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Q-Operator and Fusion Relations for $U_q(C^{(2)}(2))$

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Abstract. The construction of the Q-operator for twisted affine superalgebra $U_q(C^{(2)}(2))$ is given. It is shown that the corresponding prefundamental representations give rise to evaluation modules some of which do not have a classical limit, which nevertheless appear to be a necessary part of fusion relations.

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1. Introduction

The Q-operator and its generalizations are important ingredients in the study of quantum integrable models. Namely, eigenvalues of the transfer matrices, corresponding to various representations, can be expressed in terms of eigenvalues of the Q-operator, which has less-complicated analytic properties. These features of the Q-operators were first noticed by Baxter the early 70s in the case of vertex models [1]. Later, after the quantum group interpretation of the quantum integrable models it was realized that the original Baxter Q-operator corresponds to the integrable model based on the simplest nontrivial quantum affine algebra $U_q(\widehat{\mathfrak{sl}}(2))$. A natural question was to generalize this notion to the higher rank and give a proper representation-theoretic meaning to these fundamental building blocks for transfer matrix eigenvalues. The first idea in that direction was given in the papers of Bazhanov et al. [3,4] in the context of the construction of integrable structure of conformal field theory: the interpretation of Q-operators for $U_q(\widehat{\mathfrak{sl}}(2))$ as transfer matrices for certain infinite-dimensional representations of the Borel subalgebra of $U_q(\widehat{\mathfrak{sl}}(2))$. Later their results were generalized in [2,13] to the case of $U_q(\widehat{\mathfrak{sl}}(n))$. Finally, in the recent preprint of Frenkel and Hernandez [6] the full representation-theoretic description of Q-operators was given for large class of integrable models based on any untwisted quantum affine algebra $\mathcal{U}_q(\mathfrak{g})$ and connected to the earlier description of the transfer matrix eigenvalues via the q-characters [7]. The infinite-dimensional representations corresponding to the Q-operator, which the authors called "prefundamental representations" were constructed just before that in [9].

At the same time, some analogs of the Q-operators were constructed in this way in the case of superalgebras [5,15,17]. In this article we improve the constructions of [15,16]. In that paper an attempt to construct the Q-operator and associated fusion relation for transfer matrices was made in the case of $U_q(C^{(2)}(2)) \equiv U_q(sl^{(2)}(2|1))$. However, the construction given there leads to only partial result: half of the resulting transfer matrices were built "by hands" out of Q-operators and did not seem to correspond to any finite-dimensional representation of $U_q(C^{(2)}(2))$. In this paper we solve this ambiguity, by allowing some representations to have no classical limit ($q \to 1$). The approach we are using allows to show explicitly the similarity between $U_q(A_1^{(1)})$ and $U_q(C^{(2)}(2))$ previously noticed on the level of universal R-matrices [12].

The structure of the article is as follows. In Section 2 we fix the notations and describe the relation between finite-dimensional representations of $U_q(\mathfrak{osp}(2|1))$ and $U_q(\mathfrak{sl}(2))$, previously noticed on the level of modular double [10]. The approach, which can be generalized to higher rank superalgebras, is that we find representations of $U_q(\mathfrak{osp}(2|1))$ inside the tensor product of finite-dimensional representation of $U_{-iq}(\mathfrak{sl}(2))$ and two-dimensional Clifford algebra. Such representation splits into two irreducible representations which differ by the parity of the highest weight and have equal dimensions. It is notable that the even-dimensional irreducible representations obtained in this way do not have the classical limit. We also give explicit formulas for R-matrix in these representations. In Section 3 we consider evaluation modules for $U_q(C^{(2)}(2))$, which can be obtained in a similar fashion from evaluation modules of $U_{-iq}(A_1^{(1)})$. We explicitly find the resulting trigonometric R-matrix and its matrix coefficients (with the details of calculations in the Appendix). We also introduce in Section 3 the prefundamental representations for $U_q(C^{(2)}(2))$ and study in detail the relations in the Grothendieck ring of prefundamental representations combined with evaluation representations. The relations in the Grothendieck ring lead to relations between transfer matrices and Q-operators: in Section 4 we correct the constructions of [15], where the integrable structure of superconformal field theory was studied, now changing "fusion-like" relations by the true fusion relations.

2. Quantum Superalgebra $U_q(\mathfrak{osp}(2|1))$ and its Representations

We define the quantum superalgebra $U_q(\mathfrak{osp}(2|1))$ as follows. It is a Hopf algebra generated by even element $\mathcal{K}$ and odd elements $\mathcal{E}$ and $\mathcal{F}$ such that

$$\{\mathcal{E},\mathcal{F}\}:=\mathcal{E}\mathcal{F}+\mathcal{F}\mathcal{E}=\frac{\mathcal{K}-\mathcal{K}^{-1}}{q+q^{-1}},$$
$$\mathcal{K}\mathcal{E}=q^2\mathcal{E}\mathcal{K},$$
$$\mathcal{K}\mathcal{F}=q^{-2}\mathcal{F}\mathcal{K},$$

where the corresponding co-product is

$$\Delta(\mathcal{E})=\mathcal{E}\otimes\mathcal{K}+1\otimes\mathcal{E},$$
$$\Delta(\mathcal{F})=\mathcal{F}\otimes 1+\mathcal{K}^{-1}\otimes\mathcal{F},$$
$$\Delta(\mathcal{K})=\mathcal{K}\otimes\mathcal{K}.$$

Let us choose the (odd) Clifford generators ξ,η satisfying

$$\xi^2=\eta^2=1,\quad \xi\eta=-\eta\xi \tag{1}$$

which acts in the space $\mathbb{C}^{1|1}:=\mathrm{span}\{|+\rangle,|-\rangle\}$, where $|-\rangle$, $|+\rangle$ are odd and even vectors correspondingly, by

$$\pi(\xi)=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad \pi(\eta)=\begin{pmatrix}0&i\\-i&0\end{pmatrix}, \tag{2}$$

such that

$$\pi(i\xi\eta)=\begin{pmatrix}1&0\\0&-1\end{pmatrix}. \tag{3}$$

The following notation will play a crucial role in relating the superalgebra and the classical case via the spinor representation:

DEFINITION 2.1. We denote by

$$q_*:=-iq \tag{4}$$

and writing $q:=\mathrm{e}^{\pi i b^2}, q_*:=\mathrm{e}^{\pi i b_*^2}$, we have

$$b^2=b_*^2+\frac{1}{2}. \tag{5}$$

Then we have the following proposition observed in [10], which can be proved by direct computation.

PROPOSITION 2.2. *If* E,F,K *generate* $U_{q_*}(\mathfrak{sl}(2))$, *then*

$$\mathcal{E}=\xi E,\quad \mathcal{F}=\eta F,\quad \mathcal{K}=i\xi\eta K \tag{6}$$

generate $U_q(\mathfrak{osp}(2|1))$.

Therefore, we are now able to relate the representations of $U_{q_*}(\mathfrak{sl}(2))$ and $U_q(\mathfrak{osp}(2|1))$. Let us do it explicitly.

Consider the $s+1=2l+1$-dimensional representation V_s of $U_{q_*}(\mathfrak{sl}(2))$ with basis

$$e^l_m, \quad m=-l,\dots,l$$

and action

$$\begin{aligned}
K\cdot e^l_m &= q_*^{2m}e^l_m,\\
H\cdot e^l_m &= (2m)e^l_m,\\
E\cdot e^l_m &= [l-m]_{q_*}e^l_{m+1},\\
F\cdot e^l_m &= [l+m]_{q_*}e^l_{m-1},
\end{aligned}$$

where formally $K=q_*^H$ and $[n]_q:=\frac{q^n-q^{-n}}{q-q^{-1}}$ is the quantum number.

The generators $\mathcal{E},\mathcal{F},\mathcal{K}$ naturally act on $V_s\otimes\mathbb{C}^{1|1}$ by means of the $U_{q_*}(\mathfrak{sl}(2))$ action, and it decomposes as

$$W_s=V_s\otimes\mathbb{C}^{1|1}=W_s^+\oplus W_s^-, \tag{7}$$

where $W_s^{\pm}$ has highest weight $w_s^{\pm}=e^l_l\otimes|\pm\rangle$ and spanned by

$$W_s^{\pm}=\mathrm{span}\{w_s^{\pm},\mathcal{F}\cdot w_s^{\pm},\mathcal{F}^2\cdot w_s^{\pm},\dots,\mathcal{F}^s\cdot w_s^{\pm}\}. \tag{8}$$

Let $e^l_{m,\pm}:=e^l_m\otimes|\pm\rangle$ be the natural basis of $V_s\otimes\mathbb{C}^{1|1}$. Note that $e^l_{m,-}$ is an odd vector while $e^l_{m,+}$ is even. Then the action of $\mathcal{E},\mathcal{F},\mathcal{K}$ can be written explicitly as follows:

PROPOSITION 2.3.

$$\begin{aligned}
\mathcal{K}\cdot e^l_{m,\pm} &= \pm q_*^{2m}e^l_{m,\pm},\\
&= \pm i^{-2m}q^{2m}e^l_{m,\pm},\\
\mathcal{E}\cdot e^l_{m,\pm} &= [l-m]_{q_*}e^l_{m+1,\mp},\\
&= i^{l-m-1}\{l-m\}_q e^l_{m+1,\mp},\\
\mathcal{F}\cdot e^l_{m,\pm} &= \mp i[l+m]_{q_*}e^l_{m-1,\mp}\\
&= \mp i^{l+m}\{l+m\}_q e^l_{m-1,\mp}
\end{aligned}$$

where $\{n\}_q:=\frac{q^{-n}-(-1)^nq^n}{q+q^{-1}}=i^{1-n}[n]_{q_*}$.

We notice that the representations of even dimension is something which we do not encounter in the classical case, namely all the finite-dimensional irreducible representations of Lie superalgebra $\mathfrak{osp}(2|1)$ are odd-dimensional.

Remark 2.4. It is well known that the finite-dimensional irreducible representations of $\mathfrak{osp}(2|1)$ Lie superalgebra have odd dimension only (see e.g. [8]). One can relate $W_s^{\pm}$ for even s to those by considering classical limit. Due to our normalization, to do that one has to proceed through the following steps. First, one has to rescale e_m^l so that $F e_m^l = \mathrm{e}_{m-1}^l$ and renormalize E so that $E' = \frac{q+q^{-1}}{q-q^{-1}} E$. Then $\mathcal{E}' = \xi E'$, and $\mathcal{F}$ are such that the commutation relations on $W_s^{\pm}$ in the limit $q \longrightarrow 1$ are such that $[\mathcal{E}', \mathcal{F}] = H$, i.e. the commutation relations of $\mathfrak{osp}(2|1)$. Such limiting procedure is not possible in the case of even-dimensional $W_s^{\pm}$ as the coefficients will not converge.

EXAMPLE 2.5. For $l = \frac{1}{2}$, the representation on $(W_1^{\pm}, \pi_1)$ with basis $\{e^{1/2}_{1/2,\pm}, e^{1/2}_{-1/2,\mp}\}$ is given by

$$\pi_1(\mathcal{K}) = \begin{pmatrix} \mp iq & 0 \\ 0 & \mp iq^{-1} \end{pmatrix}, \quad \pi_1(H) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$
$$\pi_1(\mathcal{E}) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \pi_1(\mathcal{F}) = \begin{pmatrix} 0 & 0 \\ \mp i & 0 \end{pmatrix}.$$

For $l=1$, the representation on $(W_2^{\pm}, \pi_2)$ with basis $\{e^1_{1,\pm}, e^1_{0,\mp}, e^1_{-1,\pm}\}$ is given by

$$\pi_2(\mathcal{K}) = \begin{pmatrix} \mp q^2 & 0 & 0 \\ 0 & \mp 1 & 0 \\ 0 & 0 & \mp q^{-2} \end{pmatrix}, \quad \pi_2(H) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix},$$

$$\pi_2(\mathcal{E}) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & i(q^{-1}-q) \\ 0 & 0 & 0 \end{pmatrix}, \quad \pi_2(\mathcal{F}) = \begin{pmatrix} 0 & 0 & 0 \\ \pm(q^{-1}-q) & 0 & 0 \\ 0 & \pm i & 0 \end{pmatrix}.$$

Now we will find the formula for the R-matrix acting in tensor product of $W_s^{\pm}$.
Let

$$\exp_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{\lceil n \rceil_q!} \tag{9}$$

where $\lceil n \rceil_q = \frac{1-q^n}{1-q}$. The the following Theorem holds.

THEOREM 2.6. *The universal R matrix is given by*

$$\boldsymbol{R} = Q\mathcal{R}$$

where $Q := C q_*^{\frac{H\otimes H}{2}}$ *with* $C := \frac{1}{2}(1\otimes 1 + i\xi\eta\otimes 1 + 1\otimes i\xi\eta + \xi\eta\otimes\xi\eta)$ *such that*

$$C\cdot|(-1)^{\epsilon_1}\rangle\otimes|(-1)^{\epsilon_2}\rangle = (-1)^{\epsilon_1\epsilon_2}|(-1)^{\epsilon_1}\rangle\otimes|(-1)^{\epsilon_2}\rangle, \quad \epsilon_i\in\{0,1\}, \tag{10}$$

and

$$
\begin{aligned}
\mathcal{R} &:= \exp_{q_*^{-2}}(i(q_*^{-1} - q_*)\mathcal{E} \otimes \mathcal{F}) \\
&= \exp_{-q^{-2}}(-(q+q^{-1})\mathcal{E} \otimes \mathcal{F}) \\
&= \sum a_n \mathcal{E}^n \otimes \mathcal{F}^n
\end{aligned} \tag{11}
$$

where

$$
a_n = (-1)^n q^{\frac{1}{2}n(n-1)} \frac{(q+q^{-1})^n}{\{n\}_q!} \tag{12}
$$

The proof is given in Appendix.

Finally, let us give for completeness the explicit matrix coefficients of R. Namely, we find the pairing for $\mathbf{R}_{l_1,l_2} = \mathbf{R}|_{W_{s_1}^{\pm} \otimes W_{s_2}^{\pm}}$

$$
\left\langle e^{l_1}_{m_1',\epsilon_1} \otimes e^{l_2}_{m_2',\epsilon_2}, \mathbf{R}_{l_1,l_2}(e^{l_1}_{m_1,\epsilon_1'} \otimes e^{l_2}_{m_2,\epsilon_2'}) \right\rangle
$$

where $\epsilon_i \in \{0, 1\}$ indicates the parity, namely $|\pm\rangle = |(-1)^{\epsilon}\rangle$. Let us fix l_1, l_2 and write $e^l_{m,\epsilon}$ for $e^l_{m,\pm}$.

PROPOSITION 2.7.

$$
\left\langle e^{l_1}_{m_1',\epsilon_1} \otimes e^{l_2}_{m_2',\epsilon_2}, \boldsymbol{R}_{l_1,l_2}(e^{l_1}_{m_1,\epsilon_1'} \otimes e^{l_2}_{m_2,\epsilon_2'}) \right\rangle = 0
$$

if $m_1' - m_1 \neq m_2 - m_2'$ *or* $m_1' - m_1 = m_2 - m_2' < 0$.

Otherwise let $n = m_1' - m_1$, *we have*

$$
\begin{aligned}
&\left\langle e^{l_1}_{m_1',\epsilon_1'} \otimes e^{l_2}_{m_2',\epsilon_2'}, \boldsymbol{R}_{l_1,l_2}(e^{l_1}_{m_1,\epsilon_1} \otimes e^{l_2}_{m_2,\epsilon_2}) \right\rangle \\
&\quad = i^{(l_1-m_1+l_2+m_2-1)n-2m_1'm_2'}(-1)^{\epsilon_1\epsilon_2+n} q^{\frac{1}{2}n(n-1)+2m_1'm_2'} \cdot \\
&\qquad \cdot \frac{(q+q^{-1})^n}{\{n\}_q!} \frac{\{l_1-m_1\}_q!}{\{l_1-m_1-n\}_q!} \frac{\{l_2+m_2\}_q!}{\{l_2+m_2-n\}_q!}
\end{aligned}
$$

In terms of q_* *and using the standard* $[n]_{q_*}$ *instead, we get*

$$
= q_*^{\frac{1}{2}n(n-1)+2m_1'm_2'}(-1)^{\epsilon_1\epsilon_2} \frac{(q_* - q_*^{-1})^n}{[n]_{q_*}!} \frac{[l_1-m_1]_{q_*}!}{[l_1-m_1-n]_{q_*}!} \frac{[l_2+m_2]_{q_*}!}{[l_2+m_2-n]_{q_*}!}
$$

Note that there are no more i's using the q_* *notation.*

EXAMPLE 2.8. For $W_1^+ \otimes W_1^+$, let the basis be $\{e^{1/2}_{1/2,+}, e^{1/2}_{-1/2,-}\} \otimes \{e^{1/2}_{1/2,+}, e^{1/2}_{-1/2,-}\}$. Then R is given by

$$\mathbf{R}_{\frac{1}{2},\frac{1}{2}} = \begin{pmatrix} q_*^{\frac{1}{2}} & 0 & 0 & 0 \\ 0 & q_*^{-\frac{1}{2}} & (1-q_*^{-2})q_*^{\frac{1}{2}} & 0 \\ 0 & 0 & q_*^{-\frac{1}{2}} & 0 \\ 0 & 0 & 0 & -q_*^{\frac{1}{2}} \end{pmatrix}$$

EXAMPLE 2.9. For $W_2^+ \otimes W_2^+$, let the basis be $\{e^1_{1,+}, e^1_{0,-}, e^1_{-1,+}\} \otimes \{e^1_{1,+}, e^1_{0,-}, e^1_{-1,+}\}$. Then R is given by

$$\mathbf{R}_{1,1} = \begin{pmatrix} q_*^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & q_*^2 - q_*^{-2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q_*^{-2} & 0 & q_*^{-2}(q_*^{-1} - q_*) & 0 & (q_*^2 - q_*^{-2})(1 - q_*^{-2}) & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & (q_*^2 - q_*^{-2})(q_* + q_*^{-1}) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & q_*^2 - q_*^{-2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & q_*^{-2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q_*^2 \end{pmatrix}$$

Finally we give some remarks about the Casimir operator. In $U_{q_*}(\mathfrak{sl}(2))$, it is known that the center is generated by the Casimir operator given by (up to some additive constant):

$$\mathbf{C}_{\mathfrak{sl}(2)} = FE + \frac{q_* K + q_*^{-1} K^{-1}}{(q_* - q_*^{-1})^2} \tag{13}$$

By Proposition 2.2, it is obvious that $\mathbf{C}_{\mathfrak{sl}(2)}$ commutes with our generators. However, it is not an element of $U_q(\mathfrak{osp}(2|1))$. Instead, the element

$$\sqrt{\mathbf{C}_{\mathfrak{osp}(2|1)}} := \eta\xi \mathbf{C}_{\mathfrak{sl}(2)} = \mathcal{F}\mathcal{E} - \frac{q\mathcal{K} - q^{-1}\mathcal{K}^{-1}}{(q+q^{-1})^2} \tag{14}$$

will be an element in $U_q(\mathfrak{osp}(2|1))$ super-commuting with the generators $\{\mathcal{E}, \mathcal{F}, \mathcal{K}\}$. By construction, its square $\mathbf{C}_{\mathfrak{osp}(2|1)} := -\mathbf{C}^2_{\mathfrak{sl}(2)}$ is in the center of $U_q(\mathfrak{osp}(2|1))$, given by

$$\mathbf{C}_{\mathfrak{osp}(2|1)} = -\mathcal{F}^2\mathcal{E}^2 + \frac{q-q^{-1}}{(q+q^{-1})^2}(q^2\mathcal{K} + q^{-2}\mathcal{K}^{-1})\mathcal{F}\mathcal{E} + \frac{q^2\mathcal{K}^2 + q^{-2}\mathcal{K}^{-2}}{(q+q^{-1})^4} \tag{15}$$

up to an additive constant. Under a rescaling of the generators, this is precisely the Casimir element of $U_q(\mathfrak{osp}(2|1))$ found in [14].

Now it is also clear that the representations $W_s^{\pm}$ correspond to the positive and negative spectrum of the square root of the Casimir element $\sqrt{\mathbf{C}_{\mathfrak{osp}(2|1)}} = \eta\xi\mathbf{C}_{\mathfrak{sl}(2)}$.

3. Evaluation Modules for $U_q(C^{(2)}(2))$ and Prefundamental Representations

The quantum affine superalgebra $U_q(C^{(2)}(2))$ is generated by $\mathcal{E}_i, \mathcal{F}_i, \mathcal{K}_i,\ i=0,1$, where $\mathcal{E}_i$ and $\mathcal{F}_i$ are odd, with Cartan matrix given by

$$A=\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}.$$

In particular, we have

$$\mathcal{K}_i\mathcal{E}_j = q^{a_{ij}}\mathcal{E}_j\mathcal{K}_i, \quad \mathcal{K}_i\mathcal{F}_j = q^{-a_{ij}}\mathcal{F}_j\mathcal{K}_i, \tag{16}$$

and in addition the Serre relations

$$\mathcal{E}_i^3\mathcal{E}_j + \{3\}_q\mathcal{E}_i^2\mathcal{E}_j\mathcal{E}_i - \{3\}_q\mathcal{E}_i\mathcal{E}_j\mathcal{E}_i^2 - \mathcal{E}_j\mathcal{E}_i^3 = 0 \tag{17}$$
$$\mathcal{F}_i^3\mathcal{F}_j + \{3\}_q\mathcal{F}_i^2\mathcal{F}_j\mathcal{F}_i - \{3\}_q\mathcal{F}_i\mathcal{F}_j\mathcal{F}_i^2 - \mathcal{F}_j\mathcal{F}_i^3 = 0 \tag{18}$$

where $\{3\}_q = \frac{q^3+q^{-3}}{q+q^{-1}}$. Furthermore, for later convenience we modify the scaling of $\mathcal{F}_i$ and use instead the following commutation relations:

$$\{\mathcal{E}_i, \mathcal{F}_i\} = \frac{\mathcal{K}_i - \mathcal{K}_i^{-1}}{q+q^{-1}}. \tag{19}$$

3.1. EVALUATION MODULES FOR $U_q(C^{(2)}(2))$ AND TRIGONOMETRIC R-MATRIX

One check easily that we have the following spinor representation as in the $U_q(\mathfrak{osp}(2|1))$ case:

$$\mathcal{E}_j = E_j\xi, \quad \mathcal{F}_j = F_j\eta, \quad \mathcal{K}_j = i\xi\eta K_j, \tag{20}$$

and we also have the evaluation modules induced from $(A_1^{(1)})_{q*}$ given by

$$E_1 \mapsto \lambda E, \quad E_0 \mapsto \lambda F$$
$$F_1 \mapsto \lambda^{-1}F, \quad F_0 \mapsto \lambda^{-1}E$$
$$K_1 \mapsto K, \quad K_0 \mapsto K^{-1}$$

Then using the 2-dimensional representation of the Clifford algebra, we can consider its action as before on $V_s \otimes \mathbb{C}^{1|1}$ and decompose it into $W_s(\lambda) := W_s(\lambda)^+ \otimes W_s(\lambda)^-$.

PROPOSITION 3.1. *The action on the evaluation module* $W_s(\lambda)^{\pm}$ *with basis* $e^l_{m,\pm}$, $s=2l$, $m=-l,\dots,l$, *is given by*

$$\begin{aligned}
\mathcal{E}_1\cdot e^l_{m,\pm} &= \lambda[l-m]_{q_*}e^l_{m+1,\mp}\\
\mathcal{E}_0\cdot e^l_{m,\pm} &= \lambda[l+m]_{q_*}e^l_{m-1,\mp}\\
\mathcal{F}_1\cdot e^l_{m,\pm} &= \mp i\lambda^{-1}[l+m]_{q_*}e^l_{m-1,\mp}\\
\mathcal{F}_0\cdot e^l_{m,\pm} &= \mp i\lambda^{-1}[l-m]_{q_*}e^l_{m+1,\mp}\\
\mathcal{K}_1\cdot e^l_{m,\pm} &= \pm q_*^{2m}e^l_{m,\pm}\\
\mathcal{K}_0\cdot e^l_{m,\pm} &= \pm q_*^{-2m}e^l_{m,\pm}\\
\mathcal{K}_\delta\cdot e^l_{m,\pm} &= e^l_{m,\pm}
\end{aligned}$$

In the case $s=1$, one can solve for the R matrix explicitly.

PROPOSITION 3.2. *The* R *matrix for* $s=1$, $W_s(\lambda_1)^{\epsilon_1}\otimes W_s(\lambda_2)^{\epsilon_2}$, $\epsilon_i\in\{+,-\}$, *is, up to scalar, given by*

$$\boldsymbol{R}\simeq\begin{pmatrix}1-z^2q_*^2 & 0 & 0 & 0\\ 0 & \epsilon_1q_*(1-z^2) & \epsilon_2 z(1-q_*^2) & 0\\ 0 & \epsilon_1 z(1-q_*^2) & \epsilon_2 q_*(1-z^2) & 0\\ 0 & 0 & 0 & -\epsilon_1\epsilon_2(1-z^2q_*^2)\end{pmatrix}, \tag{21}$$

where $z=\frac{\lambda_2}{\lambda_1}$. *Alternatively, let* $\lambda_i=\mathrm{e}^{x_i}$, *then we can cast it in trigonometric terms:*

$$\boldsymbol{R}\simeq\begin{pmatrix}\sinh(x_1-x_2-\ln q_*) & 0 & 0 & 0\\ 0 & \epsilon_1\sinh(x_1-x_2) & \epsilon_2\sinh(\ln q_*) & 0\\ 0 & \epsilon_1\sinh(\ln q_*) & \epsilon_2\sinh(x_1-x_2) & 0\\ 0 & 0 & 0 & -\epsilon_1\epsilon_2\sinh(x_1-x_2-\ln q_*)\end{pmatrix}. \tag{22}$$

In the general case, one has to calculate the action of the generators corresponding to the imaginary roots. The explicit calculation is given in the Appendix and the explicit form of the R-matrix is presented in Theorem A.4.

3.2. PREFUNDAMENTAL REPRESENTATIONS AND THE GROTHENDIECK RING

Let us consider the Verma modules corresponding to evaluation modules of $U_q(C^{(2)}(2))$. Namely, let us start from the following representation of $U_q(\mathfrak{osp}(2|1))$:

$$\mathcal{W}_s^{\pm}=\{\mathcal{F}^k\cdot w_s^{\pm}\}_{k=0}^{\infty}, \tag{23}$$

where $w_s^{\pm}:=e^l_l\otimes|\pm\rangle$ as before such that $\mathcal{K}\cdot w_s^{\pm}=\pm q_*^s w_s^{\pm}$.

Writing $|k\rangle_\pm := \mathcal{F}^k w_s^\pm$, the basis are related to $e^l_{m,\pm}$ of the $s+1$-dimensional module $W_s^\pm$ from before by

$$\begin{aligned} &|0\rangle_\pm = w_s^\pm = e^l_{l,\pm}, \\ &|k\rangle_\pm = \mathcal{F}^k \cdot w_s^\pm = \mathcal{F}^k \cdot e^l_{l,\pm} = i^{-k} \frac{[2l]_{q_*}!}{[2l-k]_{q_*}!} e^l_{l-k,\pm(-1)^k}. \end{aligned} \tag{24}$$

Note that $|k\rangle_\pm$ is an even vector when $\pm(-1)^k = +1$.

This gives rise to the following evaluation module of the upper Borel part $\mathfrak{b}_+$ of $U_q(C^{(2)}(2))$ on $\mathcal{W}_s^\pm$:

$$\begin{aligned} &\mathcal{E}_0|k\rangle_\pm = \lambda\mathcal{F}|k\rangle_\pm = \lambda|k+1\rangle_\pm, \\ &\mathcal{E}_1|k\rangle_\pm = \lambda\mathcal{E}|k\rangle_\pm = \lambda[k]_{q_*}[s-k+1]_{q_*}|k-1\rangle_\pm, \\ &\mathcal{K}_0|k\rangle_\pm = \mathcal{K}^{-1}|k\rangle_\pm = \pm q^{2k} q_*^{-s}|k\rangle_\pm = \pm(-1)^k q_*^{2k-s}|k\rangle_\pm, \\ &\mathcal{K}_1|k\rangle_\pm = \mathcal{K}|k\rangle_\pm = \pm q^{-2k} q_*^{s}|k\rangle_\pm = \pm(-1)^k q_*^{-2k+s}|k\rangle_\pm. \end{aligned}$$

Furthermore, we see that when $s \in \mathbb{Z}_{\geq 0}$, $\mathcal{W}_s^\pm(\lambda)$ has a block diagonal form such that in the Grothendieck ring of the representation of $\mathfrak{b}_+$,

$$[\mathcal{W}_s^\pm(\lambda)] = [W_s^\pm(\lambda)] + [\mathcal{W}_{-s-2}^{\pm(-1)^{s+1}}(\lambda)]. \tag{25}$$

Let us define the prefundamental (or q-oscillator) representation of $\mathfrak{b}_+$ of $U_q(C_q^{(2)}(2))$. The q-oscillator algebra is generated by $\alpha_+, \alpha_-, \mathcal{H}$ such that

$$q\alpha_+\alpha_- + q^{-1}\alpha_-\alpha_+ = -\frac{1}{q+q^{-1}}, \quad [\mathcal{H}, \alpha_\pm] = \pm 2\alpha_\pm, \tag{26}$$

where $\alpha_\pm$ are considered as odd elements. We consider the Fock modules

$$\Pi_\pm = span\{\alpha_\pm^k|0\rangle_\pm : \mathcal{H}|0\rangle_\pm = 0, \alpha_\mp|0\rangle_\pm = 0\}_{k=0}^\infty, \tag{27}$$

where the vacuum vectors $|0\rangle_\pm$ are even. Then, we have an important lemma.

LEMMA 3.3. *The following substitution provides an infinite-dimensional representation of* $\mathfrak{b}_+$*:*

$$\rho_\pm(\lambda) : \mathcal{E}_1 = \lambda\alpha_\pm, \quad \mathcal{E}_0 = \lambda\alpha_\mp, \quad \mathcal{K}_1 = q^{\pm\mathcal{H}}, \quad \mathcal{K}_0 = q^{\mp\mathcal{H}}. \tag{28}$$

Let us consider the tensor product $\rho_+(\lambda\mu) \otimes \rho_-(\lambda\mu^{-1})$. The action of $\mathcal{E}_i$ is given by

$$\begin{aligned} &\mathcal{E}_1 = \lambda(\mu\alpha_+ \otimes q^{-\mathcal{H}} + 1 \otimes \mu^{-1}\alpha_-) =: \lambda(a_- + b_-) \\ &\mathcal{E}_0 = \lambda(\mu\alpha_- \otimes q^{\mathcal{H}} + 1 \otimes \mu^{-1}\alpha_+) =: \lambda(a_+ + b_+) \end{aligned}$$

so that we have the commutation relations

$$qa_-a_+ + q^{-1}a_+a_- = -\frac{\mu^2}{q+q^{-1}},$$
$$qb_+b_- + q^{-1}b_-b_+ = -\frac{\mu^{-2}}{q+q^{-1}},$$
$$a_{\delta_1}b_{\delta_2} = -q^{2\delta_1\delta_2}b_{\delta_2}a_{\delta_1}, \quad \delta_i \in \{\pm\},$$

or, in q_* notation we have

$$q_*a_-a_+ - q_*^{-1}a_+a_- = \frac{\mu^2}{q_*-q_*^{-1}},$$
$$q_*b_+b_- - q_*^{-1}b_-b_+ = \frac{\mu^{-2}}{q_*-q_*^{-1}},$$
$$a_{\delta_1}b_{\delta_2} = q_*^{2\delta_1\delta_2}b_{\delta_2}a_{\delta_1},$$

which is similar to the bosonic case considered in [4]. Hence as in [4], the tensor product $\rho_+(\lambda\mu)\otimes\rho_-(\lambda\mu^{-1})$ decomposes as

$$\rho_+(\lambda\mu)\otimes\rho_-(\lambda\mu^{-1}) = \bigoplus_{m=0}^{\infty}\rho^{(m)}, \tag{29}$$

where

$$\rho^{(m)} : |\rho_k^{(m)}\rangle = (a_+ + b_+)^k(a_+ - \gamma b_+)^m|0\rangle_+ \otimes |0\rangle_-, \tag{30}$$

for $k\in\mathbb{Z}_{\geq 0}$ and $\gamma \neq -q_*^{2n}, n\in\mathbb{Z}$ any constant. Note that $|\rho_k^{(m)}\rangle$ is even when $k+m$ is even.

Let $\mu = q_*^{\frac{s}{2}+\frac{1}{2}}$. Then the action of $\mathfrak{b}_+$ is given by

$$\rho_+(\lambda\mu)\otimes\rho_-(\lambda\mu^{-1})(\mathcal{K}_1)|\rho_k^{(m)}\rangle = q^{-2(k+m)}|\rho_k^{(m)}\rangle = (-1)^{k+m}q_*^{-2(k+m)}|\rho_k^{(m)}\rangle,$$
$$\rho_+(\lambda\mu)\otimes\rho_-(\lambda\mu^{-1})(\mathcal{K}_0)|\rho_k^{(m)}\rangle = q^{2(k+m)}|\rho_k^{(m)}\rangle = (-1)^{k+m}q_*^{2(k+m)}|\rho_k^{(m)}\rangle,$$
$$\rho_+(\lambda\mu)\otimes\rho_-(\lambda\mu^{-1})(\mathcal{E}_0)|\rho_k^{(m)}\rangle = \lambda|\rho_{k+1}^{(m)}\rangle,$$
$$\rho_+(\lambda\mu)\otimes\rho_-(\lambda\mu^{-1})(\mathcal{E}_1)|\rho_k^{(m)}\rangle = \lambda[k]_{q_*}[s-k+1]_{q_*}|\rho_{k-1}^{(m)}\rangle + c_k^{(m)}|\rho_k^{(m-1)}\rangle,$$

where $c_k^{(m)}$ are constants not necessary in what follows.

We observe that the representation of $\mathfrak{b}_+$ has a block diagonal form defined by $\rho^{(m)}$, which resembles the Verma module $\mathcal{W}_s^{\pm}$ with a shift in the factors of $\mathcal{K}_i$. Hence in the Grothendieck ring of representation of $\mathfrak{b}_+$ we obtain

$$[\rho_+(\lambda\mu)\otimes\rho_-(\lambda\mu^{-1})] = \sum_{m=0}^{\infty}[U_{-s-2m}\otimes\mathcal{W}_s^{(-1)^m}(\lambda)] \tag{31}$$

where U_p is the 1-dimensional representation such that $\mathcal{E}_1, \mathcal{E}_0$ act trivially as 0, while $\mathcal{K}_1, \mathcal{K}_0$ acts as q_*^p, q_*^{-p} respectively. Indeed, the action of $\mathcal{K}_1$ on $U_{-s-2m} \otimes \mathcal{W}_s^{(-1)^m}(\lambda)$ is given by multiplication by

$$(q_*^{-s-2m}) \cdot ((-1)^m (-1)^k q_*^{-2k+s}) = (-1)^{k+m} q_*^{-2(k+m)}.$$

Note that $[U_0] = [W_0^+(\lambda)] = 1$ in the Grothendieck ring.

Let us denote by

$$U_{-s-2m}^{\pm} := \mathbb{C} \cdot |\pm\rangle$$

the 1-dimensional representation with odd generator $|-\rangle$ or even generator $|+\rangle$. (Here $U_p^+ := U_p$) We have

$$U_m^{\epsilon_1} \otimes U_n^{\epsilon_2} \simeq U_{m+n}^{\epsilon_1 \epsilon_2}. \tag{32}$$

Let us introduce the parity element $\sigma := [U_0^-]$ in the Grothendieck ring. Then

$$U_0^- \otimes \mathcal{W}_s^{\pm} \simeq \mathcal{W}_s^{\mp}, \quad U_0^- \otimes U_p^{\pm} \simeq U_p^{\mp}. \tag{33}$$

Hence

$$\sigma[\mathcal{W}_s^{\pm}] = [\mathcal{W}_s^{\mp}], \quad \sigma[U_p^{\pm}] = [U_p^{\mp}], \quad \sigma^2 = 1, \tag{34}$$

and we can rewrite in the Grothendieck ring:

$$\begin{aligned}
[\rho_+(q_*^{\frac{s}{2}+\frac{1}{2}}\lambda)][\rho_-(q_*^{-\frac{s}{2}-\frac{1}{2}}\lambda)] &= \sum_{m=0}^{\infty} [U_{-s-2m} \otimes \mathcal{W}_s^{(-1)^m}(\lambda)] \\
&= \sum_{m=0}^{\infty} \sigma^m [U_{-s-2m}][\mathcal{W}_s^+(\lambda)] \\
&= [\mathcal{W}_s^+(\lambda)] \sum_{m=0}^{\infty} \sigma^m [U_{-s-2m}] \\
&= [\mathcal{W}_s^+(\lambda)] \cdot f_s
\end{aligned}$$

where

$$f_s := \sum_{m=0}^{\infty} \sigma^m [U_{-s-2m}] = [U_{-s}] \sum_{m=0}^{\infty} \sigma^m [U_{-2}]^m = \frac{[U_{-s}]}{1 - \sigma[U_{-2}]}. \tag{35}$$

For simplicity, let us always fix the highest weight of the finite-dimensional module to be even and rewrite $W_s(\lambda) := W_s^+(\lambda)$.

Now from previous observation,

$$[\mathcal{W}_s^+(\lambda)] = [W_s(\lambda)] + [\mathcal{W}_{-s-2}^{(-1)^{s+1}}(\lambda)] = [W_s(\lambda)] + \sigma^{s+1}[\mathcal{W}_{-s-2}^+(\lambda)].$$

Letting $s \mapsto -s-2$, we have

$$[\rho_+(q_*^{-\frac{s}{2}-\frac{1}{2}}\lambda)][\rho_-(q_*^{\frac{s}{2}+\frac{1}{2}}\lambda)] = [\mathcal{W}^+_{-s-2}(\lambda)]\cdot f_{-s-2}.$$

Hence we have

$$\begin{aligned}[W_s(\lambda)] &= [\mathcal{W}^+_s(\lambda)] - \sigma^{s+1}[\mathcal{W}^+_{-s-2}(\lambda)] \\ &= f_s^{-1}[\rho_+(q_*^{\frac{s}{2}+\frac{1}{2}}\lambda)][\rho_-(q_*^{-\frac{s}{2}-\frac{1}{2}}\lambda)] - f_{-s-2}^{-1}\sigma^{s+1}[\rho_+(q_*^{-\frac{s}{2}-\frac{1}{2}}\lambda)][\rho_-(q_*^{\frac{s}{2}+\frac{1}{2}}\lambda)].\end{aligned}$$

In particular, letting $s=0$, we obtain the q-Wronskian identity:

$$1 = [W_0(\lambda)] = f_0^{-1}[\rho_+(q_*^{\frac{1}{2}}\lambda)][\rho_-(q_*^{-\frac{1}{2}}\lambda)] - f_{-2}^{-1}\sigma[\rho_+(q_*^{-\frac{1}{2}}\lambda)][\rho_-(q_*^{\frac{1}{2}}\lambda)]. \tag{36}$$

On the other hand, let us consider the product of $[W_1(\lambda)]$ and $[\rho_+(\lambda)]$. Using (36) with appropriate λ:

$$\begin{aligned}[W_1(\lambda)][\rho_+(\lambda)] &= f_1^{-1}[\rho_+(q_*\lambda)][\rho_-(q_*^{-1}\lambda)][\rho_+(\lambda)] - f_{-3}^{-1}[\rho_+(q_*^{-1}\lambda)][\rho_-(q_*\lambda)][\rho_+(\lambda)] \\ &= f_1^{-1}[\rho_+(q_*\lambda)](f_0 + f_{-2}^{-1} f_0\sigma[\rho_+(q_*^{-1}\lambda)][\rho_-(\lambda)] \\ &\quad - f_{-3}^{-1}[\rho_+(q_*^{-1}\lambda)](f_0^{-1} f_{-2}\sigma[\rho_+(q_*\lambda)][\rho_-(\lambda)] - f_{-2}\sigma) \\ &= f_1^{-1} f_0[\rho_+(q_*\lambda)] + f_1^{-1} f_{-2}^{-1} f_0\sigma[\rho_+(q_*\lambda)][\rho_+(q_*^{-1}\lambda)][\rho_-(\lambda)] \\ &\quad - f_{-3}^{-1} f_0^{-1} f_{-2}\sigma[\rho_+(q_*^{-1}\lambda)][\rho_+(q_*\lambda)][\rho_-(\lambda)] - f_{-3}^{-1} f_{-2}\sigma[\rho_+(q_*^{-1}\lambda)].\end{aligned}$$

Now using

$$\begin{aligned}&f_1^{-1} f_{-2}^{-1} f_0 = f_{-3}^{-1} f_0^{-1} f_{-2} = \frac{1-\sigma[U_{-2}]}{[U_1]} = f_{-1}^{-1} \\ &f_1^{-1} f_0 = [U_1], \quad f_{-3}^{-1} f_{-2} = [U_{-1}],\end{aligned}$$

we get the Baxter relation:

$$[W_1(\lambda)][\rho_+(\lambda)] = [U_1][\rho_+(q_*\lambda)] - \sigma[U_{-1}][\rho_+(q_*^{-1}\lambda)]. \tag{37}$$

Similar relation holds for $[W_1(\lambda)]$ and $[\rho_-(\lambda)]$:

$$[W_1(\lambda)][\rho_-(\lambda)] = [U_1][\rho_-(q_*^{-1}\lambda)] - \sigma[U_{-1}][\rho_-(q_*\lambda)]. \tag{38}$$

4. Transfer Matrices for SCFT

The universal R-matrix for $U_q(C^{(2)}(2))$ belongs to a completion of $\mathcal{U}(\mathfrak{b}_+)\otimes\mathcal{U}(\mathfrak{b}_-)$. In [15] the lower Borel subalgebra $\mathfrak{b}_-$ was represented by means of vertex operators (here we use some rescaling):

$$V_\pm(u) = \int d\theta :e^{\pm\Phi(u,\theta)}: = \mp i\sqrt{2}\xi(u):e^{\pm 2\phi(u)}:,$$

where

$$\Phi(u,\theta) := \phi(u) - \frac{i}{\sqrt{2}}\theta\xi(u) \tag{39}$$

$$\phi(u) := iQ + iPu + \sum_n \frac{a_{-n}}{n} e^{inu}, \qquad \xi(u) := i^{-1/2}\sum_n \xi_n e^{-inu},$$

$$[Q,P] = \frac{ib^2}{2}, \quad [a_n, a_m] = \frac{b^2}{2} n\delta_{n+m,0}, \qquad \{\xi_n, \xi_m\} = b^2\delta_{n+m,0}.$$

$$: e^{\pm\phi(u)} := \exp\Big(\pm\sum_{n=1}^{\infty}\frac{2a_{-n}}{n}e^{inu}\Big)\exp\Big(\pm 2i(Q+Pu)\Big)\exp\Big(\mp\sum_{n=1}^{\infty}\frac{2a_n}{n}e^{-inu}\Big).$$

These are the vertex operators acting in the Fock space and according to their commutation relations, the substitution

$$H_{\alpha_1} \longrightarrow \frac{2P}{b^2}, \quad \mathcal{E}_{-\alpha_1} = \int_0^{2\pi} V_-(u)\mathrm{d}u$$

$$H_{\alpha_0} \longrightarrow -\frac{2P}{b^2}, \quad \mathcal{E}_{-\alpha_0} = \int_0^{2\pi} V_+(u)\mathrm{d}u$$

gives rise to a representation of the lower Borel subalgebra $\mathfrak{b}_-$ with $q = e^{\pi i b^2}$.

The R-matrix with $\mathfrak{b}_-$ represented as above and $\mathfrak{b}_+$ as in $W_s(\lambda)$ has the form

$$\mathbf{L}_s(\lambda) = e^{\pi i P\mathcal{H}}\mathrm{Pexp}^{(q)}\int_0^{2\pi}(\lambda V_-(u)\mathcal{E} + \lambda V_+(u)\mathcal{F})\mathrm{d}u \tag{40}$$

The letter q over the path-ordered exponential (Pexp) means certain regularization procedure, which preserves the property of Pexp (see [15] for more details).

Similarly, one can consider operators $\mathbf{L}_\pm(\lambda)$, where the upper Borel algebra $\mathfrak{b}_+$ is represented via $\rho_\pm(\lambda)$:

$$\mathbf{L}_\pm(\lambda) = e^{\pm\pi i P\mathcal{H}}\mathrm{Pexp}^{(q)}\int_0^{2\pi}(\lambda V_-(u)\alpha_\pm + \lambda V_+(u)\alpha_\mp)\mathrm{d}u \tag{41}$$

Then define

$$\begin{aligned} &\mathbf{T}_s(\lambda) := \mathrm{sTr}(e^{\pi i P\mathcal{H}}\mathbf{L}_s(\lambda)), \quad \mathbf{T}_s^+(\lambda) := \mathrm{sTr}(e^{\pi i P\mathcal{H}}\mathbf{L}_s(\lambda)) \\ &\widetilde{\mathbf{Q}}_\pm(\lambda) := \mathrm{sTr}(e^{\pm\pi i P\mathcal{H}}\mathbf{L}_\pm(\lambda)), \end{aligned} \tag{42}$$

where we consider the highest weight vector in $W_s(\lambda)$, $\rho_\pm(\lambda)$ to be even, and take the supertrace of the representation of the second tensor factor. (We ignore the convergence of the trace here, treating it as formal series in λ.)

Then from the previous decomposition and the properties of the supertrace

$$
\begin{aligned}
&\widetilde{\mathbf{Q}}_+(q_*^{\frac{s}{2}+\frac{1}{2}}\lambda)\widetilde{\mathbf{Q}}_-(q_*^{-\frac{s}{2}-\frac{1}{2}}\lambda)\\
&\quad= \mathrm{sTr}(\mathrm{e}^{\pi i P\mathcal{H}}\mathbf{L}_+(q_*^{\frac{s}{2}+\frac{1}{2}}\lambda))\mathrm{sTr}(\mathrm{e}^{-\pi i P\mathcal{H}}\mathbf{L}_-(q_*^{-\frac{s}{2}-\frac{1}{2}}\lambda))\\
&\quad= \mathrm{sTr}_{\rho_+(q_*^{\frac{s}{2}+\frac{1}{2}}\lambda)}(\mathrm{e}^{\pi i PH}\mathbf{R})\mathrm{sTr}_{\rho_-(q_*^{-\frac{s}{2}-\frac{1}{2}}\lambda)}(\mathrm{e}^{\pi i PH}\mathbf{R})\\
&\quad= \sum_{m=0}^{\infty}\mathrm{sTr}_{\mathcal{W}_s^+(\lambda)}(\mathrm{e}^{\pi i PH}\mathbf{R}))\mathrm{sTr}_{U_{-s-2m}^{(-1)^m}}(\mathrm{e}^{\pi i PH}\mathbf{R})\\
&\quad= \sum_{m=0}^{\infty}\mathrm{sTr}(\mathrm{e}^{\pi i P\mathcal{H}}\mathbf{L}_+(\lambda))\mathrm{sTr}_{U_{-s-2m}^{(-1)^m}}(\mathrm{e}^{\pi i PH}\mathbf{R})\\
&\quad= \sum_{m=0}^{\infty}\mathbf{T}_s^+(\lambda)\mathrm{sTr}_{U_{-s-2m}^{(-1)^m}}(\mathrm{e}^{2\pi i P\mathcal{H}})\\
&\quad= \sum_{m=0}^{\infty}(\mathbf{T}_s(\lambda)+(-1)^{s+1}\mathbf{T}_{-s-2}^+(\lambda))\mathrm{sTr}_{U_{-s-2m}^{(-1)^m}}(\mathrm{e}^{2\pi i P\mathcal{H}})\\
&\quad= (\mathbf{T}_s(\lambda)+(-1)^{s+1}\mathbf{T}_{-s-2}^+(\lambda))\sum_{m=0}^{\infty}(-1)^m\mathrm{e}^{2\pi i P_*(-s-2m)}\\
&\quad= \frac{\mathrm{e}^{-2\pi i P_*(s-1)}}{2\cos(2\pi P_*)}(\mathbf{T}_s(\lambda)+(-1)^{s+1}\mathbf{T}_{-s-2}^+(\lambda)),
\end{aligned}
$$

where $P_* = \frac{b_*^2}{b^2}P$. Define the rescaled operator

$$
\mathbf{Q}_\pm(\lambda) := 2\cos(2\pi P_*)\mathrm{e}^{2\pi i P_*}(\lambda)^{\pm\frac{2P_*}{b_*^2}}\widetilde{\mathbf{Q}}_\pm(\lambda). \tag{43}
$$

Then

$$
\mathbf{Q}_+(q_*^{\frac{s}{2}+\frac{1}{2}}\lambda)\mathbf{Q}_-(q_*^{-\frac{s}{2}-\frac{1}{2}}\lambda) = 2\cos(2\pi P_*)(\mathbf{T}_s(\lambda)+(-1)^{s+1}\mathbf{T}_{-s-2}^+(\lambda)).
$$

Together with the other relation by substituting $s \longrightarrow -s-2$:

$$
\mathbf{Q}_+(q_*^{-\frac{s}{2}-\frac{1}{2}}\lambda)\mathbf{Q}_-(q_*^{\frac{s}{2}+\frac{1}{2}}\lambda) = 2\cos(2\pi P_*)\mathbf{T}_{-s-2}^+(\lambda),
$$

we have

$$
2\cos(2\pi P_*)\mathbf{T}_s(\lambda) = \mathbf{Q}_+(q_*^{\frac{s}{2}+\frac{1}{2}}\lambda)\mathbf{Q}_-(q_*^{-\frac{s}{2}-\frac{1}{2}}\lambda) + (-1)^s\mathbf{Q}_+(q_*^{-\frac{s}{2}-\frac{1}{2}}\lambda)\mathbf{Q}_-(q_*^{\frac{s}{2}+\frac{1}{2}}\lambda). \tag{44}
$$

In particular, we obtain the quantum super-Wronskian relation:

$$
2\cos(2\pi P_*) = \mathbf{Q}_+(q_*^{\frac{1}{2}}\lambda)\mathbf{Q}_-(q_*^{-\frac{1}{2}}\lambda) + \mathbf{Q}_+(q_*^{-\frac{1}{2}}\lambda)\mathbf{Q}_-(q_*^{\frac{1}{2}}\lambda). \tag{45}
$$

The Baxter T–Q relations for **Q**-operator follow from previous section:

$$
\mathbf{T}_1(\lambda)\cdot\mathbf{Q}_\pm(\lambda) = \pm\mathbf{Q}_\pm(q_*\lambda) \mp \mathbf{Q}_\pm(q_*^{-1}\lambda). \tag{46}
$$

The fusion relation, which follows from the quantum super-Wronskian relation, is

$$\mathbf{T}_s(q_*^{\frac{1}{2}}\lambda)\mathbf{T}_s(q_*^{-\frac{1}{2}}\lambda) = \mathbf{T}_{s+1}(\lambda)\mathbf{T}_{s-1}(\lambda) + (-1)^s. \tag{47}$$

This relation is similar to the one considered in [15], but now all the transfer matrices correspond to the representations of $U_q(C_q^{(2)}(2))$. In particular,

$$\mathbf{T}_2(\lambda) = \mathbf{T}_1(q_*^{\frac{1}{2}}\lambda)\mathbf{T}_1(q_*^{-\frac{1}{2}}\lambda) + 1. \tag{48}$$

Therefore

$$n\mathbf{T}_2(q_*^{\frac{1}{2}}\lambda) = \mathbf{T}_1(q_*\lambda)\mathbf{T}_1(\lambda) + 1,$$

so that the Baxter relation for $\mathbf{T}_2$ is as follows.

$$\begin{aligned}\mathbf{T}_2(q_*^{\frac{1}{2}}\lambda)\mathbf{Q}_\pm(\lambda) &= \mathbf{Q}_\pm(\lambda) + \mathbf{T}_1(q_*\lambda)(\pm\mathbf{Q}_\pm(q_*\lambda) \mp \mathbf{Q}_\pm(q_*^{-1}\lambda)) \\ &= \mathbf{Q}_\pm(\lambda) + \mathbf{Q}_\pm(q_*^2\lambda) - \mathbf{Q}_\pm(\lambda) \mp \mathbf{T}_1(q_*\lambda)\mathbf{Q}_\pm(q_*^{-1}\lambda) \\ &= \mathbf{Q}_\pm(q_*^2\lambda) \mp \mathbf{T}_1(q_*\lambda)\mathbf{Q}_\pm(q_*^{-1}\lambda)\end{aligned}$$

Moreover, one can write down the expression for each $\mathbf{T}_s$ in terms of either one of $\mathbf{Q}_\pm(\lambda)$ using the quantum super-Wronskian relation:

$$\mathbf{T}_s(\lambda) = \mathbf{Q}_\pm(q_*^{\frac{s}{2}+\frac{1}{2}}\lambda)\mathbf{Q}_\pm(q_*^{-\frac{s}{2}-\frac{1}{2}}\lambda) \sum_{k=-s/2}^{s/2} \frac{(-1)^{(k\pm\frac{s}{2})}}{\mathbf{Q}_\pm(q_*^{k+\frac{1}{2}}\lambda)\mathbf{Q}_\pm(q_*^{k-\frac{1}{2}}\lambda)} \tag{49}$$

The $\mathbf{T}_2$-transfer matrix has a classical limit of the trace of monodromy matrix for super-KdV equation. The asymptotic expansion of it should produce both local and nonlocal integrals of motion for superconformal field theory (SCFT). We suppose that operators $\mathbf{Q}_\pm(\lambda)$ possess nice analytic properties as in the $A_1^{(1)}$ case [3].

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A. Appendix

Let us introduce the q-numbers:

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$$

such that

$$[n]_{q_*} = \frac{q_*^n - q_*^{-n}}{q_* - q_*^{-1}} = i^{n-1}\frac{q^{-n} - (-1)^n q^n}{q + q^{-1}} = i^{n-1}\{n\}_q$$

with the usual notation in superalgebra

$$\{n\}_q := \frac{q^{-n} - (-1)^n q^n}{q + q^{-1}}.$$

A.1. R-MATRIX FOR $U_q(\mathfrak{osp}(2|1))$

Let us prove Theorem 2.6 that the universal R matrix is given by

$$\mathbf{R} = Q\mathcal{R}, \tag{50}$$

where $Q = Cq_*^{\frac{H\otimes H}{2}}$ with $C = \frac{1}{2}(1\otimes 1 + i\xi\eta\otimes 1 + 1\otimes i\xi\eta + \xi\eta\otimes\xi\eta)$ such that

$$C\cdot|(-1)^{\epsilon_1}\rangle\otimes|(-1)^{\epsilon_2}\rangle = (-1)^{\epsilon_1\epsilon_2}|(-1)^{\epsilon_1}\rangle\otimes|(-1)^{\epsilon_2}\rangle, \quad \epsilon_i\in\{0,1\},$$

and

$$\mathcal{R} = \exp_{q_*^{-2}}(i(q_*^{-1} - q_*)\mathcal{E}\otimes\mathcal{F}) = \exp_{-q^{-2}}(-(q+q^{-1})\mathcal{E}\otimes\mathcal{F}) = \sum a_n\mathcal{E}^n\otimes\mathcal{F}^n.$$

Note that using

$$\lceil n\rceil_{q_*^{-2}} = (-q)^{1-n}\{n\}_q,$$

we have

$$a_n = (-1)^n q^{\frac{1}{2}n(n-1)}\frac{(q+q^{-1})^n}{\{n\}_q!}.$$

By definition a_n satisfies

$$\frac{a_n}{a_{n-1}} = -q^n\frac{(1+q^{-2})}{\{n\}_q}. \tag{51}$$

The properties of an R-matrix states that

$$\Delta^{op}(X)\mathbf{R} = \mathbf{R}\Delta(X), \quad X\in\mathcal{U}_q\mathfrak{osp}(2|1), \tag{52}$$

i.e. on the generators we have

$$(\mathcal{K}\otimes\mathcal{E}+\mathcal{E}\otimes 1)\mathcal{R}=\mathcal{R}(1\otimes\mathcal{E}+\mathcal{E}\otimes\mathcal{K}) \tag{53}$$

$$(1\otimes\mathcal{F}+\mathcal{F}\otimes\mathcal{K}^{-1})\mathcal{R}=\mathcal{R}(\mathcal{K}^{-1}\otimes\mathcal{F}+\mathcal{F}\otimes 1) \tag{54}$$

$$(\mathcal{K}\otimes\mathcal{K})\mathcal{R}=\mathcal{R}(\mathcal{K}\otimes\mathcal{K}) \tag{55}$$

To prove that R satisfies the properties of the R-matrix, one checks that

$$(\mathcal{K}\otimes\mathcal{E})Q=Q(1\otimes\mathcal{E}) \tag{56}$$

$$(\mathcal{E}\otimes 1)Q=Q(\mathcal{E}\otimes\mathcal{K}^{-1}) \tag{57}$$

which follows easily from the commutation relations of the Clifford algebra, and

$$(1\otimes\mathcal{E}+\mathcal{E}\otimes\mathcal{K}^{-1})\mathcal{R}=\mathcal{R}(1\otimes\mathcal{E}+\mathcal{E}\otimes\mathcal{K}) \tag{58}$$

The calculation for $\mathcal{F}$ is similar, while the relation for $\mathcal{K}$ is trivial since it commutes with every term. Using

$$\mathcal{E}\mathcal{F}^n-(-1)^n\mathcal{F}^n\mathcal{E}=\frac{q^n\{n\}_q}{1+q^2}\mathcal{K}\mathcal{F}^{n-1}+\frac{(-1)^nq^{-n}\{n\}_q}{1+q^{-2}}K^{-1}\mathcal{F}^{n-1}, \tag{59}$$

we have

$$(\mathcal{E}\otimes\mathcal{K}^{-1})(\mathcal{E}^n\otimes\mathcal{F}^n)-(\mathcal{E}^n\otimes\mathcal{F}^n)(\mathcal{E}\otimes\mathcal{K})=\mathcal{E}^{n+1}\otimes\mathcal{K}^{-1}\mathcal{F}^n-(-1)^nq^{2n}\mathcal{E}^{n+1}\otimes\mathcal{K}\mathcal{F}^n$$

$$\begin{aligned}(1\otimes\mathcal{E})(\mathcal{E}^n\otimes\mathcal{F}^n)-(\mathcal{E}^n\otimes\mathcal{F}^n)(1\otimes\mathcal{E})&=(-1)^n\mathcal{E}^n\otimes\mathcal{E}\mathcal{F}^n-\mathcal{E}^n\otimes\mathcal{F}^n\mathcal{E}\\&=(-1)^n\frac{q^n\{n\}_q}{1+q^2}\mathcal{E}^n\otimes\mathcal{K}\mathcal{F}^{n-1}\\&\quad+\frac{q^{-n}\{n\}_q}{1+q^{-2}}\mathcal{E}^n\otimes K^{-1}\mathcal{F}^{n-1}\end{aligned}$$

Hence adding up both sides, we need $a_0=1$ and

$$\begin{aligned}a_{n-1}+\frac{q^{-n}\{n\}_q}{1+q^{-2}}a_n&=0\\(-1)^{n-1}q^{2(n-1)}a_{n-1}-(-1)^n\frac{q^n\{n\}_q}{1+q^2}a_n&=0\end{aligned}$$

both of which is equivalent to

$$\frac{a_n}{a_{n-1}}=-q^n\frac{(1+q^{-2})}{\{n\}_q}$$

as required.

By writing formally

$$K=q^{H'}=i\xi\eta q_*^H,$$

the following proposition shows that up to a constant, the Cartan part of the universal R-matrix using the Clifford generators coincides with the usual expression.

PROPOSITION A.1. *On the space* $W_{s_1}^{\pm_1}\otimes W_{s_2}^{\pm_2}$, *we have the action*

$$q^{\frac{H'\otimes H'}{2}}=(-1)^{-l_1 l_2}\widetilde{q}Cq_*^{\frac{H\otimes H}{2}}, \tag{60}$$

where C is the Clifford part $\frac{1}{2}(1\otimes 1+i\xi\eta\otimes 1+1\otimes i\xi\eta+\xi\eta\otimes\xi\eta)$ *and* H' *reproduces the action of K on* $W_s^{\pm}$*:*

$$H'=\begin{cases} H-l\frac{\pi i}{\ln q} & +\\ H-(l+1)\frac{\pi i}{\ln q} & - \end{cases}$$

with $s=2l$.

Proof. For simplicity, consider the action on the basis $e_m^{l_1}\otimes e_n^{l_2}\in W_{s_1}^+\otimes W_{s_2}^+$. The action on other parity is similar. Then we have

$$\begin{aligned} q^{\frac{H'\otimes H'}{2}} &= q^{\frac{H\otimes H}{2}}(i^{-l_2 H}\otimes 1)(1\otimes i^{-l_1 H})\widetilde{q}\\ &= q^{2mn}(-1)^{-l_2 m-l_1 n}\widetilde{q} \end{aligned}$$

while

$$\begin{aligned} Cq_*^{\frac{H\otimes H}{2}} &= (-1)^{(l_1-m)(l_2-n)}q_*^{2mn}\\ &= (-1)^{-mn}q^{2mn}(-1)^{(l_1-m)(l_2-n)}\\ &= (-1)^{l_1 l_2-l_2 m-l_1 n}q^{2mn} \end{aligned}$$

□

A.2. UNIVERSAL R MATRIX FOR $U_q(C^{(2)}(2))$

Recall from (19) that we have rescaled our generator $\mathcal{F}_i$ from the usual definition by $c=\frac{q+q^{-1}}{q-q^{-1}}$. Hence modifying the constants from [11,12] accordingly, the universal R matrix in general is of the form

$$\mathbf{R}=Q\mathcal{R}_{>0}\mathcal{R}_0\mathcal{R}_{<0}, \tag{61}$$

where

$$Q=q^{\frac{H_1\otimes H_1}{2}+H_\delta\otimes H_d+H_d\otimes H_\delta}, \tag{62}$$

with $H_\delta=H_0+H_1$ and H_d the extended generators such that

$$[H_d,\mathcal{E}_0]=\mathcal{E}_0,\quad [H_d,\mathcal{E}_1]=0,$$

and

$$\mathcal{R}_{>0} = \prod_{n\geq 0} \exp_{-q^{-2}}((-1)^{n+1}(q^{-1}+q)\mathcal{E}_{\alpha+n\delta}\otimes\mathcal{F}_{\alpha+n\delta}),$$
$$\mathcal{R}_{<0} = \prod_{n\geq 0} \exp_{-q^{-2}}((-1)^{n+1}(q^{-1}+q)\mathcal{E}_{\delta-\alpha+n\delta}\otimes\mathcal{F}_{\delta-\alpha+n\delta}),$$
$$\mathcal{R}_0 = \exp\left(\sum_{n>0}\frac{n(q+q^{-1})^2}{q^{2n}-q^{-2n}}\mathcal{E}_{n\delta}\otimes\mathcal{F}_{n\delta}\right),$$

where the imaginary generators $\mathcal{E}_{n\delta\pm\alpha}, \mathcal{F}_{n\delta\pm\alpha}$ are defined below.

PROPOSITION A.2. *The Cartan term can be replaced using the Clifford part:*

$$Q = Cq_*^{\frac{H_1\otimes H_1}{2}+H_\delta\otimes H_d+H_d\otimes H_\delta}. \tag{63}$$

Proof. We just need to check that the following same commutation holds:

$$(\mathcal{E}_1\otimes 1)Q = (\mathcal{E}_1\otimes\mathcal{K}_1^{-1}), \quad (\mathcal{E}_0\otimes 1)Q = (\mathcal{E}_0\otimes\mathcal{K}_0^{-1})$$
$$(\mathcal{F}_1\otimes\mathcal{K}_1^{-1})Q = (\mathcal{F}_1\otimes 1), \quad (\mathcal{F}_0\otimes\mathcal{K}_0^{-1})Q = (\mathcal{F}_0\otimes 1)$$
$$(\mathcal{K}_1\otimes\mathcal{E}_1)Q = (1\otimes\mathcal{E}_1), \quad (\mathcal{K}_0\otimes\mathcal{E}_0)Q = (1\otimes\mathcal{E}_0)$$
$$(1\otimes\mathcal{F}_1)Q = (\mathcal{K}_1\otimes\mathcal{F}_1), \quad (1\otimes\mathcal{F}_0)Q = (\mathcal{K}_0\otimes\mathcal{F}_0)$$

Then it follows that the Clifford part C commutes correctly with the odd elements because $\mathcal{E}_i = E_i\xi, \mathcal{F}_i = F_i\eta$ and $\mathcal{K}_i = K_i i\xi\eta$ as before, and the even part follows from the relation of $U_{q^*}(A_1^{(1)})$. □

Let us define the following notations for the generators:

$$\mathcal{E}_1 := \mathcal{E}_\alpha, \quad \mathcal{E}_0 := \mathcal{E}_{\delta-\alpha}$$
$$\mathcal{F}_1 := \mathcal{F}_\alpha, \quad \mathcal{F}_0 := \mathcal{F}_{\delta-\alpha}$$
$$\mathcal{K}_1 := \mathcal{K}_\alpha, \quad \mathcal{K}_0 := \mathcal{K}_{\delta-\alpha}$$
$$\mathcal{K}_\delta := \mathcal{K}_\alpha\mathcal{K}_{\delta-\alpha}.$$

Then using

$$[e_\beta, e_{\beta'}]_q := e_\beta e_{\beta'} - (-1)^{\theta(\beta)\theta(\beta')}q^{(\beta,\beta')}e_{\beta'}e_\beta, \tag{64}$$

where $\theta(\beta)$ is the parity of e_β, we define

$$\mathcal{E}_\delta := [\mathcal{E}_\alpha, \mathcal{E}_{\delta-\alpha}]_q = \mathcal{E}_1\mathcal{E}_0 + q^{-2}\mathcal{E}_0\mathcal{E}_1,$$
$$\mathcal{F}_\delta := [\mathcal{F}_{\delta-\alpha}, \mathcal{F}_\alpha]_{q^{-1}} = \mathcal{F}_0\mathcal{F}_1 + q^2\mathcal{F}_1\mathcal{F}_0.$$

Both $\mathcal{E}_\delta, \mathcal{F}_\delta$ are even.

Next we define

$$\begin{aligned}
\mathcal{E}_{n\delta+\alpha} &:= \frac{1}{q-q^{-1}}[\mathcal{E}_{(n-1)\delta+\alpha}, \mathcal{E}_\delta],\\
\mathcal{F}_{n\delta+\alpha} &:= \frac{1}{q-q^{-1}}[\mathcal{F}_\delta, \mathcal{F}_{(n-1)\delta+\alpha}],\\
\mathcal{E}_{(n+1)\delta-\alpha} &:= \frac{1}{q-q^{-1}}[\mathcal{E}_\delta, \mathcal{E}_{n\delta-\alpha}],\\
\mathcal{F}_{(n+1)\delta-\alpha} &:= \frac{1}{q-q^{-1}}[\mathcal{F}_{n\delta-\alpha}, \mathcal{F}_\delta].
\end{aligned}$$

These are all odd.

The pure imaginary roots are harder to define. First we define

$$\begin{aligned}
&\mathcal{E}'_{n\delta} := [\mathcal{E}_\alpha, \mathcal{E}_{n\delta-\alpha}]_q = \mathcal{E}_\alpha\mathcal{E}_{n\delta-\alpha} + q^{-2}\mathcal{E}_{n\delta-\alpha}\mathcal{E}_\alpha,\\
&\mathcal{F}'_{n\delta} := [\mathcal{F}_{n\delta-\alpha}, \mathcal{F}_\alpha]_{q^{-1}} = \mathcal{F}_{n\delta-\alpha}\mathcal{F}_\alpha + q^2\mathcal{F}_\alpha\mathcal{F}_{n\delta-\alpha}.
\end{aligned}$$

Note that $\mathcal{E}'_\delta = \mathcal{E}_\delta$, $\mathcal{F}'_\delta = \mathcal{F}_\delta$. Then the pure imaginary root vectors are defined recursively by

$$\begin{aligned}
\mathcal{E}_{n\delta} &= \sum_{p_1+2p_2+\cdots+np_n=n} \frac{(q-q^{-1})^{\sum p_i - 1}(\sum p_i - 1)!}{p_1!\cdots p_n!}(\mathcal{E}'_\delta)^{p_1}\cdots(\mathcal{E}'_{n\delta})^{p_n},\\
\mathcal{F}_{n\delta} &= \sum_{p_1+2p_2+\cdots+np_n=n} \frac{(q^{-1}-q)^{\sum p_i - 1}(\sum p_i - 1)!}{p_1!\cdots p_n!}(\mathcal{F}'_{n\delta})^{p_n}\cdots(\mathcal{F}'_\delta)^{p_1}.
\end{aligned}$$

More explicitly, using generating functions:

$$\begin{aligned}
\mathbf{E}'(u) &:= -(q+q^{-1})\sum_{n\geq 1}\mathcal{E}'_{n\delta}u^{-n},\\
\mathbf{E}(u) &:= -(q+q^{-1})\sum_{n\geq 1}\mathcal{E}_{n\delta}u^{-n},\\
\mathbf{F}'(u) &:= (q+q^{-1})\sum_{n\geq 1}\mathcal{F}'_{n\delta}u^{-n},\\
\mathbf{F}(u) &:= (q+q^{-1})\sum_{n\geq 1}\mathcal{F}_{n\delta}u^{-n},
\end{aligned}$$

we have

$$\mathbf{E}'(u) = -1 + \exp\mathbf{E}(u), \quad \mathbf{E}(u) = \ln(1+\mathbf{E}'(u))$$

and similarly for $\mathbf{F}(u)$.

PROPOSITION A.3. *We have the following action of the non-simple generators on* $W_s^{\pm}(\lambda)$*:*

$$\begin{aligned}
\mathcal{E}_\delta \cdot e^l_{m,\pm} &= \lambda^2 q_*^{-m-1}\left(q_*^l[l+m+1]_{q_*} - q_*^{-l}[l-m+1]_{q_*}\right) e^l_{m,\pm}\\
\mathcal{F}_\delta \cdot e^l_{m,\pm} &= \lambda^{-2} q_*^{m+1}\left(q_*^l[l-m+1]_{q_*} - q_*^{-l}[l+m+1]_{q_*}\right) e^l_{m,\pm}\\
\mathcal{E}_{n\delta+\alpha} \cdot e^l_{m,\pm} &= i^n\lambda^{2n+1} q_*^{-2n(m+1)}[l-m]_{q_*} e^l_{m+1,\mp}\\
\mathcal{E}_{(n+1)\delta-\alpha} \cdot e^l_{m,\pm} &= i^n\lambda^{2n+1} q_*^{-2nm}[l+m]_{q_*} e^l_{m-1,\mp}\\
\mathcal{F}_{n\delta+\alpha} \cdot e^l_{m,\pm} &= \pm i^{n-1}\lambda^{-2n-1} q_*^{2nm}[l+m]_{q_*} e^l_{m-1,\mp}\\
\mathcal{F}_{(n+1)\delta-\alpha} \cdot e^l_{m,\pm} &= \pm i^{n-1}\lambda^{-2n-1} q_*^{2n(m+1)}[l-m]_{q_*} e^l_{m+1,\mp}\\
\mathcal{E}'_{n\delta} \cdot e^l_{m,\pm} &= i^{n-1}\lambda^{2n} q_*^{-2(n-1)m}\Big([l+m]_{q_*}[l-m+1]_{q_*}\\
&\quad - q_*^{-2n}[l-m]_{q_*}[l+m+1]_{q_*}\Big) e^l_{m,\pm}\\
\mathcal{F}'_{n\delta} \cdot e^l_{m,\pm} &= i^{n-1}\lambda^{-2n} q_*^{2(n-1)m}\Big([l+m]_{q_*}[l-m+1]_{q_*}\\
&\quad - q_*^{2n}[l-m]_{q_*}[l+m+1]_{q_*}\Big) e^l_{m,\pm}.
\end{aligned}$$

By the generating functions, we get

$$\begin{aligned}
\mathcal{E}_{n\delta} \cdot e^l_{m,\pm} &= i^{n-1}\frac{\lambda^{2n}}{n} N(l,m,n,q_*) e^l_{m,\pm},\\
\mathcal{F}_{n\delta} \cdot e^l_{m,\pm} &= i^{n-1}\frac{\lambda^{-2n}}{n} N(l,m,n,q_*^{-1}) e^l_{m,\pm},
\end{aligned}$$

where

$$\begin{aligned}
N(l,m,n,q) &:= q^{-n(m+1)}(q^{n(l+1)}[n(l+m)]_q - q^{-n(l+1)}[n(l-m)]_q)\\
&= \frac{q^{2nl} + q^{-2n(l+1)} - q^{-2nm} - q^{-2n(m+1)}}{q - q^{-1}}.
\end{aligned}$$

THEOREM A.4. *We have the following expression for R:*

$$\boldsymbol{R} = Q\mathcal{R}_{>0}\mathcal{R}_0\mathcal{R}_{<0},$$

where the matrix coefficients of each component are given below expressed only in terms of q_**:*

- *The matrix coefficients of* $\mathcal{R}_{>0}$ *is given by:*

$$\langle e^{l_1}_{m'_1,\epsilon'_1} \otimes e^{l_1}_{m'_2,\epsilon'_2} | \mathcal{R}_{>0} | e^{l_1}_{m_1,\epsilon_1} \otimes e^{l_2}_{m_2,\epsilon_2}\rangle = 0$$

if $m'_1 - m_1 \neq m_2 - m'_2$ *or* $m'_1 - m_1 = m_2 - m'_2 < 0$.

Otherwise let $n=m_1'-m_1$, *we have*

$$\begin{aligned}&\langle e^{l_1}_{m_1',\epsilon_1'}\otimes e^{l_2}_{m_2',\epsilon_2'}|\mathcal{R}_{>0}|e^{l_1}_{m_1,\epsilon_1}\otimes e^{l_2}_{m_2,\epsilon_2}\rangle\\&=\frac{(-1)^{n(\epsilon_1+\epsilon_2-1)}(q_*-q_*^{-1})^n(\lambda_1\lambda_2)^n}{\prod_{k=1}^n(\lambda_2^2-q_*{}^{2m_2-2m_1-2k}\lambda_1^2)}\frac{[l_1-m_1]_{q_*}!}{\lfloor n\rfloor_{q_*}![l_1-m_1-n]_{q_*}!}\frac{[l_2+m_2]_{q_*}!}{[l_2+m_2-n]_{q_*}!}\end{aligned}$$

where $\lfloor n\rfloor_{q_*}=\frac{1-q_*^{-2n}}{1-q_*^{-2}}$.

- *Similarly, the matrix coefficients of* $\mathcal{R}_{<0}$ *are given by*

$$\langle e^{l_1}_{m_1',\epsilon_1'}\otimes e^{l_2}_{m_2',\epsilon_2'}|\mathcal{R}_{<0}|e^{l_1}_{m_1,\epsilon_1}\otimes e^{l_2}_{m_2,\epsilon_2}\rangle=0$$

if $m_1'-m_1\neq m_2-m_2'$ *or* $m_1'-m_1=m_2-m_2'>0$.
Otherwise let $n=m_1-m_1'$, *we have*

$$\begin{aligned}&\langle e^{l_1}_{m_1',\epsilon_1'}\otimes e^{l_2}_{m_2',\epsilon_2'}|\mathcal{R}_{<0}|e^{l_1}_{m_1,\epsilon_1}\otimes e^{l_2}_{m_2,\epsilon_2}\rangle=0\\&=\frac{(-1)^{n(\epsilon_1+\epsilon_2-1)}(q_*-q_*^{-1})^n(\lambda_1\lambda_2)^n}{\prod_{k=1}^n(\lambda_2^2-q_*{}^{2m_2-2m_1+2(k+n-1)}\lambda_1^2)}\frac{[l_1+m_1]_{q_*}!}{\lfloor n\rfloor_{q_*}![l_1+m_1-n]_{q_*}!}\frac{[l_2-m_2]_{q_*}!}{[l_2-m_2-n]_{q_*}!}\end{aligned}$$

- *The matrix coefficients of* $\mathcal{R}_0$ *are given by*

$$\begin{aligned}\mathcal{R}_0(e^{l_1}_{m_1,\epsilon_1}\otimes e^{l_2}_{m_2,\epsilon_2})=f_q\cdot\prod_{k=1}^{l_1+m_1}\frac{\lambda_2^2-\lambda_1^2q_*^{2l_1+2l_2-2k+2}}{\lambda_2^2-\lambda_1^2q_*^{2m_2-2m_1+2k}}\\\times\prod_{k=1}^{l_2+m_2}\frac{\lambda_2^2-\lambda_1^2q_*^{2m_2-2m_1-2k}}{\lambda_2^2-\lambda_1^2q_*^{-2l_1-2l_2+2k-2}}e^{l_1}_{m_1,\epsilon_1}\otimes e^{l_2}_{m_2,\epsilon_2},\end{aligned}$$

where

$$\begin{aligned}f_q(l_1,\lambda_1,l_2,\lambda_2)&=\exp\left(\sum_{n>0}\frac{1}{n}\left(\frac{\lambda_1}{\lambda_2}\right)^{2n}\frac{(q_*^{2l_1n}-q_*{}^{-2l_1n})(q_*^{2l_2n}-q_*{}^{-2l_2n})}{q_*^{2n}-q_*^{-2n}}\right)\\&=\exp\left(\sum_{n>0}\frac{1}{n}\left(\frac{\lambda_1}{\lambda_2}\right)^{2n}[2l_1]_{q_*^n}[2l_2]_{q_*^n}\frac{q_*^n-q_*^{-n}}{q_*^n+q_*^{-n}}\right).\end{aligned}$$

- *Finally, the action of* Q *is given by*

$$Q(e^{l_1}_{m_1,\epsilon_1}\otimes e^{l_2}_{m_2,\epsilon_2})=(-1)^{\epsilon_1\epsilon_2}q_*^{2m_1m_2}e^{l_1}_{m_1,\epsilon_1}\otimes e^{l_2}_{m_2,\epsilon_2}.$$

EXAMPLE A.5. When $l_1=l_2=\frac{1}{2}$, we have

$$\mathbf{R}_{\frac12,\frac12}(\lambda_1,\lambda_2)=q_*^{\frac12}f_{q_*}\begin{pmatrix}1&0&0&0\\0&\frac{\lambda_1^2-\lambda_2^2}{\lambda_1^2q_*^{-1}-\lambda_2^2q_*}&\frac{\lambda_1\lambda_2(q_*^{-1}-q_*)}{\lambda_1^2q_*^{-1}-\lambda_2^2q_*}&0\\0&\frac{\lambda_1\lambda_2(q_*^{-1}-q_*)}{\lambda_1^2q_*^{-1}-\lambda_2^2q_*}&\frac{\lambda_1^2-\lambda_2^2}{\lambda_1^2q_*^{-1}-\lambda_2^2q_*}&0\\0&0&0&-1\end{pmatrix}$$

where

$$f_{q_*}(\lambda_1,\lambda_2):=\exp\left(\sum_{n>0}\frac{1}{n}\left(\frac{\lambda_1}{\lambda_2}\right)^{2n}\frac{q_*^n-q_*^{-n}}{q_*^n+q_*^{-n}}\right).$$

Note that up to a constant we recover our previous formula (21).

EXAMPLE A.6. Using Theorem A.4, we found for example that the universal R matrix acting on $W_2^+\otimes W_2^+$ is given by

$$\mathbf{R}_{1,1}(\lambda_1,\lambda_2)=\frac{q_*^2 f_q}{a}\begin{pmatrix} a & . & . & . & . & . & . & . & . \\ . & b & . & d & . & . & . & . & . \\ . & . & c & . & f & . & g & . & . \\ . & d & . & b & . & . & . & . & . \\ . & . & -h & . & -e & . & -h & . & . \\ . & . & . & . & . & b & . & d & . \\ . & . & g & . & f & . & c & . & . \\ . & . & . & . & . & d & . & b & . \\ . & . & . & . & . & . & . & . & a \end{pmatrix}$$

where $\lambda_1=\mathrm{e}^{x_1}$, $\lambda_2=\mathrm{e}^{x_2}$,

$$\begin{aligned}
a&=4\sinh(x_1-x_2-\ln q_*)\sinh(x_1-x_2-2\ln q_*)\\
b&=4\sinh(x_1-x_2)\sinh(x_1-x_2-\ln q_*)\\
c&=4\sinh(x_1-x_2)\sinh(x_1-x_2+\ln q_*)\\
d&=-4\sinh(x_1-x_2-\ln q_*)\sinh(2\ln q_*)\\
e&=2\cosh(2x_1-2x_2-\ln q_*)-4\cosh(\ln q_*)+2\cosh(3\ln q_*)\\
f&=4q_*^{-1}\sinh(x_1-x_2)\sinh(\ln q_*)\\
g&=4\sinh(\ln q_*)\sinh(2\ln q_*)\\
h&=8q_*\sinh(x_1-x_2)\cosh(\ln q_*)\sinh(2\ln q_*)
\end{aligned}$$

and all other entries are zero.

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