

Cluster realization of positive representations of split real quantum Borel subalgebra

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Dedicated to the memory of Ludvig D. Faddeev

Abstract

In our previous work [13], we studied the positive representations of split real quantum groups $\mathcal{U}_{q\bar{q}}(\mathfrak{g}_{\mathbb{R}})$ restricted to its Borel part, and showed that they are closed under taking tensor products. However, the tensor product decomposition was only constructed abstractly using the GNS-representation of a C^* -algebraic version of the Drinfeld-Jimbo quantum groups. In this paper, using the recently discovered cluster realization of quantum groups [14], we write down the decomposition explicitly by realizing it as a sequence of cluster mutations in the corresponding quiver diagram representing the tensor product.

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1 Introduction

1.1 Positive representations of split real quantum groups

To any finite dimensional complex simple Lie algebra \mathfrak{g} , Drinfeld [3] and Jimbo [15] defined a remarkable Hopf algebra $\mathcal{U}_q(\mathfrak{g})$ known as the quantum group. The notion of *positive representations* was introduced in [8] as a new research program devoted to the representation theory of its split real form $\mathcal{U}_{q\bar{q}}(\mathfrak{g}_{\mathbb{R}})$ which uses the concept of Faddeev's modular double [4, 5], and generalizes the case of $\mathcal{U}_{q\bar{q}}(\mathfrak{sl}(2, \mathbb{R}))$ studied extensively by Teschner *et al.* [2, 18, 19] from the physics point of view.

Explicit construction of the positive representations \mathcal{P}_{λ} of $\mathcal{U}_{q\bar{q}}(\mathfrak{g}_{\mathbb{R}})$, parametrized by the \mathbb{R}_+ -span of positive weights $\lambda \in P_{\mathbb{R}}^+$, was constructed in [11, 12] for simple Lie algebra \mathfrak{g} of all types, where the generators of the quantum groups are realized by positive essentially self-adjoint operators acting on certain Hilbert space, and compatible with the modular double structure (see Section 2.2 for a review). Although the representations involve

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unbounded operators, the algebraic relations are well-defined and are unitary equivalent to the *integrable representations* of the quantum plane in the sense of Schmüdgen [10, 24].

A long standing conjecture in the theory of positive representations is the closure under taking tensor product:

$$\mathcal{P}_\alpha \otimes \mathcal{P}_\beta \simeq \int_{\gamma \in P_{\mathbb{R}}^+} \mathcal{P}_\gamma \otimes \mathcal{M}_{\alpha\beta}^\gamma d\mu(\gamma) \quad (1.1)$$

for some Plancherel measure $d\mu(\gamma)$ and multiplicity module $\mathcal{M}_{\alpha\beta}^\gamma$. The simplest case of $\mathcal{U}_{q\bar{q}}(\mathfrak{sl}(2, \mathbb{R}))$ has been proved previously in [19], which is related to the fusion relations of quantum Liouville theory. The case in type A_n is recently solved algebraically in [23] using a cluster realization [14, 22] of the positive representations discussed below, and is related to certain quantum open Toda systems. For other types however, the question is still open, but as a first step, we showed in [13] that the restriction of \mathcal{P}_λ to the Borel part $\mathcal{U}_{q\bar{q}}(\mathfrak{b}_{\mathbb{R}})$ is indeed closed under taking tensor product:

Theorem 1.1. *Let \mathcal{P}_λ^b be the positive representations \mathcal{P}_λ restricted to the Borel part $\mathcal{U}_{q\bar{q}}(\mathfrak{b}_{\mathbb{R}})$. Then $\mathcal{P}_\lambda^b \simeq \mathcal{P}^b$ does not depend on λ , and we have the unitary equivalence*

$$\mathcal{P}^b \otimes \mathcal{P}^b \simeq \mathcal{P}^b \otimes \mathcal{M}, \quad (1.2)$$

where \mathcal{M} is a multiplicity module in which $\mathcal{U}_{q\bar{q}}(\mathfrak{b}_{\mathbb{R}})$ acts trivially.

As applications, this gives a new candidate for quantum higher Teichmüller theory, where the above unitary equivalence gives the *quantum mutation operator* satisfying the pentagon relation (see [9, 13] for a review). It also provides a major step towards proving the tensor product decomposition of \mathcal{P}_λ in general. Together with the braiding by the universal \mathcal{R} operator, the positive representations will carry a (continuous) braided tensor category structure, which may give rise to new class of TQFT's in the sense of Reshetikhin-Turaev [20, 21].

The construction of the unitary equivalence in [13] utilizes the language of *multiplier Hopf algebra* and the GNS representations of C^* -algebra, which leads to the existence of a unitary operator W called the *multiplicative unitary* giving the desired intertwiner (1.2). However, the construction of W on the C^* -algebraic level is quite abstract and it is very difficult to write down explicitly the intertwiner as a product of quantum dilogarithm functions. We presented several examples in [13] for type A_n and hint at a relationship to the Heisenberg double, but we were not able to generalize it to other types nor write down the formula in general.

1.2 Cluster realization of positive representations

In this paper, we reprove Theorem 1.1 using a new technique which comes with the discovery of the cluster realization of quantum groups in [22] for type A_n and [14] for general types, where we found an embedding of the Drinfeld's double of the Borel part $\mathcal{D}(\mathcal{U}_q(\mathfrak{b})) \hookrightarrow \mathcal{X}_{D_{2,1}}$ into a quantum cluster algebra associated to a quiver $Q_{D_{2,1}}$ on the triangulation of a once punctured disk with two marked points. The quiver itself has a geometric meaning representing the Poisson structure of the moduli space of framed local systems for general groups [7, 17]. A polarization of the quantum cluster variables, i.e. a choice of representations by the canonical variables on some Hilbert space $L^2(\mathbb{R}^N)$ through the exponential functions, recovers the positive representations \mathcal{P}_λ of $\mathcal{U}_{q\bar{q}}(\mathfrak{g}_{\mathbb{R}})$, which is the quotient of $\mathcal{D}(\mathcal{U}_q(\mathfrak{b}))$ by identifying the Cartan part $K_i K'_i = 1$.

In particular, the generators of $\mathcal{U}_q(\mathfrak{g})$ can be represented combinatorially using certain paths on $Q_{D_{2,1}}$ representing a telescoping sums of quantum cluster variables, while the coproduct is given by the quiver $Q_{D_{2,2}}$ on twice-punctured disk with two marked points, which is simply concatenating two copies of $Q_{D_{2,1}}$ along one edge. Furthermore, cluster mutations correspond to unitary equivalence of the representations \mathcal{P}_λ . Finally, to each triangulation, we constructed explicitly in [14] using the universal R matrix the sequence of cluster mutations μ realizing quiver mutations associated to the flip of triangulation as in Figure 1.

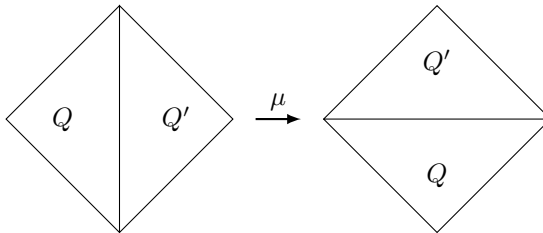


Figure 1: Flip of triangulation, with Q, Q' the associated quivers.

In this new language, it turns out that the restriction \mathcal{P}_λ^b can be easily describe by the so-called F_i -paths as shown schematically in red below in Figure 2. A flip of triangulation in this case creates a self-folded triangle, but since for the study of \mathcal{P}_λ^b , the mutation sequence does not involve the gluing edge \mathcal{E} in the quiver, we can relax by un-gluing the edges \mathcal{E} and consider only a normal flip of triangulation as in Figure 6 of Section 6, where we prove the main result:

Theorem 1.2. *The sequence of cluster mutations realizing the flip of triangulation preserves the F_i -paths representing the restriction of positive representations to $\mathcal{U}_{q\bar{q}}(\mathfrak{b}_{\mathbb{R}})$.*

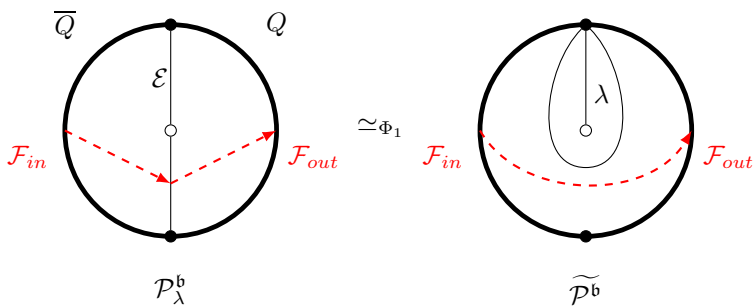


Figure 2: Flipping to self-folded triangle, where the F_i -paths are preserved.

In particular, this shows that \mathcal{P}_λ^b is unitary equivalent to a representation $\widetilde{\mathcal{P}}^b$ representing the generators on the *basic quiver* $Q_{D_{3,0}}$ associated only to a single triangle. In [13] we call this the *standard form*, which is obtained by omitting half of the operators in the explicit representation of \mathcal{P}_λ^b , and coincides with the so-called *Feigin's homomorphism*.

Finally, the tensor product realized by concatenating two copies of $Q_{D_{2,1}}$, can be decomposed using three flips of triangles as shown schematically in Figure 3 (which is the same as the first three steps in the proof of tensor product decomposition of \mathcal{P}_λ for the whole quantum group in type A_n constructed in [23]). Since we found in [14] an explicit formula for the cluster mutations using a product of quantum dilogarithms, we *completely solve the combinatorial problem* posed in [13] for explicitly writing down the tensor product decomposition of positive representations restricted to $\mathcal{U}_{q\bar{q}}(\mathfrak{b}_{\mathbb{R}})$ of *all types*, which cannot be done previously without the cluster realization.

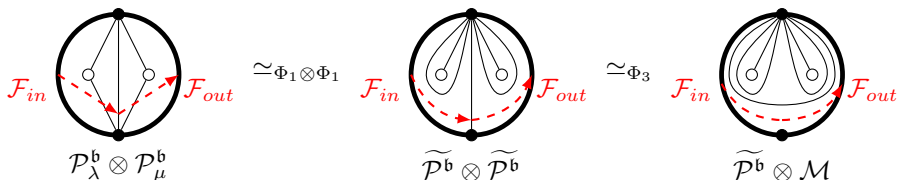


Figure 3: Tensor product decomposition $\mathcal{P}_\lambda^b \otimes \mathcal{P}_\mu^b \simeq \widetilde{\mathcal{P}}^b \otimes \mathcal{M}$.

1.3 Outline of the paper

The paper is organized as follows. In Section 2, we recall the positive representations \mathcal{P}_λ and some identities of the quantum dilogarithm function g_b . In Section 3, we recall the definitions and results concerning quantum torus algebra \mathcal{X}^1 . In Section 4, we reconstruct the basic quiver Q^i in [14] in the notation of the current paper, and in Section 5 we recall the cluster realization of positive representations. Finally in Section 6, we prove the main result and in Section 7 we give several examples to demonstrate the algorithms of the construction of the tensor product decomposition of positive representations restricted to the Borel part $\mathcal{U}_{q\bar{q}}(\mathfrak{b}_{\mathbb{R}})$.

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2 Preliminaries

In this section, let us recall several notations and definitions that will be used throughout the paper. We will follow mostly the convention used in [13] and [14].

2.1 Definition of the modular double $\mathcal{U}_{q\tilde{q}}(\mathfrak{g}_{\mathbb{R}})$

Let \mathfrak{g} be a simple Lie algebra over \mathbb{C} , $\mathcal{I} = \{1, 2, \dots, n\}$ denotes the set of nodes of the Dynkin diagram of \mathfrak{g} where $n = \text{rank}(\mathfrak{g})$. Let $\{\alpha_i\}_{i \in \mathcal{I}}$ be the set of positive simple roots, and let $w_0 \in W$ be the longest element of the Weyl group of \mathfrak{g} , where $N := l(w_0)$ is the length of the longest word. We call a sequence

$$\mathbf{i} = (i_1, \dots, i_N) \in \mathcal{I}^N$$

a reduced word of w_0 if $w_0 = s_{i_1} \dots s_{i_N}$ is a reduced expression, where s_{i_k} are the simple reflections of the root space. We will denote the reversed word by

$$\bar{\mathbf{i}} := (i_N, \dots, i_1).$$

Definition 2.1. Let q be a formal parameter. Let $(-, -)$ be the W -invariant inner product of the root lattice, and we define

$$a_{ij} := \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)},$$

such that $A := (a_{ij})$ is the Cartan matrix.

We normalize $(-, -)$ as follows: for $i \in \mathcal{I}$, we define the multipliers to be

$$\mathbf{d}_i := \frac{1}{2}(\alpha_i, \alpha_i) := \begin{cases} 1 & i \text{ is long root or in the simply-laced case,} \\ \frac{1}{2} & i \text{ is short root in type } B, C, F, \\ \frac{1}{3} & i \text{ is short root in type } G_2, \end{cases} \quad (2.1)$$

where $(\alpha_i, \alpha_j) = -1$ when i, j are adjacent in the Dynkin diagram, such that

$$\mathbf{d}_i a_{ij} = \mathbf{d}_j a_{ji}.$$

We then define $q_i := q^{\mathbf{d}_i}$, which we will also write as

$$q_l := q, \quad (2.2)$$

$$q_s := \begin{cases} q^{\frac{1}{2}} & \mathfrak{g} \text{ is of type } B_n, C_n, F_4, \\ q^{\frac{1}{3}} & \mathfrak{g} \text{ is of type } G_2, \end{cases} \quad (2.3)$$

for the q parameters corresponding to long and short roots respectively.

Definition 2.2. [3, 15] The Drinfeld-Jimbo quantum group $\mathcal{U}_q(\mathfrak{g}_{\mathbb{R}})$ is the Hopf algebra generated by $\{E_i, F_i, K_i^{\pm 1}\}_{i \in \mathcal{I}}$ over \mathbb{C} subjected to the relations for $i, j \in \mathcal{I}$:

$$K_i E_j = q_i^{a_{ij}} E_j K_i, \quad K_i F_j = q_i^{-a_{ij}} F_j K_i, \quad [E_i, F_j] = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}, \quad (2.4)$$

together with the Serre relations for $i \neq j$:

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \frac{[1-a_{ij}]_{q_i}!}{[1-a_{ij}-k]_{q_i}! [k]_{q_i}!} X_i^k X_j X_i^{1-a_{ij}-k} = 0, \quad X = E, F, \quad (2.5)$$

where $[k]_q := \frac{q^k - q^{-k}}{q - q^{-1}}$.

The Hopf algebra structure of $\mathcal{U}_q(\mathfrak{g})$ is given by

$$\Delta(E_i) = 1 \otimes E_i + E_i \otimes K_i, \quad (2.6)$$

$$\Delta(F_i) = F_i \otimes 1 + K_i^{-1} \otimes F_i, \quad (2.7)$$

$$\Delta(K_i) = K_i \otimes K_i. \quad (2.8)$$

We will not need the counit and antipode in this paper.

In the split real case, it is required that $|q| = 1$. Throughout the paper, we let

$$q := e^{\pi\sqrt{-1}b^2} \quad (2.9)$$

with $0 < b^2 < 1$ and $b^2 \in \mathbb{R} \setminus \mathbb{Q}$. We also write

$$q_i := e^{\pi\sqrt{-1}b_i^2}$$

such that

$$b_i := \begin{cases} b_l := b & \alpha_i \text{ is long root or } \mathfrak{g} \text{ is simply-laced,} \\ b_s := \sqrt{d_i}b & \alpha_i \text{ is short root.} \end{cases} \quad (2.10)$$

We define $\mathcal{U}_q(\mathfrak{g}_{\mathbb{R}})$ to be the real form of $\mathcal{U}_q(\mathfrak{g})$ induced by the star structure

$$E_i^* = E_i, \quad F_i^* = F_i, \quad K_i^* = K_i. \quad (2.11)$$

Finally, from the results of [11, 12], let $\tilde{q} := e^{\pi\sqrt{-1}b_s^{-2}}$ and we define the modular double to be

$$\mathcal{U}_{q\tilde{q}}(\mathfrak{g}_{\mathbb{R}}) := \mathcal{U}_q(\mathfrak{g}_{\mathbb{R}}) \otimes \mathcal{U}_{\tilde{q}}(\mathfrak{g}_{\mathbb{R}}) \quad \mathfrak{g} \text{ is simply-laced,} \quad (2.12)$$

$$\mathcal{U}_{q\tilde{q}}(\mathfrak{g}_{\mathbb{R}}) := \mathcal{U}_q(\mathfrak{g}_{\mathbb{R}}) \otimes \mathcal{U}_{\tilde{q}}({}^L\mathfrak{g}_{\mathbb{R}}) \quad \text{otherwise,} \quad (2.13)$$

where ${}^L\mathfrak{g}_{\mathbb{R}}$ is the Langlands dual obtained by interchanging the long and short roots of $\mathfrak{g}_{\mathbb{R}}$.

2.2 Positive representations of $\mathcal{U}_{q\tilde{q}}(\mathfrak{g}_{\mathbb{R}})$

In [8, 11, 12], a special class of representations for $\mathcal{U}_{q\tilde{q}}(\mathfrak{g}_{\mathbb{R}})$, called the positive representations, is defined. The generators of $\mathcal{U}_{q\tilde{q}}(\mathfrak{g}_{\mathbb{R}})$ are realized by positive essentially self-adjoint operators on certain Hilbert space, and satisfy the *transcendental relations* (2.15). In particular the quantum group and its modular double counterpart are represented on the same Hilbert space, generalizing the situation of $\mathcal{U}_{q\tilde{q}}(\mathfrak{sl}(2, \mathbb{R}))$ introduced in [4, 5] and studied in [19]. More precisely,

Theorem 2.3. [8, 11, 12] Define the rescaled generators to be

$$\mathbf{e}_i := 2 \sin(\pi b_i^2) E_i, \quad \mathbf{f}_i := 2 \sin(\pi b_i^2) F_i. \quad (2.14)$$

There exists a family of representations \mathcal{P}_λ of $\mathcal{U}_{q\tilde{q}}(\mathfrak{g}_{\mathbb{R}})$ parametrized by the \mathbb{R}_+ -span of the cone of positive weights $\lambda \in P_{\mathbb{R}}^+$, or equivalently by $\lambda \in \mathbb{R}_+^{\text{rank}(\mathfrak{g})}$, such that

- The generators $\mathbf{e}_i, \mathbf{f}_i, K_i$ are represented by positive essentially self-adjoint operators acting on $L^2(\mathbb{R}^N)$ where $N = l(w_0)$.

- Define the transcendental generators:

$$\tilde{\mathbf{e}}_i := \mathbf{e}_i^{\frac{1}{b_i^2}}, \quad \tilde{\mathbf{f}}_i := \mathbf{f}_i^{\frac{1}{b_i^2}}, \quad \widetilde{K}_i := K_i^{\frac{1}{b_i^2}}. \quad (2.15)$$

- if \mathfrak{g} is simply-laced, the generators $\tilde{\mathbf{e}}_i, \tilde{\mathbf{f}}_i, \widetilde{K}_i$ are obtained by replacing b with b^{-1} in the representations of the generators $\mathbf{e}_i, \mathbf{f}_i, K_i$.
- If \mathfrak{g} is non-simply-laced, then the generators $\widetilde{E}_i, \widetilde{F}_i, \widetilde{K}_i$ with $\tilde{\mathbf{e}}_i := 2 \sin(\pi b_i^{-2}) \widetilde{E}_i$ and $\tilde{\mathbf{f}}_i := 2 \sin(\pi b_i^{-2}) \widetilde{F}_i$ generate $\mathcal{U}_{\tilde{q}}(L\mathfrak{g}_{\mathbb{R}})$.

- The generators $\mathbf{e}_i, \mathbf{f}_i, K_i$ and $\tilde{\mathbf{e}}_i, \tilde{\mathbf{f}}_i, \widetilde{K}_i$ commute weakly up to a sign.

Let $\mathcal{P}_\lambda \simeq L^2(\mathbb{R}^N, du_1 \dots du_N)$, and let $u_k, p_k := \frac{1}{2\pi\sqrt{-1}} \frac{\partial}{\partial u_k}$ be the standard position and momentum operator respectively acting on $L^2(\mathbb{R}^N)$ as self-adjoint operators. Then the representations can be constructed explicitly as follows:

Theorem 2.4. [11, 12] For a fixed reduced word $\mathbf{i} = (i_1, \dots, i_N)$, the positive representation $\mathcal{P}_\lambda^{\mathbf{i}}$ parametrized by $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}_{\geq 0}^n$ is given by positive essentially self-adjoint operators

$$K_i = \exp\left(-2\pi b_i \lambda_i - \pi \sum_{k=1}^N a_{i, i_k} b_{i_k} u_k\right), \quad (2.16)$$

$$\begin{aligned} \mathbf{f}_i &= \mathbf{f}_i^- + \mathbf{f}_i^+ \\ &:= \sum_{k: i_k=i} \mathbf{f}^{k,-} + \sum_{k: i_k=i} \mathbf{f}^{k,+}, \end{aligned} \quad (2.17)$$

acting on $L^2(\mathbb{R}^N)$, where

$$\mathbf{f}^{k,\pm} := \exp\left(\pm \left(\sum_{j=1}^{k-1} \pi b_{i_j} a_{i_j, i_k} u_j + \pi b_{i_k} u_k + 2\pi b_{i_k} \lambda_{i_k}\right) + 2\pi b_{i_k} p_k\right). \quad (2.18)$$

The E_i generators corresponding to the right most root i_N of \mathbf{i} is given explicitly by

$$\begin{aligned} \mathbf{e}_{i_N} &= \mathbf{e}_{i_N}^- + \mathbf{e}_{i_N}^+ \\ &:= e^{\pi b_{i_N} (u_N - 2p_N)} + e^{\pi b_{i_N} (-u_N - 2p_N)}, \end{aligned} \quad (2.19)$$

while for the other generators we have in general

$$\mathbf{e}_i = \mathbf{e}_i^- + \mathbf{e}_i^+, \quad (2.20)$$

where \mathbf{e}_i^\pm can be obtained from conjugation by quantum dilogarithms of $\mathbf{e}_{i_N}^\pm$ above, and can be realized explicitly as the E_i -paths polynomials on the cluster realization [14] (cf. Section 5).

Theorem 2.5. For any reduced words \mathbf{i} and \mathbf{i}' , we have unitary equivalence

$$\mathcal{P}_\lambda \simeq \mathcal{P}_\lambda^{\mathbf{i}} \simeq \mathcal{P}_\lambda^{\mathbf{i}'}. \quad (2.21)$$

In particular, for any root index $i \in \mathcal{I}$, we can choose \mathbf{i} with $i_N = i$ in order to observe that

Corollary 2.6. For any $i = 1, \dots, n$, we have

$$\mathbf{f}_i^- \mathbf{f}_i^+ = q_i^{-2} \mathbf{f}_i^+ \mathbf{f}_i^-, \quad \mathbf{e}_i^- \mathbf{e}_i^+ = q_i^{-2} \mathbf{e}_i^+ \mathbf{e}_i^-, \quad (2.22)$$

$$\frac{[\mathbf{e}_i^\pm, \mathbf{f}_j^\mp]}{q_i - q_i^{-1}} = \mp \delta_{ij} K_i^{\pm 1}, \quad (2.23)$$

$$\mathbf{f}^{k, \pm} \mathbf{f}^{l, \pm} = q_i^{\pm a_{kl}} \mathbf{f}^{l, \pm} \mathbf{f}^{k, \pm}, \quad 1 \leq l < k \leq N. \quad (2.24)$$

Remark 2.7. We see that, for example, the triple $\{\mathbf{e}_i^-, \mathbf{f}_i^+, K_i^{-1}\}$ forms the commutation relation of the Heisenberg double [16], which will be needed to prove the tensor product decomposition of the positive representations restricted to the Borel part in Section 6, thus confirming the intuition discussed in [13].

2.3 Quantum dilogarithm identities

We recall the quantum dilogarithm identities needed in this paper. More details can be found in [6, 14]. The non-compact quantum dilogarithm is a meromorphic function that can be represented as an integral expression:

$$g_b(x) := \exp \left(\frac{1}{4} \int_{\mathbb{R}+i0} \frac{x^{\frac{t}{b}}}{\sinh(\pi b t) \sinh(\pi b^{-1} t)} \frac{dt}{t} \right), \quad (2.25)$$

such that by functional calculus, it is unitary when x is positive self-adjoint.

Remark 2.8. During formal algebraic manipulation, one may consider its compact version instead, which in terms of formal power series are related by

$$g_b(x) \sim \Psi^q(x)^{-1}, \quad (2.26)$$

where the Faddeev-Kashaev's quantum dilogarithm is

$$\Psi^q(x) := \prod_{r=0}^{\infty} (1 + q^{2r+1} x)^{-1} = \text{Exp}_{q^{-2}} \left(\frac{u}{q - q^{-1}} \right), \quad (2.27)$$

with

$$\text{Exp}_q(x) := \sum_{k \geq 0} \frac{x^k}{(k)_q!}, \quad (k)_q := \frac{1 - q^k}{1 - q}. \quad (2.28)$$

We will only need the following identities in this paper.

Lemma 2.9 (Quantum dilogarithm identities). *Let u, v be positive self-adjoint variables. If $uv = q^2 vu$, then we have the quantum exponential relation:*

$$g_b(u + v) = g_b(u)g_b(v), \quad (2.29)$$

and the conjugation

$$\begin{aligned} g_b(v)u g_b(v)^* &= qvu + u, \\ g_b(u)^*v g_b(u) &= v + qvu. \end{aligned} \quad (2.30)$$

We also have in the doubly-laced case ($\sqrt{2}b_s = b$)

$$g_{b_s}(u+v) = g_{b_s}(u)g_b(q^{-1}uv)g_{b_s}(v). \quad (2.31)$$

Let again u, v be positive self-adjoint and define

$$c := \frac{[u, v]}{q - q^{-1}},$$

such that $uc = q^2cu$ and $cv = q^2vc$. Then we have the generalized equation:

$$\begin{aligned} g_b(v)ug_b^*(v) &= c + u, \\ g_b(u)^*vg_b(u) &= v + c, \end{aligned} \quad (2.32)$$

in which (2.30) is a special case.

3 Quantum cluster algebra

In this section, let us recall the notation of the quantum torus algebra used in this paper. We will follow mostly the notations used in [14] and [22] but with some modifications.

3.1 Quantum torus algebra and quiver

Definition 3.1 (Cluster seed). *A cluster seed is a datum $\mathbf{i} = (I, I_0, B, D)$ where I is a finite set, $I_0 \subset I$ is a subset called the frozen subset, $B = (b_{ij})_{i,j \in I}$ a skew-symmetrizable $\frac{1}{2}\mathbb{Z}$ -valued matrix called the exchange matrix, and $D = \text{diag}(d_i)_{i \in I}$ is a diagonal \mathbb{Q} -matrix called the multiplier such that $DB = -B^T D$ is skew-symmetric.*

Notation 3.2. *In the rest of the paper, we will write $\mathbf{i}' = (I', I'_0, B', D')$ and $\bar{\mathbf{i}} = (\bar{I}, \bar{I}_0, \bar{B}, \bar{D})$ and the respective elements as $i' \in I', \bar{i} \in \bar{I}$ etc. for convenience.*

Definition 3.3 (Quantum torus algebra). *Let q be a formal parameter. We define the quantum torus algebra $\mathcal{X}^{\mathbf{i}}$ associated to a cluster seed \mathbf{i} to be an associative algebra over $\mathbb{C}[q^d]$, where $d = \min_{i \in I}(d_i)$, generated by $\{X_i\}_{i \in I}$ subject to the relations*

$$X_i X_j = q^{-2w_{ij}} X_j X_i, \quad i, j \in I, \quad (3.1)$$

where

$$w_{ij} = d_i b_{ij} = -w_{ji}. \quad (3.2)$$

The generators $X_i \in \mathcal{X}^{\mathbf{i}}$ are called the quantum cluster variables, and they are frozen if $i \in I_0$. We denote by $\mathbf{T}^{\mathbf{i}}$ the non-commutative field of fraction of $\mathcal{X}^{\mathbf{i}}$.

Alternatively, given B and D as above, let $\Lambda_{\mathbf{i}}$ be a lattice with basis $\{e_i\}_{i \in I}$, and define a skew symmetric $d\mathbb{Z}$ -valued form on $\Lambda_{\mathbf{i}}$ by $(e_i, e_j) := w_{ij}$. Then $\mathcal{X}^{\mathbf{i}}$ is generated by $\{X_{\lambda}\}_{\lambda \in \Lambda_{\mathbf{i}}}$ with $X_0 := 1$ subject to the relations

$$q^{(\lambda, \mu)} X_{\lambda} X_{\mu} = X_{\lambda + \mu}. \quad (3.3)$$

Notation 3.4. Under this realization, we shall write $X_i = X_{e_i}$, and define the notation

$$X_{i_1, \dots, i_k} := X_{e_{i_1} + \dots + e_{i_k}}, \quad (3.4)$$

or more generally for $n_1, \dots, n_k \in \mathbb{Z}$,

$$X_{i_1^{n_1}, \dots, i_k^{n_k}} := X_{n_1 e_{i_1} + \dots + n_k e_{i_k}}. \quad (3.5)$$

Let $c_{ij} \in \frac{1}{2}\mathbb{Z}$ for $i, j \in I$ be defined by $c_{ij} = \begin{cases} b_{ij} & \text{if } d_i = d_j, \\ w_{ij} & \text{otherwise.} \end{cases}$

Definition 3.5 (Quiver associated to \mathbf{i}). We associate to each seed $\mathbf{i} = (I, I_0, B, D)$ a quiver $Q^{\mathbf{i}}$ with vertices labeled by I and adjacency matrix $(c_{ij})_{i, j \in I}$. We call $i \in I$ a short (resp. long) node if $q^{d_i} = q_s$ (resp. $q^{d_i} = q$). An arrow $i \rightarrow j$ represents the algebraic relation

$$X_i X_j = q_*^{-2} X_j X_i \quad (3.6)$$

where $q_* = q_s$ if both i, j are short nodes, or $q_* = q$ otherwise.

We will use squares to denote frozen nodes $i \in I_0$ and circles otherwise. We will also use dashed arrow if $c_{ij} = \frac{1}{2}$, which only occurs between frozen nodes. We will represent the algebraic relations (3.1) by thick or thin arrows (see Figure 4) for display convenience (thickness is *not* part of the data of the quiver). Thin arrows only occur in the non-simply-laced case between two short nodes.

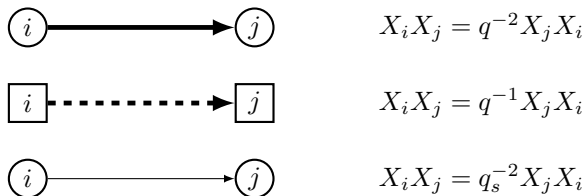


Figure 4: Arrows between nodes and their algebraic meaning.

Definition 3.6. A positive representation of the quantum torus algebra $\mathcal{X}^{\mathbf{i}}$ on a Hilbert space $\mathcal{H} = L^2(\mathbb{R}^M)$ is an assignment

$$X_i = e^{2\pi b L_i}, \quad i \in I, \quad (3.7)$$

where $L_i := L_i(u_k, p_k, \lambda_k)$ is a linear combination of the position and momentum operators $\{u_k, p_k\}_{k=1}^M$ and complex parameters λ_k with

$$[L_i, L_j] = \frac{w_{ij}}{2\pi\sqrt{-1}}, \quad (3.8)$$

such that X_i acts as a positive self-adjoint operator on \mathcal{H} .

3.2 Quantum cluster mutation

Next we define the cluster mutations of a seed and its quiver, and the quantum cluster mutations for the algebra. Here we will use the notion that keeps the indexing I of the seeds, which ensures the consistency of the relation $\mu_k^2 = \text{Id}$.

Definition 3.7 (Cluster mutation). *Given a pair of seeds \mathbf{i}, \mathbf{i}' with $I = I', I_0 = I'_0$, and an element $k \in I \setminus I_0$, a cluster mutation in direction k is an isomorphism $\mu_k : \mathbf{i} \rightarrow \mathbf{i}'$ such that $\mu_k(i) = i$ for all $i \in I$, $d'_i = d_i$, and*

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k, \\ b_{ij} + \frac{b_{ik}|b_{kj}| + |b_{ik}|b_{kj}}{2} & \text{otherwise.} \end{cases} \quad (3.9)$$

Then the quiver mutation $Q^{\mathbf{i}} \rightarrow Q^{\mathbf{i}'}$ corresponding to the mutation μ_k can be performed by the well-known rule (with obvious generalization for dashed arrows)

- (1) reverse all the arrows incident to the vertex k ;
- (2) for each pair of arrows $i \rightarrow k$ and $k \rightarrow j$, add n_{ij}^k arrows from $i \rightarrow j$, where $n_{ij}^k = d_k^{-1}$ if k is a short node and both i, j are long nodes, or $n_{ij}^k = 1$ otherwise;
- (3) delete any 2-cycles.

Definition 3.8 (Quantum cluster mutation). *The cluster mutation $\mu_k : \mathbf{i} \rightarrow \mathbf{i}'$, induces an isomorphism $\mu_k^q : \mathbf{T}^{\mathbf{i}'} \rightarrow \mathbf{T}^{\mathbf{i}}$ called the quantum cluster mutation, which can be written as a composition of two homomorphisms*

$$\mu_k^q = \mu_k^\# \circ \mu'_k, \quad (3.10)$$

where $\mu'_k : \mathbf{T}^{\mathbf{i}'} \rightarrow \mathbf{T}^{\mathbf{i}}$ is a monomial transformation defined by

$$\mu'_k(\widehat{X}_i) := \begin{cases} X_k^{-1} & \text{if } i = k, \\ X_i & \text{if } i \neq k \text{ and } b_{ki} \leq 0, \\ q_i^{b_{ik}b_{ki}} X_i X_k^{b_{ki}} & \text{if } i \neq k \text{ and } b_{ki} \geq 0, \end{cases} \quad (3.11)$$

and $\mu_k^\# : \mathbf{T}^{\mathbf{i}} \rightarrow \mathbf{T}^{\mathbf{i}}$ is a conjugation by the quantum dilogarithm function

$$\mu_k^\# := \text{Ad}_{g_{b_k}^*(X_k)}, \quad (3.12)$$

where $b_k = \sqrt{d_k}b$.

It is also useful to recall the following lemma from [22, Lemma 1.1]:

Lemma 3.9. *Let $\mu_{i_1}, \dots, \mu_{i_k}$ be a sequence of mutation, and denote the intermediate seeds by $\mathbf{i}_j := \mu_{i_j} \cdots \mu_{i_1}(\mathbf{i})$. Then the induced quantum cluster mutation $\mu_{i_1}^q \cdots \mu_{i_k}^q : \mathbf{T}^{\mathbf{i}_k} \rightarrow \mathbf{T}^{\mathbf{i}}$ can be written as*

$$\mu_{i_1}^q \cdots \mu_{i_k}^q = \Phi_k \circ M_k, \quad (3.13)$$

where $M_k : \mathbf{T}^{\mathbf{i}_k} \rightarrow \mathbf{T}^{\mathbf{i}}$ and $\Phi_k : \mathbf{T}^{\mathbf{i}} \rightarrow \mathbf{T}^{\mathbf{i}}$ are given by

$$M_k := \mu'_{i_1} \mu'_{i_2} \cdots \mu'_{i_k}, \quad (3.14)$$

$$\Phi_k := \text{Ad}_{g_{b_{i_1}}^*(X_{i_1})} \text{Ad}_{g_{b_{i_2}}^*(M_1(X_{i_2}^{(1)}))} \cdots \text{Ad}_{g_{b_{i_k}}^*(M_{k-1}(X_{i_k}^{(k-1)}))}, \quad (3.15)$$

and $X_i^{(j)} \in \mathcal{X}^{i_j}$ denotes the corresponding quantum cluster variables of the algebra \mathcal{X}^{i_j} .

3.3 Amalgamation

Finally we briefly recall the procedure of *amalgamation* of two quantum torus algebra [7]:

Definition 3.10. Let \mathbf{i}, \mathbf{i}' be two cluster seeds and let $\mathcal{X}^{\mathbf{i}}, \mathcal{X}^{\mathbf{i}'}$ be the corresponding quantum torus algebra. Let $J \subset I_0$ and $J' \subset I'_0$ be subsets of the frozen nodes with a bijection $\phi: J \rightarrow J'$ such that

$$d'_{\phi(i)} = d_i, \quad i \in J.$$

Then the amalgamation of $\mathcal{X}^{\mathbf{i}}$ and $\mathcal{X}^{\mathbf{i}'}$ along ϕ is identified with the subalgebra $\tilde{\mathcal{X}} \subset \mathcal{X}^{\mathbf{i}} \otimes \mathcal{X}^{\mathbf{i}'}$ generated by the variables $\{\tilde{X}_i\}_{i \in I \cup I'}$ where

$$\begin{aligned} \tilde{X}_i &:= \begin{cases} X_i \otimes 1 & \text{if } i \in I \setminus J, \\ 1 \otimes X'_i & \text{if } i \in I' \setminus J', \end{cases} \\ \tilde{X}_i = \tilde{X}_{\phi(i)} &:= X_i \otimes X'_{\phi(i)} \quad i \in J. \end{aligned} \tag{3.16}$$

Equivalently, the amalgamation of the corresponding quivers Q, Q' is a new quiver \tilde{Q} constructed by gluing the vertices along ϕ , defreezing those vertices that are glued, and removing any resulting 2-cycles.

4 The basic quivers

In [14], we constructed the basic quiver Q associated to a triangle such that the quantum group can be embedded into an amalgamation of Q and its mirror image \bar{Q} (See Figure 2). Let us recall its construction relevant to this paper, with a slightly different indexing.

First of all, following [1], we define the notation k^+ and k^- as follows

Notation 4.1. Given a reduced word $\mathbf{i} = (i_1, \dots, i_N)$, for $k \in \{1, \dots, N\}$, we denote by k^+ the smallest index $l > k$ such that $i_k = i_l$ if it exists, or $N + i_k$ otherwise.

Similarly, for $k \in \{1, \dots, N + n\}$, we denote by k^- the largest index $l < k$ such that $i^+ = k$ if it exists, or 0 otherwise. Finally we define $i_{N+j} := j \in \mathcal{I}$ for $j = 1, \dots, n$.

We define the set of extremal indices by

$$\mathcal{F}_{in} := \{k : k^- = 0\}, \quad \mathcal{F}_{out} := \{N + 1, \dots, N + n\}, \tag{4.1}$$

and let $k^* := (N + k)^-$ be the largest index which is not extremal and such that $i_{k^*} = k$.

By abuse of notation we will also use \mathbf{i} to denote the cluster seed corresponding to the quiver $Q^{\mathbf{i}}$, called the *basic quiver*, described as follows.

Definition 4.2. The basic quiver $Q^{\mathbf{i}}$ has nodes $I = \{1, \dots, N + 2n\}$. It contains a subquiver $Q^{\mathbf{i}}_{\mathcal{F}}$ with $N + n$ nodes $\{1, 2, \dots, N + n\}$ where the frozen nodes are $\mathcal{F}_{in} \cup \mathcal{F}_{out}$, such that for $k = 1, \dots, N$:

- there is a single arrow $k \rightarrow k^+$,
- there is a single arrow $l \rightarrow k$ if $l^- < k^- < l < k$,

and for the frozen nodes,

- there is a half arrow $k \dashrightarrow l$ if $k, l \in \mathcal{F}_{in}$ and $k > l$,
- there is a half arrow $k \dashrightarrow l$ if $k, l \in \mathcal{F}_{out}$ and $k^- > l^-$.

The multiplier $D = (d_i)$ of the cluster seed \mathbf{i} is defined to be $d_k := \mathbf{d}_{i_k}$.

We attach the frozen nodes to two sides of the triangle, and we will usually display the arrows $k \rightarrow k^+$ in Q_F^i in horizontal rows. The full quiver Q^i is constructed by adding n more frozen nodes

$$\mathcal{E} := \{N + n + 1, \dots, N + 2n\}$$

attached to the third side with multiplier $d_{N+n+k} := \mathbf{d}_k$, and with additional arrows between them and Q_F^i , but in this paper we do not need them, so we will not review the construction. Alternatively, the subquiver Q_F^i can also be constructed from building blocks called the *elementary quivers*, see e.g. [14, 17].

Example 4.3. *Our running example will be the quiver for type A_3 (Figure 5a) and type B_3 (Figure 5b). For completeness we will show the full quiver Q^i , but grayed out the part of the quiver outside Q_F^i which is not needed in the construction of this paper. We highlighted the F_i -paths (cf. Definition 4.6) in red.*

Definition 4.4. *Given two basic quivers $Q^i, Q^{i'}$ associated to two reduced words \mathbf{i}, \mathbf{i}' , we denote by $Q^{ii'}$ the quiver obtained by amalgamation along $k \mapsto \phi(k)$ where $k \in \mathcal{F}_{out} \subset Q^i$ and $\phi(k) \in \mathcal{F}_{in} \subset Q^{i'}$ such that $i'_{\phi(k)} = i_k$. We denote by*

$$\mathcal{X}^{ii'} \subset \mathcal{X}^i \otimes \mathcal{X}^{i'}$$

the corresponding quantum torus algebra. Similarly, we let $Q_F^{ii'}$ be the subquiver obtained by amalgamating Q_F^i and $Q_F^{i'}$ along the same map ϕ , and let

$$\mathcal{X}_F^{ii'} \subset \mathcal{X}^{ii'}$$

be the corresponding subalgebra.

Definition 4.5. *Given a path $\mathcal{P} = (i_1, \dots, i_m)$ on the quiver Q^i with $i_k \rightarrow i_{k+1}$ for every k , we write*

$$X_{\mathcal{P}} := X_{i_1, \dots, i_m} \tag{4.2}$$

for the product of all cluster variables along the path, and we define the path polynomial (note that we ignore the last index) by

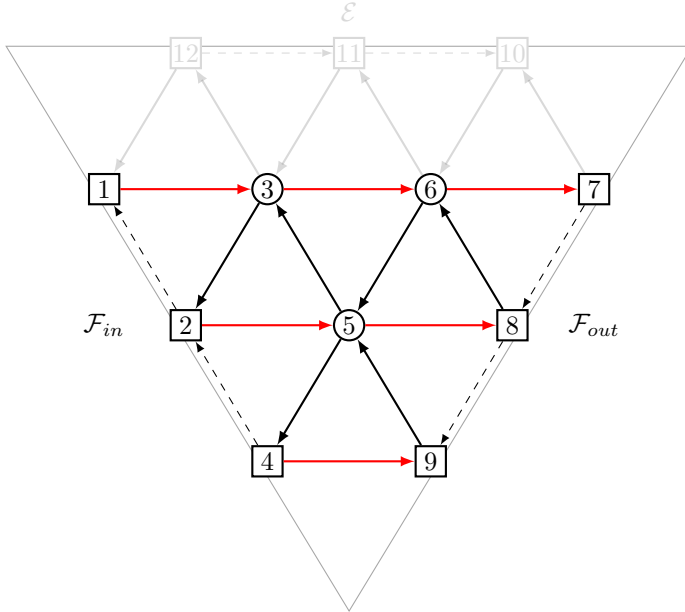
$$X(\mathcal{P}) := X(i_1, \dots, i_{m-1}) := \sum_{k=1}^{m-1} X_{i_1, \dots, i_k} \in \mathcal{X}^i. \tag{4.3}$$

Given two paths $\mathcal{P}, \mathcal{P}'$ with $i_m = i'_1$, we simply denote by

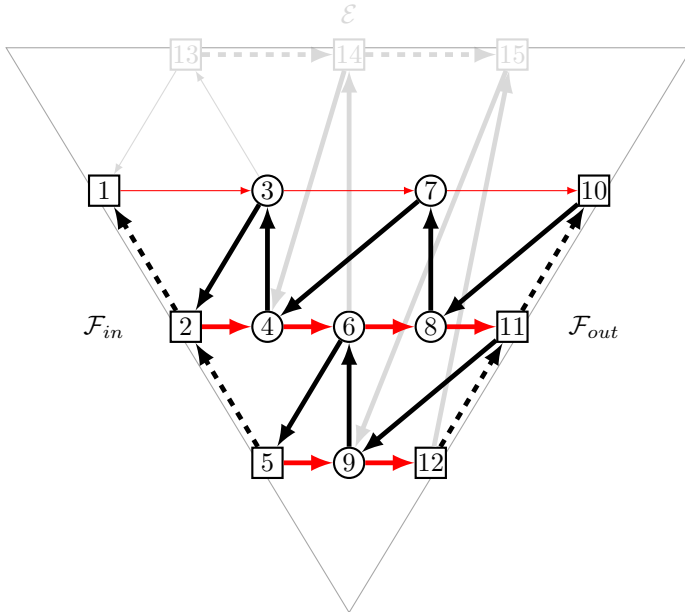
$$\mathcal{P}\mathcal{P}' = (i_1, \dots, i_m = i'_1, \dots, i'_{m'})$$

the path in $Q^{ii'}$ obtained by concatenating the two paths.

Definition 4.6. *An F_i -path is the path on the quiver Q^i which starts from \mathcal{F}_{in} and ends at \mathcal{F}_{out} along the horizontal path on the quiver with nodes $\{k : i_k = i\}$ (see e.g. Figure 5).*



(a) Type A_3 with $\mathbf{i} = (1, 2, 1, 3, 2, 1)$



(b) Type B_3 with $\mathbf{i} = (1, 2, 1, 2, 3, 2, 1, 2, 3)$

Figure 5: The quiver $Q_F^{\mathbf{i}} \subset Q^{\mathbf{i}}$ attached to a triangle, and the F_i -paths highlighted in red.

5 Cluster realization of quantum groups

Taking $Q := Q^{\mathbf{i}}$ as the basic quiver, we have an embedding of $\mathcal{U}_q(\mathfrak{g})$ into the quantum torus algebra $\mathcal{X}^{\bar{\mathbf{i}}}$ (where the subquiver $\bar{Q} := Q^{\bar{\mathbf{i}}}$ is a mirror image of $Q^{\mathbf{i}}$). In particular, the restriction to the Borel part $\mathcal{U}_{q\bar{q}}(\mathfrak{b}_{\mathbb{R}})$ generated by $\{F_i, K_i^{-1}\}_{i \in \mathcal{I}}$ only requires the subalgebra $\mathcal{X}_F^{\bar{\mathbf{i}}}$. Following [14], we will use

$$K'_i := K_i^{-1} \quad (5.1)$$

instead for typesetting purpose, which originally refers to the opposite Cartan generators of the Drinfeld's double.

Let $\mathbf{i} = (i_1, \dots, i_N)$ and $\bar{\mathbf{i}} = (\bar{i}_1, \dots, \bar{i}_N) := (i_N, \dots, i_1)$. Let $I \simeq \bar{I} \simeq \{1, \dots, N + 2n\}$ be the corresponding vertices of $Q^{\mathbf{i}}, Q^{\bar{\mathbf{i}}}$, and by abuse of notation we write

$$\bar{i}_k := \bar{i}_{\bar{k}}.$$

Let us define a bijection $\sigma : \bar{I} \rightarrow I, \bar{k} \mapsto l$ such that $\bar{i}_k = i_l$ and

$$|\{j : i_j = i_l, j < l\}| = |\{\bar{j} : \bar{i}_{\bar{j}} = \bar{i}_k, \bar{j} < \bar{k}\}|, \quad (5.2)$$

i.e. the indices appear in the same horizontal position in the quiver $Q^{\mathbf{i}}, Q^{\bar{\mathbf{i}}}$ from the left.

Recall the expression of the positive representation \mathcal{P}_λ given in Theorem 2.4. If we redefine

$$\bar{\mathbf{f}}^{\bar{k}, -} := \mathbf{f}^{\sigma(k), -}$$

and rewrite the sum as

$$\begin{aligned} \mathbf{f}_i &= \mathbf{f}_i^- + \mathbf{f}_i^+ \\ &= \sum_{\bar{k}: \bar{i}_k = i} \bar{\mathbf{f}}^{\bar{k}, -} + \sum_{k: i_k = i} \mathbf{f}^{k, +}, \end{aligned} \quad (5.3)$$

then we observe that each monomial will q_i^{-2} commute with all the terms to the right.

Define the consecutive ratios of the $\mathbf{f}^{k, \pm}$ monomials as

$$X_{\bar{k}} = \begin{cases} \bar{\mathbf{f}}^{\bar{k}, -} & \bar{k} \in \bar{\mathcal{F}}_{in}, \\ q_i^{-1} \bar{\mathbf{f}}^{\bar{k}, -} (\bar{\mathbf{f}}^{\bar{k}, -})^{-1} & \bar{k} \notin \bar{\mathcal{F}}_{in} \cup \bar{\mathcal{F}}_{out}, \end{cases} \quad (5.4)$$

$$X_k = \begin{cases} q_i^{-1} \mathbf{f}^{k, +} (\mathbf{f}^{k, +})^{-1} & k \in \mathcal{F}_{in}, \\ q_i^{-1} \mathbf{f}^{k, +} (\mathbf{f}^{k, -})^{-1} & k \notin \mathcal{F}_{in} \cup \mathcal{F}_{out}, \\ q_i^{-1} K_{i_k}^{-1} (\mathbf{f}^{k, +})^{-1} & k \in \mathcal{F}_{out}. \end{cases} \quad (5.5)$$

Let $\mathcal{P}_\lambda^{\mathbf{b}}$ be the positive representations \mathcal{P}_λ restricted to the Borel part $\mathcal{U}_{q\bar{q}}(\mathfrak{b}_{\mathbb{R}})$. Then from the explicit expression (2.17) of the operators and the commutation relations from Definition 4.2, we have

Theorem 5.1. [14] *The assignment (5.4)-(5.5) gives a positive representation of $\mathcal{X}_F^{\bar{\mathbf{i}}}$ generated by $\{X_{\bar{k}}, X_k\}_{k=1}^{N+2n}$ in the sense of Definition 3.6.*

Equivalently, let $\mathcal{P}_{F_i}, \bar{\mathcal{P}}_{F_i}$ be the F_i paths of $Q^{\mathbf{i}}$ and $Q^{\bar{\mathbf{i}}}$ respectively. Let

$$\mathbf{f}_i := X(\bar{\mathcal{P}}_{F_i} \mathcal{P}_{F_i}) \in \mathcal{X}_F^{\bar{\mathbf{i}}}, \quad (5.6)$$

$$K'_i := X_{\bar{\mathcal{P}}_{F_i} \mathcal{P}_{F_i}} \in \mathcal{X}_F^{\bar{\mathbf{i}}}$$

as in Definition 4.5. Then the assignment (5.4)-(5.5) coincides with the positive representation $\mathcal{P}_\lambda^{\mathfrak{b}}$ of the Borel part. We also have

$$\mathbf{f}_i^- = X(\overline{\mathcal{P}}_{F_i}).$$

For the tensor product $\mathcal{P}_\lambda^{\mathfrak{b}} \otimes \mathcal{P}_\mu^{\mathfrak{b}}$, the coproduct $\Delta(\mathbf{f}_i), \Delta(K'_i)$ are represented on $\mathcal{X}^{\overline{\mathfrak{ii}}} \otimes \mathcal{X}^{\overline{\mathfrak{ii}}}$ in a similar way by concatenation of the F_i -paths of two copies of the quiver $Q^{\overline{\mathfrak{ii}}}, Q^{\overline{\mathfrak{ii}}}$ amalgamated along $\mathcal{F}_{out} = \mathcal{F}'_{in}$.

On the other hand, we observe in [13] that the Borel part can alternatively be represented using just half of the sum given by

$$\mathbf{f}_i = \sum_{k:i_k=i} \mathbf{f}^{k,+}, \quad (5.7)$$

which coincides with the so-called *Feigin's homomorphism*. Let us call this representation $\widetilde{\mathcal{P}}_\lambda^{\mathfrak{b}}$ (which we called it the *standard form* in [13]).

Lemma 5.2. $\widetilde{\mathcal{P}}_\lambda^{\mathfrak{b}} \simeq \widetilde{\mathcal{P}}^{\mathfrak{b}}$ is independent of the parameter λ .

Proof. The proof is similar but easier than the one in [13]. From the explicit expression for K_i , since the Cartan matrix $A = (a_{ij})$ is invertible, we can always make a shift of variables

$$u_k \mapsto u_k - c_k$$

for some constant c_k , where $k \in \mathcal{F}_{in}$, such that the parameter λ_i vanishes in the expression of K_i , while each monomial term $\mathbf{f}^{k,+}$ picks up some new parameters c'_k . Now since each $\mathbf{f}^{k,+}$ has a unique momentum operator $2p_k$, by the unitary transformation given by multiplication by $e^{\pi i u_k c'_k}$ for some constant c'_k , which acts as

$$2p_k \mapsto 2p_k - c'_k,$$

we can make the resulting parameters c'_k vanish. \square

Modifying (5.5) slightly with

$$X_k = \begin{cases} \mathbf{f}^{k,+} & k \in \mathcal{F}_{in}, \\ q_i^{-1} \mathbf{f}^{k,+} (\mathbf{f}^{k^-,+})^{-1} & k \notin \mathcal{F}_{in} \cup \mathcal{F}_{out}, \\ q_i^{-1} K_i^{-1} (\mathbf{f}^{k^*,+})^{-1} & k \in \mathcal{F}_{out}, \end{cases} \quad (5.8)$$

we conclude that

Theorem 5.3. The assignment (5.8) gives a positive representation of the quantum torus algebra $\mathcal{X}_F^{\mathfrak{i}} \subset \mathcal{X}^{\mathfrak{i}}$ associated to $Q_F^{\mathfrak{i}}$. Equivalently, let

$$\begin{aligned} \mathbf{f}_i &:= X(\mathcal{P}_{F_i}) \in \mathcal{X}_F^{\mathfrak{i}}, \\ K'_i &:= X_{\mathcal{P}_{F_i}} \in \mathcal{X}_F^{\mathfrak{i}}, \end{aligned} \quad (5.9)$$

then the assignment (5.8) gives the representation $\widetilde{\mathcal{P}}^{\mathfrak{b}}$ of $\mathcal{U}_{q\bar{q}}(\mathfrak{b}_{\mathbb{R}})$.

We also have the notion of an E_i -path. Consider the quiver $Q_{D_{2,1}}^{\bar{\mathbf{i}}}$ associated to a once punctured disk with two marked points, obtained by gluing $Q^{\bar{\mathbf{i}}}, Q^{\mathbf{i}}$ along $\overline{\mathcal{F}}_{out} = \mathcal{F}_{in}$ as well as $\overline{\mathcal{E}} = \mathcal{E}$ (see Figure 2). Recall that due to Theorem 2.4, the generators \mathbf{e}_i can be represented as a sum of monomials

$$\mathbf{e}_i = \mathbf{e}_i^- + \mathbf{e}_i^+.$$

In particular, if \mathbf{i} is well-chosen, \mathbf{e}_i can be represented again by a path polynomial (with slight modification in type C_n, E_8, F_4 and G_2 , see [14]) of the form

$$\mathbf{e}_i = X(\mathcal{P}_{E_i} \overline{\mathcal{P}}_{E_i}) \in \mathcal{X}_{D_{2,1}}^{\bar{\mathbf{i}}}, \quad (5.10)$$

where \mathcal{P}_{E_i} and $\overline{\mathcal{P}}_{E_i}$ are respectively a path on the $Q^{\mathbf{i}}, Q^{\bar{\mathbf{i}}}$ quivers, called the E_i -paths. The path \mathcal{P}_{E_i} starts at \mathcal{F}_{out} and ends at \mathcal{E} , while $\overline{\mathcal{P}}_{E_i}$ starts at $\overline{\mathcal{E}}$ and ends at $\overline{\mathcal{F}}_{in}$. Similarly we have

$$\mathbf{e}_i^- = X(\mathcal{P}_{E_i}).$$

6 Main results

We are now ready to give an alternative proof of the main theorem in [13].

Theorem 6.1. *We have the following unitary equivalences of positive representations restricted to the Borel part $\mathcal{U}_{q\bar{q}}(\mathfrak{b}_{\mathbb{R}})$:*

$$\mathcal{P}_{\lambda}^{\mathfrak{b}} \simeq \widetilde{\mathcal{P}}^{\mathfrak{b}}, \quad (6.1)$$

$$\widetilde{\mathcal{P}}^{\mathfrak{b}} \otimes \widetilde{\mathcal{P}}^{\mathfrak{b}} \simeq \widetilde{\mathcal{P}}^{\mathfrak{b}} \otimes \mathcal{M}, \quad (6.2)$$

for some multiplicity module $\mathcal{M} \simeq L^2(\mathbb{R}^N)$ where $\mathcal{U}_{q\bar{q}}(\mathfrak{b}_{\mathbb{R}})$ acts trivially. This means that the positive representations restricted to $\mathcal{U}_{q\bar{q}}(\mathfrak{b}_{\mathbb{R}})$ is closed under taking tensor product.

Recall that given an embedding of

$$\mathcal{U}_q(\mathfrak{g}) \hookrightarrow \mathcal{X}_{D_{2,2}}^{\bar{\mathbf{i}}\bar{\mathbf{i}}\mathbf{i}} \subset \mathcal{X}_{D_{2,1}}^{\bar{\mathbf{i}}\mathbf{i}} \otimes \mathcal{X}_{D_{2,1}}^{\bar{\mathbf{i}}\mathbf{i}}$$

into the quantum torus algebra associated to a disk with 2 punctures and 2 marked points given by the coproduct, we show in [14] that the reduced \mathcal{R} operator can be decomposed into

$$\overline{\mathcal{R}} = \mathcal{R}_4 \cdot \mathcal{R}_3 \cdot \mathcal{R}_2 \cdot \mathcal{R}_1, \quad (6.3)$$

where

$$\begin{aligned} \mathcal{R}_4 &= g_{b_{i_N}}(\mathbf{e}_{i_N}^+ \otimes \mathbf{f}^{N,+}) \dots g_{b_{i_2}}(\mathbf{e}_{i_2}^+ \otimes \mathbf{f}^{2,+}) g_{b_{i_1}}(\mathbf{e}_{i_1}^+ \otimes \mathbf{f}^{1,+}), \\ \mathcal{R}_3 &= g_{b_{i_N}}(\mathbf{e}_{i_N}^- \otimes \mathbf{f}^{N,+}) \dots g_{b_{i_2}}(\mathbf{e}_{i_2}^- \otimes \mathbf{f}^{2,+}) g_{b_{i_1}}(\mathbf{e}_{i_1}^- \otimes \mathbf{f}^{1,+}), \\ \mathcal{R}_2 &= g_{b_{i_1}}(\mathbf{e}_{i_1}^+ \otimes \mathbf{f}^{1,-}) g_{b_{i_2}}(\mathbf{e}_{i_2}^+ \otimes \mathbf{f}^{2,-}) \dots g_{b_{i_N}}(\mathbf{e}_{i_N}^+ \otimes \mathbf{f}^{N,-}), \\ \mathcal{R}_1 &= g_{b_{i_1}}(\mathbf{e}_{i_1}^- \otimes \mathbf{f}^{1,-}) g_{b_{i_2}}(\mathbf{e}_{i_2}^- \otimes \mathbf{f}^{2,-}) \dots g_{b_{i_N}}(\mathbf{e}_{i_N}^- \otimes \mathbf{f}^{N,-}). \end{aligned}$$

Also \mathcal{R}_2 commute with \mathcal{R}_3 . By the correspondence in Lemma 3.9, the 4 factors correspond to sequences of quiver mutations associated to flipping of triangulation in different configurations. In this paper, we show explicitly the following fact, which is only implied implicitly in [14].

Theorem 6.2. *The quiver mutations*

$$\mu_{\mathcal{R}_k}^a := \mu_{m_1}^a \cdots \mu_{m_M}^a$$

induced by \mathcal{R}_k , $k = 1, 3$, corresponding to the flip of triangulation preserve the F_i -paths (see Figure 6). More precisely, for an amalgamation $Q^{\mathbf{i}'}$ of two quivers with $\mathbf{i}' = \bar{\mathbf{i}}$ ($k = 1$) or $\mathbf{i}' = \mathbf{i}$ ($k = 3$) along $\mathcal{F}_{out} = \mathcal{F}'_{in}$, if $\mathcal{P}_{F_i}, \mathcal{P}'_{F_i}$ are the F_i paths of the left and right triangle respectively, and $\widehat{\mathcal{P}}_{F_i}$ are the F_i paths of the bottom triangle of the mutated quiver, then

$$\mu_{\mathcal{R}_k}^a(\widehat{X}(\widehat{\mathcal{P}}_{F_i})) = (X(\mathcal{P}_{F_i}\mathcal{P}'_{F_i})), \quad k = 1, 3, \quad (6.4)$$

where \widehat{X}_j belongs to the quantum torus algebra $\widehat{\mathcal{X}}^{\mathbf{i}'}$ of the mutated quiver, amalgamated along $\widehat{\mathcal{E}} = \widehat{\mathcal{F}}'_{out}$ (see Figure 6).

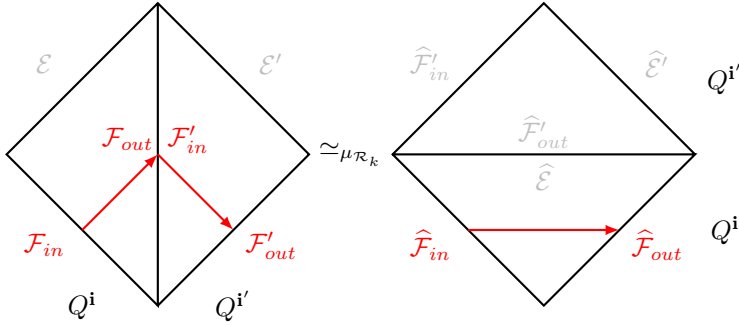


Figure 6: Configurations of the mutations $\mu_{\mathcal{R}_k}$, $k = 1, 3$, with the F_i paths schematically shown in red, and the irrelevant extremal nodes grayed out.

According to the result of [14], the quiver mutations corresponding to \mathcal{R}_k using Lemma 3.9 are of the type as in Figure 7, and in the doubly-laced case also as in Figure 8. (The case of type G_2 was treated explicitly in [14] already and will be omitted.)

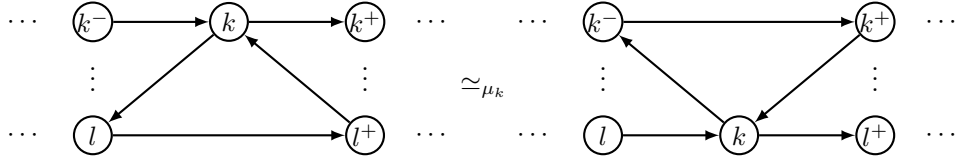


Figure 7: Mutation at simply-laced part.

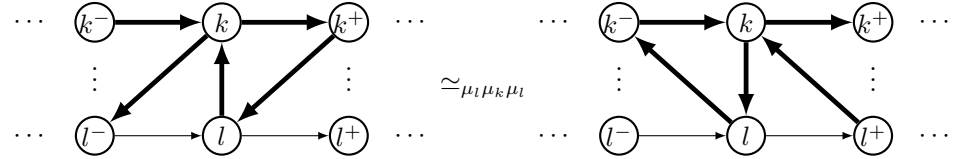


Figure 8: Mutation at doubly-laced part.

In particular, this means that the mutation of an F_i -path is again a path and its direction is preserved: in the simply-laced moves, the path polynomials are given by

$$\mu_k^q(\dots + \widehat{X}_{\dots, k^-} + \widehat{X}_{\dots, k^-, k^+} + \dots) = (\dots + X_{\dots, k^-} + X_{\dots, k^-, k} + X_{\dots, k^-, k, k^+} + \dots),$$

while the path polynomials are preserved in the doubly-laced moves.

In terms of unitary equivalence, we have

Proposition 6.3. *Let $\mathbf{i}' = \bar{\mathbf{i}}$ and consider $Q^{\bar{\mathbf{i}}}$ as in Figure 6. Let also \mathcal{P}_{E_i} be the E_i -paths for the left part $Q^{\mathbf{i}'}$ of the quiver, $\mathcal{P}_{F_i}, \overline{\mathcal{P}}_{F_i}$ be the F_i -paths for the left part $Q^{\mathbf{i}'}$ and right part $Q^{\bar{\mathbf{i}}}$ of the quiver respectively. Let*

$$\begin{aligned} X(\mathcal{P}_{E_i}) &=: \mathbf{e}_i^- \otimes 1, \\ X(\overline{\mathcal{P}}_{F_i}) &=: 1 \otimes \mathbf{f}_i^- \\ &=: 1 \otimes \sum_{k: i_k = i} \mathbf{f}_i^{k, -} \end{aligned}$$

and define

$$\Phi_1 := g_{b_{i_1}}(\mathbf{e}_{i_1}^- \otimes \mathbf{f}_i^{1, -}) g_{b_{i_2}}(\mathbf{e}_{i_2}^- \otimes \mathbf{f}_i^{2, -}) \dots g_{b_{i_N}}(\mathbf{e}_{i_N}^- \otimes \mathbf{f}_i^{N, -}). \quad (6.5)$$

Then we have

$$Ad_{\Phi_1}(X(\mathcal{P}_{F_i} \overline{\mathcal{P}}_{F_i})) = X(\mathcal{P}_{F_i}) = M_{\mathcal{R}_1}(\widehat{X}(\widehat{\mathcal{P}}_{F_i})) \in \mathcal{X}_i \otimes 1 \quad (6.6)$$

$$Ad_{\Phi_1}(X_{\mathcal{P}_{F_i} \overline{\mathcal{P}}_{F_i}}) = X_{\mathcal{P}_{F_i} \overline{\mathcal{P}}_{F_i}} = M_{\mathcal{R}_1}(\widehat{X}_{\widehat{\mathcal{P}}_{F_i}}) \in \mathcal{X}_i \otimes \mathcal{X}_{\bar{\mathbf{i}}} \quad (6.7)$$

where $M_{\mathcal{R}_1} = \mu'_{m_1} \dots \mu'_{m_M}$ is the monomial transform $\widehat{\mathcal{X}}_{\bar{\mathbf{i}}} \rightarrow \mathcal{X}_{\bar{\mathbf{i}}}$.

Proof. Let also

$$\begin{aligned} X(\mathcal{P}_{F_i}) &=: \mathbf{f}_i^+ \otimes 1, \\ X_{\mathcal{P}_{F_i}} &=: \underline{K}'_i \otimes 1. \end{aligned}$$

Then the commutation relations among $\{\mathbf{e}_i^-, \mathbf{f}_i^+, \mathbf{f}_i^{k, -}, \underline{K}'_i\}$ is exactly the same as $\{\mathbf{e}_i^-, \mathbf{f}_i^+, \mathbf{f}_i^{k, -}, K'_i\}$ from Theorem 2.4. Hence by Proposition 2.6 and (2.32), we have

$$\begin{aligned} Ad_{g_{b_{i_k}}(\mathbf{e}_{i_k} \otimes \mathbf{f}_i^{k, -})}(\mathbf{f}_i^+ \otimes 1) &= \begin{cases} \mathbf{f}_i^+ \otimes 1 + K'_i \otimes \mathbf{f}_i^{k, -} & i_k = i, \\ \mathbf{f}_i^+ \otimes 1 & i_k \neq i, \end{cases} \\ Ad_{g_{b_{i_k}}(\mathbf{e}_{i_k} \otimes \mathbf{f}_i^{k, -})}(K'_i \otimes \mathbf{f}_i^{l, -}) &= K'_i \otimes \mathbf{f}_i^{l, -}, \quad k < l, \\ Ad_{g_{b_{i_k}}(\mathbf{e}_{i_k} \otimes \mathbf{f}_i^{k, -})}(K'_i \otimes K'_i) &= K'_i \otimes K'_i. \end{aligned}$$

Since

$$X(\mathcal{P}_{F_i} \overline{\mathcal{P}}_{F_i}) = \mathbf{f}_i^+ \otimes 1 + \underline{K}'_i \otimes \mathbf{f}_i^-$$

and

$$X_{\mathcal{P}_{F_i} \overline{\mathcal{P}}_{F_i}} = \underline{K}'_i \otimes \underline{K}'_i,$$

by induction we have the first equality. Since the mutated F_i paths are the unique paths joining \mathcal{F}_{in} and \mathcal{F}_{out} of the bottom triangle, the F_i -paths are preserved under mutation. By Lemma 3.9 we have the claim for the monomial transform. \square

Note that the embedding to the quiver $Q^{\mathbf{i}\bar{\mathbf{i}}}$ gives the positive representation $\mathcal{P}_\lambda^{\bar{\mathbf{i}}}$ restricted to the Borel part $\mathcal{U}_{q\bar{q}}(\mathfrak{b}_\mathbb{R})$ for the word $\bar{\mathbf{i}}$ instead, but

$$\mathcal{P}_\lambda \simeq \mathcal{P}_\lambda^{\bar{\mathbf{i}}} \simeq \mathcal{P}_\lambda^{\mathbf{i}}.$$

Hence applying Proposition 6.3 we obtain the unitary transformation of the positive self-adjoint operators \mathbf{f}_i of $\mathcal{P}_\lambda^{\mathbf{b}}$ to $\widetilde{\mathcal{P}}_\lambda^{\mathbf{b}} \simeq \widetilde{\mathcal{P}}^{\mathbf{b}}$:

$$\mathbf{f}_i = \mathbf{f}_i^+ + \mathbf{f}_i^- \simeq \mathbf{f}_i^+. \quad (6.8)$$

It remains to show that K'_i depends only on $Q^{\mathbf{i}}$ for a unitary transformation that preserves \mathbf{f}_i^+ :

Proposition 6.4. *As positive self-adjoint operators, we have the unitary equivalence*

$$K'_i = X_{\mathcal{P}_{F_i} \overline{\mathcal{P}}_{F_i}} \simeq X_{\mathcal{P}_{F_i}} \quad (6.9)$$

Proof. The factors in the expression of K'_i involving the second path $\overline{\mathcal{P}}_{F_i}$ commute with the first factor, hence we can just treat them as parameters of the representation. Hence using exactly the same argument as in Lemma 5.2, we arrive at the conclusion. \square

This proves the first claim of Theorem 6.1.

In exactly the same way, for $Q^{\mathbf{i}\mathbf{i}}$ as in Figure 6 for $\mathbf{i}' = \mathbf{i}$, we have

Proposition 6.5. *Let \mathcal{P}_{E_i} be the E_i -paths for the left part $Q_1^{\mathbf{i}}$ of the quiver, and $\mathcal{P}_{F_i}^1, \mathcal{P}_{F_i}^2$ be the F_i -paths for the left part $Q_1^{\mathbf{i}}$ and the right part $Q_2^{\mathbf{i}}$ of the quiver respectively. Let*

$$\begin{aligned} X(\mathcal{P}_{E_i}) &=: \underline{\mathbf{e}}_i^- \otimes 1, \\ X(\mathcal{P}_{F_i}^2) &=: 1 \otimes \sum_{k:i_k=i} \underline{\mathbf{f}}^{k,+}, \end{aligned}$$

and define

$$\Phi_3 = g_{b_{i_N}}(\underline{\mathbf{e}}_{i_N}^- \otimes \underline{\mathbf{f}}^{N,+}) \dots g_{b_{i_2}}(\underline{\mathbf{e}}_{i_2}^- \otimes \underline{\mathbf{f}}^{2,+}) g_{b_{i_1}}(\underline{\mathbf{e}}_{i_1}^- \otimes \underline{\mathbf{f}}^{1,+}). \quad (6.10)$$

Then

$$Ad_{\Phi_3}(X(\mathcal{P}_{F_i}^1 \mathcal{P}_{F_i}^2)) = X(\mathcal{P}_{F_i}^1) \in \mathcal{X}_i \otimes 1, \quad (6.11)$$

$$Ad_{\Phi_3}(X_{\mathcal{P}_{F_i}^1 \mathcal{P}_{F_i}^2}) = X_{\mathcal{P}_{F_i}^1 \mathcal{P}_{F_i}^2} \in \mathcal{X}_i \otimes \mathcal{X}_i. \quad (6.12)$$

Since the coproduct on $\widetilde{\mathcal{P}}^{\mathbf{b}} \otimes \widetilde{\mathcal{P}}^{\mathbf{b}}$ is simply represented by concatenation of the F_i -paths, together with exactly the same argument as in Proposition 6.4 for the K'_i generators, we conclude that

$$\begin{aligned} \Delta(\mathbf{f}_i) &\simeq \mathbf{f}_i \otimes 1, \\ \Delta(K'_i) &\simeq K'_i \otimes 1, \end{aligned}$$

and hence

$$\widetilde{\mathcal{P}}^{\mathbf{b}} \otimes \widetilde{\mathcal{P}}^{\mathbf{b}} \simeq \widetilde{\mathcal{P}}^{\mathbf{b}} \otimes \mathcal{M}$$

for a multiplicity module $\mathcal{M} \simeq L^2(\mathbb{R}^N)$, and this concludes the proof of the main theorem. \square

In particular, since one can decompose $g_{b_i}(\mathbf{e}_i^-) = \prod g_{b_{\dots}}(X_{\dots})$ into product of quantum dilogarithms with quantum cluster monomials as arguments [14], one can rewrite the unitary transformation Φ_k as honest unitary operators acting on $L^2(\mathbb{R}^N)$ easily using the formula (5.4)-(5.5) to rewrite the quantum cluster variables X_i as positive self-adjoint operators in terms of the canonical variables $\{u_i, p_i\}$. Therefore this completely solves the combinatorial problem posed in [13] for explicitly writing down the tensor product decomposition of positive representations restricted to $\mathcal{U}_{q\bar{q}}(\mathfrak{b}_{\mathbb{R}})$ of all types other than type A_n , which cannot be done previously without the cluster realization.

7 Examples

We consider several examples to illustrate the construction above. We will describe the basic quiver, and the positive representations realized on the quiver Q^{ii} . We recall the \mathcal{P}_{E_i} paths found in [14] and its quantum dilogarithm decomposition. This allows us to describe the unitary transformation Φ_1 and Φ_3 explicitly using (6.5), (6.10) corresponding to certain sequence μ of quiver mutations on Q^{ii} and Q^{ii} respectively. We will use the standard indexing, and display the $Q^{\text{ii}'}$ quiver and its mutation, highlighting the \mathcal{P}_{F_i} paths, while grayed out the irrelevant parts incident to the frozen nodes \mathcal{E} .

Remark 7.1. *For the first flip, in the more general setting corresponding to self-folded triangles where the frozen nodes in \mathcal{E} of both triangles are identified (see Figure 2), although the Q_F^{ii} part will not be affected, the grayed out part of the resulting mutated quiver after $\mu_{\mathcal{R}_1}$ is much more complicated and interesting (see [23] for an example in type A_n), and is considered an important combinatorial part for the general problem of tensor product decomposition of positive representations of the whole quantum group $\mathcal{U}_{q\bar{q}}(\mathfrak{g}_{\mathbb{R}})$.*

7.1 Type A_1

The representation theory of $\mathcal{U}_{q\bar{q}}(\mathfrak{sl}(2, \mathbb{R}))$, first described by Faddeev [5], is studied extensively in various papers [2, 10, 18, 19]. In terms of the notations in this paper, it reads the following. The basic quiver is simply given by 3 nodes as in Figure 9. Obviously $Q^{\text{i}} \simeq Q^{\text{ii}}$, but let us still denote the nodes of Q^{i} by $\bar{k} \in \bar{\mathbb{I}}$.

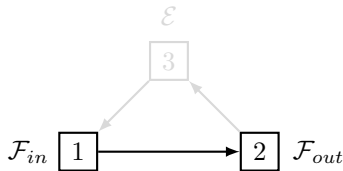


Figure 9: Basic quiver $Q_{A_1}^{\text{i}}$.

The positive representation $\mathcal{P}_{\lambda}^{\text{b}}$ is represented through \mathcal{X}^{ii} (where $\bar{\mathbb{I}} = 2$) as

$$\mathbf{f} = e^{\pi b(-u+2\lambda+2p)} + e^{\pi b(u-2\lambda+2p)} := X_1 + X_{1, \bar{\mathbb{I}}},$$

$$K' = e^{2\pi b(u-\lambda)} := X_{1, \bar{\mathbb{I}}, \bar{2}}.$$

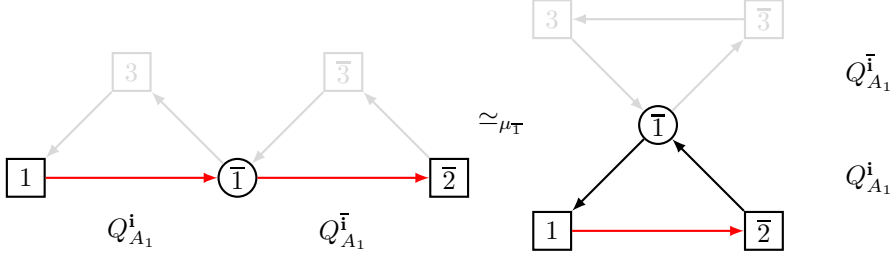


Figure 10: $\mathcal{P}_\lambda^b \simeq \widetilde{\mathcal{P}}^b$ in type A_1 with the F_i path shown in red.

The E and F -paths are $\mathcal{P}_E = (2, 3)$, $\mathcal{P}_F = (1, 2)$ respectively, hence Φ_1 is given by

$$\Phi_1 = g_b(\mathbf{e} \otimes \mathbf{f}^{1,-}) = g_b(X_2 \otimes X_{\bar{1}}) := g_b(X_{\bar{1}}) = g_b(e^{\pi b(2u-4\lambda)}),$$

where we identified $X_{\bar{1}} := X_2 \otimes X_{\bar{1}}$ in the embedding $\mathcal{X}^{\bar{i}\bar{i}} \hookrightarrow \mathcal{X}^i \otimes \mathcal{X}^{\bar{i}}$. It simply mutates at the node $\bar{1} = 2$ as in Figure 10. Using (2.30), we obviously have

$$\begin{aligned} Ad_{g_b(X_{\bar{1}})}(\mathbf{f}) &= Ad_{g_b(X_{\bar{1}})}(X_1 + X_{1,\bar{1}}) = X_1 = M_{\mathcal{R}_1}(\widehat{X}_1), \\ Ad_{g_b(X_{\bar{1}})}(K') &= Ad_{g_b(X_{\bar{1}})}(X_{1,\bar{1},\bar{2}}) = X_{1,\bar{1},\bar{2}} = M_{\mathcal{R}_1}(\widehat{X}_{1,\bar{2}}). \end{aligned}$$

Finally, we can get rid of λ by making a shift

$$(2p \mapsto 2p - \lambda) \circ (u \mapsto u + \lambda)$$

to obtain the unitary equivalence $\mathcal{P}_\lambda \simeq \widetilde{\mathcal{P}}^b$:

$$\mathbf{f} \simeq e^{\pi b(-u+2p)}, \quad K' \simeq e^{2\pi b u}.$$

For the tensor product $\widetilde{\mathcal{P}}^b \otimes \widetilde{\mathcal{P}}^b$ as in Figure 11, we use $'$ to denote the nodes of the second tensor factor, such that

$$\begin{aligned} \Delta(\mathbf{f}) &= e^{\pi(-u+2p)} + e^{\pi b(2u-u'+2p')} = X_1 + X_{1,1'}, \\ \Delta(K') &= e^{2\pi b(u+u')} = X_{1,1',\bar{2}'}. \end{aligned}$$

Up to re-indexing, we have exactly the same transformation, where

$$\Phi_3 = g_b(X_{1'}) = g_b(e^{\pi b(3u-u'-2p+2p')})$$

and

$$\begin{aligned} Ad_{g_b(X_{1'})}(\Delta(\mathbf{f})) &= Ad_{g_b(X_{1'})}(X_1 + X_{1,1'}) = X_1 = M_{\mathcal{R}_3}(\widehat{X}_1), \\ Ad_{g_b(X_{1'})}(\Delta(K')) &= Ad_{g_b(X_{1'})}(X_{1,1',\bar{2}'}) = X_{1,1',\bar{2}'} = M_{\mathcal{R}_3}(\widehat{X}_{1,\bar{2}'}). \end{aligned}$$

Again shifting by

$$(2p \mapsto 2p - u') \circ (u \mapsto u - u')$$

we arrive at

$$\Delta(\mathbf{f}) \simeq e^{\pi(-u+2p)}, \quad \Delta(K') \simeq e^{2\pi b u},$$

which gives

$$\widetilde{\mathcal{P}}^b \otimes \widetilde{\mathcal{P}}^b \simeq \widetilde{\mathcal{P}}^b \otimes \mathcal{M}$$

as required.

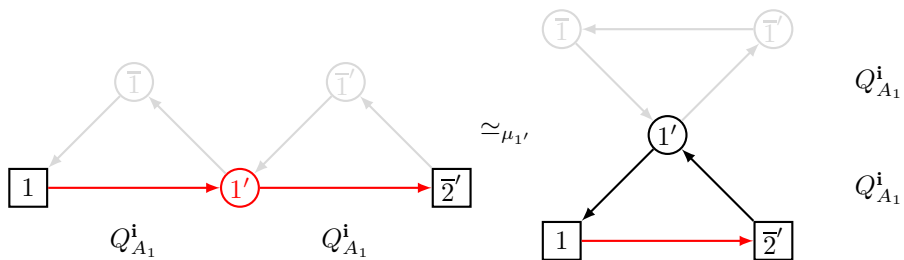


Figure 11: $\widetilde{\mathcal{P}}^b \otimes \widetilde{\mathcal{P}}^b \simeq \widetilde{\mathcal{P}}^b \otimes \mathcal{M}$ in type A_1 .

7.2 Type A_n

Let

$$\mathbf{i} = (1, 2, 1, 3, 2, 1, \dots, n, n-1, \dots, 1)$$

be the standard word. We note that $\bar{\mathbf{i}}$ differs from \mathbf{i} only by the moves $(i, j) \longleftrightarrow (j, i)$ for $|i - j| > 1$. In particular, the quiver $Q^{\mathbf{i}} \simeq Q^{\bar{\mathbf{i}}}$ is identical up to re-indexing (in fact it establishes a \mathbb{Z}_3 symmetry). Therefore the mutation sequence corresponding to \mathcal{R}_1 and \mathcal{R}_3 is again identical up to re-indexing. Two different but equivalent mutation sequences for \mathcal{R}_k has been established explicitly in [14] and [22]. Let us apply this for our running example in the case A_3 .

The basic quiver for A_3 is drawn in Figure 5a. The positive representation \mathcal{P}_λ^b is given on $\mathcal{X}^{\mathbf{i}\bar{\mathbf{i}}}$ (where $\bar{1} = 7, \bar{2} = 8, \bar{3} = 9$) as shown in the left quiver of Figure 12 by

$$\mathbf{f}_1 = X(1, 3, 6, \bar{1}, \bar{4}, \bar{6}),$$

$$\mathbf{f}_2 = X(2, 5, \bar{2}, \bar{5}),$$

$$\mathbf{f}_3 = X(4, \bar{3}),$$

$$K'_1 = X_{1,3,6,\bar{1},\bar{4},\bar{6},\bar{7}},$$

$$K'_2 = X_{2,5,\bar{2},\bar{5},\bar{8}},$$

$$K'_3 = X_{4,\bar{3},\bar{9}}.$$

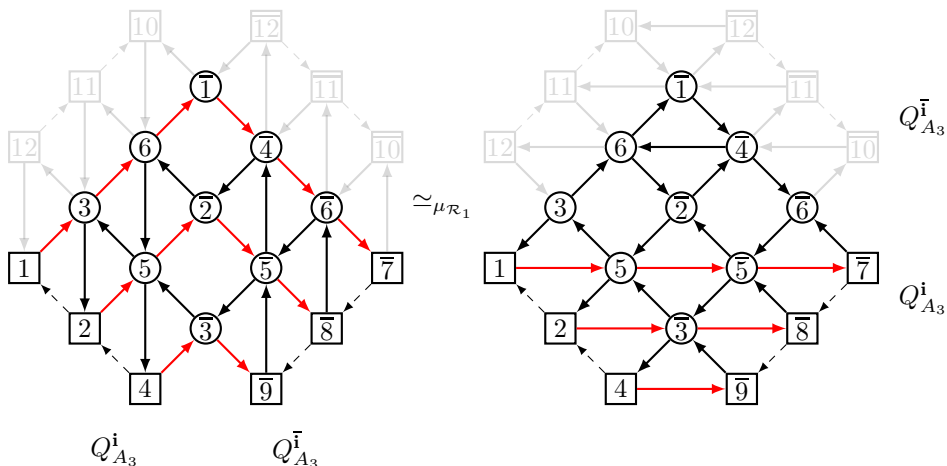


Figure 12: $\mathcal{P}_\lambda^b \simeq \widetilde{\mathcal{P}}^b$ in type A_3 drawn symmetrically, with F_i -paths shown in red

In general for the current choice of \mathbf{i} , the E_i -paths are the unique shortest path that goes from \mathcal{F}_{out} to \mathcal{E} corresponding to the root index i . In the case of type A_3 they are given by (cf. Figure 5a)

$$\begin{aligned}\mathcal{P}_{E_1} &= (7, 10), \\ \mathcal{P}_{E_2} &= (8, 6, 11), \\ \mathcal{P}_{E_3} &= (9, 5, 3, 12).\end{aligned}$$

Hence by (2.29) we easily get

$$\begin{aligned}g_b(\tilde{\mathbf{e}}_1^-) &= g_b(X_7), \\ g_b(\tilde{\mathbf{e}}_2^-) &= g_b(X_{6,8})g_b(X_8), \\ g_b(\tilde{\mathbf{e}}_3^-) &= g_b(X_{3,5,9})g_b(X_{5,9})g_b(X_9).\end{aligned}$$

The F_i -paths are given by the shortest path joining \mathcal{F}_{in} to \mathcal{F}_{out}

$$\begin{aligned}\mathcal{P}_{F_1} &= (1, 3, 6, 7), \\ \mathcal{P}_{F_2} &= (2, 5, 8), \\ \mathcal{P}_{F_3} &= (4, 9).\end{aligned}$$

Identifying $X_{\overline{1}} := X_7 \otimes X_{\overline{1}}$, $X_{\overline{2}} := X_8 \otimes X_{\overline{2}}$ and $X_{\overline{3}} := X_9 \otimes X_{\overline{3}}$, we obtain using (6.5) the unitary transformation Φ_1 corresponding to a sequence of 10 mutations:

$$\Phi_1 = g_b(X_{\overline{1},\overline{4},\overline{6}})g_b(X_{6,\overline{2},\overline{5}})g_b(X_{\overline{2},\overline{5}})g_b(X_{\overline{1},\overline{4}})g_b(X_{3,5,\overline{3}})g_b(X_{5,\overline{3}})g_b(X_{\overline{3}})g_b(X_{6,\overline{2}})g_b(X_{\overline{2}})g_b(X_{\overline{1}}).$$

According to [14], it corresponds to mutation at

$$\mu_{\mathcal{R}_1} = (\overline{1}, \overline{2}, 6, \overline{3}, 5, 3, \overline{4}, \overline{5}, \overline{2}, \overline{6}).$$

The mutation sequence for Q^{ii} and the unitary transform Φ_3 is identical up to re-indexing since $Q^{\text{i}} \simeq Q^{\overline{\text{i}}}$. Hence together with the shifting as in Lemma 5.2 gives $\widetilde{\mathcal{P}}^{\text{b}} \otimes \widetilde{\mathcal{P}}^{\text{b}} \simeq \widetilde{\mathcal{P}}^{\text{b}} \otimes \mathcal{M}$ as required.

We can further write down the transformations Φ_k explicitly as unitary operators on the Hilbert space $L^2(\mathbb{R}^N)$ easily. For example, using the notation in Theorem 2.4, the representation $\widetilde{\mathcal{P}}^{\text{b}} \simeq L^2(\mathbb{R}^6)$ for $\mathbf{i} = (1, 2, 1, 3, 2, 1)$ is given by

$$\begin{aligned}\mathbf{f}_1 &= e^{\pi b(-2u_1+u_2-2u_3+u_5-u_6+2p_6)} + e^{\pi b(-2u_1+u_2-u_3+2p_3)} + e^{\pi b(-u_1+2p_1)}, \\ \mathbf{f}_2 &= e^{\pi b(u_1-2u_2+u_3+u_4-u_5+2p_5)} + e^{\pi b(u_1-u_2+2p_2)}, \\ \mathbf{f}_3 &= e^{\pi b(u_2-u_4+2p_4)}, \\ K'_1 &= e^{\pi b(2u_1-u_2+2u_3-u_5+2u_6)}, \\ K'_2 &= e^{\pi b(-u_1+2u_2-u_3-u_4+2u_5-u_6)}, \\ K'_3 &= e^{\pi b(-u_2+2u_4-u_5)}.\end{aligned}$$

Then using ' for the second component, and

$$\Delta(\mathbf{f}_i) = \mathbf{f} \otimes 1 + K'_i \otimes \mathbf{f}_i,$$

we obtain the unitary equivalence $\widetilde{\mathcal{P}}^{\mathbf{b}} \otimes \widetilde{\mathcal{P}}^{\mathbf{b}} \simeq \widetilde{\mathcal{P}}^{\mathbf{b}} \otimes \mathcal{M}$ where

$$\begin{aligned} Ad_{\Phi_3} \cdot \Delta(\mathbf{f}_i) &= \mathbf{f}_i \otimes 1, \\ Ad_{\Phi_3} \cdot \Delta(K_i) &= K_i \otimes K_i, \end{aligned}$$

is given by

$$\begin{aligned} \Phi_3 &= g_b(e^{\pi b(3u_1 - u_2 + 2u_3 - u_5 + 2u_6 - u'_1 - 2p_1 + 2p'_1)}) g_b(e^{\pi b(-5u_1 + 2u_2 - u_4 + 2u_5 - u_6 + u'_1 - u'_2 + 2p_1 - 2p_2 - 2p_3 + 2p'_2)}) \\ &\quad g_b(e^{\pi b(-2u_1 + 3u_2 - u_3 - u_4 + 2u_5 - u_6 + u'_1 - u'_2 - 2p_2 + 2p'_2)}) g_b(e^{\pi b(3u_1 - u_2 + 2u_3 - u_5 + 2u_6 + 2u'_1 + u'_2 - u'_3 - 2p_1 + 2p'_3)}) \\ &\quad g_b(e^{\pi b(4u_1 - u_2 + 2u_4 - u_5 + u_6 + u'_2 - u'_4 + 2p_2 + 2p_3 - 2p_4 - 2p_5 - 2p_6 + 2p'_4)}) g_b(e^{\pi b(-u_2 - u_3 + 2u_4 + u'_2 - u'_4 + 2p_2 - 2p_4 - 2p_5 + 2p'_4)}) \\ &\quad g_b(e^{\pi b(-2u_2 + 3u_4 - u_5 + u'_2 - u'_4 - 2p_4 + 2p'_4)}) g_b(e^{\pi b(-5u_1 + 2u_2 - u_4 + 2u_5 - u_6 + u'_1 - 2u'_2 + u'_3 + u'_4 - u'_5 + 2p_1 - 2p_2 - 2p_3 + 2p'_5)}) \\ &\quad g_b(e^{\pi b(-2u_1 + 3u_2 - u_3 - u_4 + 2u_5 - u_6 + u'_1 - 2u'_2 + u'_3 + u'_4 - u'_5 - 2p_2 + 2p'_5)}) \\ &\quad g_b(e^{\pi b(3u_1 - u_2 + 2u_3 - u_5 + 2u_6 - 2u'_1 + u'_2 - 2u'_3 + u'_5 - u'_6 - 2p_1 + 2p'_6)}), \end{aligned}$$

while the unitary transformation \mathcal{S} in Proposition 6.4:

$$\begin{aligned} \mathcal{S} \cdot (K_i \otimes K_i) &= K_i \otimes 1, \\ \mathcal{S} \cdot (\mathbf{f}_i \otimes 1) &= \mathbf{f}_i \otimes 1 \end{aligned}$$

is given by the shifts

$$\begin{aligned} \mathcal{S} &= (2p_1 \mapsto 2p_1 - u'_1 - u'_3 - u'_6) \circ (2p_2 \mapsto 2p_2 + u'_1 - u'_2 + u'_3 - u'_5 + u'_6) \circ \\ &\quad (2p_3 \mapsto 2p_3 + 2u'_1 + u'_2 + 2u'_3 + u'_5 + 2u'_6) \circ (2p_4 \mapsto 2p_4 + u'_2 - u'_4 + u'_5) \circ \\ &\quad (2p_5 \mapsto 2p_5 + u'_1 - 2u'_2 + u'_3 + u'_4 - 2u'_5 + u'_6) \circ (2p_6 \mapsto 2p_6 - 2u'_1 + u'_2 - 2u'_3 + u'_5 - 2u'_6) \circ \\ &\quad (u_1 \mapsto u_1 - u'_1 - u'_3 - u'_6) \circ (u_2 \mapsto u_2 - u'_2 - u'_5) \circ (u_4 \mapsto u_4 - u'_4), \end{aligned}$$

where

$$2p_k \mapsto 2p_k - \lambda$$

is realized by multiplication by the unitary function $e^{\pi i \lambda u_k}$ on $L^2(\mathbb{R}^6)$.

7.3 Type B_3

Finally let us consider the situation where $Q^{\mathbf{i}} \not\cong Q^{\bar{\mathbf{i}}}$, hence we need to describe Φ_1 and Φ_3 separately. Let

$$\mathbf{i} = (1, 2, 1, 2, 3, 2, 1, 2, 3)$$

be the reduced word chosen in [14], and

$$\bar{\mathbf{i}} = (3, 2, 1, 2, 3, 2, 1, 2, 1)$$

be the reversed word. The basic quiver for B_3 is drawn in Figure 5b. The positive representation $\mathcal{P}_\lambda^{\mathbf{b}}$ is given on $\mathcal{X}^{\mathbf{i}\bar{\mathbf{i}}}$ indexed as in the top of Figure 13 by

$$\begin{aligned} \mathbf{f}_1 &= X(1, 3, 6, \bar{1}, \bar{4}, \bar{6}), & K'_1 &= X_{1,3,6,\bar{1},\bar{4},\bar{6},\bar{7}}, \\ \mathbf{f}_2 &= X(2, 5, \bar{2}, \bar{5}), & K'_2 &= X_{2,5,\bar{2},\bar{5},\bar{8}}, \\ \mathbf{f}_3 &= X(4, \bar{3}), & K'_3 &= X_{4,\bar{3},\bar{9}}. \end{aligned}$$

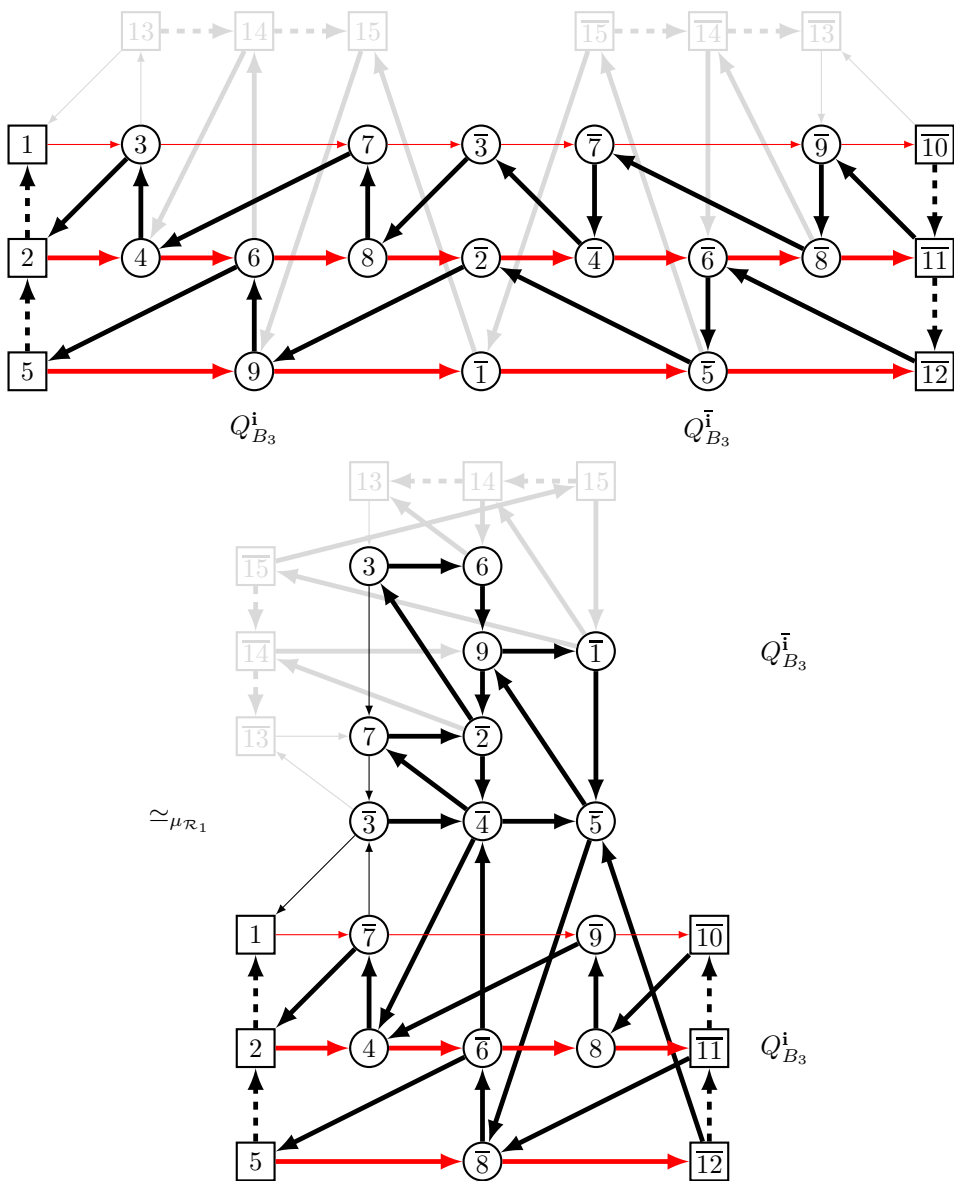


Figure 13: $\mathcal{P}_\lambda^b \simeq \widetilde{\mathcal{P}}^b$ in type B_3 with F_i paths shown in red.

The E_i -paths are described explicitly in [14] given by (cf Figure 5b)

$$\begin{aligned} \mathcal{P}_{E_1} &= (10, 8, 7, 4, 3, 13), \\ \mathcal{P}_{E_2} &= (11, 9, 6, 14), \\ \mathcal{P}_{E_3} &= (12, 15), \end{aligned}$$

and correspond to the following decomposition of g_b which follows from (2.29) and (2.31):

$$\begin{aligned} g_b(\mathbf{e}'_1) &= g_{b_s}(X_{3,4,7,8,10})g_{b_s}(X_{4,7,8,10})g_b(X_{4,7^2,8^2,10^2})g_{b_s}(X_{7,8,10})g_{b_s}(X_{8,10})g_b(X_{8,10^2})g_{b_s}(X_{10}), \\ g_b(\mathbf{e}'_2) &= g_b(X_{6,9,11})g_b(X_{9,11})g_b(X_{11}), \\ g_b(\mathbf{e}'_3) &= g_b(X_{12}). \end{aligned}$$

The F_i -paths are given by the horizontal red paths:

$$\begin{aligned} \mathcal{P}_{F_1} &= (1, 3, 7, 10), \\ \mathcal{P}_{F_2} &= (2, 4, 6, 8, 11), \\ \mathcal{P}_{F_3} &= (5, 9, 12). \end{aligned}$$

Identifying $X_{\bar{1}} := X_{12} \otimes X_{\bar{1}}$, $X_{\bar{2}} := X_{11} \otimes X_{\bar{2}}$ and $X_{\bar{3}} := X_{10} \otimes X_{\bar{3}}$ in \mathcal{X}^{ii} , we obtain the unitary transformation Φ_1 corresponding to a sequence of 35 mutations as follows:

$$\begin{aligned} \Phi_1 &= g_{b_s}(X_{3,4,7,8,\bar{3},\bar{7},\bar{9}})g_{b_s}(X_{4,7,8,\bar{3},\bar{7},\bar{9}})g_b(X_{4,7^2,8^2,\bar{3}^2,\bar{7}^2,\bar{9}^2})g_{b_s}(X_{7,8,\bar{3},\bar{7},\bar{9}})g_{b_s}(X_{8,\bar{3},\bar{7},\bar{9}})g_b(X_{8,\bar{3}^2,\bar{7}^2,\bar{9}^2}) \\ &\quad g_{b_s}(X_{\bar{3},\bar{7},\bar{9}})g_b(X_{6,9,\bar{2},\bar{4},\bar{6},\bar{8}})g_b(X_{9,\bar{2},\bar{4},\bar{6},\bar{8}})g_b(X_{\bar{2},\bar{4},\bar{6},\bar{8}})g_{b_s}(X_{3,4,7,8,\bar{3},\bar{7}})g_{b_s}(X_{4,7,8,\bar{3},\bar{7}})g_b(X_{4,7^2,8^2,\bar{3}^2,\bar{7}^2}) \\ &\quad g_{b_s}(X_{7,8,\bar{3},\bar{7}})g_{b_s}(X_{8,\bar{3},\bar{7}})g_b(X_{8,\bar{3}^2,\bar{7}^2})g_{b_s}(X_{\bar{3},\bar{7}})g_b(X_{6,9,\bar{2},\bar{4},\bar{6}})g_b(X_{9,\bar{2},\bar{4},\bar{6}})g_b(X_{\bar{2},\bar{4},\bar{6}})g_b(X_{\bar{1},\bar{5}})g_b(X_{6,9,\bar{2},\bar{4}}) \\ &\quad g_b(X_{9,\bar{2},\bar{4}})g_b(X_{\bar{2},\bar{4}})g_{b_s}(X_{3,4,7,8,\bar{3}})g_{b_s}(X_{4,7,8,\bar{3}})g_b(X_{4,7^2,8^2,\bar{3}^2})g_{b_s}(X_{7,8,\bar{3}})g_{b_s}(X_{8,\bar{3}})g_b(X_{8,\bar{3}^2})g_{b_s}(X_{\bar{3}}) \\ &\quad g_b(X_{6,9,\bar{2}})g_b(X_{9,\bar{2}})g_b(X_{\bar{2}})g_b(X_{\bar{1}}), \end{aligned}$$

which corresponds to the mutation sequence of Q^{ii} at

$$\mu = (\bar{1}, \bar{2}, 9, 6, \bar{3}, 8, \bar{3}, 7, 4, 7, 3, \bar{4}, \bar{2}, 9, \bar{5}, \bar{6}, \bar{4}, \bar{2}, \bar{7}, 8, \bar{7}, \bar{3}, 4, \bar{3}, 7, \bar{8}, \bar{6}, \bar{4}, \bar{9}, 8, \bar{9}, \bar{7}, 4, \bar{7}, \bar{3}).$$

Finally, the unitary transformation Φ_3 is constructed similarly, where we use $\mathbf{f}^{k,+}$ instead of $\mathbf{f}^{k,-}$, and \mathbf{i} instead of $\bar{\mathbf{i}}$. Using the standard indexing as in Figure 14, and identifying $X_{1'}$:= $X_{10} \otimes X_{1'}$, $X_{2'}$:= $X_{11} \otimes X_{2'}$ and $X_{3'}$:= $X_{12} \otimes X_{3'}$ in \mathcal{X}^{ii} we have

$$\begin{aligned} \Phi_3 &= g_b(X_{5',9'})g_b(X_{6,9,2',4',6',8'})g_b(X_{9,2',4',6',8'})g_b(X_{2',4',6',8'})g_{b_s}(X_{3,4,7,8,1',3',7'})g_{b_s}(X_{4,7,8,1',3',7'}) \\ &\quad g_b(X_{4,7^2,8^2,1'^2,3'^2,7'^2})g_{b_s}(X_{7,8,1',3',7'})g_{b_s}(X_{8,1',3',7'})g_b(X_{8,1'^2,3'^2,7'^2})g_{b_s}(X_{1',3',7'})g_b(X_{6,9,2',4',6'}) \\ &\quad g_b(X_{9,2',4',6'})g_b(X_{2',4',6'})g_b(X_{5'})g_b(X_{6,9,2',4'})g_b(X_{9,2',4'})g_b(X_{2',4'})g_{b_s}(X_{3,4,7,8,1',3'})g_{b_s}(X_{4,7,8,1',3'}) \\ &\quad g_b(X_{4,7^2,8^2,1'^2,3'^2})g_{b_s}(X_{7,8,1',3'})g_{b_s}(X_{8,1',3'})g_b(X_{8,1'^2,3'^2})g_{b_s}(X_{1',3'})g_b(X_{6,9,2'})g_b(X_{9,2'})g_b(X_{2'}) \\ &\quad g_{b_s}(X_{3,4,7,8,1'})g_{b_s}(X_{4,7,8,1'})g_b(X_{4,7^2,8^2,1'^2})g_{b_s}(X_{7,8,1'})g_{b_s}(X_{8,1'})g_b(X_{8,1'^2})g_{b_s}(X_{1'}), \end{aligned}$$

which corresponds to the mutation sequence of Q^{ii} at

$$\mu = (1', 8, 1', 7, 4, 7, 3, 2', 9, 6, 3', 8, 3', 1', 4, 1', 7, 4', 2', 9, 5', 6', 4', 2', 7', 8, 7', 3', 4, 3', 1', 8', 6', 4', 9').$$

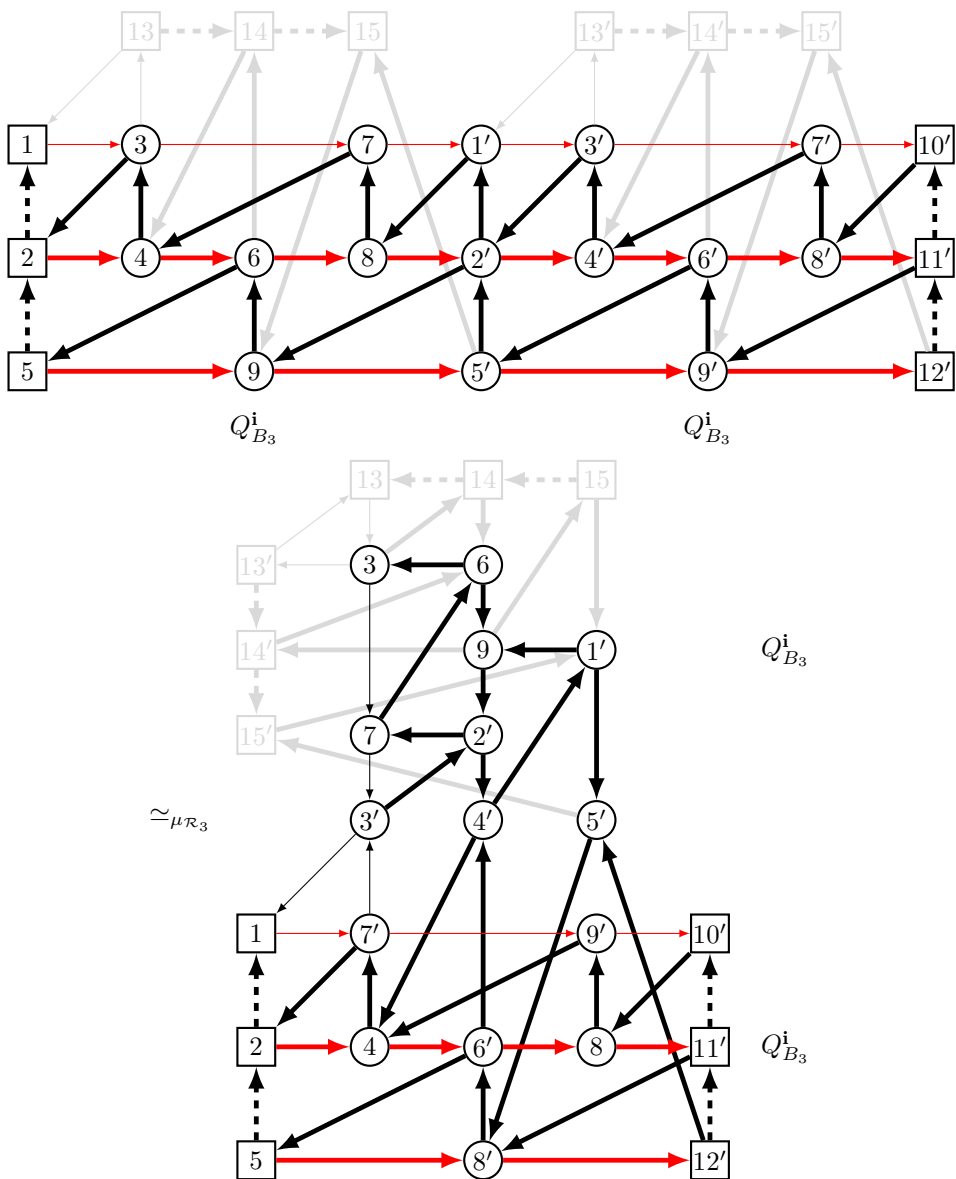


Figure 14: $\tilde{\mathcal{P}}^b \otimes \tilde{\mathcal{P}}^b \simeq \tilde{\mathcal{P}}^b \otimes \mathcal{M}$ in type B_3 with F_i paths shown in red.

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