Journal of Algebra 425 (2015) 245–276

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Positive representations of non-simply-laced split real quantum groups

Ivan C.H. Ip

Kavli Institute for the Physics and Mathematics of the Universe (WPI), The University of Tokyo, Kashiwa, Chiba 277-8583, Japan

ARTICLE INFO

Article history:
Received 29 May 2014
Available online 22 December 2014
Communicated by Nicolás Andruskiewitsch

MSC:
17B37
81R50

Keywords:
Positive representations
Non-simply-laced
Split real quantum groups
Langlands dual
Modular double

ABSTRACT

We construct the positive principal series representations for $\mathcal{U}_q(\mathfrak{g}_\mathbb{R})$ where $\mathfrak{g}$ is of type B_n, C_n, F_4 or G_2, parametrized by $\mathbb{R}^n$ where n is the rank of $\mathfrak{g}$. We show that under the representations, the generators of the Langlands dual group $\mathcal{U}_{\tilde{q}}({}^L\mathfrak{g}_\mathbb{R})$ are related to the generators of $\mathcal{U}_q(\mathfrak{g}_\mathbb{R})$ by the transcendental relations. This gives a new and very simple analytic relation between the Langlands dual pair. We define the modified quantum group $\mathbf{U}_{\mathfrak{q}\tilde{\mathfrak{q}}}(\mathfrak{g}_\mathbb{R}) = \mathbf{U}_\mathfrak{q}(\mathfrak{g}_\mathbb{R}) \otimes \mathbf{U}_{\tilde{\mathfrak{q}}}({}^L\mathfrak{g}_\mathbb{R})$ of the modular double and show that the representations of both parts of the modular double commute with each other, and there is an embedding into the q-tori polynomials.

Contents

E-mail address: ivan.ip@ipmu.jp.

http://dx.doi.org/10.1016/j.jalgebra.2014.11.019
0021-8693/

1. Introduction

In this paper, we give the construction of the positive principal series representations for the quantum group $\mathcal{U}_q(\mathfrak{g}_\mathbb{R})$ and its modular double where $\mathfrak{g}$ is of non-simply-laced type B_n, C_n, F_4 or G_2, generalizing our recent work [10] on the simply-laced case, thus completing the constructions corresponding to simple $\mathfrak{g}$ of all types. The transcendental relations that were part of the axioms in the simply-laced case, relate the quantum group $\mathcal{U}_q(\mathfrak{g}_\mathbb{R})$ with its Langlands dual $\mathcal{U}_{\widetilde{q}}({}^L\mathfrak{g}_\mathbb{R})$ in the non simply-laced case. This might be considered as the simplest realization of the Langlands dual pair, given by a single analytic relation.

The notion of the *positive principal series representations* was introduced in [7] as a new research program devoted to the representation theory of split real quantum groups. It uses the concept of modular double for quantum groups [5], and has been studied for $\mathcal{U}_{q\widetilde{q}}(\mathfrak{sl}(2,\mathbb{R}))$ by Ponsot and Teschner [14]. Let us recall the definition in the simply-laced case. Let E_i, F_i, K_i be the generators of $\mathcal{U}_q(\mathfrak{g}_\mathbb{R})$ with the standard quantum relations, where $q = e^{\pi i b^2}$, $b^2 \in \mathbb{R} \setminus \mathbb{Q}$ and $0 < b < 1$. Similarly let $\widetilde{E_i}$, $\widetilde{F_i}$, $\widetilde{K_i}$ be the generators of $\mathcal{U}_{\widetilde{q}}(\mathfrak{g}_\mathbb{R})$ by replacing b with b^{-1}, where $\widetilde{q} = e^{\pi i b^{-2}}$. Then using the rescaled variables

$$e_i = 2\sin\left(\pi b^2\right)E_i, \qquad f_i = 2\sin\left(\pi b^2\right)F_i \tag{1.1}$$

and similarly for $\widetilde{e_i}$ and $\widetilde{f_i}$ with b replaced by b^{-1}, the positive representations have the following remarkable properties:

(i) the generators e_i, f_i, $K_i^{\pm 1}$ and $\widetilde{e_i}$, $\widetilde{f_i}$, $\widetilde{K_i}^{\pm 1}$ are represented by positive essentially self-adjoint operators,
(ii) the generators satisfy the transcendental relations

$$e_i^{\frac{1}{b^2}} = \widetilde{e_i}, \qquad f_i^{\frac{1}{b^2}} = \widetilde{f_i}, \qquad K_i^{\frac{1}{b^2}} = \widetilde{K_i}. \tag{1.2}$$

Furthermore, by modifying the definition of e_i, f_i, $K_i^{\pm 1}$ and the tilde variables with certain factors of K_i's, we also obtain the compatibility with the modular double $\mathbf{U}_{\mathfrak{q}\widetilde{\mathfrak{q}}}(\mathfrak{g}_{\mathbb{R}})$:

(iii) the generators $\mathbf{e}_i$, $\mathbf{f}_i$, $\mathbf{K}_i^{\pm 1}$ commute with $\widetilde{\mathbf{e}}_i$, $\widetilde{\mathbf{f}}_i$, $\widetilde{\mathbf{K}_i}^{\pm 1}$.

Note that since the generators are represented by positive operators, the real powers $\frac{1}{b_i^2}$ defining the transcendental relations are well-defined by means of functional calculus. However, when $\mathfrak{g}$ is not simply-laced, the transcendental relations (1.2) are substantially different. The transcendental relations now relate $\mathfrak{g}$ with its Langlands dual ${}^L\mathfrak{g}$ directly with appropriate changes of parameters. This is a special feature of the non-simply-laced case for quantum groups. The starting point of this paper is the following remarkable result:

Theorem 1.1 *(Langlands duality for quantum group). For each positive simple root α_i, define*

$$q_i = q^{\frac{1}{2}(\alpha_i,\alpha_i)} = e^{\pi i b_i^2},$$

and let $b_s = b_i$ when α_i is a short root (see Definition 2.2). Let $\widetilde{q} = e^{\pi i b_s^{-2}}$. Define the operators

$$\widetilde{e_i} := (e_i)^{\frac{1}{b_i^2}}, \tag{1.3}$$

$$\widetilde{f_i} := (f_i)^{\frac{1}{b_i^2}}, \tag{1.4}$$

$$\widetilde{K_i} := (K_i)^{\frac{1}{b_i^2}}, \tag{1.5}$$

where

$$e_i = 2\sin \pi b_i^2 E_i, \qquad f_i = 2\sin \pi b_i^2 F_i, \tag{1.6}$$

$$\widetilde{e_i} = 2\sin \pi b_i^{-2} \widetilde{E_i}, \qquad \widetilde{f_i} = 2\sin \pi b_i^{-2} \widetilde{F_i}. \tag{1.7}$$

Then the generators of $\mathcal{U}_{\widetilde{q}}({}^L\mathfrak{g}_{\mathbb{R}})$ are represented by the operators $\widetilde{E_i}$, $\widetilde{F_i}$ and $\widetilde{K_i}$, where ${}^L\mathfrak{g}_{\mathbb{R}}$ is defined by replacing long roots with short roots and short roots with long roots in the Dynkin diagram of $\mathfrak{g}$.

We remark that the Langlands dual already in the simply-laced case appears as the commutant of the quantum group $\mathcal{U}_q(\mathfrak{g}_{\mathbb{R}})$ [10, Theorem 10.4], where the type of algebra is the same. The possibility of such phenomenon has been suggested in [8]. Hence Theorem 1.1 about the transcendental relations gives an explicit construction of the Langlands dual in the context of representation theory of split real quantum groups, where in the non-simply-laced case the type of algebra is different. Furthermore it cannot be obtained in the classical setting as $b \to 0$. This relation between modular duality and Langlands duality should indeed have deep consequences, as pointed out for example in [6,15].

With the above theorem, we define the (modified) modular double by

$$\mathbf{U}_{\mathfrak{q}\widetilde{\mathfrak{q}}}(\mathfrak{g}_{\mathbb{R}}) := \mathbf{U}_{\mathfrak{q}}(\mathfrak{g}_{\mathbb{R}}) \otimes \mathbf{U}_{\widetilde{\mathfrak{q}}}({}^L\mathfrak{g}_{\mathbb{R}}), \tag{1.8}$$

and the main results of the paper are the following:

Theorem 1.2. *There exists a family of positive principal series representation for $\mathcal{U}_q(\mathfrak{g}_{\mathbb{R}})$ and its (modified) modular double $\mathbf{U}_{\mathfrak{q}\widetilde{\mathfrak{q}}}(\mathfrak{g}_{\mathbb{R}})$, parametrized by $\lambda \in \mathbb{R}^n_{\geq 0}$ where $n = rank(\mathfrak{g})$, satisfying properties (i), (iii) and Theorem 1.1 above.*

More precisely, for every reduced expression for w_0, we parametrize $U^+_{>0}$ using Lusztig's coordinate, and construct explicitly the positive representations. For each change of words of w_0, we establish a unitary transformation (Theorem 4.5), so that in particular the family of positive representation is *independent of choice* of reduced expression of w_0.

Hence by choosing a "good" reduced expression for w_0, we can write down explicitly the positive representations for $\mathcal{U}_q(\mathfrak{g}_{\mathbb{R}})$. Not surprisingly, by the philosophy of folding of Dynkin diagram, we observe the following

Theorem 1.3. *The positive representations of $\mathcal{U}_q(\mathfrak{g}_{\mathbb{R}})$ of type B_n, C_n, F_4 and G_2 can be obtained essentially (up to some quantized multiples) from the positive representations of $\mathcal{U}_q(\mathfrak{g}_{\mathbb{R}})$ of type A_{2n-1}, D_{n+1}, E_6 and D_4 respectively under certain identifications of roots.*

Finally, as in the simply-laced case, by using the modified version $\mathbf{U}_{\mathfrak{q}\widetilde{\mathfrak{q}}}(\mathfrak{g}_{\mathbb{R}})$ of the modular double, we have the following properties.

Theorem 1.4. *The commutant of (the adjoint form) $\mathbf{U}_{\mathfrak{q}}(\mathfrak{g}_{\mathbb{R}})$ is the simply-connected form of the Langlands dual group $\widehat{\mathbf{U}_{\widetilde{\mathfrak{q}}}}({}^L\mathfrak{g}_{\mathbb{R}})$.*

In fact, from the proof of Theorem 1.4, the correspondence of the commutant of the quantum groups corresponding to other forms in between the simply-connected and the adjoint forms can be deduced.

Theorem 1.5. *Let s (resp. l) be the number of indices corresponding to short (resp. long) roots, so that $s + l = l(w_0)$. Then we have an embedding*

$$\mathbf{U}_{\mathfrak{q}\widetilde{\mathfrak{q}}}(\mathfrak{g}_{\mathbb{R}}) \hookrightarrow \mathbb{C}\left[\mathbb{T}^{s,l}_{\mathfrak{q}\widetilde{\mathfrak{q}}}\right], \tag{1.9}$$

of the modified modular double into the Laurent polynomials generated by s q_s-tori and l q_l-tori and their modular double counterparts. In particular each generator of $\mathbf{U}_{\mathfrak{q}\widetilde{\mathfrak{q}}}(\mathfrak{g}_{\mathbb{R}})$ is realized as a Laurent polynomial of the q-tori variables.

Theorem 1.6. *The positive representations corresponding to the parameters λ and $w(\lambda)$ are unitary equivalent, where $\lambda \in \mathbb{R}^n$, and $w \in W$ is a Weyl group element. In particular the positive representations are parametrized by $\lambda \in \mathbb{R}^n_{\geq 0}$.*

The paper is organized as follows. In Section 2, we fix some notations and recall the definition of $\mathcal{U}_q(\mathfrak{g}_\mathbb{R})$ of general type, the definition of the quantum dilogarithm function, and Lusztig's parametrization of the positive unipotent semi-subgroup $U^+_{>0}$ of G. In Section 3, we give the general construction of the positive representations for F_i and K_i, and the action of E_i for a particular choice of w_0. In Section 4, we study in detail the positive representations of $\mathcal{U}_q(\mathfrak{g}_\mathbb{R})$ where $\mathfrak{g}$ is of type B_2, and describe the transformation needed to relate different reduced expression of w_0. In Sections 5 and 6 we give the explicit action of the positive representations of all type. In Section 7 we describe the relations between the positive representations and folding of the Dynkin diagram. In Section 8, we prove the main theorem about the transcendental relations which is related to the Langlands dual quantum group. Finally in Section 9 we introduce the modified quantum group $\mathbf{U}_{\mathfrak{q}\tilde{\mathfrak{q}}}(\mathfrak{g}_\mathbb{R})$ and state the main theorems about the positive representations of the modular double, the Langlands dual as the commutant, and its embedding into the q-tori.

2. Preliminaries

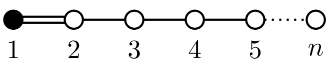
Throughout the paper, we will let $q = e^{\pi i b^2}$ with $0 < b^2 < 1$ and $b \in \mathbb{R} \setminus \mathbb{Q}$.

2.1. Definition of $\mathcal{U}_q(\mathfrak{g}_\mathbb{R})$

We recall the definition of the quantum group $\mathcal{U}_q(\mathfrak{g}_\mathbb{R})$ where $\mathfrak{g}$ is of general type [4,12].

Definition 2.1. Let I denote the set of nodes of the Dynkin diagram of $\mathfrak{g}$, with the following labeling. Here the black nodes correspond to short roots, and white nodes correspond to long roots.

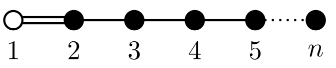
The Dynkin diagram for Type B_n is given by



and the corresponding Cartan matrix is given by ($1 \leq i, j \leq n$):

$$a_{ij} = \begin{cases} 2 & i = j, \\ -2 & (i,j) = (1,2), \\ -1 & |i-j| = 1 \text{ and } (i,j) \neq (1,2), \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

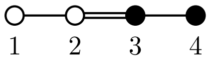
The Dynkin diagram for Type C_n is given by



and the corresponding Cartan matrix is given by ($1 \le i, j \le n$):

$$a_{ij} = \begin{cases} 2 & i = j, \\ -2 & (i,j) = (2,1), \\ -1 & |i-j| = 1 \text{ and } (i,j) \neq (2,1), \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

The Dynkin diagram for Type F_4 is given by

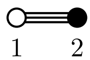


and the corresponding Cartan matrix is given by

$$A = (a_{ij}) = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}, \tag{2.3}$$

where $1, 2$ are short roots, $3, 4$ are long roots.

The Dynkin diagram for Type G_2 is given by



and the corresponding Cartan matrix is given by

$$A = (a_{ij}) = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}. \tag{2.4}$$

Definition 2.2. Let $(-,-)$ be the W-invariant inner product of the root lattice such that $(\alpha,\alpha) = 2$ for long roots α, where W is the Weyl group of the Cartan datum. Let α_i, $i \in I$ be the positive simple roots, and we define

$$q_i = q^{\frac{1}{2}(\alpha_i,\alpha_i)}. \tag{2.5}$$

In the case when $\mathfrak{g}$ is of type B_n, C_n and F_4, we define $b_l = b$, and $b_s = \frac{b}{\sqrt{2}}$ with the following normalization:

$$q_i = \begin{cases} e^{\pi i b_l^2} = q & i \text{ is long root,} \\ e^{\pi i b_s^2} = q^{\frac{1}{2}} & i \text{ is short root.} \end{cases} \tag{2.6}$$

In the case when $\mathfrak{g}$ is of type G_2, we define $b_l = b$, and $b_s = \frac{b}{\sqrt{3}}$ with the following normalization:

$$q_i = \begin{cases} e^{\pi i b_l^2} = q & i \text{ is long root}, \\ e^{\pi i b_s^2} = q^{\frac{1}{3}} & i \text{ is short root}. \end{cases} \tag{2.7}$$

Definition 2.3. Let $A = (a_{ij})$ denote the Cartan matrix. Then $\mathcal{U}_q(\mathfrak{g}_\mathbb{R})$ with $q = e^{\pi i b_l^2}$ is generated by E_i, F_i and $K_i^{\pm 1}$, $i \in I$ subject to the following relations:

$$K_i E_j = q_i^{a_{ij}} E_j K_i, \tag{2.8}$$

$$K_i F_j = q_i^{-a_{ij}} F_j K_i, \tag{2.9}$$

$$[E_i, F_j] = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}, \tag{2.10}$$

together with the Serre relations for $i \neq j$:

$$\sum_{n=0}^{1-a_{ij}} (-1)^n \frac{[1-a_{ij}]_{q_i}!}{[1-a_{ij}-n]_{q_i}![n]_{q_i}!} E_i^n E_j E_i^{1-a_{ij}-n} = 0, \tag{2.11}$$

$$\sum_{n=0}^{1-a_{ij}} (-1)^n \frac{[1-a_{ij}]_{q_i}!}{[1-a_{ij}-n]_{q_i}![n]_{q_i}!} F_i^n F_j F_i^{1-a_{ij}-n} = 0, \tag{2.12}$$

where $[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$. We also define formally the elements H_i so that $K_i = q_i^{H_i}$.

2.2. Quantum dilogarithm

Let us briefly recall the definition and some properties of the quantum dilogarithm functions [10]. Let $Q := b + b^{-1}$.

Definition 2.4. The quantum dilogarithm function $G_b(x)$ is defined on $0 \leq Re(z) \leq Q$ by

$$G_b(x) = \overline{\zeta_b} \exp\left(- \int_\Omega \frac{e^{\pi t z}}{(e^{\pi b t} - 1)(e^{\pi b^{-1} t} - 1)} \frac{dt}{t} \right), \tag{2.13}$$

where

$$\zeta_b = e^{\frac{\pi i}{2}(\frac{b^2 + b^{-2}}{6} + \frac{1}{2})}, \tag{2.14}$$

and the contour goes along $\mathbb{R}$ with a small semicircle going above the pole at $t = 0$. This can be extended meromorphically to the whole complex plane.

Definition 2.5. The function $g_b(x)$ is defined by

$$g_b(x) = \frac{\overline{\zeta_b}}{G_b(\frac{Q}{2} + \frac{\log x}{2\pi i b})}, \tag{2.15}$$

where log takes the principal branch of x.

The function $g_b(x)$, also called the quantum dilogarithm, is the crucial tool for all the transformations between self-adjoint operators. In particular it gives the unitary transformation that relates the positive representations corresponding to different expressions of the longest element. In particular, we will need the following two properties of $g_b(x)$.

Lemma 2.6. *(See [3].) $|g_b(x)| = 1$ when $x \in \mathbb{R}_+$, hence $g_b(X)$ is a unitary operator for any positive operator X.*

Lemma 2.7. *(See [3].)Let u, v be positive essentially self-adjoint operators. If $uv = q^2vu$, then*

$$g_b(u)^* v g_b(u) = q^{-1}uv + v, \tag{2.16}$$

$$g_b(v) u g_b(v)^* = u + q^{-1}uv. \tag{2.17}$$

If $uv = q^4vu$, then we apply the Lemma twice and obtain

$$g_b(u)^* v g_b(u) = v + [2]_q q^2 vu + q^4 vu^2, \tag{2.18}$$

$$g_b(v) u g_b(v)^* = u + [2]_q q^{-2} uv + q^{-4} uv^2. \tag{2.19}$$

More generally, if $uv = q^{2n}vu$, then

$$g_b(u)^* v g_b(u) = \sum_{k=0}^{n} \frac{[n]_q!}{[n-k]_q![k]_q!} q^k vu^{2k}, \tag{2.20}$$

$$g_b(v) u g_b(v)^* = \sum_{k=0}^{n} \frac{[n]_q!}{[n-k]_q![k]_q!} q^{-k} uv^{2k}. \tag{2.21}$$

As a consequence of the above Lemma, we also have Volkov's magic lemma:

Lemma 2.8. *(See [16].) If $uv = q^2vu$ where u, v are positive essentially self-adjoint operators, then $u + v$ is also a positive essentially self-adjoint operator, and*

$$(u+v)^{\frac{1}{b^2}} = u^{\frac{1}{b^2}} + v^{\frac{1}{b^2}}. \tag{2.22}$$

2.3. Lusztig's data

The following are described in detail in [13]. For simplicity, let G be a simple Lie group. Recall that for any simple root $\alpha_i \in \Delta$ there exists a homomorphism $SL_2(\mathbb{R}) \to G$ denoted by

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \mapsto x_i(a) \in U_i^+, \tag{2.23}$$

$$\begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix} \mapsto \chi_i(b) \in T, \tag{2.24}$$

$$\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \mapsto y_i(c) \in U_i^-, \tag{2.25}$$

called the *pinning* of G, where T is the split real maximal torus of G, and U_i^+ and U_i^- are the simple root subgroups of U^+ and U^- respectively. Then the positive unipotent semigroup $U_{>0}^+$ is defined by the image of the map $\mathbb{R}_{>0}^m \to U^+$ given by

$$(a_1, a_2, ..., a_m) \mapsto x_{i_1}(a_1)x_{i_2}(a_2)...x_{i_n}(a_m), \tag{2.26}$$

where $s_{i_1}s_{i_2}...s_{i_m}$ is a reduced expression for the longest element w_0 of the Weyl group W. We define $U_{>0}^-$ in a similar way.

Lemma 2.9. *(See [13].) We have the following identities:*

$$\chi_i(b)x_i(a) = x_i\big(b^2a\big)\chi_i(b), \tag{2.27}$$

$$x_i(a)y_j(c) = y_j(c)x_i(a) \qquad \text{if } i \neq j, \tag{2.28}$$

$$x_i(a)\chi_i(b)y_i(c) = y_i\left(\frac{c}{ac+b^2}\right)\chi_i\left(\frac{ac+b^2}{b}\right)x_i\left(\frac{a}{ac+b^2}\right). \tag{2.29}$$

In the simply-laced case, assume the roots α_i and α_j are joined by an edge in the Dynkin diagram. Then we have

$$\chi_i(b)x_j(a) = x_j\big(b^{-1}a\big)\chi_i(b), \tag{2.30}$$

$$x_i(a)x_j(b)x_i(c) = x_j\left(\frac{bc}{a+c}\right)x_i(a+c)x_j\left(\frac{ab}{a+c}\right). \tag{2.31}$$

3. Construction of positive representations

Let us recall the classical construction.

Proposition 3.1. *Let $\mathbb{C}[U_{>0}^+]$ be the space of continuous functions on the positive unipotent semigroup $U_{>0}^+$ defined in (2.26). The minimal principal series representation for $\mathcal{U}(\mathfrak{g}_\mathbb{R})$ can be realized as the infinitesimal action of $g \in G_\mathbb{R}$ acting on $\mathbb{C}[U_{>0}^+]$ by*

$$g \cdot f(h) = \chi_\lambda(hg)f\big([hg]_+\big). \tag{3.1}$$

Here we write the Gauss decomposition of g as

$$g = g_-g_0g_+ \in U_{>0}^-T_{>0}U_{>0}^+, \tag{3.2}$$

so that $[g]_+ = g_+$ *is the projection of* g *onto* $U_{>0}^+$, *and* $\chi_\lambda(g)$ *is the character function defined by*

$$\chi_\lambda(g) = \prod_{i=1}^n u_i^{2\lambda_i}, \tag{3.3}$$

where n *is the rank of* $\mathfrak{g}_\mathbb{R}$, $\lambda = (\lambda_i) \in \mathbb{C}^n$ *and* $u_i = \chi_i^{-1}(g_0) \in T_{>0}$. *(One can also treat* $\lambda := \sum_{i=1}^n \lambda_i \alpha_i^\vee \in \mathfrak{h}^*$ *where* $\alpha_i^\vee$ *are the dual coroots.)*

Following [10], we will use the formal Mellin transformation of the form

$$f(u) := \int F(x) x^u dx \tag{3.4}$$

on each variable, which transforms differential operators on $F(x)$ into finite difference operators on $f(u)$. Using this technique, the positive representations in the simply-laced case are constructed in [10]. We extend the construction to all types as follows.

Definition 3.2. Let us denote Lusztig's coordinates of $U_{>0}^+$ given in Section 2.3 by x_i^k, where i is the corresponding root index, and k denotes the sequence this root is appearing in w_0 from the right. Similarly we denote by u_i^k the Mellin transformed variables. We will also denote the Mellin transformed variables by v_i, $1 \leq i \leq l(w_0)$ counting from the left, and let $v(i,k)$ be the index such that $u_i^k = v_{v(i,k)}$.

Example 3.3. The coordinates for A_3 corresponding to $w_0 = s_3 s_2 s_1 s_3 s_2 s_3$ is given by

$$\left(u_3^3, u_2^2, u_1^1, u_3^2, u_2^1, u_3^1\right) = (v_1, v_2, v_3, v_4, v_5, v_6).$$

Let $q_i = e^{\pi i b_i^2}$ and $Q_i = b_i + b_i^{-1}$ (cf. Definition 2.2). We define the quantized action from the classical Mellin transformed action with the appropriate q_i-number. The quantized actions will then be unbounded, positive essentially self-adjoint operators acting on the Hilbert space $L^2(\mathbb{R}^{dim(U^+)})$.

Definition 3.4. Choose the reduced expression for $w_0 = w_{l-1} s_i$ where w_{l-1} is the reduced expression for ws_i. Then the classical (Mellin transformed) action of E_i is given by

$$E_i : f \mapsto \left(u_i^1 + 1\right) f\left(u_i^1 + 1\right), \tag{3.5}$$

and we define the positive quantized action by

$$E_i = \left[\frac{Q_i}{2b_i} - \frac{i}{b} u_i^1\right]_{q_i} e^{-2\pi b p_i^1}. \tag{3.6}$$

For the operators F_i and H_i, we slightly modify the action from [10] so that the following hold for $\mathfrak{g}$ of all type with the Cartan matrix (a_{ij}).

Definition 3.5. Let $r(j)$ be the root corresponding to the variable v_j. For any reduced expression w_0, the classical Mellin transformed action is given by

$$F_i : f \mapsto \sum_{k=1}^{n} \left(1 - \sum_{j=1}^{v(i,k)-1} a_{i,r(j)} v_j - u_i^k + 2\lambda_i\right) f\left(u_i^k - 1\right) \tag{3.7}$$

and the positive quantized action is given by (with $\lambda_i \in \mathbb{R}$):

$$F_i = \sum_{k=1}^{n} \left[\frac{Q_i}{2b_i} + \frac{i}{b}\left(\sum_{j=1}^{v(i,k)-1} a_{i,r(j)} v_j + u_i^k + 2\lambda_i\right)\right]_{q_i} e^{2\pi b p_i^k}. \tag{3.8}$$

The classical Mellin transformed action of H_i is multiplication by

$$H_i = \sum_{j=1}^{l(w_0)} -a_{i,r(j)} v_j + 2\lambda_i, \tag{3.9}$$

and the quantized action (after rescaling) is given by

$$K_i = q_i^{H_i} = e^{-\pi b_i (\sum_{k=1}^{l(w_0)} a_{i,r(k)} v_k + 2\lambda_i)}. \tag{3.10}$$

From these definitions, for each fixed choice of expression of w_0, the commutation relations between E_i, F_i and K_i are easily checked. The Serre relations between E_i will be given by the explicit rank 2 expressions in the next section and the unitary transformations of operators. For the action of F_i we have:

Theorem 3.6. *The action of F_i defined above satisfy the quantum Serre relations (2.12).*

Proof. The method of the proof is similar to the approach given in [7]. Let i corresponds to the short root and j the long root, and let us write

$$F_i = \sum_{k=1}^{n} \left[F_i^k(\mathbf{v})\right]_{q_i} e^{2\pi b p_i^k},$$

where $F_i^k(\mathbf{v})$ are linear functions in $\mathbf{v} := (v_i)$. Fix k, k_1, k_2, k_3 and denote by

$$\begin{aligned} a_{mn} &= e^{2\pi b p_i^{k_m}} \cdot F_i^{k_n}(\mathbf{v}) - F_i^{k_n}(\mathbf{v}), \\ b_n &= e^{2\pi b p_j^{k}} \cdot F_i^{k_n}(\mathbf{v}) - F_i^{k_n}(\mathbf{v}), \\ c_n &= e^{2\pi b p_i^{k_n}} \cdot F_j^{k}(\mathbf{v}) - F_j^{k}(\mathbf{v}). \end{aligned}$$

That is, for each fixed term we look at how much the factor of F_j is shifted by the action of the different components of F_i. The quantum Serre relations for simply-laced roots are proved in exactly the same way in [7]. However in the doubly-laced case the calculation is more involved. We observe from the explicit expressions that

$$a_{mn} = 2 - a_{nm}, \qquad b_n = -2 - 2c_n, \tag{3.11}$$

and furthermore a_{mn} only takes value in $\{0, 1, 2\}$ while b_n only takes values in $\{0, -2\}$. Then the quantum Serre relations for the long root F_j is equivalent to the vanishing of the following expression

$$\begin{aligned} QSE \iff & [2+b_1]_q\big([a_{12}+b_2]_q[a_{23}-a_{31}+b_3]_q + [2-a_{31}+b_3]_q[a_{12}-a_{23}+b_2]_q\big) \\ & + [2+b_2]_q\big([a_{23}+b_3]_q[a_{31}-a_{12}+b_1]_q + [2-a_{12}+b_1]_q[a_{23}-a_{31}+b_3]_q\big) \\ & + [2+b_3]_q\big([a_{31}+b_1]_q[a_{12}-a_{23}+b_2]_q + [2-a_{23}+b_2]_q[a_{31}-a_{12}+b_1]_q\big) \\ = \ & 0 \end{aligned}$$

which can be checked directly. The quantum Serre relations for type G_2 can be checked directly using the explicit expression given in Section 6. □

4. Positive representations of $\mathcal{U}_q(\mathfrak{g}_{\mathbb{R}})$ of type $B_2 = C_2$

4.1. Lusztig's data and transformation

From Section 2.3, we know that when $\mathfrak{g}$ is of type $B_2 = C_2$, the positive unipotent subgroup is parametrized by

$$x_1(a)x_2(b)x_1(c)x_2(d) = x_2(d')x_1(c')x_2(b')x_1(a'), \tag{4.1}$$

where $a, b, c, d, a', b', c', d' \in \mathbb{R}_{>0}$. Let us choose the following root subgroup on $C_2 = Sp(4, \mathbb{R})$:

$$x_s(t) = \begin{pmatrix} 1 & t & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -t & 1 \end{pmatrix}, \qquad x_l(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{4.2}$$

where x_s and x_l correspond to the short and long root respectively.

Then we have the following transformation rules:

Lemma 4.1. *(See [2, Thm 3.1].) We have*

$$a' = \frac{abc}{R}, \qquad b' = \frac{R^2}{S}, \tag{4.3}$$

$$c' = \frac{S}{R}, \qquad d' = \frac{bc^2d}{S}, \tag{4.4}$$

where

$$R = ab + ad + cd, \qquad S = a^2 b + d(a+c)^2. \tag{4.5}$$

Furthermore, the transformation

$$\phi : (a, b, c, d) \mapsto \left(a', b', c', d'\right) \tag{4.6}$$

is an involution:

$$\left(a'', b'', c'', d''\right) = (a, b, c, d). \tag{4.7}$$

4.2. Classical principal series representations

Using the above transformation, we can find the classical principal series representations for B_2 by commuting the corresponding root subgroup to the front. Under the Mellin transform, let t, v be the variables corresponding to Lusztig's parameters a, c of the short root, while u, w correspond to the parameters b, d of the long root.

Proposition 4.2. *Corresponding to $w_0 = s_1 s_2 s_1 s_2$ where 1 is short and 2 is long, we have*

$$E_1 = \frac{d}{b}\frac{\partial}{\partial a} + \frac{2d}{c}\frac{\partial}{\partial b} + \left(1 - \frac{d}{b}\right)\frac{\partial}{\partial c} - \frac{2d}{c}\frac{\partial}{\partial d}, \tag{4.8}$$

$$E_2 = \frac{\partial}{\partial d}, \tag{4.9}$$

so that the corresponding Mellin transformed action on $f(t, u, v, w)$ is given by

$$\begin{aligned} E_1 : f \mapsto\ & (1+t) f(t+1, u+1, w-1), \\ & + (1 + 2u - v) f(u+1, v+1, w-1) + (1 + v - 2w) f(v+1) \end{aligned} \tag{4.10}$$

$$E_2 : f \mapsto (1+w) f(w+1). \tag{4.11}$$

(For notational convenience, the unshifted variables are omitted in the argument.)

On the other hand, corresponding to $w_0 = s_2 s_1 s_2 s_1$ we have

$$E_1 = \frac{\partial}{\partial a}, \tag{4.12}$$

$$E_2 = -\frac{a}{b}\left(1 + \frac{a}{c}\right)\frac{\partial}{\partial a} + \left(1 - \frac{a^2}{c^2}\right)\frac{\partial}{\partial b} + \frac{a}{b}\left(\frac{a}{c} + 1\right)\frac{\partial}{\partial c} + \frac{a^2}{c^2}\frac{\partial}{\partial d}, \tag{4.13}$$

so that the corresponding Mellin transformed action on $f(t, u, v, w)$ is given by

$$E_1 : f \mapsto (1+t)f(t+1), \tag{4.14}$$

$$\begin{aligned} E_2 : f \mapsto\ & (1-t+u)f(u+1) + (2-t+v)f(a-1,u+1,v+1) \\ & + (1-u+v)f(t-2,u+1,v+2) + (1+w)f(t-2,v+2,w+1). \end{aligned} \tag{4.15}$$

The actions of F_i and H_i are given in Definition 3.5.

Proof. The action of the differential operators follows from the differentiation of the group action (3.1) by the matrices given explicitly in (4.2), and then the Mellin transformed action follows from definition (3.4). □

4.3. Explicit expressions

Following the work in [10], we will construct the positive representations for $\mathcal{U}_q(\mathfrak{g}_\mathbb{R})$ by quantizing the weights of the Mellin transformed action appropriately, and introduce certain twisting to make the actions positive.

Recall from Definition 2.2 that $q_1 = q^{\frac{1}{2}} = e^{\pi i b_s^2}$ and $q_2 = q = e^{\pi i b^2}$. Also let $Q_s = b_s + b_s^{-1}$.

Theorem 4.3. *The positive representations corresponding to $w_0 = s_1 s_2 s_1 s_2$ for $\mathcal{U}_q(\mathfrak{g}_\mathbb{R})$ where $\mathfrak{g}$ is of type B_2, acting on $L^2(\mathbb{R}^4)$, are given by*

$$\begin{aligned} E_1 = & \left[\frac{Q_s}{2b_s} - \frac{i}{b}t\right]_{q_1} e^{2\pi b(-p_t-p_u+p_w)} + \left[\frac{Q_s}{2b_s} - \frac{i}{b}(2u-v)\right]_{q_1} e^{2\pi b(-p_u-p_v+p_w)} \\ & + \left[\frac{Q_s}{2b_s} - \frac{i}{b}(v-2w)\right]_{q_1} e^{-2\pi b p_v} \end{aligned} \tag{4.16}$$

$$E_2 = \left[\frac{Q}{2b} - \frac{i}{b}w\right]_{q_2} e^{-2\pi b p_w} \tag{4.17}$$

$$F_1 = \left[\frac{Q_s}{2b_s} + \frac{i}{b}(2\lambda_1+t)\right]_{q_1} e^{2\pi b p_t} + \left[\frac{Q_s}{2b_s} + \frac{i}{b}(2\lambda_1+2t-2u+v)\right]_{q_1} e^{2\pi b p_v} \tag{4.18}$$

$$F_2 = \left[\frac{Q}{2b} + \frac{i}{b}(2\lambda_2-t+u)\right]_{q_2} e^{2\pi b p_u} + \left[\frac{Q}{2b} + \frac{i}{b}(2\lambda_2-t+2u-v+w)\right]_{q_2} e^{2\pi b p_w} \tag{4.19}$$

$$K_1 = q_1^{H_1} = e^{\pi b(-\lambda_1-t+u-v+w)} \tag{4.20}$$

$$K_2 = q_2^{H_2} = e^{\pi b(-2\lambda_2+t-2u+v-2w)}. \tag{4.21}$$

Note that for $q = e^{\pi i b^2}$,

$$\left[\frac{Q}{2b} - \frac{i}{b}u\right]_q e^{2\pi b p} = e^{\pi b(u+2p)} + e^{\pi b(-u+2p)} \tag{4.22}$$

is a positive essentially self-adjoint operator whenever $[p,u] = \frac{1}{2\pi i}$.

On the other hand, the positive representations corresponding to $w_0 = s_2s_1s_2s_1$ *are more complicated, and are given by*

$$E_1 = \left[\frac{Q_s}{2b_s} - \frac{i}{b}t\right]_{q_1} e^{-2\pi b p_t} \tag{4.23}$$

$$\begin{aligned} E_2 = &\left[\frac{Q}{2b} - \frac{i}{b}w\right]_{q_2} e^{2\pi b(2p_t - 2p_v - p_w)} + \left[\frac{Q}{2b} - \frac{i}{b}(u-t)\right]_{q_2} e^{-2\pi b p_u} \\ &+ [2]_{q_1}\left[\frac{Q}{2b} - \frac{i}{2b}(v-t)\right]_{q_2} e^{2\pi b(p_t - p_u - p_v)} \\ &+ \left[\frac{Q}{2b} - \frac{i}{b}(v-u)\right]_{q_2} e^{2\pi b(2p_t - p_u - 2p_v)} \end{aligned} \tag{4.24}$$

$$\begin{aligned} F_1 = &\left[\frac{Q_s}{2b_s} + \frac{i}{b}(2\lambda_1 + v - 2w)\right]_{q_1} e^{2\pi b p_v} \\ &+ \left[\frac{Q_s}{2b_s} + \frac{i}{b}(2\lambda_1 + t - 2u + 2v - 2w)\right]_{q_1} e^{2\pi b p_t} \end{aligned} \tag{4.25}$$

$$F_2 = \left[\frac{Q}{2b} + \frac{i}{b}(2\lambda_2 + w)\right]_{q_2} e^{2\pi b p_w} + \left[\frac{Q}{2b} + \frac{i}{b}(2\lambda_2 + u - v + 2w)\right]_{q_2} e^{2\pi b p_u} \tag{4.26}$$

$$K_1 = e^{\pi b(-\lambda_1 - t + u - v + w)} \tag{4.27}$$

$$K_2 = e^{\pi b(-2\lambda_2 + t - 2u + v - 2w)}. \tag{4.28}$$

Hence one can see that the expression resembles that of the classical formula. However, it turns out that it is more natural to consider the rescaled version, where for variables corresponding to the short root, we rescale by $\sqrt{2}$, so that

$$\begin{aligned} bu &\mapsto b\sqrt{2}u = 2b_s u, \\ bp_u &\mapsto \frac{b}{\sqrt{2}}p_u = b_s p_u. \end{aligned}$$

We will also rescale the parameters λ_s corresponding to short roots to $\sqrt{2}\lambda_s$.

Hence let us introduce the following notation. Let u_s and u_l be a linear combinations of Mellin transformed variables corresponding to the short root and long root respectively. (This also applies to the parameters λ_i). Also let p_s and p_l be the corresponding shifting operators.

Definition 4.4. We denote by

$$[u_s + u_l]e(p_s + p_l) = e^{\pi(b_s u_s + b_l u_l) + 2\pi(b_s p_s + b_l p_l)} + e^{-\pi(b_s u_s + b_l u_l) + 2\pi(b_s p_s + b_l p_l)}. \tag{4.29}$$

Then under the above rescaling, we can rewrite Theorem 4.3 as follows:

Theorem 4.3*. *Let*

$$e_i := 2\sin(\pi b_i^2)E_i = \left(\frac{i}{q_i - q_i^{-1}}\right)^{-1} E_i, \tag{4.30}$$

$$f_i := 2\sin(\pi b_i^2)F_i = \left(\frac{i}{q_i - q_i^{-1}}\right)^{-1} F_i. \tag{4.31}$$

Then the positive representations corresponding to $w_0 = s_1s_2s_1s_2$ *for* $\mathcal{U}_q(\mathfrak{g}_\mathbb{R})$ *where* $\mathfrak{g}$ *is of type* B_2*, acting on* $L^2(\mathbb{R}^4)$*, are given by*

$$e_1 = [t]e(-p_t - p_u + p_w) + [u-v]e(-p_u - p_v + p_w) + [v-w]e(-p_v) \tag{4.32}$$
$$e_2 = [w]e(-p_w) \tag{4.33}$$
$$f_1 = [2\lambda_1 - t]e(p_t) + [2\lambda_1 - 2t + u - v]e(p_v) \tag{4.34}$$
$$f_2 = [2\lambda_2 + 2t - u]e(p_u) + [2\lambda_2 + 2t - 2u + 2v - w]e(p_w) \tag{4.35}$$
$$K_1 = e^{\pi b_s(2\lambda_1 - 2t - 2v)} e^{\pi b(u+w)} \tag{4.36}$$
$$K_2 = e^{\pi b(2\lambda_2 - 2u - 2w)} e^{\pi b_s(2t+2v)}. \tag{4.37}$$

On the other hand, the positive representations corresponding to $w_0 = s_2s_1s_2s_1$ *are given by*

$$e_1 = [t]e(-p_t) \tag{4.38}$$
$$\begin{aligned} e_2 = {} & [w]e(2p_t - 2p_v - p_w) + [u-2t]e(-p_u) \\ & + [2]_{q_1}[v-t]e(p_t - p_u - p_v) + [2v-u]e(2p_t - p_u - 2p_v) \end{aligned} \tag{4.39}$$
$$f_1 = [2\lambda_1 - v + w]e(p_v) + [2\lambda_1 - t + u - 2v + w]e(p_t) \tag{4.40}$$
$$f_2 = [2\lambda_2 - w]e(p_w) + [2\lambda_2 - u + 2v - 2w]e(p_u) \tag{4.41}$$
$$K_1 = e^{\pi b_s(2\lambda_1 - 2t - 2v)} e^{\pi b(u+w)} \tag{4.42}$$
$$K_2 = e^{\pi b(2\lambda_2 - 2u - 2w)} e^{\pi b_s(2t+2v)}. \tag{4.43}$$

Proof. The commutation relations of the operators can be checked directly. Note that the action of F_i and K_i coincides with the one given in Definition 3.5. □

4.4. Transformations of operators

As in the simply-laced case, by quantizing ϕ from (4.6), there is a unitary transformation Φ that intertwines the above action corresponding to the change of reduced expression $w_0 = s_1s_2s_1s_2 = s_2s_1s_2s_1$.

Theorem 4.5. *We define the transformation*

$$\Phi : L^2(\mathbb{R}^4) \to L^2(\mathbb{R}^4)$$
$$f(t,u,v,w) \mapsto \Phi f := F(t',u',v',w')$$

by

$$\Phi := T \circ \Phi_3 \circ \Phi_2 \circ \Phi_1, \tag{4.44}$$

where

$$\Phi_1 = \frac{g_b(e^{\pi b(u+w)-2\pi b_s v+2\pi b(p_w-p_u)})}{g_b(e^{-\pi b(u+w)+2\pi b_s v+2\pi b(p_w-p_u)})}, \tag{4.45}$$

$$\Phi_2 = \frac{g_{b_s}(e^{\pi b_s(t-v)+\pi bw+2\pi b_s(p_v-p_t)+2\pi b(p_w-p_u)})}{g_{b_s}(e^{-\pi b_s(t-v)-\pi bw+2\pi b_s(p_v-p_t)+2\pi b(p_w-p_u)})}, \tag{4.46}$$

$$\Phi_3 = \frac{g_b(e^{2\pi b_s t+\pi b(w-u)+2\pi b_s(2p_v-2p_t)+2\pi b(p_w-p_u)})}{g_b(e^{-2\pi b_s t-\pi b(w-u)+2\pi b_s(2p_v-2p_t)+2\pi b(p_w-p_u)})}, \tag{4.47}$$

and T is the transformation matrix of determinant -1:

$$\begin{pmatrix} t' \\ u' \\ v' \\ w' \end{pmatrix} := T \begin{pmatrix} t \\ u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 0 & \sqrt{2} & -1 & 0 \\ 0 & 2 & -\sqrt{2} & 1 \\ 1 & -\sqrt{2} & 2 & 0 \\ 0 & -1 & \sqrt{2} & 0 \end{pmatrix} \begin{pmatrix} t \\ u \\ v \\ w \end{pmatrix}. \tag{4.48}$$

Then Φ is a unitary transformations, $\Phi^2 = 1$, and for any operators X,

$$X \longmapsto \Phi \circ X \circ \Phi^{-1} \tag{4.49}$$

gives the corresponding action on $s_1s_2s_1s_2 \longleftrightarrow s_2s_1s_2s_1$.

Proof. The ratios of g_b is well-defined since the argument commute with each other. Furthermore they are unitary by Lemma 2.6. For the proof, it suffices to apply the conjugation properties of g_b from Lemma 2.7 to the positive representation of type B_2 given by Theorem 4.3. Note that the factor $[2]_{q_1}$ from E_2 is obtained by applying Φ_2 using (2.18)–(2.19). □

5. Positive representations of $\mathcal{U}_q(\mathfrak{g}_\mathbb{R})$

From Theorem 4.5, together with the result of [10, Theorem 5.2], we can find explicit expressions of the positive representation starting from any reduced expression of w_0, followed by applying the unitary transformations to the desired reduced expression.

Consequently, all the operators will be positive essentially self-adjoint. Furthermore, by bringing $s_is_js_is_j$ to the front of w_0, all the commutation relations between E_i and E_j will be easy to check, while the commutation relations involving F_i follows from Theorem 3.6 and their explicit expressions.

Actually, we only need the following transformations rules:

Proposition 5.1. *If i, j are not connected in the Dynkin diagram, corresponding to $...s_is_j... \longleftrightarrow ...s_js_i...$, the operators transformed on $f(u,v)$ simply by*

$$u \longleftrightarrow v. \tag{5.1}$$

For i, j connected by a single edge in the Dynkin diagram, corresponding to $...s_is_js_i... \longleftrightarrow ...s_js_is_j...$, the operators given by [10, Theorem 5.2] transformed on $f(u,v,w)$ as

$$[w]e(-p_w) \longleftrightarrow [u]e(-p_u - p_v + p_w) + [v-w]e(-p_v). \tag{5.2}$$

For i, j connected by a double edge in the Dynkin diagram, corresponding to $...s_is_js_is_j... \longleftrightarrow ...s_js_is_js_i...$ the operators given by Theorem 4.5 transformed on $f(t,u,v,w)$ as

$$(4.32) \longleftrightarrow (4.38), \qquad (4.33) \longleftrightarrow (4.39). \tag{5.3}$$

Let us also rewrite the action of F_i and K_i from Definition 3.5 in terms of the rescaled variables using Definition 4.4.

Definition 3.5*. Let

$$f_i = 2\sin \pi b_i^2 F_i = \left(\frac{i}{q_i - q_i^{-1}}\right)^{-1} F_i.$$

Then the quantized action of F_i is given by

$$f_i = \sum_{k=1}^{n} \left[-a_{r(j),i}v_j + u_i^k - 2\lambda_i\right] e\left(p_i^k\right) \tag{5.4}$$

$$K_i = e^{-2\pi b_i \lambda_i - \sum_{j=1}^{l(w_0)} \pi b_j a_{r(j),i} v_j} \tag{5.5}$$

where v_j is the labeling given in Definition 3.2.

5.1. Type B_n

Let us choose the following reduced expression for w_0

$$w_0 = 1212 \quad 32123 \quad 4321234 \quad ...n(n-1)...1...n$$

where for simplicity, we denote by $i := s_i$. Then by transposing the desired index to the right, and applying the rules from Proposition 5.1 repeatedly, we obtain

Proposition 5.2. *The action of E_1 is given by*

$$\begin{aligned} e_1 = &\sum_{k=1}^{n} [u_1^k - u_2^{2k-1}] e\left(-p_1^k - \sum_{l=1}^{2k-2} (-1)^l p_2^l\right) \\ &+ \sum_{k=1}^{n-1} [u_2^{2k} - u_1^k] e\left(-u_1^k - \sum_{l=1}^{2k} (-1)^l p_2^l\right). \end{aligned} \tag{5.6}$$

Note that the variable $u_2^{2n-1} = 0$ is non-existent.

The action of E_i for $i \geq 2$ is given by

$$e_i = \sum_{k=1}^{2(n-i)+1} \left[(-1)^k \left(u_{i+1}^k - u_i^k\right)\right] e\left(\sum_{l_1=1}^{s_1(k)} (-1)^{l_1} p_i^{l_1} + \sum_{l_2=1}^{s_2(k)} (-1)^{l_2} p_{i+1}^{l_2}\right), \tag{5.7}$$

where $e_i = 2\sin \pi b_i^2 E_i$ and

$$\begin{aligned} s_1(k) &:= 2\left\lceil \frac{k}{2} \right\rceil - 1, \\ s_2(k) &:= 2\left\lfloor \frac{k}{2} \right\rfloor. \end{aligned}$$

5.2. Type C_n

Using the exact same expression for w_0 as in type B_n, we have the following action.

Proposition 5.3. *The action of E_1 is given by*

$$\begin{aligned} e_1 = &\sum_{k=1}^{n} [u_1^k - 2u_2^{2k-1}] e\left(-p_1^k - 2\sum_{l=1}^{2k-2} (-1)^l p_2^l\right) \\ &+ [2]_{q_s} \sum_{k=1}^{n-1} [u_2^{2k} - u_2^{2k-1}] e\left(-p_1^k - \sum_{l=1}^{2k} (-1)^l u_2^l\right) \\ &+ \sum_{k=1}^{n-1} [2u_2^{2k} - u_1^k] e\left(-p_1^k - 2\sum_{l=1}^{2k} (-1)^l u_2^l\right), \end{aligned} \tag{5.8}$$

while the action of E_i for $i \geq 2$ is the same as (5.7).

5.3. Type F_4

Let us choose the following reduced expression for w_0:

$$w_0 = 3232 \quad 12321 \quad 432312343213234$$

which follows from the embedding $B_2 \subset B_3 \subset F_4$. We will use the notation introduced in [11] to simplify the expressions. Let

$$P_1 = -p_1^3 + p_1^2 - p_2^4 + 2p_3^4 - 2p_3^5 + 2p_3^2 - 2p_3^3 + p_2^1 - p_1^1$$
$$P_2 = -p_2^7 + p_2^7 - p_1^4 + p_1^3 - p_2^5 + 2p_3^5 - 2p_3^6 + p_2^3 - p_1^2 + p_1^1 - p_2^2 + 2p_3^1 - 2p_3^2$$
$$P_3 = -p_3^9 - p_2^8 + p_2^7 - p_2^6 + p_2^5 + p_3^6 - p_4^3 - p_3^5 + p_3^3 - p_2^3 + p_2^2 + p_3^2 - p_3^4 + p_4^1 - p_3^1$$

and let $P_i(x)$ be the partial sum of P_i from x (ignoring the coefficient) to the right most term. For example

$$P_1\big(p_3^2\big) := 2p_3^2 - 2p_3^3 + p_2^1 - p_1^1.$$

Proposition 5.4. *The action of E_i is given as follows:*

$$\begin{aligned} e_1 = {} & \big[u_1^3\big]e(P_1) + \big[u_2^4 - u_1^2\big]e\big(P_1\big(p_2^4\big)\big) + \big[u_2^3 - 2u_3^4\big]e\big(-p_2^3 + P_1\big(p_3^2\big)\big) \\ & + [2]_{q_s}\big[u_3^5 - u_3^4\big]e\big(-p_2^3 + p_3^4 - p_3^5 + P_1\big(p_3^2\big)\big) + \big[2u_3^5 - u_2^3\big]e\big(-p_2^3 + P_1\big(p_3^4\big)\big) \\ & + \big[u_2^2 - 2u_3^2\big]e\big(-p_2^2 + p_2^1 - p_1^1\big) + [2]_{q_s}\big[p_3^3 - p_3^2\big]e\big(-p_2^2 + p_3^2 - p_3^3 + p_2^1 - p_1^1\big) \\ & + \big[2u_3^3 - u_2^2\big]e\big(-p_2^2 + P_1\big(p_3^2\big)\big) + \big[u_1^1 - u_2^1\big]e\big(-p_1^1\big), \end{aligned} \tag{5.9}$$

$$\begin{aligned} e_2 = {} & \big[u_2^7\big]e(P_2) + \big[u_1^4 - u_2^6\big]e\big(P_2\big(p_1^4\big)\big) + \big[u_2^5 - u_1^3\big]e\big(P_2\big(p_2^5\big)\big) + \big[u_2^4 - 2u_3^5\big]e\big(p_2^3 - p_2^4 + P_2\big(p_1^2\big)\big) \\ & + [2]_{q_s}\big[u_3^6 - u_3^5\big]e\big(p_3^5 - p_3^6 - p_2^4 + P_2\big(p_2^3\big)\big) + \big[2p_3^6 - p_2^4\big]e\big(-p_2^4 + P_2\big(p_3^5\big)\big) \\ & + \big[u_1^2 - u_2^3\big]e\big(P_2\big(p_1^2\big)\big) + \big[p_2^2 - p_1^1\big]e\big(P_2\big(p_2^2\big)\big) + \big[2u_3^2 - u_2^1\big]e\big(-p_2^1 + P_2\big(p_3^1\big)\big) \\ & + [2]_{q_s}\big[u_3^2 - u_3^1\big]e\big(p_3^1 - p_3^2 - p_2^1\big) + \big[u_2^1 - 2u_3^1\big]e\big(-p_2^1\big), \end{aligned} \tag{5.10}$$

$$\begin{aligned} e_3 = {} & \big[u_3^9\big]e(P_3) + \big[u_2^8 - u_3^8\big]e\big(P_3\big(p_2^8\big)\big) + \big[u_3^8 - u_2^7\big]e\big(-p_3^8 + P\big(p_2^6\big)\big) + \big[u_2^6 - u_3^7\big]e\big(-p_3^7 + P\big(p_2^6\big)\big) \\ & + \big[u_3^7 - u_2^5\big]e\big(-p_3^7 + P_3\big(p_3^6\big)\big) + \big[u_4^3 - u_3^6\big]e\big(P_3\big(p_4^3\big)\big) + \big[u_3^5 + u_3^4 - u_4^2\big]e\big(-p_4^2 + P\big(p_3^5\big)\big) \\ & + \big[u_3^5 - u_3^3\big]e\big(-p_4^2 - p_3^5 + p_3^4 + P_3\big(p_2^3\big)\big) + \big[u_2^3 - u_3^4 - u_3^3\big]e\big(-p_4^2 + P_3\big(p_2^2\big)\big) \\ & + \big[u_3^4 + u_3^3 - u_2^2\big]e\big(-p_4^2 + P_3\big(p_3^2\big)\big) + \big[u_3^4 - u_3^2\big]e\big(-p_4^2 + p_3^3 + P_3\big(p_3^4\big)\big) \\ & + \big[u_4^2 - u_3^3 - u_3^2\big]e\big(-p_4^2 + p_4^1 - p_3^1\big) + \big[u_3^1 - u_4^1\big]e\big(-p_3^1\big), \end{aligned} \tag{5.11}$$

$$e_4 = \big[u_4^1\big]e\big(-p_4^1\big). \tag{5.12}$$

6. Positive representations of $\mathcal{U}_q(\mathfrak{g}_\mathbb{R})$ for type G_2

Let $w_0 = s_2s_1s_2s_1s_2s_1$ where 1 is the long root and 2 is the short root. First let us describe the root subgroup $x_1(t)$ and $x_2(t)$ embedded in $SL(7)$, which can be found in [9].

$$x_1(t) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & t & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & t & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{6.1}$$

$$x_2(t) = \begin{pmatrix} 1 & -t & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -t & -t^2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2t & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & t \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{6.2}$$

Solving for the matrix coefficients explicitly by Mathematica, we found the relations between Lusztig's parametrization:

$$x_1(a)x_2(b)x_1(c)x_2(d)x_1(e)x_2(f) = x_2(a')x_1(b')x_2(c')x_1(d')x_2(e')x_1(f') \tag{6.3}$$

(an explicit relation can also be found in [2, Thm 3.1]). From this we can derive the classical principal series representation:

Proposition 6.1. *The classical principal series representation corresponding to $w_0 = s_2s_1s_2s_1s_2s_1$ is given by*

$$E_1 = \frac{\partial}{\partial f}, \tag{6.4}$$

$$E_2 = \frac{ef}{bc}\frac{\partial}{\partial a} + \frac{3ef}{c^2}\frac{\partial}{\partial b} + \left(\frac{2f}{d} - \frac{ef}{bc} + \frac{2ef}{cd}\right)\frac{\partial}{\partial c} + \left(\frac{3f}{e} - \frac{3ef}{c^2}\right)\frac{\partial}{\partial d} + \left(1 - \frac{2f}{d} - \frac{2ef}{cd}\right)\frac{\partial}{\partial e} - \frac{3f}{e}\frac{\partial}{\partial f}, \tag{6.5}$$

and the Mellin transformed action on $f(r,s,t,u,v,w)$ is given by

$$E_1 : f \mapsto (w+1)f(w+1), \tag{6.6}$$

$$\begin{aligned}E_2 : f \mapsto\ & (1+r)f(r+1, s+1, t+1, v-1, w-1)\\ &+ (1+3s-t)f(s+1, t+2, v-1, w-1)\\ &+ (1+3u-2v)f(u+1, v+1, w-1)\\ &+ (2+2t-2v)f(t+1, u+1, w-1)\\ &+ (1+2t-3u)f(t+2, u+1, v-1, w-1)\\ &+ (1+v-3w)f(v+1).\end{aligned} \tag{6.7}$$

Again, for notational convenience, the unshifted variables are omitted in the argument.

To get the quantized action, we again rescaled the variables corresponding to whether the index are long or short root. Using the notation from Definition 4.4 (with $b_s = \frac{b}{\sqrt{3}}$), we found the action as follows.

Theorem 6.2. *The action of $\mathcal{U}_q(\mathfrak{g}_\mathbb{R})$ on $L^2(\mathbb{R}^6)$, where $\mathfrak{g}$ is of type G_2 corresponding to $w_0 = s_2s_1s_2s_1s_2s_1$ is given by*

$$e_1 = [w]e(-p_w), \tag{6.8}$$

$$\begin{aligned}e_2 =\ & [r]e(-p_r - p_s - p_t + p_v + p_w) + [s-t]e(-p_s - 2p_t + p_v + p_w)\\ &+ [u-2v]e(-p_u - p_v + p_w) + [2]_{q_2}[t-v]e(-p_t - p_u + p_w)\\ &+ [2t-u]e(-2p_t - p_u + p_v + p_w) + [v-w]e(-p_v),\end{aligned} \tag{6.9}$$

where again $e_i = 2\sin\pi b_i^2 E_i$. The action of F_i and K_i are given by Definition 3.5 as before.*

The action corresponding to $w_0 = s_1s_2s_1s_2s_1s_2$ can also be computed, and E_1 consists of 13 terms. It can instead be obtained from the folding of the positive representations of type D_4 described in the next section.

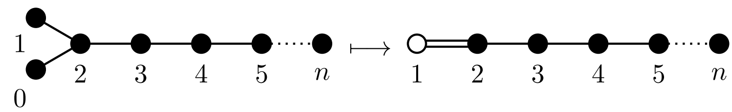
7. Folding of representations

By comparing the action of C_n given in Section 5.2 and the action of D_{n+1} given in [10], there is a strong similarity between the actions. In fact the action of E_i for $i \geq 2$ are identical. This holds in general due to the principle of folding of Dynkin diagram.

It is known that the philosophy of folding in the setting of quantum groups is more complicated [1]. Nevertheless, with the mixture of classical construction, the Mellin transformation and quantization described in this paper, we can still obtain the description of the positive representations in the non-simply-laced case from the corresponding unfolded simply-laced type by means of the folding of pinning [13, 1.5]. Explicitly:

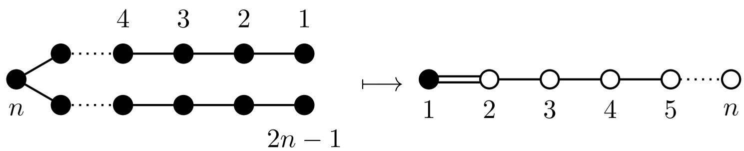
Theorem 7.1. *Let us use the labeling introduced in Section 2.1 and [10].*

The positive representations of C_n corresponding to w_0 can be obtained from the positive representations of D_{n+1} with s_1 in $w_0(C_n)$ replaced by s_0s_1 in $w_0(D_{n+1})$.



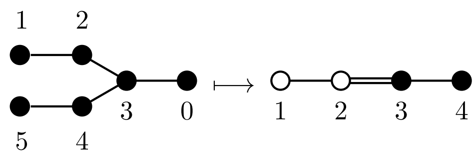
D_{n+1} C_n

The positive representations of B_n corresponding to w_0 can be obtained from the positive representations of A_{2n-1} with s_{n-k+1} in $w_0(B_n)$ replaced by s_ks_{2n-k} in $w_0(A_{2n-1})$ for $1 \leq k \leq n-1$.



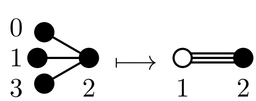
A_{2n-1} B_n

The positive representations of F_4 corresponding to w_0 can be obtained from the positive representations of E_6 with s_1, s_2, s_3, s_4 in $w_0(F_4)$ replaced by s_1s_5, s_2s_4, s_3, s_0 in $w_0(E_6)$.



E_6 F_4

Finally, the positive representations of G_2 can be obtained from the positive representations of D_4 with s_1 in $w_0(G_2)$ replaced by $s_0s_1s_3$ in $w_0(D_4)$.



D_4 G_2

Proof. First we rescaled q so that for non-paired index i, the action of E_i for $w_0 = w_{l-1}s_i$ matches with the correct q. For example, we have to use $\mathcal{U}_{q_s}(D_{n+1})$ where $q_s = q^{\frac{1}{2}}$ so that it corresponds to the short roots in $\mathcal{U}_q(C_n)$.

Next we identify the variables corresponding to the paired roots in the simply-laced case. For example we let $u_0^k = u_1^k$ and $p_0^k = p_1^k$ in D_{n+1}. These will simplify certain expressions, and occasionally produce identical terms.

Then we replace the parameter b by the appropriate b_i according to whether they correspond to short or long roots, much like the procedure described before Definition 4.4.

Finally, for the identical terms, we quantize the multiples with q_s. For example

$$2[v-t]e(p_t-p_u-p_v) \longmapsto [2]_{q_s}[v-t]e(p_t-p_u-p_v). \qquad \square$$

8. Transcendental relations

We recall from the simply-laced case that, if we define the following operators $\widetilde{e_i}$, $\widetilde{f_i}$, $\widetilde{K_i}$ as:

$$\widetilde{e_i} = (e_i)^{\frac{1}{b^2}} \tag{8.1}$$

$$\widetilde{f_i} = (f_i)^{\frac{1}{b^2}} \tag{8.2}$$

$$\widetilde{K_i} = (K_i)^{\frac{1}{b^2}}. \tag{8.3}$$

Then the operators are precisely the same as replacing b with b^{-1}, so that the operators

$$\widetilde{E_i} = \left(2\sin\pi b^{-2}\right)^{-1} e_i,$$

$$\widetilde{F_i} = \left(2\sin\pi b^{-2}\right)^{-1} f_i$$

and $\widetilde{K_i}$ generates a representation of $\mathcal{U}_{\widetilde{q}}(\mathfrak{g}_{\mathbb{R}})$ where $\widetilde{q} = e^{\pi i b^{-2}}$. In particular, the transcendental relations mean that

$$\begin{aligned}\left([w]e(-p_w)\right)^{\frac{1}{b^2}} &= \left(e^{\pi b(w-2p_w)} + e^{\pi b(-w-2p_w)}\right)^{\frac{1}{b^2}} \\ &= e^{\pi b^{-1}(w-2p_w)} + e^{\pi b^{-1}(-w-2p_w)} := \left([w]e(-p_w)\right)_* \end{aligned} \tag{8.4}$$

which is due to Lemma 2.8.

However in the case of non-simply-laced type, the tilde generators no longer generates $\mathcal{U}_{\widetilde{q}}(\mathfrak{g}_{\mathbb{R}})$, but rather short roots become long roots and vice versa. More precisely, we have the following Theorem.

Theorem 8.1. *Let $\widetilde{q} = e^{\pi i b_s^{-2}}$ and define $\widetilde{q_i} = e^{\pi i b_i^{-2}}$. Define the operators*

$$\widetilde{e_i} := (e_i)^{\frac{1}{b_i^2}} \tag{8.5}$$

$$\widetilde{f_i} := (f_i)^{\frac{1}{b_i^2}} \tag{8.6}$$

$$\widetilde{K_i} := (K_i)^{\frac{1}{b_i^2}} \tag{8.7}$$

where as before

$$e_i = 2\sin\pi b_i^2 E_i, \qquad f_i = 2\sin\pi b_i^2 F_i, \tag{8.8}$$

$$\widetilde{e_i} = 2\sin\pi b_i^{-2}\widetilde{E_i}, \qquad \widetilde{f_i} = 2\sin\pi b_i^{-2}\widetilde{F_i}. \tag{8.9}$$

Then the generators of $\mathcal{U}_{\widetilde{q}}({}^L\mathfrak{g}_\mathbb{R})$ are represented by the operators $\widetilde{E_i}$, $\widetilde{F_i}$ and $\widetilde{K_i}$, where ${}^L\mathfrak{g}_\mathbb{R}$ is defined by replacing long roots with short roots and short roots with long roots in the Dynkin diagram of $\mathfrak{g}$.

This is a nontrivial result, since other than exchanging q_i with $\widetilde{q_i}$, the corresponding quantum Serre relations also get interchanged, and there is no classical analogue of the above transcendental relations.

To give the proof of Theorem 8.1, let us introduce the following notation.

Definition 8.2. Let us denote by

$$\big([u_s + u_l]e(p_s + p_l)\big)_* \tag{8.10}$$

the same notation as in Definition 4.4, however with b and b_s replaced by b^{-1} and b_s^{-1} respectively.

We begin with a lemma.

Lemma 8.3. *Let E_i, F_i, K_i be the generators of $\mathcal{U}_q(\mathfrak{g}_\mathbb{R})$ of type B_2 where $i = 1$ corresponds to the short root, and $i = 2$ corresponds to the long root. Then we have*

$$q_1 = e^{\pi i b_s^2} = q^{\frac{1}{2}}, \qquad q_2 = e^{\pi i b^2} = q,$$

$$\widetilde{q_1} = e^{\pi i b_s^{-2}} = \widetilde{q}, \qquad \widetilde{q_2} = e^{\pi i b^{-2}} = \widetilde{q}^{\frac{1}{2}},$$

so that $\widetilde{E_i}$, $\widetilde{F_i}$, $\widetilde{K_i}$ generates $\mathcal{U}_{\widetilde{q}}(\mathfrak{g}_\mathbb{R})$ of type C_2 where $i = 1$ corresponds to the long root and $i = 2$ corresponds to the short root.

Proof. Let us choose the positive representation of B_2 corresponding to $w_0 = s_1 s_2 s_1 s_2$ given by Theorem 4.3*. Then the transcendental relations for E_2 are clear from (8.4),

$$\big([w]e(-p_w)\big)^{\frac{1}{b^2}} = e^{\pi b^{-1}(w-2p_w)} + e^{\pi b^{-1}(-w-2p_w)} := \big([w]e(-p_w)\big)_*.$$

The transcendental relations for E_1 are less trivial. Let $q_s = q_1$ and write the terms in (4.32) as

$$\begin{aligned} e_1 &= [t]e(-p_t - p_u + p_w) + [u-v]e(-p_u - p_v + p_w) + [v-w]e(-p_v) \\ &= \big(A_1^- + A_1^+\big) + \big(A_2^- + A_2^+\big) + \big(A_3^- + A_3^+\big) \end{aligned}$$

where each bracket corresponds to the two terms in $[-]e(-)$ so that

$$A_i^+ A_i^- = q_s^2 A_i^- A_i^+.$$

Then by direct calculation we can write the above sum as

$$e_1 = A_3^+ + A_2^+ + A_1^+ + A_1^- + A_2^- + A_3^-$$

so that each term q_s^2 commutes with all other terms to their right, except for A_3 and A_2 where we have instead

$$A_3^+ A_2^+ = q_s^4 A_2^+ A_3^+, \qquad A_2^- A_3^- = q_s^4 A_3^- A_2^-.$$

Hence by Lemma 2.8, we obtain

$$\begin{aligned}
(e_1)^{\frac{1}{b_s^2}} &= \left(A_3^+ + A_2^+ + A_1^+ + A_1^- + A_2^- + A_3^-\right)^{\frac{1}{b_s^2}} \\
&= \left(A_3^+ + A_2^+\right)^{\frac{1}{b_s^2}} + \left(A_1^+ + A_1^- + A_2^- + A_3^-\right)^{\frac{1}{b_s^2}} \\
&= \left(A_3^+ + A_2^+\right)^{\frac{1}{b_s^2}} + \left(A_1^+ + A_1^-\right)^{\frac{1}{b_s^2}} + \left(A_2^- + A_3^-\right)^{\frac{1}{b_s^2}} \\
&= \left(A_3^{+\,\frac{1}{2b_s^2}} + A_2^{+\,\frac{1}{2b_s^2}}\right)^2 + \left(A_1^{+\,\frac{1}{b_s^2}} + A_1^{-\,\frac{1}{b_s^2}}\right) + \left(A_2^{-\,\frac{1}{2b_s^2}} + A_3^{-\,\frac{1}{2b_s^2}}\right)^2 \\
&= A_3^{+\,\frac{1}{b_s^2}} + [2]_{\widetilde{q_2}} \widetilde{q_2}^{-1} A_3^+ A_2^+ + A_2^{+\,\frac{1}{b_s^2}} + A_1^{+\,\frac{1}{b_s^2}} \\
&\quad + A_3^{-\,\frac{1}{b_s^2}} + [2]_{\widetilde{q_2}} \widetilde{q_2} A_3^- A_2^- + A_2^{-\,\frac{1}{b_s^2}} + A_1^{-\,\frac{1}{b_s^2}} \\
&= \big([2u - v]e(2p_w - p_v - 2p_u)\big)_* + [2]_{\widetilde{q_2}} \big([u - w]e(p_w - p_v - p_u)\big)_* \\
&\quad + \big([v - 2w]e(-p_v)\big)_* + \big([t]e(2p_w - 2p_u - p_t)\big)_*
\end{aligned}$$

Upon identification $(t, u, v, w) \longleftrightarrow (w, v, u, t)$ we see that this is exactly the expression (4.39) for the long root element E_2 of C_2 with q replaced by $\widetilde{q}$.

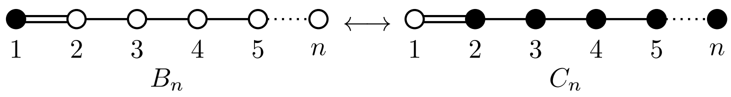
Finally, the operators $F_1, K_1 \longleftrightarrow F_2, K_2$, $q_i \longleftrightarrow \widetilde{q_i}$ under the transcendental relations. The transcendental relations of F_i follow also from simple application of Lemma 2.8 and the transcendental relations for K_i are trivial. □

Proof of Theorem 8.1. In general, the transcendental relations for F_i, K_i can be checked directly from Definition 3.5*. Since each term of f_i is q_i^2 commuting with each other, we can apply Lemma 2.8 and induction to conclude that $f_i^{\frac{1}{b_i^2}}$ has exactly the same expression with b_i replaced by b_i^{-1}. Hence under the scaling of the short root it is equivalent to the Cartan matrix (a_{ij}) being transposed.

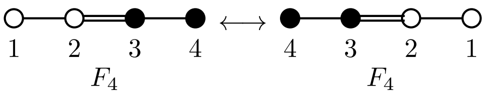
For the action of $\widetilde{E_i}$, it suffices to choose a simple reduced expression of w_0 in order to check the commutation relation, much like proof given in [10]. In particular, we can choose the reduced expression of w_0 that ends with $s_i s_j s_i$ for simply-laced connecting root indices, or $s_i s_j s_i s_j$ for doubly-connected root indices. Then we see from (8.4) and Lemma 8.3 that the relations between $\widetilde{E_i}$, $\widetilde{F_i}$, $\widetilde{K_i}$ and $\widetilde{E_j}$, $\widetilde{F_j}$, $\widetilde{K_j}$ hold as generators of $\mathcal{U}_{\widetilde{q}}({}^L\mathfrak{g}_{\mathbb{R}})$.

Finally, the action $\widetilde{E_1}$ and $\widetilde{E_2}$ for $\mathcal{U}_{\widetilde{q}}(\mathfrak{g}_{\mathbb{R}})$ of type G_2 corresponding to $w_0 = s_2s_1s_2s_1s_2s_1$ from Theorem 6.2 can be checked along the same line. One needs to apply appropriate quantum dilogarithm transformations using Lemma 2.7 to simplify the expression, and apply the transcendental relation to the corresponding terms. Then reapply the inverse transformations will expand the expression and give the 13 terms which coincide with the action of E_2 and E_1 corresponding to $w_0 = s_1s_2s_1s_2s_1s_2$ with b_i replaced by b_i^{-1}. □

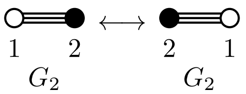
Therefore we have the correspondence of generators:



(8.11)



(8.12)



(8.13)

9. Modified quantum group and its modular double

As in the simply-laced case, we note that the generators E_i, F_i, K_i in general do not commute with $\widetilde{E_i}$, $\widetilde{F_i}$, $\widetilde{K_i}$. For example, in type B_2, the relation $K_1E_2 = q^{-1}E_2K_1$ implies

$$K_1\widetilde{E_2} = q^{\frac{1}{b^2}}\widetilde{E_2}K_1 = -\widetilde{E_2}K_1.$$

More precisely, we have:

Proposition 9.1. *The generators E_i, F_i, K_i commute with $\widetilde{E_i}$, $\widetilde{F_i}$, $\widetilde{K_i}$ up to a sign. We have*

$$\widetilde{E_i}E_j = (-1)^{a_{ij}}E_j\widetilde{E_i}$$
$$\widetilde{E_i}K_j = (-1)^{a_{ij}}K_j\widetilde{E_i}$$
$$\widetilde{K_i}E_j = (-1)^{a_{ij}}E_j\widetilde{K_i}$$

and similarly for E_i replaced by F_i.

Hence as in [10], we modify the generators with powers of K_i in order to take care of the commutation relation between the generators and the other part of the modular double. It turns out the modification is exactly the same even for non simply-laced type, and the corresponding results are as follows.

Proposition 9.2. *For each node i in the Dynkin diagram, we assign a weight $n_i \in \{0,1\}$ such that $|n_i - n_j| = 1$ if i, j are connected in the diagram, so that n_i alternates along the edges.*

We define $\mathfrak{q} := q^2 = e^{2\pi i b^2}$ and

$$\mathfrak{q}_i := \begin{cases} q_i^{-2} & \text{if } n_i = 0, \\ q_i^2 & \text{if } n_i = 1, \end{cases} \tag{9.1}$$

where q_i is defined by (2.6) and (2.7) as before, and we define the modified quantum generators by

$$\mathbf{E}_i := q_i^{n_i} E_i K_i^{n_i}, \tag{9.2}$$

$$\mathbf{F}_i := q_i^{1-n_i} F_i K_i^{n_i-1}, \tag{9.3}$$

$$\mathbf{K}_i := \mathfrak{q}_i^{H_i} = \begin{cases} K_i^{-2} & \text{if } n_i = 0, \\ K_i^2 & \text{if } n_i = 1. \end{cases} \tag{9.4}$$

Then the variables are positive self-adjoint. Let

$$[A,B]_{\mathfrak{q}} = AB - \mathfrak{q}^{-1} BA \tag{9.5}$$

be the quantum commutator. Then the quantum relations in the new variables become:

$$\mathbf{K}_i \mathbf{E}_j = \mathfrak{q}_i{}^{a_{ij}} \mathbf{E}_j \mathbf{K}_i, \tag{9.6}$$

$$\mathbf{K}_i \mathbf{F}_j = \mathfrak{q}_i{}^{-a_{ij}} \mathbf{F}_j \mathbf{K}_i, \tag{9.7}$$

$$\mathbf{E}_i \mathbf{F}_j = \mathbf{F}_j \mathbf{E}_i \quad \text{if } i \neq j, \tag{9.8}$$

$$[\mathbf{E}_i, \mathbf{F}_i]_{\mathfrak{q}_i} = \frac{1 - \mathbf{K}_i}{1 - \mathfrak{q}_i}, \tag{9.9}$$

and the quantum Serre relations become

$$\Big[\Big[\Big[[\mathbf{E}_j, \mathbf{E}_i]_{\mathfrak{q}_i^{-a_{ij}}}, \ldots \mathbf{E}_i\Big]_{\mathfrak{q}_i^2}, \mathbf{E}_i\Big]_{\mathfrak{q}_i}, \mathbf{E}_i\Big] = 0, \tag{9.10}$$

$$\Big[\Big[\Big[[\mathbf{F}_j, \mathbf{F}_i]_{\mathfrak{q}_i^{-a_{ij}}}, \ldots \mathbf{F}_i\Big]_{\mathfrak{q}_i^2}, \mathbf{F}_i\Big]_{\mathfrak{q}_i}, \mathbf{F}_i\Big] = 0. \tag{9.11}$$

We denote the modified quantum group by $\mathbf{U}_{\mathfrak{q}}(\mathfrak{g}_{\mathbb{R}})$.

Hence we can now state the main theorem for $\mathbf{U}_{\mathfrak{q}\widetilde{\mathfrak{q}}}(\mathfrak{g}_\mathbb{R})$:

Theorem 9.3. *Let* $\widetilde{\mathfrak{q}} := \widetilde{q}^2 = e^{2\pi \mathbf{i} b_s^{-2}}$. *We define the tilde part of the modified modular double by*

$$\widetilde{\mathbf{e}_i} := (\mathbf{e}_i)^{\frac{1}{b_i^2}}, \tag{9.12}$$

$$\widetilde{\mathbf{f}_i} := (\mathbf{f}_i)^{\frac{1}{b_i^2}}, \tag{9.13}$$

$$\widetilde{\mathbf{K}_i} := (\mathbf{K}_i)^{\frac{1}{b_i^2}}, \tag{9.14}$$

where as before

$$\mathbf{e}_i = 2\sin\left(\pi b_i^2\right)\mathbf{E}_i, \qquad \mathbf{f}_i = 2\sin\left(\pi b_i^2\right)\mathbf{F}_i, \tag{9.15}$$

$$\widetilde{\mathbf{e}_i} = 2\sin\left(\pi b_i^{-2}\right)\widetilde{\mathbf{E}_i}, \qquad \widetilde{\mathbf{f}_i} = 2\sin\left(\pi b_i^{-2}\right)\widetilde{\mathbf{F}_i}. \tag{9.16}$$

Then the properties of positive representations are satisfied:

(i) *the operators* $\mathbf{e}_i$, $\mathbf{f}_i$, $\mathbf{K}_i$ *and their tilde counterparts are represented by positive essentially self-adjoint operators,*
(ii) *the operators* $\widetilde{\mathbf{E}_i}$, $\widetilde{\mathbf{F}_i}$, $\widetilde{\mathbf{K}_i}$ *generates the Langlands dual* $\mathbf{U}_{\widetilde{\mathfrak{q}}}({}^L\mathfrak{g}_\mathbb{R})$,
(iii) *all the generators* $\mathbf{E}_i$, $\mathbf{F}_i$, $\mathbf{K}_i$ *commute with all* $\widetilde{\mathbf{E}_j}$, $\widetilde{\mathbf{F}_j}$, $\widetilde{\mathbf{K}_j}$.

Therefore we have constructed the positive principal series representations of the modular double

$$\mathbf{U}_{\mathfrak{q}\widetilde{\mathfrak{q}}}(\mathfrak{g}_\mathbb{R}) := \mathbf{U}_{\mathfrak{q}}(\mathfrak{g}_\mathbb{R}) \otimes \mathbf{U}_{\widetilde{\mathfrak{q}}}({}^L\mathfrak{g}_\mathbb{R}), \tag{9.17}$$

parametrized by rank$(\mathfrak{g})$ *numbers* $\lambda_i \in \mathbb{R}$.

Not surprisingly, all the additional properties in the simply-laced case [10] also hold in the general case, with slight modifications. The proof of the following result about the commutant is exactly the same as in the simply-laced case (with A^T introduced due to the factor $\frac{1}{2}(\alpha_i, \alpha_i)$ in the proof).

Theorem 9.4. *The commutant for the positive representation of* $\mathbf{U}_{\mathfrak{q}}(\mathfrak{g}_\mathbb{R})$ *is generated by* $\widetilde{\mathbf{E}_i}$, $\widetilde{\mathbf{F}_i}$ *and elements of the form*

$$\widetilde{\mathbf{K}^{\mathbf{b}^k}} := \widetilde{\mathbf{K}}_1^{b_1^k}\widetilde{\mathbf{K}}_2^{b_2^k}\cdots\widetilde{\mathbf{K}}_n^{b_n^k}, \tag{9.18}$$

for $k = 1, ..., n$, *where the vector* $\mathbf{b}_k = (b_1^k, b_2^k, ..., b_n^k)^T$ *satisfies*

$$A^T\mathbf{b}_k = \mathbf{e}_k \tag{9.19}$$

where $A = (a_{ij})$ is the Cartan matrix, A^T its transpose, and $\mathbf{e}_k$ are the standard unit vectors.

Remark 9.5. We note that the lattice generated by $\widetilde{\mathbf{K}_i}$ is a sub-lattice generated by $\widetilde{\mathbf{K}^{\mathbf{b}^k}}$. In fact one can check that the commutant of (the adjoint form) $\mathbf{U}_\mathfrak{q}(\mathfrak{g}_\mathbb{R})$ is precisely the simply-connected form of its Langlands dual quantum group, denoted by $\widehat{\mathbf{U}_{\widetilde{\mathfrak{q}}}}({}^L\mathfrak{g}_\mathbb{R})$. For example, in type B_n, $\widehat{\mathbf{U}_{\widetilde{\mathfrak{q}}}}({}^L\mathfrak{g}_\mathbb{R})$ is just adjoining $\widetilde{K_1}^{\frac{1}{2}}$ to $\mathbf{U}_{\widetilde{\mathfrak{q}}}({}^L\mathfrak{g}_\mathbb{R})$.

Let the index i corresponds to the coordinate parametrization $\{x_i\}$, and b_i as usual denotes b_s or b_l depending on whether the corresponding root is short or long. Let s (resp. l) be the number of indices corresponding to short (resp. long) roots, so that $s+l = l(w_0)$. Let $\mathbb{C}[\mathbb{T}^{s,l}_{\mathfrak{q}\widetilde{\mathfrak{q}}}]$ be the quantum torus algebra generated by $\{\mathbf{u}_i^{\pm1}, \mathbf{v}_i^{\pm1}, \widetilde{\mathbf{u}}_i^{\pm1}, \widetilde{\mathbf{v}}_i^{\pm1}\}_{i=1}^{l(w_0)}$ with the $\mathfrak{q}$-commutation relations:

$$\mathbf{u}_i\mathbf{v}_i = \mathfrak{q}_i\mathbf{v}_i\mathbf{u}_i, \qquad \widetilde{\mathbf{u}}_i\widetilde{\mathbf{v}}_i = \widetilde{\mathfrak{q}}_i\widetilde{\mathbf{v}}_i\widetilde{\mathbf{u}}_i, \tag{9.20}$$

which can be realized by positive self adjoint operators

$$\mathbf{u}_i = e^{2\pi b_i u_i}, \qquad \mathbf{v}_i = e^{2\pi b_i p_i}, \tag{9.21}$$

and similarly for $\widetilde{\mathbf{u}}_i$, $\widetilde{\mathbf{v}}_i$ with b_i replaced by b_i^{-1}. Then we have Theorem 1.5 of the introduction generalizing the results of the simply-laced case.

Theorem 9.6. *We have an embedding*

$$\mathbf{U}_{\mathfrak{q}\widetilde{\mathfrak{q}}}(\mathfrak{g}_\mathbb{R}) \hookrightarrow \mathbb{C}\big[\mathbb{T}^{s,l}_{\mathfrak{q}\widetilde{\mathfrak{q}}}\big] \tag{9.22}$$

where each generator of $\mathbf{U}_{\mathfrak{q}\widetilde{\mathfrak{q}}}(\mathfrak{g}_\mathbb{R})$ is realized as a Laurent polynomial of $\{\mathbf{u}_i, \mathbf{v}_j\}$.

In particular

$$\begin{aligned}
&\mathbf{U}_{\mathfrak{q}\widetilde{\mathfrak{q}}}(\mathfrak{g}_\mathbb{R}) \hookrightarrow \mathbb{C}\big[\mathbb{T}^{n,n^2-n}_{\mathfrak{q}\widetilde{\mathfrak{q}}}\big] && \mathfrak{g}\textit{is of type } B_n\\
&\mathbf{U}_{\mathfrak{q}\widetilde{\mathfrak{q}}}(\mathfrak{g}_\mathbb{R}) \hookrightarrow \mathbb{C}\big[\mathbb{T}^{n^2-n,n}_{\mathfrak{q}\widetilde{\mathfrak{q}}}\big] && \mathfrak{g} \textit{ is of type } C_n\\
&\mathbf{U}_{\mathfrak{q}\widetilde{\mathfrak{q}}}(\mathfrak{g}_\mathbb{R}) \hookrightarrow \mathbb{C}\big[\mathbb{T}^{12,12}_{\mathfrak{q}\widetilde{\mathfrak{q}}}\big] && \mathfrak{g} \textit{ is of type } F_4\\
&\mathbf{U}_{\mathfrak{q}\widetilde{\mathfrak{q}}}(\mathfrak{g}_\mathbb{R}) \hookrightarrow \mathbb{C}\big[\mathbb{T}^{3,3}_{\mathfrak{q}\widetilde{\mathfrak{q}}}\big] && \mathfrak{g} \textit{ is of type } G_2
\end{aligned}$$

Proof. The proof is exactly the same as in the simply-laced case, with scaling by appropriate b's. Since positive representations are irreducible, to obtain such embedding, all we need is a transformation of the representations of the modified generators $\mathbf{E}_i$, $\mathbf{F}_i$, $\mathbf{K}_i$ that shifts the p's appropriately so that the coefficients of the variables are all even. In particular, since only the parity matters, we observe that we can ignore the brackets in our notation (cf. Definition 4.4) when considering such shifts. Note that this embedding is by no mean unique.

First we do this for the expressions of $\mathbf{F}_i$. Explicitly, as before let v_i denote the coordinate used in Definition 3.5* and $r(i)$ its corresponding root. Also let

$$c_{jk} := a_{r(k)r(j)} b_k / b_j.$$

Then the unitary transformation can be obtained from multiplication by the unitary functions

$$e^{\frac{1}{2}\pi i v_i^2} : 2\pi b_i p_i \mapsto 2\pi b_i p_i + \pi b_i v_i,$$

for all i, and

$$e^{\pi i c_{jk} v_j v_k} : 2\pi b_j p_j \mapsto 2\pi b_j p_j + \pi a_{r(k)r(j)} b_k v_k, \qquad 2\pi b_k p_k \mapsto 2\pi b_k p_k + \pi a_{r(j)r(k)} b_j v_j$$

whenever $j < k$, $n_{r(j)} = 0$ and $r(j)$, $r(k)$ are adjacent in the Dynkin diagram. Note that these transformations commute with each other, so the order does not matter.

One can check from Definition 3.5* and the modified formula (9.3) that the first shifts from these transformations systematically cancel the odd coefficients for the terms of $\mathbf{F}_i$ with $n_i = 0$. The second shift then deals with the odd coefficients in the terms of $\mathbf{F}_j$ adjacent to $\mathbf{F}_i$ in the Dynkin diagram, corresponding to $n_j = 1$.

In the original Definition 3.5*, since each term of $\mathbf{F}_i$ contains only a single $e(p_{v_k})$, and commutes or $\mathfrak{q}_{r(k)}$-commutes with other terms in $\mathbf{E}_j$, the transformations defined above will automatically force all the coefficients of the terms in $\mathbf{E}_j$ to be even as well. □

Finally as in the simply-laced case, we have the following unitary equivalence of positive representations:

Theorem 9.7. *Let $\mathcal{P}_\lambda$ denote the positive principal series representations of $\mathcal{U}_q(\mathfrak{g}_\mathbb{R})$ corresponding to the parameter $\lambda = (\lambda_i)_{i=1}^n$ where $n = rank(\mathfrak{g})$. Then*

$$\mathcal{P}_\lambda \simeq \mathcal{P}_{w(\lambda)} \tag{9.23}$$

are unitary equivalent representations for any Weyl group element $w \in W$ acting on λ, namely for simple reflections,

$$s_i(\lambda_j) := \lambda_j - a_{ij}\lambda_i \tag{9.24}$$

where a_{ij} is the Cartan matrix. In particular, the positive principal series representations are parametrized by $\lambda \in \mathbb{R}^n_{\geq 0}$.

Proof. The proof is exactly the same as in the simply-laced case, slightly modified with the parameter b replaced by the appropriate b_i's when applying the intertwiner G_{λ_i} (cf. (11.4) in [10]). □

Acknowledgment

This work was partially supported by Yale University and World Premier International Research Center Initiative (WPI Initiative), MEXT, Japan.

References

[1] A. Berenstein, J. Greenstein, Quantum folding, Int. Math. Res. Not. IMRN 21 (2011) 4821–4883.
[2] A. Berenstein, A. Zelevinsky, Total positivity in Schubert varieties, Comment. Math. Helv. 72 (1997) 128–166.
[3] A.G. Bytsko, K. Teschner, R-operator, co-product and Haar-measure for the modular double of $U_q(\mathfrak{sl}(2,\mathbb{R}))$, Comm. Math. Phys. 240 (2003) 171–196.
[4] V. Chari, A. Pressley, A Guide to Quantum Groups, Cambridge Univ. Press, Cambridge, 1994.
[5] L.D. Faddeev, Modular double of quantum group, arXiv:math/9912078, 1999.
[6] V.V. Fock, A.B. Goncharov, The quantum dilogarithm and representations of quantum cluster varieties, Invent. Math. 175 (2009) 223–286.
[7] I.B. Frenkel, I. Ip, Positive representations of split real quantum groups and future perspectives, arXiv:1111.1033, 2011.
[8] A. Gerasimov, S. Kharchev, D. Lebedev, Representation theory and quantum integrability, Progr. Math. 237 (2005) 133–156.
[9] J. Hurrelbrink, U. Rehmann, Eine endliche Präsentation der Gruppe $G_2(\mathbb{Z})$, Math. Z. 141 (1975) 243–251.
[10] I. Ip, Positive representations of split real simply-laced quantum groups, arXiv:1203.2018, 2012.
[11] I. Ip, Positive representations and harmonic analysis of split real quantum groups, PhD thesis, Yale University, 2012.
[12] G. Lusztig, Introduction to Quantum Groups, Progress in Mathematics, vol. 110, Birkhäuser Boston, Inc., Boston, MA, 1993.
[13] G. Lusztig, Total positivity in reductive groups, in: Lie Theory and Geometry: In Honor of B. Kostant, in: Progr. in Math., vol. 123, Birkhäuser, 1994, pp. 531–568.
[14] B. Ponsot, J. Teschner, Liouville bootstrap via harmonic analysis on a noncompact quantum group, arXiv:hep-th/9911110, 1999.
[15] J. Teschner, Quantization of the Hitchin moduli spaces, Liouville theory, and the geometric Langlands correspondence I, arXiv:1005.2846, 2010.
[16] A.Yu. Volkov, Noncommutative hypergeometry, Comm. Math. Phys. 258 (2) (2005) 257–273.