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Polygonal-Path Approximations on the Path Spaces of Quantum-Mechanical Systems

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Properties of the subset of polygonal paths in the Hilbert space of paths referring to a d-dimensional quantum-mechanical system are examined. The results are used to discuss various types of polygonal-path approximations appearing in the functional-integration theory. The uniform approximation is applied to extend the definition of the Feynman maps from our previous paper and to prove consistency of this extension. Relations of the extended F_{-i} map to the Wiener integral are given.

1. INTRODUCTION

Functional integration often employs methods in which the path space under consideration is replaced by the subset of polygonal paths. In this way, e.g., the Wiener integral of sufficiently smooth function(al)s can be evaluated (Cameron, 1960; Truman, 1978). However, while in the mentioned case the polygonal-path approximations represent a useful calculation technique, they are of conceptual importance for the Feynman integral because of the absence of its sufficiently general and widely accepted definition.

The polygonal paths were connected closely with the very beginning of the concept of F integral (Feynman, 1948; Feynman and Hibbs, 1965). Later they have appeared in various attempts to develop a rigorous F-integral theory, to say nothing of numerous nonrigorous calculations. We have listed some of these attempts in the introduction of our previous paper (Exner and Kolerov, 1980), hereafter referred as [I]: among them the Nelson variant of

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Feynman's heuristic definition (Nelson, 1964; Reed and Simon, 1975) and its generalizations (Combe et al., 1978) as well as the method of Truman (1977, 1978, 1979) are based on variously modified polygonal-path approximations. Let us recall some other treatments in which this idea played a central role: the above-mentioned paper of Cameron (1960) and those of Gel'fand and Yaglom (1956), Babbitt (1963) [for a more complete bibliography till the middle of seventies we refer to (Albeverio and Hoegh-Krohn (1976), and further to the paper of these authors (1979), (Truman, 1979), and other papers contained in the same proceedings], or recently the treatment of F integrals on Riemannian manifolds (Elworthy and Truman, 1979) and the general cylindrical approximation of Tarski (1979) with a particular choice of the path space and the reference family.

The present paper is devoted to the study of polygonal-path approximations on the Hilbert space of paths which refers to a quantum-mechanical system with d degrees of freedom. In the following section we examine in details properties of time-interval partitions and of the corresponding polygonal paths. Results of this treatment will serve in Section 3 for discussion of different types of polygonal-path approximations, in particular those used by the above-named authors. Further we shall apply there the "strongest" one of them, the uniform polygonal-path approximation, to extend domain of the F maps introduced in [I] and to prove consistency of this extension. Among these maps the F_{-i} map is particularly interesting: we shall give some sufficient conditions under which it can be identified with the Wiener integral. For $d=1$ the F_{-i} map is closely related to the sequential Wiener integral of Cameron (1960). We shall show that the basic theorem concerning the latter must be improved: modified conditions on the order of growth under which the assertion holds are given in concluding remarks. Applications of the polygonally extended F integrals to solving Schrödinger-type equations will be discussed elsewhere (Exner and Kolerov, 1981b).

2. PARTITIONS AND POLYGONAL PATHS

We shall consider a d-dimensional quantum-mechanical system with the configuration space $\mathbb{R}^d$, the elements of which will be abbreviated as $x=(x_1,\ldots,x_d)$. Let us introduce first some notation:

$$J^t=[0,t], \qquad t>0$$

$$\gamma: J^t \to \mathbb{R}^d$$

is a $\mathbb{R}^d$-valued function, $\gamma(\tau)=(\gamma_1(\tau),\ldots,\gamma_d(\tau))$, conventionally

$$\gamma(\tau)\cdot\tilde{\gamma}(\tau)=\sum_{j=1}^{d}\gamma_j(\tau)\tilde{\gamma}_j(\tau)$$

further

$$\gamma^2(\tau)=\gamma(\tau)\cdot\gamma(\tau) \quad \text{and} \quad |\gamma(\tau)|=[\gamma^2(\tau)]^{1/2}$$

$$C_0[J^t;\mathbb{R}^d]=\{\gamma\colon J^t\to\mathbb{R}^d\colon \gamma \text{ continuous in } J^t, \gamma(t)=0\}$$

γ is said to be absolutely continuous in J^t iff γ_j, $j=1,\ldots,d$, are absolutely continuous in J^t

$$AC_0[J^t;\mathbb{R}^d]=\{\gamma\in C_0[J^t;\mathbb{R}^d]\colon \gamma \text{ absolutely continuous in } J^t,$$
$$\dot{\gamma}\in L^2(J^t;\mathbb{R}^d)\}$$

clearly $\dot{\gamma}\in L^2(J^t;\mathbb{R}^d)$ iff $\dot{\gamma}_j\in L^2(J^t;\mathbb{R})$, $j=1,\ldots,d$

$$(\gamma,\tilde{\gamma})=\int_{J^t}\dot{\gamma}(\tau)\cdot\dot{\tilde{\gamma}}(\tau)\,d\tau=\sum_{j=1}^{d}\int_{J^t}\dot{\gamma}_j(\tau)\dot{\tilde{\gamma}}_j(\tau)\,d\tau$$

We shall adopt in the following $AC_0[J^t;\mathbb{R}^d]$ as the path space; for the sake of simplicity we shall denote it often as $\mathcal{H}$. The following assertion is valid (Exner and Kolerov 1981a):

Proposition 1. (a) $AC_0[J^t;\mathbb{R}^d]$ equipped with the inner product $(.,.)$ is a real separable Hilbert space.

(a) The elements of $AC_0[J^t;\mathbb{R}^d]$ can be expressed by means of trigonometric series: if γ is an arbitrary element of $AC_0[J^t;\mathbb{R}^d]$, then there exist $\alpha_0\in\mathbb{R}^d$, $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}\subset\mathbb{R}^d$ which obey

$$\sum_{n=1}^{\infty}(\alpha_n^2+\beta_n^2)=\sum_{n=1}^{\infty}\sum_{j=1}^{d}(\alpha_{nj}^2+\beta_{nj}^2)<\infty \tag{1}$$

and such that

$$\gamma(\tau)=\alpha_0(\tau-t)+\sum_{n=1}^{\infty}\frac{\alpha_n t}{2\pi n}\sin\left(\frac{2\pi n\tau}{t}\right)+\sum_{n=1}^{\infty}\frac{\beta_n t}{2\pi n}\times\left[1-\cos\left(\frac{2\pi n\tau}{t}\right)\right] \tag{2}$$

for all $\tau \in J^t$. Conversely, any sequence of $\mathbb{R}^d$-valued coefficients which fulfills (1) determines through (2) some element of $AC_0[J^t;\mathbb{R}^d]$. Finally, if $\tilde{\alpha}_0, \{\tilde{\alpha}_n\}, \{\tilde{\beta}_n\}$ refer to $\tilde{\gamma} \in AC_0[J^t;\mathbb{R}^d]$, the inner product is given by

$$(\gamma, \tilde{\gamma}) = t\alpha_0 \cdot \tilde{\alpha}_0 + \frac{t}{2} \sum_{n=1}^{\infty} \left(\alpha_n \cdot \tilde{\alpha}_n + \beta_n \cdot \tilde{\beta}_n\right) \tag{3}$$

As mentioned above we shall deal with polygonal approximations to the elements of $\mathcal{H}$. To this purpose we introduce first some more notions. *Partition* of J^t is a set $\sigma = \{\tau_i\colon\ i = 0,\dots,n\}$, $0 = \tau_0 < \tau_1 < \cdots < \tau_n = t$. The family of all these partitions is denoted as $\mathcal{P}(J^t)$; further we introduce $\Delta_i = [\tau_i, \tau_{i+1}]$ and $\delta_i = \tau_{i+1} - \tau_i$. A partition σ' is said to be *refinement* of $\sigma, \sigma' \supset \sigma$, if each Δ'_k is contained in some Δ_i; clearly $\supset$ defines a partial ordering on $\mathcal{P}(J^t)$ without maximal elements. The symbols $\sigma \cap \sigma'$ and $\sigma \cup \sigma'$ mean the partitions obtained by natural ordering of the intersection and union of σ, σ', respectively. A partition σ is said to *decompose to subpartitions* $\sigma^{(1)},\dots,\sigma^{(r)}, \sigma = \{\sigma^{(1)},\dots,\sigma^{(r)}\}$ if $\sigma^{(1)} = \{\tau_i\colon\ i = 0,\dots,i_1\}, \sigma^{(2)} = \{\tau_i\colon i = i_1,\dots,i_2\},\dots,\sigma^{(r)} = \{\tau_i\colon i = i_{r-1},\dots,n\}$ (end points of the neighboring partitions coincide). Let $\sigma \supset \sigma'$ so that $\tau'_k = \tau_{i_k}$ for each $k = 0,1,\dots,n'$, then the decomposition $\sigma = \{\sigma^{(1)},\dots,\sigma^{(n')}\}, \sigma^{(k)} = \{\tau_i\colon\ i = i_{k-1},\dots,i_k\}$, is said to be *generated* by σ'. Decompositions $\sigma = \{\sigma^{(1)},\dots,\sigma^{(r)}\}, \tilde{\sigma} = \{\tilde{\sigma}^{(1)},\dots,\tilde{\sigma}^{(r)}\}$ are *comparable* if $\tau_{i_j} = \tilde{\tau}_{k_j}, j = 1,\dots,r$; in other words, if the subpartitions of σ and $\tilde{\sigma}$ refer to the same subintervals of J^t. Partitions $\sigma, \tilde{\sigma} \in \mathcal{P}(J^t)$ are said to be *commuting* if there exist comparable decompositions $\sigma = \{\sigma^{(1)},\dots,\sigma^{(r)}\}$ and $\tilde{\sigma} = \{\tilde{\sigma}^{(1)},\dots,\tilde{\sigma}^{(r)}\}$ such that $\sigma^{(j)} \subset \tilde{\sigma}^{(j)}$ or $\sigma^{(j)} \supset \tilde{\sigma}^{(j)}$ for each $j = 1,\dots,r$. Clearly, σ and σ' commute if one of them refines the other. Finally, we introduce $\sigma' * \sigma = \{0,t\} \cup \{\tau'_k\colon\ [\tau'_{k-1}, \tau'_{k+1}] \not\subset \Delta_i,\ i = 1,\dots,n\}$. The following auxiliary statement holds:

Proposition 2. Partitions $\sigma, \sigma' \in \mathcal{P}(J^t)$ commute iff $\sigma * \sigma' = \sigma' * \sigma$; then $\sigma * \sigma' = \sigma' * \sigma = \sigma \cap \sigma'$.

Proof. (a) Let $\sigma, \tilde{\sigma}$ commute so that there exist comparable decompositions $\sigma = \{\sigma^{(1)},\dots,\sigma^{(r)}\}, \tilde{\sigma} = \{\tilde{\sigma}^{(1)},\dots,\tilde{\sigma}^{(r)}\}$. Assume an arbitrary $j = 1,\dots,r$. The end points $\tau_{i_j} = \tilde{\tau}_{k_j}$ and $\tau_{i_{j+1}} = \tilde{\tau}_{k_{j+1}}$ of the subpartitions $\sigma^{(j)}, \tilde{\sigma}^{(j)}$ belong clearly to $\sigma \cap \tilde{\sigma}$, and also to $\sigma * \tilde{\sigma}$ and $\tilde{\sigma} * \sigma$, since their neighboring points belong to different intervals of the other partition. Let, e.g., $\sigma^{(j)} \supset \tilde{\sigma}^{(j)}$; then the points of $\sigma^{(j)} \backslash \tilde{\sigma}^{(j)}$ are contained neither in $\sigma \cap \tilde{\sigma}$, nor in $\sigma * \tilde{\sigma}$, nor of course in $\tilde{\sigma} * \sigma$. On the other hand, it can be seen easily that the subpartition $\tilde{\sigma}^{(j)}$ belongs wholly to the sets $\sigma \cap \tilde{\sigma}, \sigma * \tilde{\sigma}, \tilde{\sigma} * \sigma$. Thus these sets coincide with the more rough subpartition in every particular interval, and therefore they equal each other.

(b1) Conversely, let $\sigma*\sigma'=\sigma'*\sigma$. Assume any two neighboring points τ_{i_0}, τ_{i_0+1} of σ. If there is some $\tau'_k\in\sigma', \tau_{i_0}<\tau'_k<\tau_{i_0+1}$, then the following possibilities arise: either $\tau_{i_0}\leqslant\tau'_{k-1}<\tau'_{k+1}\leqslant\tau_{i_0+1}$ or $\tau'_k\in\sigma'*\sigma$ and in the same time $\tau'_k\notin\sigma\supset\sigma*\sigma'$; however, the latter contradicts to the assumption. As to the former one: either $\tau'_{k-1}=\tau_{i_0}$ and $\tau'_{k+1}=\tau_{i_0+1}$ or at least one of them belongs to $(\tau_{i_0}, \tau_{i_0+1})$ and the same argument as above can be applied. Since the partition σ' is finite we arrive to the following result: there exist some k_1, k_2, $k_1<k<k_2$, such that $\tau'_{k_1}=\tau_{i_0}$ and $\tau'_{k_2}=\tau_{i_0+1}$. Analogously, if $\tau'_{k_0}<\tau_i<\tau'_{k_0+1}$, then there exist $i_1<i<i_2$ such that $\tau_{i_1}=\tau'_{k_0}$ and $\tau_{i_2}=\tau'_{k_0+1}$.
(b2) Assume further the decompositions of σ and σ' generated by $\sigma\cap\sigma'$ which are clearly comparable. Let $\Delta=[\xi,\eta]$ be an arbitrary subinterval of $\sigma\cap\sigma'$, $\xi=\tau_{i_0}=\tau'_{k_0}$; then the interior of Δ contains points of at most one of the partitions σ,σ'. Suppose that this is not true; then either $\xi=\tau_{i_0}<\tau'_{k_0+1}<\tau_{i_0+1}<\eta$ or $\xi=\tau'_{k_0}<\tau_{i_0+1}<\tau'_{k_0+1}<\eta$; in both these cases, however, (b1) implies the existence of $\zeta\in(\xi,\eta)$ which belongs to $\sigma\cap\sigma'$, but according to the assumption ξ,η are neighboring points of $\sigma\cap\sigma'$. Consequently, one of the subpartitions referring to Δ is trivial (consisting of ξ,η only) and the other is therefore its refinement, i.e., σ and σ' commute.

Further we introduce for any fixed $\sigma\in\mathcal{P}(J^t)$ the mapping $P^c(\sigma)$: $C_0[J^t;\mathbb{R}^d]\to AC_0[J^t;\mathbb{R}^d]$ by

$$\big(P^c(\sigma)\gamma\big)(\tau)=\gamma^i+(\gamma^{i+1}-\gamma^i)\delta_i^{-1}(\tau-\tau_i) \tag{4}$$

for $\tau\in\Delta_i$, $i=0,1,\ldots,n-1$, where $\gamma^i\equiv\gamma(\tau_i)$. It assigns obviously to each continuous path $\gamma\in C_0[J^t;\mathbb{R}^d]$ the polygonal path going through the points γ^i, $i=0,1,\ldots,n$. In what follows we shall deal mainly with the restrictions

$$P(\sigma)=P^c(\sigma)\restriction AC_0[J^t;\mathbb{R}^d] \tag{5}$$

Properties of the operators $P(\sigma)$ can be derived easily by means of the reproduction kernel technique. Let us denote

$$g\colon J^t\times J^t\to\mathbb{R}: g(\tau,\xi)=t-\max(\tau,\xi) \tag{6}$$

$$G\colon J^t\times J^t\to\mathcal{L}(\mathbb{R}^d)\colon G(\tau,\xi)=g(\tau,\xi)I_d \tag{7}$$

where I_d is the unit operator on $\mathbb{R}^d$. As noticed by Albeverio and Hoegh-Krohn (1976), $G(.,.)$ represents the kernel of the operator $-(d^2/d\tau^2)I_d$

with boundary conditions $\varphi(t)=\dot{\varphi}(0)=0$. For our purpose the following property is important:

Proposition 3. $G(.,.)$ is a reproducing kernel of $AC_0[J^t;\mathbb{R}^d]$ in the sense that $G(\tau,.)\beta\in AC_0[J^t;\mathbb{R}^d]$ for all $\tau\in J^t, \beta\in\mathbb{R}^d$ and the relation

$$(\gamma, G(\tau,.)\beta)=\beta\cdot\gamma(\tau) \tag{8}$$

holds for each $\gamma\in AC_0[J^t;\mathbb{R}^d]$.

Proof. If $\beta\in\mathbb{R}^d$, then $G(\tau,.)\beta=\beta g(\tau,.)$ belongs obviously to $AC_0[J^t;\mathbb{R}^d]$. Further

$$(\gamma, G(\tau,.)\beta)=\int_{J^t}\dot{\gamma}(\xi)\cdot\beta\frac{\partial g(\tau,\xi)}{\partial\xi}d\xi=-\int_\tau^t\dot{\gamma}(\xi)\cdot\beta\, d\xi=\gamma(\tau)\cdot\beta$$

because $\gamma(t)=0$.

Theorem 1. (a) $P(\sigma)$ is an orthogonal projection for any $\sigma\in\mathcal{P}(J^t)$.
(b) $P(\sigma)$ commutes with $P(\sigma')$ iff the partitions σ,σ' commute.
(c) $P(\sigma)\geqslant P(\sigma')$ iff $\sigma\supset\sigma'$.
(d) $\dim P(\sigma)=dn(\sigma)$, especially the d-dimensional subspace of linear paths ending in the origin corresponds to the trivial partition $\sigma_0=\{0,t\}$.

Proof. (a) For an arbitrary $\gamma\in\mathcal{H}$ we have $(P(\sigma)\gamma)(\tau_i)=\gamma^i=\gamma(\tau_i)$ so $(P(\sigma))^2=P(\sigma)$. The relation

$$g(\tau_{i+1},\tau)-g(\tau_i,\tau)=\begin{cases}-\delta_i, & \tau\leqslant\tau_i\\ \tau-\tau_{i+1}, & \tau\in\Delta_i\\ 0, & \tau\geqslant\tau_{i+1}\end{cases} \tag{9}$$

implies easily the following identity:

$$(P(\sigma)\gamma)(\tau)=\sum_{i=0}^{n-1}(G(\tau_{i+1},\tau)-G(\tau_i,\tau))\delta_i^{-1}(\gamma^{i+1}-\gamma^i) \tag{10}$$

Thus we can write

$$\begin{aligned}(\tilde{\gamma},P(\sigma)\gamma)=&\sum_{i=0}^{n-1}\left(\tilde{\gamma},G(\tau_{i+1},.)(\gamma^{i+1}-\gamma^i)\right)\delta_i^{-1}\\ &-\sum_{i=0}^{n-1}\left(\tilde{\gamma},G(\tau_i,.)(\gamma^{i+1}-\gamma^i)\right)\delta_i^{-1}\end{aligned}$$

and the reproducing kernel property (8) yields

$$(\tilde{\gamma}, P(\sigma)\gamma) = \sum_{i=0}^{n-1} (\tilde{\gamma}^{i+1} - \tilde{\gamma}^{i}) \cdot (\gamma^{i+1} - \gamma^{i}) = (P(\sigma)\tilde{\gamma}, \gamma) \tag{11}$$

for all $\gamma, \tilde{\gamma} \in \mathcal{H}$, where $\tilde{\gamma}^k$ denotes again $\tilde{\gamma}(\tau_k)$. Consequently, the operator $P(\sigma)$ is symmetric, idempotent, and defined everywhere in $\mathcal{H}$, i.e., an orthogonal projection.

(b) The mapping $\sigma \mapsto P(\sigma)$ is obviously injective. If σ, σ' do not commute, then $\sigma * \sigma' \neq \sigma' * \sigma$ due to Proposition 2, and the mentioned injectivity together with the relation $\operatorname{Ran} P(\sigma)P(\sigma') = \operatorname{Ran} P(\sigma * \sigma')$ show that $P(\sigma), P(\sigma')$ do not commute too. Conversely, let σ, σ' commute. A simple calculation using relation (10) yields

$$(\tilde{\gamma}, P(\sigma')P(\sigma)\gamma) = \sum_{k=0}^{n'-1} \sum_{i=0}^{n-1} \Gamma_{ik} (\tilde{\gamma}(\tau'_{k+1}) - \tilde{\gamma}(\tau'_k)) \cdot (\gamma(\tau_{i+1}) - \gamma(\tau_i)) \delta_i^{-1} (\delta'_k)^{-1} \tag{12}$$

where

$$\Gamma_{ik} = g(\tau_{i+1}, \tau'_{k+1}) - g(\tau_{i+1}, \tau'_k) - g(\tau_i, \tau'_{k+1}) + g(\tau_i, \tau'_k)$$

for arbitrary $\gamma, \tilde{\gamma} \in \mathcal{H}$, and analogously

$$(\tilde{\gamma}, P(\sigma)P(\sigma')\gamma) = \sum_{i=0}^{n-1} \sum_{k=0}^{n'-1} \Gamma'_{ki} (\tilde{\gamma}(\tau_{i+1}) - \tilde{\gamma}(\tau_i)) \cdot (\gamma(\tau'_{k+1}) - \gamma(\tau'_k)) \delta_i^{-1} (\delta'_k)^{-1} \tag{13}$$

where

$$\Gamma'_{ki} = g(\tau'_{k+1}, \tau_{i+1}) - g(\tau'_{k+1}, \tau_i) - g(\tau'_k, \tau_{i+1}) + g(\tau'_k, \tau_i)$$

The interval J' can be decomposed due to the assumption into subintervals $\Delta^{(j)}$, $j = 1, \ldots, r$, such that in each of them the corresponding subpartitions fulfill either $\sigma^{(j)} \supset (\sigma')^{(j)}$ or $\sigma^{(j)} \subset (\sigma')^{(j)}$. In the first case the relation (9) implies

$$\Gamma_{ik}=\Gamma'_{ki}=\begin{cases}\delta_i, & \Delta_i\subset\Delta'_k\\ 0, & \text{otherwise}\end{cases}\qquad (*)$$

for each $\Delta'_k\subset\Delta^{(j)}$. Similarly, if $\sigma^{(j)}\subset(\sigma')^{(j)}$, we obtain

$$\Gamma_{ik}=\Gamma'_{ki}=\begin{cases}\delta'_k, & \Delta'_k\subset\Delta_i\\ 0, & \text{otherwise}\end{cases}\qquad (**)$$

for each $\Delta'_k\subset\Delta^{(j)}$. In particular, $\Gamma_{ik}=\Gamma'_{ki}=0$ if the intervals Δ_i and Δ'_k are disjoint; then (12) may be rewritten as follows:

$$(\tilde\gamma,P(\sigma')P(\sigma)\gamma)=\sum_{j=1}^{r}(\tilde\gamma,P(\sigma')P(\sigma)\gamma)_j \qquad (14)$$

where

$$(\tilde\gamma,P(\sigma')P(\sigma)\gamma)_j=\sum_{\Delta'_k\subset\Delta^{(j)}}\sum_{\Delta_i\subset\Delta'_k}\Gamma_{ik}(\tilde\gamma(\tau'_{k+1})-\tilde\gamma(\tau'_k))\cdot(\gamma(\tau_{i+1})-\gamma(\tau_i))\delta_i^{-1}(\delta'_k)^{-1}$$

in the case that $\sigma^{(j)}\supset(\sigma')^{(j)}$. The last relation can be simplified using $(*)$ to the form

$$(\tilde\gamma,P(\sigma')P(\sigma)\gamma)_j=\sum_{\Delta'_k\subset\Delta^{(j)}}(\tilde\gamma(\tau'_{k+1})-\tilde\gamma(\tau'_k))\cdot(\gamma(\tau'_{k+1})-\gamma(\tau'_k))(\delta'_k)^{-1} \qquad (15)$$

On the other hand, if $\sigma^{(j)}\subset(\sigma')^{(j)}$, then $(**)$ yields

$$(\tilde\gamma,P(\sigma')P(\sigma)\gamma)_j=\sum_{\Delta_i\subset\Delta^{(j)}}(\tilde\gamma(\tau_{i+1})-\tilde\gamma(\tau_i))\cdot(\gamma(\tau_{i+1})-\gamma(\tau_i))\delta_i^{-1} \qquad (16)$$

Further $\Gamma_{ik}=\Gamma'_{ki}$ so $(\tilde\gamma,P(\sigma)P(\sigma')\gamma)$ is expressed again by the formulas (14)–(16). Consequently, $(\tilde\gamma,P(\sigma')P(\sigma)\gamma)=(\tilde\gamma,P(\sigma)P(\sigma')\gamma)$ for all $\gamma,\tilde\gamma\in\mathcal{H}$, i.e., the projections commute.

(c) If $\sigma \supset \sigma'$, then $P(\sigma)$ and $P(\sigma')$ commute according to (b). The relations (14)–(16) now read

$$(\tilde{\gamma}, P(\sigma')P(\sigma)\gamma)$$
$$= \sum_{\Delta'_k \subset J'} (\tilde{\gamma}(\tau'_{k+1}) - \tilde{\gamma}(\tau'_k)) \cdot (\gamma(\tau'_{k+1}) - \gamma(\tau'_k))(\delta'_k)^{-1} = (\tilde{\gamma}, P(\sigma')\gamma)$$

[cf. (11)] for arbitrary $\gamma, \tilde{\gamma} \in \mathcal{H}$ so that $P(\sigma')P(\sigma) = P(\sigma)P(\sigma') = P(\sigma')$, i.e., $P(\sigma) \geqslant P(\sigma')$. Another equivalent formulation of the last inequality is $\operatorname{Ran} P(\sigma) \supset \operatorname{Ran} P(\sigma')$. If $\sigma \not\supset \sigma'$, then there exist Δ_{i_0} and τ'_{k_0} such that $\tau_{i_0} < \tau'_{k_0} < \tau_{i_0+1}$. Each function from $\operatorname{Ran} P(\sigma)$ is linear in Δ_{i_0}; it is not true for $\operatorname{Ran} P(\sigma')$ which contains paths having a "corner" at $\tau = \tau'_{k_0}$. Thus $\operatorname{Ran} P(\sigma) \not\supset \operatorname{Ran} P(\sigma')$ or equivalently $P(\sigma) \not\geqslant P(\sigma')$.

(d) Let $\{e_j\}_{j=1}^{d}$ be an orthonormal basis in $\mathbb{R}^d$ and $\sigma = \{\tau_i\}_{i=0}^{n} \in \mathcal{P}(J')$. The functions

$$\gamma_{ij}\colon \gamma_{ij}(\tau) = e_j(g(\tau_{i+1}, \tau) - g(\tau_i, \tau))\delta_i^{-1/2} \tag{17}$$

$i = 0, \ldots, n-1, j = 1, \ldots, d$, are orthonormal and span $P(\sigma)\mathcal{H}$ due to (10).

A sequence $\{\sigma_m\}_{m=1}^{\infty}$ of partitions is said to be *crumbling* if the lengths of all subintervals $\Delta_i(\sigma_m)$ tend to zero with $m \to \infty$, i.e., if

$$\lim_{m \to \infty} \delta(\sigma_m) = 0, \qquad \delta(\sigma_m) = \max_{0 \leqslant i \leqslant n(\sigma_m)-1} \delta_i(\sigma_m) \tag{18}$$

Such sequences are of central importance for polygonal-path approximations because of the following property:

Theorem 2. Let a sequence $\{\sigma_m\}_{m=1}^{\infty} \subset \mathcal{P}(J')$ be crumbling; then $\text{s-lim}_{m\to\infty} P(\sigma_m) = I$. Furthermore, the convergence is uniform in the set $\mathcal{C}(J')$ of all crumbling sequences: to each $\gamma \in \mathcal{H}$, $\varepsilon > 0$ there exists $\delta[\varepsilon] > 0$ such that $\|P(\sigma)\gamma - \gamma\| < \varepsilon$ for all $\sigma \in \mathcal{P}(J')$ with $\delta(\sigma) < \delta[\varepsilon]$, or symbolically

$$\operatorname*{s-lim}_{\delta(\sigma)\to 0} P(\sigma) = I \tag{19}$$

Remark. On the other hand, it is clear that if $\text{s-lim}_{m\to\infty} P(\sigma_m)$ exists for a noncrumbling $\{\sigma_m\}_{m=1}^{\infty}$, it cannot be equal to the unit operator. Let us take a suitable $\gamma \in \mathcal{H}$, say $\gamma(\tau) = (2t - \tau)^2 - t^2$; then

$$\|P(\sigma)\gamma - \gamma\|^2 \geqslant \int_{\tau_i}^{\tau_{i+1}} |2\tau - \tau_i - \tau_{i+1}|^2 \, d\tau$$

for each subinterval $\Delta_i \in \sigma$ so that $\|P(\sigma)\gamma-\gamma\| \geqslant [\frac{1}{3}\delta(\sigma)]^{1/2}$

Proof of the theorem. We have to show that

$$V \equiv \left\{ \gamma \in AC_0[J';\mathbb{R}^d] : \lim_{\delta(\sigma)\to 0} \|P(\sigma)\gamma-\gamma\|=0 \right\} = AC_0[J';\mathbb{R}^d]$$

We shall prove first that V is a closed subspace in $\mathcal{H}$. Linearity is obvious, closedness follows from the $\frac{1}{3}\varepsilon$ trick: an arbitrary Cauchy sequence $\{\gamma^{(r)}\}_{r=1}^{\infty} \subset V$ converges in the norm to some $\gamma \in \mathcal{H}$, further

$$\|P(\sigma)\gamma-\gamma\| \leqslant 2\|\gamma^{(r)}-\gamma\| + \|P(\sigma)\gamma^{(r)}-\gamma^{(r)}\| \qquad (*)$$

To any $\varepsilon>0$ there exist $r_0(\varepsilon)$ and $\delta_0(r,\varepsilon)$ such that

$$\|\gamma^{(r)}-\gamma\| < \tfrac{1}{3}\varepsilon, \qquad \|P(\sigma)\gamma^{(r)}-\gamma^{(r)}\| < \tfrac{1}{3}\varepsilon \qquad (**)$$

for all $r>r_0(\varepsilon)$ and $\sigma \in \mathcal{P}(J')$ with $\delta(\sigma)<\delta_0(r,\varepsilon)$. For an arbitrary partition σ with $\delta(\sigma)<\delta_0(r_0(\varepsilon)+1,\varepsilon)$ the relations $(*)$, $(**)$ give $\|P(\sigma)\gamma-\gamma\|<\varepsilon$ so that γ belongs to V, which is therefore closed.

According to Proposition 1 the elements of $\mathcal{H}$ can be expressed by trigonometric series (2). Let $\{e_j\}_{j=1}^{d}$ be some orthonormal basis in $\mathbb{R}^d$. One can check easily that the functions u_{jk}: $u_{jk}(\tau)=e_j v_k(\tau)$, $j=1,\ldots,d$, $k=1,2,\ldots$, where

$$v_1(\tau)=\tau-t, \qquad v_{2N}(\tau)=\sin\left(\frac{2\pi N\tau}{t}\right), \qquad v_{2N+1}(\tau)=1-\cos\left(\frac{2\pi N\tau}{t}\right)$$

form an orthonormal basis in $\mathcal{H}$. It is sufficient therefore to verify that $\beta v_k(\cdot)$ is contained in V for all $\beta \in \mathbb{R}^d$ and $k=1,2,\ldots$. This is trivial for $k=1$. Assume further $k=2N$ and $\tau \in \Delta_i^m \equiv [\tau_i^m, \tau_{i+1}^m]$. We have

$$(P(\sigma_m)\beta v_{2N})(\tau) = \beta \sin\left(\frac{2\pi N}{t}\tau_i^m\right) + \beta\left[\sin\left(\frac{2\pi N}{t}\tau_{i+1}^m\right) - \sin\left(\frac{2\pi N}{t}\tau_i^m\right)\right](\tau-\tau_i^m)(\delta_i^m)^{-1}$$

where $\delta_i^m \equiv \delta_i(\sigma_m)$, further a simple calculation gives

$$F(\tau) \equiv -\frac{d}{d\tau}\left[P(\sigma_m)\beta v_{2N}\right](\tau) + \beta\dot{v}_{2N}(\tau)$$

$$= -2\beta\cos\left[\frac{\pi N}{t}(2\tau_i^m + \delta_i^m)\right]\sin\left(\frac{\pi N}{t}\delta_i^m\right)(\delta_i^m)^{-1}$$

$$+\frac{2\pi N\beta}{t}\cos\left(\frac{2\pi N\tau}{t}\right)$$

$$= -\beta\cos\left(\frac{2\pi N}{t}\tau_i^m\right)\sin\left(\frac{2\pi N}{t}\delta_i^m\right)(\delta_i^m)^{-1}$$

$$+2\beta\sin\left(\frac{2\pi N}{t}\tau_i^m\right)\sin^2\left(\frac{\pi N}{t}\delta_i^m\right)(\delta_i^m)^{-1} + \frac{2\pi N\beta}{t}\cos\left(\frac{2\pi N\tau}{t}\right)$$

$$= \frac{2\pi N\beta}{t}\left\{\left[\cos\left(\frac{2\pi N}{t}\tau\right) - \cos\left(\frac{2\pi N}{t}\tau_i^m\right)\right]\right.$$

$$+\cos\left(\frac{2\pi N}{t}\tau_i^m\right)\left[1 - \frac{\sin\left(\frac{2\pi N}{t}\delta_i^m\right)}{\frac{2\pi N}{t}\delta_i^m}\right]$$

$$\left.+\sin\left(\frac{2\pi N}{t}\tau_i^m\right)\frac{\sin^2\left(\frac{\pi N}{t}\delta_i^m\right)}{\frac{\pi N}{t}\delta_i^m}\right\}$$

The last expression can be estimated by means of the inequalities

$$|\cos x - \cos y| \leqslant |x-y|, \qquad \left|\frac{\sin^2 x}{x}\right| \leqslant |x|, \qquad \left|1 - \frac{\sin x}{x}\right| \leqslant \frac{1}{6}x^2$$

we obtain in this way

$$|F(\tau)| \leqslant \frac{2\pi N|\beta|}{t}\left\{\frac{2\pi N}{t}(\tau - \tau_i^m) + \frac{1}{6}\left(\frac{2\pi N}{t}\delta_i^m\right)^2 + \frac{\pi N}{t}\delta_i^m\right\}$$

Further the estimates

$$\frac{2\pi N}{t}\delta_i^m \leq 2\pi N, \qquad \frac{\pi N}{3}+\frac{1}{2}<\frac{2\pi N}{3}<\pi N$$

give

$$|F(\tau)|<\left(\frac{2\pi N}{t}\right)^2|\beta|\left[(\tau-\tau_i^m)+\pi N\delta_i^m\right]$$

so

$$\int_{\tau_i^m}^{\tau_{i+1}^m}|F(\tau)|^2\,d\tau<\left(\frac{2\pi N}{t}\right)^4\beta^2\left(\tfrac{1}{3}+\pi N+\pi^2N^2\right)(\delta_i^m)^3$$

$$<\left(\frac{2\pi N}{t}\right)^4\beta^2(\pi N+1)^2(\delta_i^m)^3$$

It holds $\delta_i^m \leq \delta(\sigma_m)$ according to (18), thus we finally obtain

$$\|P(\sigma_m)\beta v_{2N}-\beta v_{2N}\|^2=\sum_{i=0}^{n(\sigma_m)}\int_{\tau_i^m}^{\tau_{i+1}^m}|F(\tau)|^2\,d\tau$$

$$<(2\pi N)^4t^{-3}\beta^2(\pi N+1)^2[\delta(\sigma_m)]^2$$

Since the sequence $\{\sigma_m\}_{m=1}^{\infty}$ is assumed to be crumbling, we obtain $\lim_{m\to\infty}P(\sigma_m)\beta v_{2N}=\beta v_{2N}$; further the last inequality shows that this convergence is uniform in $\mathcal{C}(J^t)$, i.e., $\beta v_{2N}\in V$ for any natural N. A similar argument applies to v_{2N+1}, $N=1,2,\ldots$.

3. POLYGONAL-PATH APPROXIMATIONS

The main purpose of the polygonal-path methods is to determine the Feynman integral and the relation objects. The term "approximation" is thus a little misleading, because it means at the same time definition of the "approximated quantity" too. The ideology of polygonal-path definitions of the F integral is essentially that of the Riemann-integral theory; however, since there is no analogy to the Darboux sums (Shilov and Gurevich, 1967) here the definitions must be formulated in terms of limits with respect to sequences or nets of partitions.

Let us consider now in more detail the problem of how to define the F maps, i.e., the one-parameter family of complex-valued maps $f\mapsto\Phi_s$ from a suitable set of functions f on $\mathcal{H}$, which correspond to the formal functional integrals

$$\Phi_s = \int_{\mathcal{H}} \exp\left(\frac{i}{2s}\|\gamma\|^2\right) f(\gamma)\mathcal{D}_\gamma \tag{20}$$

with s nonzero, $\operatorname{Im} s \leq 0$ (this subset of s in $\mathbb{C}$ was denoted as $\mathbb{C}_F$ in [I]). The F integral clearly refers to the case $s=1$, if we set for simplicity the Planck constant $\hbar$ as well as the mass(es) equal to 1. Each polygonal-path approach to this problem starts from a choice of mappings φ_s: $\mathcal{P}' \to \mathbb{C}$ which assign to every partition σ of some subset $\mathcal{P}'$ in $\mathcal{P}(J^t)$ "finite-dimensional approximations" $\varphi_s(\sigma)$ to the functional integrals (20). The following step consists of taking a limit of $\varphi_s(\sigma)$, which corresponds in some sense to gradual refining of σ; if this limit exists it is identified with Φ_s.

It is therefore clear that there exist at least two points of view for classification of polygonal-path methods: (i) according to a choice of φ_s, and (ii) according to a choice of the limiting procedure. The first one will be discussed only briefly here (see also remarks in the following section); we limit ourselves to the case which is physically the most interesting, i.e., $s=1$ and

$$f(\gamma) = \exp\left\{-i\int_0^t V(\gamma(\tau)+x)\,d\tau\right\} u(\gamma(0)+x) \tag{21}$$

where V is a potential on $\mathbb{R}^d$ and u belongs to some subset of $L^2(\mathbb{R}^d)$. Then different choices of $\varphi_1(\sigma)$ are possible (of course, for those V, u for which the corresponding expressions make sense), e.g.,

$$\varphi_1(\sigma) := I_1(f \circ P(\sigma)) \tag{22a}$$

where $I_1(\cdot)$ is the F integral of Albeverio and Hoegh-Krohn (1976, 1979), or

$$\varphi_1(\sigma) := (2\pi i)^{-n/2} \int_{P(\sigma)\mathcal{H}} \exp\left(\frac{i}{2}\|\gamma'\|^2\right) f(\gamma')\,dm(\gamma')$$

$$= (2\pi i)^{-n/2} \int_{P(\sigma)\mathcal{H}} \exp[iS(\gamma'+x)]\, u(\gamma(0)+x)\,dm(\gamma') \tag{22b}$$

where $\gamma' = P(\sigma)\gamma$, $n = \dim P(\sigma)$, further $S(\gamma)$ is the action along the path γ and the integral in (22b) is understood in the improper sense (see Truman, 1977, 1978, 1979; Elworthy and Truman, 1979), or finally

$$\varphi_1(\sigma) := (2\pi i)^{-n/2} \int_{P(\sigma)\mathcal{H}} \exp\{iS_\sigma(\gamma'+x)\}\, u(\gamma(0)+x)\,dm(\gamma') \tag{22c}$$

where the action S is replaced by a Riemannian approximation S_σ (in fact, it concerns the potential part only) corresponding to the partition σ, and the integral is again the improper one (see Babbitt, 1963; Nelson, 1964; Combe et al., 1978; Tarski, 1979). The last choice admits use of the Lie–Trotter–Kato formula and gives thus stronger results than the previous two (in the sense that the corresponding functional integral exists and expresses dynamics for a much wider class of potentials). The expression (22b) is in turn applicable to more potentials than (22a); if the latter exists and the integral in (22b) converges in the proper Lebesgue sense, then they equal each other (cf. Truman, 1978; and [I]). On the other hand, (22c) does not correspond exactly to the heuristic prescription of Feynman (1948) and choice of S_σ burdens the definition with an additional arbitrariness. Except for that, the improper integrals in (22b) and (22c) are sensitive to the defining prescription (see DeWitt-Morette et al., 1979; and [I]) so it is desirable to get rid of them.

Let us turn now to discussion of possible limits with respect to the partitions σ. Again various possibilities arise:

(*a*) *The* n^{-1} *and* 2^{-n} *Approximations*. The simplest choice is to take the sequence $\{\sigma_n^e\}_{n=1}^{\infty}$ of "equidistant" partitions (Gel'fand and Yaglom, 1956; Babbitt, 1963; Nelson, 1964; Truman, 1977, 1978; Combe et al., 1978), i.e., with $\tau_i^{(n)}=it/n$, and to set $\Phi_s:=\lim_{n\to\infty}\varphi_s(\sigma_n^e)$. The same can be performed with any subsequence of $\{\sigma_n^e\}_{n=1}^{\infty}$, in particular that one with $\delta_i^{(m)}=2^{-m}t$.

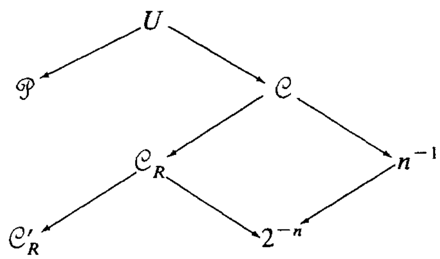
(*b*) *The* $\mathcal{C}$ *Approximation*. One assumes all crumbling sequences and sets $\Phi_s:=\lim_{m\to\infty}\varphi_s(\sigma_m)$ if the limit exists for each $\{\sigma_m\}_{m=1}^{\infty}\in\mathcal{C}(J')$ and does not depend on a particular choice of the sequence.

(*c*) *The* $\mathcal{C}_R$ *and* $\mathcal{C}'_R$ *Approximations*. These are analogous to (b) with $\mathcal{C}(J')$ replaced by $\mathcal{C}_R$, the subset of refining sequences in $\mathcal{C}(J')$, i.e., with $\sigma_{m+1}\supset\sigma_m$ for all m, or by some subset $\mathcal{C}'_R\subset\mathcal{C}_R$. This is the method of Tarski (1979) if we identify his path space with our $\mathcal{H}$ and his projections with $P(\sigma),\sigma\in\mathcal{P}(J')$. He employs increasing sequences of projections with unit limit; it means just the assumptions formulated above in view of Theorems 1 and 2. His reference families correspond, of course, to subsets $\mathcal{C}'_R\subset\mathcal{C}_R$.

(*d*) *The* $\mathcal{P}$ *Approximation*. The set $\mathcal{P}(J')$ is partially ordered by $\supset$; further to each $\sigma,\sigma'\in\mathcal{P}(J')$ there exists $\sigma''\in\mathcal{P}(J')$, say $\sigma''=\sigma\cup\sigma'$, so that $\sigma\subset\sigma''$ and $\sigma'\subset\sigma''$. In other words, $\mathcal{P}(J')$ with $\supset$ is a directed set, and one can define Φ_s as a limit along it, $\Phi_s:=\lim_{\mathcal{P}}\varphi_s(\sigma)$. More explicitly, there exists $\sigma_\varepsilon\in\mathcal{P}(J')$ to each $\varepsilon>0$ such that $|\Phi_s-\varphi_s(\sigma)|<\varepsilon$ for all $\sigma\supset\sigma_\varepsilon$.

(*e*) *The Uniform Approximation*. One assumes again all sequences $\{\sigma_m\}_{m=1}^{\infty}\in\mathcal{C}(J')$, but requires now the convergence to be uniform with respect to the "norm" $\delta(\sigma)$ of σ, i.e. $\Phi_s:=\lim_{\delta(\sigma)\to 0}\varphi_s(\sigma)$, where the limit is understood in the same sense as in Theorem 2. This approach belongs to Cameron (1960) and was further used, e.g., by Johnson and Skoug (1973); a similar approximation, however, with a not very clearly specified subset of $\mathcal{C}(J')$ was recently used by Elworthy and Truman (1979).

Proposition 4. Mutual relations of the above-listed limiting prescriptions are given by the following diagram:



where the arrows denote implications.

Proof. $(U)\Rightarrow(\mathcal{P})$: There exists $\delta[\varepsilon]$ to each $\varepsilon>0$; we choose $\sigma_\varepsilon\in\mathcal{P}(J')$ such that $\delta(\sigma_\varepsilon)<\delta[\varepsilon]$; then we have $\delta(\sigma)<\delta[\varepsilon]$ for all $\sigma\supset\sigma_\varepsilon$, and therefore $|\Phi_s^u-\varphi_s(\sigma)|<\varepsilon$ for all $\sigma\supset\sigma_\varepsilon$, i.e., $\lim_{\mathcal{P}}\varphi_s(\sigma)=\Phi_s^u\equiv\lim_{\delta(\sigma)\to0}\varphi_s(\sigma)$.

$(U)\Rightarrow(\mathcal{C})$: Let us take $\varepsilon>0$, to which some $\delta[\varepsilon]$ corresponds, and an arbitrary $\{\sigma_m\}_{m=1}^{\infty}\in\mathcal{C}(J')$; since the latter is crumbling there exists $m_0(\delta[\varepsilon])$ such that $\delta(\sigma_m)<\delta[\varepsilon]$ for all $m>m_0$, and consequently $|\Phi_s^u-\varphi_s(\sigma_m)|<\varepsilon$ for $m>m_0$, i.e., $\lim_{m\to\infty}\varphi_s(\sigma_m)=\phi_s^u$ independently of $\{\sigma_m\}_{m=1}^{\infty}$

The remaining implications are trivial.

We postpone commenting on these relations till the next section. Now we shall use the uniform polygonal-path approximation in order to extend domain of the F maps introduced in the paper [I], to which we refer for the notation.

A function f: $\mathcal{H}\to\mathbf{C}$ is said to belong to $\mathcal{F}_s^u(\mathcal{H})$ for a given $s\in\mathbf{C}_F$ if the following conditions are fulfilled:

(i) the "cylindrical projections" $f\circ P(\sigma)$ belong to $\mathcal{F}(\mathcal{H})$, the B algebra of F integrable functions, for all $\sigma\in\mathcal{P}(J')$;

(ii) the uniform limit $\lim_{\delta(\sigma)\to0}I_s(f\circ P(\sigma))$ exists.

Then we define naturally the *uniformly extended F_s map* in the following way:

$$I_s^u:\ \mathcal{F}_s^u(\mathcal{H})\to\mathbf{C},\qquad I_s^u(f):=\lim_{\delta(\sigma)\to0}I_s(f\circ P(\sigma)) \tag{23}$$

in particular, $I_1^u(\cdot)$ will be again called the *Feynman integral.*

Of course, one must check consistency of this extension. Another problem which arises here concerns relations between $I^{u}_{-i}(\cdot)$ and the Wiener integral; it can be solved under suitable smoothness and boundedness assumptions. Let us denote $\mathcal{P}[J';\mathbb{R}^d]=\{\gamma_\sigma\colon \gamma_\sigma=P(\sigma)\gamma, \sigma\in\mathcal{P}(J'), \gamma\in\mathcal{H}\}$, the set of all polygonal paths in $\mathcal{H}$; further w and $\|\cdot\|_\infty$ will be the Wiener measure understood as the n-fold product measure of W measures with unit dispersion on $C_0[J';\mathbb{R}]$—cf. Kuo (1975) and the uniform norm on $C_0[J';\mathbb{R}^d]$, respectively.

Theorem 3. (a) Let $s\in\mathbb{C}_F$, then $\mathcal{F}_s^u(\mathcal{H})\supset\mathcal{F}(\mathcal{H})$ and $I_s^u(f)=I_s(f)$ for each $f\in\mathcal{F}(\mathcal{H})$.

(b) Let $f\in\mathcal{F}^u_{-i}(\mathcal{H})$ be a restriction to $\mathcal{H}$ of a w measurable function $F\colon C_0[J';\mathbb{R}^d]\to\mathbb{C}$ which is uniformly continuous with exception of a w zero subset of $C_0[J';\mathbb{R}^d]\setminus\mathcal{P}[J';\mathbb{R}^d]$. If there exists $K>0$ such that $|f(\gamma)|\leq K$ for all $\gamma\in\mathcal{P}[J';\mathbb{R}^d]$, then

$$I^u_{-i}(f)=\int_{C_0[J';\mathbb{R}^d]}F(\gamma)\,dw(\gamma) \tag{24}$$

and $|I^u_{-i}(f)|\leq K$. In particular, if

$$f\in\mathcal{F}(\mathcal{H}),\quad f(\gamma)=\int_{\mathcal{H}}\exp[i(\gamma,\gamma')]\,d\mu(\gamma'), \mu\in\mathcal{M}(\mathcal{H})$$

then

$$\int_{C_0[J';\mathbb{R}^d]}F(\gamma)\,dw(\gamma)=\int_{\mathcal{H}}\exp(-\tfrac{1}{2}\|\gamma\|^2)\,d\mu(\gamma) \tag{25}$$

Proof. (a) Let $f\in\mathcal{F}(\mathcal{H})$, then the same argument as in the proof of Theorem 2 in [I] shows that the condition (i) is fulfilled. As to (ii), it follows immediately from the above Theorem 2 together with Theorem 1(c) of [I].

(b) The set $\mathcal{P}[J';\mathbb{R}^d]$ is $\|\cdot\|_\infty$-dense in $C_0[J';\mathbb{R}^d]$: assume some $\gamma\in C_0[J';\mathbb{R}^d]$ and an arbitrary $\varepsilon>0$; then according to the Weierstrass theorem there exist polynomials π_j such that $|\pi_j(\tau)-\gamma_j(\tau)|<\frac{1}{3}d^{-1/2}\varepsilon$ for all $\tau\in J', j=1,\ldots,d$, so the path $\pi^\varepsilon\colon \pi^\varepsilon(\tau)=(\pi_1(\tau),\ldots,\pi_d(\tau))$ obeys

$$\|\pi^\varepsilon-\gamma\|_\infty<\tfrac{1}{3}\varepsilon \tag{$*$}$$

Further each π_j can be approximated by $P^c(\sigma)\pi_j$ [cf. (4)]: if $\pi_j(\tau)=\sum_{k=0}^n\alpha_k\tau^k$, one obtains easily $|\pi_j(\tau)-(P^c(\sigma)\pi_j)(\tau)|\leq 2\delta(\sigma)\sum_{k=1}^n k|\alpha_k|t^{k-1}$, thus there exists $\delta_0(\varepsilon)$ such that

$$\|\pi^\varepsilon-P^c(\sigma)\pi^\varepsilon\|_\infty<\tfrac{1}{3}\varepsilon \tag{$**$}$$

for all σ with $\delta(\sigma)<\delta_0(\varepsilon)$. Finally, the inequality

$$\|P^c(\sigma)\gamma\|_\infty \leqslant \|\gamma\|_\infty, \qquad \gamma\in C_0[J';\mathbb{R}^d] \tag{26}$$

together with $(*)$ give $\|P^c(\sigma)\pi^\varepsilon - P^c(\sigma)\gamma\|_\infty < \frac{1}{3}\varepsilon$; combining it with $(*)$, $(**)$ we arrive at the relation

$$\lim_{\delta(\sigma)\to 0} \|\gamma - P^c(\sigma)\gamma\|_\infty = 0, \qquad \gamma\in C_0[J';\mathbb{R}^d] \tag{27}$$

Now the assumed continuity of F implies

$$\lim_{\delta(\sigma)\to 0} F(P^c(\sigma)\gamma) = F(\gamma) \qquad w\text{—almost everywhere in } C_0[J';\mathbb{R}^d] \tag{28}$$

and

$$|F(\gamma)|\leqslant K \qquad \text{for } w\text{—almost all } \gamma \tag{29}$$

Further we take $F(P^c(\sigma)\gamma)=F(P^c(\sigma)\gamma_1,\dots,P^c(\sigma)\gamma_d)$; owing to the definition of w as a product measure and using the Fubini theorem we get

$$\int_{C_0[J';\mathbb{R}^d]} F(P^c(\sigma)\gamma)\,dw(\gamma)$$

$$= \int_{C_0[J';\mathbb{R}]} dw_1(\gamma_1)\cdots \int_{C_0[J';\mathbb{R}]} dw_1(\gamma_d) F(P^c(\sigma)\gamma_1,\dots,P^c(\sigma)\gamma_d)$$

Applying now d times the standard formula to the above cylindrical integrals with respect to w_1 and using the Fubini theorem once more we can rewrite the last expression as follows:

$$\left[(2\pi)^n \prod_{k=0}^{n-1} \delta_k\right]^{-d/2} \int_{\mathbb{R}^{nd}} \exp\left\{-\frac{1}{2}\sum_{k=0}^{n-1} |\gamma^{k+1}-\gamma^k|^2 \delta_k^{-1}\right\}$$

$$\times f_\sigma(\gamma^0,\dots,\gamma^{n-1})\,d\gamma^0\cdots d\gamma^{n-1}$$

where $f_\sigma(\gamma^0,\dots,\gamma^{n-1}) := F(P^c(\sigma)\gamma)$. On the other hand, one has $f_\sigma(\gamma^0,\dots,\gamma^{n-1}) = f(P(\sigma)\gamma)$ for $\gamma\in\mathcal{H}$. Further the relation (10) makes it possible to express $P(\sigma)\gamma$ in terms of the orthonormal basis (17):

$$P(\sigma)\gamma = \sum_{k=0}^{n-1} \sum_{j=1}^{d} \delta_k^{-1/2}(\gamma^{k+1}-\gamma^k)_j \gamma_{kj}$$

so we can make the substitution $(\gamma^0,\ldots,\gamma^{n-1}) \to \gamma_\sigma \equiv P(\sigma)\gamma$ in the last integral and obtain

$$\int_{C_0[J';\mathbb{R}^d]} F(P^c(\sigma)\gamma)\,dw(\gamma) = (2\pi)^{-nd/2} \int_{P(\sigma)\mathcal{K}} \exp\{-\tfrac{1}{2}\|\gamma_\sigma\|^2\} f(\gamma_\sigma)\,dm(\gamma_\sigma)$$
$$= I_{-i}(f \circ P(\sigma)) \tag{30}$$

where m is the Lebesgue measure on $P(\sigma)\mathcal{K}$; the last equality follows from Sections 3(ii, iv) in [I]. By the assumption, the right-hand side of (30) tends to $I^u_{-i}(f)$ with $\delta(\sigma) \to 0$, further (24) and the related bound follow from the dominated convergence theorem, (27), (28) and the normalization of w. Finally, if $f \in \mathcal{F}(\mathcal{K})$, then $|f(\gamma)| \leq \|f\|_0$ for all $\gamma \in \mathcal{K}$ due to Proposition 2 of [I] and the assertion (a) together with (24) and the definition of $I_{-i}(\cdot)$ prove (25).

4. CONCLUDING REMARKS

The polygonal-path methods are not, of course, the only tool of the F-integral theory. Their results must be compared with the results of other approaches, in particular with the methods of Itô (1967), DeWitt-Morette (1972, 1974), and those based on analytic continuation of the Wiener integral [e.g., Cameron (1960, 1962–63, 1968), Cameron and Storvick (1966, 1968), Johnson and Skoug (1973)]. Anyhow we feel that, though the situation on this field is a little better now than that described by Dyson (1972) nine years ago, the existence of different "weakly interacting" concepts and of many scattered results represents the challenge to deal with for both mathematicians and physicists.

In conclusion, let us make some comments on the matters discussed in the previous section:

(*a*) *On the Choice of* $\varphi_s(\sigma)$. Starting from (22a) one can avoid complications with improper integrals in definition of the F integral. Except for that, the analogous approach to the F maps [cf. condition (i) of the above definition] allows us to treat them on an equal footing for all $s \in \mathbb{C}_F$ including the real ones. On the other hand, the definition under consideration applies to those f only for which all $f \circ P(\sigma)$ are continuous (cf.

Proposition 2 of [I]), and this seems to be too restrictive from the viewpoint of physical interest. An alternative way is to consider the case of real s, i.e., the F integrals, separately; it was pursued, e.g., by Cameron (1960, 1962–63), Cameron and Storvick (1968), or by Johnson and Skoug (1973). In this approach one defines the F_s maps for $\operatorname{Im} s<0$ by polygonal-path approximation based on $\varphi_s(\sigma)$ defined on analogy with (22b); obviously improper integrals are not needed for a reasonable class of functions f. The F integral ϕ_1 of f is then defined as $\lim_{\varepsilon\to 0+}\phi_{1-i\varepsilon}$. The idea of this definition is thus close to that of Gel'fand and Yaglom (1956), however, with replacement of the erroneous measure-theoretical determination of $\varphi_s(\sigma)$ by the sequential one. Let us mention finally that a similar procedure can be applied to $\varphi_s(\sigma)$ defined on analogy with (22c). Such a method could be promising, if only independence on a chosen Riemannian approximation to the action has been established. A certain progress in this direction was achieved by Cameron (1968).

(*b*) *On the Limiting Prescriptions.* Proposition 4 illustrates the dominating role of the uniform approximation. As to the $\mathcal{P}$ approximation, we have not found it used in the literature; however, it represents one of the natural choices. Let us recall in this connection Itô's definition (1967) of the F integral, where the limit is taken along the directed set of all trace-class covariance operators. Let us further stress that there is no direct correspondence between the $\mathcal{P}$ and $\mathcal{C}_R$ approximations, because in general convergence of a net, the index set of which is not fully ordered, does not imply convergence of its subnets (in particular, subsequences) and vice versa. In the same sense one cannot assert that Itô's definition yields a sequential approximation (cf. P.6, sec. 2 in Albeverio and Hoegh-Krohn, 1979) without an extra proof.

(*c*) *On Theorem* 3. A somewhat stronger assertion can be formulated, namely, instead of bounded functions in part (b) one can assume those with limited order of growth. For this purpose one has to know the distribution of $\|\gamma\|_\infty$ with respect to w: if $d=1$ then the deduction of Cameron (1960) can be adapted (see below), the general case will be discussed elsewhere. However, as presently stated, the assumptions of the part (b) cover most of the physically interesting functions [cf. the right-hand side of the Feynman–Kac formula (Reed and Simon, 1975)]. Let us further notice that the function F can be discontinuous (on a w-zero set), but only outside $\mathcal{P}[J^t;\mathbb{R}^d]$. The analogous assumption is not stated explicitly in the mentioned paper of Cameron; however, it is clear from the proof. Finally the relation (25) was first obtained (for $d=1$, and in a slightly weaker form) by Truman (1978).

(*d*) *On Cameron's Sequential Wiener Integral.* It is defined (for $d=1$) by the uniform polygonal-path limit of $\varphi_{-i}(\sigma)$ analogous to (22b). Cameron

(1960) deduced sufficient conditions under which it can be identified with the usual Wiener integral; they are like our Theorem 3(b) with exception of the boundedness condition, which is replaced by the following one: there exists a measurable nondecreasing $\varphi: \mathbb{R}_+ \to \mathbb{R}_+$ such that $|F(\gamma)| \leq \varphi(\|\gamma\|_\infty)$ for all $\gamma \in \mathcal{P}[J'; \mathbb{R}^d]$ and $u \mapsto \varphi(u)\exp\{-u^2/2t\}$ is integrable on $(0, \infty)$. Proof of this assertion depends essentially on the distribution ω of $\|\gamma\|_\infty$ with respect to the Wiener measure w_1 borrowed from Erdős and Kac (1946). Cameron's argument is wrong at this point: one can check easily that it is not Theorem I, but Theorem II of Erdős and Kac which gives ω. Consequently, a slightly different assertion can be proved [and the assumptions of Theorems 2, 3, and 5 of Cameron (1960) must be correspondingly modified]: one has to demand integrability of

$$u \mapsto \pi t \varphi(u) u^{-3} \sum_{m=0}^{\infty} (-1)^m (2m+1) \exp\{-\tfrac{1}{8}\pi^2 t u^{-2}(2m+1)^2\}$$

[cf. also the theorem of Fernique quoted by Kuo (1975)].

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NOTE ADDED IN PROOF

Formal application of the Euler-Maclaurin formula to the last sum together with numerical analysis suggest that its asymptotics could be $(8/\pi t)^{1/2}\varphi(u)\exp(-u^2/2t)$ for large u, i.e., that $(2t)^{-1}$ might be the upper bound for α in the Fernique's theorem. In such a case, the results of Cameron would be recovered.

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