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The positive contractive part of a noncommutative L^p-space is a complete Jordan invariant

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Let $1 \leq p \leq +\infty$. We show that the positive part of the closed unit ball of a non-commutative L^p-space, as a metric space, is a complete Jordan *-invariant for the underlying von Neumann algebra.

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1. Introduction

Given a von Neumann algebra M, celebrated results of R.V. Kadison showed that several partial structures of M can recover the von Neumann algebra up to Jordan *-isomorphisms. In particular, each of the following is a complete Jordan *-invariant of M: the Banach space structure of the self-adjoint part M_{sa} of M ([4, Theorem 2]), the ordered vector space structure of M_{sa} ([4, Corollary 5]) and the topological convex set structure of the normal state space of M ([5, Theorem 4.5]).

Let $p \in [1, +\infty]$, and let $L^p(M)$ be the non-commutative L^p-space associated to M with the canonical cone $L^p(M)_+$. If M is semi-finite, P.-K. Tam showed in [14] that the ordered Banach space $(L^p(M)_{\mathrm{sa}}, L^p(M)_+)$ characterises M up to Jordan *-isomorphisms. In the case when M is σ-finite (but not necessarily semi-finite) and $p = 2$, the corresponding result follows from a result of A. Connes (namely, [2, Théorème 3.3]). For a general W^*-algebra M, results of L.M. Schmitt in [12] show that the ordered Banach space $(L^p(M)_{\mathrm{sa}}, L^p(M)_+)$ determines the real Lie algebra $M/Z(M)$, where $Z(M)$ is the center of M. On the other hand, extending results of B. Russo ([11]) and F.J. Yeadon ([16]), D. Sherman showed in [13] that the Banach space $L^p(M)$ is also a complete Jordan *-invariant for a general von Neumann algebra M when $p \neq 2$.

Along these lines, we will show in this article that the underlying metric space structure of the positive contractive part

$$L^p(M)^1_+ := L^p(M)_+ \cap L^p(M)^1 \qquad (1 \leq p \leq +\infty)$$

of $L^p(M)$ is also a complete Jordan *-invariant of M, where $L^p(M)^1$ is the closed unit ball. More precisely, we obtain in Theorem 3.1 that two arbitrary von Neumann algebras M and N are Jordan *-isomorphic whenever there exists a bijection Φ from $L^p(M)^1_+$ onto $L^p(N)^1_+$ which is isometric in the sense that

$$\|\Phi(x) - \Phi(y)\| = \|x - y\| \qquad (x, y \in L^p(M)^1_+).$$

Notice that, when $p = 2$, the closed unit ball $L^2(M)^1$ itself is not a complete Jordan *-invariant (since for any infinite dimensional von Neumann algebra M with a separable predual, one has $L^2(M) \cong \ell^2$), but its positive part is a Jordan *-invariant.

The ideas of our proof go as follows. In the case of $p = +\infty$, we employ a strong form of the Mazur–Ulam theorem (which was first proved by P. Mankiewicz) to show that a "shifting" Ψ of Φ extends to a linear bijective isometry from M_{sa} onto N_{sa} (see Proposition 3.2), and the map $x \mapsto \Psi(1)\Psi(x)$ induces a Jordan *-isomorphism from M to N (thanks to a result of R.V. Kadison). In the case of $p = 1$, we use Lemma 3.6 to show that $\Phi(0) = 0$, except for a few finite dimensional cases. We then use our previous result in [6] concerning normal state spaces to obtain the conclusion. For the remaining few finite dimensional cases, we use a Hausdorff dimension argument to show that M and N are *-isomorphic. In the case of $p \in (1, +\infty)$, we first use the strict convexity of $L^p(N)_{\mathrm{sa}}$

to show that Φ is affine (Lemma 3.10) and hence $\Phi(0) = 0$ (Proposition 3.9). Then we use several properties of non-commutative L^p-spaces (see Statement (S1)–(S4)) to relate the restriction of Φ on the positive part of the unit sphere of $L^p(M)_{\mathrm{sa}}$ to a biorthogonality preserving bijection between the normal state spaces of M and N. Finally, we use our previous results in [6] to finish the proof.

2. Preliminaries

Throughout this article, if E is a subset of a normed space X and $\lambda > 0$, we set

$$E^\lambda := \{x \in E : \|x\| \leq \lambda\}.$$

In the following, we will briefly recall (mainly from [15] and [10]) notations concerning non-commutative L^p-spaces. Let M be a (complex) von Neumann algebra on a (complex) Hilbert space $\mathfrak{H}$. Let φ be a fixed normal semi-finite faithful weight on M and α : $\mathbb{R} \to \mathrm{Aut}(M)$ be the modular automorphism group corresponding to φ. Then the von Neumann algebra crossed product $\check{M} := M \bar{\rtimes}_\alpha \mathbb{R}$ is semi-finite and we fix a normal faithful semi-finite trace τ on $\check{M}$. The *measure topology* on $\check{M}$ (as introduced by E. Nelson in [8]) is given by a neighborhood basis at 0 of the form

$$U(\epsilon, \delta) := \{x \in \check{M} : \|xp\| \leq \epsilon \text{ and } \tau(1-p) \leq \delta, \text{ for a projection } p \in \check{M}\}.$$

The completion, $L_0(\check{M}, \tau)$, of $\check{M}$ with respect to this topology is a *-algebra extending the *-algebra structure on $\check{M}$.

One may identify $L_0(\check{M}, \tau)$ with a collection of closed and densely defined operators on $L^2(\mathbb{R}; \mathfrak{H})$ affiliated with $\check{M}$. More precisely, suppose that T is such a closed operator on $L^2(\mathbb{R}; \mathfrak{H})$ and that $|T|$ is the absolute value of T with the spectral measure $E_{|T|}$. Then T corresponds (uniquely) to an element in $L_0(\check{M}, \tau)$ if and only if $\tau\big(1 - E_{|T|}([0, \lambda])\big) < +\infty$ when λ is large enough. In this case, the *-operation on $L_0(\check{M}, \tau)$ coincides with the adjoint. Moreover, the addition and the multiplication on $L_0(\check{M}, \tau)$ are the closures of the corresponding operations for densely defined closed operators. We denote by $L_0(\check{M}, \tau)_+$ the set of all positive self-adjoint (but not necessarily bounded) operators in $L_0(\check{M}, \tau)$.

The dual action $\hat{\alpha} : \mathbb{R} \to \mathrm{Aut}(\check{M})$ of α extends to an action on $L_0(\check{M}, \tau)$ by *-automorphisms. For any $p \in [1, +\infty]$, we set (with the convention that $e^{-s/p} = 1$ when $p = +\infty$)

$$L^p(M) := \{T \in L_0(\check{M}, \tau) : \hat{\alpha}_s(T) = e^{-s/p} T, \text{ for all } s \in \mathbb{R}\}.$$

Denote by $L^p(M)_{\mathrm{sa}}$ the set of all self-adjoint operators in $L^p(M)$ and put

$$L^p(M)_+ := L^p(M) \cap L_0(\check{M}, \tau)_+.$$

If $T \in L_0(\check{M}, \tau)$ and $T = u|T|$ is the polar decomposition, then $T \in L^p(M)$ if and only if $|T| \in L^p(M)$.

In the case when $p \in (1, +\infty)$, the map that sends $x \in \check{M}_+$ to x^p extends to a map

$$\Lambda_p : L_0(\check{M}, \tau)_+ \to L_0(\check{M}, \tau)_+.$$

For any $T \in L_0(\check{M}, \tau)_+$, one has $T \in L^p(M)$ if and only if $\Lambda_p(T) \in L^1(M)$. There is a canonical identification of M_* with $L^1(M)$ that sends the positive part $M_{*,+}$ of M_* onto $L^1(M)_+$, and this induces a Banach space norm $\|\cdot\|_1$ on $L^1(M)$ (see e.g. Theorem 7 in Chapter II of [15]). The function defined by

$$\|T\|_p := \|\Lambda_p(|T|)\|_1^{1/p} \tag{2.1}$$

is a norm on $L^p(M)$ that turns it into a Banach space. Let us also denote

$$\mathfrak{S}^p(M) := \{T \in L^p(M)_+ : \|T\|_p = 1\}. \tag{2.2}$$

It is known that $(L^p(M), L^p(M)_+)$ is independent of the choice of φ up to an isometric order isomorphism (see e.g. Theorem 37 and Corollary 38 in Chapter II of [15]). On the other hand, one may identify M with $L^\infty(M)$ (as ordered Banach spaces) through the canonical inclusion $M \subseteq \check{M} \subseteq L_0(\check{M}, \tau)$ (see Proposition 10 in Chapter II of [15]).

3. The main result

Theorem 3.1. *Let M and N be two von Neumann algebras and let $p \in [1, +\infty]$. If there is a bijective isometry $\Phi : L^p(M)^1_+ \to L^p(N)^1_+$, then M and N are Jordan *-isomorphic.*

In order to prove this result, we shall give some preparations in Propositions 3.2, 3.5 and 3.9 for the cases $p = +\infty$, $p = 1$ and $p \in (1, +\infty)$, respectively.

3.1. The case of $p = +\infty$

Proposition 3.2. *If $\Phi : M^1_+ \to N^1_+$ is a bijective isometry, then $\Psi : M^{1/2}_{\mathrm{sa}} \to N^{1/2}_{\mathrm{sa}}$ given by $\Psi(x) := \Phi(x + \frac{1}{2}) - \frac{1}{2}$ extends to a linear isometry from M_{sa} onto N_{sa}.*

It then follows from [4, Theorem 2] that $\Psi(1)$ is a self-adjoint unitary and Ψ further induces a Jordan isomorphism $x \mapsto \Psi(1)\Psi(x)$ from M_{sa} onto N_{sa}.

Example 3.3. Let $M = \mathbb{C} \oplus_\infty \mathbb{C}$. The set M^1_+ equals the square in $\mathbb{R} \oplus_\infty \mathbb{R}$ with vertices $(0, 0)$, $(0, 1)$, $(1, 1)$ and $(1, 0)$. If $\Phi_0 : \mathbb{R} \oplus_\infty \mathbb{R} \to \mathbb{R} \oplus_\infty \mathbb{R}$ is the clockwise rotation by 90 degree about the center $(\frac{1}{2}, \frac{1}{2})$, then the restriction Φ of Φ_0 on M^1_+ is a bijective isometry onto M^1_+ that sends $(0, 0)$ to $(0, 1)$. Hence, Φ itself cannot be extended to a linear map. However, if Ψ is defined as in Proposition 3.2, then $\Psi(1, 1) = \Phi\left(\frac{3}{2}, \frac{3}{2}\right) - \left(\frac{1}{2}, \frac{1}{2}\right) = (1, -1)$ and the map

$$(x, y) \mapsto \Psi(1,1)\Psi(x, y) = (1, -1)\big(\Phi_0(x + 1/2, y + 1/2) - (1/2, 1/2)\big) = (y, x)$$

extends to a *-automorphism of M.

In order to establish Proposition 3.2, we need the following stronger version of the Mazur–Ulam theorem, which was first proved in [7, Theorem 2] (see also [1, Theorem 14.1]).

Lemma 3.4. *Let U be a non-empty open connected subset of a normed space X and W be an open subset of a normed space Y. Then every isometry from U onto W can be extended uniquely to an affine isometry from X onto Y.*

Proof of Proposition 3.2. Let us first note that for any $x \in M_{\rm sa}$, one has $x \in M_+^1$ if and only if $\|x - \frac{1}{2}\| \leq \frac{1}{2}$ (by considering the C^*-subalgebra generated by x and 1). Thus, $x \mapsto x - \frac{1}{2}$ is a bijective isometry from M_+^1 onto $M_{\rm sa}^{\frac{1}{2}}$ and the map Ψ in the statement is a bijective isometry from $M_{\rm sa}^{\frac{1}{2}}$ onto $N_{\rm sa}^{\frac{1}{2}}$.

If $x \in M_{\rm sa}^{\frac{1}{2}}$, then $\|x\| = \frac{1}{2}$ if and only if there exists $x' \in M_{\rm sa}^{\frac{1}{2}}$ with $\|x - x'\| = 1$. This implies

$$\Psi\big(\{x \in M_{\rm sa} : \|x\| = 1/2\}\big) = \{y \in N_{\rm sa} : \|y\| = 1/2\}.$$

Consequently, $\Psi(0) = 0$ and Ψ will send the interior, $B_M\left(0, \frac{1}{2}\right)$, of $M_{\rm sa}^{\frac{1}{2}}$ onto the interior of $N_{\rm sa}^{\frac{1}{2}}$. By Lemma 3.4, the map $\Psi|_{B_M(0,\frac{1}{2})}$ extends to a linear isometry $\tilde{\Psi}$ from $M_{\rm sa}$ onto $N_{\rm sa}$ and the continuity of Ψ tells us that $\tilde{\Psi}|_{M_{\rm sa}^{\frac{1}{2}}} = \Psi$. $\square$

3.2. The case of $p = 1$

Proposition 3.5. *If there exists a bijective isometry Φ from $M_{*,+}^1$ onto $N_{*,+}^1$, then M and N are Jordan *-isomorphic.*

Note, first of all, that one cannot use Lemma 3.4 for this case, because the interior of $M_{*,+}^1$ could be an empty set; e.g., when $M = L^\infty([0, 1])$.

For any $\mu \in M_{*,+}$, we denote by $\operatorname{supp}\mu$ the support projection of μ in M. Recall that for any $\mu, \nu \in M_{*,+}$, we have

$$\|\mu - \nu\| = \|\mu\| + \|\nu\| \quad \text{if and only if} \quad \operatorname{supp}\mu \cdot \operatorname{supp}\nu = 0. \tag{3.1}$$

In order to obtain Proposition 3.5, we need the following lemma.

Lemma 3.6. *If N contains three non-zero projections, q_1, q_2 and q_3, orthogonal to each other, then the bijective isometry Φ in Proposition 3.5 will send 0 to 0.*

Proof. Suppose on the contrary that $\Phi(0) \neq 0$. Let us first show that $\operatorname{supp}\Phi(0) = 1$. Indeed, if it is not the case, one can find $\mu \in M^1_{*,+}$ such that $\|\Phi(\mu)\| = 1$ and $\operatorname{supp}\Phi(\mu) \leq 1 - \operatorname{supp}\Phi(0)$, which, together with (3.1), gives the contradiction that

$$1 \geq \|\mu - 0\| = \|\Phi(\mu) - \Phi(0)\| = \|\Phi(\mu)\| + \|\Phi(0)\| > 1.$$

As a result, $\Phi(0)(q_k) > 0$ for $k = 1, 2, 3$. We may also assume, without loss of generality, that $\Phi(0)(q_1) \leq \|\Phi(0)\|/3$ because

$$\sum\nolimits_{k=1}^{3} \Phi(0)(q_k) \leq \|\Phi(0)\|.$$

Now, pick any $\nu \in M^1_{*,+}$ with $\|\Phi(\nu)\| = 1$ and $\operatorname{supp}\Phi(\nu) \leq q_1$. Since $2q_1 - 1$ is a unitary and $\|\Phi(\nu) - \Phi(0)\| = \|\nu\| \leq 1$, one arrives at the following contradiction:

$$\begin{aligned} 1 &\geq |(\Phi(\nu) - \Phi(0))(q_1 - (1 - q_1))| = |1 - \Phi(0)(q_1) + \Phi(0)(1 - q_1)| \\ &= 1 + \|\Phi(0)\| - 2\Phi(0)(q_1) > 1. \qquad \square \end{aligned}$$

By Lemma 3.6, if N contains three non-zero projections orthogonal to one another, then Φ induces an isometric bijection from the normal state space of M to that of N, and hence, we may conclude that M and N are Jordan *-isomorphic by using [6, Theorem 3.4]. For the benefit of the readers, we will instead go through briefly the argument of [6, Theorem 3.4] by recalling the following two lemmas. These two lemmas will also be needed in the case of $p \in (1, +\infty)$ below.

Let us recall that a bijection Γ from the lattice of projections in M to that of N is an *orthoisomorphism* if for any projections p and q in M, one has

$$pq = 0 \quad \text{if and only if} \quad \Gamma(p)\Gamma(q) = 0.$$

Lemma 3.7. *([6, Lemma 3.1(a)]) Suppose that Ψ is a bijection from the normal state space of M to that of N, which is* biorthogonality preserving *in the sense that for any normal states μ and ν of M, one has*

$$\operatorname{supp}\mu \cdot \operatorname{supp}\nu = 0 \quad \textit{if and only if} \quad \operatorname{supp}\Psi(\mu) \cdot \operatorname{supp}\Psi(\nu) = 0.$$

Then there is an orthoisomorphism $\check{\Psi}$ from the lattice of projections in M to that of N satisfying $\check{\Psi}(\operatorname{supp}\mu) = \operatorname{supp}\Psi(\mu)$ for any normal state μ on M.

A second lemma that we need is the following possibly well-known variant of a theorem of H.A. Dye in [3] (see e.g. [6, Lemma 2.2(a)]). Note that an assumption of not having type I_2 summand is needed for the original version of Dye's theorem. However, the variant here has a weaker conclusion and does not need the assumption concerning the absence of type I_2 summand.

Lemma 3.8. *If there exists an orthoisomorphism from the lattice of projections in M to that of N, then M and N are Jordan *-isomorphic.*

Proof of Proposition 3.5. Let us first consider the case when N contains three non-zero projections orthogonal to each other. Then by Lemma 3.6, the map Φ restricts to an isometric bijection Ψ from the normal state space of M to that of N. Hence, (3.1) implies that Ψ is biorthogonality preserving. The conclusion now follows from Lemmas 3.7 and 3.8. In the case when M contains three non-zero projections orthogonal to one another, one may obtain the same conclusion by considering the bijective isometry Φ^{-1}.

Suppose that neither M nor N contains three non-zero projections orthogonal to one another. Then M and N can only be $\mathbb{C}$, $\mathbb{C} \oplus_\infty \mathbb{C}$ or $M_2(\mathbb{C})$. Observe that the Hausdorff dimensions of the quasi-state space of $\mathbb{C}$, $\mathbb{C} \oplus_\infty \mathbb{C}$ and $M_2(\mathbb{C})$ are 1, 2 and 4 respectively. Since a bijective isometry preserves Hausdorff dimensions, we conclude that M and N are *-isomorphic. □

3.3. A preparation for the case of $p \in (1, +\infty)$

Proposition 3.9. *Let $p \in (1, +\infty)$. Suppose that M and N are two von Neumann algebras such that $M \neq \mathbb{C}$ or $N \neq \mathbb{C}$. If $\Phi : L^p(M)^1_+ \to L^p(N)^1_+$ is a bijective isometry, then Φ is an affine map sending 0 to 0.*

Notice that $L^p(M)_{\rm sa}$ and $L^p(N)_{\rm sa}$ are strictly convex Banach spaces for $p \in (1, +\infty)$ (see e.g., Section 5 of [9]). The following lemma is possibly well-known, but we give its simple proof here for completeness.

Lemma 3.10. *Let X_1 and X_2 be Banach spaces such that X_2 is strictly convex. Then every isometry from a convex subset K of X_1 to X_2 is automatically an affine map.*

Proof. We need to verify that $f\big((x+y)/2\big) = \big(f(x)+f(y)\big)/2$ for every $x, y \in K$. Suppose that x and y are two arbitrarily chosen fixed elements in K with $x \neq y$. By replacing K with $K - y$ and f with the map from $K - y$ to X_2 that sends z to $f(z + y) - f(y)$, one may assume that $y = 0$ and $f(0) = 0$. Thus, we have $\|f(z)\| = \|z\|$ $(z \in K)$ and

$$\|f(x) - f(x/2)\| = \|x/2\| = \|f(x)\|/2 = \|f(x)\| - \|x\|/2 = \|f(x)\| - \|f(x/2)\|.$$

Now, the strict convexity of X_2 gives $f(x) - f(x/2) = t \cdot f(x/2)$ for some $t \in \mathbb{R}_+$. This, together with $\|f(x)\|/2 = \|f(x)\| - \|f(x/2)\|$, will produce $t = 1$. Hence we have the required relation $f(x)/2 = f(x/2)$. □

Proof of Proposition 3.9. We note, first of all, that if $M = \mathbb{C}$, then the Hausdorff dimension of $L^p(M)^1_+$ is one and hence so is the Hausdorff dimension of $L^p(N)^1_+$, which gives $N = \mathbb{C}$. Therefore, we may assume that the dimensions of both $M_{\rm sa}$ and $N_{\rm sa}$ are at least two.

Since $L^p(M)_{\rm sa}$ is strictly convex, the set of extreme points of $L^p(M)_+^1$ is $\mathfrak{S}^p(M) \cup \{0\}$ (see (2.2)). The same is true for $L^p(N)_+^1$. By Lemma 3.10, the map Φ is affine and hence $\Phi(0) \in \mathfrak{S}^p(N) \cup \{0\}$. Suppose on the contrary that $\Phi(0) \in \mathfrak{S}^p(N)$. Then, as $\dim L^p(N)_{\rm sa} > 1$, there is a sequence $\{v_k\}_{k\in\mathbb{N}}$ in $\mathfrak{S}^p(N) \setminus \{\Phi(0)\}$ with $\|v_k - \Phi(0)\| \to 0$, and hence $\{\Phi^{-1}(v_k)\}_{k\in\mathbb{N}}$ is a sequence in $\mathfrak{S}^p(M)$ norm-converging to 0, which is absurd. □

3.4. The finishing of the proof

For any $T \in L^p(M)_{\rm sa}$, we denote by $\operatorname{supp} T$ the support projection of T, i.e. $\operatorname{supp} T$ is the smallest projection p in M satisfying $T \cdot p = T$ (or equivalently, $p \cdot T = T$). Let us recall the following statements concerning $S, T \in L^p(M)_+$ from Fact 1.2 and Fact 1.3 of [10]:

S1). $\operatorname{supp} \Lambda_p(T) = \operatorname{supp} T$;
S2). $S \cdot T = 0$ if and only if $\operatorname{supp} S \cdot \operatorname{supp} T = 0$;
S3). if $\operatorname{supp} S \cdot \operatorname{supp} T = 0$, then $\|S + T\|_p^p = \|S - T\|_p^p = \|S\|_p^p + \|T\|_p^p$;
S4). if $p \neq 2$ and $\|S + T\|_p^p = \|S - T\|_p^p = \|S\|_p^p + \|T\|_p^p$, then $\operatorname{supp} S \cdot \operatorname{supp} T = 0$.

Proof of Theorem 3.1. The cases of $p = +\infty$ and $p = 1$ are proved in Proposition 3.2 (together with [4, Theorem 2]) and Proposition 3.5, through the canonical identifications of $L^1(M)$ and $L^\infty(M)$ with M_* and M, respectively. Moreover, the case of $p = 2$ is already established in [6, Corollary 3.11] (due to Proposition 3.9 and [6, Proposition 3.7]).

Now, we consider $p \in (1, +\infty) \setminus \{2\}$. Without loss of generality, we may assume that $M \neq \mathbb{C}$ or $N \neq \mathbb{C}$. By Proposition 3.9, the map Φ is affine and $\Phi(0) = 0$. On the other hand, it follows from Relation (2.1) that Λ_p restricts to a bijection from $L^p(M)_+^1$ onto $L^1(M)_+^1$ with $\Lambda_p(\mathfrak{S}^p(M)) = \mathfrak{S}^1(M)$ (see (2.2)). Therefore, Φ induces a bijection $\hat{\Phi} : \mathfrak{S}^1(M) \to \mathfrak{S}^1(N)$ with $\hat{\Phi}(A) = \Lambda_p(\Phi(\Lambda_p^{-1}(A))$. For any $A, B \in \mathfrak{S}^1(M)$, it follows from (S1), (S3) and (S4) that

$$\operatorname{supp} A \cdot \operatorname{supp} B = 0 \text{ if and only if } \left\| \frac{\Lambda_p^{-1}(A)}{2} + \frac{\Lambda_p^{-1}(B)}{2} \right\|_p^p = \left\| \frac{\Lambda_p^{-1}(A)}{2} - \frac{\Lambda_p^{-1}(B)}{2} \right\|_p^p = 2^{1-p}.$$

As Φ is an isometric affine map satisfying $\Phi(0) = 0$, the map $\hat{\Phi}$ can be regarded as a biorthogonality preserving bijection between the normal state spaces of M and N (through the identification $L^1(M) = M_*$ and (S2)). The conclusion now follows from Lemmas 3.7 and 3.8. □

Remark 3.11. Suppose that $M \ncong \mathbb{C}$. When $p = 2$, one can use [6, Theorem 3.8] and the argument of [6, Proposition 3.7] to obtain a Jordan *-isomorphism $\Theta : N \to M$ with

$\Lambda_2 \circ \Phi = \Theta^* \circ \Lambda_2|_{L^2(M)^1_+}$. In the case of $p \in [1, +\infty) \setminus \{2\}$, let us state the following question:

> Does there exist a Jordan *-isomorphism $\Theta : N \to M$ satisfying $\Lambda_p \circ \Phi = \Theta^* \circ \Lambda_p|_{L^p(M)^1_+}$?

In the case of $p = +\infty$, we have already seen in Example 3.3 that this stronger version does not hold.

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