UNIFORM PRODUCT FORMULAE WITH APPLICATION TO THE FEYNMAN–NELSON INTEGRAL FOR OPEN SYSTEMS

P. EXNER★ and G.I. KOLEROV
Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, 141980 Dubna, U.S.S.R.

ABSTRACT. The product formula for perturbations of propagators by Faris is generalized: we show that one can use arbitrary partitions of a given interval and perform the limit uniformly with respect to the partition norm. The results are applied to express solutions of the Schrodinger equation, in particular for some complex and time-dependent potentials, by means of Nelson-type path integrals.

1. INTRODUCTION

Among approaches to defining the Feynman integral, various sequential methods are probably the most popular ones. Here we consider the one based on product formulae, the original idea of Nelson [1]. This method is powerful in the sense that it enables us to prove the strict version of Feynman's basic dynamical formula for a wide class of potentials [2, 3], including time-dependent ones [4], in comparison with other known methods (see, e.g., [5, 6]). On the other hand, the method does not correspond exactly to the Feynman's heuristic prescription because it replaces the action along polygonal paths by its Riemannian approximation. Moreover, one can regard as an unsatisfactory feature the fact that it only employs the 'equidistant' time-interval partitions.

In the present paper we show that the last-mentioned source of arbitrariness can be removed: the product formula for perturbations of propagators by Faris [4] remains valid if the limit is carried out with any 'crumbling' sequence of partitions uniformly w.r.t. the partition 'norm' given by the maximal subinterval length. In particular, it gives the uniform version of the Trotter formula [7] in the important special case considered in Reference [1], when the sum of the generators of continuous contractive semigroups (CCSG) involved is closed and itself generates a CCSG.

Applying these results to the *F*-integral we make one more generalization: we assume potentials not only time-dependent but *complex* (obeying the dissipativity condition). This is an alternative way of treating non-isolated systems. Such a description is not merely useful phenomenologically but can be embedded into the standard quantum-mechanical framework within the pseudo-Hamiltonian approach [8]. Our goal is to establish the validity of the Feynman–Ito formula with

★On leave of absence from Nuclear Centre, Charles University, Areál Troja, Povltavská ul., 18000 Prague, Czechoslovakia.

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the F-integral defined in terms of the mentioned product formula for three classes of complex potentials (cf. Theorems 2 and 3 below). Let us remark that for some particular cases analogous results have been recently obtained using other definitions of the F-integral [9].

The last introductory item concerns notation: we shall use $J^t = [0, t]$, $t > 0$, $\sigma = \{\tau_i : 0 = \tau_0 < \tau_1 < \ldots < \tau_n = t\}$ partition of J^t; the set of all these partitions is denoted as $\mathscr{P}(J^t)$, $\delta_k = \tau_{k+1} - \tau_k$, $k = 0, 1, \ldots, n-1$, $\delta(\sigma) = \max\{\delta_k : k = 0, 1, \ldots, n-1\}$.

2. THE PRODUCT FORMULAE

THEOREM 1. *Let X with a norm $\|\cdot\|$ be a Banach space, and assume that for each $t \in J^b$, $A(t)$ and $B(t)$ are generators of CCSG's $\{e^{-A(t)s} : s \geqslant 0\}$ and $\{e^{-B(t)s} : s \geqslant 0\}$, respectively. Let, further, $C(t) = A(t) + B(t)$ be a closed operator for each $t \in J^b$, the domain of which is a dense subspace D in X independent of t. Assume that for every $u \in D$, $A(\cdot)u$ and $B(\cdot)u$ are C^0 on J^b. Let there exist a contraction-valued propagator $V(.,.)$ on X (cf. [4]) such that $V(t, s)D \subset D$ for all $t, s \in J^b$. Let us denote $u(t) = V(t, 0)u$ for $u \in D$ and assume that $C(\cdot)u(\cdot)$ is C^0 on J^b and that $u(\cdot)$ is C^1 on $(0, b)$ and there obeys*

$$\frac{du(t)}{dt} + C(t)u(t) = 0. \tag{1}$$

Finally, let $u(\cdot)$ be C^0 on J^b w.r.t. the Banach norm $\|\cdot\|_0$ in D, $\|u\|_0 = \|u\| + \|C(t_0)u\|$ for some $t_0 \in J^b$. Then, for any $t \in J^b$,

$$V(t, 0) = s - \lim_{\delta(\sigma) \to 0} R(\tau_{n-1}, \delta_{n-1}) R(\tau_{n-2}, \delta_{n-2}) \ldots R(0, \delta_0), \tag{2}$$

where $R(\tau, \delta) = e^{-A(\tau)\delta}\, e^{-B(\tau)\delta}$ for $\tau \in J^b$, $\delta \geqslant 0$ and τ_k, δ_k in (2) refer to a partition $\sigma \in \mathscr{P}(J^t)$.

REMARK. The operator $C(t_0)$ above can be replaced by any closed operator C on X, the domain of which is D. If the resolvent set $\rho(C)$ is nonempty, then in order to check the $\|\cdot\|_0$-continuity of $u(\cdot)$ it is enough to show that $Cu(\cdot)$ is $\|\cdot\|$-continuous on J^b (cf. [4], Prop. 1 and the remark following Theorem 1)

Proof of the Theorem. We abbreviate $P_k = \exp(-A(\tau_k)\delta_k)$, $Q_k = \exp(-B(\tau_k)\delta_k)$, $R_k = P_k Q_k = R(\tau_k, \delta_k)$, $k = 0, 1, \ldots, n-1$, and $S(\sigma) = R_{n-1} R_{n-2} \ldots R_0 - V(t, 0)$. Relation (2) now reads $s - \lim_{\delta(\sigma) \to 0} S(\sigma) = 0$. We shall show first that for any $u \in D$, $S(\sigma)u \to 0$ with $\delta(\sigma) \to 0$. Using the equality

$$S(\sigma) = \sum_{k=0}^{n-1} R_{n-1} \ldots R_{k+1}(R_k - V(\tau_{k+1}, \tau_k))V(\tau_k, 0)$$

together with $\|R_k\| \leqslant 1$ and $A(\tau_k) + B(\tau_k) = C(\tau_k)$ we obtain

$$\|S(\sigma)u\| \leqslant \sum_{k=0}^{n-1} \delta_k \|(E_1(\tau_k, \delta_k) + E_2(\tau_k, \delta_k) - E_3(\tau_k, \delta_k))V(\tau_k, 0)u\|$$

where

$$E_1(\tau, \delta) = (\mathrm{e}^{-A(\tau)\delta} - I)\delta^{-1} + A(\tau),$$

$$E_2(\tau, \delta) = \mathrm{e}^{-A(\tau)\delta}(\mathrm{e}^{-B(\tau)\delta} - I)\delta^{-1} + B(\tau),$$

$$E_3(\tau, \delta) = (V(\tau + \delta, \tau) - I)\delta^{-1} + C(\tau).$$

Now $\Sigma_{k=0}^{n-1}\delta_k = t$ so that we have

$$\|S(\sigma)u\| \leqslant t \sum_{j=1}^{3} \sup \{\|E_j(\tau, \delta)V(\tau, 0)u\| : (\tau, \delta) \in M(\sigma)\},$$

where $M(\sigma) = \{(\tau, \delta) : 0 < \delta < \delta(\sigma), 0 \leqslant \tau \leqslant t - \delta\}$. Faris [4] proved that the under-stated continuity assumptions

$$\lim_{\delta \to 0} E_j(\tau, \delta)V(\tau, 0)u = 0, \quad j = 1, 2, 3,$$

for each $u \in D$ uniformly in $\tau \in J^b$. This implies easily that

$$\lim_{\delta(\sigma) \to 0} \sup_{M(\sigma)} \|E_j(\tau, \delta)V(\tau, 0)u\| = 0, \quad j = 1, 2, 3,$$

i.e., $\lim_{\delta(\sigma) \to 0} S(\sigma)u = 0$. Finally, D is assumed to be dense in X and $\|S(\sigma)\| \leqslant \|R_{n-1} \dots R_0\| + \|V(t, 0)\| \leqslant$ for each partition σ; thus $\lim_{\delta(\sigma) \to 0} S(\sigma)u = 0$ for $u \in X$ as well, i.e., relation (2) holds. ■

In particular, if A, B are t-independent, we obtain the following uniform version of the 'special' Trotter formula:

COROLLARY. Let A, B be generators of CCSG's on a Banach space X. If the sum $C = A + B$ generates a CCSG, then for each $t > 0$,

$$\mathrm{e}^{-Ct} = s - \lim_{\delta(\sigma) \to 0} R(\delta_{n-1})R(\delta_{n-2}) \dots R(\delta_0), \tag{3}$$

where $R(s) = \mathrm{e}^{-As}\,\mathrm{e}^{-Bs}$ and δ_k refer to a partition $\sigma \in \mathscr{P}(J^t)$.

3. APPLICATION TO THE FEYNMAN INTEGRAL

Now we specify the Banach space X to be $\mathcal{H}_s = L^2(\mathbb{R}^d)$ and set $A = -(i/2)\Delta$ (independently of t)

and $B(t) = iV_t$ (for each $t \in J^b$, where $(V_t\varphi)(x) = v(x, t)\varphi(x)$. We assume that (a) the Borel function $v(.,.)$ on $\mathbb{R}^d \times J^b$ is complex-valued and such that $v(., t)$ is almost regular (i.e., it can assume infinite values at most on a Borel set of Lebesgue measure zeros – cf. [8]) for each $t \in J^b$, $t \mapsto v(., t)$ is $\|\cdot\|_\infty$-continuous and the dissipativity condition

$$\mathrm{Im}\, v(x, t) \leqslant 0 \tag{4}$$

holds for each $t \in J^b$ and almost all $x \in \mathbb{R}^d$.

The right-hand side of (2) can be expressed now more explicitly: using the free propagator corresponding to $H_0 = -\frac{1}{2}\Delta$ we obtain for $\psi_t = V(t, 0)\varphi$ the following relation

$$\psi_t(x) = \lim_{\delta(\sigma)\to 0} ((2\pi i)^n \delta_0 \delta_1 \ldots \delta_{n-1})^{-d/2} \int_{\mathbb{R}^d} \ldots \int_{\mathbb{R}^d} \exp\{iS(\sigma; \gamma_0, \ldots, \gamma_n)\} \times \\ \times \varphi(\gamma_0)\, d\gamma_0\, d\gamma_1 \ldots d\gamma_{n-1}, \tag{5}$$

where $\gamma_n = x$, the integrals are, in general, improper ones,

$$\int_{\mathbb{R}^d} d\gamma = \lim_{m\to\infty} \int_{|\gamma| \leqslant m} d\gamma,$$

all limits are understood in the L^2-sense, and

$$S(\sigma; \gamma_0, \ldots, \gamma_n) = \sum_{k=0}^{n-1} \left[\frac{1}{2}\left(\frac{\gamma_{k+1} - \gamma_k}{\delta_k} \right)^2 - v(\gamma_k, \tau_k) \right] \delta_k . \tag{6}$$

One can interpret the last expression as the Riemannian approximation to

$$S_v(\gamma) = \int_0^t [\tfrac{1}{2} |\dot{\gamma}(\tau)|^2 - v(\gamma(\tau), \tau)]\, d\tau \tag{7}$$

if the latter makes sense. A particular case of the relation (5) first appeared in Reference 1; this is why we call the right-hand side of (5) the *uniform Feynman–Nelson integral* and abbreviate it as

$$\int^{\mathrm{ufn}} \exp\{iS_v(\gamma)\} \varphi(\gamma(0))\, \mathcal{D}\gamma$$

(for discussion of relations to other definitions of the F-integral cf. [10]).

With these prerequisites Theorem 1 can be reformulated for the considered particular case. Clearly

$$\left\{ \exp\left(\frac{is}{2} \Delta \right) : s \geqslant 0 \right\}$$

is a CCSG and the same is true for $\{\exp(-isV_t): s \geqslant 0\}$ due to (4). Further, the assumption (a) implies that $t \to V_t\varphi$ is continuous for each $\varphi \in D$. Thus we have:

PROPOSITION. In addition to (a), assume that (b) for each $t \in J^b$, the domain D of $H(t) = H_0 + V_t$ is dense in $\mathcal{H}_s$, independent of t, and the operator $H(t)$ is closed, (c) there exists a contraction-valued propagator $V(.,.)$ on $\mathcal{H}_s$, which preserves D for all $t, s \in J^b$, and with the following property: let $\psi_t = V(t, 0)\varphi$ for an arbitrary $\varphi \in D$, then the function $t \mapsto H(t)\psi_t$ is C^0 on J^b, further $t \mapsto \psi_t$ is C^1 on $(0, b)$ and there satisfies

$$i\frac{\mathrm{d}}{\mathrm{d}t}\psi_t = H(t)\psi_t = -\tfrac{1}{2}\Delta\psi_t + V_t\psi_t, \tag{8}$$

(d) the function $t \to \psi_t$ is continuous w.r.t. $\|\cdot\|_0$ corresponding to $H(t_0)$ for some $t_0 \in J^b$. Then for each $\varphi \in \mathcal{H}_s$, ψ_t is given by (5), i.e.,

$$\psi_t(x) = \int^{\mathrm{ufn}} \exp\{iS_v(\gamma)\}\varphi(\gamma(0))\,\mathcal{D}_\gamma. \tag{9}$$

Let us exhibit now some classes of potentials for which the conditions (b) – (d) are fulfilled. As to (b), it can be easily verified in the following cases:

$$v(.,t) \text{ bounded for each } t \in J^b \tag{10}$$

trivially with $D = D(H_0)$, further

$$v(.,t) \in L^2(\mathbb{R}^d) + L^\infty(\mathbb{R}^d) \text{ for each } t \text{ and } d \leqslant 3, \tag{11}$$

where again $D = D(H_0)$ (for proof see [8], theorem 7), and, finally,

$$v(x,t) = x \cdot B(t)x, \qquad B(t) = A(t) - iW(t), \tag{12}$$

are positive symmetric $d \times d$ matrices (linear operators on $\mathbb{R}^d$) for each $t \in J^b$, $A(t)$ strictly positive. As to the last case, one verifies first that (for fixed t) $H_0 + x \cdot A(t)x$ is self-adjoint on $D(H_0) \cap D(Q^2)$, Q^2 being the operator of multiplication by $r^2 = x_1^2 + \ldots + x_d^2$ on $\mathcal{H}_s$, then the Kato–Rellich-type lemma (cf. [2], section X.8; [8]) is applied sucessively to prove that $iH(t)$ is closed (on $D = D(H_0) \cap D(Q^2)$) and generates a CCSG. Details of this proof will be given elsewhere.

Each operator $iH(t_0)$ corresponding to some of the potentials (10) – (12) with fixed $t_0 \in J^b$ generates a CCSG, and therefore its resolvent set is nonempty, due to Hille–Yosida theorem. Using now the remark following Theorem 1, together with the fact that the operator $V_{t_0} \restriction D$ is closable, one can show easily that condition (d) is satisfied for the above classes of potentials, whenever (c) is satisfied.

As to condition (c), let us consider first the case (10), where its validity can be established under a slightly strengthened smoothness assumption:

THEOREM 2. *Assume* (a) *and* (10), *further let* $t \mapsto v(.,t)$ *be a* $\|\cdot\|_\infty$*-continuously differentiable function with the derivative bounded in* $(0, b)$. *Then for each* $\varphi \in D(H_0)$, *the right-hand side of* (9) *makes sense and expresses the solution* $t \mapsto \psi_t$ *of Equation* (8) *corresponding to the initial data* φ.

Proof. In view of the Proposition and the above considerations, it is sufficient to verify (c). This can be accomplished by virtue of Theorem X.70 from Reference [2]; standard arguments show that its assumptions are fulfilled under the stated requirements on v. ∎

As to the unbounded potentials (11), (12), we limit ourselves in the present paper to the simplest possibility when they are time-independent. The above-mentioned results imply that $H = H_0 + V$ generates a CCSG in this case, and therefore the condition is fulfilled automatically. We thus obtain the following result:

THEOREM 3. *Let* V *be an operator of multiplication by a complex-valued almost regular Borel function* v *which obeys the dissipativity condition* (4) *a.e. in* $\mathbb{R}^d$. *Assume further that either* $d \leqslant 3$ *and* $v \in L^2(\mathbb{R}^d) + L^\infty(\mathbb{R}^d)$ *or* $v(x) = x \cdot (A - iW)x$ *with* A *strictly positive and* W *positive. Then the right-hand side of* (8) *makes sense and expresses the solution* $t \mapsto \psi_t$ *of the equation*

$$i\frac{\mathrm{d}}{\mathrm{d}t}\psi_t = -\tfrac{1}{2}\Delta\psi_t + V\psi_t \tag{13}$$

corresponding to initial data $\varphi \in D$, *where* $D = D(H_0)$ *in the first case and* $D = D(H_0) \cap D(Q^2)$ *in the second one.*

4. CONCLUDING REMARKS

The assertions formulated in the previous section remain valid if we use some other reasonable polygonal-path approximation instead of the uniform one, because the latter is the 'strongest' one among them [10]. There are several ways in which the presented results could be generalized, e.g.:

- to treat unbounded time-dependent potentials of the type (11) under suitably strengthened t-smoothness conditions,
- to drop the restriction $d \leqslant 3$ in (11) with the replacement of L^2 by a suitable L^p,
- to prove (and apply) the uniform version of the general Trotter formula (where the sum is not assumed closed – cf. [7]) and of its generalizations [11, 12].

Let us finally remark that according to the mentioned results of Reference [9] the *UFN*-integral coincides with the *F*-integral in the sense of References [5, 6] for potentials $v \in \mathscr{F}(\mathbb{R}^d)$, i.e., those which are Fourier transforms of finite measures on $\mathbb{R}^d$, as well as for the damped harmonic oscillator. It is desirable to find some weaker sufficient conditions under which a value of the *F*-integral will be independent of the used Riemannian approximation to the action.

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