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A remark on rotationally symmetric solutions to the centroaffine Minkowski problem ☆

Jian Lu

Department of Applied Mathematics, Zhejiang University of Technology, Hangzhou 310023, China

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Abstract

In this paper we study the solvability of the rotationally symmetric centroaffine Minkowski problem. By delicate blow-up analyses, we remove a technical condition in the existence result obtained by Lu and Wang [30].

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1. Introduction

Given a convex body X in the Euclidean space $\mathbb{R}^{n+1}$ containing the origin, the *centroaffine curvature* of ∂X at point p is by definition equal to K/d^{n+2}, where K is the Gauss curvature and d is the distance from the origin to the tangent hyperplane of ∂X at p. The centroaffine curvature is invariant under unimodular linear transforms in $\mathbb{R}^{n+1}$ and has received much attention in geometry [36,37]. The *centroaffine Minkowski problem* [11] is a prescribed centroaffine curvature problem, which in the smooth case is equivalent to solving the following Monge–Ampère type equation

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E-mail address: lj-tshu04@163.com.

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$$\det(\nabla^2 H + HI) = \frac{f}{H^{n+2}} \quad \text{on } S^n, \tag{1}$$

where f is a given positive function, H is the support function of a bounded convex body X in $\mathbb{R}^{n+1}$, I is the unit matrix, $\nabla^2 H = (\nabla_{ij} H)$ is the Hessian matrix of covariant derivatives of H with respect to an orthonormal frame on S^n. When f is a constant, this equation describes affine hyperspheres of elliptic type, and all its solutions are ellipsoids centered at the origin [8].

Equation (1) is also the special case of the L_p-Minkowski problem with $p = -n - 1$. The L_p-Minkowski problem, introduced by Lutwak [31], is an important generalization of the classical Minkowski problem, and is a basic problem in the L_p-Brunn–Minkowski theory in modern convex geometry. It has attracted great attention over the last two decades, see e.g. [5,6,10,11,14,16,18–21,26,32–34,39,40,42,44,46] and references therein.

Equation (1) naturally arises in anisotropic Gauss curvature flows and describes their self-similar solutions [4,7,12,17,41]. Besides, its parabolic form can be used for image processing [2]. Eq. (1) can be reduced to a singular Monge–Ampère equation in the half Euclidean space $\mathbb{R}^{n+1}_+$, the regularity of which was strongly studied in [22,23].

Equation (1) corresponds to the critical case of the famous Blaschke–Santaló inequality in convex geometry [35]:

$$\operatorname{vol}(X) \inf_{\xi \in X} \frac{1}{n+1} \int_{S^n} \frac{dS(x)}{(H(x) - \xi \cdot x)^{n+1}} \leq \kappa_{n+1}^2, \tag{2}$$

where X is any convex body in $\mathbb{R}^{n+1}$, $\operatorname{vol}(X)$ is the volume of the convex body X, H is the support function of X, and κ_{n+1} is the volume of the unit ball in $\mathbb{R}^{n+1}$. Also Eq. (1) remains invariant under projective transforms on S^n [11,30]. When f is a constant function, it only has constant solutions up to projective transformations. This result has been known for a long time, see e.g. [8], which implies that there is no a priori estimates on solutions for general f without additional assumptions. Besides, Chou and Wang [11] found an obstruction for solutions to Eq. (1), which means it may have no solution for some f. On the other hand, it may also have many solutions for some f [15]. This situation is similar, in some aspects, to the prescribed scalar curvature problem on S^n, which involves critical exponents of Sobolev inequalities and the Kazdan–Warner obstruction [9,38]. So the solvability of Eq. (1) is a rather complicated problem due to these features.

For $n = 1$, the existence of solutions to Eq. (1) was investigated in [1,3,10,12,13,24,25,40,43]. In general, one needs to impose some non-degenerate and topological degree conditions on f to obtain an existence result.

For higher n-dimension, only several special cases were studied, see [29,30] for the rotationally symmetric case, [27] for a generalized rotationally symmetric case, [21] for the mirror-symmetric case, and [45] for the discrete case. In these papers, sufficient conditions for the existence of solutions can be found. However, the solvability of Eq. (1) for a general f is still open.

In this paper, we are only concerned about the rotationally symmetric case of Eq. (1). That is, the given function f and solutions H are assumed to be rotationally symmetric with respect to the x_{n+1}-axis in $\mathbb{R}^{n+1}$ with $n \geq 1$. In the spherical coordinates, a rotationally symmetric function f on S^n can be regarded as a function on $[0, \pi]$, such that

$$f(\theta) := f(x_1, \cdots, x_{n+1}) \text{ with } x_{n+1} = \cos\theta.$$

In particular, $f(0)$ and $f(\pi)$ are values of f at the north and south poles respectively. By the correspondence $x_{n+1} = \cos\theta$, one can naturally extend $f(\theta)$ on $[0, \pi]$ to be a 2π-periodic and even function on $\mathbb{R}$. Observe that if $f \in C^m(S^n)$ for some integer m, then $f \in C^m(\mathbb{R})$. Using the superscript $'$ denotes $\frac{\mathrm{d}}{\mathrm{d}\theta}$, we have $f'(0) = f'(\pi) = 0$ if it is differentiable. Throughout this paper, we will always use these conventions.

A typical existence result about the rotationally symmetric case of Eq. (1) was first established in [30] and then supplemented in [29]. To state this result, we introduce two quantities:

$$ni(f) = \begin{cases} -f''(\frac{\pi}{2}), & n \geq 2, \\ \int_0^\pi [f'(\theta) - f'(\frac{\pi}{2})] \tan\theta \, \mathrm{d}\theta, & n = 1, \end{cases}$$

and

$$pi(f) = \int_0^\pi f'(\theta) \cot\theta \, \mathrm{d}\theta.$$

Theorem A *([29,30]). Assume that $f \in C^2(S^n)$ (requiring C^6 for $n = 2$), and that f is positive and rotationally symmetric. If $f'(\frac{\pi}{2}) = 0$ and $ni(f) \cdot pi(f) < 0$, then Eq. (1) admits a rotationally symmetric solution.*

The assumption $f'(\frac{\pi}{2}) = 0$ in the above theorem is not essential, but used to reduce some difficulties in blow-up analyses. It was showed in [29] that this assumption can be removed when f is very close to a positive constant. The aim of this paper is to remove this technical assumption in a general case.

For $n = 1, 2$, we follow the arguments in [29,30], carry out more delicate analyses, and then remove the condition $f'(\frac{\pi}{2}) = 0$ completely.

Theorem 1. *Assume that $f \in C^2(S^1)$ or $f \in C^{2,\alpha}(S^2)$ for some $\alpha \in (0, 1)$, and that f is positive and rotationally symmetric. If $ni(f) \cdot pi(f) < 0$, then Eq. (1) admits a rotationally symmetric solution.*

For $n \geq 3$, the above method is no longer applicable. Inspired by [27], we carry out blow-up analyses for a variational method to obtain the following

Theorem 2. *Assume that $f \in C^2(S^n)$ with $n \geq 3$, and that f is positive and rotationally symmetric. If $ni(f) < -\frac{n+1}{n+2} f'(\frac{\pi}{2})^2 / f(\frac{\pi}{2})$ and $pi(f) > 0$, then Eq. (1) admits a rotationally symmetric solution.*

We see in the case $n \geq 3$, a little more restriction on $ni(f)$ will be needed when the assumption $f'(\frac{\pi}{2}) = 0$ is removed. However if $f'(\frac{\pi}{2}) = 0$, Theorem 2 just becomes into the existence theorem [30, Theorem 1.3].

The paper is organized as follows. In section 2, we provide some basic facts about Eq. (1) and convex bodies. Then we prove Theorem 1 and 2 in section 3 and section 4 respectively.

2. Preliminaries

In this section we state some properties about Eq. (1) and a few facts in convex geometry, which will be used throughout this paper. One can consult [37] for more knowledge about convex geometry.

An obstruction for solutions to Eq. (1) was found by Chou and Wang [11].

Lemma 1 *([11]). Let H be a C^3-solution to equation* (1)*. Then we have*

$$\int_{S^n} \frac{\nabla_\xi f}{H^{n+1}} = 0 \tag{3}$$

for any projective vector field ξ, given by

$$\xi(x) = Bx - (x^T Bx)x, \quad x \in S^n,$$

where B is an arbitrary matrix of order $n+1$.

In the rotationally symmetric case, (3) is reduced to

$$\int_0^\pi \frac{f'(\theta)\sin^n\theta\cos\theta}{H^{n+1}(\theta)} d\theta = 0. \tag{4}$$

See [30, Proposition 3.1].

We have a volume estimate for any solution to Eq. (1).

Lemma 2 *([30]). There exist positive constants $C_n, \tilde{C}_n$, depending only on n, such that for any solution H to Eq.* (1)*, we have*

$$C_n\sqrt{f_{\min}} \le \mathrm{vol}(H) \le \tilde{C}_n\sqrt{f_{\max}},$$

where $f_{\min} = \inf_{S^n} f$, $f_{\max} = \sup_{S^n} f$, and $\mathrm{vol}(H)$ is the volume of the convex body determined by H.

Let X be any convex body in $\mathbb{R}^{n+1}$, and H be its support function. Under the action of a unimodular linear transform $A^T \in \mathrm{SL}(n+1)$, X becomes into another convex body $X_A := A^T X$. Denote the support function of X_A by H_A. Then

$$H_A(x) = |Ax| \cdot H\Big(\frac{Ax}{|Ax|}\Big), \quad x \in S^n. \tag{5}$$

See e.g. [30, (2.11)].

We remark that if H is a solution to Eq. (1), then H_A is a solution to the following equation

$$\det(\nabla^2 H_A + H_A I) = \frac{f_A}{H_A^{n+2}}, \quad f_A(x) = f\Big(\frac{Ax}{|Ax|}\Big). \tag{6}$$

See [11] for more details.

Related to the linear transform, there is an integral variable substitution formula.

Lemma 3 *([28]). For any integral function g on S^n, and any matrix $A \in \mathrm{GL}(n+1)$, we have the following variable substitution for integration:*

$$\int_{S^n} g(y)\,\mathrm{d}S(y) = \int_{S^n} g\left(\frac{Ax}{|Ax|}\right) \cdot \frac{|\det A|}{|Ax|^{n+1}}\,\mathrm{d}S(x).$$

By this lemma and (5), we see for any unimodular linear transform $A \in \mathrm{SL}(n+1)$, there is

$$\int_{S^n} \frac{f}{H^{n+1}} = \int_{S^n} \frac{f_A}{H_A^{n+1}}, \tag{7}$$

where f_A is the same as in (6).

John's Lemma in convex geometry says that for any non-degenerate convex body X in $\mathbb{R}^{n+1}$, there is a unique ellipsoid E which attains the minimum volume among all ellipsoids containing X. This ellipsoid E is called the *minimum ellipsoid* of X. It satisfies

$$\frac{1}{n+1} E \subset X \subset E,$$

where $\lambda E = \{x_0 + \lambda(x - x_0) : x \in E\}$ with x_0 the center of E. We say X is *normalized* if the E is a ball.

We denote the area of S^n by σ_n, and the unit vector along x_i-axis by e_i for $i = 1, 2, \cdots, n+1$.

3. Proof of Theorem 1

In this section, we prove Theorem 1. To achieve this, one needs an improvement of [30, Theorem 1.2].

Theorem 3. *Assume that $f \in C^2(S^1)$ or $f \in C^{2,\alpha}(S^2)$ for some $\alpha \in (0, 1)$, and that f is positive and rotationally symmetric. If $ni(f) \cdot pi(f) \neq 0$, then there exist positive constants C, $\tilde{C}$ depending only on n and f, such that for any rotationally symmetric solution H to Eq. (1), we have*

$$C \le H \le \tilde{C}.$$

Once we have Theorem 3, we can repeat the arguments of [29] to prove that Eq. (1) admits a rotationally symmetric solution provided $ni(f) \cdot pi(f) < 0$, yielding Theorem 1. One can consult [29] for details. So in the rest of this section, we are only concerned about Theorem 3, and give its proof.

Let $\{H_k\}$ be any sequence of rotationally symmetric solutions to Eq. (1). For each H_k, define $a_k \in \mathbb{R}$ and $A_k \in \mathrm{SL}(n+1)$ as

$$\begin{aligned} a_k &= f(\tfrac{\pi}{2})^{\frac{1}{2}} / H_k(\tfrac{\pi}{2})^{n+1}, \\ A_k &= \mathrm{diag}\big(a_k^{\frac{1}{n+1}}, \cdots, a_k^{\frac{1}{n+1}}, a_k^{-\frac{n}{n+1}}\big). \end{aligned} \tag{8}$$

Let

$$H_{A_k}(x) = |A_k x| \cdot H_k\left(\frac{A_k x}{|A_k x|}\right), \quad x \in S^n. \tag{9}$$

Then H_{A_k} is a rotationally symmetric solution to Eq. (6) with A replaced by A_k. Note that

$$H_{A_k}(\tfrac{\pi}{2}) = a_k^{\frac{1}{n+1}} H_k(\tfrac{\pi}{2}) = f(\tfrac{\pi}{2})^{\frac{1}{2n+2}}. \tag{10}$$

Lemma 4. *There exist positive constants C, $\tilde{C}$ depending only on n, $f_{\max}$ and $f_{\min}$, such that*

$$C \le H_{A_k} \le \tilde{C}. \tag{11}$$

Proof. By the rotational symmetry of H_{A_k}, one can easily see that

$$\mathrm{vol}(H_{A_k}) \ge \frac{\kappa_n}{n+1} H_{A_k}(\tfrac{\pi}{2})^n [H_{A_k}(0) + H_{A_k}(\pi)], \tag{12}$$

and

$$\max H_{A_k} \le \sqrt{H_{A_k}(\tfrac{\pi}{2})^2 + [H_{A_k}(0) + H_{A_k}(\pi)]^2}. \tag{13}$$

Recalling H_{A_k} satisfies equation (6), by the volume estimate given in Lemma 2, we have

$$\mathrm{vol}(H_{A_k}) \le C_n \sqrt{\max f_{A_k}} = C_n \sqrt{f_{\max}}, \tag{14}$$

which together with (12) yields

$$\begin{aligned} H_{A_k}(0) + H_{A_k}(\pi) &\le \tilde{C}_n \sqrt{f_{\max}} \cdot H_{A_k}(\tfrac{\pi}{2})^{-n} \\ &= \tilde{C}_n \sqrt{f_{\max}} \cdot f(\tfrac{\pi}{2})^{-\frac{n}{2n+2}} \\ &\le \tilde{C}_n f_{\max}^{\frac{1}{2}} f_{\min}^{-\frac{n}{2n+2}}, \end{aligned}$$

where we have used (10) for the equality. Now from (13) we obtain

$$\max H_{A_k} \le f_{\max}^{\frac{1}{2n+2}} + \tilde{C}_n f_{\max}^{\frac{1}{2}} f_{\min}^{-\frac{n}{2n+2}}, \tag{15}$$

which means the second inequality in (11) is true.

On the other hand, by virtue of [30, Lemma 2.3], there is

$$\min H_{A_k} \cdot (\max H_{A_k})^n \cdot \mathrm{vol}(H_{A_k}) \ge C_n f_{\min}.$$

Combining it with (14) and (15), we easily obtain the first inequality in (11). □

To obtain uniform upper and lower bounds for $\{H_k\}$, by (9) and Lemma 4, we should exclude two cases, namely $a_k \to +\infty$ or $a_k \to 0^+$ when $k \to +\infty$. The second case can be still solved by the method developed in [30]. But for the first case where $a_k \to +\infty$, one needs more delicate analyses to deal with. The following are details.

First note that in the rotationally symmetric case, f_{A_k} defined in (6) can be written as

$$f_{A_k}(\theta) = f(\gamma_{a_k}(\theta)), \tag{16}$$

where

$$\gamma_{a_k}(\theta) = \arccos\Big(\frac{\cos\theta}{i_{a_k}(\theta)}\Big), \quad i_{a_k}(\theta) = \sqrt{a_k^2 \sin^2\theta + \cos^2\theta}, \tag{17}$$

see [30, (3.3)–(3.4)].

Lemma 5. *Assume $a_k \to +\infty$ when $k \to +\infty$. Then H_{A_k} converges to the constant function $f(\frac{\pi}{2})^{\frac{1}{2n+2}}$ uniformly on $[0, \pi]$.*

Proof. From Lemma 4, we see $\{H_{A_k}\}$ is uniformly bounded. By the Blaschke selection theorem, one may assume that $\{H_{A_k}\}$ converges uniformly to some support function H_∞ on S^n, which is also rotationally symmetric. It remains to prove that

$$H_\infty \equiv f(\tfrac{\pi}{2})^{\frac{1}{2n+2}} \quad \text{on } S^n. \tag{18}$$

Recall Eq. (6), namely

$$\det(\nabla^2 H_{A_k} + H_{A_k} I) = \frac{f_{A_k}}{H_{A_k}^{n+2}} \quad \text{on } S^n. \tag{19}$$

Note when $a_k \to +\infty$, f_{A_k} converges to $f(\frac{\pi}{2})$ almost everywhere on $[0, \pi]$, see (16). Passing to the limit in Eq. (19), we see H_∞ is a generalized solution to

$$\det(\nabla^2 H_\infty + H_\infty I) = \frac{f(\frac{\pi}{2})}{H_\infty^{n+2}} \quad \text{on } S^n.$$

So H_∞ is an elliptic affine sphere, which must be an ellipsoid [8]. By the rotational symmetry of H_∞, it should be expressed as

$$H_\infty(x) = f(\tfrac{\pi}{2})^{\frac{1}{2n+2}} |\Lambda x|, \quad x \in S^n \tag{20}$$

for some $\Lambda \in \mathrm{SL}(n+1)$ of form

$$\Lambda = \mathrm{diag}\big(\lambda^{\frac{1}{n+1}}, \cdots, \lambda^{\frac{1}{n+1}}, \lambda^{-\frac{n}{n+1}}\big) \text{ with } \lambda > 0.$$

Then

$$H_\infty(\tfrac{\pi}{2}) = f(\tfrac{\pi}{2})^{\frac{1}{2n+2}} \lambda^{\frac{1}{n+1}}.$$

On the other hand, recalling (10), we have

$$H_\infty(\tfrac{\pi}{2}) = f(\tfrac{\pi}{2})^{\frac{1}{2n+2}}.$$

Hence $\lambda = 1$, namely Λ is the identity matrix of order $n+1$. Now (20) is simplified into (18). The proof of this lemma is completed. □

Recall H_{A_k} satisfies equation (19), which in the rotationally symmetric case can be simplified into the following form:

$$(H''_{A_k} + H_{A_k})(H'_{A_k}\cot\theta + H_{A_k})^{n-1} = \frac{f_{A_k}}{H_{A_k}^{n+2}} \quad \text{on } [0,\pi], \tag{21}$$

see [29, (2)].

Lemma 6.

(a) *There exist positive constants C, $\tilde{C}$ depending only on n, $f_{\max}$ and $f_{\min}$, such that*

$$C \le H'_{A_k}\cot\theta + H_{A_k} \le \tilde{C}, \tag{22}$$

$$C \le H''_{A_k} + H_{A_k} \le \tilde{C}. \tag{23}$$

(b) *If $a_k \to +\infty$ when $k \to +\infty$, then $\left\{H''_{A_k}\sin^{\frac{1}{4}}\theta\right\}$ converges to 0 uniformly on $[0,\pi]$.*

Proof. (a) Recalling Lemma 4, we obtain from (21) that

$$C_1 \le (H''_{A_k} + H_{A_k})(H'_{A_k}\cot\theta + H_{A_k})^{n-1} \le C_2 \tag{24}$$

for some positive constants C_1, C_2 depending only on n, $f_{\max}$ and $f_{\min}$. Note

$$(H'_{A_k}\cos\theta + H_{A_k}\sin\theta)' = (H''_{A_k} + H_{A_k})\cos\theta,$$

the above inequality can be written as

$$C_1 \le \frac{1}{n\sin^{n-1}\theta\cos\theta}\cdot\frac{\mathrm{d}}{\mathrm{d}\theta}(H'_{A_k}\cos\theta + H_{A_k}\sin\theta)^n \le C_2. \tag{25}$$

When $\theta \in [0,\pi/2]$, we have by (25) that

$$\frac{\mathrm{d}}{\mathrm{d}\theta}C_1\sin^n\theta \le \frac{\mathrm{d}}{\mathrm{d}\theta}(H'_{A_k}\cos\theta + H_{A_k}\sin\theta)^n \le \frac{\mathrm{d}}{\mathrm{d}\theta}C_2\sin^n\theta,$$

which together with $H'_{A_k}(0) = 0$ implies

$$C_1^{\frac{1}{n}}\sin\theta \le H'_{A_k}\cos\theta + H_{A_k}\sin\theta \le C_2^{\frac{1}{n}}\sin\theta, \quad \forall\theta \in [0,\pi/2].$$

Similarly, by (25) and $H'_{A_k}(\pi) = 0$, we also have

$$C_1^{\frac{1}{n}} \sin\theta \le H'_{A_k} \cos\theta + H_{A_k} \sin\theta \le C_2^{\frac{1}{n}} \sin\theta, \quad \forall \theta \in [\pi/2, \pi].$$

Therefore

$$C_1^{\frac{1}{n}} \le H'_{A_k} \cot\theta + H_{A_k} \le C_2^{\frac{1}{n}}, \quad \forall \theta \in [0, \pi],$$

which is just (22). Now recalling (24), one can obtain (23).

(b) We first note that by (11) and (23), there is

$$|H''_{A_k}| \le C_3 \tag{26}$$

for some positive constant C_3 depending only on n, $f_{\max}$ and $f_{\min}$.

Now assume $a_k \to +\infty$ as $k \to +\infty$. We claim that for any $\delta \in (0, \pi/2)$,

$$H''_{A_k} \rightrightarrows 0 \text{ uniformly on } [\delta, \pi - \delta]. \tag{27}$$

In fact, by (16), $f_{A_k} \rightrightarrows f(\frac{\pi}{2})$ uniformly on $[\delta, \pi - \delta]$. By Lemma 5, $H_{A_k} \rightrightarrows f(\frac{\pi}{2})^{\frac{1}{2n+2}}$ uniformly on $[0, \pi]$, which implies that $H'_{A_k} \rightrightarrows 0$ uniformly on $[0, \pi]$. Then by (21), when $\theta \in [\delta, \pi - \delta]$, we have

$$\begin{aligned} H''_{A_k} &= f_{A_k} H_{A_k}^{-n-2} (H'_{A_k} \cot\theta + H_{A_k})^{1-n} - H_{A_k} \\ &\rightrightarrows f(\tfrac{\pi}{2}) \cdot f(\tfrac{\pi}{2})^{-\frac{n+2}{2n+2}} \cdot f(\tfrac{\pi}{2})^{\frac{1-n}{2n+2}} - f(\tfrac{\pi}{2})^{\frac{1}{2n+2}} \\ &= 0. \end{aligned}$$

Thus (27) is true.

We now prove

$$H''_{A_k} \sin^{\frac{1}{4}}\theta \rightrightarrows 0 \text{ uniformly on } [0, \pi]. \tag{28}$$

Given any $\epsilon > 0$. By (26), there exists some $\delta \in (0, \pi/2)$, such that

$$\sup_{[0,\delta]\cup[\pi-\delta,\pi]} |H''_{A_k} \sin^{\frac{1}{4}}\theta| < \epsilon, \quad \forall k. \tag{29}$$

Then by virtue of (27), there exists a k_0, such that

$$\sup_{[\delta,\pi-\delta]} |H''_{A_k}| < \epsilon, \quad \forall k \ge k_0. \tag{30}$$

Combining (29) and (30), we have

$$\sup_{[0,\pi]} |H''_{A_k} \sin^{\frac{1}{4}}\theta| < \epsilon, \quad \forall k \ge k_0.$$

Thus (28) is true. □

With a more detailed analysis, we can strengthen Lemma 5 for $n = 1, 2$.

Lemma 7. *Assume $a_k \to +\infty$ as $k \to +\infty$. For sufficiently large k, we have*

$$\max_{[0,\pi]} \left| H_{A_k} - f(\tfrac{\pi}{2})^{\frac{1}{2n+2}} \right| \le C \begin{cases} \int_0^\pi |f_{A_k} - f(\tfrac{\pi}{2})| \, \mathrm{d}\theta, & \text{if } n = 1, \\ \int_0^\pi |f_{A_k} - f(\tfrac{\pi}{2})| \sin^{\frac{1}{2}}\theta \, \mathrm{d}\theta, & \text{if } n = 2, \end{cases} \tag{31}$$

where C is a positive constant depending only on $f(\frac{\pi}{2})$.

Proof. For simplicity, let

$$\beta := f(\tfrac{\pi}{2})^{\frac{1}{2n+2}} \quad \text{and} \quad h_k(\theta) := H_{A_k}(\theta) - \beta.$$

Also we will drop the subscript k in the following proof if no confusion arises. Recall by Lemma 5, h converges uniformly to 0 on $[0, \pi]$ as $k \to +\infty$.

(a) When $n = 1$. Now Eq. (21) is simplified as

$$h'' + h + \beta = \frac{f_A}{H_A^3}. \tag{32}$$

Observing that

$$\begin{aligned} H_A^{-3} &= (\beta + h)^{-3} \\ &= \beta^{-3} - 3\beta^{-4}h + 6\tau^{-5}h^2, \end{aligned}$$

where τ is between β and $H_A(\theta)$, and that $\beta = f(\frac{\pi}{2})^{1/4}$, we have

$$\frac{f(\frac{\pi}{2})}{H_A^3} = \beta - 3h + 6\beta^4\tau^{-5}h^2.$$

Then (32) can be written as

$$h'' + h + 3h - 6\beta^4\tau^{-5}h^2 = \frac{f_A - f(\frac{\pi}{2})}{H_A^3},$$

namely

$$h'' + 4h = \frac{f_A - f(\frac{\pi}{2})}{H_A^3} + 6\beta^4\tau^{-5}h^2. \tag{33}$$

Recalling $h(\frac{\pi}{2}) = 0$ by (10), we can apply Lemma 8 to equation (33) and then obtain

$$\max |h| \le \left\| \frac{f_A - f(\frac{\pi}{2})}{H_A^3} \right\|_{L^1[0,\pi]} + \left\| 6\beta^4 \tau^{-5} h^2 \right\|_{L^1[0,\pi]}. \tag{34}$$

Since $H_A \rightrightarrows \beta > 0$ uniformly on $[0, \pi]$ as $k \to +\infty$, there exists a large integer k_0, such that

$$\max |H_A - \beta| \le \frac{\beta}{2}, \quad \forall k \ge k_0.$$

Then when $k \ge k_0$ we have

$$\max |h| \le \frac{\beta}{2} \quad \text{and} \quad H_A, \tau \in \left[\frac{\beta}{2}, \frac{3\beta}{2}\right]. \tag{35}$$

Thus (34) is simplified into

$$\max |h| \le 8\beta^{-3} \left\| f_A - f(\tfrac{\pi}{2}) \right\|_{L^1[0,\pi]} + 192\beta^{-1}\pi (\max |h|)^2.$$

By virtue of $\max |h| \to 0$ as $k \to +\infty$, we also can assume

$$192\beta^{-1}\pi \cdot \max |h| < \frac{1}{2} \text{ when } k \ge k_0.$$

Hence

$$\max |h| \le 16\beta^{-3} \left\| f_A - f(\tfrac{\pi}{2}) \right\|_{L^1[0,\pi]}, \quad \forall k \ge k_0,$$

which is just (31) for $n = 1$.

(b) When $n = 2$. Now Eq. (21) is written as

$$(h'' + h + \beta)(h' \cot\theta + h + \beta) = \frac{f_A}{H_A^4}, \tag{36}$$

namely

$$\beta(h'' + h' \cot\theta + 2h) + \beta^2 + (h'' + h)(h' \cot\theta + h) = \frac{f_A}{H_A^4}. \tag{37}$$

Observing that

$$\begin{aligned} H_A^{-4} &= (\beta + h)^{-4} \\ &= \beta^{-4} - 4\beta^{-5} h + 10\tau^{-6} h^2, \end{aligned}$$

where τ is between β and $H_A(\theta)$, and that $\beta = f(\frac{\pi}{2})^{1/6}$, we have

$$\frac{f(\frac{\pi}{2})}{H_A^4} = \beta^2 - 4\beta h + 10\beta^6\tau^{-6}h^2. \tag{38}$$

Then (37) can be written as

$$\beta(h'' + h'\cot\theta + 6h) + (h'' + h)(h'\cot\theta + h) - 10\beta^6\tau^{-6}h^2 = \frac{f_A - f(\frac{\pi}{2})}{H_A^4},$$

namely

$$h'' + h'\cot\theta + 6h = \frac{f_A - f(\frac{\pi}{2})}{\beta H_A^4} + R_a(\theta), \tag{39}$$

where

$$R_a(\theta) = 10\beta^5\tau^{-6}h^2 - \beta^{-1}(h'' + h)(h'\cot\theta + h). \tag{40}$$

Applying Lemma 10 to equation (39), we have

$$\begin{aligned} \max|h| &\le 2\int_0^\pi \frac{|f_A - f(\frac{\pi}{2})|}{\beta H_A^4}(2 - \log\sin\theta)\sin\theta\,\mathrm{d}\theta \\ &\quad + 2\int_0^\pi |R_a(\theta)|(2 - \log\sin\theta)\sin\theta\,\mathrm{d}\theta \\ &\le 4\int_0^\pi \frac{|f_A - f(\frac{\pi}{2})|}{\beta H_A^4}\sin^{\frac{1}{2}}\theta\,\mathrm{d}\theta + 6\int_0^\pi |R_a(\theta)|\sin^{\frac{3}{4}}\theta\,\mathrm{d}\theta. \end{aligned}$$

Recalling (35), we obtain

$$\max|h| \le 64\beta^{-5}\int_0^\pi |f_A - f(\tfrac{\pi}{2})|\sin^{\frac{1}{2}}\theta\,\mathrm{d}\theta + 6\int_0^\pi |R_a(\theta)|\sin^{\frac{3}{4}}\theta\,\mathrm{d}\theta. \tag{41}$$

We see R_a involves derivatives of h. To deal with them, we need to explore (36) more carefully. Note that

$$(h'\cos\theta + h\sin\theta + \beta\sin\theta)' = (h'' + h + \beta)\cos\theta,$$

then Eq. (36) is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}\theta}(h'\cos\theta + h\sin\theta + \beta\sin\theta)^2 = \frac{f_A}{H_A^4}\cdot 2\sin\theta\cos\theta.$$

Therefore we have

$$(h' \cos\theta + h \sin\theta + \beta \sin\theta)^2 = \int_0^\theta \frac{f_A}{H_A^4} \cdot 2 \sin t \cos t \,\mathrm{d}t, \quad \forall \theta \in [0, \pi/2]. \tag{42}$$

Since $h' \cot\theta + h + \beta > 0$, there is

$$h' \cos\theta + h \sin\theta + \beta \sin\theta = \left(\int_0^\theta \frac{f_A}{H_A^4} \cdot 2 \sin t \cos t \,\mathrm{d}t \right)^{1/2}.$$

Thus we have

$$\begin{aligned} \left| h' \cos\theta + h \sin\theta \right| &= \left| \left(\int_0^\theta \frac{f_A}{H_A^4} \cdot 2 \sin t \cos t \,\mathrm{d}t \right)^{1/2} - \beta \sin\theta \right| \\ &= \frac{\left| \int_0^\theta \frac{f_A}{H_A^4} \cdot 2 \sin t \cos t \,\mathrm{d}t - \beta^2 \sin^2\theta \right|}{\left(\int_0^\theta \frac{f_A}{H_A^4} \cdot 2 \sin t \cos t \,\mathrm{d}t \right)^{1/2} + \beta \sin\theta} \\ &\le \frac{1}{\beta \sin\theta} \left| \int_0^\theta \frac{f_A}{H_A^4} \cdot 2 \sin t \cos t \,\mathrm{d}t - \beta^2 \sin^2\theta \right|. \end{aligned} \tag{43}$$

Recalling (38), there is

$$\frac{f_A}{H_A^4} = \frac{f_A - f(\frac{\pi}{2})}{H_A^4} + \beta^2 - 4\beta h + 10\beta^6 \tau^{-6} h^2,$$

which implies that

$$\begin{aligned} \int_0^\theta \frac{f_A}{H_A^4} \cdot 2 \sin t \cos t \,\mathrm{d}t &= \int_0^\theta \frac{f_A - f(\frac{\pi}{2})}{H_A^4} \cdot 2 \sin t \cos t \,\mathrm{d}t + \beta^2 \int_0^\theta 2 \sin t \cos t \,\mathrm{d}t \\ &\quad + \int_0^\theta (-4\beta h + 10\beta^6 \tau^{-6} h^2) \cdot 2 \sin t \cos t \,\mathrm{d}t \\ &= \int_0^\theta \frac{f_A - f(\frac{\pi}{2})}{H_A^4} \cdot 2 \sin t \cos t \,\mathrm{d}t + \beta^2 \sin^2\theta \\ &\quad + \int_0^\theta (-4\beta + 10\beta^6 \tau^{-6} h) h \cdot 2 \sin t \cos t \,\mathrm{d}t. \end{aligned}$$

Recalling (35), we obtain from the above equality that

$$\left|\int_0^\theta \frac{f_A}{H_A^4}\cdot 2\sin t\cos t\,\mathrm{d}t-\beta^2\sin^2\theta\right|\le 32\beta^{-4}\int_0^\theta |f_A-f(\tfrac{\pi}{2})|\sin t\,\mathrm{d}t+324\beta(\max|h|)\sin^2\theta.$$

Then (43) is simplified into

$$\left|h'\cos\theta+h\sin\theta\right|\le\frac{32\beta^{-5}}{\sin\theta}\int_0^\theta |f_A-f(\tfrac{\pi}{2})|\sin t\,\mathrm{d}t+324(\max|h|)\sin\theta,$$

namely

$$\left|h'\cot\theta+h\right|\sin^{\frac{1}{2}}\theta\le\frac{32\beta^{-5}}{\sin^{\frac{3}{2}}\theta}\int_0^\theta |f_A-f(\tfrac{\pi}{2})|\sin t\,\mathrm{d}t+324(\max|h|)\sin^{\frac{1}{2}}\theta.$$

Integrating both sides over $[0,\pi/2]$, we have

$$\begin{aligned}&\int_0^{\frac{\pi}{2}}|h'\cot\theta+h|\sin^{\frac{1}{2}}\theta\,\mathrm{d}\theta\\ &\quad\le 32\beta^{-5}\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}\theta}{\sin^{\frac{3}{2}}\theta}\int_0^\theta|f_A-f(\tfrac{\pi}{2})|\sin t\,\mathrm{d}t+162\pi(\max|h|)\\ &\quad=32\beta^{-5}\int_0^{\frac{\pi}{2}}|f_A-f(\tfrac{\pi}{2})|\sin t\,\mathrm{d}t\int_t^{\frac{\pi}{2}}\frac{\mathrm{d}\theta}{\sin^{\frac{3}{2}}\theta}+162\pi(\max|h|).\end{aligned}\tag{44}$$

Note that

$$\begin{aligned}\int_t^{\frac{\pi}{2}}\frac{\mathrm{d}\theta}{\sin^{\frac{3}{2}}\theta}&\le\left(\frac{\pi}{2}\right)^{\frac{3}{2}}\int_t^{\frac{\pi}{2}}\frac{\mathrm{d}\theta}{\theta^{\frac{3}{2}}}\\ &=\left(\frac{\pi}{2}\right)^{\frac{3}{2}}\cdot 2[t^{-\frac{1}{2}}-(\pi/2)^{-\frac{1}{2}}]\\ &<4\sin^{-\frac{1}{2}}t,\end{aligned}$$

then (44) is reduced into

$$\int_0^{\frac{\pi}{2}}|h'\cot\theta+h|\sin^{\frac{1}{2}}\theta\,\mathrm{d}\theta\le 128\beta^{-5}\int_0^{\frac{\pi}{2}}|f_A-f(\tfrac{\pi}{2})|\sin^{\frac{1}{2}}t\,\mathrm{d}t+162\pi(\max|h|).\tag{45}$$

Now similar to (42), we have

$$(h'\cos\theta + h\sin\theta + \beta\sin\theta)^2 = \int_\theta^\pi \frac{f_A}{H_A^4}\cdot 2\sin t\,|\cos t|\,\mathrm{d}t, \quad \forall\theta\in[\pi/2,\pi].$$

Then following almost the same arguments used to obtain (45), one can get

$$\int_{\frac{\pi}{2}}^{\pi} |h'\cot\theta + h|\sin^{\frac12}\theta\,\mathrm{d}\theta \le 128\beta^{-5}\int_{\frac{\pi}{2}}^{\pi} |f_A - f(\tfrac{\pi}{2})|\sin^{\frac12} t\,\mathrm{d}t + 162\pi(\max|h|). \tag{46}$$

Adding (45) and (46) together, we have

$$\int_0^\pi |h'\cot\theta + h|\sin^{\frac12}\theta\,\mathrm{d}\theta \le 128\beta^{-5}\int_0^\pi |f_A - f(\tfrac{\pi}{2})|\sin^{\frac12} t\,\mathrm{d}t + 324\pi\cdot\max|h|. \tag{47}$$

Now we can estimate the integral about R_a in (41). By the definition of R_a in (40), there is

$$\begin{aligned}\int_0^\pi |R_a(\theta)|\sin^{\frac34}\theta\,\mathrm{d}\theta &\le \int_0^\pi 10\beta^5\tau^{-6}h^2\sin^{\frac34}\theta\,\mathrm{d}\theta \\ &\quad + \int_0^\pi \beta^{-1}|h''+h|\,|h'\cot\theta + h|\sin^{\frac34}\theta\,\mathrm{d}\theta \\ &\le 640\pi\beta^{-1}(\max|h|)^2 + m_k\int_0^\pi |h'\cot\theta + h|\sin^{\frac12}\theta\,\mathrm{d}\theta,\end{aligned}$$

where (35) is used, and m_k is defined as

$$m_k := \beta^{-1}\max_{\theta\in[0,\pi]}|h''(\theta)+h(\theta)|\sin^{\frac14}\theta.$$

By estimate (47), the above inequality becomes into

$$\begin{aligned}\int_0^\pi |R_a(\theta)|\sin^{\frac34}\theta\,\mathrm{d}\theta &\le 640\pi\beta^{-1}(\max|h|)^2 + 324\pi m_k\cdot\max|h| \\ &\quad + 128\beta^{-5}m_k\int_0^\pi |f_A - f(\tfrac{\pi}{2})|\sin^{\frac12} t\,\mathrm{d}t.\end{aligned} \tag{48}$$

Recall Lemma 6 (b), $|h''(\theta)+h(\theta)|\sin^{\frac14}\theta$ converges uniformly to 0 on $[0,\pi]$ when $k\to+\infty$, which implies

$$m_k \to 0 \text{ as } k\to+\infty.$$

Also recall $\max|h| \to 0$. We can assume when $k \geq k_0$ that

$$640\pi\beta^{-1}\max|h| + 324\pi m_k < \frac{1}{12}.$$

Then (48) is simplified into

$$\int_0^\pi |R_a(\theta)|\sin^{\frac{3}{4}}\theta\,\mathrm{d}\theta \leq \frac{1}{12}\max|h| + \frac{1}{12}\beta^{-5}\int_0^\pi |f_A - f(\tfrac{\pi}{2})|\sin^{\frac{1}{2}} t\,\mathrm{d}t. \tag{49}$$

Now combining (41) and (49), we obtain

$$\max|h| \leq 129\beta^{-5}\int_0^\pi |f_A - f(\tfrac{\pi}{2})|\sin^{\frac{1}{2}}\theta\,\mathrm{d}\theta,$$

which is just (31) for $n = 2$. □

The following Lemmas 8 and 10 have been used in the proof of the above Lemma 7.

Lemma 8. *Assume $h \in C^2(\mathbb{R})$ is 2π-periodic and even. If it satisfies the following differential equation*

$$h'' + 4h = g, \tag{50}$$

and $h(\frac{\pi}{2}) = 0$, then there is

$$\max_{\mathbb{R}}|h| \leq \|g\|_{L^1[0,\pi]}.$$

Proof. One can easily solve equation (50) to obtain

$$h(\theta) = c_1\cos 2\theta + c_2\sin 2\theta - \frac{1}{2}\cos 2\theta\int_0^\theta g(t)\sin 2t\,\mathrm{d}t + \frac{1}{2}\sin 2\theta\int_0^\theta g(t)\cos 2t\,\mathrm{d}t,$$

where c_1 and c_2 are constants to be determined. Then we have

$$h'(\theta) = -2c_1\sin 2\theta + 2c_2\cos 2\theta + \sin 2\theta\int_0^\theta g(t)\sin 2t\,\mathrm{d}t + \cos 2\theta\int_0^\theta g(t)\cos 2t\,\mathrm{d}t.$$

From $h'(0) = 0$, we get $c_2 = 0$. And $h(\frac{\pi}{2}) = 0$ implies

$$c_1 = \frac{1}{2}\int_0^{\frac{\pi}{2}} g(t)\sin 2t\,\mathrm{d}t.$$

Therefore h is given by

$$h(\theta)=\frac{1}{2}\cos 2\theta\int_{\theta}^{\frac{\pi}{2}} g(t)\sin 2t\,\mathrm{d}t+\frac{1}{2}\sin 2\theta\int_{0}^{\theta} g(t)\cos 2t\,\mathrm{d}t.$$

Hence when $\theta\in[0,\pi]$,

$$\begin{aligned}|h(\theta)|&\le\frac{1}{2}\left|\int_{\theta}^{\frac{\pi}{2}} g(t)\sin 2t\,\mathrm{d}t\right|+\frac{1}{2}\left|\int_{0}^{\theta} g(t)\cos 2t\,\mathrm{d}t\right|\\&\le\frac{1}{2}\int_{0}^{\pi}|g(t)|\,\mathrm{d}t+\frac{1}{2}\int_{0}^{\pi}|g(t)|\,\mathrm{d}t\\&=\int_{0}^{\pi}|g(t)|\,\mathrm{d}t,\end{aligned}$$

which leads to the conclusion of this lemma. □

Lemma 9. *The homogeneous differential equation*

$$h''+h'\cot\theta+6h=0\ \ in\ (0,\pi)$$

has the following two fundamental solutions:

$$\begin{aligned}h_1(\theta)&=1-3\cos^2\theta,\\h_2(\theta)&=-\frac{3}{4}\cos\theta+\frac{1}{8}(1-3\cos^2\theta)\log\frac{1-\cos\theta}{1+\cos\theta}.\end{aligned}$$

These two solutions have the following properties:

(a) $h_1(\frac{\pi}{2})=1,\ h_1'(\frac{\pi}{2})=0$ *and* $h_2(\frac{\pi}{2})=0,\ h_2'(\frac{\pi}{2})=1$.
(b) *Abel's identity:* $h_1h_2'-h_1'h_2=\csc\theta,\ \forall\theta\in(0,\pi)$.
(c) $h_1'(\theta)=6\sin\theta\cos\theta$.
(d) $|h_2(\theta)|\le 2-\log\sin\theta,\ \forall\theta\in(0,\pi)$.
(e) $|h_2'(\theta)\sin\theta|\le 5/2,\ \forall\theta\in(0,\pi)$.
(f) *As* $\theta\to 0^+$ *or* $\theta\to\pi^-$*, there is*

$$h_2'(\theta)=\frac{-1/2+o(1)}{\sin\theta}.$$

Proof. Direct computations show that h_1 and h_2 are solutions to the differential equation in the lemma. And one can easily check (a), (b) and (c).

We note that

$$\frac{1}{2}\left|\log\frac{1-\cos\theta}{1+\cos\theta}\right| \le -\log\sin\theta+\log 2,\quad \forall\theta\in(0,\pi).$$

Since both sides are symmetric with respect to $\theta=\pi/2$, we only need to verify it for $\theta\in(0,\pi/2]$, which is a direct corollary of the following equality:

$$\frac{1}{2}\left|\log\frac{1-\cos\theta}{1+\cos\theta}\right| = \frac{1}{2}\left|\log\frac{1-\cos^2\theta}{(1+\cos\theta)^2}\right| = \left|\log\frac{\sin\theta}{1+\cos\theta}\right|.$$

Now by the expression of h_2, there is

$$\begin{aligned}|h_2(\theta)| &\le \frac{3}{4}+\frac{1}{2}\left|\log\frac{1-\cos\theta}{1+\cos\theta}\right|\\ &\le \frac{3}{4}-\log\sin\theta+\log 2\\ &\le 2-\log\sin\theta,\end{aligned}$$

which is just (d).

Computing h_2', we have

$$h_2'(\theta)=\frac{3}{4}\sin\theta+\frac{3}{4}\sin\theta\cos\theta\,\log\frac{1-\cos\theta}{1+\cos\theta}+\frac{1}{4}(1-3\cos^2\theta)\csc\theta.$$

Then

$$\begin{aligned}|h_2'(\theta)| &\le \frac{3}{4}+\frac{3}{4}\sin\theta\left|\log\frac{1-\cos\theta}{1+\cos\theta}\right|+\frac{1}{2}\csc\theta\\ &\le \frac{3}{4}+\frac{3}{2}\sin\theta\cdot(-\log\sin\theta+\log 2)+\frac{1}{2}\csc\theta\\ &\le 2+\frac{1}{2}\csc\theta,\end{aligned}$$

which implies (e).

By the expression of h_2', we see as $\theta\to 0^+$ or $\theta\to\pi^-$ that

$$h_2'(\theta)\sin\theta\to -\frac{1}{2},$$

yielding (f). □

Lemma 10. *Assume $h\in C^2(\mathbb{R})$ is 2π-periodic and even. If it satisfies the following differential equation*

$$h''+h'\cot\theta+6h=g, \tag{51}$$

and $h(\frac{\pi}{2}) = 0$*, then there is*

$$\max_{\mathbb{R}} |h| \leq 2 \int_0^{\pi} |g(\theta)|(2 - \log \sin\theta) \sin\theta \, \mathrm{d}\theta. \tag{52}$$

Proof. Recalling Lemma 9, h_1 and h_2 are two fundamental solutions to the homogeneous differential equation:

$$h'' + h' \cot\theta + 6h = 0 \text{ in } (0, \pi).$$

By method of variation of parameters and Lemma 9 (b), we solve (51) in $(0, \pi)$ and obtain

$$h(\theta) = c_1 h_1 + c_2 h_2 - h_1 \int_{\pi/2}^{\theta} h_2(t) g(t) \sin t \, \mathrm{d}t + h_2 \int_{\pi/2}^{\theta} h_1(t) g(t) \sin t \, \mathrm{d}t, \tag{53}$$

where c_1 and c_2 are constants to be determined. Note the assumption $h(\frac{\pi}{2}) = 0$, and by (53)

$$h(\tfrac{\pi}{2}) = c_1 h_1(\tfrac{\pi}{2}) + c_2 h_2(\tfrac{\pi}{2}) = c_1,$$

there is $c_1 = 0$. Then

$$h'(\theta) = c_2 h_2' - h_1' \int_{\pi/2}^{\theta} h_2(t) g(t) \sin t \, \mathrm{d}t + h_2' \int_{\pi/2}^{\theta} h_1(t) g(t) \sin t \, \mathrm{d}t. \tag{54}$$

To determine c_2, we need to compute $h'(0)$.

By Lemma 9 (d), $|h_2|$ is an integrable function in $(0, \pi/2]$. Then

$$\int_{\pi/2}^{0} h_2(t) g(t) \sin t \, \mathrm{d}t$$

is a finite number. For small $\theta > 0$, one can rewrite (54) as

$$\frac{1}{h_2'(\theta)} \left[h'(\theta) + h_1'(\theta) \int_{\pi/2}^{\theta} h_2(t) g(t) \sin t \, \mathrm{d}t \right] = c_2 + \int_{\pi/2}^{\theta} h_1(t) g(t) \sin t \, \mathrm{d}t. \tag{55}$$

Letting $\theta \to 0^+$, and recalling $h'(0) = 0$, $h_1'(0) = 0$ and Lemma 9 (f), we obtain

$$0 = c_2 + \int_{\pi/2}^{0} h_1(t) g(t) \sin t \, \mathrm{d}t,$$

namely

$$c_2 = \int_0^{\pi/2} h_1(t)g(t)\sin t\,\mathrm{d}t.$$

Therefore (53) is simplified into

$$h(\theta) = -h_1 \int_{\pi/2}^{\theta} h_2(t)g(t)\sin t\,\mathrm{d}t + h_2 \int_0^{\theta} h_1(t)g(t)\sin t\,\mathrm{d}t. \tag{56}$$

Recalling Lemma 9 (d) and the expression of h given in (56), we obtain for any $\theta \in (0, \pi/2]$ that

$$|h(\theta)| \le 2 \int_{\theta}^{\pi/2} (2 - \log\sin t)|g(t)|\sin t\,\mathrm{d}t + (2 - \log\sin\theta) \int_0^{\theta} 2\,|g(t)|\sin t\,\mathrm{d}t.$$

Observing

$$(2 - \log\sin\theta) \int_0^{\theta} 2\,|g(t)|\sin t\,\mathrm{d}t \le 2 \int_0^{\theta} (2 - \log\sin t)|g(t)|\sin t\,\mathrm{d}t,$$

we have

$$|h(\theta)| \le 2 \int_0^{\pi/2} (2 - \log\sin t)|g(t)|\sin t\,\mathrm{d}t, \quad \forall \theta \in (0, \pi/2].$$

Namely

$$\max_{[0,\pi/2]} |h| \le 2 \int_0^{\pi} (2 - \log\sin t)|g(t)|\sin t\,\mathrm{d}t. \tag{57}$$

Again by Lemma 9 (d), we see

$$\int_{\pi/2}^{\pi} h_2(t)g(t)\sin t\,\mathrm{d}t$$

is a finite number. Since (55) is also true when θ is close to π^-, letting $\theta \to \pi^-$, and recalling $h'(\pi) = 0$, $h_1'(\pi) = 0$ and Lemma 9 (f), we obtain

$$0 = c_2 + \int_{\pi/2}^{\pi} h_1(t)g(t)\sin t\,\mathrm{d}t.$$

Recall the expression of c_2, there is

$$\int_0^{\pi} h_1(t)g(t)\sin t\,\mathrm{d}t = 0.$$

Now h in (56) can be also expressed as

$$h(\theta) = -h_1 \int_{\pi/2}^{\theta} h_2(t)g(t)\sin t\,\mathrm{d}t + h_2 \int_{\pi}^{\theta} h_1(t)g(t)\sin t\,\mathrm{d}t. \tag{58}$$

By Lemma 9 (d), we obtain for any $\theta \in [\pi/2, \pi)$ that

$$|h(\theta)| \le 2 \int_{\pi/2}^{\theta} (2 - \log\sin t)|g(t)|\sin t\,\mathrm{d}t + (2 - \log\sin\theta) \int_{\theta}^{\pi} 2\,|g(t)|\sin t\,\mathrm{d}t.$$

Observing

$$(2 - \log\sin\theta) \int_{\theta}^{\pi} 2\,|g(t)|\sin t\,\mathrm{d}t \le 2 \int_{\theta}^{\pi} (2 - \log\sin t)|g(t)|\sin t\,\mathrm{d}t,$$

we have

$$|h(\theta)| \le 2 \int_{\pi/2}^{\pi} (2 - \log\sin t)|g(t)|\sin t\,\mathrm{d}t, \quad \forall\theta \in [\pi/2, \pi).$$

Namely

$$\max_{[\pi/2,\pi]} |h| \le 2 \int_0^{\pi} (2 - \log\sin t)|g(t)|\sin t\,\mathrm{d}t. \tag{59}$$

Now combining (57) and (59), we obtain (52). □

Based on Lemma 7, one can easily find out the asymptotic behavior of H_{A_k}.

Lemma 11. *Assume $a_k \to +\infty$ as $k \to +\infty$. Then we have*

$$H_{A_k} - f(\tfrac{\pi}{2})^{\frac{1}{2n+2}} = O(1) \begin{cases} a_k^{-1} \log a_k, & \text{if } n = 1, \\ a_k^{-1}, & \text{if } n = 2, \end{cases} \tag{60}$$

where the bounds of $O(1)$ depend only on $\|f\|_{C^1}$.

Proof. Let

$$\Lambda_k = \int_0^{\pi} \left| f(\gamma_{a_k}(\theta)) - f(\tfrac{\pi}{2}) \right| \sin^{2\delta}\theta \, \mathrm{d}\theta, \quad \delta = 0 \text{ or } 1/4.$$

Consider the variable substitution

$$\theta = \gamma_{1/a_k}(t) = \arccos\left(\frac{\cos t}{i_{1/a_k}(t)}\right),$$

see (17) for its definition. Direct computations show that

$$\sin\theta = \frac{\sin t}{(\sin^2 t + a_k^2 \cos^2 t)^{1/2}},$$
$$\mathrm{d}\theta = \frac{a_k}{\sin^2 t + a_k^2 \cos^2 t} \, \mathrm{d}t.$$

Then we have

$$\begin{aligned} \Lambda_k &= \int_0^{\pi} \left| f(t) - f(\tfrac{\pi}{2}) \right| \frac{\sin^{2\delta} t \cdot a_k \, \mathrm{d}t}{(\sin^2 t + a_k^2 \cos^2 t)^{1+\delta}} \\ &\le \|f\|_{C^1} \int_0^{\pi} \frac{|t - \pi/2| \cdot a_k \, \mathrm{d}t}{(\sin^2 t + a_k^2 \cos^2 t)^{1+\delta}} \\ &= 2\|f\|_{C^1} a_k \int_0^{\frac{\pi}{2}} \frac{|t - \pi/2| \, \mathrm{d}t}{(\sin^2 t + a_k^2 \cos^2 t)^{1+\delta}} \\ &= 2\|f\|_{C^1} a_k \int_0^{\frac{\pi}{2}} \frac{t \, \mathrm{d}t}{(\cos^2 t + a_k^2 \sin^2 t)^{1+\delta}}. \end{aligned} \tag{61}$$

Since $a_k \to +\infty$ as $k \to +\infty$, we can assume $a_k > 2$ without loss of generality. For $t \in [0, \pi/2]$, we have

$$\begin{aligned}\cos^2 t + a_k^2 \sin^2 t &= 1 + (a_k^2 - 1)\sin^2 t \\ &\geq 1 + \frac{a_k^2}{4}\cdot\frac{4}{\pi^2}t^2 \\ &= \frac{1}{\pi^2}(\pi^2 + a_k^2 t^2).\end{aligned} \tag{62}$$

Then (61) can be simplified as

$$\begin{aligned}\Lambda_k &\leq 2\pi^3 \|f\|_{C^1} a_k \int_0^{\frac{\pi}{2}} \frac{t\,\mathrm{d}t}{(\pi^2 + a_k^2 t^2)^{1+\delta}} \\ &\leq 2\pi^3 \|f\|_{C^1} \begin{cases} a_k^{-1}\log a_k, & \text{if } \delta = 0, \\ 2a_k^{-1}, & \text{if } \delta = 1/4. \end{cases}\end{aligned}$$

Now note $f_{A_k}(\theta) = f(\gamma_{a_k}(\theta))$, (31) is reduced into

$$\max_{[0,\pi]} \left| H_{A_k} - f(\tfrac{\pi}{2})^{\frac{1}{2n+2}} \right| \leq C \begin{cases} a_k^{-1}\log a_k, & \text{if } n = 1, \\ a_k^{-1}, & \text{if } n = 2, \end{cases}$$

where $C > 0$ depends only on $\|f\|_{C^1}$. This inequality immediately leads to (60). □

We can prove the following

Lemma 12. *Assume $a_k \to +\infty$ as $k \to +\infty$. Then we have*

$$\int_0^{\pi} \frac{1}{H_{A_k}^{n+1}} \cdot \frac{a_k \sin^n\theta \cos\theta}{i_{a_k}^2(\theta)}\,\mathrm{d}\theta = O(1) \begin{cases} a_k^{-2}\log^2 a_k, & \text{if } n = 1, \\ a_k^{-2}, & \text{if } n = 2, \end{cases} \tag{63}$$

where the bounds of $O(1)$ depend only on $\|f\|_{C^1}$.

Proof. Let Λ_k denote the integral on the left hand side of (63), and

$$h_k := H_{A_k} - f(\tfrac{\pi}{2})^{\frac{1}{2n+2}}.$$

Observe that

$$\begin{aligned}H_{A_k}^{-n-1} &= \left[f(\tfrac{\pi}{2})^{\frac{1}{2n+2}} + h_k\right]^{-n-1} \\ &= f(\tfrac{\pi}{2})^{-\frac{1}{2}} - (n+1)\tau^{-n-2}h_k,\end{aligned}$$

where τ is between $f(\frac{\pi}{2})^{\frac{1}{2n+2}}$ and H_{A_k}. Then

$$\Lambda_k = f(\tfrac{\pi}{2})^{-\frac{1}{2}} \int_0^\pi \frac{a_k \sin^n\theta \cos\theta}{i_{a_k}^2(\theta)}\,\mathrm{d}\theta - (n+1)\int_0^\pi \tau^{-n-2} h_k \cdot \frac{a_k \sin^n\theta\cos\theta}{i_{a_k}^2(\theta)}\,\mathrm{d}\theta$$
$$= -(n+1)\int_0^\pi \tau^{-n-2} h_k \cdot \frac{a_k \sin^n\theta\cos\theta}{i_{a_k}^2(\theta)}\,\mathrm{d}\theta.$$

Recall $H_{A_k} \rightrightarrows f(\frac{\pi}{2})^{\frac{1}{2n+2}}$ uniformly on $[0,\pi]$, we can assume that

$$\frac{1}{2} f(\tfrac{\pi}{2})^{\frac{1}{2n+2}} \le \tau \le \frac{3}{2} f(\tfrac{\pi}{2})^{\frac{1}{2n+2}}$$

for sufficiently large k. Therefore

$$|\Lambda_k| \le C \int_0^\pi |h_k| \cdot \frac{a_k \sin^n\theta}{i_{a_k}^2(\theta)}\,\mathrm{d}\theta \tag{64}$$

for some positive constant C depending only on n and $f(\frac{\pi}{2})$.

(a) When $n = 1$. By Lemma 11,

$$h_k = O(1) a_k^{-1} \log a_k.$$

Then we obtain from (64) that

$$|\Lambda_k| \le C \log a_k \int_0^\pi \frac{\sin\theta}{a_k^2 \sin^2\theta + \cos^2\theta}\,\mathrm{d}\theta, \tag{65}$$

where $C > 0$ depends only on $\|f\|_{C^1}$. Assume $a_k > 2$ and recall (62), we have

$$\int_0^\pi \frac{\sin\theta\,\mathrm{d}\theta}{a_k^2\sin^2\theta + \cos^2\theta} = 2\int_0^{\frac{\pi}{2}} \frac{\sin\theta\,\mathrm{d}\theta}{a_k^2\sin^2\theta + \cos^2\theta}$$
$$\le 2\pi^2 \int_0^{\frac{\pi}{2}} \frac{\theta\,\mathrm{d}\theta}{\pi^2 + a_k^2\theta^2}$$
$$\le 2\pi^2 a_k^{-2} \log a_k.$$

Thus (65) says

$$|\Lambda_k| \le C a_k^{-2} \log^2 a_k,$$

which is just (63) for $n = 1$.

(b) When $n = 2$. By Lemma 11,

$$h_k = O(1)a_k^{-1}.$$

Then we obtain by (64) that

$$|\Lambda_k| \le C \int_0^\pi \frac{\sin^2\theta}{a_k^2\sin^2\theta + \cos^2\theta}\,\mathrm{d}\theta \le C\pi a_k^{-2},$$

where $C > 0$ depends only on $\|f\|_{C^1}$. Thus (63) with $n = 2$ is true. □

Now we can strengthen [30, Lemma 3.2] when $n = 1, 2$.

Lemma 13. *Assume $a_k \to +\infty$ as $k \to +\infty$. Then we have*

$$\int_0^\pi \frac{f'(\gamma_{a_k}(\theta))}{H_{A_k}^{n+1}} \cdot \frac{a_k \sin^n\theta\cos\theta}{i_{a_k}^2(\theta)}\,\mathrm{d}\theta = f(\tfrac{\pi}{2})^{-\frac{1}{2}}[ni(f) + o(1)] \begin{cases} a_k^{-1}, & \text{if } n = 1, \\ a_k^{-2}\log a_k^2, & \text{if } n = 2. \end{cases} \tag{66}$$

Proof. Let Λ_k denote the integral on the left hand side of (66). Then

$$\Lambda_k = \int_0^\pi \frac{f'(\gamma_{a_k}(\theta)) - f'(\frac{\pi}{2})}{H_{A_k}^{n+1}} \cdot \frac{a_k \sin^n\theta\cos\theta}{i_{a_k}^2(\theta)}\,\mathrm{d}\theta + \int_0^\pi \frac{f'(\frac{\pi}{2})}{H_{A_k}^{n+1}} \cdot \frac{a_k \sin^n\theta\cos\theta}{i_{a_k}^2(\theta)}\,\mathrm{d}\theta =: I_k + II_k.$$

(a) When $n = 1$. Applying [30, Lemma 3.2] to I_k and Lemma 12 to II_k, we have

$$\Lambda_k = f(\tfrac{\pi}{2})^{-\frac{1}{2}}[ni(f) + o(1)]a_k^{-1} + f'(\tfrac{\pi}{2}) \cdot O(1)a_k^{-2}\log^2 a_k = f(\tfrac{\pi}{2})^{-\frac{1}{2}}[ni(f) + o(1)]a_k^{-1}.$$

(b) When $n = 2$. Applying [30, Lemma 3.2][1] to I_k and Lemma 12 to II_k, we have

$$\Lambda_k = f(\tfrac{\pi}{2})^{-\frac{1}{2}}[ni(f) + o(1)]a_k^{-2}\log a_k^2 + f'(\tfrac{\pi}{2}) \cdot O(1)a_k^{-2} = f(\tfrac{\pi}{2})^{-\frac{1}{2}}[ni(f) + o(1)]a_k^{-2}\log a_k^2.$$

The proof of this lemma is completed. □

We are in position to complete the proof of Theorem 3.

[1] One can check that the conclusion for $n = 2$ is still true under the weaker assumption $f \in C^{2,\alpha}(S^2)$.

Proof of Theorem 3. By [30, Theorem 1.1], we only need to obtain a uniform positive lower bound for rotationally symmetric solutions. Suppose to the contrary that there exists a sequence of rotationally symmetric solutions $\{H_k\}$ to equation (1) such that $\min_{S^n} H_k \to 0^+$ as $k \to +\infty$. For each k, we define a_k, A_k and H_{A_k} as in (8) and (9). By Lemma 4, H_{A_k} is uniformly bounded from above and below. Then we have either $a_k \to +\infty$ or $a_k \to 0^+$.

Recall H_{A_k} is a rotationally symmetric solution to equation (6) with A replaced by A_k. Applying the obstruction condition (4), we have the following

$$\begin{aligned} 0 &= \int_0^\pi \frac{f'_{A_k}(\theta)\sin^n\theta\cos\theta}{H_{A_k}^{n+1}(\theta)}\,\mathrm{d}\theta \\ &= \int_0^\pi \frac{f'(\gamma_{a_k}(\theta))}{H_{A_k}^{n+1}(\theta)}\cdot\frac{a_k\sin^n\theta\cos\theta}{i_{a_k}^2(\theta)}\,\mathrm{d}\theta. \end{aligned} \tag{67}$$

For the case when $a_k \to +\infty$, applying Lemma 13 to (67), we have $ni(f)=0$. For the case when $a_k \to 0^+$, since by Blaschke selection theorem a subsequence of $\{H_{A_k}\}$ converges uniformly to some positive support function on S^n, we apply [30, Lemma 3.3] to (67), and see $pi(f)=0$. In both cases we reach a contradiction with our assumptions on f in Theorem 3. The proof of this theorem is completed. □

4. Proof of Theorem 2

In this section, we prove Theorem 2, which dealing with the case when $n \geq 3$. The method given in the previous section is not applicable to the higher dimensional case. Instead, we use the variational method and blow-up analyses posted in [27].

By arguments in [27], in order to obtain a rotationally symmetric solution to Eq. (1), we only need to find a maximizer of

$$\sup_{|X|=\kappa_{n+1}} \inf_{\xi\in X} J[H(x)-\xi\cdot x], \tag{68}$$

where the supremum is taken among all rotationally symmetric bounded convex bodies X in $\mathbb{R}^{n+1}$ containing the origin with volume κ_{n+1}, the infimum is taken among all points $\xi \in X$, H is the support function of X, and the functional J is given by

$$J[H] = \frac{1}{n+1}\int_{S^n} \frac{f}{H^{n+1}}. \tag{69}$$

Note that for each H, $\inf_{\xi\in X} J[H(x)-\xi\cdot x]$ is attained at a unique point $\xi \in X$. By the Blaschke–Santaló inequality (2), the maximizing problem (68) has an upper bound. But it may not admit a maximizer for some f, see [28]. So we need to impose additional conditions on f to obtain the existence of a maximizer. A class of these conditions can be found by the method of blow-up analysis.

Let $\{H_k\}$ be a maximizing sequence to (68). If it is uniformly bounded, by the Blaschke selection theorem, a subsequence of $\{H_k\}$ converges uniformly to a support function H_∞ which would be a maximizer. If not, namely

$$\sup_{S^n} H_k \to +\infty \text{ as } k \to \infty, \tag{70}$$

then we will deduce a contradiction by the assumptions of Theorem 2, and thus complete the proof of this theorem.

Let X_k be the convex body determined by H_k. For each k choose a unimodular linear transformation $A_k^T \in \mathrm{SL}(n+1)$ that normalizes X_k. Namely the convex body

$$X_{A_k} := A_k^T(X_k)$$

is normalized. Denote its support function by H_{A_k}. Since X_{A_k} has the same volume κ_{n+1}, they are uniformly bounded. On account of Blaschke selection theorem, we assume without loss of generality that X_{A_k} converges to some normalized convex body $\hat{X}$, namely H_{A_k} converges uniformly on S^n to $\hat{H}$, the support function of $\hat{X}$. One can prove that $\hat{H}$ is positive on S^n. Applying formula (7) and the bounded convergence theorem, one gets

$$\begin{aligned} J_{\sup} &:= \lim_{k\to\infty} J[H_k] \\ &= \lim_{k\to\infty} \frac{1}{n+1} \int_{S^n} \frac{f_{A_k}}{H_{A_k}^{n+1}} \\ &= \frac{1}{n+1} \int_{S^n} \frac{\hat{f}}{\hat{H}^{n+1}}, \end{aligned} \tag{71}$$

where $\hat{f}$ is the limit function of f_{A_k}. We want to find some rotationally symmetric H with volume κ_{n+1}, such that

$$J_{\sup} < \inf_{\xi} J[H(x) - \xi \cdot x]. \tag{72}$$

This is a contradiction, from which we will know (70) is false and then complete the proof of the theorem.

To construct (72), we need to find out the expression of $\hat{f}$ first. Note by the rotational symmetry of X_k, the normalizing matrix A_k^T can be chosen as

$$A_k^T = \mathrm{diag}\left(\lambda_k^{\frac{1}{n+1}}, \cdots, \lambda_k^{\frac{1}{n+1}}, \lambda_k^{-\frac{n}{n+1}}\right) \text{ with } \lambda_k > 0.$$

Recalling the definition in (7), we have

$$f_{A_k}(x_1, \cdots, x_n, x_{n+1}) = f\left(\frac{\lambda_k x_1, \cdots, \lambda_k x_n, x_{n+1}}{\sqrt{\lambda_k^2(x_1^2 + \cdots + x_n^2) + x_{n+1}^2}}\right).$$

By the assumption (70), there are only two cases:

$$\lambda_k \to 0 \text{ or } \lambda_k \to \infty, \text{ as } k \to \infty.$$

Correspondingly, we have

$$\hat{f}(x_1, \cdots, x_n, x_{n+1}) = \begin{cases} f(e_{n+1}), & \text{if } x_{n+1} > 0; \\ f(-e_{n+1}), & \text{if } x_{n+1} < 0 \end{cases} \quad \text{when } \lambda_k \to 0, \tag{73}$$

or

$$\hat{f}(x_1, \cdots, x_n, x_{n+1}) = f\left(\frac{x_1, \cdots, x_n, 0}{\sqrt{x_1^2 + \cdots + x_n^2}}\right) \text{ when } \lambda_k \to \infty. \tag{74}$$

For the case when $\lambda_k \to 0$, we can still use the arguments in [27, Section 4.1] to show (72) under the assumption $pi(f) > 0$.

It remains to consider the case when $\lambda_k \to \infty$. Now the analyses in [21,27] are no longer suitable. We provide new blow-up analyses in the following. Since f is rotationally symmetric, one can see from (74) that

$$\hat{f} \text{ is a constant function on } S^n \text{ when } \lambda_k \to \infty.$$

This fact is crucial in our following proof.

A good upper bound of $J_{\sup}$ will be needed.

Lemma 14. *Assume $\lambda_k \to \infty$. There is $J_{\sup} \le \hat{f}\,\kappa_{n+1}$.*

Proof. Recall [27, (3.12)]:

$$J_{\sup} = \inf_{\xi \in \hat{X}} \frac{1}{n+1} \int_{S^n} \frac{\hat{f}(x)\, \mathrm{d}S(x)}{(\hat{H}(x) - \xi \cdot x)^{n+1}}.$$

Note $\hat{f}$ is now a constant, by the Blaschke–Santaló inequality (2), we have

$$\begin{aligned} J_{\sup} &= \hat{f} \inf_{\xi \in \hat{X}} \frac{1}{n+1} \int_{S^n} \frac{\mathrm{d}S(x)}{(\hat{H}(x) - \xi \cdot x)^{n+1}} \\ &\le \hat{f}\, \kappa_{n+1}^2 / \operatorname{vol}(\hat{X}) \\ &= \hat{f}\, \kappa_{n+1}, \end{aligned}$$

which is just our lemma. □

To prove (72), we consider a family of ellipsoids:

$$E_a = \left\{ \xi \in \mathbb{R}^{n+1} : |A(a)\xi| \le 1 \right\},$$

where $A(a) \in \mathrm{SL}(n+1)$ is given by

$$A(a) = \operatorname{diag}\left(a^{\frac{1}{n+1}}, \cdots, a^{\frac{1}{n+1}}, a^{-\frac{n}{n+1}}\right), \quad a > 0.$$

Note each E_a is a rotationally symmetric ellipsoid with volume κ_{n+1}. And its support function, H_a, is given by

$$H_a(x) = |A(a)^{-1}x|, \quad \forall x \in S^n.$$

Now we define

$$J(a) := \inf_{\xi \in E_a} J[H_a(x) - \xi \cdot x]. \tag{75}$$

By (7), we have

$$\begin{aligned} J(a) &= \inf_{\xi \in E_a} \frac{1}{n+1} \int_{S^n} \frac{f}{(H_a - \xi \cdot x)^{n+1}} \\ &= \inf_{|\xi| \le 1} \frac{1}{n+1} \int_{S^n} \frac{f_{A(a)}}{(1 - \xi \cdot x)^{n+1}} \\ &=: \frac{1}{n+1} \int_{S^n} \frac{f_a}{(1 - \xi_a \cdot x)^{n+1}}, \end{aligned} \tag{76}$$

where the infimum is attained at ξ_a, and $f_a = f_{A(a)}$ is defined as

$$f_a(x_1, \cdots, x_n, x_{n+1}) = f\left(\frac{ax_1, \cdots, ax_n, x_{n+1}}{\sqrt{a^2(x_1^2 + \cdots + x_n^2) + x_{n+1}^2}}\right). \tag{77}$$

Recalling (74), we see when $a \to \infty$ that

$$f_a \to \hat{f} \ \text{ a.e. on } \ S^n. \tag{78}$$

For the function f defined on S^n, one can extend it to $\mathbb{R}^{n+1}$ such that it is homogeneous of degree zero. Note that f remains rotationally symmetric in the whole $\mathbb{R}^{n+1}$. For a point $x \in \mathbb{R}^{n+1}$, we write $x = (x', z)$ where

$$x' = (x_1, \cdots, x_n), \quad z = x_{n+1}.$$

Then we can use the standard notations in Euclidean space such as f_z', f_{zz}'' for partial derivatives of f with respect to z.

The following analysis about f_a will be needed.

Lemma 15. *For any $\varphi \in C(S^n)$, we have as $a \to \infty$ that*

$$\int_{S^n} \varphi(x)[f_a(x) - \hat{f}\,]\, dS(x) = \frac{1}{a} \cdot f_z'(e_1) \int_{S^n} \frac{\varphi(x) z}{|x'|}\, dS(x) + \frac{1}{a^2} \cdot f_{zz}''(e_1) \int_{S^n} \frac{\varphi(x) z^2}{2|x'|^2}\, dS(x) + \frac{o(1)}{a^2}. \tag{79}$$

Proof. Let Λ_a denote the integral on the left hand side of (79). By virtue of the Taylor's expansion, for each $x = (x', z) \in S^n$ with $x' \neq 0$, there exists a $t(x) \in (0, 1/a)$ such that

$$f_a(x) - \hat{f} = f(x', z/a) - f(x', 0) = f_z'(x', 0)\frac{z}{a} + \frac{1}{2} f_{zz}''(x', tz)\frac{z^2}{a^2}.$$

Then

$$\begin{aligned}\Lambda_a &= \frac{1}{a}\int_{S^n} \varphi(x) f_z'(x', 0) z \,\mathrm{d}S(x) + \frac{1}{2a^2}\int_{S^n} \varphi(x) f_{zz}''(x', tz) z^2 \,\mathrm{d}S(x) \\ &=: \frac{1}{a} I + \frac{1}{2a^2} I\!I.\end{aligned} \tag{80}$$

To deal with these integrals, we need the following formula:

$$\int_{S^n} g(x)\,\mathrm{d}S(x) = \sigma_{n-1}\int_0^\pi g(\cdot\,, \cos\theta)\sin^{n-1}\theta \,\mathrm{d}\theta \tag{81}$$

for any rotationally symmetric and integrable function g defined on S^n. One can easily check it by the coarea formula.

Now for I, since f_z' is homogeneous of degree -1, then

$$f_z'(x', 0) = \frac{1}{|x'|} f_z'\left(\frac{x'}{|x'|}, 0\right) = \frac{1}{|x'|} f_z'(e_1).$$

Therefore

$$I = f_z'(e_1)\int_{S^n} \frac{\varphi(x) z}{|x'|}\,\mathrm{d}S(x). \tag{82}$$

We remark that I is well defined, since when $n \geq 3$,

$$\int_{S^n} \frac{z}{|x'|}\,\mathrm{d}S(x) = \sigma_{n-1}\int_0^\pi \cos\theta \sin^{n-2}\theta \,\mathrm{d}\theta = C(n) < +\infty.$$

For $I\!I$, note that f_{zz}'' is homogeneous of degree -2, then

$$\begin{aligned}\left|\varphi(x) f_{zz}''(x', tz) z^2\right| &= \left|\varphi(x) f_{zz}''\left(\frac{x', tz}{\sqrt{|x'|^2 + t^2 z^2}}\right)\frac{z^2}{|x'|^2 + t^2 z^2}\right| \\ &\leq \|\varphi\|_{C^0} \cdot \|f\|_{C^2} \cdot \frac{z^2}{|x'|^2},\end{aligned}$$

which is integrable on S^n, since when $n \geq 3$,

$$\int_{S^n} \frac{z^2}{|x'|^2}\,\mathrm{d}S(x) = \sigma_{n-1}\int_0^{\pi} \cos^2\theta \sin^{n-3}\theta\,\mathrm{d}\theta = C(n) < +\infty.$$

Applying the dominated convergence theorem to $I\!I$, we obtain

$$\begin{aligned}\lim_{a\to\infty} I\!I &= \int_{S^n} \varphi(x) f''_{zz}(x',0)z^2\,\mathrm{d}S(x)\\ &= \int_{S^n} \varphi(x) f''_{zz}\left(\frac{x'}{|x'|},0\right)\frac{z^2}{|x'|^2}\,\mathrm{d}S(x)\\ &= f''_{zz}(e_1)\int_{S^n} \frac{\varphi(x)z^2}{|x'|^2}\,\mathrm{d}S(x).\end{aligned}$$

Namely

$$I\!I = f''_{zz}(e_1)\int_{S^n} \frac{\varphi(x)z^2}{|x'|^2}\,\mathrm{d}S(x) + o(1) \quad \text{as } a\to\infty. \tag{83}$$

Now combining (80), (82) and (83), we will obtain (79). □

We also need to analyze ξ_a defined in (76). Since f_a is rotationally symmetric, by [27, (3.9)], ξ_a can be written as

$$\xi_a = \eta_a e_{n+1} \quad \text{for some } \eta_a \in \mathbb{R}. \tag{84}$$

The following asymptotic behavior of η_a will be needed.

Lemma 16. *When $a\to\infty$, we have*

$$\eta_a = \left(\frac{-b_1 f'_z(e_1)}{(n+2)b_0 \hat{f}} + o(1)\right)\frac{1}{a}, \tag{85}$$

where

$$b_0 = \int_{S^n} z^2\,\mathrm{d}S(x), \qquad b_1 = \int_{S^n} \frac{z^2}{|x'|}\,\mathrm{d}S(x). \tag{86}$$

Proof. Since $|\xi_a| \le 1$, we assume without loss of generality that $\xi_a \to \xi_\infty$ as $a\to\infty$. By the definition of ξ_a in (76), for each $|\xi|<1$, there is

$$\int_{S^n} \frac{f_a}{(1-\xi_a\cdot x)^{n+1}} \le \int_{S^n} \frac{f_a}{(1-\xi\cdot x)^{n+1}}.$$

Passing to the limit and recalling (78), we obtain

$$\int_{S^n} \frac{\hat{f}}{(1-\xi_\infty \cdot x)^{n+1}} \le \int_{S^n} \frac{\hat{f}}{(1-\xi \cdot x)^{n+1}}, \quad \forall |\xi| < 1.$$

Note $\hat{f}$ is a constant, there is

$$\int_{S^n} \frac{1}{(1-\xi_\infty \cdot x)^{n+1}} = \inf_{|\xi|<1} \int_{S^n} \frac{1}{(1-\xi \cdot x)^{n+1}}.$$

Thus $\xi_\infty = 0$. Namely $\xi_a \to 0$ as $a \to \infty$, which implies

$$\eta_a \to 0 \text{ as } a \to \infty. \tag{87}$$

By definition, ξ_a is the unique minimum point of

$$\int_{S^n} \frac{f_a}{(1-\xi \cdot x)^{n+1}},$$

which is a strictly convex function with respect to ξ. The vanishing first order derivatives yield

$$\int_{S^n} \frac{f_a}{(1-\xi_a \cdot x)^{n+2}} x_i = 0, \quad i = 1, 2, \cdots, n+1.$$

Recall (84) and that f_a is rotationally symmetric, these equalities are equivalent to

$$\int_{S^n} \frac{f_a x_{n+1}}{(1-\eta_a x_{n+1})^{n+2}} = 0. \tag{88}$$

For simplicity, we write

$$\phi(t) = -\frac{1}{t^{n+2}}, \quad \forall t > 0.$$

Recall $x = (x', z)$, then (88) says

$$\int_{S^n} \phi(1-\eta_a z) f_a z \, \mathrm{d}S(x) = 0. \tag{89}$$

By (87), for sufficiently large a, there is $|\eta_a| < 1/2$. Then

$$\frac{1}{2} < 1 - \eta_a z < \frac{3}{2}.$$

Thus

$$\phi(1-\eta_a z)=\phi(1)-\phi'(1)\eta_a z+\frac{1}{2}\phi''(\tau)\eta_a^2 z^2,$$

where τ varies in $(1/2, 3/2)$. Inserting it into (89), we obtain

$$\phi(1)\int\limits_{S^n} f_a z\,\mathrm{d}S(x)-\phi'(1)\eta_a\int\limits_{S^n} f_a z^2\,\mathrm{d}S(x)+\frac{1}{2}\eta_a^2\int\limits_{S^n}\phi''(\tau)f_a z^3\,\mathrm{d}S(x)=0,$$

which obviously can be written as

$$\phi(1)\int\limits_{S^n} f_a z\,\mathrm{d}S(x)-\phi'(1)\eta_a\int\limits_{S^n} f_a z^2\,\mathrm{d}S(x)+O(1)\eta_a^2=0. \tag{90}$$

Recalling $\hat{f}$ is a constant, and applying Lemma 15, we have as $a\to\infty$ that

$$\begin{aligned}\int\limits_{S^n} f_a z\,\mathrm{d}S(x)&=\int\limits_{S^n} z(f_a-\hat{f})\,\mathrm{d}S(x)\\&=\frac{1}{a}\left(f_z'(e_1)\int\limits_{S^n}\frac{z^2}{|x'|}\,\mathrm{d}S(x)+o(1)\right)\\&=\frac{1}{a}[b_1 f_z'(e_1)+o(1)].\end{aligned} \tag{91}$$

By (78), there is

$$\begin{aligned}\int\limits_{S^n} f_a z^2\,\mathrm{d}S(x)&=\hat{f}\int\limits_{S^n} z^2\,\mathrm{d}S(x)+o(1)\\&=b_0\hat{f}+o(1).\end{aligned} \tag{92}$$

Now combining (90), (91) and (92), we obtain as $a\to\infty$ that

$$\phi(1)[b_1 f_z'(e_1)+o(1)]\frac{1}{a}-\phi'(1)\eta_a[b_0\hat{f}+o(1)]+O(1)\eta_a^2=0,$$

which yields

$$\begin{aligned}\eta_a&=\frac{\phi(1)[b_1 f_z'(e_1)+o(1)]}{\phi'(1)[b_0\hat{f}+o(1)]}\cdot\frac{1}{a}\\&=\left(\frac{\phi(1)b_1 f_z'(e_1)}{\phi'(1)b_0\hat{f}}+o(1)\right)\frac{1}{a}.\end{aligned}$$

Observing $\phi(1)=-1$ and $\phi'(1)=n+2$, we obtain (85). □

Now we can obtain the asymptotic behavior of $J(a)$ defined in (75)–(76).

Lemma 17. *When $a \to \infty$, we have*

$$J(a) = \hat{f}\kappa_{n+1} + \left(\frac{b_2 f''_{zz}(e_1)}{2(n+1)} - \frac{b_1^2 f'_z(e_1)^2}{2(n+2)b_0 \hat{f}} + o(1)\right)\frac{1}{a^2}, \tag{93}$$

where b_0 and b_1 are given in (86), and

$$b_2 = \int_{S^n} \frac{z^2}{|x'|^2}\, \mathrm{d}S(x). \tag{94}$$

Proof. For simplicity, we write

$$\phi(t) = \frac{1}{n+1} t^{-n-1}, \quad \forall t > 0.$$

Then (76) says

$$\begin{aligned} J(a) &= \int_{S^n} \phi(1 - \xi_a \cdot x) f_a\, \mathrm{d}S(x) \\ &= \int_{S^n} \phi(1 - \eta_a z) f_a\, \mathrm{d}S(x), \end{aligned} \tag{95}$$

where (84) and $x = (x', z)$ have been used for the second equality. By Lemma 16, one can assume

$$\frac{1}{2} < 1 - \eta_a z < \frac{3}{2}$$

for sufficiently large a. Then

$$\phi(1 - \eta_a z) = \phi(1) - \phi'(1)\eta_a z + \frac{1}{2}\phi''(1)\eta_a^2 z^2 - \frac{1}{6}\phi'''(\tau)\eta_a^3 z^3,$$

where τ varies in $(1/2, 3/2)$. Inserting it into (95), we obtain

$$\begin{aligned} J(a) &= \phi(1)\int_{S^n} f_a - \phi'(1)\eta_a \int_{S^n} f_a z + \frac{1}{2}\phi''(1)\eta_a^2 \int_{S^n} f_a z^2 - \frac{1}{6}\eta_a^3 \int_{S^n} \phi'''(\tau) f_a z^3 \\ &= \phi(1)\int_{S^n} f_a - \phi'(1)\eta_a \int_{S^n} f_a z + \frac{1}{2}\phi''(1)\eta_a^2 \int_{S^n} f_a z^2 + O(1)\eta_a^3. \end{aligned}$$

Recalling (87), (91) and (92), we have as $a \to \infty$ that

$$J(a) = \phi(1)\int_{S^n} f_a - \phi'(1)\eta_a [b_1 f'_z(e_1) + o(1)]\frac{1}{a} + \frac{1}{2}\phi''(1)\eta_a^2 [b_0 \hat{f} + o(1)].$$

Note by Lemma 16,

$$\eta_a = \left(\frac{-b_1 f_z'(e_1)}{(n+2)b_0 \hat{f}} + o(1) \right) \frac{1}{a},$$

one gets

$$J(a) = \phi(1) \int_{S^n} f_a - \phi'(1) \left(\frac{-b_1^2 f_z'(e_1)^2}{(n+2)b_0 \hat{f}} + o(1) \right) \frac{1}{a^2} + \frac{1}{2}\phi''(1) \left(\frac{b_1^2 f_z'(e_1)^2}{(n+2)^2 b_0 \hat{f}} + o(1) \right) \frac{1}{a^2}.$$

Observe $\phi(1) = \frac{1}{n+1}$, $\phi'(1) = -1$ and $\phi''(1) = n+2$, then $J(a)$ is simplified as

$$\begin{aligned} J(a) &= \frac{1}{n+1} \int_{S^n} f_a + \left(\frac{-b_1^2 f_z'(e_1)^2}{(n+2)b_0 \hat{f}} + o(1) \right) \frac{1}{a^2} + \frac{1}{2} \left(\frac{b_1^2 f_z'(e_1)^2}{(n+2)b_0 \hat{f}} + o(1) \right) \frac{1}{a^2} \\ &= \frac{1}{n+1} \int_{S^n} f_a + \left(\frac{-b_1^2 f_z'(e_1)^2}{2(n+2)b_0 \hat{f}} + o(1) \right) \frac{1}{a^2}. \end{aligned} \tag{96}$$

By Lemma 15, when $a \to \infty$,

$$\begin{aligned} \int_{S^n} [f_a(x) - \hat{f}\,] \mathrm{d}S(x) &= \frac{1}{a^2} \cdot f_{zz}''(e_1) \int_{S^n} \frac{z^2}{2|x'|^2} \mathrm{d}S(x) + \frac{o(1)}{a^2} \\ &= \frac{1}{a^2} \left(\frac{1}{2} b_2 f_{zz}''(e_1) + o(1) \right), \end{aligned}$$

namely

$$\frac{1}{n+1} \int_{S^n} f_a = \hat{f} \kappa_{n+1} + \frac{1}{a^2} \left(\frac{b_2 f_{zz}''(e_1)}{2(n+1)} + o(1) \right). \tag{97}$$

Inserting (97) into (96), we obtain when $a \to \infty$ that

$$J(a) = \hat{f} \kappa_{n+1} + \left(\frac{b_2 f_{zz}''(e_1)}{2(n+1)} - \frac{b_1^2 f_z'(e_1)^2}{2(n+2)b_0 \hat{f}} + o(1) \right) \frac{1}{a^2},$$

which is just (93). □

Now by Lemma 17, if

$$\frac{b_2 f_{zz}''(e_1)}{2(n+1)} - \frac{b_1^2 f_z'(e_1)^2}{2(n+2)b_0 \hat{f}} > 0, \tag{98}$$

then for sufficiently large a there is

$$J(a) > \hat{f}\kappa_{n+1}.$$

Recalling Lemma 14, for the case $\lambda_k \to \infty$, we have $J_{\sup} \leq \hat{f}\kappa_{n+1}$. Thus

$$J(a) > J_{\sup}$$

for sufficiently large a. Recalling the definition of $J(a)$ in (75), we see this inequality is just (72).

So to obtain (72) for the case when $\lambda_k \to \infty$, it remains to check (98). Recalling our notations, we have

$$f(\theta) = f(\cdot, \cos\theta) = f(\sin\theta, 0, \cdots, 0, \cos\theta).$$

Note that $f(\frac{\pi}{2}) = f(e_1) = \hat{f}$. Also there is

$$\begin{aligned} f'(\theta) &= \cos\theta f_1' - \sin\theta f_z' \\ &= -\cos\theta f_z' \cot\theta - \sin\theta f_z' \\ &= -\frac{f_z'}{\sin\theta}, \end{aligned}$$

where that $\nabla f(x) \cdot x = 0$ has been used for the second equality. Therefore one immediately gets that $f'(\frac{\pi}{2}) = -f_z'(e_1)$, and that

$$-ni(f) = f''(\tfrac{\pi}{2}) = f_{zz}''(e_1).$$

Now (98) is equivalent to

$$-\frac{b_2\, ni(f)}{2(n+1)} - \frac{b_1^2 f'(\frac{\pi}{2})^2}{2(n+2)b_0 f(\frac{\pi}{2})} > 0,$$

namely

$$ni(f) < -\frac{(n+1)b_1^2}{(n+2)b_0 b_2} f'(\tfrac{\pi}{2})^2 / f(\tfrac{\pi}{2}). \tag{99}$$

Here we recall that b_0, b_1 and b_2 are given in (86) and (94), which depend only on n and can be easily worked out by formula (81). Observe that

$$\begin{aligned} b_1^2 &= \left(\int_{S^n} \frac{z^2}{|x'|} \, \mathrm{d}S(x) \right)^2 \\ &< \int_{S^n} z^2 \, \mathrm{d}S(x) \cdot \int_{S^n} \frac{z^2}{|x'|^2} \, \mathrm{d}S(x) \\ &= b_0 b_2, \end{aligned}$$

then the assumption on $ni(f)$ in Theorem 2 implies (99), namely (98).

Now we have obtained (72) in both possible blow-up cases under assumptions of Theorem 2. According to our previous discussion, the proof of this theorem is completed.

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