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IRRATIONAL FACTOR OF ORDER k AND ITS CONNECTIONS WITH k-FREE INTEGERS

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Abstract. We introduce an irrational factor of order k defined by $I_k(n) = \prod_{i=1}^{l} p_i^{\beta_i}$, where $n = \prod_{i=1}^{l} p_i^{\alpha_i}$ is the factorization of n and $\beta_i = \begin{cases} \alpha_i, & \text{if } \alpha_i < k \\ \frac{1}{\alpha_i}, & \text{if } \alpha_i \geqq k \end{cases}$. It turns out that the function $\frac{I_k(n)}{n}$ well approximates the characteristic function of k-free integers. We also derive asymptotic formulas for $\prod_{v=1}^{n} I_k(v)^{\frac{1}{n}}$, $\sum_{n \leqq x} I_k(n)$ and $\sum_{n \leqq x} \left(1 - \frac{n}{x}\right) I_k(n)$.

1. Introduction

The study of k-power free integers arises naturally in number theory (see [8] or Ch. VI in [9] for a nice survey). In [2], Atanassov defined the irrational factor function $I(n) = \prod_{i=1}^{l} p_i^{\frac{1}{\alpha_i}}$, where $n = \prod_{i=1}^{l} p_i^{\alpha_i}$ is the factorization of n. Alkan et al. in [1] noticed that $I(n)$ may be used to measure how far n is away from being k-power free or k-power full. More precisely, large values of $I(n)$ correspond to k-power free integers and small values of $I(n)$ correspond to k-power full integers. However, since k is not involved in the definition, $I(n)$ is a rough measurement. Motivated by Alkan et al.'s observation, we introduced a parameter k to the function I so that the new functions I_k can do a better job. For any integer $k \geqq 2$, we define the *irrational factor function of order* k as follows:

$$I_k(n) = \prod_{i=1}^{l} p_i^{\beta_i}, \quad \text{where} \quad \beta_i = \begin{cases} \alpha_i, & \text{if } \alpha_i < k; \\ \dfrac{1}{\alpha_i}, & \text{if } \alpha_i \geqq k. \end{cases}$$

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Let $\mu_k(n)$ denote the characteristic function of k-free integers (this notation is motivated by the formula $\mu_k(n) = \sum_{d^k|n} \mu(d)$, where $\mu(n)$ is the Möbius function). We can see that $\frac{I_k(n)}{n}$ is very close to $\mu_k(n)$ in the following sense. On one hand,

$$\left|\frac{I_k(n)}{n} - \mu_k(n)\right| \leqq \frac{1}{2^{k-\frac{1}{k}}},$$

which is small for large k. The difference is even smaller in the following important special cases:

$$\left|\frac{I_k(n)}{n} - \mu_k(n)\right| \leqq \begin{cases} 0, & \text{if } n \text{ is } k\text{-free;} \\ \dfrac{1}{n^{1-\frac{1}{k^2}}}, & \text{if } n \text{ is } k\text{-power full.} \end{cases}$$

On the other hand, the average values of $\frac{I_k(n)}{n}$ and $\mu_k(n)$ are also very close to each other (see the remark after Theorem 2).

Our goal is to obtain some properties of $I_k(n)$. Note that $I_k(n) = I(n)$ when $k = 2$. We first recall some results on $I(n)$. Panaitopol [7] proved that $e^{-7} < G(n) := \prod_{v=1}^{n} I(v)^{\frac{1}{n}} < n$. Moreover, Alkan et al. [1] established the following asymptotic formulas for averages of $I(n)$.

Theorem. *There are absolute constants c_1, c_2, $c_3 > 0$ such that*

(i) $G(n) = \prod_{v=1}^{n} I(v)^{\frac{1}{n}} = c_1 n + O\big(\sqrt{n}\big)$.

(ii) $\sum_{n \leqq x} I(n) = c_2 x^2 + O\big(x^{\frac{3}{2}} (\log x)^{\frac{9}{4}}\big)$.

(iii) $\sum_{n \leqq x} \big(1 - \frac{n}{x}\big) I(n) = \frac{c_2}{3} x^2 + O\big(x^{\frac{3}{2}} e^{-c_3 (\log x)^{3/5} (\log\log x)^{-1/5}}\big)$.

It is natural to ask if similar results hold for $I_k(n)$. We then define

$$G_k(n) = \prod_{v=1}^{n} I_k(v)^{\frac{1}{n}}.$$

Inspired by the techniques used in [1], we derived the following corresponding results for $k \geqq 3$ (all the implied constants in big O depend on k).

Theorem 1. *For $k \geqq 2$, we have*

$$G_k(n) = e^{-1-c} n + O\big(n^{\frac{1}{k}}\big),$$

where

$$c = \left(k - \frac{1}{k}\right) \sum_p \frac{\log p}{p^k} + \sum_{k+1 \leqq l} \left(1 + \frac{1}{l(l+1)}\right) \sum_p \frac{\log p}{p^l}.$$

THEOREM 2. *For $k \geqq 3$, we have*

$$\sum_{n \leqq x} I_k(n) = \frac{K_k(2)}{2\zeta(k)} x^2 + O\big(x^{\nu(k)} \log^2 x\big),$$

where $K_k(2)$ is a constant depending only on k, and

$$\nu(k) = \frac{2(2k-1)}{3k-2}.$$

REMARK. It is known that (see [12]) the number of k-free integers that are $\leqq x$ is

$$\sum_{n \leqq x} \mu_k(n) = \frac{x}{\zeta(k)} + O\Big(x^{\frac{1}{k}} \exp\big\{-ck^{-\frac{8}{5}} \log^{\frac{3}{5}} x (\log\log x)^{-\frac{1}{5}}\big\}\Big).$$

Using partial summation, it follows from Theorem 2 that the average of $\frac{I_k(n)}{n}$ is $\frac{K_k(2)}{\zeta(k)}$. It is easy to see that $\lim_{k\to\infty} K_k(2) = 1$ (see the definition of $K_k(s)$ in the proof). Therefore, for large k, $\frac{I_k(n)}{n}$ and $\mu_k(n)$ are very close in average.

THEOREM 3. *For $k \geqq 3$, we have*

$$\sum_{n \leqq x} \Big(1 - \frac{n}{x}\Big) I_k(n) = \frac{K_k(2)}{6\zeta(k)} x^2 + O\bigg(\frac{x^{1+\frac{1}{k}}}{\Delta(x)}\bigg),$$

where

$$\Delta(x) = e^{c(\log x)^{\frac{3}{5}} (\log\log x)^{-\frac{1}{5}}}$$

for some constant $c > 0$.

We remark that we get better estimates of error terms for larger k.

2. Asymptotic formula for the geometric mean

In this section, we prove Theorem 1. By the definition of $I_k(n)$ and $G_k(n)$,

$$G_k(n)^n = \prod_{v=1}^{n} \prod_{p \mid v} p^{\beta_p(v)} = \prod_{p \leqq n} \sum_{\substack{v=1 \\ p \mid v}}^{n} p^{\beta_p(v)} = \prod_{p \leqq n} p^{\alpha(n,p)}, \tag{2.1}$$

where

$$\alpha(n,p)=\sum_{\substack{v=1\\ p\mid v}}^{n}\beta_p(v).$$

Then,

$$\alpha(n,p)=\sum_{l=1}^{k-1}\sum_{\beta_p(v)=l}l+\sum_{l=k}^{\infty}\sum_{\substack{v=1\\ \beta_p(v)=l}}^{n}\frac{1}{l}$$

$$=\sum_{l=1}^{k-1}\#\{1\leqq v\leqq n:\,p^l\mid v,\ p^{l+1}\nmid v\}+\sum_{l=k}^{\infty}\frac{1}{l}\#\{1\leqq v\leqq n:\,p^l\mid v,\ p^{l+1}\nmid v\}$$

$$=\sum_{l=1}^{k-1}l\left(\left[\frac{n}{p^l}\right]-\left[\frac{n}{p^{l+1}}\right]\right)+\sum_{l=k}^{\infty}\frac{1}{l}\left(\left[\frac{n}{p^l}\right]-\left[\frac{n}{p^{l+1}}\right]\right)$$

$$=\sum_{l=1}^{k}\left[\frac{n}{p^l}\right]-\left(k-\frac{1}{k}\right)\left[\frac{n}{p^k}\right]-\sum_{l=k+1}^{\infty}\frac{1}{l(l-1)}\left[\frac{n}{p^l}\right].$$

And we know that (see [4], Theorem 416, p. 342)

$$n!=\prod_{p\leqq n}p^{\sum_{l=1}^{\infty}\left[\frac{n}{p^l}\right]}.$$

Thus, by (2.1) and the above formula,

$$\log n!-n\log G_k(n)$$

$$=\sum_{p\leqq n}\left(\left(k-\frac{1}{k}\right)\left[\frac{n}{p^l}\right]+\sum_{l=k+1}^{\infty}\left(1+\frac{1}{l(l-1)}\right)\left[\frac{n}{p^l}\right]\right)\log p$$

$$=\sum_{p^k\leqq n}\left(k-\frac{1}{k}\right)\left[\frac{n}{p^k}\right]\log p+\sum_{k+1\leqq l\leqq\left[\frac{\log n}{\log 2}\right]+1}\left(1+\frac{1}{l(l-1)}\right)\sum_{p^{k+1}\leqq n}\left[\frac{n}{p^l}\right]\log p$$

$$=\left(k-\frac{1}{k}\right)n\sum_{p^k\leqq n}\frac{\log p}{p^k}+O\left(\left(k-\frac{1}{k}\right)\sum_{p^k\leqq n}\log p\right)$$

$$+ n \sum_{k+1 \leqq l \leqq \left[\frac{\log n}{\log 2}\right]+1} \left(1 + \frac{1}{l(l+1)}\right) \sum_{p^{k+1} \leqq n} \frac{\log p}{p^l}$$

$$+ O\Bigg(\sum_{k+1 \leqq l \leqq \left[\frac{\log n}{\log 2}\right]+1} \sum_{p^{k+1} \leqq n} \log p \Bigg)$$

$$= \left(k - \frac{1}{k}\right) n \sum_{p} \frac{\log p}{p^k} - \left(k - \frac{1}{k}\right) \sum_{p > n^{\frac{1}{k}}} \frac{\log p}{p^k}$$

$$+ n \sum_{k+1 \leqq l \leqq \left[\frac{\log n}{\log 2}\right]+1} \left(1 + \frac{1}{l(l+1)}\right) \sum_{p} \frac{\log p}{p^l}$$

$$- n \sum_{k+1 \leqq l \leqq \left[\frac{\log n}{\log 2}\right]+1} \left(1 + \frac{1}{l(l+1)}\right) \sum_{p > n^{\frac{1}{k+1}}} \frac{\log p}{p^l} + O\big(n^{\frac{1}{k}}\big).$$

We have

$$n \sum_{p > n^{k+1}} \frac{\log p}{p^l} = O\big(n^{\frac{1}{k+1}}\big), \quad \left(k - \frac{1}{k}\right) \sum_{p > n^{\frac{1}{k}}} \frac{\log p}{p^k} = O\big(n^{\frac{1}{k}}\big),$$

and so

$$n \sum_{k+1 \leqq l \leqq \left[\frac{\log n}{\log 2}\right]+1} \left(1 + \frac{1}{l(l+1)}\right) \sum_{p > n^{\frac{1}{k+1}}} \frac{\log p}{p^l} = O\big(n^{\frac{1}{k+1}} \log n\big).$$

Let

$$c = \left(k - \frac{1}{k}\right) \sum_{p} \frac{\log p}{p^k} + \sum_{k+1 \leqq l} \left(1 + \frac{1}{l(l+1)}\right) \sum_{p} \frac{\log p}{p^l}.$$

Then,

(2.2)

$$\log n! - n \log G_k(n) = cn - n \sum_{l > \left[\frac{\log n}{\log 2}\right]+1} \left(1 + \frac{1}{l(l+1)}\right) \sum_{p} \frac{\log p}{p^l} + O\big(n^{\frac{1}{k}}\big).$$

Further,

$$\sum_{p} \frac{\log p}{p^l} = O\left(\frac{1}{2^l}\right),$$

and

$$n\sum_{l>\left[\frac{\log n}{\log 2}\right]+1}\left(1+\frac{1}{l(l+1)}\right)\sum_{p}\frac{\log p}{p^{l}}=O\left(n\sum_{l>\left[\frac{\log n}{\log 2}\right]+1}\frac{1}{2^{l}}\right)=O(1).$$

Thus, by (2.2), we deduce

$$\log G_k(n)=\frac{\log n!}{n}-c+O\big(n^{\frac{1-k}{k}}\big). \tag{2.3}$$

Hence, by (2.3) and Stirling's formula (see [6], Ch. III, Theorem 5, pp. 44),

$$G_k(n)=e^{-1-c}n\left(1+O\left(\frac{n^{\frac{1}{k}}}{n}\right)\right)=e^{-1-c}n+O\big(n^{\frac{1}{k}}\big).$$

3. Asymptotic formula for $\sum_{n\leqq x} I_k(n)$

We will first consider the Dirichlet series with $I_k(n)$ as coefficients and then prove Theorem 2. Let $L(s)=\sum_{n=1}^{\infty}\frac{I_k(n)}{n^s}$. From our definition, we can see that $I_k(n)$ is multiplicative. Then, $L(s)$ has Euler product,

$$L(s)=\prod_{p}\left(1+\frac{I_k(p)}{p^s}+\frac{I_k(p^2)}{p^{2s}}+\cdots+\frac{I_k(p^m)}{p^{ms}}+\cdots\right)=\frac{\zeta(s-1)}{\zeta(ks-k)}K_k(s),$$

where $K_k(s)=\prod_{p}\big(1+A_p(s)\big)$, and

$$A_p(s)=\frac{\sum_{m=k}^{\infty}\frac{1}{p^{ms-\frac{1}{m}}}}{1+\frac{1}{p^{s-1}}+\cdots+\frac{1}{p^{(k-1)(s-1)}}}=\frac{\frac{1}{p^{k-\frac{1}{k}}}\sum_{m=0}^{\infty}\frac{1}{p^{ms+\frac{1}{k}-\frac{1}{m+k}}}}{1+\frac{1}{p^{s-1}}+\cdots+\frac{1}{p^{(k-1)(s-1)}}}.$$

Let $s=\sigma+it$. Then

$$\left|1+\frac{1}{p^{s-1}}+\cdots+\frac{1}{p^{(k-1)(s-1)}}\right|=\left|\frac{1-\frac{1}{p^{k(s-1)}}}{1-\frac{1}{p^{s-1}}}\right|\geqq\frac{1-\frac{1}{p^{k(\sigma-1)}}}{1+\frac{1}{p^{\sigma-1}}}\geqq\frac{1-\frac{1}{2^{k(\sigma-1)}}}{1+\frac{1}{2^{\sigma-1}}},$$

$$\left|\frac{1}{p^{ks-\frac{1}{k}}}\right|=\frac{1}{p^{k\sigma-\frac{1}{k}}},$$

and

$$\left|\sum_{m=0}^{\infty}\frac{1}{p^{ms+\frac{1}{k}-\frac{1}{m+k}}}\right|\leqq 1+\frac{1}{p^{\sigma}}+\frac{1}{p^{2\sigma}}+\cdots=\frac{1}{1-\frac{1}{p^{\sigma}}}\leqq 2.$$

So that

$$\left| A_p(s) \right| \leqq \frac{2\left(1 - \frac{1}{2^{\sigma-1}}\right)}{1 + \frac{1}{2^{k(\sigma-1)}}} \frac{1}{p^{k\sigma - \frac{1}{k}}},$$

then,

$$\sum_p \left| A_p(s) \right| \leqq \frac{2\left(1 - \frac{1}{2^{\sigma-1}}\right)}{1 + \frac{1}{2^{k(\sigma-1)}}} \sum_p \frac{1}{p^{k\sigma - \frac{1}{k}}} < \infty.$$

It follows that $K_k(s) = \prod_p \left(1 + A_p(s)\right)$ is absolutely convergent and defines an analytic function on $\Re(s) > 1$. Moreover, for any fixed $\sigma_0 > 1$,

$$\left| K_k(s) \right| \leqq \prod_p \left(1 + \left| A_p(s) \right| \right) \leqq \prod_p \left(1 + \frac{2\left(1 - \frac{1}{2^{\sigma_0-1}}\right)}{1 + \frac{1}{2^{k(\sigma_0-1)}}} \frac{1}{p^{k\sigma_0 - \frac{1}{k}}}\right) = c_{k,\sigma_0} < \infty.$$

Hence, $K_k(s)$ is uniformly bounded on $\Re(s) \geqq \sigma_0$. The fact that $K_k(s)$ is analytic on $\Re(s) > 1$ shows that $L(s)$ has meromorphic continuation to this half-plane. Further, since $\zeta(s-1)$ has a simple pole at $s = 2$ and $\zeta(ks-k)$ has no zeros on $\Re(s) > 1 + \frac{1}{k}$, it follows that $L(s)$ has analytic continuation to $\Re(s) > 1 + \frac{1}{k}$, with the exception of a simple pole at $s = 2$. Under the Riemann Hypothesis, this function would have analytic continuation to $\Re(s) > 1 + \frac{1}{2k}$.

Now we turn to the proof of Theorem 2. By Perron's formula (see [10], pp. 300–303; and [11], pp. 60–62),

$$\sum_{n \leqq x} a(n) = \frac{1}{2\pi i} \int_{\alpha - iT}^{\alpha + iT} \frac{x^s}{s} A(s)\, ds + R(x, \alpha, T),$$

where $A(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$, and

$$\left| R(x, \alpha, T) \right| \leqq \frac{x^\alpha}{T} \sum_{n=1}^{\infty} \frac{\left| a(n) \right|}{n^\alpha \left| \log x/n \right|}.$$

Fix $0 < T \leqq x^2$. Put $\alpha = 2 + \frac{c}{\log x}$ for some $c > 0$ and $a(n) = I_k(n)$, and

$$A(s) = \sum_{n=1}^{\infty} \frac{I_k(n)}{n^s} = \frac{\zeta(s-1)}{\zeta(ks-k)} K_k(s).$$

We have

$$\sum_{n \leqq x} I_k(n) = \frac{1}{2\pi i} \int_{\alpha - iT}^{\alpha + iT} \frac{x^s \zeta(s-1)}{s\zeta(ks-k)} K_k(s)\, ds + R(x, \alpha, T), \tag{3.1}$$

where

$$\left| R(x, \alpha, T) \right| = O\left(\frac{x^\alpha}{T} \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \frac{n^{1-\alpha}}{|\log x/n|} \right).$$

To treat the sum in the error term, we decompose it into three subsums (see [3], pp. 106–107): $n \leqq \frac{x}{2}$, $\frac{x}{2} < n \leqq \frac{3x}{2}$, and $n > \frac{3x}{2}$. We can get

$$\left| R(x, \alpha, T) \right| = O\left(\frac{x^2 \log x}{T} \right).$$

By residue theorem, we obtain

$$\frac{1}{2\pi i} \int_{\alpha - iT}^{\alpha + iT} \frac{x^s \zeta(s-1)}{s\zeta(ks-k)} K_k(s)\, ds = \frac{K_k(2)}{2\zeta(k)} x^2 + \sum_{v=1}^{3} J_v, \tag{3.2}$$

where J_i represents the integral along the path C_i and $C_1 = \left\{ \sigma - iT : \alpha \leqq \sigma \leqq 1 + \frac{1}{k} \right\}$, $C_2 = \left\{ 1 + \frac{1}{k} + it : -T \leqq t \leqq T \right\}$, and $C_3 = \left\{ \sigma + iT : 1 + \frac{1}{k} \leqq \sigma \leqq \alpha \right\}$.

It is well known that (see [5], Theorem 1.9, pp. 25)

$$\left| \zeta(\sigma + it) \right| = \begin{cases} O\left(t^{\frac{1-\sigma}{2}} \log t \right), & \text{if } \ 0 \leqq \sigma \leqq 1 \\ O(\log t), & \text{if } \ 1 \leqq \sigma \leqq 2, \\ O(1), & \text{if } \ \sigma \geqq 2. \end{cases}$$

On the line segments on which $s = \sigma \pm iT$, $1 + \frac{1}{k} \leqq \sigma \leqq \alpha$, by the above estimates, we have

$$\left| \zeta(\sigma - 1 + iT) \right| = \begin{cases} O\left(T^{1 - \frac{\sigma}{2}} \log T \right) & \text{if } \ 1 + \dfrac{1}{k} \leqq \sigma \leqq 2, \\ O(\log T) & \text{if } \ 2 \leqq \sigma \leqq \alpha, \end{cases}$$

and by (3.6.1) in [10]

$$\frac{1}{\left| \zeta(k\sigma - k + ikT) \right|} = O_k(\log T).$$

So that

$$(3.3)\qquad |J_1|, |J_3| = O\left(\int_{1+\frac{1}{k}}^{\alpha} \frac{\left|x^{\sigma+iT}\right|\left|\zeta(\sigma-1+iT)\right|}{|\sigma+iT|\left|\zeta(k\sigma-k+ikT)\right|}\left|K_k(\sigma+iT)\right| dt \right)$$

$$= O\left(\log^2 x \left[\int_{1+\frac{1}{k}}^{2} \left(\frac{x}{\sqrt{T}}\right)^{\sigma} d\sigma + \int_2^{\alpha} \frac{x^{\alpha}}{T}\, d\sigma \right] \right) = O\left(\frac{x^2 \log^2 x}{T}\right).$$

On the line segment $s = 1 + \frac{1}{k} + it$, $t \in [-T, T]$,

$$\frac{1}{\left|\zeta(1+ikt)\right|} = O(\log T) = O(\log x).$$

So that

$$|J_2| = O\left(\int_0^T \frac{\left|x^{1+\frac{1}{k}+it}\right|\left|\zeta\left(\frac{1}{k}+it\right)\right|}{\left|1+\frac{1}{k}+it\right|\left|\zeta(1+ikt)\right|} \left|K_k\left(1+\frac{1}{k}+it\right)\right| dt \right)$$

$$= O\left(x^{1+\frac{1}{k}} \log x \int_0^T \frac{\left|\zeta\left(\frac{1}{k}+it\right)\right|}{\left|1+\frac{1}{k}+it\right|} dt \right)$$

$$= O\left(x^{1+\frac{1}{k}} \log x \left[1 + \int_1^T \frac{\left|\zeta\left(\frac{1}{k}+it\right)\right|}{t} dt \right] \right).$$

By the mean square of $\zeta(s)$ (see [5], Theorem 1.11, p. 28),

$$(3.4)\qquad \int_0^T \left|\zeta(\sigma+it)\right|^2 dt = O(T), \quad \text{for} \quad \frac{1}{2} < \sigma < 1.$$

For $0 < \sigma < \frac{1}{2}$, we need the functional equation $\zeta(s) = \chi(s)\zeta(1-s)$ where $\chi(s) = \pi^{s-\frac{1}{2}}\Gamma\left(\frac{1}{2}(1-s)\right)/\Gamma\left(\frac{1}{2}s\right)$. By Stirling's formula ([6], Ch. III, Corollary 2, pp. 45), $\left|\chi(\sigma+it)\right| \asymp \left(|t|+2\right)^{\frac{1}{2}-\sigma}$. So we get the estimate

$$\left|\zeta(\sigma+it)\right| = O\left(\left(|t|+2\right)^{\frac{1}{2}-\sigma}\left|\zeta(1-\sigma+it)\right|\right).$$

Thus, by (3.4), we get

$$\int_{\frac{T}{2}}^{T} \left|\zeta\left(\frac{1}{k}+it\right)\right|^2 dt = O\left(T^{2-\frac{2}{k}}\right).$$

Then, by Cauchy–Schwarz,

$$\int_{\frac{T}{2}}^{T} \frac{\left|\zeta\left(\frac{1}{k}+it\right)\right|}{t}\,dt \leqq \left(\int_{\frac{T}{2}}^{T} \frac{1}{t^2}\,dt\right)^{\frac{1}{2}} \left(\int_{\frac{T}{2}}^{T} \left|\zeta\left(\frac{1}{k}+it\right)\right|^2 dt\right)^{\frac{1}{2}} = O\left(T^{\frac{1}{2}-\frac{1}{k}}\right).$$

Hence,

$$|J_2| = O\left(x^{1+\frac{1}{k}} \log x \log T \cdot T^{\frac{1}{2}-\frac{1}{k}}\right). \tag{3.5}$$

At last, we combine all the estimates (3.3), (3.5) and formulas (3.1), (3.2), and take $a = \frac{2(k-1)}{3k-2}$ and $T = x^a$ to get the desired result.

4. Asymptotic formula for $\sum_{n\leqq x} \left(1 - \frac{n}{x}\right) I_k(n)$

We use another form of Perron's formula to derive Theorem 3:

$$\sum_{n\leqq x} \left(1 - \frac{n}{x}\right) I_k(n) = \frac{1}{2\pi i} \int_{\alpha-i\infty}^{\alpha+i\infty} \frac{x^s \zeta(s-1)}{s(s+1)\zeta(ks-k)} K_k(s)\,ds.$$

We want to use Vinogradov–Korobov zero-free region (see [11], p. 135)

$$\sigma \geqq 1 - \frac{c_5}{(\log t)^{\frac{2}{3}} (\log\log t)^{\frac{1}{3}}}, \quad t \geqq t_0.$$

And

$$\frac{1}{\left|\zeta(s)\right|} = O\left((\log t)^{\frac{2}{3}} (\log\log t)^{\frac{1}{3}}\right).$$

Fix $T, U > 0$, such that $T < U < x^2$. Let

$$\alpha = 2 + \frac{c_5}{\log x}, \quad \text{and} \quad \beta = 1 + \frac{1}{k} - \frac{c_6}{(\log 2T)^{\frac{2}{3}} (\log\log 2T)^{\frac{1}{3}}}.$$

We know that (see [11], p. 45)

$$\frac{1}{\left|\zeta(s)\right|} = O\left(|s|\right), \quad \sigma \geqq 1.$$

By the residue theorem,

$$\sum_{n\leqq x} \left(1 - \frac{n}{x}\right) I_k(n) = \frac{K_k(2)}{6\zeta(k)} x^2 + \sum_{v=1}^{9} J_v, \tag{4.1}$$

where J_i denotes the integral along the path C_i and

$$C_1 = \{\alpha + it : t \geqq U\}, \quad C_2 = \left\{\sigma + iU : 1 + \frac{1}{k} \leqq \sigma \leqq \alpha\right\},$$

$$C_3 = \left\{1 + \frac{1}{k} + it : T \leqq t \leqq U\right\}, \quad C_4 = \left\{\sigma + iT : \beta \leqq \sigma \leqq 1 + \frac{1}{k}\right\},$$

$$C_5 = \{\beta + it : |t| \leqq T\}, \quad C_6 = \left\{\sigma - iT : \beta \leqq \sigma \leqq 1 + \frac{1}{k}\right\},$$

$$C_7 = \left\{1 + \frac{1}{k} + it : -U \leqq t \leqq -T\right\}, \quad C_8 = \left\{\sigma - iU : 1 + \frac{1}{k} \leqq \sigma \leqq \alpha\right\},$$

$$C_9 = \{\alpha + it : t \leqq -U\}.$$

In the following, we will estimate these J_i's $(i = 1, 2, \dots, 9)$.

First, we estimate J_1 and J_9, $s = \alpha + it$, $|t| \geqq U$, and we know

$$\left|\zeta(\alpha - 1 + it)\right| \leqq \left|\zeta(\alpha - 1)\right| = O(\log x),$$

and

$$\frac{1}{\left|\zeta(k\alpha - k + ikt)\right|} = O(1).$$

So,

$$|J_1|, |J_9| = O\left(\int_U^\infty \frac{\left|x^{\alpha+it}\right|\left|\zeta(\alpha - 1 + it)\right|\left|K_k(\alpha + it)\right|}{|\alpha + it| \cdot |\alpha + 1 + it|\left|\zeta(k\alpha - k + ikt)\right|}\, dt\right) \tag{4.2}$$

$$= O\left(x^2 \log x \int_U^\infty \frac{dt}{t^2}\right) = O\left(\frac{x^2 \log x}{U}\right).$$

J_2, and J_8, $s = \sigma \pm iU$, $1 + \frac{1}{k} \leqq \sigma \leqq \alpha$.

$$\left|\zeta(\sigma - 1 + iU)\right| = O\big(U^{1-\frac{\sigma}{2}} \log U\big), \quad 1 + \frac{1}{k} \leqq \sigma \leqq 2.$$

$$\left|\zeta(\sigma - 1 + iU)\right| = O(\log U), \quad 2 \leqq \sigma \leqq \alpha.$$

and

$$\frac{1}{\left|\zeta(k\sigma - k + ikU)\right|} = O(\log kU) = O_k(\log x).$$

Then,

$$(4.3)\qquad |J_2|,|J_8| = O\left(\int_{1+\frac{1}{k}}^{\alpha} \frac{x^\sigma\big|\zeta(\sigma-1+iU)\big|\,\big|K_k(\sigma+iU)\big|}{|\sigma+iU|\,|\sigma+1+iU|\big|\zeta(k\sigma-k+ikU)\big|}\,d\sigma\right)$$

$$= O\left(\int_{1+\frac{1}{k}}^{2} \frac{x^\sigma U^{1-\frac{\sigma}{2}}\log U}{U^2}\log x\,d\sigma + \int_2^{\alpha}\frac{x^\sigma\log U\log x}{U^2}\,d\sigma\right)$$

$$= O\left(\log^2 x\left[\int_{1+\frac{1}{k}}^{2}\frac{1}{U}\left(\frac{x}{\sqrt{U}}\right)^{\sigma} d\sigma + \int_2^{\alpha}\frac{x^\sigma}{U^2}\,d\sigma\right]\right) = O\left(\frac{x^2\log^2 x}{U^2}\right).$$

J_3, J_7: $s = 1+\frac{1}{k}+it$, $T \leqq |t| \leqq U$.

$$\frac{1}{\big|\zeta(1+ikt)\big|} = O(\log kt) = O_k(\log U).$$

By the mean square theorem and functional equation,

$$\int_1^X \big|\zeta(\sigma+it)\big|^2\,dt = O\big(X^{2-2\sigma}\big),\quad \text{for}\quad 0<\sigma<\frac{1}{2}.$$

Then,

$$|J_3|,|J_7| = O\left(\int_T^U \frac{x^{1+\frac{1}{k}}\left|\zeta\left(\frac{1}{k}+it\right)\right|\left|K_k\left(1+\frac{1}{k}+it\right)\right|}{\left|1+\frac{1}{k}+it\right|\left|2+\frac{1}{k}+it\right|\left|\zeta(1+ikt)\right|}\,dt\right)$$

$$= O\left(x^{1+\frac{1}{k}}\log U\int_T^U\frac{\left|\zeta\left(\frac{1}{k}+it\right)\right|}{t^2}\,dt\right).$$

For the last integral,

$$\int_T^U \frac{\left|\zeta\left(\frac{1}{k}+it\right)\right|}{t^2}\,dt = O\left(\sum_{\frac{T}{2}\leqq 2^l\leqq U}\int_{2^l}^{2^{l+1}}\frac{\left|\zeta\left(\frac{1}{k}+it\right)\right|}{t^2}\,dt\right)$$

$$= O\left(\sum_{\frac{T}{2}\leqq 2^l\leqq U}\left(\int_{2^l}^{2^{l+1}}\frac{1}{t^4}\,dt\right)^{\frac{1}{2}}\left(\int_{2^l}^{2^{l+1}}\left|\zeta\left(\frac{1}{k}+it\right)\right|^2 dt\right)^{\frac{1}{2}}\right)$$

$$= O\left(\sum_{2^{l+1}\geqq T}\frac{1}{\left(2^l\right)^{\frac{3}{2}}}\cdot\left(2^l\right)^{1-\frac{1}{k}}\right) = O\left(\frac{1}{T^{\frac{1}{2}+\frac{1}{k}}}\right).$$

Thus,

$$|J_3|, |J_7| = O\left(\frac{x^{1+\frac{1}{k}}\log x}{T^{\frac{1}{2}+\frac{1}{k}}}\right). \tag{4.4}$$

J_4, J_6: $s = \sigma + iT$, $\beta \leqq \sigma \leqq 1 + \frac{1}{k}$.

$$\left|\zeta(\sigma - 1 + iT)\right| = O\big(T^{1-\frac{\sigma}{2}}\log T\big).$$

$$\frac{1}{\left|\zeta(k\sigma - k + ikT)\right|} = O_k\big((\log T)^{\frac{2}{3}}(\log\log T)^{\frac{1}{3}}\big).$$

Then,

$$|J_4|, |J_6| = O\left(\int_\beta^{1+\frac{1}{k}} \frac{x^\sigma\left|\zeta(\sigma - 1 + iT)\right|\,\left|K_k(\sigma + iT)\right|}{|\sigma + iT|\,|\sigma + 1 + iT|\left|\zeta(k\sigma - k + ikt)\right|}\,d\sigma\right) \tag{4.5}$$

$$= O\left(\frac{(\log T)^{\frac{5}{3}}(\log\log T)^{\frac{1}{3}}}{T}\int_\beta^{1+\frac{1}{k}}\left(\frac{x}{\sqrt{T}}\right)^\sigma d\sigma\right)$$

$$= O\left(\frac{x^{1+\frac{1}{k}}(\log T)^{\frac{5}{3}}(\log\log T)^{\frac{1}{3}}}{T^{\frac{3}{2}+\frac{1}{2k}}}\right).$$

J_5: $s = \beta + it$, $|t| \leqq T$.

$$\left|\zeta(\beta - 1 + it)\right| = O\Big(\big(|t| + 2\big)^{1-\frac{\beta}{2}}\log\big(|t| + 2\big)\Big).$$

$$\frac{1}{\left|\zeta(k\beta - k + ikt)\right|} = O\Big(\big(\log\big(|t| + 2\big)\big)^{\frac{2}{3}}\big(\log\log\big(|t| + 2\big)\big)^{\frac{1}{3}}\Big).$$

Then,

$$|J_5| = O\left(\int_{-T}^{T}\frac{\left|x^{\beta+it}\right|\,\left|\zeta(\beta - 1 + it)\right|\,\left|K_k(\beta + it)\right|}{|\beta + it|\,|\beta + 1 + it|\left|\zeta(k\beta - k + ikt)\right|}\,dt\right) \tag{4.6}$$

$$= O\left(x^\beta\int_{-T}^{T}\frac{\big(|t| + 2\big)^{1-\frac{\beta}{2}}\big(\log\big(|t| + 2\big)\big)^{\frac{5}{3}}\big(\log\log\big(|t| + 2\big)\big)^{\frac{1}{3}}}{t^2 + 1}\,dt\right) = O\big(x^\beta\big).$$

We combine (4.1) and all the estimates (4.2–4.6), and take $U = x$, $T = \exp\left\{c_7\frac{(\log x)^{\frac{3}{5}}}{(\log\log x)^{\frac{1}{5}}}\right\}$, and $(c_7)^{\frac{5}{3}} = c_6$. Then we get the desired result.

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References

[1] E. Alkan, A. H. Ledoan and A. Zaharescu, Asymptotic behavior of the irrational factor, *Acta Math. Hungar.,* **121** (2008), 293–305.

[2] K. T. Atanassov, Irrational factor: definition, properties and problems, *Notes on Number Theory Discrete Math.,* **2** (1996), 42–44.

[3] H. Davenport, *Multiplicative Number Theory,* 3rd ed. (revised and with a preface by H. L. Montgomery), Graduate Texts in Mathematics, 74, Springer-Verlag (New York, 2000).

[4] G. H. Hardy and E. M. Wright, *An Introduction tot he Theory of Numbers,* 5th ed., The Clarendon Press, Oxford University Press (New York, 1979).

[5] A. Ivić, *The Riemann Zeta-Function: Theory and Applications,* Reprint edition, Dover Publications (2003).

[6] A. A. Karatsuba, *Basic Analytic Number Theory,* Springer-Verlag (Berlin, Heidelberg, 1993).

[7] L. Panaitopol, Properties of the Atanassov functions, *Adv. Stud. Contemp. Math. (Kyungshang),* **8** (2004), 55–58.

[8] F. Pappalardi, A survey on k-freeness, *Number Theory, Ramanujan Math. Soc. Lect. Notes Ser.,* vol. 1, pp. 71–88. Ramanujan Math. Soc. (Mysor, 2005).

[9] J. Sándor, D. S. Mitrinović and B. Crstici, *Handbook of Number Theory I,* 2nd printing, Springer (2006).

[10] E. C. Titchmarsh, *The Theory of Functions,* 2nd ed., Oxford University Press (London, 1939).

[11] E. C. Titchmarsh, *The Theory of the Riemann Zeta-Function,* 2nd ed. (Edited and with a preface by D. R. Heath-Brown), The Clarendon Press, Oxford University Press (New York, 1986).

[12] A. Walfisz, *Weylsche Exponentialsummen in der neueren Zahlentheorie,* Mathematische Forschungsberichte, XV. VEB Deutscher Verlag der Wissenschaften, (Berlin, 1963).