} := (\text{dom}(S), (\cdot, \cdot)_{\mathcal{W}}), \quad (2.32)$$

with associated scalar product

$$(f, g)_{\mathcal{W}} := \mathbf{a}(f, g) = (Sf, Sg)_{\mathcal{H}}, \quad f, g \in \text{dom}(S). \quad (2.33)$$

In addition, we note that $\iota_{\mathcal{W}} : \mathcal{W} \hookrightarrow \mathcal{H}$, the embedding operator of \mathcal{W} into \mathcal{H} , is continuous due to $S \geq \varepsilon I_{\mathcal{H}}$. Hence, precise notation would be using

$$(w_1, w_2)_{\mathcal{W}} = \mathbf{a}(\iota_{\mathcal{W}} w_1, \iota_{\mathcal{W}} w_2) = (S \iota_{\mathcal{W}} w_1, S \iota_{\mathcal{W}} w_2)_{\mathcal{H}}, \quad w_1, w_2 \in \mathcal{W}, \quad (2.34)$$

but in the interest of simplicity of notation we will omit the embedding operator $\iota_{\mathcal{W}}$ in the following.

With the sesquilinear forms \mathbf{a} and \mathbf{b} and the Hilbert space \mathcal{W} as above, given $w_2 \in \mathcal{W}$, the map $\mathcal{W} \ni w_1 \mapsto (w_1, Sw_2)_{\mathcal{H}} \in \mathbb{C}$ is continuous. This allows us to define the operator Tw_2 as the unique element in \mathcal{W} such that

$$(w_1, Tw_2)_{\mathcal{W}} = (w_1, Sw_2)_{\mathcal{H}} \text{ for all } w_1 \in \mathcal{W}. \quad (2.35)$$

This implies

$$\mathbf{a}(w_1, Tw_2) = (w_1, Tw_2)_{\mathcal{W}} = (w_1, Sw_2)_{\mathcal{H}} = \mathbf{b}(w_1, w_2) \quad (2.36)$$

for all $w_1, w_2 \in \mathcal{W}$. In addition, the operator T satisfies

$$0 \leq T = T^* \in \mathcal{B}(\mathcal{W}) \quad \text{and} \quad \|T\|_{\mathcal{B}(\mathcal{W})} \leq \varepsilon^{-1}. \quad (2.37)$$

We will call T the *abstract buckling problem operator* associated with the Krein–von Neumann extension S_K of S .

Next, recalling the notation $\widehat{\mathcal{H}} = (\ker(S^*))^\perp$ (cf. (2.16)), we introduce the operator

$$\widehat{S} : \mathcal{W} \rightarrow \widehat{\mathcal{H}}, \quad w \mapsto Sw. \quad (2.38)$$

Clearly, $\text{ran}(\widehat{S}) = \text{ran}(S)$ and since $S \geq \varepsilon I_{\mathcal{H}}$ for some $\varepsilon > 0$ and S is closed in \mathcal{H} , $\text{ran}(S)$ is also closed, and hence coincides with $(\ker(S^*))^\perp$. This yields

$$\text{ran}(\widehat{S}) = \text{ran}(S) = \widehat{\mathcal{H}}. \quad (2.39)$$

In fact, it follows that $\widehat{S} \in \mathcal{B}(\mathcal{W}, \widehat{\mathcal{H}})$ maps \mathcal{W} unitarily onto $\widehat{\mathcal{H}}$ (cf. [6]).

Continuing, we briefly recall the polar decomposition of S ,

$$S = U_S |S|, \quad (2.40)$$

where, with $\varepsilon > 0$ as in Hypothesis 2.2,

$$|S| = (S^*S)^{1/2} \geq \varepsilon I_{\mathcal{H}} \text{ and } U_S \in \mathcal{B}(\mathcal{H}, \widehat{\mathcal{H}}) \text{ unitary.} \quad (2.41)$$

Then the principal unitary equivalence result proved in [6] reads as follows:

Theorem 2.6. *Assume Hypothesis 2.2. Then the inverse of the reduced Krein–von Neumann extension \widehat{S}_K in $\widehat{\mathcal{H}}$ and the abstract buckling problem operator T in \mathcal{W} are unitarily equivalent. Specifically,*

$$(\widehat{S}_K)^{-1} = \widehat{S}T(\widehat{S})^{-1}. \quad (2.42)$$

In particular, the nonzero eigenvalues of S_K are reciprocals of the eigenvalues of T . Moreover, one has

$$(\widehat{S}_K)^{-1} = U_S[|S|^{-1}S|S|^{-1}](U_S)^{-1}, \quad (2.43)$$

where $U_S \in \mathcal{B}(\mathcal{H}, \widehat{\mathcal{H}})$ is the unitary operator in the polar decomposition (2.40) of S and the operator $|S|^{-1}S|S|^{-1} \in \mathcal{B}(\mathcal{H})$ is self-adjoint and strictly positive in \mathcal{H} .

We emphasize that the unitary equivalence in (2.42) is independent of any spectral assumptions on S_K (such as the spectrum of S_K consists of eigenvalues only) and applies to the restrictions of S_K to its pure point, absolutely continuous, and singularly continuous spectral subspaces, respectively.

Equation (2.43) is motivated by rewriting the abstract linear pencil buckling eigenvalue problem (2.26), $S^*Sw = \lambda Sw$, $\lambda \in \mathbb{C} \setminus \{0\}$, in the form

$$|S|^{-1}Sw = (S^*S)^{-1/2}Sw = \lambda^{-1}(S^*S)^{1/2}w = \lambda^{-1}|S|w \quad (2.44)$$

and hence in the form of a standard eigenvalue problem

$$|S|^{-1}S|S|^{-1}v = \lambda^{-1}v, \quad \lambda \in \mathbb{C} \setminus \{0\}, \quad v := |S|w. \quad (2.45)$$

Again, self-adjointness and strict positivity of $|S|^{-1}S|S|^{-1}$ imply $\lambda \in (0, \infty)$.

We conclude this section with an elementary result recently noted in [8] that relates the nonzero eigenvalues of S_K directly with the sesquilinear forms \mathbf{a} and \mathbf{b} :

Lemma 2.7 ([8]). *Assume Hypothesis 2.2 and introduce*

$$\sigma_p(\mathbf{a}, \mathbf{b}) := \left\{ \lambda \in \mathbb{C} \mid \text{there exists } g_\lambda \in \text{dom}(S) \setminus \{0\} \right. \\ \left. \text{such that } \mathbf{a}(f, g_\lambda) = \lambda \mathbf{b}(f, g_\lambda) \text{ for all } f \in \text{dom}(S) \right\}. \quad (2.46)$$

Then

$$\sigma_p(\mathbf{a}, \mathbf{b}) = \sigma_p(S_K) \setminus \{0\} \quad (2.47)$$

(counting multiplicity), in particular, $\sigma_p(\mathbf{a}, \mathbf{b}) \subset (0, \infty)$, and $g_\lambda \in \text{dom}(S) \setminus \{0\}$ in (2.46) actually satisfies

$$g_\lambda \in \text{dom}(S^*S), \quad S^*Sg_\lambda = \lambda Sg_\lambda. \quad (2.48)$$

In addition,

$$\lambda \in \sigma_p(\mathbf{a}, \mathbf{b}) \text{ if and only if } \lambda^{-1} \in \sigma_p(T) \quad (2.49)$$

(counting multiplicity). Finally,

$$T \in \mathcal{B}_\infty(\mathcal{W}) \iff (\widehat{S}_K)^{-1} \in \mathcal{B}_\infty(\widehat{\mathcal{H}}) \iff \sigma_{ess}(S_K) \subseteq \{0\}, \quad (2.50)$$

and hence,

$$\sigma_p(\mathbf{a}, \mathbf{b}) = \sigma(S_K) \setminus \{0\} = \sigma_d(S_K) \setminus \{0\} \quad (2.51)$$

if (2.50) holds. In particular, if one of S_F or $|S|$ has purely discrete spectrum (i.e., $\sigma_{ess}(S_F) = \emptyset$ or $\sigma_{ess}(|S|) = \emptyset$), then (2.50) and (2.51) hold.

One notices that $f \in \text{dom}(S)$ in the definition (2.46) of $\sigma_p(\mathbf{a}, \mathbf{b})$ can be replaced by $f \in C(S)$ for any (operator) core $C(S)$ for S (equivalently, by any form core for the form \mathbf{a}).

3. AN UPPER BOUND FOR THE EIGENVALUE COUNTING FUNCTION FOR HIGHER-ORDER KREIN LAPLACIANS ON BOUNDED DOMAINS

In this section we derive an upper bound for the eigenvalue counting function for higher-order Krein Laplacians on open bounded domains $\Omega \subset \mathbb{R}^n$. In fact, we will also permit certain classes of unbounded finite volume domains and no assumptions on the boundary of Ω will be made.

Before introducing the class of constant coefficient partial differential operators in $L^2(\Omega)$ at hand, we recall a few auxiliary facts to be used in the proof of Theorem 3.7.

Lemma 3.1. *Suppose that S is a densely defined, symmetric, closed operator in \mathcal{H} . Then $|S|$ and hence S is infinitesimally bounded with respect to S^*S , more precisely, one has*

$$\text{for all } \varepsilon > 0, \quad \|Sf\|_{\mathcal{B}(\mathcal{H})} = \| |S|f \|_{\mathcal{B}(\mathcal{H})} \leq \varepsilon \|S^*Sf\|_{\mathcal{H}}^2 + (4\varepsilon)^{-1} \|f\|_{\mathcal{H}}^2, \quad (3.1) \\ f \in \text{dom}(S^*S).$$

In addition, S is relatively compact with respect to S^*S if $|S|$, or equivalently, S^*S , has compact resolvent. In particular,

$$\sigma_{ess}(S^*S - \lambda S) = \sigma_{ess}(S^*S), \quad \lambda \in \mathbb{R}. \quad (3.2)$$

Proof. Employing the polar decomposition of S , $S = U_S|S|$, where U_S is a partial isometry and $|S| = (S^*S)^{1/2}$ (cf. [30, Sect. VI.2.7]), one obtains

$$\|Sf\|_{\mathcal{B}(\mathcal{H})} = \||S|f\|_{\mathcal{B}(\mathcal{H})}, \quad f \in \text{dom}(S) = \text{dom}(|S|), \quad (3.3)$$

and hence the spectral theorem applied to $|S|$, together with the elementary inequality $\lambda \leq \varepsilon\lambda^2 + (4\varepsilon)^{-1}$, $\varepsilon > 0$, $\lambda \geq 0$, proves inequality (3.1).

The relative compactness assertion then follows from

$$S(S^*S + I_{\mathcal{H}})^{-1} = \left[S(|S|^2 + I_{\mathcal{H}})^{-1/2} \right] (|S|^2 + I_{\mathcal{H}})^{-1/2} \in \mathcal{B}_{\infty}(\mathcal{H}), \quad (3.4)$$

since $S(|S|^2 + I_{\mathcal{H}})^{-1/2} \in \mathcal{B}(\mathcal{H})$ and $(|S|^2 + I_{\mathcal{H}})^{-1/2} \in \mathcal{B}_{\infty}(\mathcal{H})$. \square

Given a lower semibounded, self-adjoint operator $T \geq c_T I_{\mathcal{H}}$ in \mathcal{H} , we denote by \mathfrak{q}_T its uniquely associated form, that is,

$$\mathfrak{q}_T(f, g) = (|T|^{1/2}f, \text{sgn}(T)|T|^{1/2}g)_{\mathcal{H}}, \quad f, g \in \text{dom}(\mathfrak{q}) = \text{dom}(|T|^{1/2}), \quad (3.5)$$

and by $\{E_T(\lambda)\}_{\lambda \in \mathbb{R}}$ the family of spectral projections of T . We recall the following well-known variational characterization of dimensions of spectral projections $\{E_T([c_T, \mu])\}$, $\mu > c_T$.

Lemma 3.2. *Assume that $c_T I_{\mathcal{H}} \leq T$ is self-adjoint in \mathcal{H} and $\mu > c_T$. Suppose that $\mathcal{F} \subset \text{dom}(|T|^{1/2})$ is a linear subspace such that*

$$\mathfrak{q}_T(f, f) < \mu \|f\|_{\mathcal{H}}^2, \quad f \in \mathcal{F} \setminus \{0\}. \quad (3.6)$$

Then,

$$\dim(\text{ran}(E_T([c_T, \mu]))) = \sup_{\mathcal{F} \subset \text{dom}(|T|^{1/2})} (\dim(\mathcal{F})). \quad (3.7)$$

We add the following elementary observation: Let $c \in \mathbb{R}$ and $B \geq cI_{\mathcal{H}}$ be a self-adjoint operator in \mathcal{H} , and introduce the sesquilinear form b in \mathcal{H} associated with B via

$$\begin{aligned} b(u, v) &= ((B - cI_{\mathcal{H}})^{1/2}u, (B - cI_{\mathcal{H}})^{1/2}v)_{\mathcal{H}} + c(u, v)_{\mathcal{H}}, \\ u, v \in \text{dom}(b) &= \text{dom}(|B|^{1/2}). \end{aligned} \quad (3.8)$$

Given B and b , one introduces the Hilbert space $\mathcal{H}_b \subseteq \mathcal{H}$ by

$$\begin{aligned} \mathcal{H}_b &= (\text{dom}(|B|^{1/2}), (\cdot, \cdot)_{\mathcal{H}_b}), \\ (u, v)_{\mathcal{H}_b} &= b(u, v) + (1 - c)(u, v)_{\mathcal{H}} \\ &= ((B - cI_{\mathcal{H}})^{1/2}u, (B - cI_{\mathcal{H}})^{1/2}v)_{\mathcal{H}} + (u, v)_{\mathcal{H}} \\ &= ((B + (1 - c)I_{\mathcal{H}})^{1/2}u, (B + (1 - c)I_{\mathcal{H}})^{1/2}v)_{\mathcal{H}}. \end{aligned} \quad (3.9)$$

One observes that

$$(B + (1 - c)I_{\mathcal{H}})^{1/2}: \mathcal{H}_b \rightarrow \mathcal{H} \text{ is unitary.} \quad (3.10)$$

Lemma 3.3 (see, e.g., [21]). *Let \mathcal{H} , B , b , and \mathcal{H}_b be as in (3.8)–(3.10). Then B has purely discrete spectrum, that is, $\sigma_{\text{ess}}(B) = \emptyset$, if and only if \mathcal{H}_b embeds compactly into \mathcal{H} .*

Next we turn to higher-order Laplacians $(-\Delta)^m$ in $L^2(\Omega)$ and hence introduce the following set of assumptions on $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$.

Hypothesis 3.4. Let $n \in \mathbb{N}$ and $\emptyset \neq \Omega \subset \mathbb{R}^n$ be open such that:

- (i) Ω has finite width $\ell_0 \in (0, \infty)$, that is, Ω lies between two hyperplanes in \mathbb{R}^n which are a distance ℓ_0 apart.
- (ii) Ω has finite Euclidean volume (denoted by $|\Omega| < \infty$).
- (iii) Fix $m \in \mathbb{N}$ and suppose that the Sobolev space $\mathring{W}^{2m}(\Omega)$ embeds compactly into $L^2(\Omega)$.

Here $W^k(\Omega)$ is defined as usual by

$$W^k(\Omega) := \{u \in L^2(\Omega) \mid \partial^\alpha u \in L^2(\Omega), 0 \leq |\alpha| \leq k\}, \quad k \in \mathbb{N}_0, \quad (3.11)$$

with $\alpha \in \mathbb{N}_0^n$ and $\partial^\alpha u$ denoting weak derivatives of u . The space $W^k(\Omega)$ is endowed with the norm

$$\|u\|_{k,\Omega} := \left(\sum_{0 \leq |\alpha| \leq k} \|\partial^\alpha u\|_{L^2(\Omega)}^2 \right)^{1/2}, \quad u \in W^k(\Omega). \quad (3.12)$$

In addition, define

$$\mathring{W}^k(\Omega) = \overline{C_0^\infty(\Omega)}^{W^k(\Omega)}, \quad k \in \mathbb{N}_0, \quad (3.13)$$

and note that $\mathring{W}^k(\Omega)$ is a closed linear subspace of $W^k(\Omega)$. Hypothesis 3.4 (i) then implies the Poincaré-type inequality (cf., e.g., [17, p. 242]),

$$\|u\|_{q,\Omega} \leq \ell_0^{q-p} \|u\|_{p,\Omega}, \quad u \in \mathring{W}^p(\Omega), \quad q \in \mathbb{N}_0, \quad p \in \mathbb{N}, \quad q \leq p, \quad (3.14)$$

where we introduced the abbreviation

$$\|u\|_{k,\Omega} := \left(\sum_{|\alpha|=k} \|\partial^\alpha u\|_{L^2(\Omega)}^2 \right)^{1/2}, \quad u \in W^k(\Omega), \quad k \in \mathbb{N}_0. \quad (3.15)$$

Thus, $\|u\|_{k,\Omega}$, $u \in W^k(\Omega)$, represents an equivalent norm on $\mathring{W}^k(\Omega)$.

Necessary and sufficient conditions for Hypothesis 3.4 (iii) to hold in terms of appropriate capacities can be found, for instance, in [2], [3, Ch. 6], [40, Ch. 6].

We note that Hypothesis 3.4 is always satisfied for $\Omega \subset \mathbb{R}^n$ nonempty, open, and bounded.

We proceed with the following useful identity:

Lemma 3.5. Let $k \in \mathbb{N}$ and assume $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ is open. Then,

$$\sum_{|\alpha|=k} \frac{k!}{\alpha!} \int_{\Omega} |(\partial^\alpha \phi)(x)|^2 d^n x = \int_{\Omega} \overline{(\Delta^k \phi)(x)} \phi(x) d^n x, \quad \phi \in C_0^\infty(\Omega). \quad (3.16)$$

Proof. Using the fact that $\text{supp}(\phi) \subset \Omega$ and employing the Plancherel identity, one obtains

$$\int_{\Omega} \overline{(\Delta^k \phi)(x)} \phi(x) d^n x = \int_{\mathbb{R}^n} \overline{(\Delta^k \phi)(x)} \phi(x) d^n x = \int_{\mathbb{R}^n} |\xi|^{2k} |\widehat{\phi}(\xi)|^2 d^n \xi. \quad (3.17)$$

Similarly,

$$\begin{aligned} \sum_{|\alpha|=k} \frac{k!}{\alpha!} \int_{\Omega} |(\partial^\alpha \phi)(x)|^2 d^n x &= \sum_{|\alpha|=k} \frac{k!}{\alpha!} \int_{\mathbb{R}^n} |(\partial^\alpha \phi)(x)|^2 d^n x \\ &= \sum_{|\alpha|=k} \frac{k!}{\alpha!} \int_{\mathbb{R}^n} \xi^{2\alpha} |\widehat{\phi}(\xi)|^2 d^n \xi. \end{aligned} \quad (3.18)$$

Since in general, $\left(\sum_{j=1}^n x_j\right)^N = \sum_{|\alpha|=N} \frac{|\alpha|!}{\alpha!} x^\alpha$, with $x := (x_1, \dots, x_n)$, by the Multinomial Theorem, one concludes that

$$\sum_{|\alpha|=k} \frac{k!}{\alpha!} \xi^{2\alpha} = \left(\sum_{j=1}^n \xi_j^2\right)^k = |\xi|^{2k}, \quad \xi \in \mathbb{R}^n. \quad (3.19)$$

Therefore, using (3.17)–(3.19), we may write

$$\sum_{|\alpha|=k} \frac{k!}{\alpha!} \int_{\Omega} |(\partial^\alpha \phi)(x)|^2 d^n x = \int_{\mathbb{R}^n} |\xi|^{2k} |\widehat{\phi}(\xi)|^2 d^n \xi = \int_{\Omega} \overline{(\Delta^k \phi)(x)} \phi(x) d^n x, \quad (3.20)$$

completing the proof of (3.16). \square

Lemma 3.5 is a key input for the next result.

Theorem 3.6. *Assume Hypothesis 3.4(i) and let $m \in \mathbb{N}$. Consider the minimal operator*

$$A_{min,\Omega,m} := (-\Delta)^m, \quad \text{dom}(A_{min,\Omega,m}) := C_0^\infty(\Omega), \quad (3.21)$$

in $L^2(\Omega)$. Then the closure of $A_{min,\Omega,m}$ in $L^2(\Omega)$ is given by

$$A_{\Omega,m} = (-\Delta)^m, \quad \text{dom}(A_{\Omega,m}) = \dot{W}^{2m}(\Omega). \quad (3.22)$$

In addition, $A_{\Omega,m}$ is a strictly positive operator, that is, there exists $\varepsilon > 0$ such that

$$A_{\Omega,m} \geq \varepsilon I_{\Omega}. \quad (3.23)$$

Proof. Clearly $A_{min,\Omega,m}$ is symmetric and hence closable. Assuming $\phi \in C_0^\infty(\Omega)$, repeatedly integrating by parts and an application of Lemma 3.5 yield,

$$\begin{aligned} \int_{\Omega} |((-\Delta)^m \phi)(x)|^2 d^n x &= \int_{\Omega} \overline{((-\Delta)^m \phi)(x)} ((-\Delta)^m \phi)(x) d^n x \\ &= \int_{\Omega} \overline{((-\Delta)^{2m} \phi)(x)} \phi(x) d^n x \\ &= \sum_{|\alpha|=2m} \frac{(2m)!}{\alpha!} \int_{\Omega} |(\partial^\alpha \phi)(x)|^2 d^n x. \end{aligned} \quad (3.24)$$

By density of $C_0^\infty(\Omega)$ in $\dot{W}^{2m}(\Omega)$, identity (3.24) extends to

$$\int_{\Omega} |((-\Delta)^m u)(x)|^2 d^n x = \sum_{|\alpha|=2m} \frac{(2m)!}{\alpha!} \int_{\Omega} |(\partial^\alpha u)(x)|^2 d^n x, \quad u \in \dot{W}^{2m}(\Omega). \quad (3.25)$$

Next, combining the Poincaré inequality (3.14) with (3.25) implies that for some constant $C_{m,\Omega} > 0$,

$$\int_{\Omega} |((-\Delta)^m u)(x)|^2 d^n x \geq C_{m,\Omega} \sum_{0 \leq |\beta| \leq 2m} \|\partial^\beta u\|_{L^2(\Omega)}^2 \approx \|u\|_{m,\Omega}^2, \quad u \in \dot{W}^{2m}(\Omega). \quad (3.26)$$

Finally, consider $\{f_j\}_{j \in \mathbb{N}} \subset \dot{W}^{2m}(\Omega)$, $f, g \in L^2(\Omega)$, such that

$$\lim_{j \rightarrow \infty} \|f_j - f\|_{L^2(\Omega)} = 0 \quad \text{and} \quad \lim_{j \rightarrow \infty} \|(-\Delta)^m f_j - g\|_{L^2(\Omega)} = 0. \quad (3.27)$$

Applying (3.26) to $u := (f_j - f_k) \in \mathring{W}^{2m}(\Omega)$, one infers for some $c_{m,\Omega} > 0$,

$$\int_{\Omega} |((-\Delta)^m (f_j - f_k))(x)|^2 d^n x \geq c_{m,\Omega} \|f_j - f_k\|_{m,\Omega}^2, \quad j, k \in \mathbb{N}, \quad (3.28)$$

implying that actually, $\{f_j\}_{j \in \mathbb{N}}$ is a Cauchy sequence in $\mathring{W}^{2m}(\Omega)$. By completeness of the latter space one concludes that $f \in \mathring{W}^{2m}(\Omega)$. Taking arbitrary $\psi \in C_0^\infty(\Omega)$, and introducing the standard distributional pairing, $\mathcal{D}(\Omega)' \langle \cdot, \cdot \rangle_{\mathcal{D}(\Omega)}$, with $\mathcal{D}(\Omega) := C_0^\infty(\Omega)$ equipped with the usual inductive limit topology, one concludes that

$$\begin{aligned} (g, \psi)_{L^2(\Omega)} &= \mathcal{D}(\Omega)' \langle g, \psi \rangle_{\mathcal{D}(\Omega)} = \lim_{j \rightarrow \infty} \mathcal{D}(\Omega)' \langle (-\Delta)^m f_j, \psi \rangle_{\mathcal{D}(\Omega)} \\ &= \lim_{j \rightarrow \infty} \int_{\Omega} \overline{f_j(x)} ((-\Delta)^m \psi)(x) d^n x = \int_{\Omega} \overline{f(x)} ((-\Delta)^m \psi)(x) d^n x \\ &= \mathcal{D}(\Omega)' \langle (-\Delta)^m f, \psi \rangle_{\mathcal{D}(\Omega)}. \end{aligned} \quad (3.29)$$

Hence, $g = (-\Delta)^m f$, implying closedness of $A_{\Omega,m}$. By the definition of $\mathring{W}^{2m}(\Omega)$ (cf. (3.13)), $A_{\Omega,m}$ is the closure of $A_{min,\Omega,m}$.

Strict positivity of $A_{min,\Omega,m}$, and hence that of $A_{\Omega,m}$, follows from (3.16) and the Poincaré-type inequalities (3.14) (choosing $j = 0$). \square

In the following we pick $m \in \mathbb{N}$ as in Hypothesis 3.4 (iii) and denote by $A_{K,\Omega,m}$ and $A_{F,\Omega,m}$ the Krein and Friedrichs extension of $A_{\Omega,m}$ in $L^2(\Omega)$, respectively. By Hypothesis 3.4 (iii), $\text{dom}(A_{\Omega,m}) = \mathring{W}^{2m}(\Omega)$ embeds compactly into $L^2(\Omega)$ and hence by Lemma 3.3, the operator $A_{\Omega,m}^* A_{\Omega,m}$ has purely discrete spectrum, equivalently, the resolvent of $A_{\Omega,m}^* A_{\Omega,m}$ is compact, in particular,

$$[A_{\Omega,m}^* A_{\Omega,m}]^{-1} \in \mathcal{B}_\infty(L^2(\Omega)), \quad (3.30)$$

as the form associated with $A_{\Omega,m}^* A_{\Omega,m}$ is given by

$$\mathfrak{a}_{\Omega,m}(f, g) := (A_{\Omega,m} f, A_{\Omega,m} g)_{L^2(\Omega)}, \quad f, g \in \text{dom}(\mathfrak{a}_{\Omega,m}) := \text{dom}(A_{\Omega,m}). \quad (3.31)$$

Consequently, also

$$|A_{\Omega,m}|^{-1} = [A_{\Omega,m}^* A_{\Omega,m}]^{-1/2} \in \mathcal{B}_\infty(L^2(\Omega)), \quad (3.32)$$

implying

$$(\widehat{A}_{K,\Omega,m})^{-1} \in \mathcal{B}_\infty(L^2(\Omega)) \quad (3.33)$$

by (2.43). Thus,

$$\sigma_{ess}(A_{K,\Omega,m}) \subseteq \{0\}. \quad (3.34)$$

Let $\{\lambda_{K,\Omega,j}\}_{j \in \mathbb{N}} \subset (0, \infty)$ be the strictly positive eigenvalues of $A_{K,\Omega,m}$ enumerated in nondecreasing order, counting multiplicity, and let

$$N(\lambda, A_{K,\Omega,m}) := \#\{j \in \mathbb{N} \mid 0 < \lambda_{K,\Omega,j} < \lambda\}, \quad \lambda > 0, \quad (3.35)$$

be the eigenvalue distribution function for $A_{K,\Omega,m}$. Recalling the standard notation

$$x_+ := \max(0, x), \quad x \in \mathbb{R}, \quad (3.36)$$

then $N(\cdot, A_{K,\Omega,m})$ permits the following estimate following the approach in [35].

Theorem 3.7. *Assume Hypothesis 3.4. Then for any $\alpha > 0$, one has the estimate,*

$$\begin{aligned} N(\lambda, A_{K,\Omega,m}) &\leq (2\pi)^{-n} |\Omega| \lambda^{n/(2m)} \min_{\alpha > 0} \left(\alpha^{-1} \int_{\mathbb{R}^n} [\alpha - |\xi|^{4m} + |\xi|^{2m}]_+ d^n \xi \right), \\ &\lambda > 0. \end{aligned} \quad (3.37)$$

Proof. Following our abstract Section 2, we introduce in addition to the symmetric form $\mathfrak{a}_{\Omega,m}$ in $L^2(\Omega)$ (cf. (3.31)), the form

$$\mathfrak{b}_{\Omega,m}(f, g) := (f, A_{\Omega,m}g)_{L^2(\Omega)}, \quad f, g \in \text{dom}(\mathfrak{b}_{\Omega,m}) := \text{dom}(A_{\Omega,m}). \quad (3.38)$$

By Lemma 2.7, particularly, by (2.49), one concludes that

$$N(\lambda, A_{K,\Omega,m}) \leq \max \left(\dim \{f \in \text{dom}(A_{\Omega,m}) \mid \mathfrak{a}_{\Omega,m}(f, f) - \lambda \mathfrak{b}_{\Omega,m}(f, f) < 0\} \right), \quad (3.39)$$

by also employing (2.51) and the fact that

$$\mathfrak{a}_{\Omega,m}(f_{K,\Omega,j}, f_{K,\Omega,j}) - \lambda \mathfrak{b}_{\Omega,m}(f_{K,\Omega,j}, f_{K,\Omega,j}) = (\lambda_{K,\Omega,j} - \lambda) \|f_{K,\Omega,j}\|_{L^2(\Omega)}^2 < 0, \quad (3.40)$$

where $f_{K,\Omega,j} \in \text{dom}(A_{\Omega,m}) \setminus \{0\}$ additionally satisfies

$$\begin{aligned} f_{K,\Omega,j} &\in \text{dom}(A_{\Omega,m}^* A_{\Omega,m}) \text{ and} \\ A_{\Omega,m}^* A_{\Omega,m} f_{K,\Omega,j} &= \lambda_{K,\Omega,j} A_{\Omega,m} f_{K,\Omega,j}. \end{aligned} \quad (3.41)$$

To further analyze (3.39) we now fix $\lambda \in (0, \infty)$ and introduce the auxiliary operator

$$\begin{aligned} L_{\Omega,m,\lambda} &:= A_{\Omega,m}^* A_{\Omega,m} - \lambda A_{\Omega,m}, \\ \text{dom}(L_{\Omega,m,\lambda}) &:= \text{dom}(A_{\Omega,m}^* A_{\Omega,m}). \end{aligned} \quad (3.42)$$

By Lemma 3.1, $L_{\Omega,m}$ is self-adjoint, bounded from below, with purely discrete spectrum as its form domain

$$\text{dom}(|L_{\Omega,m,\lambda}|^{1/2}) = \text{dom}(A_{\Omega,m}) = \mathring{W}^{2m}(\Omega) \quad (3.43)$$

embeds compactly into $L^2(\Omega)$ by Hypothesis 3.4 (iii) (cf. Lemma 3.3). We will study the auxiliary eigenvalue problem,

$$L_{\Omega,m,\lambda} \varphi_j = \mu_j \varphi_j, \quad \varphi_j \in \text{dom}(L_{\Omega,m,\lambda}), \quad (3.44)$$

where $\{\varphi_j\}_{j \in \mathbb{N}}$ represents an orthonormal basis of eigenfunctions in $L^2(\Omega)$ and for simplicity of notation we repeat the eigenvalues μ_j of $L_{\Omega,m}$ according to their multiplicity, assuming φ_j to be linearly independent in the following. Since $\varphi_j \in \mathring{W}^{2m}(\Omega)$, we denote by

$$\tilde{\varphi}_j(x) := \begin{cases} \varphi_j(x), & x \in \Omega, \\ 0, & x \in \mathbb{R} \setminus \Omega, \end{cases} \quad (3.45)$$

their zero-extension of φ_j to all of \mathbb{R}^n and note that

$$\partial^\alpha \tilde{\varphi}_j = \widetilde{\partial^\alpha \varphi_j}, \quad 0 \leq |\alpha| \leq 2m. \quad (3.46)$$

Next, given $\mu > 0$, one estimates

$$\mu^{-1} \sum_{\substack{j \in \mathbb{N} \\ \mu_j < \mu}} (\mu - \mu_j) \geq \mu^{-1} \sum_{\substack{j \in \mathbb{N}, \\ \mu_j < 0, \mu_j < \mu}} (\mu - \mu_j) \geq \mu^{-1} \sum_{\substack{j \in \mathbb{N}, \\ \mu_j < 0, \mu_j < \mu}} \mu = n_-(L_{\Omega,m,\lambda}), \quad (3.47)$$

where $n_-(L_{\Omega,m,\lambda})$ denotes the number of strictly negative eigenvalues of $L_{\Omega,m}$. Combining, Lemma 3.2 and (3.39) one concludes that

$$\begin{aligned} N(\lambda, A_{K,\Omega,m}) &\leq \max \left(\dim \{f \in \text{dom}(A_{\min,\Omega,m}) \mid \mathfrak{a}_{\Omega,m}(f, f) - \lambda \mathfrak{b}_{\Omega,m}(f, f) < 0\} \right) \\ &= n_-(L_{\Omega,m,\lambda}) \leq \mu^{-1} \sum_{\substack{j \in \mathbb{N} \\ \mu_j < \mu}} (\mu - \mu_j) = \mu^{-1} \sum_{j \in \mathbb{N}} [\mu - \mu_j]_+, \quad \mu > 0. \end{aligned} \quad (3.48)$$

Next we focus on estimating the right-hand side of (3.48).

$$\begin{aligned}
N(\lambda, A_{K,\Omega,m}) &\leq \mu^{-1} \sum_{j \in \mathbb{N}} (\mu - \mu_j)_+ = \mu^{-1} \sum_{j \in \mathbb{N}} [(\varphi_j, (\mu - \mu_j) \varphi_j)_{L^2(\Omega)}]_+ \\
&= \sum_{j \in \mathbb{N}} \left[\mu \|\varphi_j\|_{L^2(\Omega)}^2 - \|(-\Delta)^m \varphi_j\|_{L^2(\Omega)}^2 + \lambda (\varphi_j, (-\Delta)^m \varphi_j)_{L^2(\Omega)} \right]_+ \\
&= \sum_{j \in \mathbb{N}} \left[\mu \|\tilde{\varphi}_j\|_{L^2(\mathbb{R}^n)}^2 - \|(-\Delta)^m \tilde{\varphi}_j\|_{L^2(\mathbb{R}^n)}^2 + \lambda (\tilde{\varphi}_j, (-\Delta)^m \tilde{\varphi}_j)_{L^2(\mathbb{R}^n)} \right]_+ \\
&= \sum_{j \in \mathbb{N}} \left[\int_{\mathbb{R}^n} [\mu - (|\xi|^{4m} - \lambda |\xi|^{2m})] |\widehat{\tilde{\varphi}}_j(\xi)|^2 d^n \xi \right]_+ \\
&\leq \sum_{j \in \mathbb{N}} \int_{\mathbb{R}^n} [\mu - (|\xi|^{4m} - \lambda |\xi|^{2m})]_+ |\widehat{\tilde{\varphi}}_j(\xi)|^2 d^n \xi \\
&= \sum_{j \in \mathbb{N}} \int_{\mathbb{R}^n} [\mu - |\xi|^{4m} + \lambda |\xi|^{2m}]_+ |\widehat{\tilde{\varphi}}_j(\xi)|^2 d^n \xi \\
&= \int_{\mathbb{R}^n} [\mu - |\xi|^{4m} + \lambda |\xi|^{2m}]_+ \sum_{j \in \mathbb{N}} |\widehat{\tilde{\varphi}}_j(\xi)|^2 d^n \xi. \tag{3.49}
\end{aligned}$$

Here we used unitarity of the Fourier transform on $L^2(\mathbb{R}^n)$, the fact that $[\mu - |\xi|^{4m} + \lambda |\xi|^{2m}]_+$ has compact support (rendering the integral over a compact subset of \mathbb{R}^n), and the monotone convergence theorem in the final step.

Next, one observes that

$$\begin{aligned}
\sum_{j \in \mathbb{N}} |\widehat{\tilde{\varphi}}_j(\xi)|^2 &= (2\pi)^{-n} \sum_{j \in \mathbb{N}} |(e^{i\xi \cdot}, \tilde{\varphi}_j)_{L^2(\mathbb{R}^n)}|^2 = (2\pi)^{-n} \sum_{j \in \mathbb{N}} |(e^{i\xi \cdot}, \varphi_j)_{L^2(\Omega)}|^2 \\
&= (2\pi)^{-n} \|e^{i\xi \cdot}\|_{L^2(\Omega)}^2 = (2\pi)^{-n} |\Omega|, \tag{3.50}
\end{aligned}$$

employing the fact that $\{\varphi_j\}_{j \in \mathbb{N}}$ represents an orthonormal basis in $L^2(\Omega)$.

Combining (3.49) and (3.50), introducing $\alpha = \lambda^{-2}\mu$, changing variables, $\xi = \lambda^{1/m}\eta$, and minimizing with respect to $\alpha > 0$, proves (3.37). \square

4. COMPARISONS WITH OTHER BOUNDS, WEYL ASYMPTOTICS, AND SOME NUMERICAL RESULTS

In our final section we briefly offer a discussion of the bound (3.37) on the eigenvalue counting function $N(\lambda, A_{K,\Omega,m})$ supported by some numerical results.

For smooth, bounded domains $\Omega \subset \mathbb{R}^n$, and smooth lower-order coefficients (not necessarily constant), Weyl asymptotics for $N(\lambda, A_{K,\Omega,m})$ as $\lambda \rightarrow \infty$ was first derived by Grubb [25],

$$N(\lambda, A_{K,\Omega,m}) \underset{\lambda \rightarrow \infty}{=} (2\pi)^{-n} v_n |\Omega| \lambda^{n/(2m)} + O(\lambda^{(n-\theta)/(2m)}), \tag{4.1}$$

where $v_n := \pi^{n/2}/\Gamma((n+2)/2)$ denotes the Euclidean volume of the unit ball in \mathbb{R}^n ($\Gamma(\cdot)$ being the Gamma function, cf. [1, Sect. 6.1]), and

$$\theta := \max \left\{ \frac{1}{2} - \varepsilon, \frac{2m}{2m+n-1} \right\}, \text{ with } \varepsilon > 0 \text{ arbitrary.} \tag{4.2}$$

We also refer to [42], [43], and more recently, [27], where the authors derive a sharpening of the remainder in (4.1) to any $\theta < 1$. In the case $m = 1$, Weyl asymptotics

for $N(\lambda, A_{K,\Omega,1})$ was derived in [5] for (bounded) quasi-convex domains, and most recently, in [8] for bounded Lipschitz domains.

The power law behavior $\lambda^{n/(2m)}$ of the estimate (3.37) for general domains governed by Hypothesis 3.4 (no smoothness of Ω being assumed at all in the case of bounded domains), coincides with that in the known Weyl asymptotics (4.1) and is of course consistent with the abstract estimate (2.24). In this connection we note that Weyl-type asymptotics and estimates for $N(\lambda, A_{F,\Omega,m})$, and hence upper bounds for $N(\lambda, A_{K,\Omega,m})$, without regularity assumptions on Ω can be found, for instance, in [10], [11], [12], [13], [14], [15], [19], [20], [28], [29], [35], [38], [37], [41], [44], [45], [46], [47], [48], [52]. We mention, in particular, the bound for $N(\lambda, A_{F,\Omega,m})$ derived in [35] (extending earlier results in [37] in the case $m = 1$) which reads

$$N(\lambda, A_{F,\Omega,m}) \leq (2\pi)^{-n} v_n |\Omega| \lambda^{n/(2m)} [1 + (2m/n)]^{n/(2m)}, \quad \lambda > 0. \quad (4.3)$$

It is illustrative to compare the bound (3.37) with the leading asymptotic Weyl formula (4.1), as well as with the bound in terms of the eigenvalue counting function $N(\lambda, A_{F,\Omega,m})$ since by (2.24), one has

$$N(\lambda, A_{K,\Omega,m}) \leq N(\lambda, A_{F,\Omega,m}), \quad \lambda > 0. \quad (4.4)$$

For simplicity we focus on the special case $m = 1$. We start by introducing the ratio

$$r_{K/W,n} := v_n^{-1} \min_{\alpha > 0} \left(\alpha^{-1} \int_{\mathbb{R}^n} [\alpha - |\xi|^4 + |\xi|^2]_+ d^n \xi \right), \quad n \in \mathbb{N}, \quad (4.5)$$

of the right-hand side in our buckling problem induced bound (3.37) and the leading-order term in the Weyl-type asymptotics (4.1). Similarly, according to (4.4), we consider the ratio

$$r_{F/W,n} := [1 + (2/n)]^{n/2}, \quad n \in \mathbb{N}, \quad (4.6)$$

of the right-hand side in (4.3) induced by the Dirichlet Laplacian and the leading-order term in the Weyl-type asymptotics (4.1). One observes that

$$\lim_{n \rightarrow \infty} r_{F/W,n} = e = 2.71828 \dots \quad (4.7)$$

In addition, we directly computed the ratio of the right-hand sides of equations (3.37) and (4.3), that is,

$$r_{K/F,n} := v_n^{-1} \min_{\alpha > 0} \left(\alpha^{-1} \int_{\mathbb{R}^n} [\alpha - |\xi|^4 + |\xi|^2]_+ d^n \xi \right) / [1 + (2/n)]^{n/2}. \quad (4.8)$$

Employing Wolfram's *Mathematica* (<http://www.wolfram.com/mathematica/>), the following numerical results have been obtained:

	$r_{K/W,n}$	$r_{F/W,n}$	$r_{K/F,n}$
$n = 1$	1.29	1.73	0.75
$n = 2$	1.50	2.00	0.75
$n = 3$	1.66	2.15	0.77
$n = 4$	1.78	2.25	0.79
$n = 20$	2.39	2.59	0.92
$n = 200$	2.68	2.70	0.99
$n = 1000$	2.71	2.71	0.99

Table 4.1

Thus, our approach estimating $N(\cdot, A_{K,\Omega,m})$ with the help of the buckling problem, numerically, yields better estimates than that obtained from estimating $N(\cdot, A_{F,\Omega,m})$ as in (4.3) and then using (4.4).

We also note that an explicit evaluation of the integral in our bound (3.37) yields

$$\begin{aligned} & \alpha^{-1} \int_{\mathbb{R}^n} [\alpha - |\xi|^{4m} + |\xi|^{2m}]_+ d^n \xi \\ &= \frac{nv_n}{\alpha} \int_0^\infty r^{n-1} [\alpha - r^{4m} + r^{2m}]_+ dr \\ &= \frac{nv_n}{\alpha} \int_0^{r_\alpha} r^{n-1} [\alpha - r^{4m} + r^{2m}] dr \\ &= \frac{nv_n r_\alpha^n}{\alpha} \left[\frac{r_\alpha^{2m}}{n+2m} + \frac{\alpha}{n} - \frac{r_\alpha^{4m}}{n+4m} \right], \quad m \in \mathbb{N}, \end{aligned} \quad (4.9)$$

where

$$r_\alpha = \left[[1 + (4\alpha + 1)^{1/2}] / 2 \right]^{1/(2m)}, \quad \alpha > 0, \quad m \in \mathbb{N}. \quad (4.10)$$

Analytically minimizing the right-hand side of (4.9) with respect to $\alpha > 0$ appears to be a daunting task. However, choosing $m = 1$ and introducing

$$f_n(\alpha) := \frac{1}{v_n \alpha} \int_{\mathbb{R}^n} [\alpha - |\xi|^4 + |\xi|^2]_+ d^n \xi, \quad \alpha > 0, \quad n \in \mathbb{N}, \quad (4.11)$$

$$g_n := [1 + (2/n)]^{n/2}, \quad n \in \mathbb{N}, \quad (4.12)$$

one verifies, for instance, that the particular choice $\alpha = 3/4$ yields that

$$\min_{\alpha > 0} f_1(\alpha) \leq f_1(3/4) = (5/3)^{1/2} < 3^{1/2} = g_1, \quad (4.13)$$

$$\min_{\alpha > 0} f_2(\alpha) \leq f_2(3/4) = 3/2 < 2 = g_2, \quad (4.14)$$

$$\min_{\alpha > 0} f_3(\alpha) \leq f_3(3/4) = (7/5)^{3/2} < (5/3)^{3/2} = g_3, \quad (4.15)$$

$$\min_{\alpha > 0} f_4(\alpha) \leq f_4(3/4) = 16/9 < 9/4 = g_4, \quad (4.16)$$

and hence for $m = 1$ one concludes that the right-hand side of our bound (3.37) is strictly less than the right-hand side of (4.3) for $1 \leq n \leq 4$.

Acknowledgments. We are grateful to Sergey Naboko for very helpful discussions. F.G. and A.L. are indebted to all organizers of QMath12, and particularly, to Pavel Exner, Wolfgang König, and Hagen Neidhardt, for fostering a very stimulating atmosphere during the conference, leading to this collaboration.

REFERENCES

- [1] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, Dover, New York, 1972.
- [2] R. A. Adams, *Capacity and compact imbeddings*, J. Math. Mech. **19**, 923–929 (1970).
- [3] R. A. Adams and J. J. F. Fournier, *Sobolev Spaces*, second edition, Academic Press, 2003.
- [4] A. Alonso and B. Simon, *The Birman-Krein-Vishik theory of selfadjoint extensions of semi-bounded operators*, J. Operator Th. **4**, 251–270 (1980); Addenda: **6**, 407 (1981).
- [5] M. S. Ashbaugh, F. Gesztesy, M. Mitrea, and G. Teschl, *Spectral theory for perturbed Krein Laplacians in nonsmooth domains*, Adv. Math. **223**, 1372–1467 (2010).
- [6] M. S. Ashbaugh, F. Gesztesy, M. Mitrea, R. Shterenberg, and G. Teschl, *The Krein—von Neumann extension and its connection to an abstract buckling problem*, Math. Nachr. **283**, 165–179 (2010).

- [7] M. S. Ashbaugh, F. Gesztesy, M. Mitrea, R. Shterenberg, and G. Teschl, *A survey on the Krein–von Neumann extension, the corresponding abstract buckling problem, and Weyl-type spectral asymptotics for perturbed Krein Laplacians in non smooth domains*, in *Mathematical Physics, Spectral Theory and Stochastic Analysis*, M. Demuth and W. Kirsch (eds.), Operator Theory: Advances and Applications, Vol. 232, Birkhäuser, Springer, Basel, 2013, pp. 1–106.
- [8] J. Behrndt, F. Gesztesy, T. Micheler, and M. Mitrea, *Sharp Boundary Trace Theory on Bounded Lipschitz Domains and Applications to the Krein–Von Neumann extension*, in preparation.
- [9] M. Sh. Birman, *On the theory of self-adjoint extensions of positive definite operators*, Mat. Sbornik **38**, 431–450 (1956) (Russian).
- [10] M. Sh. Birman and M. Z. Solomyak, *Leading term in the asymptotic spectral formula for “non-smooth” elliptic problems*, Funkcional. Anal. i Priložen **4**, no. 4, 1–13 (1970) (Russian); Engl. transl. in Funct. Anal. Appl. **4**, 265–275 (1970).
- [11] M. Sh. Birman and M. Z. Solomyak, *On the asymptotic spectrum of “non-smooth” elliptic equations*, Funkcional. Anal. i Priložen **5**, no. 1, 69–70 (1971) (Russian); Engl. transl. in Funct. Anal. Appl. **5**, 56–57 (1971).
- [12] M. Š. Birman and M. Z. Solomjak, *Spectral asymptotics of nonsmooth elliptic operators. I*, Trans. Moscow Math. Soc. **27**, 1–52 (1972).
- [13] M. Š. Birman and M. Z. Solomjak, *Spectral asymptotics of nonsmooth elliptic operators. II*, Trans. Moscow Math. Soc. **28**, 1–32 (1973).
- [14] M. Sh. Birman and M. Z. Solomyak, *Asymptotic behavior of the spectrum of differential equations*, Itogi Nauki i Tekhniki, Matematicheskii Analiz., **14**, 5–58 (1977) (Russian); Engl. transl. in J. Soviet Math. **12**, no. 3, 247–283 (1979).
- [15] M. S. Birman and M. Z. Solomjak, *Quantitative Analysis in Sobolev Imbedding Theorems and Applications to Spectral Theory*, AMS Translations, Series 2, Vol. 114, Providence, RI, 1980, pp. 1–132.
- [16] V. A. Derkach and M. M. Malamud, *Generalized resolvents and the boundary value problems for Hermitian operators with gaps*, J. Funct. Anal. **95**, 1–95 (1991).
- [17] D. E. Edmunds and W. D. Evans, *Spectral Theory and Differential Operators*, Clarendon Press, Oxford, 1989.
- [18] W. G. Faris, *Self-Adjoint Operators*, Lecture Notes in Mathematics, Vol. 433, Springer, Berlin, 1975.
- [19] L. Geisinger, *A short proof of Weyl’s law for fractional differential operators*, preprint, arXiv:1309.1867.
- [20] L. Geisinger, A. Laptev, and T. Weidl, *Geometrical versions of improved Berezin–Li–Yau inequalities*, J. Spectral Theory **1**, 87–109 (2011).
- [21] F. Gesztesy and M. Mitrea, *Generalized Robin Laplacians and some remarks on a paper by Filonov on eigenvalue inequalities*, J. Diff. Eq. **247**, 2871–2896 (2009).
- [22] F. Gesztesy and M. Mitrea, *A description of all self-adjoint extensions of the Laplacian and Krein-type resolvent formulas on non-smooth domains*, J. Analyse Math. **113**, 53–172 (2011).
- [23] G. Grubb, *A characterization of the non-local boundary value problems associated with an elliptic operator*, Ann. Scuola Norm. Sup. Pisa (3), **22**, 425–513 (1968).
- [24] G. Grubb, *Les problèmes aux limites généraux d’un opérateur elliptique, provenant de la théorie variationnelle*, Bull. Sci. Math. (2), **94**, 113–157 (1970).
- [25] G. Grubb, *Spectral asymptotics for the “soft” selfadjoint extension of a symmetric elliptic differential operator*, J. Operator Th. **10**, 9–20 (1983).
- [26] G. Grubb, *Distributions and Operators*, Graduate Texts in Mathematics, Vol. 252, Springer, New York, 2009.
- [27] G. Grubb, *Krein-like extensions and the lower boundedness problem for elliptic operators*, J. Diff. Eq. **252**, 852–885 (2012).
- [28] E. M. Harrell II and L. Hermi, *Differential inequalities for Riesz means and Weyl-type bounds for eigenvalues*, J. Funct. Anal. **254**, 3173–3191 (2008).
- [29] E. M. Harrell II and L. Hermi, *On Riesz means of eigenvalues*, Commun. Partial Diff. Eq. **36**, 1521–1543 (2011).
- [30] T. Kato, *Perturbation Theory for Linear Operators*, corr. printing of the 2nd ed., Springer, Berlin, 1980.

- [31] V. A. Kozlov, *Estimation of the remainder in a formula for the asymptotic behavior of the spectrum of nonsemibounded elliptic systems*, Vestnik Leningrad. Univ. Mat. Mekh. Astronom. **1979**, no 4., 112–113, 125 (Russian).
- [32] V. A. Kozlov, *Estimates of the remainder in formulas for the asymptotic behavior of the spectrum for linear operator bundles*, Funktsional. Anal. i Prilozhen **17**, no. 2, 80–81 (1983). Engl. transl. in Funct. Anal. Appl. **17**, no. 2, 147–149 (1983).
- [33] V. A. Kozlov, *Remainder estimates in spectral asymptotic formulas for linear operator pencils*, Linear and Nonlinear Partial Differential Equations. Spectral Asymptotic Behavior, pp. 34–56, Probl. Mat. Anal. **9**, Leningrad Univ., Leningrad, 1984; Engl. transl. in J. Sov. Math. **35**, 2180–2193 (1986).
- [34] M. G. Krein, *The theory of self-adjoint extensions of semi-bounded Hermitian transformations and its applications. I*, Mat. Sbornik **20**, 431–495 (1947) (Russian).
- [35] A. Laptev, *Dirichlet and Neumann eigenvalue problems on domains in Euclidean spaces*, J. Funct. Anal. **151**, 531–545 (1997).
- [36] M. Levendorskiĭ, *Asymptotic Distribution of Eigenvalues of Differential Operators*, Kluwer, Dordrecht, 1990.
- [37] P. Li and S.-T. Yau, *On the Schrödinger equation and the eigenvalue problem*, Commun. Math. Phys. **88**, 309–318 (1983).
- [38] E. H. Lieb, *The number of bound states of one-body Schroedinger operators and the Weyl problem*, in *Geometry of the Laplace Operator*, R. Osserman and A. Weinstein (eds.), Proc. Symposia Pure Math., Vol. 36, Amer. Math. Soc., Providence, RI, 1980, pp. 241–252.
- [39] M. M. Malamud, *Certain classes of extensions of a lacunary Hermitian operator*, Ukrainian Math. J. **44**, No. 2, 190–204 (1992).
- [40] V. Maz'ya, *Sobolev Spaces with Applications to Elliptic Partial Differential Equations*, 2nd, rev. and augmented ed., Springer, Berlin, 2011.
- [41] G. Métivier, *Valeurs propres de problèmes aux limites elliptiques irrégulières*, Mém. Soc. Math. France **51–52**, 125–219 (1977).
- [42] V. A. Mikhailets, *Distribution of the eigenvalues of finite multiplicity of Neumann extensions of an elliptic operator*, Differentsial'nye Uravneniya **30**, 178–179 (1994) (Russian); Engl. transl. in Diff. Eq. **30**, 167–168 (1994).
- [43] V. A. Mikhailets, *Discrete spectrum of the extreme nonnegative extension of the positive elliptic differential operator*, in *Proceedings of the Ukrainian Mathematical Congress–2001, Section 7, Nonlinear Analysis*, Kyiv, 2006, pp. 80–94.
- [44] Yu. Netrusov and Yu. Safarov, *Weyl asymptotic formula for the Laplacian on domains with rough boundaries*, Comm. Math. Phys. **253**, no. 2, 481–509 (2005).
- [45] G. V. Rozenbljum, *On the distribution of eigenvalues of the first boundary value problem in unbounded regions*, Sov. Math. Dokl. **12**, 1539–1542 (1971).
- [46] G. V. Rozenbljum, *On the eigenvalues of the first boundary value problem in unbounded domains*, Math. USSR Sb. **18**, 235–248 (1972).
- [47] G. V. Rozenblyum, *Distribution of the discrete spectrum of singular differential operators*, Sov. Math. **20**, 63–71 (1976).
- [48] Yu. Safarov, *Fourier Tauber theorems and applications*, J. Funct. Anal. **185**, 111–128 (2001).
- [49] Yu. Safarov and D. Vassiliev, *The Asymptotic Distribution of Eigenvalues of Partial Differential Operators*, Transl. of Math. Monographs, Vol. 155, Amer. Math. Soc., Providence, RI, 1997.
- [50] K. Schmüdgen, *Unbounded Self-Adjoint Operators on Hilbert Space*, Graduate Texts in Mathematics, Vol. 265, Springer, Dordrecht, 2012.
- [51] M. L. Višik, *On general boundary problems for elliptic differential equations*, Trudy Moskov. Mat. Obsc. **1**, 187–246 (1952) (Russian); Engl. transl. in Amer. Math. Soc. Transl. (2), **24**, 107–172 (1963).
- [52] T. Weidl, *Improved Berezin–Li–Yau inequalities with a remainder term*, in *Spectral Theory of Differential Operators. M. Sh. Birman 80th Anniversary Collection*, Adv. Math. Sci., Vol. 62, Amer. Math. Soc. Transl. (2) **225** (2008), pp. 253–263.

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