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*Research Article*

# A Generalized Hybrid Steepest-Descent Method for Variational Inequalities in Banach Spaces

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The hybrid steepest-descent method introduced by Yamada (2001) is an algorithmic solution to the variational inequality problem over the fixed point set of nonlinear mapping and applicable to a broad range of convexly constrained nonlinear inverse problems in real Hilbert spaces. Lehdili and Moudafi (1996) introduced the new prox-Tikhonov regularization method for proximal point algorithm to generate a strongly convergent sequence and established a convergence property for it by using the technique of variational distance in Hilbert spaces. In this paper, motivated by Yamada's hybrid steepest-descent and Lehdili and Moudafi's algorithms, a generalized hybrid steepest-descent algorithm for computing the solutions of the variational inequality problem over the common fixed point set of sequence of nonexpansive-type mappings in the framework of Banach space is proposed. The strong convergence for the proposed algorithm to the solution is guaranteed under some assumptions. Our strong convergence theorems extend and improve certain corresponding results in the recent literature.

## 1. Introduction

Let $\mathscr{H}$ be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, respectively. Let $C$ be a nonempty closed convex subset of $\mathscr{H}$ and $D$ a nonempty closed convex subset of $C$.

It is well known that the standard smooth convex optimization problem [1], given a convex, Fréchet-differentiable function $f : \mathscr{H} \to \mathbb{R}$ and a closed convex subset $C$ of $\mathscr{H}$, find a point $x^* \in C$ such that

$$f(x^*) = \min\{x \in C : f(x)\} \tag{1.1}$$

can be formulated equivalently as the variational inequality problem VIP$(\nabla f, \mathcal{H})$ over $C$ (see [2, 3]):

$$\langle \nabla f x^*, v - x^* \rangle \geq 0 \quad \forall v \in C, \tag{1.2}$$

where $\nabla f : \mathcal{H} \to \mathcal{H}$ is the gradient of $f$.

In general, for a nonlinear mapping $\mathcal{F} : \mathcal{H} \to \mathcal{H}$ over $C$, the variational inequality problem VIP$(\mathcal{F}, C)$ over $D$ is to find a point $x^* \in D$ such that

$$\langle \mathcal{F} x^*, v - x^* \rangle \geq 0 \quad \forall v \in D. \tag{1.3}$$

It is important to note that the theory of variational inequalities has been playing an important role in the study of many diverse disciplines, for instance, partial differential equations, optimal control, optimization, mathematical programming, mechanics, finance, and so forth, see, for example, [1, 2, 4–6] and references therein.

It is also known that if $\mathcal{F}$ is Lipschitzian and strongly monotone, then for small $\mu > 0$, the mapping $P_C(I - \mu\mathcal{F})$ is a contraction, where $P_C$ is the metric projection from $\mathcal{H}$ onto $C$ (see Section 2.3). In this case, the Banach contraction principle guarantees that VIP$(\mathcal{F}, C)$ has a unique solution $x^*$ and the sequence of Picard iteration process, given by,

$$x_{n+1} = P_C(I - \mu\mathcal{F}) x_n \quad \forall n \in \mathbb{N} \tag{1.4}$$

converges strongly to $x^*$. This simplest iterative method for approximating the unique solution of VIP$(\mathcal{F}, C)$ over $C$ is called the *projected gradient method* [1]. It has been used widely in many practical problems, due, partially, to its fast convergence.

The projected gradient method was first proposed by Goldstein [7] and Levitin and Polyak [8] for solving convexly constrained minimization problems. This method is regarded as an extension of the steepest-descent or Cauchy algorithm for solving unconstrained optimization problems. It now has many variants in different settings, and supplies a prototype for various more advanced projection methods. In [9], the first author introduced the normal $S$-iteration process and studied an iterative method for approximating the unique solution of VIP$(\mathcal{F}, C)$ over $C$ as follows:

$$x_{n+1} = P_C(I - \mu\mathcal{F})\left[(1 - \alpha_n)x_n + \alpha_n P_C(I - \mu\mathcal{F})x_n\right] \quad \forall n \in \mathbb{N}. \tag{1.5}$$

Note that the rate of convergence of iterative method (1.5) is faster than projected gradient method (1.4), see [9].

The projected gradient method requires repetitive use of $P_C$, although the closed form expression of $P_C$ is not always known in many situations. In order to reduce the complexity probably caused by the projection mapping $P_C$, Yamada (see [6]) introduced a hybrid steepest-descent method for solving the problem VIP$(\mathcal{F}, \mathcal{H})$. Here is the idea. Suppose $T$ (e.g., $T = P_C$) is a mapping from a Hilbert space $\mathcal{H}$ into itself with a nonempty fixed point set $F[T]$, and $\mathcal{F}$ is a Lipschitzian and strongly monotone over $\mathcal{H}$. Starting with an arbitrary initial guess $x_1$ in $\mathcal{H}$, one generates a sequence $\{x_n\}$ by the following algorithm:

$$x_{n+1} := T[x_n - \lambda_n \mathcal{F}(x_n)] \quad \forall n \in \mathbb{N}, \tag{1.6}$$

where $\{\lambda_n\}$ is a slowly diminishing sequence. Yamada [6, Theorem 3.3, page 486] proved that the sequence $\{x_n\}$ defined by (1.6) converges strongly to a unique solution of VIP($\mathcal{F}, \mathcal{K}$) over $F[T]$.

Let $X$ be a real Banach space with dual space $X^*$. We denote by $J$ the normalized duality mapping from $X$ into $2^{X^*}$ defined by

$$J(x) := \left\{ f^* \in X^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2 \right\}, \quad x \in X, \tag{1.7}$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. It is well known that the normalized duality mapping is single-valued if $X$ smooth, see [10]. Let $C$ be a nonempty subset of a real Banach space $X$. A mapping $T : X \to X$ is said to be

(1) *pseudocontractive over* $C$ if for each $x, y \in C$, there exists $j(x-y) \in J(x-y)$ satisfying

$$\langle Tx - Ty, j(x-y) \rangle \leq \|x-y\|^2, \tag{1.8}$$

(2) $\delta$*-strongly accretive over* $C$ if for each $x, y \in C$, there exist a constant $\delta > 0$ and $j(x-y) \in J(x-y)$ satisfying

$$\langle Tx - Ty, j(x-y) \rangle \geq \delta \|x-y\|^2. \tag{1.9}$$

We consider the following general variational inequality problem over the fixed point set of nonlinear mapping in the framework of Banach space.

*Problem 1.1.* (*general variational inequality problem over the fixed point set of nonlinear mapping*). Let $C$ be a nonempty closed convex subset of a real smooth Banach space $X$. Let $T : C \to C$ be a (possibly nonlinear) mapping of which fixed point set $F[T]$ is a nonempty closed convex set. Then for a given strongly accretive operator $\mathcal{F} : X \to X$ over $C$, the general variational inequality problem VIP($\mathcal{F}, C$) over $F[T]$ is

$$\text{find a point } x^* \in F[T] \quad \text{such that } \langle \mathcal{F}x^*, J(v - x^*) \rangle \geq 0 \ \forall v \in F[T]. \tag{1.10}$$

Recently, the method (1.6) has been applied successfully to signal processing, inverse problems, and so on [11–13]. This situation induces a natural question.

*Question 1.2.* Does sequence $\{x_n\}$, defined by (1.6), converges strongly a solution to a general variational inequality problem in the Banach space setting, that is, Problem 1.1 in a case where $T : C \to C$ is given as such a nonexpansive mapping?

We now consider the following variational inclusion problem:

$$\text{find } z \in C \quad \text{such that } 0 \in Az, \tag{P}$$

in the framework of Banach space $X$, where $A : X \to 2^X$ is a multivalued operator acting on $C \subseteq X$. In the sequel, we assume that $\mathbb{S} = A^{-1}(0)$, the set of solutions of Problem $(P)$ is nonempty.

The Problem $(P)$ can be regarded as a unified formulation of several important problems. For an appropriate choice of the operator $A$, Problem $(P)$ covers a wide range of mathematical applications; for example, variational inequalities, complementarity problems, and nonsmooth convex optimization. Problem $(P)$ has applications in physics, economics, and in several areas of engineering. In particular, if $\psi : \mathcal{H} \to \mathbb{R} \cup \{\infty\}$ is a proper, lower semicontinuous convex function, its subdifferential $\partial\psi = A$ is a maximal monotone operator, and a point $z \in \mathcal{H}$ minimizes $\psi$ if and only if $0 \in \partial\psi(z)$.

One of the most interesting and important problems in the theory of maximal monotone operators is to find an efficient iterative algorithm to compute approximately zeroes of maximal monotone operators. One method for solving zeros of maximal monotone operators is *proximal point algorithm*. Let $A$ be a maximal monotone operator in a Hilbert space $\mathcal{H}$. The proximal point algorithm generates, for starting $x_1 \in \mathcal{H}$, a sequence $\{x_n\}$ in $\mathcal{H}$ by

$$x_{n+1} = J_{c_n} x_n \quad \forall n \in \mathbb{N}, \tag{1.11}$$

where $J_{c_n} := (I + c_n A)^{-1}$ is the resolvent operator associated with the operator $A$, and $\{c_n\}$ is a regularization sequence in $(0,\infty)$. This iterative procedure is based on the fact that the proximal map $J_{c_n}$ is single-valued and nonexpansive. This algorithm was first introduced by Martinet [14]. If $\psi : \mathcal{H} \to \mathbb{R} \cup \{\infty\}$ is a proper lower semicontinuous convex function, then the algorithm reduces to

$$x_{n+1} = \operatorname*{argmin}_{y \in \mathcal{H}} \left\{ \psi(y) + \frac{1}{2c_n} \|x_n - y\|^2 \right\} \quad \forall n \in \mathbb{N}. \tag{1.12}$$

Rockafellar [15] studied the proximal point algorithm in the framework of Hilbert space and he proved the following.

**Theorem 1.3.** *Let $\mathcal{H}$ be a Hilbert space and $A \subset \mathcal{H} \times \mathcal{H}$ a maximal monotone operator. Let $\{x_n\}$ be a sequence in $\mathcal{H}$ defined by* (1.11), *where $\{c_n\}$ is a sequence in $(0,\infty)$ such that* $\liminf_{n\to\infty} c_n > 0$. *If $\mathbb{S} \neq \emptyset$, then the sequence $\{x_n\}$ converges weakly to an element of $\mathbb{S}$.*

Such weak convergence is global; that is, the just announced result holds in fact for any $x_1 \in \mathcal{H}$.

Further, Rockafellar [15] posed an open question of whether the sequence generated by (1.11) converges strongly or not. This question was solved by Güler [16], who constructed an example for which the sequence generated by (1.11) converges weakly but not strongly. This brings us to a natural question of how to modify the proximal point algorithm so that strongly convergent sequence is guaranteed. The *Tikhonov method* which generates a sequence $\{\tilde{x}_n\}$ by the rule

$$\tilde{x}_n = J^A_{\mu_n} u \quad \forall n \in \mathbb{N}, \tag{1.13}$$

where $u \in \mathcal{H}$ and $\mu_n > 0$ such that $\mu_n \to \infty$ is studied by several authors (see, e.g., Takahashi [17] and Wong et al. [18]) to answer the above question.

In [19], Lehdili and Moudafi combined the technique of the proximal map and the Tikhonov regularization to introduce the *prox-Tikhonov method* which generates the sequence $\{x_n\}$ by the algorithm

$$x_{n+1} = J_{\lambda_n}^{A_n} x_n \quad \forall n \in \mathbb{N}, \tag{1.14}$$

where $A_n = \mu_n I + A$, $\mu_n > 0$ is viewed as a Tikhonov regularization of $A$. Note that $A_n$ is strongly monotone, that is, $\langle x - x', y - y' \rangle \geq \mu_n \|x - x'\|^2$ for all $(x, y), (x', y') \in G(A_n)$, where $G(A_n)$ is graph of $A_n$.

Using the technique of variational distance, Lehdili and Moudafi [19] were able to prove strong convergence of the algorithm (1.14) for solving Problem $(P)$ when $A$ is maximal monotone operator on $\mathscr{H}$ under certain conditions imposed upon the sequences $\{\lambda_n\}$ and $\{\mu_n\}$.

It should be also noted that $A_n$ is now a maximal monotone operator, hence $\{J_{\lambda_n}^{A_n}\}$ is a sequence of nonexpansive mappings.

The main objective of this article is to solve the proposed Problem 1.1. To achieve this goal, we present an existence theorem for Problem 1.1. Motivated by Yamada's hybrid steepest-descent and Lehdili and Moudafi's algorithms (1.6) and (1.14), we also present an iterative algorithm and investigate the convergence theory of the proposed algorithm for solving Problem 1.1. The outline of this paper is as follows. In Section 2, we present some theoretical tools which are needed in the sequel. In Section 3, we present (Theorem 3.3) the existence and uniqueness of solution of Problem 1.1 in a case when $T : C \to C$ is not necessarily nonexpansive mapping. In Section 4, we propose an iterative algorithm (Algorithm 4.1), as a generalization of Yamada's hybrid steepest-descent and Lehdili and Moudafi's algorithms (1.6) and (1.14), for computing to a unique solution of the variational inequality VIP$(\mathcal{F}, C)$ over $\bigcap_{n\in\mathbb{N}} F[T_n]$ in the framework of Banach space. In Section 5, we apply our result to the problem of finding a common fixed point of a countable family of nonexpansive mappings and the solution of Problem $(P)$. Our strong convergence theorems extend and improve corresponding results of Ceng et al. [20]; Ceng et al. [21]; Lehdili and Moudafi [19]; Sahu [9]; and Yamada [6].

## 2. Preliminaries and Notations

### *2.1. Derivatives of Functionals*

Let $X$ be a real Banach space. In the sequel, we always use $S_X$ to denote the unit sphere $S_X = \{x \in X : \|x\| = 1\}$. Then X is said to be

(i) *strictly convex* if $x, y \in S_X$ with $x \neq y \Rightarrow \|(1-\lambda)x + \lambda y\| < 1$ for all $\lambda \in (0,1)$;

(ii) *smooth* if the limit $\lim_{t\to 0}((\|x + ty\| - \|x\|)/t)$ exists for each $x$ and $y$ in $S_X$. In this case, the norm of $X$ is said to be *Gâteaux differentiable*.

The norm of $X$ is said to be *uniformly Gâteaux differentiable* if for each $y \in S_X$, this limit is attained uniformly for $x \in S_X$.

It is well known that every uniformly smooth space (e.g., $L_p$ space, $1 < p < \infty$) has a uniformly Gâteaux-differentiable norm (see, e.g., [10]).

Let $U$ be an open subset of a real Hilbert space $\mathscr{H}$. Then, a function $\Theta : \mathscr{H} \to \mathbb{R} \cup \{\infty\}$ is called Gâteaux differentiable [22, page 135] on $U$ if for each $u \in U$, there exists $a(u) \in \mathscr{H}$ such that

$$\lim_{t \to 0} \frac{\Theta(u + th) - \Theta(u)}{t} = \langle a(u), h \rangle \quad \forall h \in \mathscr{H}. \tag{2.1}$$

Then, $\Theta' : U \to \mathscr{H} : u \to a(u)$ is called the *Gâteaux derivative* of $\Theta$ on $U$.

*Example 2.1* (see [6]). Suppose that $h \in \mathscr{H}$, $\beta \in \mathbb{R}$ and $Q : \mathscr{H} \to \mathscr{H}$ is a bounded linear, self-adjoint, that is, $\langle Q(x), y \rangle = \langle x, Q(y) \rangle$ for all $x, y \in \mathscr{H}$, and strongly positive mapping, that is, $\langle Q(x), x \rangle \geq \alpha \|x\|^2$ for all $x \in \mathscr{H}$ and for some $\alpha > 0$. Define the quadratic function $\Theta : \mathscr{H} \to \mathbb{R}$ by

$$\Theta(x) := \frac{1}{2}\langle Q(x), x \rangle - \langle h, x \rangle + \beta \quad \forall x \in \mathscr{H}. \tag{2.2}$$

Then, the Gâteaux derivative $\Theta'(x) = Q(x) - \beta$ is $\|Q\|$-Lipschitzian and $\alpha$-strongly monotone on $\mathscr{H}$.

### *2.2. Lipschitzian Type Mappings*

Let $C$ be a nonempty subset of a real Banach space $X$ and let $S_1, S_2 : C \to X$ be two mappings. We denote $\mathcal{B}(C)$, the collection of all bounded subsets of $C$. The deviation between $S_1$ and $S_2$ on $B \in \mathcal{B}(C)$, denoted by $\mathfrak{D}_B(S_1, S_2)$, is defined by

$$\mathfrak{D}_B(S_1, S_2) = \sup\{\|S_1 x - S_2 x\| : x \in B\}. \tag{2.3}$$

A mapping $T : C \to X$ is said to be

(1) *L-Lipschitzian* if there exists a constant $L \in (0, \infty)$ such that $\|Tx - Ty\| \leq L\|x - y\|$ for all $x, y \in C$;

(2) *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$;

(3) *strongly pseudocontractive* if for each $x, y \in C$, there exist a constant $k \in (0, 1)$ and $j(x - y) \in J(x - y)$ satisfying

$$\langle Tx - Ty, j(x - y) \rangle \leq k\|x - y\|^2, \tag{2.4}$$

(4) $\lambda$-strictly pseudocontractive (see [23]) if for each $x, y \in C$, there exist a constant $\lambda > 0$ and $j(x - y) \in J(x - y)$ such that

$$\langle Tx - Ty, j(x - y) \rangle \leq \|x - y\|^2 - \lambda\|x - y - (Tx - Ty)\|^2. \tag{2.5}$$

The inequality (2.5) can be restated as

$$\langle x - y - (Tx - Ty), j(x - y) \rangle \geq \lambda\|x - y - (Tx - Ty)\|^2. \tag{2.6}$$

In Hilbert spaces, (2.5) (and so (2.6)) is equivalent to the following inequality

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|x - y - (Tx - Ty)\|^2, \tag{2.7}$$

where $k = 1 - 2\lambda$. From (2.6), one can prove that if $T$ is $\lambda$-strict pseudocontractive, then $T$ is Lipschitz continuous with the Lipschitz constant $L = (1 + \lambda)/\lambda$ (see, Proposition 3.1). Throughout the paper, we assume that $L_{\lambda,\delta} := \sqrt{(1 - \delta)/\lambda}$.

*Fact 2.2* (see [10, Corollary 5.7.15]). Let $C$ be a nonempty closed convex subset of a Banach space $X$ and $T : C \to C$ a continuous strongly pseudocontractive mapping. Then $T$ has a unique fixed point in $C$.

Fix a sequence $\{a_n\}$ in $[0, \infty)$ with $a_n \to 0$ and let $\{T_n\}$ be a sequence of mappings from $C$ into $X$. Then $\{T_n\}$ is called a sequence of asymptotically nonexpansive mappings if there exists a sequence $\{k_n\}$ in $[1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T_n x - T_n y\| \leq k_n\|x - y\| \quad \forall x, y \in C,\ n \in \mathbb{N}. \tag{2.8}$$

Motivated by the notion of nearly nonexpansive mappings (see [10, 24]), we say $\{T_n\}$ is a sequence of nearly nonexpansive mappings if

$$\|T_n x - T_n y\| \leq \|x - y\| + a_n \quad \forall x, y \in C,\ n \in \mathbb{N}. \tag{2.9}$$

*Remark 2.3.* If $\{T_n\}$ is a sequence of asymptotically nonexpansive mappings with bounded domain, then $\{T_n\}$ is a sequence of nearly nonexpansive mappings. To see this, let $\{T_n\}$ be a sequence of asymptotically nonexpansive mappings with sequence $\{k_n\}$ defined on a bounded set $C$ with diameter $\operatorname{diam}(C)$. Fix $a_n := (k_n - 1)\operatorname{diam}(C)$. Then,

$$\|T_n x - T_n y\| \leq \|x - y\| + (k_n - 1)\|x - y\| \leq \|x - y\| + a_n \tag{2.10}$$

for all $x, y \in C$ and $n \in \mathbb{N}$.

We prove the following proposition.

*Proposition 2.4.* Let $C$ be a closed bounded set of a Banach space $X$ and $\{T_n\}$ a sequence of nearly nonexpansive self-mappings of $C$ with sequence $\{a_n\}$ such that $\sum_{n=1}^{\infty} \mathfrak{D}_C(T_n, T_{n+1}) < \infty$. Then, for each $x \in C$, $\{T_n x\}$ converges strongly to some point of $C$. Moreover, if $T$ is a mapping of $C$ into itself defined by $Tz = \lim_{n\to\infty} T_n z$ for all $z \in C$, then $T$ is nonexpansive and $\lim_{n\to\infty} \mathfrak{D}_C(T_n, T) = 0$.

*Proof.* The assumption $\sum_{n=1}^{\infty} \mathfrak{D}_C(T_n, T_{n+1}) < \infty$ implies that $\sum_{n=1}^{\infty} \|T_n x - T_{n+1} x\| < \infty$ for all $z \in C$. Hence $\{T_n z\}$ is a Cauchy sequence for each $z \in C$. Hence, for $x \in C$, $\{T_n x\}$ converges strongly to some point in $C$. Let $T$ be a mapping of $C$ into itself defined by $Tz = \lim_{n\to\infty} T_n z$

for all $z \in C$. It is easy to see that $T$ is nonexpansive. For $z \in C$ and $m, n \in \mathbb{N}$ with $m > n$, we have

$$\begin{aligned}
\|T_n x - T_m x\| &\le \sum_{k=n}^{m-1} \|T_k x - T_{k+1} x\| \\
&\le \sum_{k=n}^{m-1} \mathfrak{D}_C(T_k, T_{k+1}) \\
&\le \sum_{k=n}^{\infty} \mathfrak{D}_C(T_k, T_{k+1}).
\end{aligned} \tag{2.11}$$

Then

$$\|T_n x - Tx\| = \lim_{m\to\infty} \|T_n x - T_m x\| \le \sum_{k=n}^{\infty} \mathfrak{D}_C(T_k, T_{k+1}) \quad \forall x \in C,\ n \in \mathbb{N}, \tag{2.12}$$

which implies that

$$\mathfrak{D}_C(T_n, T) \le \sum_{k=n}^{\infty} \mathfrak{D}_C(T_k, T_{k+1}) \quad \forall n \in \mathbb{N}. \tag{2.13}$$

Therefore, $\lim_{n\to\infty} \mathfrak{D}_C(T_n, T) = 0$. □

### *2.3. Nonexpansive Mappings and Fixed Points*

A closed convex subset $C$ of a Banach space $X$ is said to have the fixed-point property for nonexpansive self-mappings if every nonexpansive mapping of a nonempty closed convex bounded subset $M$ of $C$ into itself has a fixed point in $M$.

A closed convex subset $C$ of a Banach space $X$ is said to have normal structure if for each closed convex bounded subset of $D$ of $C$ which contains at least two points, there exists an element $x \in D$ which is not a diametral point of $D$. It is well known that a closed convex subset of a uniformly smooth Banach space has normal structure, see [10] for more details.

The following result was proved by Kirk [25].

*Fact 2.5* (Kirk [25]). Let X be a reflexive Banach space and let $C$ be a nonempty closed convex bounded subset of $X$ which has normal structure. Let $T$ be a nonexpansive mapping of $C$ into itself. Then $F[T]$ is nonempty.

A subset $C$ of a Banach space $X$ is called a *retract* of $X$ if there exists a continuous mapping $P$ from $X$ onto $C$ such that $Px = x$ for all $x$ in $C$. We call such $P$ a *retraction* of $X$ onto $C$. It follows that if a mapping $P$ is a retraction, then $Py = y$ for all $y$ in the range of $P$. A retraction $P$ is said to be *sunny* if $P(Px + t(x - Px)) = Px$ for each $x$ in $X$ and $t \ge 0$. If a sunny retraction $P$ is also nonexpansive, then $C$ is said to be a *sunny nonexpansive retract* of $X$.

Let $C$ be a nonempty subset of a Banach space $X$ and let $x \in X$. An element $y_0 \in C$ is said to be a *best approximation* to $x$ if $\|x - y_0\| = d(x, C)$, where $d(x, C) = \inf_{y \in C}\|x - y\|$. The set of all best approximations from $x$ to $C$ is denoted by

$$P_C(x) = \{y \in C : \|x - y\| = d(x, C)\}. \tag{2.14}$$

This defines a mapping $P_C$ from $X$ into $2^C$ and is called the *nearest point projection mapping* (*metric projection mapping*) onto $C$. It is well known that if $C$ is a nonempty closed convex subset of a real Hilbert space $\mathscr{H}$, then the nearest point projection $P_C$ from $\mathscr{H}$ onto $C$ is the unique sunny nonexpansive retraction of $\mathscr{H}$ onto $C$. It is also known that $P_C x \in C$ and

$$\langle x - P_C x, P_C x - y\rangle \geq 0 \quad \forall x \in \mathscr{H},\ y \in C. \tag{2.15}$$

Let $\mathcal{F}$ be a monotone mapping of $\mathscr{H}$ into $\mathscr{H}$ over $C \subseteq \mathscr{H}$. In the context of the variational inequality problem, the characterization of projection (2.15) implies

$$x^* \in \mathrm{VIP}(\mathcal{F}, C) \Longleftrightarrow x^* = P_C(x^* - \mu A x^*) \quad \forall \mu > 0. \tag{2.16}$$

We know the following fact concerning nonexpansive retraction.

*Fact 2.6* (Goebel and Reich [26, Lemma 13.1]). Let $C$ be a convex subset of a real smooth Banach space $X$, $D$ a nonempty subset of $C$, and $P$ a retraction from $C$ onto $D$. Then the following are equivalent:

(a) $P$ is a sunny and nonexpansive.

(b) $\langle x - Px, J(z - Px)\rangle \leq 0$ for all $x \in C$, $z \in D$.

(c) $\langle x - y, J(Px - Py)\rangle \geq \|Px - Py\|^2$ for all $x, y \in C$.

*Fact 2.7* (Wong et al. [18, Proposition 6.1]). Let $C$ be a nonempty closed convex subset of a strictly convex Banach space X and let $\lambda_i > 0$ ($i = 1, 2, \ldots, N$) such that $\sum_{i=1}^{N} \lambda_i = 1$. Let $T_1, T_2, \ldots, T_N : C \to C$ be nonexpansive mappings with $\bigcap_{i=1}^{N} F(T_i) \neq \emptyset$ and let $T = \sum_{i=1}^{N} \lambda_i T_i$. Then $T$ is nonexpansive from C into itself and $F(T) = \bigcap_{i=1}^{N} F(T_i)$.

*Fact 2.8* (Bruck [27]). Let $C$ be a nonempty closed convex subset of a strictly convex Banach space $X$. Let $\{S_k\}$ be a sequence nonexpansive mappings of $C$ into itself with $\bigcap_{k=1}^{\infty} F[S_k] \neq \emptyset$ and $\{\beta_k\}$ sequence of positive real numbers such that $\sum_{k=1}^{\infty} \beta_k = 1$. Then the mapping $T = \sum_{k=1}^{\infty} \beta_k S_k$ is well defined on $C$ and $F[T] = \bigcap_{k=1}^{\infty} F[S_k]$.

### 2.4. Accretive Operators and Zero

Let $X$ be a real Banach space $X$. For an operator $A : X \to 2^X$, we define its domain, range, and graph as follows:

$$\begin{gathered} D(A) = \{x \in X : Ax \neq \emptyset\}, \qquad R(A) = \cup\{Az : z \in D(A)\}, \\ G(T) = \{(x, y) \in X \times X : x \in D(A), y \in Ax\}, \end{gathered} \tag{2.17}$$

respectively. Thus, we write $A : X \to 2^X$ as follows: $A \subset X \times X$. The inverse $A^{-1}$ of $A$ is defined by

$$x \in A^{-1}y \Longleftrightarrow y \in Ax. \tag{2.18}$$

The operator $A$ is said to be *accretive* if, for each $x_i \in D(A)$ and $y_i \in Ax_i$ $(i = 1, 2)$, there is $j \in J(x_1 - x_2)$ such that $\langle y_1 - y_2, j\rangle \geq 0$. An accretive operator $A$ is said to be *maximal accretive* if there is no proper accretive extension of $A$ and *m-accretive* if $R(I + A) = X$ (it follows that $R(I + rA) = X$ for all $r > 0$). If $A$ is $m$-accretive, then it is maximal accretive (see Fact 2.10), but the converse is not true in general. If $A$ is accretive, then we can define, for each $\lambda > 0$, a nonexpansive single-valued mapping $J_\lambda : R(1+\lambda A) \to D(A)$ by $J_\lambda = (I+\lambda A)^{-1}$. It is called the *resolvent* of $A$. An accretive operator $A$ defined on $X$ is said to satisfy the *range condition* if $\overline{D(A)} \subset R(1 + \lambda A)$ for all $\lambda > 0$, where $\overline{D(A)}$ denotes the closure of the domain of $A$. It is well known that for an accretive operator $A$ which satisfies the range condition, $A^{-1}(0) = F(J_\lambda^A)$ for all $\lambda > 0$. We also define the *Yosida approximation* $A_r$ by $A_r = (I - J_r^A)/r$. We know that $A_r x \in AJ_r^A x$ for all $x \in R(I + rA)$ and $\|A_r x\| \leq |Ax| = \inf\{\|y\| : y \in Ax\}$ for all $x \in D(A) \cap R(I + rA)$. We also know the following [28]: for each $\lambda, \mu > 0$ and $x \in R(I + \lambda A) \cap R(I + \mu A)$, it holds that

$$\|J_\lambda x - J_\mu x\| \leq \frac{|\lambda - \mu|}{\lambda}\|x - J_\lambda x\|. \tag{2.19}$$

Let $f$ be a continuous linear functional on $\ell_\infty$. We use $f_n(x_{n+m})$ to denote

$$f(x_{m+1}, x_{m+2}, x_{m+3}, \ldots, x_{m+n}, \ldots), \tag{2.20}$$

for $m = 0, 1, 2, \ldots$. A continuous linear functional $j$ on $l_\infty$ is called a *Banach limit* if $\|j\|_* = j(1) = 1$ and $j_n(x_n) = j_n(x_{n+1})$ for each $x = (x_1, x_2, \ldots)$ in $l_\infty$.

Fix any Banach limit and denote it by LIM. Note that $\|\text{LIM}\|_* = 1$,

$$\begin{aligned} \liminf_{n\to\infty} t_n \leq \text{LIM}_n t_n \leq \limsup_{n\to\infty} t_n, \\ \text{LIM}_n t_n = \text{LIM}_n t_{n+1}, \quad \forall (t_n) \in l_\infty. \end{aligned} \tag{2.21}$$

The following facts will be needed in the sequel for the proof of our main results.

*Fact 2.9* (Ha and Jung [29, Lemma 1]). Let $X$ be a Banach space with a uniformly Gâteaux-differentiable norm, $C$ a nonempty closed convex subset of $X$, and $\{x_n\}$ a bounded sequence in $X$. Let LIM be a Banach limit and $y \in C$ such that $\text{LIM}_n\|y_n - y\|^2 = \inf_{x\in C}\text{LIM}_n\|y_n - x\|^2$. Then $\text{LIM}_n\langle x - y, J(x_n - y)\rangle \leq 0$ for all $x \in C$.

*Fact 2.10* (Cioranescu [30]). Let $X$ be a Banach space and let $A : X \to 2^X$ be an $m$-accretive operator. Then $A$ is maximal accretive. If $\mathscr{H}$ is a Hilbert space, then $A : \mathscr{H} \to 2^{\mathscr{H}}$ is maximal accretive if and only if it is $m$-accretive.

## 3. Existence and Uniqueness of Solutions of VIP($\mathcal{F}, C$)

In this section, we deal with the existence and uniqueness of the solution of Problem 1.1 in a case where $T : C \to C$ is given as such a pseudocontractive mapping.

The following propositions will be used frequently throughout the paper.

*Proposition 3.1.* Let $C$ be a nonempty subset of a real smooth Banach space $X$ and $\mathcal{F} : X \to X$ an operator over $C$. Then

(a) if $\mathcal{F}$ is $\lambda$-strictly pseudocontractive, then $\mathcal{F}$ is Lipschitzian with constant $1 + 1/\lambda$;

(b) if $\mathcal{F}$ is both $\delta$-strongly accretive and $\lambda$-strictly pseudocontractive over $C$ with $\lambda+\delta > 1$, then $I - \mathcal{F}$ is a contraction with Lipschitz constant $L_{\lambda,\delta}$;

(c) if $\tau \in (0,1)$ is a fixed number and $\mathcal{F}$ is both $\delta$-strongly accretive and $\lambda$-strictly pseudocontractive over $C$ with $\lambda + \delta > 1$ and $R(I - \tau\mathcal{F}) \subseteq C$, then $I - \tau\mathcal{F} : C \to C$ is a contraction mapping with Lipschitz constant $1 - (1 - L_{\lambda,\delta})\tau$.

*Proof.* (a) Let $x, y \in C$. From (2.6), we have

$$\begin{aligned} \lambda\|x - y - (\mathcal{F}x - \mathcal{F}y)\|^2 &\le \langle x - y - (\mathcal{F}x - \mathcal{F}y), J(x - y)\rangle \\ &\le \|x - y - (\mathcal{F}x - \mathcal{F}y)\|\|x - y\|, \end{aligned} \tag{3.1}$$

which gives us

$$\|x - y - (\mathcal{F}x - \mathcal{F}y)\| \le \frac{1}{\lambda}\|x - y\|. \tag{3.2}$$

Thus,

$$\|\mathcal{F}x - \mathcal{F}y\| \le \|x - y\| + \|x - y - (\mathcal{F}x - \mathcal{F}y)\| \le \left(1 + \frac{1}{\lambda}\right)\|x - y\|. \tag{3.3}$$

Hence, $\mathcal{F}$ is Lipschitzian with constant $1 + 1/\lambda$.

(b) Let $x, y \in C$. Further, from (2.6), we have

$$\begin{aligned} \lambda\|x - y - (\mathcal{F}x - \mathcal{F}y)\|^2 &\le \|x - y\|^2 - \langle \mathcal{F}x - \mathcal{F}y, J(x - y)\rangle \\ &\le (1 - \delta)\|x - y\|^2. \end{aligned} \tag{3.4}$$

Observe that

$$\lambda + \delta > 1 \Longleftrightarrow L_{\lambda,\delta} \in (0,1). \tag{3.5}$$

Hence

$$\|x - y - (\mathcal{F}x - \mathcal{F}y)\| \le \sqrt{\frac{1 - \delta}{\lambda}}\|x - y\| = L_{\lambda,\delta}\|x - y\|. \tag{3.6}$$

(c) Let $x, y \in C$ and fixed a number $\tau \in (0,1)$. Assume that $\lambda+\delta > 1$ and $R(I-\tau\mathcal{F}) \subseteq C$. Since $I - \mathcal{F}$ is a contraction with Lipschitz constant $L_{\lambda,\delta}$, we have

$$\begin{aligned}
\|(I-\tau\mathcal{F})x-(I-\tau\mathcal{F})y\| &\leq \|x-y-\tau(\mathcal{F}x-\mathcal{F}y)\| \\
&= \|(1-\tau)(x-y)+\tau[(I-\mathcal{F})x-(I-\mathcal{F})y]\| \\
&\leq (1-\tau)\|x-y\|+\tau\|(I-\mathcal{F})x-(I-\mathcal{F})y\| \\
&\leq (1-(1-L_{\lambda,\delta})\tau)\|x-y\|.
\end{aligned} \tag{3.7}$$

Therefore, $I-\tau\mathcal{F} : C \to C$ is a contraction mapping with Lipschitz constant $1-(1-L_{\lambda,\delta})\tau$. □

*Proposition 3.2.* Let $C$ be a nonempty closed convex subset of a real smooth Banach space $X$. Let $T : C \to C$ be a continuous pseudocontractive mapping and let $\mathcal{F} : X \to X$ be both $\delta$-strongly accretive and $\lambda$-strictly pseudocontractive over $C$ with $\lambda+\delta > 1$ and $R(I-\tau\mathcal{F}) \subseteq C$ for each $\tau \in (0,1)$. Assume that $C$ has the fixed-point property for nonexpansive self-mappings. Then one has the following.

(a) For each $t \in (0,1)$, one chooses a number $\mu_t \in (0,1)$ arbitrarily, there exists a unique point $v_t$ of $C$ defined by

$$v_t = (1-t)Tv_t + t(I-\mu_t\mathcal{F})v_t. \tag{3.8}$$

(b) If $F[T] \neq \emptyset$ and $v_t$ is a unique point of $C$ defined by (3.8), then

(i) $\{v_t\}$ is bounded,
(ii) $\langle \mathcal{F}(v_t), J(v_t-v)\rangle \leq 0$ for all $v \in F[T]$.

*Proof.* (a) For each $t \in (0,1)$, we choose a number $\mu_t \in (0,1)$ arbitrarily and then the mapping $G_t : C \to C$ defined by

$$G_t v = (1-t)Tv + t(I-\mu_t\mathcal{F})v \quad \forall v \in C \tag{3.9}$$

is continuous and strongly pseudocontractive with constant $1-(1-L_{\lambda,\delta})t\mu_t$. Indeed, for all $x, y \in C$, by Proposition 3.1 we have

$$\begin{aligned}
\langle G_t x - G_t y, J(x-y)\rangle &= (1-t)\langle Tx-Ty, J(x-y)\rangle + t\langle (I-\mu_t\mathcal{F})x-(I-\mu_t\mathcal{F})y, J(x-y)\rangle \\
&\leq (1-t)\|x-y\|^2 + t\|(I-\mu_t\mathcal{F})x-(I-\mu_t\mathcal{F})y\|\|x-y\| \\
&\leq [1-(1-L_{\lambda,\delta})t\mu_t]\|x-y\|^2.
\end{aligned} \tag{3.10}$$

By Fact 2.2, there exists a unique fixed point $v_t$ of $G_t$ in $C$ defined by

$$v_t = (1-t)Tv_t + t(I-\mu_t\mathcal{F})v_t. \tag{3.11}$$

(b) Assume that $F[T] \neq \emptyset$. Take any $p \in F[T]$. Using (3.8), we have

$$\begin{aligned}
&\langle v_t - [(1-t)p + t(I - \mu_t \mathcal{F})v_t], J(v_t - p)\rangle \\
&\quad = \langle (1-t)Tv_t + t(I-\mu_t\mathcal{F})v_t - [(1-t)p + t(I-\mu_t\mathcal{F})v_t], J(v_t - p)\rangle \\
&\quad = (1-t)\langle Tv_t - p, J(v_t - p)\rangle \\
&\quad \leq (1-t)\|v_t - p\|^2.
\end{aligned} \tag{3.12}$$

Observe that

$$\begin{aligned}
\langle v_t - [(1-t)p + t(I-\mu_t\mathcal{F})v_t], J(v_t - p)\rangle &= \langle (1-t)(v_t - p) + t[v_t - (I-\mu_t\mathcal{F})v_t], J(v_t - p)\rangle \\
&= (1-t)\|v_t - p\|^2 + t\mu_t\langle \mathcal{F}(v_t), J(v_t - p)\rangle.
\end{aligned} \tag{3.13}$$

Thus,

$$\begin{aligned}
(1-t)\|v_t - p\|^2 + t\mu_t\langle \mathcal{F}(v_t), J(v_t - p)\rangle &= \langle v_t - [(1-t)p + t(I-\mu_t\mathcal{F})v_t], J(v_t - p)\rangle \\
&\leq (1-t)\|v_t - p\|^2,
\end{aligned} \tag{3.14}$$

which implies that

$$\langle \mathcal{F}(v_t), J(v_t - p)\rangle \leq 0. \tag{3.15}$$

Since $\mathcal{F}$ is $\delta$-strongly accretive, we have

$$\begin{aligned}
\delta\|v_t - p\|^2 &\leq \langle \mathcal{F}(v_t) - \mathcal{F}(p), J(v_t - p)\rangle \\
&= \langle \mathcal{F}(v_t), J(v_t - p)\rangle - \langle \mathcal{F}(p), J(v_t - p)\rangle \\
&\leq \langle -\mathcal{F}(p), J(v_t - p)\rangle \\
&\leq \|\mathcal{F}(p)\|\|v_t - p\|,
\end{aligned} \tag{3.16}$$

which implies that

$$\delta\|v_t - p\| \leq \|\mathcal{F}(p)\|. \tag{3.17}$$

It shows that $\{v_t\}$ is bounded. □

Now, we are ready to present the main result of this section.

**Theorem 3.3.** *Let $C$ be a nonempty closed convex subset of a real reflexive Banach space $X$ with a uniformly Gâteaux-differentiable norm. Let $T : C \to C$ be a continuous pseudocontractive mapping* *with* $F[T] \neq \emptyset$ *and let* $\mathcal{F} : X \to X$ *be both* $\delta$*-strongly accretive and* $\lambda$*-strictly pseudocontractive over* $C$ *with* $\lambda + \delta > 1$ *and* $R(I - \tau\mathcal{F}) \subseteq C$ *for each* $\tau \in (0,1)$*. Assume that* $C$ *has the fixed-point property for nonexpansive self-mappings. Then* $\{v_t\}$ *converges strongly as* $t \to 0^+$ *to a unique solution* $x^*$ *of* $VIP(\mathcal{F}, C)$ *over* $F[T]$*.*

*Proof.* By Proposition 3.2, $\{v_t : t \in (0,1)\}$ is bounded. Since $\mathcal{F}$ is a Lipschitzian mapping, it follows that $\{\mathcal{F}v_t : t \in (0,1)\}$ is bounded. From (3.8), we have

$$Tv_t = v_t + \frac{t\mu_t}{1-t}\mathcal{F}(v_t) \quad \forall t \in (0,1). \tag{3.18}$$

and hence

$$\|Tv_t\| \leq \|v_t\| + \frac{t\mu_t}{1-t}\|\mathcal{F}(v_t)\| \leq \|v_t\| + \frac{t}{1-t}\|\mathcal{F}(v_t)\| \quad \forall t \in (0,1). \tag{3.19}$$

Noticing that $\lim_{t\to 0^+}(t/(1-t)) = 0$, there exists $t_0 \in (0,1)$ that $\{Tv_t : t \in (0,t_0]\}$ is bounded. This implies from (3.18) that $\|v_t - Tv_t\| \to 0$ as $t \to 0^+$. The key is to show that $\{v_t : t \in (0,t_0]\}$ is relatively compact as $t \to 0^+$. We may choose a sequence $\{t_n\}$ in $(0,t_0]$ such that $\lim_{n\to\infty} t_n = 0$. Set $v_n := v_{t_n}$. We will show that $\{v_n\}$ contains a subsequence converging strongly to an element of $C$. Define the function $\varphi : C \to \mathbb{R}^+$ by $\varphi(x) := \mathrm{LIM}_n\|v_n - x\|^2$, $x \in C$ and let

$$M := \left\{ y \in C : \varphi(y) = \inf_{x\in C}\varphi(x) \right\}. \tag{3.20}$$

Since $X$ is reflexive, $\varphi(x) \to \infty$ as $\|x\| \to \infty$, and $\varphi$ is a continuous convex function. By Barbu and Precupanu [31, Theorem 1.2, page 79], we have that the set $M$ is nonempty. By Takahashi [28], we see that $M$ is also closed, convex, and bounded.

From [32, Theorem 6], we know that the mapping $2I - T$ has a nonexpansive inverse, denoted by $g$, which maps $C$ into itself with $F[T] = F[g]$. Note that $\lim_{n\to\infty}\|v_n - Tv_n\| = 0$ implies that $\lim_{n\to\infty}\|v_n - gv_n\| = 0$. Moreover, $M$ is invariant under $g$, that is, $R(g) \subseteq M$. In fact, for each $y \in M$, we have

$$\varphi(gy) = \mathrm{LIM}_n\|v_n - gy\|^2 \leq \mathrm{LIM}_n\|gv_n - gy\|^2 \leq \mathrm{LIM}_n\|v_n - y\|^2 = \varphi(y), \tag{3.21}$$

and hence $gy \in M$. By assumption, we have $M \cap F[g] \neq \emptyset$. Let $y^* \in M \cap F[g]$. By Fact 2.9, we have

$$\mathrm{LIM}_n\langle z - y^*, J(v_n - y^*)\rangle \leq 0 \quad \forall z \in C. \tag{3.22}$$

In particular, by taking $z = y^* - \mathcal{F}(y^*)$, we have

$$\mathrm{LIM}_n\langle -\mathcal{F}(y^*), J(v_n - y^*)\rangle \leq 0. \tag{3.23}$$

Using (3.16) and (3.23), we have

$$\delta \mathrm{LIM}_n \|v_n - y^*\|^2 \le \mathrm{LIM}_n \langle -\mathcal{F}(y^*), J(v_n - y^*) \rangle \le 0. \tag{3.24}$$

Thus, there exists a subsequence $\{v_{n_i}\}$ of $\{v_n\}$ such that $v_{n_i} \to y^*$.

Assume that $\{v_{n_j}\}$ is another subsequence of $\{v_n\}$ such that $v_{n_j} \to z^* \ne y^*$. It is easy to see that $z^* \in F[T]$. Since $v_{n_i} \to y^*$ and $J$ is norm to weak* uniform continuous, we obtain from Proposition 3.2(b) that

$$\langle \mathcal{F}(y^*), J(y^* - z^*) \rangle \le 0. \tag{3.25}$$

Similarly, we have

$$\langle \mathcal{F}(z^*), J(z^* - y^*) \rangle \le 0. \tag{3.26}$$

Adding the above two inequalities yields

$$\langle \mathcal{F}(y^*) - \mathcal{F}(z^*), J(y^* - z^*) \rangle \le 0, \tag{3.27}$$

which implies that

$$\delta \|y^* - z^*\|^2 \le \langle \mathcal{F}(y^*) - \mathcal{F}(z^*), J(y^* - z^*) \rangle \le 0, \tag{3.28}$$

a contradiction. Hence, $\{v_{t_n}\}$ converges strongly to $y^*$.

To see that the entire net $\{v_t\}$ actually converges strongly as $t \to 0^+$, we assume that there is another sequence $\{s_n\}$ with $s_n \in (0, t_0]$ and $s_n \to 0$ as $n \to \infty$ such that $v_{s_n} \to z$ as $n \to \infty$, then, $z \in F[T]$. From Proposition 3.2(b), we conclude that $z = y^*$. Therefore, $\{v_t\}$ converges strongly as $t \to 0^+$ to $y^* \in F[T]$. Noticing that $y^* \in F[T]$ is a solution of $\mathrm{VIP}(\mathcal{F}, C)$ over $F[T]$. Indeed, from Proposition 3.2(b), we have

$$\langle \mathcal{F}(y^*), J(y^* - v) \rangle \le 0 \quad \forall v \in F[T]. \tag{3.29}$$

One can easily see that $y^*$ is the unique solution of $\mathrm{VIP}(\mathcal{F}, C)$ over $F[T]$. □

As the domain of operators considered in Theorem 3.3 is not necessarily the entire space $X$, Theorem 3.3 is more general in nature. It improves Ceng et al. [20, Proposition 4.3] significantly and provides solutions of Problem 1.1.

We now replace the fixed-point property assumption, mentioned in Theorem 3.3 by imposing strict convexity on the underlying space.

**Theorem 3.4.** *Let $C$ be a nonempty closed convex subset of a real strictly convex reflexive Banach space $X$ with a uniformly Gâteaux-differentiable norm. Let $T : C \to C$ be a continuous pseudocontractive mapping with $F[T] \ne \emptyset$ and let $\mathcal{F} : X \to X$ be both $\delta$-strongly accretive and $\lambda$-strictly pseudocontractive over $C$ with $\lambda + \delta > 1$ and $R(I - \tau\mathcal{F}) \subseteq C$ for each $\tau \in (0, 1)$. Then $\{v_t\}$ converges strongly as $t \to 0^+$ to a unique solution $x^*$ of $\mathrm{VIP}(\mathcal{F}, C)$ over $F[T]$.*

*Proof.* To be able to use the argument of the proof of Theorem 3.3, we just need to show that the set $M$ defined by (3.20) has a fixed point of $g$. Since $F[g] \neq \emptyset$, let $v \in F[g]$. Since $X$ is strictly convex, it follows from [10, Proposition 2.1.10] that the set $M_0$ defined by $M_0 = \{u \in M : \|u - v\| = \inf_{x \in M} \|x - v\|\}$ is a singleton. Let $M_0 = \{u_0\}$ for some $u_0 \in M$. Observe that

$$\|gu_0 - v\| = \|gu_0 - gv\| \leq \|u_0 - v\| = \inf_{x \in M} \|x - v\|. \tag{3.30}$$

Therefore, $gu_0 = u_0$. □

## 4. Generalized Hybrid Steepest-Descent Algorithm

Motivated by Yamada's hybrid steepest-descent and Lehdili and Moudafi's algorithms, (1.6) and (1.14), we introduce the following generalized hybrid steepest-descent algorithm for computing a unique solution $x^*$ of VIP$(\mathcal{F}, C)$ over $\bigcap_{n \in \mathbb{N}} F[T_n]$.

*Algorithm 4.1.* Let $C$ be a nonempty closed convex subset of a real smooth Banach space $X$ and let $\mathcal{F} : X \to X$ be an accretive operator over $C$ such that $R(I - \tau\mathcal{F}) \subseteq C$ for each $\tau \in (0,1)$. Assume that $\{T_n\}$ is a sequence of nearly nonexpansive mappings from $C$ into itself with sequence $\{a_n\}$ such that $\bigcap_{n \in \mathbb{N}} F[T_n] \neq \emptyset$. Starting with an arbitrary initial guess $x_1 \in C$, a sequence $\{x_n\}$ in $C$ is generated via the following iterative scheme:

$$x_{n+1} = T_n[x_n - \alpha_n \mathcal{F}(x_n)] \quad \forall n \in \mathbb{N}, \tag{4.1}$$

where $\{\alpha_n\}$ is a sequence in $(0,1]$.

We will study our Algorithm 4.1 under the conditions:

(C1) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, and either $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty$ or $\lim_{n\to\infty} |1 - \alpha_n/\alpha_{n+1}| = 0$;

(C2) either $\sum_{n=1}^{\infty} \mathfrak{D}_D(T_n, T_{n+1}) < \infty$ or $\lim_{n\to\infty} (\mathfrak{D}_D(T_n, T_{n+1})/\alpha_{n+1}) = 0$ for each $D \in \mathcal{B}(C)$;

(C3) $\lim_{n\to\infty} (a_n/\alpha_n) = 0$.

Now, we are ready to prove the main theorem for computing solution of VIP$(\mathcal{F}, C)$ over $\bigcap_{n \in \mathbb{N}} F[T_n]$ in the framework of Banach space.

**Theorem 4.2.** *Let $C$ be a nonempty closed convex subset of a reflexive Banach space $X$ with a uniformly Gâteaux-differentiable norm and $\{T_n\}$ a sequence of nearly nonexpansive mappings from $C$ into itself with sequence $\{a_n\}$ such that $\bigcap_{n \in \mathbb{N}} F[T_n] \neq \emptyset$. Let $T$ be a mapping of $C$ into itself defined by $Tz = \lim_{n\to\infty} T_n z$ for all $z \in C$ and let $\mathcal{F} : X \to X$ be both $\delta$-strongly accretive and $\lambda$-strictly pseudocontractive over $C$ with $\lambda + \delta > 1$ and $R(I - \tau\mathcal{F}) \subseteq C$ for each $\tau \in (0,1)$. Assume that $C$ has the fixed-point property for nonexpansive self-mappings. For a given $x_1 \in C$, let $\{x_n\}$ be a sequence in $C$ generated by (4.1), where $\{\alpha_n\}$ is a sequence in $(0,1]$ satisfying conditions (C1)~(C3). Then, $\{x_n\}$ converges strongly to a unique solution $x^*$ of VIP$(\mathcal{F}, C)$ over $\bigcap_{n \in \mathbb{N}} F[T_n]$.*

*Proof.* Let $T$ be a mapping of $C$ into itself defined by $Tz = \lim_{n\to\infty} T_n z$ for all $z \in C$. It is clear that $T$ is a nonexpansive mapping and $\bigcap_{n\in\mathbb{N}} F[T_n] \subseteq F[T]$. So, we have $F[T] \neq \emptyset$. For each $t \in (0,1)$, we choose a number $\mu_t \in (0,1)$ arbitrarily, let $x_t$ be a unique point of $C$ such that

$$x_t = (1-t)Tx_t + t\big(I - \mu_t \mathcal{F}\big)(x_t). \tag{4.2}$$

It follows from Theorem 3.3 that $\{x_t\}$ converges strongly as $t \to 0^+$ to a unique solution $x^*$ of $\mathrm{VIP}(\mathcal{F}, C)$ over $\bigcap_{n\in\mathbb{N}} F[T_n]$. Set $y_n := x_n - \alpha_n \mathcal{F}(x_n)$. We now proceed with the following steps.

*Step 1.* $\{x_n\}$ and $\{y_n\}$ are bounded.

Observe that

$$\begin{aligned}
\|y_n - x^*\| &\leq \|x_n - x^*\| + \alpha_n \|\mathcal{F}(x_n)\| \\
&\leq \|x_n - x^*\| + \|\mathcal{F}(x_n) - \mathcal{F}(x^*)\| + \|\mathcal{F}(x^*)\| \\
&\leq \left(2 + \frac{1}{\lambda}\right)\|x_n - x^*\| + \|\mathcal{F}(x^*)\| \quad \forall n \in \mathbb{N}.
\end{aligned} \tag{4.3}$$

Invoking (4.3), we have

$$\begin{aligned}
\|x_{n+1} - x^*\| &= \|T_n[x_n - \alpha_n \mathcal{F}(x_n)] - x^*\| \\
&\leq \|x_n - \alpha_n \mathcal{F}(x_n) - x^*\| + a_n \\
&\leq \|(I - \alpha_n \mathcal{F})x_n - (I - \alpha_n \mathcal{F})x^*\| + \alpha_n \|\mathcal{F}(x^*)\| + a_n \\
&\leq (1 - (1 - L_{\lambda,\delta})\alpha_n)\|x_n - x^*\| + \alpha_n \|\mathcal{F}(x^*)\| + a_n.
\end{aligned} \tag{4.4}$$

Note that $\lim_{n\to\infty}(a_n/\alpha_n) = 0$, so there exists a constant $K > 0$ such that

$$\frac{\alpha_n \|\mathcal{F}(x^*)\| + a_n}{\alpha_n} \leq K \quad \forall n \in \mathbb{N}. \tag{4.5}$$

By (4.4), we have

$$\begin{aligned}
\|x_{n+1} - x^*\| &\leq (1 - (1 - L_{\lambda,\delta})\alpha_n)\|x_n - x^*\| + \alpha_n K \\
&\leq \max\left\{ \|x_n - x^*\|, \frac{K}{1 - L_{\lambda,\delta}} \right\} \quad \forall n \in \mathbb{N}.
\end{aligned} \tag{4.6}$$

Hence, $\{x_n\}$ is bounded and hence, from (4.3), $\{y_n\}$ is bounded.

*Step 2.* $\|y_n - Ty_n\| \to 0$ as $n \to \infty$.

Note that the condition $\lim_{n\to\infty}\alpha_n = 0$ implies that $\|y_n - x_n\| = \alpha_n\|\mathcal{F}(x_n)\| \to 0$ as $n \to \infty$. Observe that

$$\begin{aligned}\|y_n - y_{n-1}\| &= \|(I-\alpha_n\mathcal{F})x_n - (I-\alpha_n\mathcal{F})x_{n-1} + (I-\alpha_n\mathcal{F})x_{n-1} - (I-\alpha_{n-1}\mathcal{F})x_{n-1}\| \\ &\le (1-(1-L_{\lambda,\delta})\alpha_n)\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|\mathcal{F}(x_{n-1})\| \\ &\le (1-(1-L_{\lambda,\delta})\alpha_n)\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|K_1\end{aligned} \tag{4.7}$$

for some constant $K_1 > 0$. Set $B := \{y_n\}$. Then $B \in \mathcal{B}(C)$. It follows from (4.1) that

$$\begin{aligned}\|x_{n+1} - x_n\| &= \|T_n y_n - T_{n-1}y_{n-1}\| \\ &\le \|T_n y_n - T_n y_{n-1}\| + \|T_n y_{n-1} - T_{n-1}y_{n-1}\| \\ &\le \|y_n - y_{n-1}\| + \mathfrak{D}_B(T_n, T_{n-1}) + a_n \\ &\le (1-(1-L_{\lambda,\delta})\alpha_n)\|x_n - x_{n-1}\| + \mathfrak{D}_B(T_n, T_{n-1}) + |\alpha_n - \alpha_{n-1}|K_1 + a_n.\end{aligned} \tag{4.8}$$

By conditions (C1)~(C3) and Xu [33, Lemma 2.5], we obtain that $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$. Hence,

$$\begin{aligned}\|x_{n+1} - T_n x_n\| = \|T_n y_n - T_n x_n\| \le \|y_n - x_n\| + a_n \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \\ \|x_n - T_n x_n\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - T_n x_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.\end{aligned} \tag{4.9}$$

Moreover,

$$\begin{aligned}\|y_n - T_n y_n\| &\le \|y_n - x_n\| + \|x_n - T_n x_n\| + \|T_n x_n - T_n y_n\| \\ &\le 2\|y_n - x_n\| + \|x_n - T_n x_n\| + a_n \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.\end{aligned} \tag{4.10}$$

The definition of $T$ implies that

$$\begin{aligned}\|Ty_n - y_n\| &\le \|Ty_n - T_n y_n\| + \|x_{n+1} - x_n\| + \|x_n - y_n\| \\ &\le \mathfrak{D}_B(T, T_n) + \|x_{n+1} - x_n\| + \|x_n - y_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.\end{aligned} \tag{4.11}$$

*Step 3.* $\limsup_{n\to\infty}\langle \mathcal{F}(x^*), J(x^* - y_n)\rangle \le 0$.

Since $x_t - y_n = (1-t)(Tx_t - y_n) + t[(I-\mu_t\mathcal{F})(x_t) - y_n]$, we have

$$\begin{aligned}
\|x_t - y_n\|^2 &= (1-t)\langle Tx_t - y_n, J(x_t - y_n)\rangle + t\langle (I-\mu_t\mathcal{F})(x_t) - y_n, J(x_t - y_n)\rangle \\
&\le (1-t)\langle Tx_t - Ty_n + Ty_n - y_n, J(x_t - y_n)\rangle \\
&\quad + t\Big[\langle (I-\mu_t\mathcal{F})(x_t) - x_t, J(x_t - y_n)\rangle + \|x_t - y_n\|^2\Big] \\
&\le \|x_t - y_n\|^2 + (1-t)\langle Ty_n - y_n, J(x_t - y_n)\rangle - t\mu_t\langle \mathcal{F}(x_t), J(x_t - y_n)\rangle \\
&\le \|x_t - y_n\|^2 + (1-t)\|Ty_n - y_n\|\|x_t - y_n\| - t\mu_t\langle \mathcal{F}(x_t), J(x_t - y_n)\rangle,
\end{aligned} \tag{4.12}$$

which implies that

$$\langle \mathcal{F}(x_t), J(x_t - y_n)\rangle \le \frac{1-t}{t\mu_t}\|Ty_n - y_n\|\|x_t - y_n\|. \tag{4.13}$$

Since $\{x_t\}$ and $\{y_n\}$ are bounded and $\|y_n - Ty_n\| \to 0$ as $n \to \infty$, taking the superior limit in (4.13), we obtain that

$$\limsup_{n\to\infty}\langle \mathcal{F}(x_t), J(x_t - y_n)\rangle \le 0. \tag{4.14}$$

Further, since $x_t \to x^*$ as $t \to 0^+$, the set $\{x_t - y_n\}$ is bounded, and the duality mapping $J$ is norm-to-weak$^*$ uniformly continuous on bounded subsets of X, it follows that

$$\begin{aligned}
&\left|\langle \mathcal{F}(x^*), J(y_n - x^*)\rangle - \langle \mathcal{F}(x_t), J(y_n - x_t)\rangle\right| \\
&\quad = \left|\langle \mathcal{F}(x^*), J(y_n - x^*) - J(y_n - x_t)\rangle + \langle \mathcal{F}(x^*) - \mathcal{F}(x_t), J(y_n - x_t)\rangle\right| \\
&\quad \le \left|\langle \mathcal{F}(x^*), J(y_n - x^*) - J(y_n - x_t)\rangle\right| \\
&\qquad + \|\mathcal{F}(x^*) - \mathcal{F}(x_t)\|\|y_n - x_t\| \longrightarrow 0 \quad \text{as } t \longrightarrow 0^+.
\end{aligned} \tag{4.15}$$

Let $\varepsilon > 0$. Then there exists $\delta_1 > 0$ such that

$$\big\langle \mathcal{F}(x^*), J(x^* - y_n)\big\rangle < \big\langle F(x_t), J(x_t - y_n)\big\rangle + \varepsilon \quad \forall n \in \mathbb{N},\ t \in (0, \delta_1). \tag{4.16}$$

Using (4.14), we get

$$\begin{aligned}
\limsup_{n\to\infty}\langle \mathcal{F}(x^*), J(x^* - y_n)\rangle &\le \limsup_{n\to\infty}\langle \mathcal{F}(x_t), J(x^* - y_n)\rangle + \varepsilon \\
&\le \varepsilon.
\end{aligned} \tag{4.17}$$

Since $\varepsilon$ is arbitrary, we obtain that $\limsup_{n\to\infty}\langle \mathcal{F}(x^*), J(x^* - y_n)\rangle \le 0$.

*Step 4.* $\{x_n\}$ converges strongly to $x^*$.

Observe that

$$\begin{aligned}\|y_n - x^*\|^2 &= \langle (I-\alpha_n \mathcal{F})x_n - (I-\alpha_n \mathcal{F})x^* + (I-\alpha_n \mathcal{F})x^* - x^*, J(y_n - x^*)\rangle \\ &\leq (1-(1-L_{\lambda,\delta})\alpha_n)\|x_n - x^*\|\|y_n - x^*\| - \alpha_n \langle \mathcal{F}(x^*), J(y_n - x^*)\rangle \\ &\leq \frac{(1-(1-L_{\lambda,\delta})\alpha_n)\left[\|x_n - x^*\|^2 + \|y_n - x^*\|^2\right]}{2} - \alpha_n \langle \mathcal{F}(x^*), J(y_n - x^*)\rangle.\end{aligned} \tag{4.18}$$

Hence,

$$\|y_n - x^*\|^2 \leq (1-(1-L_{\lambda,\delta})\alpha_n)\|x_n - x^*\|^2 - 2\alpha_n \langle \mathcal{F}(x^*), J(y_n - x^*)\rangle. \tag{4.19}$$

From (4.1), we have

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &= \|T_n y_n - x^*\|^2 \\ &\leq \left(\|y_n - x^*\| + a_n\right)^2 \\ &\leq \|y_n - x^*\|^2 + K_2 a_n.\end{aligned} \tag{4.20}$$

for some $K_2 \geq 0$. Thus, we obtain

$$\|x_{n+1} - x^*\|^2 \leq (1-(1-L_{\lambda,\delta})\alpha_n)\|x_n - x^*\|^2 + 2\alpha_n \langle \mathcal{F}(x^*), J(x^* - y_n)\rangle + K_2 a_n \tag{4.21}$$

for all $n \in \mathbb{N}$. Note $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty}(a_n/\alpha_n) = 0$ and $\limsup_{n\to\infty}\langle \mathcal{F}(x^*), J(x^* - y_n)\rangle \leq 0$. Therefore, we conclude from Xu [33, Lemma 2.5] that $\{x_n\}$ converges strongly to $x^*$. □

**Corollary 4.3.** *Let $C$ be a nonempty closed convex subset of a strictly convex reflexive Banach space $X$ with a uniformly Gâteaux-differentiable norm and $\{T_n\}$ a sequence of nonexpansive mappings from $C$ into itself such that $\bigcap_{n\in\mathbb{N}} F[T_n] \neq \emptyset$. Let $T$ be a mapping of $C$ into itself defined by $Tz = \lim_{n\to\infty} T_n z$ for all $z \in C$ and let $\mathcal{F} : X \to X$ be both $\delta$-strongly accretive and $\lambda$-strictly pseudocontractive over $C$ with $\lambda + \delta > 1$ and $R(I - \tau\mathcal{F}) \subseteq C$ for each $\tau \in (0,1)$. For a given $x_1 \in C$, let $\{x_n\}$ be a sequence in $C$ generated by* (4.1)*, where $\{\alpha_n\}$ is a sequence in $(0,1]$ satisfying conditions (C1)~(C2). Then, $\{x_n\}$ converges strongly to a unique solution $x^*$ of VIP$(\mathcal{F}, C)$ over $\bigcap_{n\in\mathbb{N}} F[T_n]$.*

**Theorem 4.4.** *Let $C$ be a nonempty closed convex subset of a real strictly convex reflexive Banach space $X$ with a uniformly Gâteaux-differentiable norm and $T$ a nonexpansive mapping from $C$ into itself such that $F[T] \neq \emptyset$. Let $\mathcal{F} : X \to X$ be both $\delta$-strongly accretive and $\lambda$-strictly pseudocontractive over $C$ with $\lambda + \delta > 1$ and $R(I - \tau\mathcal{F}) \subseteq C$ for each $\tau \in (0,1)$. For given $x_1 \in C$, let $\{x_n\}$ be a sequence in $C$ generated by*

$$x_{n+1} = T[x_n - \alpha_n \mathcal{F}(x_n)] \quad \forall n \in \mathbb{N}, \tag{4.22}$$

*where $\{\alpha_n\}$ is a sequence in $(0,1]$ satisfying condition (C1). Then, $\{x_n\}$ converges strongly to a unique solution $x^*$ of VIP$(\mathcal{F}, C)$ over $F[T]$.*

*Remark 4.5.* $a_n := 1/(n+1)^a$ (for all $n \in \mathbb{N}$ and $a \in (0,1]$) satisfies the condition (C1).

**Corollary 4.6.** *Let $C$ be a nonempty closed convex subset of a reflexive Banach space $X$ with a uniformly Gâteaux-differentiable norm and $T$ a nonexpansive mapping from $C$ into itself such that $F[T] \neq \emptyset$. Let $\digamma : C \to C$ be both $\kappa$-strongly pseudocontractive and $\lambda$-strictly pseudocontractive with $\lambda > \kappa$. Assume that $C$ has the fixed-point property for nonexpansive self-mappings. For given $x_1 \in C$, let $\{x_n\}$ be a sequence in $C$ generated by*

$$x_{n+1} = T[(1-\alpha_n)x_n + \alpha_n \digamma(x_n)] \quad \forall n \in \mathbb{N}, \tag{4.23}$$

*where $\{\alpha_n\}$ is a sequence in $(0,1]$ satisfying condition (C1). Then, $\{x_n\}$ converges strongly to a unique solution $x^*$ of VIP$(I - \digamma, C)$ over $F[T]$.*

Corollary 4.6 is an improvement upon Sahu [9, Theorem 5.6] in a Banach space without uniform convexity.

## 5. Applications

### *5.1. Applications to the Common Fixed Point Problems for Nonexpansive Mappings*

**Theorem 5.1.** *Let $C$ be a nonempty closed convex subset of a strictly convex reflexive Banach space $X$ with a uniformly Gâteaux-differentiable norm. Let $\lambda_i > 0$ $(i = 1, 2, \ldots, N)$ such that $\sum_{i=1}^{N} \lambda_i = 1$ and let $T_1, T_2, \ldots, T_N : C \to C$ be nonexpansive mappings with $\bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. Let $\digamma : X \to X$ be both $\delta$-strongly accretive and $\lambda$-strictly pseudocontractive over $C$ with $\lambda + \delta > 1$ and $R(I - \tau\digamma) \subseteq C$ for each $\tau \in (0,1)$. For a given $x_1 \in C$, let $\{x_n\}$ be a sequence in $C$ generated by*

$$x_{n+1} = \sum_{i=1}^{N} \lambda_i T_i [x_n - \alpha_n \digamma(x_n)] \quad \forall n \in \mathbb{N}, \tag{5.1}$$

*where $\{\alpha_n\}$ is a sequence in $(0,1]$ satisfying condition (C1). Then, $\{x_n\}$ converges strongly to a unique solution $x^*$ of VIP$(\digamma, C)$ over $\bigcap_{i=1}^{N} F(T_i)$.*

*Proof.* Define $T = \sum_{i=1}^{N} \lambda_i T_i$. Then $T$ is nonexpansive from $C$ into itself and, hence, from Fact 2.7, we have $F(T) = \bigcap_{i=1}^{N} F(T_i)$. Therefore, Theorem 5.1 follows from Theorem 4.4. □

**Theorem 5.2.** *Let $C$ be a nonempty closed convex subset of a strictly convex reflexive Banach space $X$ with a uniformly Gâteaux-differentiable norm. Let $\{S_n\}$ be a sequence of nonexpansive mappings from $C$ into itself such that $\bigcap_{n\in\mathbb{N}} F[S_n] \neq \emptyset$ and let $\digamma : X \to X$ be both $\delta$-strongly accretive and $\lambda$-strictly pseudocontractive over $C$ with $\lambda + \delta > 1$ and $R(I - \tau\digamma) \subseteq C$ for each $\tau \in (0,1)$. Let $\{\beta_{n,k}\}$ be a family of nonnegative numbers with indices $n, k \in \mathbb{N}$ with $k \le n$ such that*

(i) $\sum_{k=1}^{n} \beta_{n,k} = 1$ *for each* $n \in \mathbb{N}$;

(ii) $\lim_{n\to\infty} \beta_{n,k} > 0$ *for each* $k \in \mathbb{N}$;

(iii) $\sum_{n=1}^{\infty} \sum_{k=1}^{n} |\beta_{n+1,k} - \beta_{n,k}| < \infty$.

*For a given $x_1 \in C$, let $\{x_n\}$ be a sequence in $C$ generated by*

$$x_{n+1} = \sum_{k=1}^{n} \beta_{n,k} S_k [x_n - \alpha_n \mathcal{F}(x_n)] \quad \forall n \in \mathbb{N}, \tag{5.2}$$

*where $\{\alpha_n\}$ is a sequence in $(0,1]$ satisfying conditions (C1)~(C2). Then, $\{x_n\}$ converges strongly to a unique solution $x^*$ of VIP$(\mathcal{F}, C)$ over $\bigcap_{n\in\mathbb{N}} F[S_n]$.*

*Proof.* Define a sequence $\{T_n\}$ of mappings on $C$ by $T_n x = \sum_{k=1}^{n} \beta_{n,k} S_k x$ for all $x \in C$ and $n \in \mathbb{N}$. It is easy to see, from condition (i) and Fact 2.7, that each $T_n$ is also a nonexpansive mapping from $C$ into itself and $F[T_n] = \bigcap_{k=1}^{n} F[S_k]$. Note that $\bigcap_{k\in\mathbb{N}} F[S_k] \subseteq \bigcap_{n\in\mathbb{N}} F[T_n]$. Moreover, by (ii) we have that for every $k \in N$, there exists $n_0 \in \mathbb{N}$ such that $\beta_{n_0,k} > 0$. Thus, we have that $F[T_{n_0}] \subseteq F[S_k]$ for $k \in \mathbb{N}$ by Fact 2.8, which implies that $\bigcap_{n\in\mathbb{N}} F[T_n] \subseteq F[S_k]$ for all $k \in \mathbb{N}$. Therefore, we obtain that $\bigcap_{k\in\mathbb{N}} F[S_k] = \bigcap_{n\in\mathbb{N}} F[T_n] \neq \emptyset$. Now, let $B \in \mathcal{B}(C)$. The nonemptiness of $\bigcap_{k\in\mathbb{N}} F[S_k]$ implies that $\{S_k x : x \in B, k \in \mathbb{N}\}$ is bounded. By using the argument of [34], we see that $Tz = \lim_{n\to\infty} T_n z$ for all $z \in C$. Hence, Theorem 5.2 follows from Corollary 4.3. □

### *5.2. Applications to the Zero Point Problems for Accretive Operators*

Consider $C$ a closed convex subset of a Banach space $X$ and $A \subset X \times X$ is an accretive operator such that $\mathbb{S} \neq \emptyset$ and $\overline{D(A)} \subset C \subset \bigcap_{t>0} R(I + tA)$. From Takahashi [28], we know that $J_r^A$ is a nonexpansive mapping of $C$ into itself and $F[J_r^A] = \mathbb{S}$ for each $r > 0$.

Motivated and inspired by two well-known methods, Yamada's hybrid steepest-descent method and Lehdili and Moudafi's method, we introduce the following algorithm which we call *prox-Tikhonov regularized hybrid steepest-descent algorithm*.

*Algorithm 5.3.* For a given $x_1 \in C$, let $\{x_n\}$ be a sequence in $C$ generated by

$$x_{n+1} = J_{r_n}^A [x_n - \alpha_n \mathcal{F}(x_n)] \quad \forall n \in \mathbb{N}, \tag{5.3}$$

where $\{\alpha_n\}$ is a sequence in $(0,1]$ and $\{r_n\}$ is a regularization sequence in $(0,\infty)$.

One can easily see that the prox-Tikhonov regularized hybrid steepest-descent algorithm is a special case of generalized hybrid steepest-descent algorithm.

The following theorem gives sufficient conditions for strong convergence of the prox-Tikhonov regularized hybrid steepest-descent algorithm (5.3) to a solution of Problem $(P)$.

**Theorem 5.4.** *Let $X$ be a reflexive Banach space with a uniformly Gâteaux-differentiable norm and $C$ a nonempty closed convex subset of $X$ which has the fixed-point property for nonexpansive self-mappings. Let $A \subset X \times X$ be an accretive operator such that $A^{-1}0 \neq \emptyset$ and $\overline{D(A)} \subset C \subset \bigcap_{t>0} R(I + tA)$. Let $\mathcal{F} : X \to X$ be both $\delta$-strongly accretive and $\lambda$-strictly pseudocontractive over $C$ with $\lambda + \delta > 1$ and $R(I - \tau\mathcal{F}) \subseteq C$ for each $\tau \in (0,1)$. For a given $x_1 \in C$, let $\{x_n\}$ be a prox-Tikhonov regularized hybrid steepest-descent iterative sequence in $C$ generated by (5.3), where $\{\alpha_n\}$ is a sequence in $(0,1]$ satisfying condition (C1) and $\{r_n\}$ is a regularization sequence in $(0,\infty)$ such that $\inf_{n\in\mathbb{N}} r_n > 0$ and $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$. Then $\{x_n\}$ converges strongly to a unique solution $x^*$ of VIP$(\mathcal{F}, C)$ over $A^{-1}0$.*

*Proof.* Set $T_n := J_{r_n}^A$. Then $\{T_n\}$ is a sequence of nonexpansive mappings from $C$ into itself such that $F[T_n] = A^{-1}0 \neq \emptyset$ for every $n \in \mathbb{N}$. We first verify that $\sum_{n=1}^{\infty} \mathfrak{D}_B(T_n, T_{n+1}) < \infty$ for every $B \in \mathcal{B}(C)$. Let $B \in \mathcal{B}(C)$. Since $F[T_n] = A^{-1}0 \neq \emptyset$ for every $n \in \mathbb{N}$, it follows that $\{T_n z : z \in B, n \in \mathbb{N}\}$ is bounded. Set $K_3 := \sup\{\|z - J_{r_{n+1}} z\| : z \in B, n \in \mathbb{N}\}$. By the assumptions for $\{r_n\}$, we may assume that $r_n \geq \varepsilon$ for all $n \in \mathbb{N}$ and $r_n \to r$ for some $r, \varepsilon > 0$. From (2.19), we have

$$\begin{aligned} \mathfrak{D}_B(T_{n+1}, T_n) &= \sup\{\|J_{r_{n+1}} z - J_{r_n} z\| : z \in B\} \\ &\leq \sup\left\{ \frac{|r_{n+1} - r_n|}{r_{n+1}} \|z - J_{r_{n+1}} z\| : z \in B \right\} \\ &\leq \frac{|r_{n+1} - r_n|}{\varepsilon} K_3 \quad \forall n \in \mathbb{N}. \end{aligned} \tag{5.4}$$

Hence, $\sum_{n=1}^{\infty} \mathfrak{D}_B(T_n, T_{n+1}) < \infty$. Set $T := J_r$. Again, from (2.19), we have

$$\|Tx - T_n x\| \leq \frac{|r - r_n|}{r} \|x - Tx\| \quad \forall x \in C, \tag{5.5}$$

which indicates that $Tx = \lim_{n\to\infty} T_n x$ for all $x \in C$. Therefore, by Theorem 4.2, $\{x_n\}$ converges strongly to a unique solution $x^*$ of VIP$(\mathcal{F}, C)$ over $A^{-1}0$. □

**Corollary 5.5.** *Let $X$ be a reflexive Banach space with a uniformly Gâteaux-differentiable norm and $C$ a nonempty closed convex subset of $X$ which has the fixed-point property for nonexpansive self-mappings. Let $A \subset X \times X$ be an accretive operator such that $A^{-1}0 \neq \emptyset$ and $\overline{D(A)} \subset C \subset \bigcap_{t>0} R(I + tA)$. Let $\mathcal{F} : C \to C$ be both $\kappa$-strongly pseudocontractive and $\lambda$-strictly pseudocontractive with $\lambda > \kappa$. For a given $x_1 \in C$, let $\{x_n\}$ be a prox-Tikhonov regularized hybrid steepest-descent iterative sequence in $C$ generated by*

$$x_{n+1} = J_{r_n}^A[(1 - \alpha_n)x_n + \alpha_n \mathcal{F}(x_n)] \quad \forall n \in \mathbb{N}, \tag{5.6}$$

*where $\{\alpha_n\}$ is a sequence in $(0,1]$ satisfying condition (C1) and $\{r_n\}$ is a regularization sequence in $(0, \infty)$ such that $\inf_{n\in\mathbb{N}} r_n > 0$ and $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$. Then $\{x_n\}$ converges strongly to a unique solution $x^*$ of VIP$(I - \mathcal{F}, C)$ over $A^{-1}0$.*

## 6. Numerical Results

In order to demonstrate the effectiveness, performance, and convergence of the proposed algorithm, we discuss the following.

*Example 6.1.* Let $\mathcal{H} = \mathbb{R}$ and $C = [0,1]$. Let $T, \mathcal{F} : C \to \mathcal{H}$ be two mappings defined by $Tx = 1 - x$ for all $x \in C$ and $\mathcal{F}x = x - 1$ for all $x \in C$. For each $\tau \in (0,1)$, we have $(I - \tau\mathcal{F})x = x - \tau(x - 1) = (1 - \tau)x + \tau$ for all $x \in C$. Define $\{\alpha_n\}$ in $[0,1]$ by $\alpha_n = 1/(n+1)^a$ for all $n \in \mathbb{N}$, where $a \in (0,1]$. The sequence $\{x_n\}$ defined by (4.22) is given by the relation

$$x_{n+1} = (1 - \alpha_n)(1 - x_n) \quad \forall n \in \mathbb{N}. \tag{6.1}$$

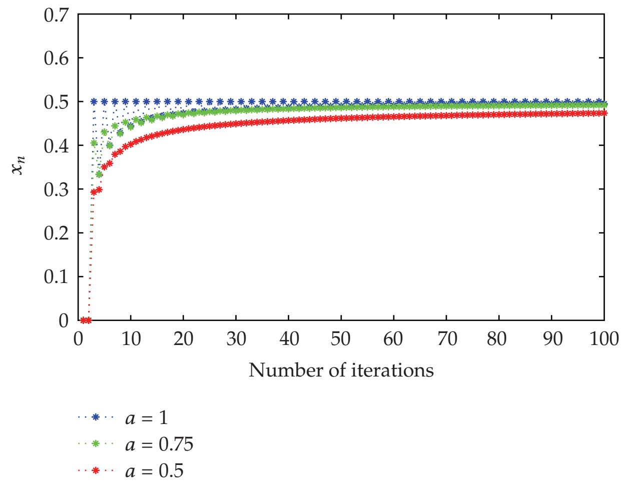


**Figure** 1

For $x_1 = 0$ and $a = 1$, the sequence $\{x_n\}$ defined by (6.1) can be explicitly written as

$$x_n = \begin{cases} \dfrac{1}{2}, & n = 2, 4, \ldots; \\ \dfrac{n-1}{2n}, & n = 3, 5, \ldots. \end{cases} \tag{6.2}$$

Observe that

(1) $T$ is nonexpansive,

(2) $\mathcal{F}$ is both 1-strongly accretive and $\lambda$-strictly pseudocontractive over $C$ for each $\lambda > 0$,

(3) $R(I - \tau\mathcal{F}) \subseteq C$ for each $\tau \in (0,1)$, and

(4) $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\lim_{n\to\infty}|1 - \alpha_n/\alpha_{n+1}| = 0$.

Thus, all the assumptions of Theorem 4.4 are satisfied. Therefore, the conclusion of Theorem 4.4 holds, that is, $x_n \to 1/2 \in F[T]$.

It is seen from Figure 1 that if $a = 1$, $a = 0.75$, and $a = 0.5$, then the corresponding iterations of sequence $\{x_n\}$ with $x_1 = 0$ defined by (6.1) are convergent to $1/2$.

*Example 6.2.* Let $\mathcal{H}$, $C$, $T$, and $\mathcal{F}$ be as in Example 6.1. Clearly $T$ is nonexpansive and $\mathcal{F}$ is both 1-strongly accretive and $\lambda$-strictly pseudocontractive over $C$ for each $\lambda > 0$. Assume that $\{a_n\}$ is a sequence in $[0,1]$ such that $\sum_{n=1}^{\infty} |a_n - a_{n+1}| < \infty$. Without loss of generality we may assume that $a_n = 1/n^{3/2}$ for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, define $T_n : C \to C$ by

$$T_n x = \begin{cases} 1 - x, & \text{if } x \in [0,1), \\ a_n, & \text{if } x = 1. \end{cases} \tag{6.3}$$

Define a sequence $\{\alpha_n\}$ in $[0,1]$ by $\alpha_n = 1/n$ for all $n \in \mathbb{N}$.

We now show that, under the assumptions of Theorem 4.2, the sequence $\{x_n\}$ generated by the proposed Algorithm 4.1 converges to a unique solution 1/2 of VIP($\mathcal{F}, C$) over $\bigcap_{n\in\mathbb{N}} F[T_n]$. We proceed with the following steps.

*Step 1.* $\{T_n\}$ is a sequence of nearly nonexpansive mappings from $C$ into itself such that $\bigcap_{n\in\mathbb{N}} F[T_n] \neq \emptyset$.

For $x, y \in [0, 1)$, we have

$$\|T_n x - T_n y\| \leq \|x - y\| \quad \forall n \in \mathbb{N}. \tag{6.4}$$

Moreover, for $x \in [0, 1)$ and $y = 1$, we have

$$\|T_n x - T_n 1\| = \|1 - x - a_n\| \leq \|x - 1\| + a_n \quad \forall n \in \mathbb{N}. \tag{6.5}$$

Thus,

$$\|T_n x - T_n y\| \leq \|x - y\| + a_n \quad \forall x, y \in C, \ n \in \mathbb{N}, \tag{6.6}$$

that is, $\{T_n\}$ is a sequence of nearly nonexpansive mappings from $C$ into itself such that $\bigcap_{n\in\mathbb{N}} F[T_n] = \{1/2\}$.

*Step 2.* $\lim_{n\to\infty} T_n z = Tz$ for all $z \in C$.

For each $n \in \mathbb{N}$, we have

$$T_n x - T_{n+1} x = \begin{cases} 0, & \text{if } x \in [0, 1), \\ a_n - a_{n+1}, & \text{if } x = 1, \end{cases} \tag{6.7}$$

and hence $\sup\{\|T_n x - T_{n+1} x\| : x \in C\} = |a_n - a_{n+1}|$. One can easily see that

$$\sum_{n=1}^{\infty} \mathfrak{D}_C(T_n, T_{n+1}) = \sum_{n=1}^{\infty} \sup\{\|T_n x - T_{n+1} x\| : x \in C\} = \sum_{n=1}^{\infty} |a_n - a_{n+1}| < \infty. \tag{6.8}$$

Since $\{T_n\}$ is a sequence of nearly nonexpansive self-mappings of $C$ with sequence $\{a_n\}$ such that $\sum_{n=1}^{\infty} \mathfrak{D}_C(T_n, T_{n+1}) < \infty$, it follows from Proposition 2.4 that for each $x \in C$, $\{T_n x\}$ converges to some point of $C$. It can be readily seen that $\lim_{n\to\infty} T_n z = Tz$ for all $z \in C$.

*Step 3.* The sequence $\{x_n\}$ defined Algorithm 4.1 converges to $1/2 \in F[T]$.

For an arbitrary $x_1 \in C$, the sequence $\{x_n\}$ defined by (4.1) can be explicitly written as

$$x_n = \begin{cases} 1, & n = 2; \\ \dfrac{1}{n-1}\left[\dfrac{n-3}{2} + 2a_2\right], & n = 3, 5 \ldots; \\ \dfrac{1}{n-1}\left[\dfrac{n}{2} - 2a_2\right], & n = 4, 6 \ldots. \end{cases} \tag{6.9}$$

As in Example 6.1, we have $\lambda + \delta > 1$ and $R(I - \tau\mathcal{F}) \subseteq C$ for each $\tau \in (0,1)$. Observe that

(1) $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}\alpha_n = \infty$ and $\lim_{n\to\infty}|1 - \alpha_n/\alpha_{n+1}| = 0$;

(2) $\sum_{n=1}^{\infty}\mathfrak{D}_C(T_n, T_{n+1}) < \infty$;

(3) $\lim_{n\to\infty}a_n/\alpha_n = 0$.

Noticing that conditions (C1)~(C3) are satisfied, $\lim_{n\to\infty}T_n z = Tz$ for all $z \in C$ and $T$ is a nonexpansive mapping. Thus, all the assumptions of Theorem 4.2 are satisfied. Therefore, the conclusion of Theorem 4.2 holds. Indeed, the sequence $\{x_n\}$ defined by (6.9) converges to $1/2 \in \bigcap_{n\in\mathbb{N}} F[T_n]$.

## 7. Concluding Remarks

(I) Theorems 3.3, 4.2, and 5.4 apply to all uniformly convex and uniformly smooth Banach spaces and in particular, to all $L_p$ spaces, $1 < p < \infty$.

(II) Theorem 4.2 appears to be a new result for solving Problem 1.1. In particular, Theorem 4.2 improves Yamada [6] in the framework of Banach space.

(III) The prox-Tikhonov algorithm (1.11) studied in Lehdili and Moudafi [19] deals essentially with a special case of the algorithm (5.6) in the framework of Hilbert space. In fact, if we set $\mathcal{F}x = 0$ for all $x \in \mathcal{H}$, then (5.6) becomes

$$x_{n+1} = J_{c_n}^{A}((1 - \alpha_n)x_n) \quad \forall n \in \mathbb{N}. \tag{7.1}$$

Setting $\lambda_n := c_n/(1 - \alpha_n)$ and $\mu_n := \alpha_n/c_n$, (7.1) can be written as

$$x_{n+1} = J_{\lambda_n}^{A_n}x_n \quad \forall n \in \mathbb{N}, \tag{7.2}$$

where $A_n = \mu_n I + A$. Thus, (7.2) is the prox-Tikhonov algorithm (1.11) considered in Lehdili and Moudafi [19]. The argument given in [19] depends heavily on the concept of the variational distance (see [35]) between two maximal monotone operators. Our argument is simple and more straightforward in the Banach space setting. Therefore, Corollary 5.5 improves and extends the convergence result presented in Lehdili and Moudafi [19] in the Banach space setting.

(IV) Our approach is simple and different from new iterative methods for finding solutions of Problem 1.1 and zero of $m$-accretive operators proposed in Ceng et al. [20] and Ceng et al. [21].

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