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A generalized rotationally symmetric case of the centroaffine Minkowski problem

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Abstract

In this paper the centroaffine Minkowski problem, a critical case of the L_p-Minkowski problem in the $n+1$ dimensional Euclidean space, is studied. By its variational structure and the method of blow-up analyses, we obtain two sufficient conditions for the existence of solutions, for a generalized rotationally symmetric case of the problem.

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1. Introduction

Given a convex body X in the Euclidean space $\mathbb{R}^{n+1}$ containing the origin, the centroaffine curvature of ∂X at point p is by definition equal to K/d^{n+2}, where K is the Gauss curvature and d is the distance from the origin to the tangent hyperplane of ∂X at p. The centroaffine curvature is invariant under unimodular linear transforms in $\mathbb{R}^{n+1}$ and has received much attention in geometry [46,48]. The centroaffine Minkowski problem [14] is to find sufficient and necessary conditions for a given positive function $\tilde{f}$, such that $\tilde{f}$ is the centroaffine curvature of a convex

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body containing the origin in $\mathbb{R}^{n+1}$. In the smooth case, the centroaffine Minkowski problem is equivalent to solving the following Monge–Ampère type equation

$$\det(\nabla^2 H + HI) = \frac{f}{H^{n+2}} \quad \text{on } S^n, \tag{1.1}$$

where f is the given positive function (in fact $f = 1/\tilde{f}$), H is the support function of a bounded convex body X in $\mathbb{R}^{n+1}$, I is the unit matrix, $\nabla^2 H = (\nabla_{ij} H)$ is the Hessian matrix of covariant derivatives of H with respect to an orthonormal frame on S^n. When f is a constant, this equation describes affine hyperspheres of elliptic type, and all its solutions are ellipsoids centered at the origin [10].

Equation (1.1) is a special case of the L_p-Minkowski problem introduced by Lutwak [40]. The L_p-Minkowski problem is an important generalization of the classical Minkowski problem, and is a basic problem in the L_p-Brunn–Minkowski theory in modern convex geometry. It has attracted great attention over the last two decades, see for example [5–7,13,14,18,22,24,26–29, 36,37,41–43,51,52,55,59,61–65] and references therein. Solutions to the L_p-Minkowski problem can be also used to prove various inequalities, e.g. [16,22,42,60]. For more related work, one can see e.g. [11,17,19–21,38,39,44,47,50,53,57,58]. In the smooth case, the L_p-Minkowski problem is equivalent to the following Monge–Ampère type equation

$$\det(\nabla^2 H + HI) = f H^{p-1} \quad \text{on } S^n.$$

So Eq. (1.1) is the special case of this equation with $p = -n-1$.

Equation (1.1) also arises in anisotropic Gauss curvature flows and describes their self-similar solutions [4,8,15,25,56]. Besides, its parabolic form can be used for image processing [2]. Eq. (1.1) can be reduced to a singular Monge–Ampère equation in the half Euclidean space $\mathbb{R}^{n+1}_+$, the regularity of which was strongly studied in [30,31].

Equation (1.1) corresponds to the critical case of the famous Blaschke–Santaló inequality in convex geometry [45]:

$$\operatorname{vol}(X) \inf_{\xi \in X} \frac{1}{n+1} \int_{S^n} \frac{dS(x)}{(H(x) - \xi \cdot x)^{n+1}} \le \omega_{n+1}^2, \tag{1.2}$$

where X is any convex body in $\mathbb{R}^{n+1}$, $\operatorname{vol}(X)$ is the volume of the convex body X, H is the support function of X, and ω_{n+1} is the volume of the unit ball in $\mathbb{R}^{n+1}$. Also Eq. (1.1) remains invariant under projective transforms on S^n [14,37]. When f is a constant function, it only has constant solution up to a projective transformation. This result has been known for a long time, see [10] for example, which implies that there is no a priori estimates on solutions for general f. Besides, Chou and Wang [14] found an obstruction for solutions to Eq. (1.1). By this obstruction, one can easily construct a smooth f such that Eq. (1.1) has no solution. Also Eq. (1.1) may have many solutions for some f [23]. This situation is similar, in some aspects, to the prescribed scalar curvature problem on S^n, which involves critical exponents of Sobolev inequalities and the Kazdan–Warner obstruction for solutions [12,49]. So the existence of solutions to equation (1.1) is a rather complicated problem due to these features.

For $n = 1$, the existence of solutions to Eq. (1.1) was investigated in [1,3,13,15,18,32,33,55, 59]. For general f, one needs to impose some nondegenerate and topological degree conditions additionally to obtain an existence result, see e.g. [32,33]. But when f is a periodic function with period $\frac{2\pi}{k}$ for $k \geq 3$, Eq. (1.1) has at least a periodic solution without additional assumptions, see e.g. [1,13,18,55,59].

For higher n-dimension, only several special cases were studied, see [36,37] for the rotationally symmetric case, [29] for the mirror-symmetric case, and [62] for the discrete case. Some sufficient conditions for the existence of solutions were found in these papers. There are no existence results of solutions about Eq. (1.1) for general f.

The major method to obtain these existence results for equation (1.1) when $n \geq 1$ is the variational method, see [29,36,37,62]. This method is essentially to consider the following maximizing problem

$$\sup_{|X|=\omega_{n+1}} \inf_{\xi \in X} J[H(x) - \xi \cdot x], \tag{1.3}$$

where the supremum is taken among a certain class of bounded convex bodies X in $\mathbb{R}^{n+1}$ containing the origin with volume ω_{n+1}, the infimum is taken among all points $\xi \in X$, H is the support function of X, and the functional J is given by

$$J[H] = \frac{1}{n+1} \int_{S^n} \frac{f}{H^{n+1}}. \tag{1.4}$$

The maximizing problem is closely related to the Blaschke–Santaló inequality (1.2). When $f \equiv 1$, (1.3) is just the left hand side of the inequality. So for any given continuous positive function f on S^n, the maximizing problem has an upper bound. If there exists a maximizer H, we expect it to be a solution to Eq. (1.1) after multiplied by a proper constant [29,37,62].

Very recently it was proved in [35] that the maximizing problem (1.3) has no maximizer when f is even, namely $f(-x) = f(x)$ for all $x \in S^n$. There the supremum is taken among all convex bodies or all origin symmetric convex bodies. This result implies that, in order to obtain a maximizer of (1.3), the class of convex bodies X should not be too large, especially can not be the set of all convex bodies. It also partly explains why all the known existence results for $n \geq 1$ are about special cases. In this paper, we further investigate the variational method for this problem. In [29], the mirror symmetric case of Eq. (1.1) was studied and two sufficient conditions for the existence were found. Here, we study another common symmetric case of Eq. (1.1), namely a generalized rotationally symmetric case.

Assume $n \geq 2$. Let $SO(n)$ be the special orthogonal group in dimension n. Assume G is a discrete (or equivalently finite) subgroup of $SO(n)$, such that for every nonzero $x' \in \mathbb{R}^n$, the orbit of x' under G spans $\mathbb{R}^n$. A function F defined on $S^n \subset \mathbb{R}^{n+1}$ is called to be *G-rotationally symmetric* with respect to the x_{n+1}-axis, if for any $x = (x', x_{n+1}) \in S^n$, there is

$$F(\Omega x', x_{n+1}) = F(x', x_{n+1}), \quad \forall \Omega \in G.$$

Correspondingly, a convex body X in $\mathbb{R}^{n+1}$ is called *G-rotationally symmetric* with respect to the x_{n+1}-axis, if

$$\begin{pmatrix} \Omega & \\ & 1 \end{pmatrix} X = X, \quad \forall \Omega \in G.$$

Obviously, the support function H of a convex body X is G-rotationally symmetric if and only if X is G-rotationally symmetric.

A trivial G-rotational symmetry is the $SO(n)$-rotational symmetry, namely $G = SO(n)$. The existence of solutions to Eq. (1.1) for this case was already studied in [36,37]. In $\mathbb{R}^3$, a typical G-rotational symmetry is the k-fold rotational symmetry. Namely, for any integer $k \geq 3$, let G be the group consisting of

$$\begin{pmatrix} \cos\frac{2\pi i}{k} & -\sin\frac{2\pi i}{k} \\ \sin\frac{2\pi i}{k} & \cos\frac{2\pi i}{k} \end{pmatrix}, \quad i = 0, 1, \cdots, k-1.$$

Then the G-rotational symmetry of a convex body in $\mathbb{R}^3$ is equivalent to that it remains unchanged under a rotation through an angle of $\frac{2\pi}{k}$ around the x_3-axis. So the G-rotational symmetry for $n \geq 2$ is a generalization of the $\frac{2\pi}{k}$-periodicity for $n = 1$. The latter case of Eq. (1.1) was intensively studied, as mentioned above. While there are no results about the former case for $n \geq 2$.

We study the G-rotationally symmetric case of Eq. (1.1) in this paper, and obtain two sufficient conditions for existence of solutions. We state our results in the following.

Denote the area of the unit n-sphere by σ_n, and

$$S^n_+ := S^n \cap \{x_{n+1} > 0\}, \quad S^n_- := S^n \cap \{x_{n+1} < 0\}.$$

Let

$$\begin{aligned} \bar{f} &= \frac{1}{\sigma_n} \int\limits_{S^n} f(x)dS(x), \\ \bar{f}_N &= \frac{2}{\sigma_n} \int\limits_{S^n_+} f(x)dS(x), \\ \bar{f}_S &= \frac{2}{\sigma_n} \int\limits_{S^n_-} f(x)dS(x), \end{aligned} \tag{1.5}$$

be the mean values of f on the sphere and on the northern and southern hemispheres respectively.

Theorem 1.1. *If f is a G-rotationally symmetric positive continuous function on S^n, such that the maximum value of f restricted to the equator and the north and south poles is not greater than $\bar{f}/2^{n+1}$ or $\frac{1}{2}\min\{\bar{f}_N, \bar{f}_S\}$, then equation (1.1) admits a G-rotationally symmetric solution.*

When f is additionally symmetric with respect to the equatorial hyperplane, from the proof of Theorem 1.1, we can easily improve it as the following

Theorem 1.2. *Assume $f \in C(S^n)$ is positive and G-rotationally symmetric, and also symmetric with respect to the equatorial hyperplane. If the maximum value of f restricted to the equator and the north and south poles is less than $\bar{f}$, then equation (1.1) admits a G-rotationally symmetric solution.*

Our another existence result involves derivatives of f. We introduce some notations first. Let

$$e_{n+1} = (0, 0, \cdots, 0, 1)$$

be the north pole, and $e_{n+1}^{\perp}$ be the equatorial hyperplane. For each $x \in S^n \cap e_{n+1}^{\perp}$, we define

$$PI(x) = \int_0^{\pi} \tilde{f}'(\theta) \cot\theta \, d\theta, \tag{1.6}$$

where $\tilde{f}$ is the restriction of f on the half great circle on S^n through the three points x and $\pm e_{n+1}$, parameterized by an arc parameter $\theta \in [0, \pi]$. When $f \in C^1(S^n)$ is G-rotationally symmetric, one can check $PI(x)$ is well defined, see Lemma 4.1. Note that for given x, $PI(x)$ is just $pi(f)$ defined in [37] for a rotationally symmetric f.

Let $M_{1/2}[\alpha, \beta]$ be the $\frac{1}{2}$-*power mean* of two positive numbers α and β, namely

$$M_{1/2}[\alpha, \beta] = \Big(\frac{\sqrt{\alpha} + \sqrt{\beta}}{2}\Big)^2. \tag{1.7}$$

Now we can state the following

Theorem 1.3. *Assume $f \in C^2(S^n)$ is positive and G-rotationally symmetric. If the maximum value of f on the equator is less than $M_{1/2}[f(e_{n+1}), f(-e_{n+1})]$, and $PI \geq 0$ with at least one positive value, then equation (1.1) admits a G-rotationally symmetric solution.*

In Theorems 1.1–1.3, we require f is G-rotationally symmetric on S^n with $n \geq 2$. Here we note that these theorems also hold when $n = 1$ and f is rotationally symmetric. We also note that even when f is $SO(n)$-rotationally symmetric, which was studied in [37], our theorems provide new existence conditions.

The blow up analyses in our proofs of these theorems are inspired by [29], which treats the mirror symmetric case of equation (1.1). The mirror symmetry has two special advantages. One is that the infimum $\inf_{\xi \in X} J[H(x) - \xi \cdot x]$ in the maximizing problem (1.3) must be attained at $\xi = 0$. The other is that the first order derivatives of f vanish on the coordinate hyperplanes since f is even with respect to each component of x. These two features play a key role in the proofs of [29]. However there are no such advantages for a G-rotationally symmetric f. To overcome these difficulties, we estimate the supremum of (1.3) more carefully by a special construction, and combine assumptions about derivatives and values of f. Besides, we observe that the blow up analyses depend only on the shape of minimum ellipsoids of convex bodies.

Note that any generalized solution to equation (1.1) must be positive on S^n, see Corollary 2.4 in [37]. Therefore, the regularity of solutions obtained in our theorems follows the standard regularity theory about Monge–Ampère equation, see [14] for example.

The paper is organized as follows. In Section 2, we investigate a variational method, which provides a solution to Eq. (1.1) whenever there exists a maximizer. Then we use blow up analyses to prove that a maximizer exists under suitable assumptions, and thus complete proofs of Theorems 1.1 and 1.3 in Section 3 and Section 4 respectively.

2. A variational method

As mentioned before, Eq. (1.1) admits a variational structure. We will study it in this section, and prove that its maximizer (if exists) is a solution to (1.1) after rescaling by a proper constant.

Let $\mathcal{S}$ denote the set of all support functions of convex bodies in $\mathbb{R}^{n+1}$, namely

$$\mathcal{S} = \left\{ H \in C(S^n) : H \text{ is the restriction of a sublinear function in } \mathbb{R}^{n+1} \right\}.$$

Let

$$\mathcal{S}_+ := \{H \in \mathcal{S} : H > 0\},$$
$$\mathcal{S}_+^G := \{H \in \mathcal{S}_+ : H \text{ is } G\text{-rotationally symmetric}\}.$$

We use $\mathcal{K}_o$ to denote the set of all convex bodies in $\mathbb{R}^{n+1}$ containing the origin in their interiors, and $\mathcal{K}_o^G$ to denote the subset of $\mathcal{K}_o$ consisting of only G-rotationally symmetric ones. Note that there is a one-to-one correspondence between $\mathcal{K}_o^G$ and $\mathcal{S}_+^G$.

Now we consider the following maximizing problem

$$\sup_{\substack{X \in \mathcal{K}_o^G \\ |X| = \omega_{n+1}}} \inf_{\xi \in X} J[H(x) - \xi \cdot x], \tag{2.1}$$

where the supremum is taken among all G-rotationally symmetric bounded convex bodies X in $\mathbb{R}^{n+1}$ containing the origin with volume ω_{n+1}, the infimum is taken among all points $\xi \in X$, H is the support function of X, and the functional J is given by

$$J[H] = \frac{1}{n+1} \int_{S^n} \frac{f}{H^{n+1}}. \tag{2.2}$$

Here we require f is G-rotationally symmetric. Note that $J[H(x) - \xi \cdot x]$ is strictly convex with respect to ξ, and goes to infinity as ξ tends to the boundary of X, so $\inf_{\xi \in X} J[H(x) - \xi \cdot x]$ is attained at a unique point ξ.

By the Blaschke–Santaló inequality, the maximizing problem (2.1) has an upper bound. But it may not admit a maximizer, see [35]. In the following of this section, we always assume that h is a maximizer of (2.1), namely

$$J[h] = \inf_{\xi \in X_h} J[h(x) - \xi \cdot x] = J_{\sup}, \tag{2.3}$$

where X_h denotes the convex body determined by h, and $J_{\sup}$ the supremum of the problem (2.1). Note that X_h is G-rotationally symmetric with respect to the x_{n+1}-axis and its volume is ω_{n+1}.

We shall first prove X_h is C^1 and strictly convex, then h provides a solution to (1.1) after rescaling by a proper constant. The proof follows that of [14], where the maximizing problem (2.1) defined on the set $\mathcal{K}_o$ was studied. Here (2.1) is restricted on a smaller subset $\mathcal{K}_o^G$, so one must be cautious when dealing with the variation.

Let $|G|$ denote the order of the group G, namely the number of elements in G. Since G is finite, $|G|$ is a positive integer. For any $\Omega \in G$, write correspondingly

$$\tilde{\Omega} := \begin{pmatrix} \Omega & \\ & 1 \end{pmatrix}. \tag{2.4}$$

For any subset $U \subset \mathbb{R}^{n+1}$, we use GU to denote the image of U under the group action of G, namely

$$GU := \left\{ \tilde{\Omega} y \mid \Omega \in G,\ y \in U \right\}. \tag{2.5}$$

Lemma 2.1. *The Gauss curvature of ∂X_h is bounded from below in the generalized sense by a positive constant C.*

Proof. Fix a point p on ∂X_h. Let $\bar{B}_r(p)$ be the closed ball in $\mathbb{R}^{n+1}$ centered at p with radius r. Note that G is a finite subgroup of $SO(n)$, for sufficiently small $r > 0$, $G\bar{B}_r(p)$ (see (2.5) for the notation) must be a finite disjoint union of closed balls with the same radius r. For such an r, let

$$\omega = \partial X_h \cap \bar{B}_r(p).$$

Since ∂X_h is G-rotationally symmetric, there is

$$G\omega = \partial X_h \cap G\bar{B}_r(p),$$

which is also G-rotationally symmetric. Let $\omega^* = \nu(\omega)$, where ν is the Gauss mapping of ∂X_h. Then $G\omega^* = \nu(G\omega)$.

For any small $t > 0$, let X_t be the convex hull of $X_h \cup N_t(G\omega)$ where

$$N_t(G\omega) = \{p : \mathrm{dist}(p, G\omega) < t\}.$$

Obviously, X_t is G-rotationally symmetric. Denote its support function and volume by h_t and $\mathrm{vol}(h_t)$ respectively. Then

$$h_t(x) = h(x) + t,\ \forall x \in G\omega^*,$$

and

$$\lim_{t \to 0^+} \frac{\mathrm{vol}(h_t) - \mathrm{vol}(h)}{t} \geq |G\omega|.$$

Observe that for any $x \notin G\omega^*$, $h_t(x) = h(x)$ for sufficiently small t. Therefore

$$\lim_{t \to 0^+} \frac{h_t(x) - h(x)}{t} = \chi(x),$$

where χ is the characteristic function of $G\omega^*$, namely $\chi(x) = 1$ for $x \in G\omega^*$ and $\chi(x) = 0$ for $x \in S^n \backslash G\omega^*$.

Let h_0 be h, and $\alpha(t) > 0$ be such that

$$\mathrm{vol}(\alpha(t)h_t) = \omega_{n+1}. \tag{2.6}$$

Then $\alpha(0) = 1$, and

$$\begin{aligned}\alpha'(0) &= -\frac{1}{(n+1)\omega_{n+1}} \lim_{t\to 0^+} \frac{\mathrm{vol}(h_t) - \mathrm{vol}(h)}{t} \\ &\le -\frac{|G\omega|}{(n+1)\omega_{n+1}},\end{aligned}$$

where the limit can be taken for any convergent subsequence. Let

$$\mathcal{J}(t) = \inf_{\xi} J[\alpha(t)h_t(x) - \xi \cdot x], \tag{2.7}$$

where the infimum is taken among all points inside the convex body determined by $\alpha(t)h_t$. Suppose the infimum is attained at the unique point $\xi(t)$. Then $\xi(t)$ is Lipschitz continuous. Without loss of generality, we can assume $\xi'(0)$ exists. Recalling h_t is G-rotationally symmetric, and h is a maximizer of (2.1), we have

$$\mathcal{J}(0) \ge \mathcal{J}(t).$$

Therefore

$$\lim_{t\to 0^+} \frac{\mathcal{J}(t) - \mathcal{J}(0)}{t} \le 0,$$

where again the limit may be taken for any convergent subsequence. By

$$\mathcal{J}(t) = J[\alpha(t)h_t(x) - \xi(t) \cdot x],$$

we have

$$-\int_{S^n} \frac{f}{h^{n+2}}(\alpha'(0)h + \chi - \xi'(0) \cdot x) \le 0.$$

Recall (2.3) says the infimum of $J[h(x) - \xi \cdot x]$ is attained at $\xi = 0$. We have

$$\int_{S^n} \frac{f}{h^{n+2}} x_i = 0, \quad i = 1, 2, \cdots, n+1.$$

Therefore

$$-\int_{S^n} \frac{f}{h^{n+2}}(\alpha'(0)h + \chi) \le 0. \tag{2.8}$$

By the estimate of $\alpha'(0)$ and the definition of χ, the above inequality becomes into

$$\frac{|G\omega|}{(n+1)\omega_{n+1}} \int_{S^n} \frac{f}{h^{n+1}} \leq \int_{G\omega^*} \frac{f}{h^{n+2}},$$

which implies

$$\frac{|G\omega^*|}{|G\omega|} \geq C > 0,$$

where C depends only on the bounds of h, f and n.

By our construction of ω, there is $|G\omega| = k|\omega|$ for some integer $1 \leq k \leq |G|$, and $|G\omega^*| \leq |G| \cdot |\omega^*|$. Hence

$$\frac{|\omega^*|}{|\omega|} \geq \frac{|G|^{-1} \cdot |G\omega^*|}{k^{-1}|G\omega|} \geq \frac{k}{|G|} C \geq \frac{C}{|G|}. \qquad \square$$

Lemma 2.2. *The Gauss curvature of ∂X_h is bounded from above in the generalized sense by a positive constant C.*

Proof. By the argument of [14, Lemma 5.6], it suffices to prove

$$\frac{|\omega^*|}{|\nu^{-1}(\omega^*)|} \leq C \tag{2.9}$$

for any closed subset $\omega^* \subset S^n$. Here ν is the Gauss mapping of ∂X_h as before.

Fix a vector $\bar{x} \in S^n$. Again let $\bar{B}_r(\bar{x})$ be the closed ball in $\mathbb{R}^{n+1}$ centered at $\bar{x}$ with radius r. As explained in the previous lemma, $G\bar{B}_r(\bar{x})$ is a finite disjoint union of closed balls with the same radius r when r is sufficiently small. For such an r, let

$$\omega^* = S^n \cap \bar{B}_r(\bar{x}).$$

For each $\Omega \in G$,

$$\tilde{\Omega}\omega^* = S^n \cap \bar{B}_r(\tilde{\Omega}\bar{x}),$$

where $\tilde{\Omega}$ is defined in (2.4). Then

$$G\omega^* = S^n \cap G\bar{B}_r(\bar{x}).$$

Let $\omega = \nu^{-1}(\omega^*)$, then $G\omega = \nu^{-1}(G\omega^*)$.

For a small $t > 0$, let

$$X_{t,\Omega} := \left\{ p \in \mathbb{R}^{n+1} : p \cdot x \leq h(x) - t(\tilde{\Omega}\bar{x}) \cdot x, \ \forall x \in \tilde{\Omega}\omega^* \right\} \cap X_h,$$

where the notation "$\cdot$" denotes the standard inner product in $\mathbb{R}^{n+1}$. Note h is positive on S^n, we see the origin is an interior point of $X_{t,\Omega}$ for a small t. Let

$$X_t := \bigcap_{\Omega \in G} X_{t,\Omega}.$$

We claim X_t is a G-rotationally symmetric convex body. To see this, we first note that

$$\widetilde{\Omega'} X_{t,\Omega} = X_{t,\Omega'\Omega}, \quad \forall\, \Omega', \Omega \in G. \tag{2.10}$$

In fact, for $p \in \widetilde{\Omega'} X_{t,\Omega}$, we have $\widetilde{\Omega'}^{-1} p \in X_{t,\Omega}$. Namely,

$$\widetilde{\Omega'}^{-1} p \in X_h, \tag{2.11}$$

and

$$\widetilde{\Omega'}^{-1} p \cdot x \le h(x) - t(\tilde{\Omega}\bar{x}) \cdot x, \quad \forall\, x \in \tilde{\Omega}\omega^*. \tag{2.12}$$

Since X_h is G-rotationally symmetric, (2.11) implies $p \in X_h$. Note $\widetilde{\Omega'}^{-1} p \cdot x = p \cdot \widetilde{\Omega'} x$ and $h(\widetilde{\Omega'} x) = h(x)$, then (2.12) is equivalent to

$$p \cdot \widetilde{\Omega'} x \le h(\widetilde{\Omega'} x) - t(\tilde{\Omega}\bar{x}) \cdot x, \quad \forall\, x \in \tilde{\Omega}\omega^*.$$

Let $y = \widetilde{\Omega'} x$, then

$$p \cdot y \le h(y) - t(\tilde{\Omega}\bar{x}) \cdot (\widetilde{\Omega'}^{-1} y), \quad \forall\, y \in \widetilde{\Omega'}\tilde{\Omega}\omega^*,$$

which is equivalent to

$$p \cdot y \le h(y) - t(\widetilde{\Omega'\Omega}\bar{x}) \cdot y, \quad \forall\, y \in \widetilde{\Omega'\Omega}\omega^*.$$

Thus $p \in X_{t,\Omega'\Omega}$. Therefore

$$\widetilde{\Omega'} X_{t,\Omega} \subset X_{t,\Omega'\Omega}.$$

Observing the above argument is reversible, we also have that

$$X_{t,\Omega'\Omega} \subset \widetilde{\Omega'} X_{t,\Omega}.$$

Hence (2.10) is true. Now for any $\Omega' \in G$,

$$\begin{aligned}\widetilde{\Omega}' X_t &= \bigcap_{\Omega \in G} \widetilde{\Omega}' X_{t,\Omega} \\ &= \bigcap_{\Omega \in G} X_{t,\Omega'\Omega} \\ &= \bigcap_{\Omega'\Omega \in G} X_{t,\Omega'\Omega} \\ &= \bigcap_{\Omega \in G} X_{t,\Omega} \\ &= X_t,\end{aligned}$$

which implies X_t is G-rotationally symmetric.

Let h_t be the support function of X_t. We also write $X_0 = X_h$ and $h_0 = h$. Given an $x \in G\omega^*$, there exists some $\Omega \in G$ such that $x \in \tilde{\Omega}\omega^*$. Note $X_t \subset X_{t,\Omega}$, we have

$$h_t(x) \le h(x) - t(\tilde{\Omega}\bar{x}) \cdot x. \tag{2.13}$$

Since $\tilde{\Omega}^{-1}x \in \omega^* = S^n \cap \bar{B}_r(\bar{x})$, and we can require $r \le 1$, there is

$$\bar{x} \cdot (\tilde{\Omega}^{-1}x) \ge \frac{1}{2}.$$

By $(\tilde{\Omega}\bar{x}) \cdot x = \bar{x} \cdot (\tilde{\Omega}^{-1}x)$, we obtain

$$(\tilde{\Omega}\bar{x}) \cdot x \ge \frac{1}{2}.$$

Now (2.13) is simplified as

$$h_t(x) \le h(x) - \frac{1}{2}t.$$

Note this inequality holds for all $x \in G\omega^*$. Observe that h_t is non-increasing in t and $h_t \le h$. Thus

$$\lim_{t\to 0^+} \frac{h_t(x) - h(x)}{t} \le -\frac{1}{2}\chi(x), \quad \forall x \in S^n. \tag{2.14}$$

Here χ is the characteristic function of $G\omega^*$, and the limit is taken for any convergent subsequence as before.

Now we estimate the volume of X_t. For each $\Omega \in G$, there is $\tilde{\Omega}\omega = \nu^{-1}(\tilde{\Omega}\omega^*)$. Therefore for any $p \in \partial X_h \backslash \tilde{\Omega}\omega$, $\nu(p) \cap \tilde{\Omega}\omega^* = \emptyset$. Hence $p \in \partial X_{t,\Omega}$ for sufficiently small t. Thus

$$\lim_{t\to 0^+} \frac{\mathrm{vol}(X_h) - \mathrm{vol}(X_{t,\Omega})}{t} \le |\tilde{\Omega}\omega| = |\omega|.$$

By the definition of X_t, we have

$$\lim_{t\to 0^+} \frac{\mathrm{vol}(X_h)-\mathrm{vol}(X_t)}{t} \le |G|\cdot|\omega|,$$

namely

$$\lim_{t\to 0^+} \frac{\mathrm{vol}(h_t)-\mathrm{vol}(h)}{t} \ge -|G|\cdot|\omega|. \tag{2.15}$$

Again let $\alpha(t)$, $\xi(t)$ and $\mathcal{J}(t)$ be defined as in the previous lemma. Then

$$\lim_{t\to 0^+} \frac{\mathcal{J}(t)-\mathcal{J}(0)}{t} \le 0.$$

Similar to (2.8), this inequality can be reduced into the following

$$-\int_{S^n} \frac{f}{h^{n+2}}\Big(\alpha'(0)h + \lim_{t\to 0^+}\frac{h_t-h}{t}\Big) \le 0. \tag{2.16}$$

By (2.15), there is

$$\alpha'(0) \le \frac{|G|\cdot|\omega|}{(n+1)\omega_{n+1}}.$$

Recalling (2.14), we obtain from (2.16) that

$$\frac{1}{2}\int_{G\omega^*} \frac{f}{h^{n+2}} \le \frac{|G|\cdot|\omega|}{(n+1)\omega_{n+1}} \int_{S^n} \frac{f}{h^{n+1}},$$

which implies

$$|G\omega^*| \le C|G|\cdot|\omega|,$$

where C depends only on the bounds of h, f and n. Note $|G\omega^*| \ge |\omega^*|$, we obtain

$$\frac{|\omega^*|}{|\omega|} \le C|G|.$$

Thus (2.9) holds, which completes the proof. □

In this section, a Borel measure μ on S^n is called to be G-rotationally symmetric, if

$$\mu(\tilde{\Omega}U) = \mu(U)$$

for every $\Omega \in G$ and every Borel subset $U \subset S^n$.

Lemma 2.3. *Let μ_1 and μ_2 be two G-rotationally symmetric σ-finite Borel measures on S^n. If for every G-rotationally symmetric $\eta \in C^\infty(S^n)$,*

$$\int_{S^n} \eta d\mu_1 = \int_{S^n} \eta d\mu_2,$$

then $\mu_1 = \mu_2$ on S^n.

Proof. For $x \in S^n$, let $B_{x,r}$ be the geodesic ball on S^n with center x and radius r. We first prove that for every $x \in S^n$, there exists some $\bar{r} > 0$ such that

$$\mu_1(B_{x,r}) = \mu_2(B_{x,r}), \quad \forall 0 < r < \bar{r}. \tag{2.17}$$

Given a point $x \in S^n$. Note G is a finite subgroup of $SO(n)$, there exists an $\bar{r} > 0$ such that for $0 < r < \bar{r}$, $GB_{x,r}$ is a finite disjoint union of geodesic balls with the same radius r. Denote the number of points in the orbit Gx by k. Since μ_1 and μ_2 are G-rotationally symmetric, we have

$$\mu_i(GB_{x,r}) = k\mu_i(B_{x,r}), \quad i = 1, 2. \tag{2.18}$$

We shall prove $\mu_1(GB_{x,r}) = \mu_2(GB_{x,r})$. Using smooth cut-off functions, one can easily construct a family of G-rotationally symmetric smooth functions $\{\eta_m\}$ on S^n, which is uniformly bounded and converges pointwise to the characteristic function, χ, of $GB_{x,r}$. By assumptions of this lemma,

$$\int_{S^n} \eta_m d\mu_1 = \int_{S^n} \eta_m d\mu_2.$$

By the bounded convergence theorem, we obtain

$$\int_{S^n} \chi d\mu_1 = \int_{S^n} \chi d\mu_2,$$

namely

$$\mu_1(GB_{x,r}) = \mu_2(GB_{x,r}).$$

Recalling (2.18), we have (2.17). Then one can easily obtain $\mu_1 = \mu_2$ on S^n. $\square$

Proposition 2.4. *∂X_h is C^1, and for some positive constant λ, λh is a generalized solution to (1.1).*

Proof. By Lemmas 2.1 and 2.2, the Gauss curvature of ∂X_h is pinched between two positive constants. By [9], ∂X_h is C^1 and strictly convex.

Now for any G-rotationally symmetric $\eta \in C^\infty(S^n)$, let

$$X_t := \left\{ p \in \mathbb{R}^{n+1} : p \cdot x \le (h + t\eta)(x),\ \forall x \in S^n \right\},$$

and h_t be its support function. Since h is C^1 and strictly convex, we have for small $t \ge 0$ that

$$h_t = h + t\eta. \tag{2.19}$$

Therefore

$$\lim_{t \to 0^+} \frac{h_t - h}{t} = \eta,$$

and

$$\lim_{t \to 0^+} \frac{\mathrm{vol}(h_t) - \mathrm{vol}(h)}{t} = \int_{S^n} \eta d\mu,$$

where μ is the area measure of ∂X_h.

Again let $\alpha(t)$ and $\mathcal{J}(t)$ be defined as in Lemma 2.1. Note h_t is G-rotationally symmetric and h is a maximizer of (2.1), we have

$$\lim_{t \to 0^+} \frac{\mathcal{J}(t) - \mathcal{J}(0)}{t} \le 0,$$

which again implies

$$-\int_{S^n} \frac{f}{h^{n+2}} (\alpha'(0) h + \eta) \le 0.$$

Since

$$\alpha'(0) = -\frac{1}{(n+1)\omega_{n+1}} \int_{S^n} \eta d\mu,$$

we have

$$c \int_{S^n} \eta d\mu - \int_{S^n} \frac{f}{h^{n+2}} \eta \le 0,$$

where

$$c = \frac{1}{(n+1)\omega_{n+1}} \int_{S^n} \frac{f}{h^{n+1}}.$$

Replacing η by $-\eta$, we see that

$$c\int_{S^n}\eta d\mu-\int_{S^n}\frac{f}{h^{n+2}}\eta=0$$

for all G-rotationally symmetric $\eta\in C^\infty(S^n)$. Note f, h and $d\mu$ are all G-rotationally symmetric, by Lemma 2.3, there is

$$cd\mu=\frac{f}{h^{n+2}}dx,$$

where dx is the standard measure on S^n. But $d\mu=\det(\nabla^2 h+hI)dx$ in the generalized sense, we obtain

$$c\det(\nabla^2 h+hI)=\frac{f}{h^{n+2}} \text{ on } S^n.$$

Now let

$$\lambda=c^{1/(2n+2)},$$

then λh is a generalized solution to (1.1). □

3. Proof of Theorem 1.1

We will prove Theorems 1.1 and 1.3 in this and next sections. By arguments in the previous section, in order to prove the existence results in these theorems, one only need to find a maximizer of

$$\sup_{\substack{X\in\mathcal{K}_o^G\\|X|=\omega_{n+1}}}\inf_{\xi\in X} J[H(x)-\xi\cdot x] \tag{3.1}$$

under assumptions of these theorems. We use blow-up analyses to find a maximizer in this paper. Several notions and properties will be needed.

First, the concept of minimum ellipsoids is needed. Recall John's Lemma in convex geometry, see [54] for example. It says that for any convex body X in $\mathbb{R}^{n+1}$, there is a minimum ellipsoid of X, denoted by E, such that

$$\frac{1}{n+1}E\subset X\subset E,$$

where $\lambda E=\{x_0+\lambda(x-x_0): x\in E\}$ and x_0 is the center of E. We say X is *normalized* if the E is a ball.

The minimum ellipsoid of a G-rotationally symmetric convex body is described by the following

Lemma 3.1. *Assume $n\ge 2$. If an ellipsoid E is the minimum ellipsoid of a G-rotationally symmetric convex body in $\mathbb{R}^{n+1}$, then it must have the following form*

$$E=\left\{x\in\mathbb{R}^{n+1}: |S(x-te_{n+1})|\le 1\right\},$$

where $t \in \mathbb{R}$ and

$$S = \mathrm{diag}(a, \cdots, a, b)$$

is a diagonal matrix of order $n+1$ with $a, b > 0$.

Proof. Since the minimum ellipsoid of a given convex body is unique, we see that E is G-rotationally symmetric. Write E as

$$E = \left\{ x \in \mathbb{R}^{n+1} : |S(x - x_0)| \le 1 \right\}, \tag{3.2}$$

where S is a positive definite symmetric matrix of order $n+1$, and x_0 is the center of E. Then for $\Omega \in G$,

$$\begin{aligned} \tilde{\Omega} E &= \left\{ y : y = \tilde{\Omega} x,\ |S(x - x_0)| \le 1 \right\} \\ &= \left\{ y : |S(\tilde{\Omega}^{-1} y - x_0)| \le 1 \right\} \\ &= \left\{ y : |\tilde{\Omega} S \tilde{\Omega}^{-1}(y - \tilde{\Omega} x_0)| \le 1 \right\}. \end{aligned}$$

Note $\tilde{\Omega} E = E$ and the expression (3.2) of E is unique, we have

$$\tilde{\Omega} x_0 = x_0 \ \text{ and } \ \tilde{\Omega} S \tilde{\Omega}^{-1} = S. \tag{3.3}$$

We recall

$$\tilde{\Omega} = \begin{pmatrix} \Omega & \\ & 1 \end{pmatrix}.$$

Write $x_0 = (x_0', x_{n+1})$, we have $\tilde{\Omega} x_0 = (\Omega x_0', x_{n+1})$. By (3.3), there is

$$\Omega x_0' = x_0', \quad \forall \Omega \in G.$$

Since the orbit of each nonzero $z \in \mathbb{R}^n$ under G spans the whole space $\mathbb{R}^n$, there must be $x_0' = 0$. Therefore we can write

$$x_0 = t e_{n+1}, \quad t \in \mathbb{R}. \tag{3.4}$$

Now partition the symmetric matrix S as

$$S = \begin{pmatrix} S' & \alpha \\ \alpha^T & b \end{pmatrix},$$

where S' is a square matrix of order n, and b is a number. By (3.3), we have $\tilde{\Omega} S = S \tilde{\Omega}$, namely

$$\begin{pmatrix} \Omega S' & \Omega \alpha \\ \alpha^T & b \end{pmatrix} = \begin{pmatrix} S' \Omega & \alpha \\ \alpha^T \Omega & b \end{pmatrix},$$

which implies

$$\Omega S' = S'\Omega,$$

and

$$\Omega\alpha = \alpha, \quad \alpha^T\Omega = \alpha^T.$$

Again we have $\alpha = 0$. Note S' is symmetric, we fix an eigenvector $z \in \mathbb{R}^n$ and its corresponding eigenvalue a, namely

$$S'z = az.$$

For any point Ωz is the orbit Gz, there is

$$S'(\Omega z) = \Omega S'z = \Omega(az) = a(\Omega z).$$

Then Ωz is also an eigenvector corresponding to a. By our assumptions, Gz spans $\mathbb{R}^n$. That is, the eigenspace corresponding to the eigenvalue a is the whole space $\mathbb{R}^n$. Then we must have

$$S' = aI_n,$$

where I_n is the identity matrix of order n. Now S is written as

$$S = \begin{pmatrix} aI_n & \\ & b \end{pmatrix}.$$

Note S is positive definite, we have $a, b > 0$. The lemma is proved. $\square$

Lemma 3.1 says that a linear transform like $cS \in GL(n+1)$ with $c > 0$ can transforms the corresponding G-rotationally symmetric convex body into a normalized one.

Another property that will be used is the invariance of the functional J under unimodular linear transforms, see [29,35]. For any convex body X in $\mathbb{R}^{n+1}$, after performing a unimodular linear transform $A^T \in SL(n+1)$, it becomes into another convex body X_A. In the following, we use H_A to denote the support function of X_A. Then

$$H_A(x) = |Ax| \cdot H\left(\frac{Ax}{|Ax|}\right), \quad x \in S^n, \tag{3.5}$$

where H is the support function of X. See [37] for more details about this type of transforms. Associating with linear transforms, we recall an integral variable substitution formula given in [29, 35].

Lemma 3.2. *For any integral function g on S^n, and any matrix $A \in GL(n+1)$, we have the following variable substitution for integration:*

$$\int_{S^n} g(y)\, dS(y) = \int_{S^n} g\left(\frac{Ax}{|Ax|}\right) \cdot \frac{|\det A|}{|Ax|^{n+1}}\, dS(x). \tag{3.6}$$

The proof of this lemma can be found in [35]. And the invariance of J follows directly from (3.5) and (3.6). That is, for any unimodular linear transform $A \in SL(n+1)$,

$$\int_{S^n} \frac{f}{H^{n+1}} = \int_{S^n} \frac{f_A}{H_A^{n+1}}, \quad f_A(x) = f\left(\frac{Ax}{|Ax|}\right). \tag{3.7}$$

Recall that for any support function H, there exists a unique interior point in the convex body X determined by H, denoted by ξ_H, such that

$$J[H(x) - \xi_H \cdot x] = \inf_{\xi \in X} J[H(x) - \xi \cdot x]. \tag{3.8}$$

Here we note that for G-rotationally symmetric H, ξ_H must be located in the x_{n+1}-axis, namely

$$\xi_H = t_H e_{n+1}, \quad t_H \in \mathbb{R}. \tag{3.9}$$

In fact, this is easily obtained by (3.6). For each $\Omega \in G$, we have

$$\begin{aligned} J[H(x) - \xi_H \cdot x] &= \frac{1}{n+1} \int_{S^n} \frac{f(x) dS(x)}{(H(x) - \xi_H \cdot x)^{n+1}} \\ &= \frac{1}{n+1} \int_{S^n} \frac{f(\tilde{\Omega} x) dS(x)}{(H(\tilde{\Omega} x) - \xi_H \cdot \tilde{\Omega} x)^{n+1}} \\ &= \frac{1}{n+1} \int_{S^n} \frac{f(x) dS(x)}{(H(x) - \tilde{\Omega}^T \xi_H \cdot x)^{n+1}} \\ &= J[H(x) - \tilde{\Omega}^T \xi_H \cdot x]. \end{aligned}$$

Since ξ_H is unique for an H, there is

$$\tilde{\Omega}^T \xi_H = \xi_H.$$

Similar to (3.3) and (3.4), we have (3.9).

After these preparations, we now start blow-up analyses to find a maximizer of (3.1). Since (3.1) has a finite upper bound, let $\{H_k\}$ be a maximizing sequence. If it is uniformly bounded, by the Blaschke's selection theorem, a subsequence of $\{H_k\}$ converges uniformly to a support function H_∞ which is a maximizer of (3.1). If not, namely

$$\sup_{S^n} H_k \to \infty \text{ as } k \to \infty. \tag{3.10}$$

Then we will deduce a contradiction by our assumptions, and thus complete proofs of our theorems.

Let X_k be the convex body determined by H_k. For each k choose a unimodular linear transformation $A_k^T \in SL(n+1)$ that normalizes X_k. Namely the convex body

$$X_{A_k} := A_k^T(X_k)$$

is normalized. Denote its support function by H_{A_k}. Since X_{A_k} has the same volume ω_{n+1}, they are uniformly bounded. On account of Blaschke's selection theorem, we assume without loss of generality that X_{A_k} converges to some normalized convex body $\hat{X}$, namely H_{A_k} converges uniformly on S^n to $\hat{H}$, the support function of $\hat{X}$. By virtue of formula (3.7) and (2.17) in [37], we see H_{A_k} also has a uniform positive lower bound. So $\hat{H}$ is positive on S^n. Applying again formula (3.7) and the bounded convergence theorem, one gets

$$\begin{aligned} J_{\sup} &:= \lim_{k\to\infty} J[H_k] \\ &= \lim_{k\to\infty} \frac{1}{n+1}\int_{S^n} \frac{f_{A_k}}{H_{A_k}^{n+1}} \\ &= \frac{1}{n+1}\int_{S^n} \frac{\hat{f}}{\hat{H}^{n+1}}, \end{aligned} \tag{3.11}$$

where $\hat{f}$ is the limit function of f_{A_k}.

We note that

$$J_{\sup} = \inf_{\xi\in\hat{X}} \frac{1}{n+1}\int_{S^n} \frac{\hat{f}(x)dS(x)}{(\hat{H}(x)-\xi\cdot x)^{n+1}}. \tag{3.12}$$

In fact, given $\xi \in \hat{X}$, by $X_{A_k} \to \hat{X}$, there exists $\xi_{A_k} \in X_{A_k}$ for each k, such that $\xi_{A_k} \to \xi$. Recalling formula (3.7), and denoting $\xi_k := A_k^{-T}\xi_{A_k}$, we have

$$\begin{aligned} \frac{1}{n+1}\int_{S^n} \frac{f_{A_k}(x)dS(x)}{\left(H_{A_k}(x)-\xi_{A_k}\cdot x\right)^{n+1}} &= \frac{1}{n+1}\int_{S^n} \frac{f(x)dS(x)}{(H_k(x)-\xi_k\cdot x)^{n+1}} \\ &= J[H_k(x)-\xi_k\cdot x] \\ &\geq J[H_k]. \end{aligned}$$

Passing to the limit as $k\to\infty$, we have

$$\frac{1}{n+1}\int_{S^n} \frac{\hat{f}(x)dS(x)}{(\hat{H}(x)-\xi\cdot x)^{n+1}} \geq J_{\sup}.$$

Therefore (3.12) holds.

We want to find some $H \in \mathcal{S}_+^G$ with volume ω_{n+1}, such that

$$J_{\sup} < \inf_{\xi} J[H(x) - \xi \cdot x]. \tag{3.13}$$

This is a contradiction, and then completes proofs. To achieve this goal, we need to explore $\hat{f}$ more carefully. The G-rotational symmetry is critical to reducing the possibilities of $\hat{f}$. By Lemma 3.1, we can choose the normalizing matrix A_k^T as

$$A_k^T = \mathrm{diag}\big(\lambda_k^{\frac{1}{n+1}}, \cdots, \lambda_k^{\frac{1}{n+1}}, \lambda_k^{-\frac{n}{n+1}}\big),$$

where $\lambda_k > 0$. Recall the definition in (3.7), we have

$$f_{A_k}(x_1, \cdots, x_n, x_{n+1}) = f\left(\frac{\lambda_k x_1, \cdots, \lambda_k x_n, x_{n+1}}{\sqrt{\lambda_k^2(x_1^2 + \cdots + x_n^2) + x_{n+1}^2}}\right).$$

By the assumption (3.10), there must be

$$\lambda_k \to 0 \text{ or } \lambda_k \to \infty, \text{ as } k \to \infty. \tag{3.14}$$

For the case when $\lambda_k \to 0$, we have

$$\hat{f}(x_1, \cdots, x_n, x_{n+1}) = \begin{cases} f(e_{n+1}), & \text{if } x_{n+1} > 0; \\ f(-e_{n+1}), & \text{if } x_{n+1} < 0. \end{cases} \tag{3.15}$$

And when $\lambda_k \to \infty$, we have

$$\hat{f}(x_1, \cdots, x_n, x_{n+1}) = f\left(\frac{x_1, \cdots, x_n, 0}{\sqrt{x_1^2 + \cdots + x_n^2}}\right). \tag{3.16}$$

The above arguments are applicable to both Theorems 1.1 and 1.3. We continue the proof of Theorem 1.1 in the following of this section, and Theorem 1.3 in the next section.

Proof of Theorem 1.1. As mentioned above, we only need to check (3.13). We first provide an upper bound of $J_{\sup}$. Recall the Blaschke–Santaló inequality (1.2), for $\hat{X}$, there exists a point $\tilde{\xi} \in \hat{X}$ such that

$$\frac{1}{n+1} \int_{S^n} \frac{dS(x)}{(\hat{H}(x) - \tilde{\xi} \cdot x)^{n+1}} \leq \omega_{n+1}.$$

By virtue of (3.12),

$$
\begin{aligned}
J_{\sup} &\le \frac{1}{n+1} \int\limits_{S^n} \frac{\hat{f}(x)dS(x)}{(\hat{H}(x) - \tilde{\xi} \cdot x)^{n+1}} \\
&\le \max \hat{f} \cdot \frac{1}{n+1} \int\limits_{S^n} \frac{dS(x)}{(\hat{H}(x) - \tilde{\xi} \cdot x)^{n+1}} \\
&\le \max \hat{f} \cdot \omega_{n+1}.
\end{aligned} \tag{3.17}
$$

Since $\hat{f}$ is given by (3.15) if $\lambda_k \to 0$ or (3.16) if $\lambda_k \to \infty$, we see

$$
\max \hat{f} \le \max \left\{ f(x) : x \in e_{n+1}^{\perp} \text{ or } x = \pm e_{n+1} \right\}.
$$

Therefore, under the assumption (3.10), we obtain

$$
J_{\sup} \le \max \left\{ f(x) : x \in e_{n+1}^{\perp} \text{ or } x = \pm e_{n+1} \right\} \cdot \omega_{n+1}. \tag{3.18}
$$

On the other hand, let $H \equiv 1$ in the following proof. It is obviously that H is G-rotationally symmetric, and $\mathrm{vol}(H) = \omega_{n+1}$. Recall (3.8) and (3.9), we have

$$
\begin{aligned}
\inf_{\xi} J[H - \xi \cdot x] &= J[1 - t_H x_{n+1}] \\
&= \frac{1}{n+1} \int\limits_{S^n} \frac{f(x)dS(x)}{(1 - t_H x_{n+1})^{n+1}}.
\end{aligned} \tag{3.19}
$$

Since $t_H \in (-1, 1)$, and note $\sigma_n = (n+1)\omega_{n+1}$,

$$
\begin{aligned}
\inf_{\xi} J[H - \xi \cdot x] &> \frac{1}{2^{n+1}} \cdot \frac{1}{n+1} \int\limits_{S^n} f \\
&= \frac{1}{2^{n+1}} \bar{f} \cdot \omega_{n+1}.
\end{aligned} \tag{3.20}
$$

We can also estimate (3.19) as follows. If $t_H \ge 0$, then

$$
\begin{aligned}
\inf_{\xi} J[H - \xi \cdot x] &> \frac{1}{n+1} \int\limits_{S^n_+} \frac{f(x)dS(x)}{(1 - t_H x_{n+1})^{n+1}} \\
&\ge \frac{1}{n+1} \int\limits_{S^n_+} f \\
&= \frac{1}{2} \bar{f}_N \cdot \omega_{n+1}.
\end{aligned}
$$

Similarly, if $t_H \le 0$, we can obtain

$$
\inf_{\xi} J[H - \xi \cdot x] > \frac{1}{2} \bar{f}_S \cdot \omega_{n+1}.
$$

Thus there is

$$\inf_{\xi} J[H - \xi \cdot x] > \frac{1}{2} \min\left\{\bar{f}_N, \bar{f}_S\right\} \cdot \omega_{n+1}. \tag{3.21}$$

Now recalling our assumption of Theorem 1.1, and combining (3.18), (3.20) and (3.21), we see

$$J_{\sup} < \inf_{\xi} J[H - \xi \cdot x].$$

This contradiction completes the proof of this theorem. □

Remark. If f is additionally symmetric with respect to the equatorial hyperplane, then one can easily see $t_H = 0$ in (3.19), which implies that

$$\inf_{\xi} J[H - \xi \cdot x] = \bar{f} \cdot \omega_{n+1}.$$

Hence Theorem 1.2 holds.

4. Proof of Theorem 1.3

In this section, we complete the proof of Theorem 1.3. After the arguments in the previous section, we only need to find an H such that (3.13) holds, namely

$$J_{\sup} < \inf_{\xi} J[H(x) - \xi \cdot x], \tag{4.1}$$

where $J_{\sup}$ is given by (3.11). There are two cases of $\hat{f}$ to deal with, namely $\lambda_k \to 0$ and $\lambda_k \to \infty$. For the case $\lambda_k \to 0$, we observe that the first derivatives of a G-rotationally symmetric f vanish at the north and south poles, the same as an $(n+1)$-mirror symmetric function in [29]. Then we can follow the blow-up analyses in [29], with a few modifications. For the case $\lambda_k \to \infty$, however, the first derivatives may not vanish on the equator, so the blow-up analyses in [29] are not applicable to the G-rotationally symmetric f. To overcome this difficulty, we carefully make use of the invariance of J, and construct a special support function to directly estimate the lower bound of $J_{\sup}$. The following are details.

4.1. When $\lambda_k \to 0$

Recall $\hat{f}$ is given by (3.15), namely

$$\hat{f}(x_1, \cdots, x_n, x_{n+1}) = \begin{cases} f(e_{n+1}), & \text{if } x_{n+1} > 0; \\ f(-e_{n+1}), & \text{if } x_{n+1} < 0. \end{cases}$$

We consider the family of convex bodies $\hat{X}_{A(a)}$ with $A(a) \in SL(n+1)$ given by

$$A(a) = \mathrm{diag}\left(a^{-\frac{1}{n+1}}, \cdots, a^{-\frac{1}{n+1}}, a^{\frac{n}{n+1}}\right), \quad a > 0.$$

Since $\hat{X}$ is G-rotationally symmetric, so is $\hat{X}_{A(a)}$. Also

$$\mathrm{vol}(\hat{X}_{A(a)}) = \mathrm{vol}(\hat{X}) = \omega_{n+1}.$$

Note the support function of $\hat{X}_{A(a)}$ is $\hat{H}_{A(a)}$. Let

$$J(a) := \inf_{\xi} J[\hat{H}_{A(a)} - \xi \cdot x]. \tag{4.2}$$

By (3.7), we have

$$\begin{aligned} J(a) &= \inf_{\xi} \frac{1}{n+1} \int_{S^n} \frac{f}{(\hat{H}_{A(a)} - \xi \cdot x)^{n+1}} \\ &= \inf_{\xi} \frac{1}{n+1} \int_{S^n} \frac{f_{A(a)^{-1}}}{(\hat{H} - \xi \cdot x)^{n+1}} \\ &=: \frac{1}{n+1} \int_{S^n} \frac{f_a}{(\hat{H} - \xi_a \cdot x)^{n+1}}, \end{aligned} \tag{4.3}$$

where the infimum is attained at $\xi_a \in \hat{X}$, and $f_a = f_{A(a)^{-1}}$ is defined as

$$f_a(x_1, \cdots, x_n, x_{n+1}) = f\left(\frac{ax_1, \cdots, ax_n, x_{n+1}}{\sqrt{a^2(x_1^2 + \cdots + x_n^2) + x_{n+1}^2}} \right). \tag{4.4}$$

We allow $a = 0$ in (4.3) and (4.4). Then $f_0 = \hat{f}$. Note (3.12), there is $\xi_0 = 0$ and

$$J(0) = J_{\sup}.$$

Now if we can find some $a > 0$ such that

$$J(a) > J(0), \tag{4.5}$$

then (4.1) holds. So it remains to check (4.5). To achieve this, we shall analyze the asymptotic behavior of $J(a)$ when $a \to 0^+$.

Following [29], for the function f defined on S^n, one can extend it to $\mathbb{R}^{n+1}$ such that it is homogeneous of degree zero. Note that f remains G-rotationally symmetric in the whole $\mathbb{R}^{n+1}$. For a point $x \in \mathbb{R}^{n+1}$, we write $x = (x', x_{n+1})$ where

$$x' = (x_1, \cdots, x_n).$$

Then we can use the standard notations in Euclidean space such as $f'_{x'}$ for the gradient and $f''_{x'x'}$ for the Hessian of f with respect to x'. From now on, we always use these conventions unless explicitly stated otherwise.

We need the following observation about the G-rotationally symmetric f.

Lemma 4.1. *If $f \in C^1$, then $f'_{x'}(0, x_{n+1}) = 0$ whenever $x_{n+1} \neq 0$.*

Proof. Fix any $x_{n+1} \neq 0$ and let

$$g(z) := f(z, x_{n+1}), \quad z \in \mathbb{R}^n.$$

Then $g \in C^1(\mathbb{R}^n)$. Note for any $\Omega \in G$,

$$g(\Omega z) = f(\Omega z, x_{n+1}) = f(z, x_{n+1}) = g(z).$$

Differentiating both sides of this equality with respect to z yields

$$(\nabla g)(\Omega z) \cdot \Omega = \nabla g(z).$$

Let $z = 0$, we obtain

$$\nabla g(0) \cdot \Omega = \nabla g(0).$$

Recall that the orbit of each nonzero $z \in \mathbb{R}^n$ under G spans $\mathbb{R}^n$, and $n \geq 2$, there must be

$$\nabla g(0) = 0,$$

which is equivalent to

$$f'_{x'}(0, x_{n+1}) = 0. \qquad \square$$

Once we have Lemma 4.1, the second part of [29, Lemma 3.2] is true for a G-rotationally symmetric f. Here we provide it as the following lemma.

Lemma 4.2 *([29]). Let $\varphi \in C(S^n)$ be a continuous function. Assume $f \in C^2(S^n)$, and f_a is given by (4.4). Then as $a \to 0^+$, there is*

$$\int_{S^n} \varphi(x)(f_a(x) - f_0(x))dS(x) = a\Big(\int_{S^n \cap e_{n+1}^{\perp}} \varphi(x) PI(x) d\sigma(x) + o(1)\Big).$$

Here $PI(x)$ is given in (1.6).

We also have the following useful observation, which can be easily seen from the proof of [29, Lemma 3.2 (b)] as well.

Lemma 4.3. *Assume $f \in C^2(S^n)$. Then*

$$\int_{S^n} |f_a(x) - f_0(x)| dS(x) \leq Ca, \tag{4.6}$$

where C only depends on $\|f\|_{C^2(S^n)}$ and n.

But we can not use Lemma 4.2 directly, due to the existence of ξ_a in $J(a)$, see (4.3). To deal with this problem, we need study ξ_a in more detail. Recall (4.3), ξ_a is the unique minimum point of

$$\frac{1}{n+1}\int_{S^n}\frac{f_a}{(\hat{H}-\xi\cdot x)^{n+1}},$$

which is strictly convex as a function of ξ. So ξ_a is continuous with respect to a. Also the vanishing first order derivatives yield

$$\int_{S^n}\frac{f_a}{(\hat{H}-\xi_a\cdot x)^{n+2}}x_i=0,\quad i=1,2,\cdots,n+1. \tag{4.7}$$

If f_a is a constant function, then ξ_a is actually the Santaló point of the convex body $\hat{X}$. The Santaló map, which maps a convex body to its Santaló point, is Lipschitz continuous at each convex body, see Proposition 1 in Kim–Reisner [34]. Now for ξ_a given in (4.7), one can still prove its Lipschitz continuity at $\hat{X}$ in a similar way. In fact, we have the following

Lemma 4.4. *There exists a sufficiently small $\bar{a}>0$, such that*

$$|\xi_a|\le Ca,\qquad \forall 0\le a\le\bar{a} \tag{4.8}$$

for some positive constant C depending only on $\hat{H}$, f and n.

Proof. For simplicity, we write

$$\phi(t)=-\frac{1}{t^{n+2}},\qquad \forall t>0.$$

Then (4.7) says

$$\int_{S^n}\phi(\hat{H}-\xi_a\cdot x)f_a\,x\,dS(x)=0. \tag{4.9}$$

In particular, for $a=0$,

$$\int_{S^n}\phi(\hat{H})f_0\,x\,dS(x)=0. \tag{4.10}$$

Combining (4.9) and (4.10), we get

$$\int_{S^n}\phi(\hat{H}-\xi_a\cdot x)(f_a-f_0)x\,dS(x)=\int_{S^n}[\phi(\hat{H})-\phi(\hat{H}-\xi_a\cdot x)]f_0\,x\,dS(x).$$

Then

$$\int\limits_{S^n} \phi(\hat{H} - \xi_a \cdot x)(f_a - f_0)x\, dS(x) = \int\limits_{S^n} \phi'(\tau)(\xi_a \cdot x) f_0\, x\, dS(x),$$

where $\tau \in \mathbb{R}$ is between $\hat{H}(x) - \xi_a \cdot x$ and $\hat{H}(x)$. Taking the inner product with ξ_a, we have

$$\int\limits_{S^n} \phi(\hat{H} - \xi_a \cdot x)(f_a - f_0)(\xi_a \cdot x) dS(x) = \int\limits_{S^n} \phi'(\tau)(\xi_a \cdot x)^2 f_0\, dS(x). \tag{4.11}$$

Since $\hat{H} > 0$, and $\hat{H}(x) - \xi_a \cdot x$ converges to $\hat{H}(x)$ uniformly on S^n when $a \to 0^+$, we can assume for $a \in [0, \bar{a}]$ that

$$\hat{H}(x) - \xi_a \cdot x, \quad \hat{H}(x), \quad \tau \in [2^{-1}\hat{H}_{\min}, 2\hat{H}_{\max}] =: [C_1, C_2].$$

Thus we can estimate (4.11) as

$$\begin{aligned} \phi'(C_2) f_{\min} \int\limits_{S^n} (\xi_a \cdot x)^2 dS(x) &\le \int\limits_{S^n} \phi(\hat{H} - \xi_a \cdot x)(f_a - f_0)(\xi_a \cdot x) dS(x) \\ &\le \int\limits_{S^n} |\phi(\hat{H} - \xi_a \cdot x)| \cdot |f_a - f_0| \cdot |\xi_a \cdot x| dS(x) \\ &\le |\phi(C_1)| \cdot |\xi_a| \int\limits_{S^n} |f_a - f_0| dS(x) \\ &\le |\phi(C_1)| \cdot |\xi_a|\, C_3 a, \end{aligned} \tag{4.12}$$

where C_3 is the positive constant in (4.6), which depends only on f and n. Note that

$$\int\limits_{S^n} (\xi_a \cdot x)^2 dS(x) = \omega_{n+1} |\xi_a|^2.$$

Now (4.12) is simplified into

$$|\xi_a| \le \frac{|\phi(C_1)| C_3}{\phi'(C_2) f_{\min} \omega_{n+1}} a.$$

We complete the proof of this lemma. □

Now we can analyze $J(a) - J(0)$. Recall (4.3), we have

$$\begin{aligned} (n+1)[J(a) - J(0)] &= \int\limits_{S^n} \frac{f_a}{(\hat{H} - \xi_a \cdot x)^{n+1}} - \int\limits_{S^n} \frac{f_0}{\hat{H}^{n+1}} \\ &= \int\limits_{S^n} \left(\frac{1}{(\hat{H} - \xi_a \cdot x)^{n+1}} - \frac{1}{\hat{H}^{n+1}} \right) f_a + \int\limits_{S^n} \frac{f_a - f_0}{\hat{H}^{n+1}}. \end{aligned} \tag{4.13}$$

To estimate the first integral above, we use Taylor expansion to obtain

$$\Big(\frac{1}{(\hat{H}-\xi_a\cdot x)^{n+1}}-\frac{1}{\hat{H}^{n+1}}\Big)f_a=\frac{(n+1)f_a}{(\hat{H}-\xi_a\cdot x)^{n+2}}(\xi_a\cdot x)+O(1)(\xi_a\cdot x)^2,$$

where the bounds of $O(1)$ are independent of $x\in S^n$ and a, when $0\le a\le\bar{a}$ for example. Therefore, by (4.7), we have

$$\begin{aligned}\int_{S^n}\Big(\frac{1}{(\hat{H}-\xi_a\cdot x)^{n+1}}-\frac{1}{\hat{H}^{n+1}}\Big)f_a&=\int_{S^n}O(1)(\xi_a\cdot x)^2dS(x)\\&=O(1)a^2,\end{aligned}$$

where the second equality holds due to (4.8). Now applying Lemma 4.2, (4.13) is simplified as

$$\begin{aligned}(n+1)[J(a)-J(0)]&=O(1)a^2+\int_{S^n}\frac{f_a-f_0}{\hat{H}^{n+1}}\\&=O(1)a^2+a\Big(\int_{S^n\cap e_{n+1}^{\perp}}\hat{H}(x)^{-n-1}PI(x)d\sigma(x)+o(1)\Big)\\&=a\Big(\int_{S^n\cap e_{n+1}^{\perp}}\hat{H}(x)^{-n-1}PI(x)d\sigma(x)+o(1)\Big).\end{aligned}$$

By the assumption of $PI(x)$ in the theorem, we see for sufficiently small $a>0$, $J(a)>J(0)$. Namely (4.1) holds, which is impossible.

4.2. When $\lambda_k\to\infty$

In this case, $\hat{f}$ is given in (3.16), namely

$$\hat{f}(x_1,\cdots,x_n,x_{n+1})=f\left(\frac{x_1,\cdots,x_n,0}{\sqrt{x_1^2+\cdots+x_n^2}}\right).$$

Similar to (3.17), we have

$$J_{\sup}\le\max\hat{f}\cdot\omega_{n+1}=\max_{S^n\cap e_{n+1}^{\perp}}f\cdot\omega_{n+1}.\tag{4.14}$$

To achieve (4.1), we make the following construction. Denote

$$M=f(e_{n+1}),\quad m=f(-e_{n+1}),$$

and

$$\alpha = \Big(\frac{2}{\sqrt{M}+\sqrt{m}}\Big)^{\frac{1}{n+1}}.$$

Let $h \in C(S^n)$ be given as

$$h(x) = \begin{cases} \alpha M^{\frac{1}{2n+2}}, & x_{n+1} \ge 0, \\ \alpha M^{-\frac{n}{2n+2}} \sqrt{M(x_1^2+\cdots+x_n^2)+mx_{n+1}^2}, & x_{n+1} < 0. \end{cases} \tag{4.15}$$

One can easily see h is the support function of a G-rotationally symmetric convex body K, which consists of a semi-ball in the north and a semi-ellipsoid in the south. And its volume

$$\begin{aligned} \mathrm{vol}(K) &= \frac{1}{2}\omega_{n+1}\big(\alpha M^{\frac{1}{2n+2}}\big)^{n+1} + \frac{1}{2}\omega_{n+1}\big(\alpha M^{\frac{1}{2n+2}}\big)^{n} \cdot \alpha M^{-\frac{n}{2n+2}}\sqrt{m} \\ &= \omega_{n+1}\alpha^{n+1} \cdot \frac{\sqrt{M}+\sqrt{m}}{2} \\ &= \omega_{n+1}. \end{aligned}$$

Recall for any $A \in SL(n+1)$, $H(x) = |Ax|$ is a solution to

$$\det(\nabla^2 H + HI) = \frac{1}{H^{n+2}} \quad \text{on } S^n.$$

One can check that h in (4.15) is a generalized solution to

$$\det(\nabla^2 h + hI) = \frac{\eta}{h^{n+2}} \quad \text{on } S^n, \tag{4.16}$$

where

$$\eta(x) = \alpha^{2n+2}(M\chi_{\{x_{n+1}>0\}} + m\chi_{\{x_{n+1}<0\}}).$$

Here χ is the characteristic function. Recall the necessary condition for the classical Minkowski problem, we have

$$\int_{S^n} \frac{\eta}{h^{n+2}} x_i = 0, \quad i = 1, 2, \cdots, n+1. \tag{4.17}$$

Let $\xi_h \in K$ be the unique point such that

$$\int_{S^n} \frac{\eta}{(h-\xi_h\cdot x)^{n+1}} = \inf_{\xi\in K} \int_{S^n} \frac{\eta}{(h-\xi\cdot x)^{n+1}}.$$

By the vanishing derivatives with respect to ξ, we have

$$\int_{S^n} \frac{\eta}{(h-\xi_h\cdot x)^{n+2}} x_i = 0, \quad i = 1, 2, \cdots, n+1.$$

Comparing it with (4.17), we obtain $\xi_h = 0$. Then

$$\int_{S^n} \frac{\eta}{h^{n+1}} = \inf_{\xi \in K} \int_{S^n} \frac{\eta}{(h - \xi \cdot x)^{n+1}}. \tag{4.18}$$

Now we consider the family of convex bodies $K_{A(a)}$ with $A(a) \in SL(n+1)$ given by

$$A(a) = \mathrm{diag}\big(a^{-\frac{1}{n+1}}, \cdots, a^{-\frac{1}{n+1}}, a^{\frac{n}{n+1}}\big), \quad a > 0.$$

Namely $K_{A(a)}$ is obtained by performing the linear transform $A(a)^T$ on K. Its support function is given by $h_{A(a)}$, see (3.5). Note that $K_{A(a)}$ is G-rotationally symmetric, and

$$\mathrm{vol}(K_{A(a)}) = \mathrm{vol}(K) = \omega_{n+1}.$$

Let

$$J(a) := \inf_{\xi} J[h_{A(a)} - \xi \cdot x] = J[h_{A(a)} - \xi_a \cdot x]. \tag{4.19}$$

By (3.7), we have

$$\begin{aligned} J(a) &= \frac{1}{n+1} \int_{S^n} \frac{f}{(h_{A(a)} - \xi_a \cdot x)^{n+1}} \\ &= \frac{1}{n+1} \int_{S^n} \frac{f_{A(a)^{-1}}}{(h - A(a)^{-T} \xi_a \cdot x)^{n+1}} \\ &=: \frac{1}{n+1} \int_{S^n} \frac{f_a}{(h - \tilde{\xi}_a \cdot x)^{n+1}}, \end{aligned}$$

where $f_a = f_{A(a)^{-1}}$ is given by

$$f_a(x_1, \cdots, x_n, x_{n+1}) = f\left(\frac{a x_1, \cdots, a x_n, x_{n+1}}{\sqrt{a^2(x_1^2 + \cdots + x_n^2) + x_{n+1}^2}} \right).$$

Since $\tilde{\xi}_a \in K$, we can assume that $\tilde{\xi}_a \to \tilde{\xi}$ as $a \to 0^+$. Also note that

$$f_a(x) \to \alpha^{-2n-2} \eta(x), \quad \text{a.e. } x \in S^n.$$

Therefore we have

$$\lim_{a\to 0^+} J(a) = \frac{1}{n+1}\int_{S^n} \frac{\alpha^{-2n-2}\eta}{(h-\tilde{\xi}\cdot x)^{n+1}}$$
$$\geq \alpha^{-2n-2}\cdot\frac{1}{n+1}\int_{S^n}\frac{\eta}{h^{n+1}},$$

where the inequality is due to (4.18). Note h is a solution to (4.16), we have

$$\frac{1}{n+1}\int_{S^n}\frac{\eta}{h^{n+1}} = \frac{1}{n+1}\int_{S^n} h\det(\nabla^2 h + hI)$$
$$= \mathrm{vol}(h)$$
$$= \omega_{n+1}.$$

Thus

$$\lim_{a\to 0^+} J(a) \geq \alpha^{-2n-2}\omega_{n+1} = \left(\frac{\sqrt{M}+\sqrt{m}}{2}\right)^2 \omega_{n+1}. \tag{4.20}$$

By our assumption in the theorem:

$$\max_{S^n\cap e_{n+1}^{\perp}} f < \left(\frac{\sqrt{M}+\sqrt{m}}{2}\right)^2,$$

combining (4.14) and (4.20), we have

$$\lim_{a\to 0^+} J(a) > J_{\sup}.$$

This implies $J(a) > J_{\sup}$ for sufficiently small $a > 0$. Recalling $J(a)$ is given by (4.19), we see (4.1) holds, which is a contradiction. Now we complete the proof of Theorem 1.3.

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