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KAPLANSKY THEOREM FOR COMPLETELY REGULAR SPACES

LEI LI AND NGAI-CHING WONG

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Abstract. Let X, Y be realcompact spaces or completely regular spaces consisting of G_δ-points. Let ϕ be a linear bijective map from $C(X)$ (resp. $C^b(X)$) onto $C(Y)$ (resp. $C^b(Y)$). We show that if ϕ preserves nonvanishing functions, that is,

$$f(x) \neq 0, \forall\, x \in X, \quad \Longleftrightarrow \quad \phi(f)(y) \neq 0, \forall\, y \in Y,$$

then ϕ is a weighted composition operator

$$\phi(f) = \phi(1) \cdot f \circ \tau,$$

arising from a homeomorphism τ from Y onto X. This result is applied also to other nice function spaces, e.g., uniformly or Lipschitz continuous functions on metric spaces.

1. Introduction

The problem here is how to recover a topological space X from the set $C(X)$ (resp. $C^b(X)$) of continuous (resp. bounded continuous) (real- or complex-valued) functions on X. We say that a net $\{x_\lambda\} \subset X$ converges to x in the *weak topology* $\sigma(X, C(X))$ if $f(x_\lambda) \to f(x)$ for all f in $C(X)$. It is easy to see that the weak topology $\sigma(X, C^b(X))$ coincides with $\sigma(X, C(X))$. A well-known fact states that X carries the weak topology $\sigma(X, C(X))$ if and only if X is completely regular (see, e.g., [9, Theorem 3.6]). In this sense, a completely regular topological space is determined by all its continuous functions.

Assume X is completely regular throughout this paper. The sets $C(X)$ and $C^b(X)$ carry the natural algebraic, lattice, and Banach space (for $C^b(X)$) structures. It is plausible that the algebra, the vector lattice, or the metric structures of $C(X)$ or $C^b(X)$ can also determine the topology of X.

Question 1.1. Suppose that there is an algebra (or lattice, or isometrically linear) isomorphism $\phi : C(X) \to C(Y)$ or $\phi : C^b(X) \to C^b(Y)$. Can we conclude that the completely regular spaces X and Y are homeomorphic?

In the literature, there are several well-known results in this line. For example, every ring isomorphism $\phi : C(X) \to C(Y)$ (resp. $\phi : C^b(X) \to C^b(Y)$) gives rise to a (surjective) homeomorphism $\tau^\upsilon : \upsilon Y \to \upsilon X$ (resp. $\tau^\beta : \beta Y \to \beta X$) between

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the Hewitt-Nachbin realcompactifications υX and υY (resp. Stone-Čech compactifications βX and βY) of the completely regular spaces X and Y, respectively. However, X and Y might be nonhomeomorphic in both cases, unless they are both realcompact or compact to start with (see Example 1.2 below).

Let us sketch a proof here. Recall that every f in $C(X)$ gives rise to a *zero set*

$$z(f) = \{x \in X : f(x) = 0\},$$

and write

$$Z(\mathcal{A}(X)) = \{z(f) : f \in \mathcal{A}(X)\}$$

for any subset $\mathcal{A}(X)$ of $C(X)$. In particular, $Z(C(X)) = Z(C^b(X))$, and denote it by $Z(X)$ for simplicity. A *z-filter* $\mathcal{F}$ on X is a filter of zero sets in $Z(X)$. Call $\mathcal{F}$ a *z-ultrafilter* if it is a maximal z-filter and call $\mathcal{F}$ *prime* if $A \in \mathcal{F}$ or $B \in \mathcal{F}$ whenever $X = A \cup B$ and $A, B \in Z(X)$. Associated to each z-ultrafilter $\mathcal{F}$ is a maximal ideal I of $C(X)$ consisting of all continuous functions f such that $z(f) \in \mathcal{F}$. Call $\mathcal{F}$ *fixed* if $\bigcap \mathcal{F}$ is a singleton, and call $\mathcal{F}$ *real* if the quotient field $C(X)/I$ is isomorphic to $\mathbb{R}$ (assuming the underlying field is $\mathbb{R}$). The Stone-Čech compactification βX can be identified with the set of *all* z-ultrafilters on X. In this setting, X consists of all fixed z-ultrafilters. The Hewitt-Nachbin realcompactification υX consists of all *real* z-ultrafilters. Clearly, X is compact if and only if $X = \beta X$. Call X a *realcompact* space if $X = \upsilon X$. In fact, X is realcompact if and only if every prime z-filter with the countable intersection property is fixed. For instance, the Lindelöf (and thus separable metric) spaces are realcompact, and discrete spaces of nonmeasurable cardinality are some other examples. Especially, all subspaces of the Euclidean spaces $\mathbb{R}^n$ (and $\mathbb{C}^n$ as well) are realcompact. In general, X is realcompact if and only if X is homeomorphic to a closed subspace of a product of real lines. However, the order interval $[0, \omega_1)$ is not realcompact, where ω_1 is the first uncountable ordinal.

As ring isomorphisms preserve z-ultrafilters and real z-ultrafilters, βX and βY and υX and υY are homeomorphic when $C(X)$ and $C(Y)$ are ring isomorphic. Because $C^b(X)$ and $C(\beta X)$ are isomorphic as Banach algebras, every ring isomorphism between $C^b(X)$ and $C^b(Y)$ also provides a homeomorphism of the compact spaces βX and βY. We refer to the books [9] and [18] for more information about z-ultrafilters and realcompact spaces. The case of the underlying field $\mathbb{K} = \mathbb{C}$ follows easily, too.

On the other hand, the classical Banach-Stone theorem tells us that the geometric structure of the Banach space $C^b(X)$ determines the topology of its Stone-Čech compactification βX. In the special case when X, Y are compact, if $\phi : C(X) \to C(Y)$ is a surjective linear isometry, then there is a homeomorphism τ from Y onto X and a unimodular continuous weight function h in $C(Y)$ such that ϕ is the weighted composition operator $\phi(f) = h \cdot f \circ \tau$. In general, when X, Y are completely regular spaces, since $C^b(X) \cong C(\beta X)$ and $C^b(Y) \cong C(\beta Y)$ as Banach spaces, there exists a surjective linear isometry between $C^b(X)$ and $C^b(Y)$ if and only if βX and βY are homeomorphic (see, e.g., [9]).

When X, Y are compact Hausdorff spaces, Kaplansky obtained in [14] yet another criterion: every lattice isomorphism $\phi : C(X) \to C(Y)$ also gives rise to a homeomorphism $\tau : Y \to X$; and he also showed in [15] that if ϕ is, in addition, additive, then $\phi(f) = h \cdot f \circ \tau$ with a strictly positive weight function h in $C(Y)$. Moreover, he showed that a positive linear map $\phi : C(X) \to C(Y)$ is a lattice isomorphism if and only if ϕ *preserves nonvanishing functions* (in two directions), that is,

$$z(f) = \emptyset \quad \Leftrightarrow \quad z(\phi(f)) = \emptyset, \quad \forall\, f \in C(X).$$

This starts a popular research subject of studying invertibility or spectrum preserving linear maps of Banach algebras (see, e.g., [4, 5]).

Nevertheless, the following example tells us that the algebraic, geometric and lattice structures of the Banach algebra $C^b(X)$ altogether are still not enough to determine the topology of a realcompact space X, even when X consists of G_δ points.

Example 1.2 (see [9, 4M])**.** Let Σ be $\mathbb{N} \cup \{\sigma\}$ (where $\sigma \in \beta\mathbb{N}\backslash\mathbb{N}$). Clearly, $\mathbb{N}$ is dense in Σ, and every function f in $C^b(\mathbb{N})$ can be extended uniquely to a function f^σ in $C^b(\Sigma)$. Although the bijective linear map ϕ from $C^b(\mathbb{N})$ onto $C^b(\Sigma)$ defined by $f \mapsto f^\sigma$ provides an isometric, algebraic and lattice isomorphism, the realcompact spaces $\mathbb{N}$ and Σ are not homeomorphic.

Notice that the map ϕ in Example 1.2 does not preserve nonvanishing functions. In Theorems 2.2 and 2.9 below, we will show that every bijective linear nonvanishing preserver between some nice subspaces of continuous functions, including uniformly continuous and Lipschitz continuous functions on metric spaces, is a weighted composition operator $f \mapsto h \cdot f \circ \tau$ arising from a homeomorphism τ between the realcompactifications of the underlying completely regular spaces. This in particular tells us that the property of a linear map preserving nonvanishing functions is stronger than those being multiplicative, lattice isomorphic, and isometric, and thus supplements many results in the literature, e.g., [1, 2, 7, 11, 12, 17].

2. Main results

Throughout, the underlying scalar field $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$. Suppose that X and Y are completely regular spaces. Let $\mathcal{A}(X)$ and $\mathcal{A}(Y)$ be vector sublattices of $C(X)$ and $C(Y)$ (self-adjoint if $\mathbb{K} = \mathbb{C}$) containing all constant functions, respectively.

Denote by $\mathcal{A}^b(X) := \mathcal{A}(X) \cap C^b(X)$ the vector sublattice of $\mathcal{A}(X)$ consisting of bounded functions, and by $\mathcal{A}(X)_+$ the subset of $\mathcal{A}(X)$ consisting of nonnegative real-valued functions. For any f in $\mathcal{A}(X)$, we can decompose $f = f_1 - f_2 + i(f_3 - f_4)$ in a unique way such that $f_1, f_2, f_3, f_4 \in \mathcal{A}(X)_+$ and $f_1 f_2 = f_3 f_4 = 0$. Write $|f| := f_1 + f_2 + f_3 + f_4$. Clearly, $|f| \geq 0$ and $z(|f|) = z(f)$.

Definition 2.1. We say that $\mathcal{A}(X)$ is

(1) *completely regular* if for every point x and closed subset F of X with $x \notin F$, there is an f in $\mathcal{A}(X)$ such that $x_0 \notin z(f)$ and $F \subseteq z(f)$;
(2) *full* if $Z(\mathcal{A}(X)) = Z(X)$;
(3) *nice* if for any sequence $\{f_n\}$ in $\mathcal{A}^b(X)_+$, there exists a sequence of strictly positive numbers $\{\lambda_n\}$ such that $\sum_{n=1}^\infty \lambda_n f_n$ converges pointwisely to a function f in $\mathcal{A}(X)$.

Note that a full subspace of $C(X)$ is completely regular, but might not be normal, i.e., separating disjoint closed sets. For instance, the space $\mathrm{Lip}(X)$ of all Lipschitz continuous functions on the metric space $X = (-1, 0) \cup (0, 1)$ is full but not normal.

The following Kaplansky type theorem can be considered as a generalization of the Gleason-Kahane-Zelazko Theorem [10, 13].

Theorem 2.2. *Let X and Y be realcompact spaces, and let $\phi : \mathcal{A}(X) \to \mathcal{A}(Y)$ be a bijective linear map preserving nonvanishing functions. Assume $\mathcal{A}(X)$ is nice and completely regular, and $\mathcal{A}(Y)$ is full. Then there is a dense subset Y_1 of Y, containing all G_δ points in Y, and a homeomorphism $\tau : Y_1 \to X$ such that*

$$\phi(f)(y) = \phi(1)(y)f(\tau(y)), \quad \forall f \in \mathcal{A}(X), \forall y \in Y_1. \tag{2.1}$$

In case all points of Y are G_δ, or in case $\mathcal{A}(X)$ is full and $\mathcal{A}(Y)$ is nice, we have $Y_1 = Y$.

We will establish the proof of Theorem 2.2 in several lemmas.

Lemma 2.3. *ϕ is biseparating, i.e.,*

$$fg = 0 \text{ on } X \quad \Leftrightarrow \quad \phi(f)\phi(g) = 0 \text{ on } Y.$$

Proof. Suppose that f and g belong to $\mathcal{A}(X)$ with $fg = 0$, but $\phi(f)\phi(g) \neq 0$. Without loss of generality, we can assume that there exists a y_0 in Y such that $\phi(f)(y_0) = \phi(g)(y_0) = 1$.

Define h in $\mathcal{A}(Y)$ by

$$h(y) = \max\left\{0, \frac{1}{2} - \operatorname{Re}\phi(f)(y), \frac{1}{2} - \operatorname{Re}\phi(g)(y)\right\}, \quad \forall y \in Y,$$

and put

$$k = \phi^{-1}(h).$$

Claim. $z(\phi(f) + \phi(k)) = \emptyset$.

Indeed, assume on the contrary that y belongs to $z(\phi(f) + \phi(k))$, that is,

$$\phi(f)(y) + \phi(k)(y) = \phi(f)(y) + h(y) = 0.$$

This provides a contradiction:

$$h(y) \geq \frac{1}{2} - \operatorname{Re}\phi(f)(y) = \frac{1}{2} + h(y).$$

It follows from $z(\phi(f) + \phi(k)) = \emptyset$ that $z(f + k) = \emptyset$. In a similar way, we also have $z(g + k) = \emptyset$. Notice that $z(f) \cap z(k) \subseteq z(f + k)$ and $z(g) \cap z(k) \subseteq z(g + k)$. We thus have $z(f) \cap z(k) = z(g) \cap z(k) = \emptyset$. By the assumption $z(f) \cup z(g) = X$, one can conclude $z(k) = \emptyset$. This is a contradiction since $(\phi k)(y_0) = h(y_0) = 0$ and ϕ is nonvanishing preserving. Hence, $\phi(f)\phi(g) = 0$, as asserted.

Similarly, we can derive that ϕ^{-1} is also separating, and hence ϕ is a biseparating map. □

We note that a biseparating mapping might not be nonvanishing preserving as shown in Example 1.2. The following lemma is motivated by the results in [6, 17].

Lemma 2.4. *ϕ sends exactly functions without common zeros to functions without common zeros. That is, for any m in $\mathbb{N}$ and $f_1, \ldots, f_m$ in $\mathcal{A}(X)$, we have*

$$\bigcap_{k=1}^{m} z(f_k) = \emptyset \quad \Longleftrightarrow \quad \bigcap_{k=1}^{m} z(\phi(f_k)) = \emptyset.$$

Proof. Note first that $\phi(1)$ is nonvanishing on Y. Define $\psi(f) := \phi(f)/\phi(1)$. It is easy to see that ψ is an injective linear map from $\mathcal{A}(X)$ into $C(Y)$, and $z(\psi(f)) = z(\phi(f))$ for all f in $\mathcal{A}(X)$.

Claim. ψ sends nonnegative real functions to nonnegative real functions.

Let $f \geq 0$ be in $\mathcal{A}(X)$, that is, $f(x) \geq 0$ for all x in X, and let λ be a nonpositive scalar in $\mathbb{K} \setminus [0, +\infty)$. As $f - \lambda$ is nonvanishing on X, we can see that $\phi(f) - \lambda\phi(1)$ is nonvanishing on Y. Therefore, $\psi(f) - \lambda$ is also nonvanishing on Y. Since λ is an arbitrary nonpositive real number, we see that $\psi(f)$ assumes values from $[0, +\infty)$.

Inherited from ϕ, the new map ψ is also biseparating. It follows that $\psi(|f|) = |\psi(f)|$ for all f in $\mathcal{A}(X)$. Now, suppose that $f_1, \ldots, f_m$ belong to $\mathcal{A}(X)$ with

$$\emptyset = \bigcap_{i=1}^{m} z(f_i) = \bigcap_{i=1}^{m} z(|f_i|) = z(\sum_{i=1}^{m} |f_i|).$$

Observe that

$$\begin{aligned}\bigcap_{k=1}^{m} z(\phi(f_k)) = \bigcap_{k=1}^{m} z(\psi(f_k)) &= \bigcap_{k=1}^{m} z(|\psi(f_k)|) \\ &= \bigcap_{k=1}^{m} z(\psi(|f_k|)) = z(\sum_{k=1}^{m} \psi(|f_k|)) \\ &= z(\psi(\sum_{k=1}^{m} |f_k|)) = z(\phi(\sum_{k=1}^{m} |f_k|)) = \emptyset.\end{aligned}$$

The proof for the other direction is similar. □

Lemma 2.5. *ϕ preserves zero set containments, i.e.,*

$$z(f) \subseteq z(g) \iff z(\phi(f)) \subseteq z(\phi(g)), \quad \forall\, f, g \in \mathcal{A}(X).$$

Proof. Assume $z(f) \subseteq z(g)$. Let y in Y be such that $\phi(g)(y) \neq 0$. As in the proof of Lemma 2.3, we can find a function k in $\mathcal{A}(X)$ such that

$$z(\phi(g) + \phi(k)) = \emptyset \quad \text{and} \quad \phi(k)(y) = 0.$$

By the assumption,

$$z(f) \cap z(k) \subseteq z(g) \cap z(k) \subseteq z(g + k) = \emptyset.$$

It follows from Lemma 2.4 that

$$z(\phi(f)) \cap z(\phi(k)) = \emptyset.$$

In particular, $\phi(f)(y) \neq 0$, as asserted. The other direction is similar. □

For any x_0 in X, let

$$\mathcal{K}_{x_0} = \{f \in \mathcal{A}(X) : f(x_0) = 0\}$$

and

$$\mathcal{Z}_{x_0} = Z(\phi(\mathcal{K}_{x_0})) = \{z(\phi f) : f \in \mathcal{K}_{x_0}\}.$$

Lemma 2.6. *$\mathcal{Z}_{x_0}$ is a prime z-filter on Y with the countable intersection property.*

Proof. We first note that by the fullness of $\mathcal{A}(Y) = \phi(\mathcal{A}(X))$, every zero set A in $Z(Y)$ can be written as $A = z(\phi(f))$ for some f in $\mathcal{A}(X)$.

Because ϕ is nonvanishing preserving, the empty set is not in $\mathcal{Z}_{x_0}$. Let $f \in \mathcal{K}_{x_0}$ and $C = z(\phi(g)) \in Z(Y)$ such that $z(\phi(f)) \subseteq C$. Then $z(f) \subseteq z(g)$ since ϕ preserves zero set containments by Lemma 2.5, and hence $g \in \mathcal{K}_{x_0}$. This means that $C \in \mathcal{Z}_{x_0}$. Let $\{f_n\}$ be a sequence of functions in $\mathcal{K}_{x_0}$. Set $g_n = \min\{1, |f_n|\}$ in $\mathcal{A}^b(X)$. Clearly, $z(g_n) = z(f_n)$. Since $\mathcal{A}(X)$ is nice, we can find a sequence $\{\lambda_n\}$ of strictly positive scalars such that the pointwise limit $g_0 = \sum_{n=1}^{\infty} \lambda_n g_n$ is in $\mathcal{A}(X)$. Obviously,

$$x_0 \in z(g_0) = \bigcap_{n=1}^{\infty} z(g_n) = \bigcap_{n=1}^{\infty} z(f_n).$$

It follows from Lemma 2.5 that

$$\emptyset \neq z(\phi g_0) \subseteq \bigcap_{n=1}^{\infty} z(\phi(f_n)).$$

This establishes that $\mathcal{Z}_{x_0}$ is a z-filter with the countable intersection property.

Finally, we check the primeness of the z-filter $\mathcal{Z}_{x_0}$. Let f, g in $\mathcal{A}(X)$ be such that $z(\phi f) \cup z(\phi g) = Y$. Then $z(f) \cup z(g) = X$ since ϕ is biseparating by Lemma 2.3. As a result, x_0 must be in $z(f)$ or $z(g)$. This means that f or g belongs to $\mathcal{K}_{x_0}$, and thus proves $\mathcal{Z}_{x_0}$ is prime. ☐

Since Y is realcompact, by Lemma 2.6 we see that the intersection of $\mathcal{Z}_{x_0}$ is a singleton, and denote it by $\{\sigma(x_0)\}$. In other words,

$$f(x_0) = 0 \quad \Longrightarrow \quad \phi(f)(\sigma(x_0)) = 0, \quad \forall f \in \mathcal{A}(X).$$

Lemma 2.7. *For any f in $\mathcal{A}(X)$, we have*

$$(\phi f)(\sigma(x)) = (\phi 1)(\sigma(x)) f(x), \quad \forall x \in X. \tag{2.2}$$

Proof. For any f in $\mathcal{A}(X)$ and x in X, the function $f - f(x)$ is in $\mathcal{K}_x$. It follows that

$$\phi(f - f(x))(\sigma(x)) = 0,$$

and thus $(\phi f)(\sigma(x)) = \phi(1)(\sigma(x)) \cdot f(x)$. ☐

Proof of Theorem 2.2. Firstly, we shall see that $\sigma : X \to Y$ is one-to-one. Suppose that $x \neq x' \in X$ and $\sigma(x) = \sigma(x')$. Choose a function f from $\mathcal{A}(X)$ such that $f(x) = 0$ and $f(x') \neq 0$. By (2.2), we have the following contradiction. Note that $\phi 1$ is nonvanishing:

$$(\phi f)(\sigma(x)) = (\phi 1)(\sigma(x)) f(x) = 0$$

and

$$(\phi f)(\sigma(x')) = (\phi 1)(\sigma(x')) f(x') \neq 0.$$

Secondly, we claim that $\sigma(X)$ is dense in Y. Indeed, if there exists a y in $Y \setminus \overline{\sigma(X)}$, then we can choose a function f_1 from $\mathcal{A}(X)$ such that $(\phi f_1)(y) = 1$ and $\phi(f_1) \equiv 0$ on $\sigma(X)$ by the fullness of $\mathcal{A}(Y) = \phi(\mathcal{A}(X))$. For any x in X, we have

$$(\phi f_1)(\sigma(x)) = (\phi 1)(\sigma(x)) f_1(x) = 0.$$

This forces $f_1 = 0$. In turn, $(\phi f_1)(y) = 0$, which is impossible.

Thirdly, σ induces a homeomorphism from X onto $\sigma(X)$. Suppose on the contrary that a net $\{x_\lambda\}$ converges to x_0 in X but $\{\sigma(x_\lambda)\}$ does not converge to $\sigma(x_0)$ in Y. Without loss of generality, we can assume that all $\sigma(x_\lambda)$ lie outside an open neighborhood of $\sigma(x)$. Find a function g in $\mathcal{A}(X)$ such that $(\phi g)(\sigma(x_\lambda)) = 0$ for all λ and $(\phi g)(\sigma(x_0)) \neq 0$. Since

$$0 = (\phi g)(\sigma(x_\lambda)) = (\phi 1)(\sigma(x_\lambda)) g(x_\lambda)$$

and $\phi 1$ is nonvanishing, $g(x_\lambda) = 0$ for all λ, and hence $g(x_0) = 0$. This forces

$$(\phi g)(\sigma(x_0)) = (\phi 1)(\sigma(x_0)) g(x_0) = 0.$$

This is a contradiction. Similarly, we can prove that σ^{-1} is continuous from $\sigma(X)$ into X. Setting $Y_1 = \sigma(X)$ and $\tau = \sigma^{-1} : \sigma(X) \to X$, we get the desired assertion (2.1).

Now we verify that Y_1 contains all G_δ points in Y. Suppose y in $Y \setminus Y_1$ is a G_δ point. It follows from the fullness of $\mathcal{A}(Y) = \phi(\mathcal{A}(X))$ that there is an f in $\mathcal{A}(X)$ such that $z(\phi(f)) = \{y\}$. In particular, $\phi(f)$ is nonvanishing on Y_1. Then, the representation (2.2) ensures that $z(f) = \emptyset$. This contradicts the nonvanishing preserving property of ϕ. Hence, $y \in Y_1$. In the case Y consists of G_δ points, $Y = Y_1$.

Lastly, we show that $\sigma : X \to Y$ is surjective when $\mathcal{A}(X)$ is full and $\mathcal{A}(Y)$ is nice. In this case, we have $Z(\mathcal{A}(X)) = Z(X)$. For any y_0 in Y, set

$$\mathcal{Z}_{y_0} = \{z(f) : (\phi f)(y_0) = 0\}.$$

Arguing as in Lemma 2.6, we see that $\mathcal{Z}_{y_0}$ is also a prime z-filter on X with the countable intersection property. Since X is realcompact, $\bigcap \mathcal{Z}_{y_0}$ is a singleton and is denoted by $\{x_0\}$. It is then easy to see that $\sigma(x_0) = y_0$. □

Remark 2.8. (1) If $\mathcal{A}(X)$ is a uniformly closed unital subalgebra of $C^b(X)$, then $\mathcal{A}(X)$ is a nice sublattice. See, e.g., [9, Lemma 16.2].

(2) When $\mathcal{A}(X) \subseteq C(X)$ and $\mathcal{A}(Y) \subseteq C(Y)$ are endowed with the compact-open topology, or $\mathcal{A}(X) \subseteq C^b(X)$ and $\mathcal{A}(Y) \subseteq C^b(Y)$ endowed with the uniform topology, ϕ is automatically continuous. A proof for these facts make use of the weighted composition representation (2.1) and is left to the readers.

Note that every continuous map $\psi : X \to Y$ between completely regular spaces can be lifted uniquely to a continuous map $\psi^\upsilon : \upsilon X \to \upsilon Y$ between their realcompactifications. In particular, every f in $C(X)$ can be lifted uniquely to an f^υ in $C(\upsilon X)$ with the same range $f^\upsilon(\upsilon X) = f(X)$ (see, e.g., [9, Theorem 8.7 and 8B]). Consequently, f is nonvanishing if and only if f^υ is nonvanishing.

Theorem 2.9. *Suppose that X, Y are completely regular spaces with realcompactifications υX, υY, respectively. Let $\mathcal{A}(X), \mathcal{A}(Y)$ be nice and full. Assume that $\phi : \mathcal{A}(X) \to \mathcal{A}(Y)$ is a bijective linear nonvanshing preserver. Then, there exists a homeomorphism $\tau^\upsilon : \upsilon Y \to \upsilon X$ such that*

$$(\phi f)^\upsilon(y) = (\phi 1)^\upsilon(y) f^\upsilon(\tau^\upsilon(y)), \quad \forall\, f \in \mathcal{A}(X), y \in \upsilon Y.$$

In case both X and Y consist of G_δ-points, τ^υ restricts to a homeomorphism $\tau : Y \to X$ such that

$$\phi(f)(y) = \phi(1)(y) f(\tau(y)), \quad \forall f \in \mathcal{A}(X), y \in Y.$$

Proof. Denote by $\mathcal{A}(\upsilon X)$ the nice and full vector sublattice of $C(\upsilon X)$ consisting of the unique extensions $f^\upsilon : \upsilon X \to \mathbb{K}$ of all f in $\mathcal{A}(X)$. Since $\phi : \mathcal{A}(X) \to \mathcal{A}(Y)$ is nonvanishing preserving, $\phi^\upsilon : \mathcal{A}(\upsilon X) \to \mathcal{A}(\upsilon Y)$ defined by $\phi^\upsilon(f^\upsilon) = (\phi f)^\upsilon$ is also nonvanishing preserving. By Theorem 2.2, there is a homeomorphism $\tau^\upsilon : \upsilon Y \to \upsilon X$ such that

$$(\phi^\upsilon f^\upsilon)(y) = (\phi^\upsilon 1^\upsilon)(y) f^\upsilon(\tau^\upsilon(y)), \quad \forall\, f^\upsilon \in \mathcal{A}(\upsilon X), y \in \upsilon Y.$$

Finally, since $\upsilon X \backslash X$ and $\upsilon Y \backslash Y$ contain no G_δ-points (see, e.g., [9, p. 132]), $\tau^\upsilon(Y) = X$ when both X, Y consist of G_δ-points. □

Recall that a metric space (X, d) is said to be *quasi-convex* if there is a constant $C > 0$ such that for any points x, y in X there is a continuous curve joining x to y in X with length not greater than $Cd(x, y)$ (see [8]). The following corollary demonstrates the applicability of our main results. We do not claim the full originality, and some content can be seen in other papers, e.g., [3] for part (c) in the case where X, Y are complete metric spaces.

Corollary 2.10. *Suppose ϕ is a bijective linear nonvanishing preserver between the following function spaces. Then there is a homeomorphism $\tau : Y \to X$ such that*

$$\phi(f)(y) = \phi(1)(y)f(\tau(y)), \quad \forall y \in Y. \tag{2.3}$$

(a) $\phi : C(X) \to C(Y)$ or $\phi : C^b(X) \to C^b(Y)$, where X, Y are both realcompact spaces, or are both completely regular spaces such that all points of X, Y are G_δ-points.
(b) $\phi : UC(X) \to UC(Y)$ or $\phi : UC^b(X) \to UC^b(Y)$, where $UC(X), UC(Y)$ consist of uniformly continuous functions on the metric spaces X, Y, respectively. In this case, τ is a uniform homeomorphsim from Y onto X.
(c) $\phi : Lip(X) \to Lip(Y)$ or $\phi : Lip^b(X) \to Lip^b(Y)$, where $Lip(X), Lip(Y)$ consist of Lipschitz continuous functions on the metric spaces X, Y, respectively. In the case $\phi : Lip(X) \to Lip(Y)$, τ is a Lipschitz homeomorphism from Y onto X. We get the same conclusion in the other case, provided that X, Y are quasi-convex.

Proof. Note that all function spaces here are full and nice, closed in the lattice operations, and contain constant functions. So Theorems 2.2 and 2.9 apply.

For (b), it follows from (2.3) that $\phi(1)(y)\phi^{-1}(1)(\tau(y)) = 1$ for all y in Y. Define a linear map $\psi(f) = \phi(\phi^{-1}(1)f) = f \circ \tau$ from $UC^b(X)$ into $UC(Y)$. Using the arguments in [16, Theorem 2.3], we can show that τ is uniformly continuous. Similarly, τ^{-1} is also uniformly continuous.

In a similar manner, assertion (c) follows from [8, Theorems 3.9 and 3.12]. $\square$

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School of Mathematical Sciences and LPMC, Nankai University, Tianjin, 300071, People's Republic of China

E-mail address: leilee@nankai.edu.cn

Department of Applied Mathematics, National Sun Yat-sen University, Kaohsiung 80424, Taiwan

E-mail address: wong@math.nsysu.edu.tw