

ON TENSOR PRODUCTS OF POSITIVE REPRESENTATIONS OF SPLIT REAL QUANTUM BOREL SUBALGEBRA $\mathcal{U}_{q\bar{q}}(\mathfrak{b}_{\mathbb{R}})$

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ABSTRACT. We study the positive representations \mathcal{P}_{λ} of split real quantum groups $\mathcal{U}_{q\bar{q}}(\mathfrak{g}_{\mathbb{R}})$ restricted to the Borel subalgebra $\mathcal{U}_{q\bar{q}}(\mathfrak{b}_{\mathbb{R}})$. We prove that the restriction is independent of the parameter λ . Furthermore, we prove that it can be constructed from the GNS-representation of the multiplier Hopf algebra $\mathcal{U}_{q\bar{q}}^{C^*}(\mathfrak{b}_{\mathbb{R}})$ defined earlier, which allows us to decompose their tensor product using the theory of the “multiplicative unitary”. In particular, the quantum mutation operator can be constructed from the multiplicity module, which will be an essential ingredient in the construction of quantum higher Teichmüller theory from the perspective of representation theory, generalizing earlier work by Frenkel-Kim.

1. INTRODUCTION

To any finite dimensional complex simple Lie algebra \mathfrak{g} , Drinfeld [5] and Jimbo [22] defined a remarkable Hopf algebra $\mathcal{U}_q(\mathfrak{g})$ known as the quantum group. Its representation theory has evolved into an important area with many applications in various fields of mathematics and physics (cf. [6] and references therein).

In classical Lie theory, one is interested in certain real subalgebras of \mathfrak{g} known as *real forms*, two important cases being \mathfrak{g}_c corresponding to compact groups (e.g. $SU(n)$) and $\mathfrak{g}_{\mathbb{R}}$ corresponding to split real groups (e.g. $SL(n, \mathbb{R})$). The finite-dimensional representation theory in the compact case is well-behaved, and it is generalized nicely to the corresponding quantum group $\mathcal{U}_q(\mathfrak{g}_c)$. On the contrary, representation theory in the split-real case is much more complicated, as was shown by the monumental works of Harish-Chandra. Its generalization to the quantum group level - involving self-adjoint operators on Hilbert spaces - is physically more relevant but is still largely open due to various analytic difficulties coming from non-compactness and the use of unbounded operators.

1.1. Positive representations. The notion of the *positive principal series representations*, or simply *positive representations*, was introduced in [13] as a new research program devoted to the representation theory of split real quantum groups $\mathcal{U}_{q\bar{q}}(\mathfrak{g}_{\mathbb{R}})$. It uses the concept of modular double for quantum groups [7, 8] and generalizes the case of $\mathcal{U}_{q\bar{q}}(\mathfrak{sl}(2, \mathbb{R}))$ studied extensively by Teschner et al. [3, 29, 30]. Explicit construction of the positive representations \mathcal{P}_{λ} of $\mathcal{U}_{q\bar{q}}(\mathfrak{g}_{\mathbb{R}})$ associated to a simple Lie algebra \mathfrak{g} has been obtained for the simply-laced case in [17] and non-simply-laced case in [18], where the generators of the quantum groups are realized

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by positive essentially self-adjoint operators. Furthermore, since the generators are represented by positive operators, we can take real powers by means of functional calculus, and we obtain the so-called *transcendental relations* of the (rescaled) generators:

$$(1.1) \quad \widetilde{\mathbf{e}}_i = \mathbf{e}_{i^{\frac{1}{b^2}}}, \quad \widetilde{\mathbf{f}}_i = \mathbf{f}_{i^{\frac{1}{b^2}}}, \quad \widetilde{K}_i = K_{i^{\frac{1}{b^2}}}.$$

These important relations give the self-duality between different parts of the modular double, representing them simultaneously on the same Hilbert space. Moreover, in the non-simply-laced case, these provide new explicit analytic relations between the quantum group and its Langlands dual [18].

In the case of the modular double $\mathcal{U}_{q\bar{q}}(\mathfrak{sl}(2, \mathbb{R}))$, the positive representations are shown to be closely related to the space of conformal blocks of Liouville theory, and there are direct relations among the two and quantum Teichmüller theory described below, related by the fusion and braiding operations [28, 34]. In particular, the family of positive representations \mathcal{P}_λ of $\mathcal{U}_{q\bar{q}}(\mathfrak{sl}(2, \mathbb{R}))$ is closed under taking tensor product, with the Plancherel measure $d\mu(\lambda)$ given by the quantum dilogarithm [30]

$$(1.2) \quad \mathcal{P}_{\lambda_1} \otimes \mathcal{P}_{\lambda_2} \simeq \int_{\mathbb{R}_+} \mathcal{P}_\lambda d\mu(\lambda).$$

The fusion relations of Liouville theory are then provided precisely by the decomposition of the triple tensor products [29]. Together with the existence of a universal R -operator (constructed for general $\mathcal{U}_{q\bar{q}}(\mathfrak{g}_{\mathbb{R}})$ in [19]) giving the braiding structure, the braided tensor category structure may give rise to a new class of topological quantum field theory (TQFT) in the sense of Reshetikhin-Turaev [31, 32, 38]. Therefore one of the major remaining unsolved problems in the theory of positive representations for higher rank is the structure under taking tensor products, and we expect that the decomposition is related to the corresponding fusion relations of more general non-compact CFT's such as the Toda conformal field theory [15, 39].

1.2. Quantum Teichmüller theory. The *Teichmüller space* \mathcal{T}_S of an oriented surface S is the space of all complex structures on S modulo diffeomorphisms isotopic to identity. It is a very important space closely related to the moduli space of Riemann surfaces and carries a natural action of the *mapping class group* Γ_S , i.e., the group of all orientation-preserving diffeomorphisms modulo isotopy that preserves a canonical Poisson structure called the *Weil-Petersson form* [1].

Hence, quantum Teichmüller theory is, roughly speaking, the quantization \mathcal{T}_S^q of the Poisson manifold \mathcal{T}_S , such that the non-commutative algebra of function is represented on some Hilbert space of states \mathcal{H} , and automorphisms of \mathcal{T}_S^q associated to $g \in \Gamma_S$ are represented by unitary operators $\rho(g)$ on \mathcal{H} . Thus, the main goal is to construct new projective unitary representations of the mapping class group Γ_S .

In [14], Frenkel-Kim constructed the quantum Teichmüller space from the perspective of representation theory of certain Hopf algebras, namely, the modular double of the quantum plane. More precisely, let $0 < b < 1, b \in \mathbb{R} \setminus \mathbb{Q}$, and

$$(1.3) \quad q = e^{\pi i b^2}, \quad \tilde{q} = e^{\pi i b^{-2}}.$$

Then the modular double of the quantum plane $\mathcal{B}_{q\bar{q}}$ is a non-compact version of the quantum torus generated by two sets of commuting generators $\{E, K\}$ and $\{\tilde{E}, \tilde{K}\}$

such that

$$(1.4) \quad KE = q^2EK, \quad \tilde{K}\tilde{E} = \tilde{q}^2\tilde{E}\tilde{K}.$$

The quantum plane has a canonical representation on $\mathcal{H} \simeq L^2(\mathbb{R})$ such that the generators E, K are represented by positive self-adjoint operators

$$(1.5) \quad E = e^{-2\pi bp}, \quad K = e^{-2\pi bx}, \quad \tilde{E} = e^{-2\pi b^{-1}p}, \quad \tilde{K} = e^{-2\pi b^{-1}x},$$

with $p = \frac{1}{2\pi i} \frac{\partial}{\partial x}$. Using this fact, Frenkel-Kim [14] showed that \mathcal{H} is closed under taking the tensor product and decomposes as

$$(1.6) \quad \mathcal{H} \otimes \mathcal{H} \simeq M \otimes \mathcal{H},$$

where $M \simeq Hom_{\mathcal{B}_{q\tilde{q}}}(\mathcal{H}, \mathcal{H} \otimes \mathcal{H})$ is the multiplicity module with $\mathcal{B}_{q\tilde{q}}$ acting trivially.

Upon identification of \mathcal{H} and M with $L^2(\mathbb{R})$, this can be expressed by a transformation using the quantum dilogarithm (cf. (6.2)). Then the canonical isomorphism

$$(1.7) \quad (\mathcal{H}_1 \otimes \mathcal{H}_2) \otimes \mathcal{H}_3 \simeq \mathcal{H}_1 \otimes (\mathcal{H}_2 \otimes \mathcal{H}_3)$$

yields an operator \mathbf{T} on the multiplicity modules of the corresponding tensor product decomposition, called the *quantum mutation operator*, which by construction satisfies the pentagon relation [14, Proposition 4.8]

$$(1.8) \quad \mathbf{T}_{23}\mathbf{T}_{12} = \mathbf{T}_{12}\mathbf{T}_{13}\mathbf{T}_{23},$$

where the standard legged notation has been used. Together with another operator \mathbf{A} where $\mathbf{A}^3 = 1$ coming from the identification of the dual representations from the antipode S of $\mathcal{B}_{q\tilde{q}}$, one recovers Kashaev’s projective representation of the mapping class groupoid \mathfrak{G} [24] which descends to a projective representation of Γ_S , and we can apply it to quantization of Teichmüller space for various surfaces as shown e.g. in [4, 10, 24]. The operators \mathbf{T} and \mathbf{A} correspond essentially to the fusion and braiding relations on the conformal field theory side.

1.3. Main results. We observe that the quantum plane is just the Borel subalgebra of $\mathcal{U}_{q\tilde{q}}(\mathfrak{sl}(2, \mathbb{R}))$, and the canonical representation on \mathcal{H} is equivalent to the restriction of the positive representation. Therefore in order to generalize to higher rank, a natural candidate is to consider the Borel subalgebra $\mathcal{U}_{q\tilde{q}}(\mathfrak{b}_{\mathbb{R}})$ of the modular double of more general split real quantum groups $\mathcal{U}_{q\tilde{q}}(\mathfrak{g}_{\mathbb{R}})$. It follows that one of the main ingredients needed to construct a version of quantum higher Teichmüller theory [11, 12] is the construction of the quantum mutation operator \mathbf{T} described above, coming from the decomposition of tensor products of positive representations of the Borel part.

In this paper, we prove the following theorem (Theorem 3.1, Corollary 5.6):

Main Theorem 1. *The positive representations \mathcal{P}_λ of $\mathcal{U}_{q\tilde{q}}(\mathfrak{g}_{\mathbb{R}})$ restricted to the Borel part $\mathcal{U}_{q\tilde{q}}(\mathfrak{b}_{\mathbb{R}})$ is independent of the parameters λ . Moreover, the tensor product of $\mathcal{P}_\lambda \simeq \mathcal{P}$ decomposes as*

$$(1.9) \quad \mathcal{P} \otimes \mathcal{P} \simeq M \otimes \mathcal{P},$$

where $M \simeq L^2(\mathbb{R}^N)$ is the multiplicity module with $\mathcal{U}_{q\tilde{q}}(\mathfrak{b}_{\mathbb{R}})$ acting trivially. Here N is the dimension of $\mathfrak{b}_{\mathbb{R}}$.

In particular, this decomposition provides strong evidence for the closure of positive representation of the full quantum group $\mathcal{U}_{q\bar{q}}(\mathfrak{g}_{\mathbb{R}})$ under taking tensor product. Using $\mathcal{P}_{\lambda} \simeq \mathcal{P}$, we also generalize in Conjecture 3.2 the Stone-von Neumann Theorem to the case of Borel subalgebras of higher rank split real quantum groups.

To prove the Main Theorem, we give a new construction of positive representations of the Borel part by means of *multiplicator Hopf algebras* introduced in [25, 36]. We use the C^* -algebraic version of the Borel part $\mathcal{A} := \mathcal{U}_{q\bar{q}}^{C^*}(\mathfrak{b}_{\mathbb{R}})$ constructed in [19, 20] and apply the theory of Gelfand-Naimark-Segal (GNS) representations of C^* -algebras. This generalizes the harmonic analysis of the quantum plane studied in [16], which was introduced for the harmonic analysis of the quantum space of functions $L^2(SL_{q\bar{q}}^+(2, \mathbb{R}))$. A useful consequence of the GNS construction is the existence of a unitary operator W , known as the *multiplicative unitary* [2], which gives the desired intertwiner

$$(1.10) \quad W\Delta(x)W^* = 1 \otimes x, \quad x \in \mathcal{A},$$

between the tensor product and its decomposition with a trivial multiplicity module. Therefore in order to apply this to the positive representations $\mathcal{P}_{\lambda} \simeq \mathcal{P}$ restricted to the Borel part, it suffices to show that \mathcal{P} is equivalent to the GNS representation. The second main result of this paper is then the following (Theorem 5.5):

Main Theorem 2. *For any simple Lie algebra \mathfrak{g} , the GNS representation \mathcal{P}_{GNS} constructed by left multiplication on $\mathcal{A} = \mathcal{U}_{q\bar{q}}^{C^*}(\mathfrak{b}_{\mathbb{R}})$ is unitarily equivalent to the positive representation $\mathcal{P}_{\lambda} \simeq \mathcal{P}$ of $\mathcal{U}_{q\bar{q}}(\mathfrak{g}_{\mathbb{R}})$ restricted to its Borel part.*

Finally, we provide concrete examples for types A_1 and A_2 , where the unitary transformations by the remarkable quantum dilogarithm functions also provide rich combinatorial insight into the tensor category structure of the positive representations. In particular, these equip us with explicit expressions for the quantum mutation operators \mathbf{T} needed in order to construct candidates for the quantum higher Teichmüller theory. Together with the (unitary) antipode R_S defined in [16] and the identification of the dual representations [14], this gives us the desired construction of the projective representation of Kashaev's mapping class groupoid \mathfrak{G} which acts on a Hilbert space $L^2(\mathbb{R}^N)$ with higher functional dimension. This is expected to be a new class of representations for \mathfrak{G} , and details will appear elsewhere.

The paper is organized as follows. In Section 2 we fix the notation and recall some background preliminaries needed in the calculation of this paper. In Section 3, we consider the restriction of the positive representations \mathcal{P}_{λ} to the Borel subalgebra and show that the representation is independent of the parameters λ . In Section 4, we consider the GNS-representation obtained from the action by left multiplication on the C^* -algebraic version of the Borel subalgebra and bring the expression into a canonical form. In Section 5, we prove the Main Theorems by showing that the representation \mathcal{P}_{GNS} constructed from the GNS representation is unitarily equivalent to the one constructed earlier in [17, 18]. Finally, in Section 6, we construct explicitly the tensor product decompositions for types A_1 and A_2 .

2. PRELIMINARIES

Throughout the paper, we will fix once and for all $q = e^{\pi i b^2}$ with $\mathbf{i} = \sqrt{-1}$, $0 < b^2 < 1$, and $b^2 \in \mathbb{R} \setminus \mathbb{Q}$. We also denote $Q = b + b^{-1}$.

2.1. Definition of the modular double $\mathcal{U}_{q\bar{q}}(\mathfrak{g}_{\mathbb{R}})$. We follow the convention used in [19]. Let \mathfrak{g} be a simple Lie algebra over \mathbb{C} , and let $I = \{1, 2, \dots, n\}$ denote the set of nodes of the Dynkin diagram of \mathfrak{g} where $n = \text{rank}(\mathfrak{g})$. Let W denote the Weyl group, and let $w_0 \in W$ be its longest element with length $l(w_0)$.

Let $q_i := q$ if $i \in I$ corresponds to the long root, and let $q_i := e^{\pi i b_s^2}$ if $i \in I$ corresponds to the short root, where $b_s := \frac{b}{\sqrt{2}}$ if \mathfrak{g} is doubly-laced, and $b_s := \frac{b}{\sqrt{3}}$ if \mathfrak{g} is of type G_2 . We assume the roots are long for simply-laced \mathfrak{g} and denote $b_l := b$ as well. We let $1 \in I$ correspond to the short root in type B_n and the long root in other types.

Definition 2.1 ([5, 22]). The Drinfeld-Jimbo quantum group $\mathcal{U}_q(\mathfrak{g}_{\mathbb{R}})$ is the algebra generated by $\{E_i, F_i, K_i^{\pm 1}\}_{i \in I}$ over \mathbb{C} subjected to the relations for $i, j \in I$:

$$(2.1) \quad K_i E_j = q_i^{a_{ij}} E_j K_i, \quad K_i F_j = q_i^{-a_{ij}} F_j K_i, \quad [E_i, F_j] = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}},$$

together with the Serre relations for $i \neq j$:

$$(2.2) \quad \sum_{k=0}^{1-a_{ij}} (-1)^k \frac{[1-a_{ij}]_{q_i}!}{[1-a_{ij}-k]_{q_i}! [k]_{q_i}!} X_i^k X_j X_i^{1-a_{ij}-k} = 0, \quad X = E, F,$$

where $[k]_q := \frac{q^k - q^{-k}}{q - q^{-1}}$ and (a_{ij}) denotes the Cartan matrix.

We choose the Hopf algebra structure of $\mathcal{U}_q(\mathfrak{g})$ to be given by (we will not need the counit and antipode in this paper)

$$(2.3) \quad \Delta(E_i) = 1 \otimes E_i + E_i \otimes K_i,$$

$$(2.4) \quad \Delta(F_i) = K_i^{-1} \otimes F_i + F_i \otimes 1,$$

$$(2.5) \quad \Delta(K_i) = K_i \otimes K_i.$$

We define $\mathcal{U}_q(\mathfrak{g}_{\mathbb{R}})$ to be the real form of $\mathcal{U}_q(\mathfrak{g})$ induced by the star structure

$$(2.6) \quad E_i^* = E_i, \quad F_i^* = F_i, \quad K_i^* = K_i.$$

Finally, from the results of [17, 18], we let

$$(2.7) \quad \tilde{q} := e^{\pi i b_s^{-2}}$$

and we define the modular double $\mathcal{U}_{q\bar{q}}(\mathfrak{g}_{\mathbb{R}})$ to be

$$(2.8) \quad \mathcal{U}_{q\bar{q}}(\mathfrak{g}_{\mathbb{R}}) := \mathcal{U}_q(\mathfrak{g}_{\mathbb{R}}) \otimes \mathcal{U}_{\bar{q}}(\mathfrak{g}_{\mathbb{R}}) \quad \mathfrak{g} \text{ is simply-laced,}$$

$$(2.9) \quad \mathcal{U}_{q\bar{q}}(\mathfrak{g}_{\mathbb{R}}) := \mathcal{U}_q(\mathfrak{g}_{\mathbb{R}}) \otimes \mathcal{U}_{\bar{q}}({}^L\mathfrak{g}_{\mathbb{R}}) \quad \text{otherwise,}$$

where ${}^L\mathfrak{g}_{\mathbb{R}}$ is the Langlands dual of $\mathfrak{g}_{\mathbb{R}}$ obtained by interchanging the long roots and short roots of $\mathfrak{g}_{\mathbb{R}}$.

2.2. Positive representations of $\mathcal{U}_{q\bar{q}}(\mathfrak{g}_{\mathbb{R}})$. In [13, 17, 18], a special class of representations for $\mathcal{U}_{q\bar{q}}(\mathfrak{g}_{\mathbb{R}})$, called the positive representations, is defined. The generators of $\mathcal{U}_{q\bar{q}}(\mathfrak{g}_{\mathbb{R}})$ are realized by positive essentially self-adjoint operators. They also satisfy the *transcendental relations* (2.11) below; in particular the quantum group

and its modular double counterpart are represented on the same Hilbert space, generalizing the situation of $\mathcal{U}_{q\bar{q}}(\mathfrak{sl}(2, \mathbb{R}))$ introduced in [7, 8] and studied in [30]. More precisely,

Theorem 2.2 ([13, 17, 18]). *Let the rescaled generators be*

$$(2.10) \quad \mathbf{e}_i := 2 \sin(\pi b_i^2) E_i, \quad \mathbf{f}_i := 2 \sin(\pi b_i^2) F_i.$$

There exists a family of representations \mathcal{P}_λ of $\mathcal{U}_{q\bar{q}}(\mathfrak{g}_{\mathbb{R}})$ parametrized by the \mathbb{R}_+ -span of the cone of positive weights $\lambda \in P_{\mathbb{R}}^+$, or equivalently by $\lambda \in \mathbb{R}_+^{\text{rank}(\mathfrak{g})}$, such that:

- *The generators $\mathbf{e}_i, \mathbf{f}_i, K_i$ are represented by positive essentially self-adjoint operators acting on $L^2(\mathbb{R}^N)$ where $N = l(w_0)$.*
- *Define the transcendental generators:*

$$(2.11) \quad \widetilde{\mathbf{e}}_i := \mathbf{e}_i^{\frac{1}{b_i^2}}, \quad \widetilde{\mathbf{f}}_i := \mathbf{f}_i^{\frac{1}{b_i^2}}, \quad \widetilde{K}_i := K_i^{\frac{1}{b_i^2}}.$$

- *If \mathfrak{g} is simply-laced, the generators $\widetilde{\mathbf{e}}_i, \widetilde{\mathbf{f}}_i, \widetilde{K}_i$ are obtained by replacing b with b^{-1} in the representations of the generators $\mathbf{e}_i, \mathbf{f}_i, K_i$.*
- *If \mathfrak{g} is non-simply-laced, then the generators $\widetilde{E}_i, \widetilde{F}_i, \widetilde{K}_i$ with $\widetilde{\mathbf{e}}_i := 2 \sin(\pi b_i^{-2}) \widetilde{E}_i$ and $\widetilde{\mathbf{f}}_i := 2 \sin(\pi b_i^{-2}) \widetilde{F}_i$ generate $\mathcal{U}_{\bar{q}}(L\mathfrak{g}_{\mathbb{R}})$.*
- *The generators $\mathbf{e}_i, \mathbf{f}_i, K_i$ and $\widetilde{\mathbf{e}}_i, \widetilde{\mathbf{f}}_i, \widetilde{K}_i$ commute weakly up to a sign.*

The positive representations are constructed for each reduced expression of the longest element $w_0 \in W$ of the Weyl group, and representations corresponding to different reduced expressions are unitarily equivalent.

Definition 2.3. Fix a reduced expression of $w_0 = s_{i_1} \dots s_{i_N}$. Let the coordinates of $L^2(\mathbb{R}^N)$ be denoted by $\{u_i^k\}$ so that i is the corresponding root index, and k denotes the sequence this root is appearing in w_0 from the right. Also denote by $\{v_j\}_{j=1}^N$ the same set of coordinates counting from the left, $v(i, k)$ the index such that $u_i^k = v_{v(i,k)}$, and $r(k)$ the root index corresponding to v_k .

Example 2.4. The coordinates of $L^2(\mathbb{R}^6)$ for A_3 corresponding to $w_0 = s_3 s_2 s_1 s_3 s_2 s_3$ is given by $(u_3^3, u_2^2, u_1^1, u_3^2, u_2^1, u_3^1) = (v_1, v_2, v_3, v_4, v_5, v_6)$.

Definition 2.5. We denote by $p_u = \frac{1}{2\pi i} \frac{\partial}{\partial u}$ and

$$(2.12) \quad e(u) := e^{\pi b u}, \quad [u] := q^{\frac{1}{2}} e(u) + q^{-\frac{1}{2}} e(-u).$$

Then the following is positive whenever $[p, u] = \frac{1}{2\pi i}$:

$$(2.13) \quad [u]e(-2p) := (q^{\frac{1}{2}} e^{\pi b u} + q^{-\frac{1}{2}} e^{-\pi b u}) e^{-2\pi b p} = e^{\pi b(u-2p)} + e^{\pi b(-u-2p)}.$$

Note that we changed the notation slightly for $e(u)$ in this paper from previous references [17]-[19] for later convenience.

Definition 2.6. By abuse of notation, we denote by

$$(2.14) \quad [u_s + u_l]e(-2p_s - 2p_l) := e^{\pi b_s(-u_s - 2p_s) + \pi b_l(-u_l - 2p_l)} + e^{\pi b_s(u_s - 2p_s) + \pi b_l(u_l - 2p_l)},$$

where u_s (resp. u_l) is a linear combination of the variables corresponding to short roots (resp. long roots). The parameters λ_i are also considered in both cases. Similarly p_s (resp. p_l) are linear combinations of the p shifting of the short root (resp. long root) variables.

We will occasionally write in the form $[X]e(-2p_Y)$, where by abuse of notation, if $Y = \sum a_i u_i, a_i \in \mathbb{Z}$, then $p_Y = \sum a_i p_{u_i}$.

Theorem 2.7 ([17, 18]). *For a fixed reduced expression of w_0 , \mathcal{P}_λ is given by*

$$(2.15) \quad \mathbf{f}_i = \sum_{k=1}^n \left[- \sum_{j=1}^{v(i,k)-1} a_{i,r(j)} v_j - u_i^k - 2\lambda_i \right] e(2p_i^k),$$

$$(2.16) \quad K_i = e \left(- \sum_{k=1}^{l(w_0)} a_{i,r(k)} v_k + 2\lambda_i \right).$$

By taking $w_0 = w' s_i$ so that the simple reflection for root i appears on the right, the action of \mathbf{e}_i is given by

$$(2.17) \quad \mathbf{e}_i = [u_i^1] e(-2p_i^1).$$

In this paper, it is instructive to recall the explicit expression in the case of ranks 1 and 2. For details of the construction and the other cases please refer to [17, 18].

Proposition 2.8 ([3, 30]). *The positive representation P_λ of $\mathcal{U}_{q\bar{q}}(\mathfrak{sl}(2, \mathbb{R}))$ acting on $L^2(\mathbb{R})$ is given by*

$$\begin{aligned} \mathbf{e} &= [u - \lambda] e(-2p) = e^{\pi b(-u + \lambda - 2p)} + e^{\pi b(u - \lambda - 2p)}, \\ \mathbf{f} &= [-u - \lambda] e(2p) = e^{\pi b(u + \lambda + 2p)} + e^{\pi b(-u - \lambda + 2p)}, \\ K &= e(-2u) = e^{-2\pi b u}. \end{aligned}$$

Note that it is unitarily equivalent to (2.15)-(2.17) by $u \mapsto u + \lambda$.

Proposition 2.9 ([17]). *The positive representation \mathcal{P}_λ of $\mathcal{U}_{q\bar{q}}(\mathfrak{sl}(3, \mathbb{R}))$ with parameters $\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}_+^2$, corresponding to the reduced expression $w_0 = s_2 s_1 s_2$, acting on $f(u, v, w) \in L^2(\mathbb{R}^3)$, is given by*

$$\begin{aligned} \mathbf{e}_1 &= [v - w] e(-2p_v) + [u] e(-2p_v + 2p_w - 2p_u), \\ \mathbf{e}_2 &= [w] e(-2p_w), \\ \mathbf{f}_1 &= [-v + u - 2\lambda_1] e(2p_v), \\ \mathbf{f}_2 &= [-2u + v - w - 2\lambda_2] e(2p_w) + [-u - 2\lambda_2] e(2p_u), \\ K_1 &= e(u - 2v + w - 2\lambda_1), \\ K_2 &= e(-2u + v - 2w - 2\lambda_2). \end{aligned}$$

Proposition 2.10 ([18]). *The positive representation \mathcal{P}_λ of $\mathcal{U}_{q\bar{q}}(\mathfrak{g}_\mathbb{R})$ with parameters $\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}_+^2$, where $\mathfrak{g}_\mathbb{R}$ is of type B_2 , corresponding to the reduced expression $w_0 = s_1 s_2 s_1 s_2$, acting on $f(t, u, v, w) \in L^2(\mathbb{R}^4)$, is given by*

$$\begin{aligned} \mathbf{e}_1 &= [t] e(-2p_t - 2p_u + 2p_w) + [u - v] e(-2p_u - 2p_v + 2p_w) + [v - w] e(-2p_v), \\ \mathbf{e}_2 &= [w] e(-2p_w), \\ \mathbf{f}_1 &= [-t - 2\lambda_1] e(2p_t) + [-2t + u - v - 2\lambda_1] e(2p_v), \\ \mathbf{f}_2 &= [2t - u - 2\lambda_2] e(2p_u) + [2t - 2u + 2v - w - 2\lambda_2] e(2p_w), \\ K_1 &= e(-2t + u - 2v + w - 2\lambda_1), \\ K_2 &= e(2t - 2u + 2v - 2w - 2\lambda_2). \end{aligned}$$

In this case (cf. Definition 2.6), u_s is linear combinations of $\{t, v\}$, while u_l is linear combinations of $\{u, w\}$. Similarly for p_s and p_l .

We will omit the case of type G_2 in this paper for simplicity.

2.3. Quantum dilogarithms $G_b(x)$ and $g_b(x)$. First introduced by Faddeev and Kashaev [7, 9], the quantum dilogarithm $G_b(x)$ and its variants $S_b(x)$ and $g_b(x)$ play a crucial role in the study of positive representations of split real quantum groups and also appear in many other areas of mathematics and physics, most notably cluster algebras, quantum Teichmüller theory, and Liouville CFT. In this subsection, we recall the definition and some properties of the quantum dilogarithm functions [3, 16, 30] that are needed in the calculations of this paper.

Definition 2.11. The quantum dilogarithm function $G_b(x)$ is defined on $0 \leq \text{Re}(x) \leq Q = b + b^{-1}$ by

$$(2.18) \quad G_b(x) = \bar{\zeta}_b \exp \left(- \int_{\Omega} \frac{e^{\pi t x}}{(e^{\pi b t} - 1)(e^{\pi b^{-1} t} - 1)} \frac{dt}{t} \right),$$

where $\zeta_b = e^{\frac{\pi i}{2}(\frac{b^2+b^{-2}}{6} + \frac{1}{2})}$, and the contour Ω goes along \mathbb{R} and goes above the pole at $t = 0$. This can be extended meromorphically to the whole complex plane with poles at $x = -nb - mb^{-1}$ and zeros at $x = Q + nb + mb^{-1}$, for $n, m \in \mathbb{Z}_{\geq 0}$.

Definition 2.12. The function $S_b(x)$ is defined by

$$(2.19) \quad S_b(x) := e^{\frac{\pi i}{2}x(Q-s)} G_b(x).$$

Proposition 2.13. The quantum dilogarithms satisfy the self-duality relations

$$(2.20) \quad S_b(x) = S_{b^{-1}}(x), \quad G_b(x) = G_{b^{-1}}(x),$$

as well as the functional equations

$$(2.21) \quad S_b(x + b^{\pm 1}) = i(e^{-\pi i b^{\pm 1} x} - e^{\pi i b^{\pm 1} x}) S_b(x),$$

$$(2.22) \quad G_b(x + b^{\pm 1}) = (1 - e^{2\pi i b^{\pm 1} x}) G_b(x).$$

We will also need another important variant of the quantum dilogarithm.

Definition 2.14. The function $g_b(x)$ is defined by

$$(2.23) \quad g_b(x) = \frac{\bar{\zeta}_b}{G_b(\frac{Q}{2} + \frac{\log x}{2\pi i b})},$$

where \log takes the principal branch of x .

We will need the following properties of $g_b(x)$:

Lemma 2.15 ([3, (3.31), (3.32)]). We have the following Fourier transformation formula:

$$(2.24) \quad \int_{\mathbb{R}+i0} \frac{e^{-\pi i t^2}}{G_b(Q + it)} X^{i b^{-1} t} dt = g_b(X),$$

$$(2.25) \quad \int_{\mathbb{R}+i0} \frac{e^{-\pi Q t}}{G_b(Q + it)} X^{i b^{-1} t} dt = g_b^*(X),$$

where X is a positive operator and the contour goes above the pole at $t = 0$.

Lemma 2.16. $g_b(X)$ is a unitary operator for any positive operator X .

Lemma 2.17. *If $UV = q^2VU$ where U, V are positive self-adjoint operators, then*

$$(2.26) \quad g_b(U)g_b(V) = g_b(U + V),$$

$$(2.27) \quad g_b(U)^*Vg_b(U) = q^{-1}UV + V,$$

$$(2.28) \quad g_b(V)Ug_b(V)^* = U + q^{-1}UV.$$

Note that (2.26) and (2.27) together imply the pentagon relation

$$(2.29) \quad g_b(V)g_b(U) = g_b(U)g_b(q^{-1}UV)g_b(V).$$

The above relations can be generalized to the non-simply-laced case [19, Prop. 3.3], which gives relations between g_{b_s} and g_{b_l} .

2.4. Lusztig’s isomorphism. We recall several definitions and results concerning Lusztig’s isomorphism in the positive representation setting needed in this paper. See [19] for more details.

Fix a positive representation \mathcal{P}_λ of $\mathcal{U}_{q\bar{q}}(\mathfrak{g}_\mathbb{R})$ and denote

$$(2.30) \quad [u, v]_q := quv - q^{-1}vu.$$

Proposition 2.18. *In the simply-laced case, the operators*

$$(2.31) \quad e_{ij} := \frac{[e_j, e_i]_{q^{\frac{1}{2}}}}{q - q^{-1}} = \frac{q^{\frac{1}{2}}e_j e_i - q^{-\frac{1}{2}}e_i e_j}{q - q^{-1}}$$

are positive essentially self-adjoint on \mathcal{P}_λ .

Proposition 2.19. *In the non-simply-laced case, the operators*

$$(2.32) \quad e_{ij} := (-1)^{a_{ij}} \left[\left[e_i, \dots, [e_i, e_j]_{q_i^{\frac{a_{ij}}{2}}} \right]_{q_i^{\frac{a_{ij}+2}{2}}} \dots \right]_{q_i^{\frac{-a_{ij}-2}{2}}} \prod_{k=1}^{-a_{ij}} (q_i^k - q_i^{-k})^{-1}$$

are positive essentially self-adjoint on \mathcal{P}_λ .

Similarly we define f_{ij} by (2.31) and (2.32) with e replaced by f .

Theorem 2.20. *For any root $i \in I$, there exists a unitary operator T_i such that*

$$(2.33) \quad T_i(e_i) = q_i f_i K_i^{-1} = q_i^{-1} K_i^{-1} f_i,$$

$$(2.34) \quad T_i(f_i) = q_i^{-1} K_i e_i = q_i e_i K_i,$$

$$(2.35) \quad T_i(e_j) = e_{ij}, \quad T_i(f_j) = f_{ij}, \quad \text{for } i, j \text{ adjacent,}$$

$$(2.36) \quad T_i(e_k) = e_k, \quad T_i(f_k) = f_k, \quad \text{for } a_{ik} = 0,$$

$$(2.37) \quad T_i(K_j) = K_j K_i^{-a_{ij}}.$$

Proposition 2.21 ([26, 27]). *The operators T_i satisfy the Coxeter relations*

$$(2.38) \quad \underbrace{T_i T_j T_i \dots}_{-a'_{ij}+2} = \underbrace{T_j T_i T_j \dots}_{-a'_{ij}+2},$$

where $-a'_{ij} = \max\{-a_{ij}, -a_{ji}\}$. Furthermore, for α_i, α_j simple roots and an element $w = s_{i_1} \dots s_{i_k} \in W$ such that $w(\alpha_i) = \alpha_j$, we have for $X = e, f, K$:

$$(2.39) \quad T_{i_1} \dots T_{i_k}(X_i) = X_j.$$

In this paper we will use instead the inverse version of the following definition.

Definition 2.22. Let $w_0 = s_{i_1} s_{i_2} \dots s_{i_N}$ be the longest element of W . We define

$$(2.40) \quad \mathbf{e}_{\alpha_k} := T_{i_1}^{-1} T_{i_2}^{-1} \dots T_{i_{k-1}}^{-1} \mathbf{e}_{i_k}.$$

These are all positive essentially self-adjoint.

Finally we record some commutation relations among the non-simple generators.

Proposition 2.23. *For the simply-laced case, if $a_{ij} = -1$, then we have*

$$(2.41) \quad \mathbf{e}_i \mathbf{e}_{ij} = q^{-1} \mathbf{e}_{ij} \mathbf{e}_i, \quad \mathbf{e}_j \mathbf{e}_{ij} = q \mathbf{e}_{ij} \mathbf{e}_j,$$

$$(2.42) \quad \mathbf{e}_i \mathbf{e}_j^n = q^n \mathbf{e}_j^n \mathbf{e}_i + q^{\frac{1}{2}} (q^{-n} - q^n) \mathbf{e}_j^{n-1} \mathbf{e}_{ij}.$$

Proposition 2.24. *For the doubly-laced case, let $a_{ij} = -2$. Denote $\mathbf{e}_{ij} := T_i \mathbf{e}_j$ and $\mathbf{e}_X := T_i T_j \mathbf{e}_i$. We have*

$$(2.43) \quad \mathbf{e}_i \mathbf{e}_{ij} = q^{-1} \mathbf{e}_{ij} \mathbf{e}_i, \quad \mathbf{e}_{ij} \mathbf{e}_X = q^{-1} \mathbf{e}_X \mathbf{e}_{ij}, \quad \mathbf{e}_X \mathbf{e}_j = q^{-1} \mathbf{e}_j \mathbf{e}_X,$$

$$(2.44) \quad \mathbf{e}_i^n \mathbf{e}_j = q^{-n} \mathbf{e}_j \mathbf{e}_i^n + q^{\frac{1}{2}} (q^n - q^{-n}) \mathbf{e}_i^{n-1} \mathbf{e}_X,$$

$$(2.45) \quad \mathbf{e}_i^n \mathbf{e}_X = \mathbf{e}_X \mathbf{e}_i^n + q^{\frac{1}{2}} (q^{-n} - 1) \mathbf{e}_{ij} \mathbf{e}_i^{n-1},$$

$$(2.46) \quad \mathbf{e}_{ij}^n \mathbf{e}_j = q^n \mathbf{e}_j \mathbf{e}_{ij} + (q^{-2n} - 1) \mathbf{e}_X^2.$$

Note that by means of [19, Prop. 6.8], we can take n to be a complex power.

2.5. GNS representation, multiplicative unitary, and multiplier Hopf algebras. Finally, we recall the definition of Gelfand-Naimark-Segal (GNS) representation of a C^* -algebra and its corresponding unitary operator called the *multiplicative unitary* that is needed in this paper. These are fundamental to the theory of locally compact quantum groups in the setting of C^* -algebras and von Neumann algebras, and to the generalization of Pontryagin duality. In particular, a multiplicative unitary encodes all the structure maps of a quantum group and its dual. See [16], [25], or [35] for more discussions.

Definition 2.25. A Gelfand-Naimark-Segal (GNS) representation of a C^* -algebra \mathcal{A} with a weight ϕ is a triple $(\mathcal{H}, \pi, \Lambda)$, where \mathcal{H} is a Hilbert space, $\Lambda : \mathcal{A} \rightarrow \mathcal{H}$ is a linear map, and $\pi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$ is a representation of \mathcal{A} on \mathcal{H} as bounded linear operators, such that $\Lambda(\mathcal{N})$ is dense in \mathcal{H} , where $\mathcal{N} = \{a \in \mathcal{A} : \phi(a^*a) < \infty\}$ and

$$(2.47) \quad \pi(a)\Lambda(b) = \Lambda(ab) \quad \forall a \in \mathcal{A}, b \in \mathcal{N},$$

$$(2.48) \quad \langle \Lambda(a), \Lambda(b) \rangle = \phi(b^*a) \quad \forall a, b \in \mathcal{N}.$$

Definition 2.26 ([2]). Let \mathcal{H} be a Hilbert space. A unitary operator $W \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$ is called a *multiplicative unitary* if it satisfies the pentagon equation

$$(2.49) \quad W_{23}W_{12} = W_{12}W_{13}W_{23},$$

where the standard leg notation is used.

Given a GNS representation $(\mathcal{H}, \pi, \Lambda)$ of a locally compact quantum group \mathcal{A} , we can define a unitary operator on $\mathcal{H} \otimes \mathcal{H}$ by

$$(2.50) \quad W^*(\Lambda(a) \otimes \Lambda(b)) := (\Lambda \otimes \Lambda)(\Delta(b)(a \otimes 1)).$$

It is known that W is a multiplicative unitary [25, Thms. 3.16 and 3.18], and the coproduct on \mathcal{A} defining it can be recovered from W :

Proposition 2.27. *Let $x \in \mathcal{A} \hookrightarrow \mathcal{B}(\mathcal{H})$ be an operator. Then*

$$(2.51) \quad W^*(1 \otimes x)W = \Delta(x)$$

as operators on $\mathcal{H} \otimes \mathcal{H}$.

Proof. See [16, Prop. 2.18]. □

In particular, Proposition 2.27 provides a unitary equivalence

$$(2.52) \quad W : \mathcal{H} \otimes \mathcal{H} \simeq M \otimes \mathcal{H}$$

of the tensor product of the GNS representations, where M is the multiplicity module equipped with the trivial action. This is the key step to proving the tensor product decomposition of the positive representations of $\mathcal{U}_{q\bar{q}}(\mathfrak{b}_{\mathbb{R}})$.

Finally, we recall the definition of a multiplier Hopf algebra.

Definition 2.28. The multiplier algebra $M(\mathcal{A})$ of a C^* -algebra $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ is the C^* -algebra of operators

$$(2.53) \quad M(\mathcal{A}) = \{b \in \mathcal{B}(\mathcal{H}) : b\mathcal{A} \subset \mathcal{A}, \mathcal{A}b \subset \mathcal{A}\}.$$

In particular, \mathcal{A} is an ideal of $M(\mathcal{A})$.

Definition 2.29. A multiplier Hopf $*$ -algebra is a C^* -algebra \mathcal{A} together with the antipode S , the counit ϵ , and the coproduct map

$$(2.54) \quad \Delta : \mathcal{A} \longrightarrow M(\mathcal{A} \otimes \mathcal{A}),$$

all of which can be extended to a map from $M(\mathcal{A})$ such that the usual properties of a Hopf algebra hold on the level of $M(\mathcal{A})$.

3. POSITIVE REPRESENTATIONS RESTRICTED TO BOREL PART

In the case of $\mathfrak{g} = \mathfrak{sl}_2$, the Borel subalgebra $\mathcal{U}_{q\bar{q}}(\mathfrak{b}_{\mathbb{R}})$ is generated by the operators K and \mathbf{e} satisfying the quantum plane relation $K\mathbf{e} = q^2\mathbf{e}K$, in the sense of integrable representations [33], i.e.,

$$(3.1) \quad K^{ib^{-1}s} \mathbf{e}^{ib^{-1}t} = e^{-2\pi i s t} \mathbf{e}^{ib^{-1}t} K^{ib^{-1}s}, \quad \forall s, t \in \mathbb{R},$$

as unitary operators. Thus the positive representation $\mathcal{P}_{\lambda} \simeq L^2(\mathbb{R})$ restricted to the Borel part is nothing but an irreducible representation of the quantum plane such that the operators are represented by positive self-adjoint operators. By Stone-von Neumann’s Theorem, we know that every such representation is unitarily equivalent to the canonical representation $\mathcal{H} \simeq L^2(\mathbb{R})$ given by

$$(3.2) \quad K = e^{-2\pi b x}, \quad \mathbf{e} = e^{-2\pi b p}.$$

In particular, the representation does not depend on any parameters. In fact, from Proposition 2.8, it is easily seen that $\Phi : \mathcal{P}_{\lambda} \longrightarrow \mathcal{H}$ given by multiplication by

$$(3.3) \quad \Phi = e^{-\frac{\pi i(x-\lambda)^2}{2}} g_b(e^{-2\pi b(x-\lambda)})$$

is the required unitary equivalence. This result also holds in the higher rank case:

Theorem 3.1. *In the higher rank case, we have*

$$(3.4) \quad \mathcal{P}_\lambda \simeq \mathcal{P}_{\lambda'}$$

as representations of $\mathcal{U}_{q\bar{q}}(\mathfrak{b}_\mathbb{R})$ for any $\lambda, \lambda' \in \mathbb{R}^{\text{rank}(\mathfrak{g})}$.

Proof. From [17, Section 11] for the simply-laced case or [18, Section 9] for the non-simply-laced case, we know that $\mathcal{P}_\lambda \simeq \mathcal{P}_{w(\lambda)}$, where $w \in W$ is any Weyl group element acting on λ , namely, for simple reflections,

$$(3.5) \quad s_i(\lambda_j) := \lambda_j - a_{ij}\lambda_i,$$

where a_{ij} is the Cartan matrix. In particular, this means that for each root $i \in I$, there exists a unitary transformation B_i such that it fixes the action of \mathbf{e}_i but modifies the action of K_j by

$$K_j \mapsto K_j e^{a_{ij}\pi b\lambda_i}.$$

However, when we restrict to the Borel subalgebra, we are released from the restriction of the action of the lower Borel generated by \mathbf{f}_i , in which the proof above allows us to define the unitary transformation B_i such that it fixes the action of \mathbf{e}_i but modifies the action of K_i by an arbitrary constant c_i :

$$K_j \mapsto K_j e^{a_{ij}\pi bc_i}.$$

Therefore, it suffices to solve the equation $\sum a_{ij}c_i = \lambda_i$ in order for all λ_i in the representation to cancel. This clearly can be done because the Cartan matrix is invertible. In particular, all representations \mathcal{P}_λ are unitarily equivalent to $\mathcal{P}_{\vec{0}}$. \square

However, we observe that $\mathcal{P}_{\vec{0}}$ is not irreducible in general. For example, in the case of type A_2 , the positive operator $K_1^{\frac{2}{3}}K_2^{-\frac{2}{3}}\mathbf{e}_{12}^{-1}\mathbf{e}_{21}$ commutes with all the generators. Hence $\mathcal{P}_{\vec{0}}$ decomposes as a direct integral $\int_{\mathbb{R}_{>0}}^\oplus P_s ds$ with respect to the spectrum $s \in \mathbb{R}_{>0}$ of this central operator, where each $P_s \simeq L^2(\mathbb{R}^2)$ is irreducible.

Conjecture 3.2. *Any irreducible representations of $\mathcal{U}_{q\bar{q}}(\mathfrak{b}_\mathbb{R})$ satisfying the conditions of a positive representation act on $P \simeq L^2(\mathbb{R}^{\text{rank}(\mathfrak{g})})$ and are unitarily equivalent to a direct integral summand of $\mathcal{P}_{\vec{0}}$. More specifically, using the notion of “integrable representation” of quantum plane [33], we require as unitary operators*

$$K_i^{ib^{-1}s} \mathbf{e}_j^{ib^{-1}t} = e^{-\pi a_{ij}st} \mathbf{e}_j^{ib^{-1}t} K_i^{ib^{-1}s} \quad \forall s, t \in \mathbb{R},$$

and in the simply-laced case,

$$\mathbf{e}_i^{ib^{-1}s} \mathbf{e}_{ij}^{ib^{-1}t} = e^{\pi st} \mathbf{e}_{ij}^{ib^{-1}t} \mathbf{e}_i^{ib^{-1}s} \quad \forall s, t \in \mathbb{R},$$

$$\mathbf{e}_j^{ib^{-1}s} \mathbf{e}_{ij}^{ib^{-1}t} = e^{-\pi st} \mathbf{e}_{ij}^{ib^{-1}t} \mathbf{e}_j^{ib^{-1}s} \quad \forall s, t \in \mathbb{R},$$

which corresponds to the Serre relations. We have similar equations for the non-simply-laced case (cf. Proposition 2.24). This generalizes the Stone von-Neumann Theorem to the case of the Borel subalgebra of a split real quantum group.

4. GNS REPRESENTATION OF $\mathcal{U}_{q\bar{q}}^{C^*}(\mathfrak{b}_\mathbb{R})$

In this section, we recall the definition of $\mathcal{U}_{q\bar{q}}^{C^*}(\mathfrak{b}_\mathbb{R})$ and construct the GNS-representation by left multiplication. We show that it is equivalent to a positive representation.

4.1. Definition of $\mathcal{U}_{q\bar{q}}^{C^*}(\mathfrak{b}_{\mathbb{R}})$. Let us denote $\mathcal{P} := \mathcal{P}_{\vec{0}}$. In [19], we used the language of multiplier Hopf algebras and defined a C^* -algebraic version of the Borel subalgebra using a continuous basis. Explicitly, consider the action of $\mathcal{U}_{q\bar{q}}(\mathfrak{b}_{\mathbb{R}})$ on $\mathcal{P}_\lambda \simeq \mathcal{P} \simeq L^2(\mathbb{R}^N)$, where $N = l(w_0)$.

Definition 4.1. Let $n = \text{rank}(\mathfrak{g})$. We define the C^* -algebraic version of the Borel subalgebra $\mathbf{U}\mathfrak{b} := \mathcal{U}_{q\bar{q}}^{C^*}(\mathfrak{b}_{\mathbb{R}})$ as the operator norm closure of the linear span of all bounded operators on $L^2(\mathbb{R}^N)$ of the form

$$(4.1) \quad \vec{F} := \left(\prod_{k=1}^N \int_C \frac{F_k(t_k)}{G_{b_{i_k}}(Q_{i_k} + \mathbf{i}t_k)} \mathbf{e}_{\alpha_k}^{\mathbf{i}b_{i_k}^{-1}t_k} dt_k \right) \left(\prod_{i=1}^n \int_{\mathbb{R}} \widehat{F}_i(s_i) K_i^{\mathbf{i}b^{-1}s_i} ds_i \right),$$

where \mathbf{e}_{α_k} is given by (2.40) with the order of multiplication as $\mathbf{e}_{\alpha_1} \dots \mathbf{e}_{\alpha_N}$ and $F_k(t_k)$ and $\widehat{F}_i(s_i)$ are both entire analytic functions that rapidly decay along the real direction; i.e., for fixed y_0 , $F_k(x + \mathbf{i}y_0)$ decays faster than any exponential function in x . Finally the contour C goes along the real axis and above the pole of G_b at $t_k = 0$.

It was shown in [19] that $\mathbf{U}\mathfrak{b}$ is a multiplier Hopf algebra (cf. Definition 2.29).

Remark 4.2. We modified the definition slightly from [19]. We have used the opposite multiplication order to be consistent with the definition of \mathbf{e}_{α_k} using $T_{i_k}^{-1}$ (cf. (2.40)) instead of T_{i_k} . The two definitions are identical. Moreover, to simplify calculations, we have used the functions $\widehat{F}_i(s_i)$ parametrized by the powers of K_i instead of a general compact function $F_0(\mathbf{H})$ considered in [19], where $K_i = q_i^{H_i}$. These are simply related by Fourier transform \mathcal{F} :

$$\int (\mathcal{F}\widehat{F})(s) K^{\mathbf{i}b^{-1}s} ds = \iint e^{-2\pi \mathbf{i}st} \widehat{F}(t) K^{\mathbf{i}b^{-1}s} dt ds = \widehat{F}\left(\frac{1}{2}\mathbf{i}bH\right).$$

4.2. GNS-representation. It turns out the definition above does not necessarily define a positive action. In order to obtain an action of $\mathcal{U}_{q\bar{q}}(\mathfrak{b}_{\mathbb{R}})$ that is positive, we modify $\vec{F} \in \mathcal{A}$ as follows.

Definition 4.3. We define the GNS representation $(\mathcal{H}, \pi, \Lambda)$ of $\mathcal{A} := \mathcal{U}_{q\bar{q}}^{C^*}(\mathfrak{b}_{\mathbb{R}})$ on $\mathcal{H} \simeq L^2(\mathbb{R}^{N+n})$ as follows. We let

$$(4.2) \quad \vec{F} := \prod_{k=1}^N \left(\int_C \frac{F_k(t_k)}{S_{b_{i_k}}(Q_{i_k} + \mathbf{i}t_k)} \mathbf{e}_{\alpha_k}^{\mathbf{i}b_{i_k}^{-1}t_k} dt_k K_{\alpha_k}^{\frac{Q}{2b}} \right) \left(\prod_{i=1}^n \int_{\mathbb{R}} \widehat{F}_i(s_i) K_i^{\mathbf{i}b^{-1}s_i} e^{\pi Q s_i} ds_i \right)$$

and define the representation $\Lambda : \mathcal{A} \rightarrow \mathcal{H}$ by

$$(4.3) \quad \vec{F} \mapsto \prod_{k=1}^N F_k(t_k) \prod_{i=1}^n \widehat{F}_i(s_i)$$

and extend by linearity. Finally, the L^2 -norm of \mathcal{H} (cf. [16, Thm. 4.9]) will be defined for each variable as

$$(4.4) \quad \|F(\dots, t_k, \dots)\|^2 := \int_{\mathbb{R}^N} \left| F\left(\dots, t_k + \frac{\mathbf{i}Q_{i_k}}{2}, \dots\right) \right|^2 dt_1 \dots dt_N,$$

$$(4.5) \quad \|\widehat{F}(\dots, s_i, \dots)\|^2 := \int_{\mathbb{R}^n} |\widehat{F}(\dots, s_i, \dots)|^2 ds_1 \dots ds_n.$$

Note that we have used $S_b(x)$ (cf. Definition 2.12) instead in the denominator. Moreover we slipped K_{α_k} in between the products of \mathbf{e}_{α_k} ; this is well-defined since $K_i^{\frac{Q}{2b}}$ is in the multiplier $M(\mathcal{A})$ of \mathcal{A} . Finally the L^2 -norm (4.4) can be thought of as shifting the arguments by $\frac{iQ}{2}$ and using the usual L^2 -norm. More precisely,

$$(4.6) \quad \mathcal{H} \longrightarrow L^2(\mathbb{R}^{N+n}, dt_1 \dots dt_N ds_1 \dots ds_n) \\ F(\dots, t_k, \dots) \mapsto F'(\dots, t_k, \dots) := F\left(\dots, t_k + \frac{iQ_{i_k}}{2}, \dots\right)$$

is an isomorphism of Hilbert space. Therefore, to obtain a representation on the usual L^2 -norm, we work on F and at the end shift all the variables t_k by this map.

Proposition 4.4. *The GNS representation is unitarily equivalent to a representation that is independent on the s_i variables.*

Proof. The action of multiplication by \mathbf{e}_i from the left will not depend on s_i . Moreover, multiplication of K_i from the left, after commuting with all the \mathbf{e}_{α_k} , gives us an extra factor $e^{-2\pi b p_{s_i}}$, which is independent from the t_i variables. Therefore these p_{s_i} can be treated as constants λ_i . Once we have shown that the t_k part is unitarily equivalent to the representation \mathcal{P} in Section 5, by Theorem 3.1 there exists a unitary transformation that can remove the dependence on s_i completely. \square

In particular the GNS representation is unitarily equivalent to

$$(4.7) \quad \mathcal{H} = L^2(\mathbb{R}^{N+n}) \simeq \mathcal{P} \otimes M_K$$

for a trivial multiplicity module $M_K \simeq L^2(\mathbb{R}^n)$. Therefore, without loss of generality, we will ignore the $K_i^{ib^{-1}s_i}$ from now on, and the GNS representation calculated below will be assumed to act trivially on the s_i variables. Note that the \mathbf{e}_{α_k} part already gives the correct functional dimension.

4.3. Rank 2 case. We illustrate the representation in the rank 2 case, which is essential to generalize to the higher rank. Let us first consider the simply-laced case and fix the longest element to be $w_0 = s_2 s_1 s_2$. Also using Definitions 2.3 and 4.1, we name the variables

$$(u, v, w) := (v_1, v_2, v_3) = (t_3, t_2, t_1) = (u_2^1, u_1^1, u_2^1).$$

Then a typical element of $\vec{F} \in \mathcal{A}$ (ignoring the ending K_i terms) is given by

$$(4.8) \quad \vec{F} = \int_C F(u, v, w) \frac{\mathbf{e}_2^{ib^{-1}w} K_2^{\frac{Q}{2b}}}{S_b(Q + iw)} \frac{\mathbf{e}_{12}^{ib^{-1}v} (K_1 K_2)^{\frac{Q}{2b}}}{S_b(Q + iw)} \frac{\mathbf{e}_1^{ib^{-1}u} K_1^{\frac{Q}{2b}}}{S_b(Q + iw)} dudvdw \\ = \int_C F(u, v, w) e^{-\frac{\pi Q v}{2}} \frac{\mathbf{e}_2^{ib^{-1}w}}{S_b(Q + iw)} \frac{\mathbf{e}_{12}^{ib^{-1}v}}{S_b(Q + iw)} \frac{\mathbf{e}_1^{ib^{-1}u}}{S_b(Q + iw)} dudvdw,$$

where in the last line we moved the K_i to the right and absorbed into the (ignored) K terms. Note that we have reversed the order of the variables since we used the opposite multiplication order (reading from left to right).

Let us illustrate how to find the actions of the generators \mathbf{e}_i in general. The action of \mathbf{e}_2 is easy. It only shifts the variables w . We have (ignoring other variables)

$$\begin{aligned} \mathbf{e}_2 \cdot \vec{F} &= \int F(u, v, w) \dots \frac{\mathbf{e}_2^{ib^{-1}w+1}}{S_b(Q + \mathbf{i}w)} \dots dw \\ &= \int F(u, v, w + \mathbf{i}b) \dots \frac{\mathbf{e}_2^{ib^{-1}w}}{S_b(Q + \mathbf{i}w - b)} \dots dw \\ &= -i \int (e^{\pi bw} - e^{-\pi bw}) F(u, v, w + \mathbf{i}b) \dots \frac{\mathbf{e}_2^{ib^{-1}w}}{S_b(Q + \mathbf{i}w)} \dots dw, \end{aligned}$$

where we have used property (2.21) of S_b . Hence the action of \mathbf{e}_2 on \mathcal{H} (i.e., on the coefficient function) is given by

$$(4.9) \quad \mathbf{e}_2 = -\mathbf{i}(e^{\pi bw} - e^{-\pi bw})e^{-2\pi bp_w}.$$

Finally, the shift $w \mapsto \frac{iQ}{2}$ gives us the positive expression on $L^2(\mathbb{R}^3)$:

$$\begin{aligned} \mathbf{e}_2 &= -\mathbf{i}(e^{\pi b(w + \frac{iQ}{2})} - e^{-\pi b(w + \frac{iQ}{2})})e^{-2\pi bp_w} \\ &= -\mathbf{i}(iq^{\frac{1}{2}}e^{\pi bw} + iq^{-\frac{1}{2}}e^{-\pi bw})e^{-2\pi bp_w} \\ (4.10) \quad &= e(w - 2p_w) + e(-w - 2p_w). \end{aligned}$$

For the action of \mathbf{e}_1 , we use Proposition 2.23 and proceed as above:

$$\begin{aligned} \mathbf{e}_1 \mathbf{e}_2^{ib^{-1}w} &= q^{ib^{-1}w} \mathbf{e}_2^{ib^{-1}w} \mathbf{e}_1 + q^{\frac{1}{2}}(q^{-ib^{-1}w} - q^{ib^{-1}w}) \mathbf{e}_2^{ib^{-1}w-1} \mathbf{e}_{12} \\ (4.11) \quad &= e^{-\pi bw} \mathbf{e}_2^{ib^{-1}w} \mathbf{e}_1 + q^{\frac{1}{2}}(e^{\pi bw} - e^{-\pi bw}) \mathbf{e}_2^{ib^{-1}w-1} \mathbf{e}_{12}. \end{aligned}$$

In the first term of (4.11) \mathbf{e}_1 commutes again with $\mathbf{e}_{12}^{ib^{-1}v}$ and picks up the factor $q^{-ib^{-1}v} = e^{\pi bv}$ before being absorbed into $\mathbf{e}_1^{ib^{-1}u}$, producing the same S_b factor as above. Hence after shifting by $\frac{iQ}{2}$ we get overall

$$e(-w + v)(e(u - 2p_u) + e(-u - 2p_u)).$$

The second term of (4.11) gives an opposite shifting $e(2p_w)$ in the variable w , which cancels the factor in front:

$$q^{\frac{1}{2}}(e^{\pi b(w-ib)} - e^{-\pi b(w-ib)}) \frac{S_b(Q + \mathbf{i}w)}{S_b(Q + \mathbf{i}w + b)} = iq^{\frac{1}{2}}.$$

The shifting $e(-2p_v)$ in v by \mathbf{e}_{12} then provides the same factor as before, as well as the extra number $e^{-\frac{\pi Q(ib)}{2}} = (\mathbf{i}q^{\frac{1}{2}})^{-1}$ from the auxiliary term $e^{-\frac{\pi Qv}{2}}$ which cancels the above number. After shifting by $\frac{iQ}{2}$ we then obtain

$$e(v - 2p_v + 2p_w) + e(-v - 2p_v + 2p_w).$$

Overall, the action of \mathbf{e}_1 is then given by

$$(4.12) \quad \mathbf{e}_1 = e(-w + v + u - 2p_u) + e(-w + v - u - 2p_u) + e(v - 2p_v + 2p_w) + e(-v - 2p_v + 2p_w).$$

Finally, the action of K_i is easy to read out. Multiplying from the left, each K_i commutes with the root vectors \mathbf{e}_j and picks up a factor of the form $(q^k)^{ib^{-1}u} = e(-ku)$, before being absorbed into the K terms. We find that the action is given by

$$K_1 = e(-2u - v + w - 2p_{s_1} + \mathbf{i}Q), \quad K_2 = e(u - v - 2w - 2p_{s_2} + \mathbf{i}Q).$$

After shifting (u, v, w) by $\frac{iQ}{2}$, we have a positive action of K_i . Then by Proposition 4.4, we can ignore the p_{s_i} factors.

In summary, we obtained

$$\begin{aligned} \mathbf{e}_1 &= e(-w + v + u - 2p_u) + e(-w + v - u - 2p_u) + e(v - 2p_v + 2p_w) \\ &\quad + e(-v - 2p_v + 2p_w), \\ \mathbf{e}_2 &= e(w - 2p_w) + e(-w - 2p_w), \\ K_1 &= e(-2u - v + w), \\ K_2 &= e(u - v - 2w), \end{aligned}$$

and we will show that this is unitarily equivalent to the positive representation \mathcal{P} .

We can do the same procedure similarly for the non-simply-laced case. For Type B_2 , using $w_0 = s_1 s_2 s_1 s_2$, we define

$$\begin{aligned} \vec{F} = \int_C F(u, v, w) e^{-\pi Q_s v - \frac{\pi Q u}{2}} &\frac{\mathbf{e}_2^{ib^{-1}w}}{S_b(Q + iw)} \frac{\mathbf{e}_{12}^{ib^{-1}v}}{S_{b_s}(Q_s + iv)} \frac{\mathbf{e}_X^{ib^{-1}u}}{S_b(Q + iu)} \\ &\times \frac{\mathbf{e}_1^{ib^{-1}t}}{S_{b_s}(Q_s + it)} dt du dv dw, \end{aligned}$$

where $\mathbf{e}_X := T_1 T_2(\mathbf{e}_1)$. Then using successively Proposition 2.24, we arrive at

$$\begin{aligned} \mathbf{e}_1 &= e(-w + u + t - 2p_t) + e(-w + u - t - 2p_t) \\ &\quad + e(-w + v + u - 2p_u + 2p_v) + e(-w + v - u - 2p_u + 2p_v) \\ &\quad + e(v - 2p_v + 2p_w) + e(-v - 2p_v + 2p_w), \\ \mathbf{e}_2 &= e(w - 2p_w) + e(-w - 2p_w), \\ K_1 &= e(-2t - u + w), \\ K_2 &= e(2t - 2v - 2w). \end{aligned}$$

From now on we will ignore M_K (cf. (4.7)) and only work on the $L^2(\mathbb{R}^N)$ part.

4.4. Twisting. We have seen from the last section that the action is positive self-adjoint and each term is very closely related to the standard form $[u]e(-2p_u)$ except for a small twist (cf. first and second term of (4.12)). In the simply-laced case this comes from the commutation relation arising from any relations that are equivalent (i.e., Lusztig transformed) to $\mathbf{e}_i \cdot \mathbf{e}_j^{ib^{-1}w}$, and locally the action looks exactly like (4.12). Now using Lemma 2.17 successively, we transform the action by $\mathbf{e}_i \mapsto \Phi_1 \mathbf{e}_i \Phi_1^*$:

$$\Phi_1 = g_b(e(w - u - 2p_u + 2p_v - 2p_w)) g_b^*(e(-w - u - 2p_u + 2p_v - 2p_w)).$$

This absorbs the second term of \mathbf{e}_1 into the third term and spits out the symmetric term from the fourth term, giving

$$(4.13) \quad \mathbf{e}_1 = [-w + v + u]e(-2p_u) + [v]e(-2p_v + 2p_w),$$

$$(4.14) \quad \mathbf{e}_2 = [w]e(-2p_w),$$

$$(4.15) \quad K_1 = e(-2u - v + w),$$

$$(4.16) \quad K_2 = e(u - v - 2w),$$

which is now symmetric in the quantum variables. On the other hand, one checks that this action preserves the action of \mathbf{e}_2 : it spits out an auxiliary term from the first term and then reabsorbs it back into the second term.

For the doubly-laced case, this transformation is given by

$$\begin{aligned} \Phi_2 &= g_{b_s}(e(-t + v - 2p_t + 2p_u - 2p_v)g_{b_s}^*(e(-t - v - 2p_t + 2p_u - 2p_v) \\ &\quad \circ g_b(e(w - u - 2p_u + 4p_v - 2p_w))g_b^*(e(-w - u - 2p_u + 4p_v - 2p_w))), \end{aligned}$$

which gives us

$$(4.17) \quad \mathbf{e}_1 = [t + u - w]e(-2p_t) + [u + v - w]e(-2p_u + 2p_v) + [v]e(-2p_v + 2p_w),$$

$$(4.18) \quad \mathbf{e}_2 = [w]e(-2p_w),$$

$$(4.19) \quad K_1 = e(-2t - u + w),$$

$$(4.20) \quad K_2 = e(2t - 2v - 2w).$$

In general, we can define transformations Φ_k which preserve the action of all $\mathbf{e}_i, i \neq k$, and bring the action of \mathbf{e}_k to the standard form (2.13). This transformation depends on the fact that there is a variable v_i of minimal index, corresponding to \mathbf{e}_{ij} in (2.42), which does not pick up any $e^{\pi b x}$ factor, hence in the standard form (2.13). Then one successively applies the above transformation to remove the twisting within \mathbf{e}_k in the order where the commutation relation takes place between adjacent pairs of root vectors (or their Lusztig transforms).

Finally, we note that under the following change of variables T , with the corresponding action on p given by the transpose inverse:

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} \mapsto \begin{pmatrix} v - u \\ u \\ w \end{pmatrix}, \quad \begin{pmatrix} p_u \\ p_v \\ p_w \end{pmatrix} \mapsto \begin{pmatrix} p_u + p_v \\ p_u \\ p_w \end{pmatrix}$$

the representation is exactly the same as that from Proposition 2.9. In particular

Proposition 4.5. *The GNS representation is independent of the choice of the reduced expression of the longest element $w_0 \in W$.*

Proof. For a change of reduced expression of w_0 , it means the change in Lusztig’s transformation $\dots T_i T_j T_i \dots \rightarrow \dots T_j T_i T_j \dots$ which locally is equivalent to the transformation given by the case of type A_2 . Hence any such transformation is given by the unitary equivalence $T^* \circ \Phi \circ T$, where Φ is the unitary equivalence from the positive representation [17], while T is the transformation matrix above. One can then calculate explicitly the map as

$$(4.21) \quad \Psi = T' \circ g_b(e(w - u + 2p_u - 2p_v + 2p_w)g_b^*(e(-w + u + 2p_u - 2p_v + 2p_w)),$$

where

$$T' : \begin{pmatrix} u \\ v \\ w \end{pmatrix} \mapsto \begin{pmatrix} -u + v + w \\ u \\ v \end{pmatrix}$$

is a unitary transformation. Finally note that the transformation Ψ is essentially the inverse transpose of Φ (interchanging the role of p_i ’s and v_i ’s); hence it is consistent with the choice of paths as established in [17]. Similarly we have a generalized transformation for the doubly-laced case given by ratios of 6 quantum dilogarithms followed by a linear transformation, which is again the inverse transpose of the transformation found in [18]. □

With the procedure above, we can construct the GNS representation \mathcal{P}_{GNS} for $\mathcal{U}_{q\bar{q}}(\mathfrak{b}_{\mathbb{R}})$ of arbitrary type (using Proposition 2.24, etc.). In particular, \mathcal{P}_{GNS} satisfies all properties of a positive representation (positivity and transcendental relations (2.11)); hence this gives us another family of positive representations for $\mathcal{U}_{q\bar{q}}(\mathfrak{b}_{\mathbb{R}})$. From the expression of Ψ , which is essentially the inverse transpose of Φ , we have

Theorem 4.6. *The expressions of e_i obtained from the GNS construction (after twisting) is precisely the positive representations with all the terms $[X]e(-2p_Y)$ replaced by $[Y]e(-2p_X)$ (cf. Definition 2.6). The action of K_i is simply obtained from a linear change of variables by Lusztig’s isomorphism corresponding to reflections by simple roots.*

In the rank 2 case, one should compare the expressions (4.13)-(4.20) with the expressions from Propositions 2.9 and 2.10.

5. EQUIVALENCE OF REPRESENTATION $\mathcal{P}_{GNS} \simeq \mathcal{P}$

So far we have constructed a family of representations \mathcal{P}_{GNS} from the GNS-representation. In this section we will show that the representation is unitarily equivalent to the one constructed earlier in [17, 18].

For type A_n , it turns out we can do this directly by a change of variables.

Theorem 5.1. *For type A_n , the GNS representation \mathcal{P}_{GNS} constructed by left multiplication on $\mathcal{A} = \mathcal{U}_{q\bar{q}}^{C^*}(\mathfrak{b}_{\mathbb{R}})$ is unitarily equivalent to the positive representation $\mathcal{P}_{\lambda} \simeq \mathcal{P}$ of $\mathcal{U}_{q\bar{q}}(\mathfrak{g}_{\mathbb{R}})$ restricted to its Borel part.*

Proof. It suffices to consider a particular choice of reduced expression of w_0 . We fixed the longest element $w_0 \in W$ to be the standard expression given in [17]:

$$w_0 = s_n s_{n-1} \dots s_1 s_n s_{n-1} \dots s_2 \dots s_n s_{n-1} s_n.$$

Then the GNS representation constructed in this section is unitarily equivalent to the positive representation \mathcal{P} by a change of variables (using the notation from Definition 2.3):

$$\begin{aligned} u_k^1 &\mapsto u_n^{n+1-k}, \\ u_k^m &\mapsto u_{n+1-m}^{n+1-k} - u_{n+2-m}^{n+2-k}, \quad m > 1. \end{aligned}$$

□

For other types, we use the following trick together with Lusztig’s isomorphisms.

Lemma 5.2. *There exists a unitary transformation such that the action of e_i is transformed into $q_i^{-1}K_i e_i$.*

Proof. If we add $\prod_{k=1}^N K_{\alpha_k}^{t_k}$ in front of $\vec{F} \in \mathcal{A}$ in Definition 4.3, then the action of e_i will pick up the correct factors according to K_i^{-1} . But if we commute these factors through the e_{α_k} to the right and absorb them into the K terms, then on the representation space \mathcal{P} this is equivalent to multiplying by an exponential function of the form $e^{\pi i h(t_1, \dots, t_N)}$, where $h(t_1, \dots, t_N)$ is a homogeneous quadratic function in the variables t_k . In particular this is a unitary transformation. □

Lemma 5.3. *Lusztig’s isomorphism (cf. Theorem 2.20) corresponding to the longest word of the Weyl group $w_0 = s_{i_1} \dots s_{i_N}$ given by $T_{i_1} \dots T_{i_N}$ transforms*

$$q_i^{-1} K_i \mathbf{e}_i \mapsto \mathbf{f}_{\sigma(i)},$$

where \mathbf{f}_i is the lower Borel generators, σ is the identity automorphism for \mathfrak{g} of types $B_n, C_n, D_{2n}, E_7, E_8, F_4, G_2$, and the unique index 2 automorphisms on the simple roots for \mathfrak{g} of types A_n, D_{2n+1} , and E_6 .

Proof. It follows directly from the action given by Theorem 2.20, the properties from Proposition 2.21, and the fact that the longest Weyl element acts on simple roots as $\alpha_i \mapsto -\alpha_{\sigma(i)}$. □

Since we have seen from Theorem 4.6 that the action given by \mathcal{P}_{GNS} is precisely obtained by flipping $[X]e(-2p_Y) \rightarrow [Y]e(-2p_X)$, Lusztig’s isomorphism T_i modified according to this flip will map the action of \mathbf{e}_i on \mathcal{P}_{GNS} to the corresponding action of \mathbf{f}_i given by Theorem 2.7. More precisely, we obtain

$$(5.1) \quad \mathbf{e}_i = \sum_{k=1}^n [-u_i^k] e \left(\sum_{j=1}^{v(i,k)-1} a_{i,r(j)} p_j + p_i^k \right),$$

with the corresponding roots and reduced expression of w_0 relabeled. Recall that a_{ij} is the Cartan matrix, $r(j)$ is the root label corresponding to the variable p_j (which is the shift in the variable v_j), and $v(i, k)$ is the index such that $u_i^k = v_{v(i,k)}$ (i.e., we sum all the terms appearing to the left of p_i^k) and we use the abused notation from Definition 2.6.

Theorem 5.4. *There exists a unitary transformation such that (5.1) is unitarily equivalent to (2.15).*

Proof. Notice that for each term in the expression of \mathbf{f}_i , the variable p_i^k with the maximal index $v(i, k)$ is unique, and these cover all variables (p_1, \dots, p_N) . Therefore we apply the transformation successively from the largest $v(i, k)$ index by

$$(5.2) \quad b_i p_i^k \mapsto b_i p_i^k - \sum_{j=1}^{v(i,k)-1} a_{i,r(j)} b_j p_j,$$

which induces the corresponding change of variables

$$(5.3) \quad b_j v_j \mapsto b_j v_j + a_{i,r(j)} b_i v_i^k$$

for $1 \leq j \leq v(i, k) - 1$. It is clear that under this transformation, we reduce all exponential terms to $e(2p_i^k)$. On the other hand, it is easy to see that under (5.3) the quantum factors are changed to

$$[-u_i^k] \mapsto \left[-u_i^k - \sum_{j=v(i,k)+1}^N a_{i,r(j)} v_j \right].$$

But this is precisely the action of \mathbf{f}_i corresponding to the longest element w_0 with the reversed reduced expression $w_0 = s_{i_N} \dots s_{i_1}$. Since the positive representation is independent of the reduced expression of w_0 , we proved the unitary equivalence. □

Finally, since K_i depends only linearly on the variables (inside an exponential), it is uniquely determined by the shifts $e(2p_i^k)$ appearing in the action of \mathbf{f}_i . In particular, under Lusztig’s transformation and the above unitary equivalence, K_i from \mathcal{P}_{GNS} transforms precisely to $K_{\sigma(i)}^{-1}$ as given in Theorem 2.7.

Therefore we can now state our main theorem.

Theorem 5.5. *For any simple Lie algebra \mathfrak{g} , the GNS representation \mathcal{P}_{GNS} constructed by left multiplication on $\mathcal{A} = \mathcal{U}_{q\bar{q}}^{C^*}(\mathfrak{b}_{\mathbb{R}})$ is unitarily equivalent to the positive representation $\mathcal{P}_{\lambda} \simeq \mathcal{P}$ of $\mathcal{U}_{q\bar{q}}(\mathfrak{g}_{\mathbb{R}})$ restricted to its Borel part.*

Corollary 5.6. *The tensor product of the positive representations \mathcal{P} restricted to the Borel part is unitarily equivalent to the tensor product of itself with a trivial multiplicity module:*

$$(5.4) \quad W : \mathcal{P} \otimes \mathcal{P} \simeq M \otimes \mathcal{P},$$

where the underlying Hilbert spaces are all $\mathcal{P} \simeq M \simeq L^2(\mathbb{R}^N)$, $N = l(w_0)$.

Proof. Since we know that $\mathcal{P} \simeq \mathcal{P}_{GNS}$ and the GNS-space $\mathcal{H} \simeq \mathcal{P}_{GNS} \otimes M_K$, this follows immediately from Proposition 2.27 of the property of the multiplicative unitary. The unitary transformation will be given by the multiplicative unitary W after factoring the trivial intertwiners on M_K , for example, by supplying the delta distributions on the corresponding variables. □

6. EXAMPLE

In this section, we write explicitly the representations and decompositions for types A_1 and A_2 , which provide us a useful basic framework when we try to deal with the explicit construction of quantum higher Teichmüller theory. The computation for type A_3 can be found in [21].

6.1. Type A_1 . Let us first consider the simplest case where \mathfrak{g} is of type A_1 . The Borel part of $\mathcal{U}_{q\bar{q}}(\mathfrak{sl}(2, \mathbb{R}))$ in this case is nothing but the quantum plane $\mathcal{B}_{q\bar{q}}$, which is studied extensively in [14, 16]. Nonetheless, it will be illustrative to see how the analysis works with respect to the theory of multiplicative unitary from Proposition 2.27.

First, we can do a transformation to bring the representation \mathcal{P}_{GNS} into a canonical form. By applying $e^{\frac{\pi i u^2}{2}} g_b(e(-2u))$, the action $\mathbf{e} = [u]e(-2p_u)$ is transformed to $\mathbf{e} = e(-2p_u)$. Also taking into account the positivity of K under the shift $u \mapsto u + \frac{iQ}{2}$, in terms of the C^* -algebraic Borel part \mathcal{A} , the expression is given by

$$\mathcal{A} \ni f = \int_C f(u, s) e^{ib^{-1}u} K^{ib^{-1}s} e^{\pi Qs} dud s.$$

Now recall from (2.50) that the decomposition is given by the multiplicative unitary

$$W^*(\Lambda(f) \otimes \Lambda(g)) = (\Lambda \otimes \Lambda)(\Delta(g)(f \otimes 1)).$$

One can compute the action of W^* on $f(u, s)g(v, t)$ to be

$$\begin{aligned} &W^*(f \otimes g) \\ &= \int f(u - \tau, s - t + \tau)g(v + \tau, t - \tau) \frac{G_b(Q + \mathbf{i}v + \mathbf{i}\tau)e^{-\pi Q t}}{G_b(Q + \mathbf{i}v)G_b(Q + \mathbf{i}\tau)} e^{-2\pi \mathbf{i}(u-\tau)(t-\tau)} d\tau \\ &= g_b^*(e(-2v))g_b^*(e(-2p_u - 2p_t + 2p_v)) \circ g_b(e(-2v))e^{-2\pi \mathbf{i}t(p_s+u)}(f \otimes g), \\ &= g_b^*(e(-2p_u - 2p_t + 2p_v - 2v)) \circ g_b^*(e(-2p_u - 2p_t + 2p_v)) \circ e^{-2\pi \mathbf{i}t(p_s+u)}(f \otimes g), \end{aligned}$$

where we shifted u, v by $\frac{iQ}{2}$, according to the Haar measure (4.4) to make the map unitary, and we used Lemma 2.15 and the pentagon relation (2.29). Now, recall that we can transform it such that the action on \mathcal{P}_{GNS} does not depend on s and t . This can be achieved by taking the Fourier transform on s and t and shifting the variables by $u \mapsto u - s, v \mapsto v - t$. We obtain finally

$$(6.1) \quad W^* = g_b^*(e(-2p_u + 2p_v - 2v))e^{2\pi \mathbf{i}p_v u} g_b^*(e(-2p_u - 2t)) \circ e^{2\pi \mathbf{i}p_t u}.$$

Since the variables s, t correspond to the trivial module M_K , we can obtain the required unitary transformation on \mathcal{P}_{GNS} by supplying the delta distribution $\delta(s)\delta(t)$, such that upon acting by $e^{2\pi \mathbf{i}p_t u}$, we get

$$W^* = g_b^*(e(-2p_u + 2p_v - 2v))e^{2\pi \mathbf{i}p_v u} g_b^*(e(-2p_u + 2u)).$$

Finally, acting on $1 \otimes \mathcal{B}_q$ by W^* , the term $g_b^*(e(-2p_u + 2u))$ has no effects and acts as a trivial intertwiner. Hence factorizing this action, we simplify \mathcal{P}_{GNS} and obtain

$$(6.2) \quad W^* = g_b^*(e(-2p_u + 2p_v - 2v))e^{2\pi \mathbf{i}p_v u},$$

$$\begin{aligned} 1 \otimes \mathbf{e} &\mapsto \Delta(\mathbf{e}) = \mathbf{e} \otimes K + 1 \otimes \mathbf{e}, \\ 1 \otimes K &\mapsto \Delta(K) = K \otimes K, \end{aligned}$$

which is precisely the integral transformation described in [14, 16].

6.2. Type A_2 . As we have seen, in general it is hard to calculate the equivalence transformation from the multiplicative unitary. However, using the functional properties of the quantum dilogarithms, the required transformation can be explicitly calculated for Type A_n . Let us consider the next simplest case $\mathfrak{g} = \mathfrak{sl}_3$. First we simplify the expression of $\mathcal{P}_\lambda \simeq \mathcal{P}$ from Proposition 2.9.

Lemma 6.1. *\mathcal{P} is unitarily equivalent to the following expressions:*

$$\begin{aligned} \mathbf{e}_1 &= e(v - w - 2p_v) + e(u - 2p_u - 2p_v + 2p_w), & K_1 &= e(u - 2v + w), \\ \mathbf{e}_2 &= e(w - 2p_w), & K_2 &= e(-2u + v - 2w). \end{aligned}$$

Proof. Apply Φ_1 (i.e., $\mathbf{e}_i \mapsto \Phi_1 \mathbf{e}_i \Phi_1^*$), where

$$\begin{aligned} \Phi_1 &= g_b(e(-u - v + w + 2p_u - 2p_w))g_b(e(u - v - w - 2p_u + 2p_w)) \\ &\quad \times g_b(e(-2u))g_b(e(-2w)). \end{aligned}$$

□

Using this expression, let us consider the tensor product representation $\mathcal{P} \otimes \mathcal{P}$:

$$\Delta(\mathbf{e}_1) = e(v - w + u' - 2v' + w' - 2p_v) + e(u + u' - 2v' + w' - 2p_u - 2p_v + 2p_w) + e(v' - w' - 2p'_v) + e(u' - 2p'_u - 2p'_v + 2p'_w),$$

$$\Delta(\mathbf{e}_2) = e(w - 2u' + v' - 2w' - 2p_w) + e(w' - 2p'_w),$$

$$\Delta(K_1) = e(u - 2v + w + u' - 2v' + w'),$$

$$\Delta(K_2) = e(-2u + v - 2w - 2u' + v' - 2w'),$$
 where the $'$ variables denote the second component.

Lemma 6.2. *The above action on $\mathcal{P} \otimes \mathcal{P}$ is unitarily equivalent to*

$$\Delta(\mathbf{e}_i) \simeq 1 \otimes \mathbf{e}_i, \\ \Delta(K_i) \simeq K_i \otimes K_i, \quad i = 1, 2.$$

Proof. Apply Φ_2 where

$$\begin{aligned} \Phi_2 = & g_b(e(w - u' - 2w' - 2p_w - 2p'_u + 4p'_w)) \\ & \circ g_b(e(u - 2v' + w' - 2p_u - 2p_v + 2p_w + 2p'_u + 2p'_v - 2p'_w)) \\ & \circ g_b(e(v - w - 2v' + w' - 2p_v + 2p'_u + 2p'_v - 2p'_w)) \\ & \circ g_b(e(w - 2u' + v' - 3w' - 2p_w + 2p'_w)). \end{aligned}$$

□

Corollary 6.3. *From the explicit expression of Lemma 6.1, the transformation Φ_2 can be written in a way independent of the representation:*

$$\begin{aligned} \Phi_2 = & g_b(q^2 \mathbf{e}_2 \otimes K_2 \mathbf{e}_{21} \mathbf{e}_{12}^{-1} \mathbf{e}_2^{-1}) g_b(q^2 \mathbf{e}_{21} \mathbf{e}_2^{-1} \otimes K_1 \mathbf{e}_{21}^{-1} \mathbf{e}_2) g_b(q \mathbf{e}_{12} \mathbf{e}_2^{-1} \otimes K_1 \mathbf{e}_{21}^{-1} \mathbf{e}_2) \\ & \times g_b(q \mathbf{e}_2 \otimes K_2 \mathbf{e}_2^{-1}). \end{aligned}$$

Hence this transformation gives directly the intertwiners

$$1 \otimes \mathbf{e}_i \simeq \Delta(\mathbf{e}_i), \\ K_i \otimes K_i \simeq K_i \otimes K_i \quad i = 1, 2.$$

Here we use the appropriate q factors to make the arguments positive. Note that the inverse operator is defined in the natural way. Since it is positive and all the terms q -commute, the expression of the transformation is still well-defined.

Finally, note that u, v, w from the first component of K_i now acts as constant, so we can treat them as λ 's and use Theorem 3.1 to remove them. We obtain finally

$$\begin{aligned} \Delta(\mathbf{e}_1) \simeq & 1 \otimes \mathbf{e}_1 = e(v' - w' - 2p'_v) + e(u' - 2p'_u - 2p'_v + 2p'_w), \\ \Delta(\mathbf{e}_2) \simeq & 1 \otimes \mathbf{e}_2 = e(w' - 2p'_w), \\ \Delta(K_1) \simeq & 1 \otimes K_1 = e(u' - 2v' + w'), \\ \Delta(K_2) \simeq & 1 \otimes K_1 = e(-2u' + v' - 2w'). \end{aligned}$$

In other words, when restricting to the Borel part, we have

$$\mathcal{P}_{\lambda_1} \otimes \mathcal{P}_{\lambda_2} \simeq \mathcal{P} \otimes \mathcal{P} = L^2(\mathbb{R}^3) \otimes \mathcal{P}.$$

Remark 6.4. For type A_n , there is a pattern of this series of transformations by g_b computed inductively by Mathematica, and we obtain

$$\mathcal{P}_{\lambda_1} \otimes \mathcal{P}_{\lambda_2} \simeq L^2(\mathbb{R}^{\frac{n(n+1)}{2}}) \otimes \mathcal{P}.$$

This procedure seems to be related to the Heisenberg double [23]. For other types however we did not establish any explicit construction of such transformations.

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