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Asymptotic Behavior for Retarded Parabolic Equations with Superlinear Perturbations

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Abstract We obtain the existence and uniqueness of solutions for a class of retarded parabolic equations with superlinear perturbations. The asymptotic behavior result is studied by using the pullback attractor framework.

Keywords Retarded equations · Semilinear parabolic equations · Superlinear perturbations · Nonautonomous systems · Pullback attractors

1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. We deal with the following problem:

$$\begin{aligned}
&\frac{\partial u}{\partial t} + Au = F(u, u_t) + h(x,t), \quad x \in \Omega, t > \tau,\\
&u(x,\tau) = u_0(x), \qquad u(x,\tau+\theta) = \phi(x,\theta), \quad x \in \Omega,\ \theta \in (-\rho, 0).
\end{aligned} \tag{1}$$

Here A is a self-adjoint positive linear operator with domain $D(A) \subset L^2(\Omega)$ and with compact resolvent, $h \in L^2_{loc}(\mathbb{R}, L^2(\Omega))$, $\tau \in \mathbb{R}$ and $F(u, u_t)$ has the following form:

$$F(u, u_t) = g(u_t) - f(u), \tag{2}$$

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in which the functions $f, g \in C(\mathbb{R})$ satisfy the conditions below:

(F1) There exists $C_f > 0$ such that $|f(u)| \leqslant C_f(1+|u|^{p-1}),\ p > 2$, for all $u \in \mathbb{R}$.
(F2) $f(u)u \geqslant C_0|u|^p - C_1$ for some $C_0, C_1 > 0$, for every $u \in \mathbb{R}$.
(F3) $(f(u)-f(v))(u-v) \geqslant -\ell(u-v)^2$ for some $\ell > 0$ and for all $u, v \in \mathbb{R}$.
(G1) $g : L^2(-\rho, 0; L^2(\Omega)) \to L^2(\Omega)$, $g(0) = 0$ and

$$\|g(\xi) - g(\eta)\|_{L^2(\Omega)} \leqslant C_g \|\xi - \eta\|_{L^2(-\rho,0;L^2(\Omega))},$$

for $C_g > 0$ and for all $\xi, \eta \in L^2(-\rho, 0; L^2(\Omega))$.
(G2) There exist $M_0, M_1 > 0$ such that

$$|(g(\xi), \eta)_{L^2(\Omega)}| \leqslant M_0 \|\xi\|_{L^2(-\rho,0;L^2(\Omega))} \|\eta\|_{L^2(\Omega)} + M_1,$$

for any $(\xi, \eta) \in L^2(-\rho, 0; L^2(\Omega)) \times L^2(\Omega)$.

Throughout this work, for $t \in \mathbb{R}$, u_t stands for the function in $L^2(-\rho, 0; L^2(\Omega))$ such that $u_t(\theta) = u(t+\theta)$.

Retarded differential equations arise in many realistic models of problems in science and engineering where there is a time lag or after-effect. In particular, the parabolic case represents some issues in mathematical biology and the time lags are often seen as maturation time for population dynamics. Let us introduce some relevant literatures in Hale [1, 2], Kolmanovskii and Myshkis [3], Hino et al. [4], Agarwal et al. [5], Erbe et al. [6]. In the last decade, the growth of theory for finite and infinite dynamical systems has lead to many studies for a wide class of evolution equations, for which the asymptotic behavior of the solutions is considered in different frameworks. The theory of global attractors has been developed originally to deal with autonomous evolution equations (see e.g. [7, 8]). In the latter, to treat nonautonomous equations, some approaches have been proposed such as the uniform attractor, the trajectory attractor (see [9–11]), the pullback attractor (see, e.g., [12, 13]). Recently, retarded partial differential equations have become an important subject in light of physical and biological motivations, some natural extensions, and the rich history of retarded ordinary differential equations. We refer the readers to [14–17], among others. As far as we know, the existence and long-time behavior of solutions to retarded semilinear parabolic equations have been analyzed when the nonlinearity contains only the retarded term. In this work, we study a model in which the nonlinearity has both retarded term and superlinear perturbation such as in (2). On the other hand, we employ the analysis of the pullback attractor to investigate the long-time behavior for our problem.

The rest of the paper is organized as follows. In Sect. 2, we study the existence and uniqueness of solution for our problem. Sect. 3 is devoted to the results on long-time behavior of solutions. In Sect. 4, we discuss some special cases when the operator A is in degenerate form and assumes some classes of nonlinearities.

2 Existence and Uniqueness Results

By the assumptions on A, we see that $A : D(A) \to L^2(\Omega)$ has a discrete spectrum that contains only positive eigenvalues $\{\lambda_k\}_{k=1}^{\infty}$ and that the corresponding eigenfunctions

$\{e_k\}_{k=1}^{\infty}$ compose an orthogonal basis of the Hilbert space $L^2(\Omega)$. Then, one can define the operator A^{α} for $\alpha \in \mathbb{R}$. Now, we introduce some notations which will be used in this paper.

- $H = L^2(\Omega)$.
- $V = D(A^{\frac{1}{2}})$ with associated product $(u, v)_V = (A^{\frac{1}{2}}u, A^{\frac{1}{2}}v)_H$.
- $C_H = C([-\rho, 0]; H)$, $C_V = C([-\rho, 0]; V)$, $L^2_H = L^2(-\rho, 0; H)$, $L^2_V = L^2(-\rho, 0; V)$.

It is worth noting that, if $\alpha < \beta$ then $D(A^{\beta}) \subset D(A^{\alpha})$ and this embedding is compact (see [8]). In particular, we have $V \subset H \equiv H' \subset V'$ and all injections are dense and compact. Here, H' and V' are dual spaces of H and V, respectively.

Let us mention that, though the operator A generates an analytic semigroup, we cannot apply the theory of fixed points to prove the existence and uniqueness result, since the nonlinearity does not satisfy the local Lipschitz property. In order to prove the existence result, we make use of the iteration schemes similar to those as in [14]. We first recall the existence and uniqueness results for the following problem

$$\begin{aligned} &\frac{\partial u}{\partial t} + Au = h(x,t) - f(u), \quad x \in \Omega, t \in (\tau, T), \\ &u(x,\tau) = u_0(x), \end{aligned} \tag{3}$$

where the nonlinearity f satisfies (F1)–(F3).

Let $T > \tau$ and let $Q_{\tau,T} = \Omega \times (\tau, T)$. By weak solution of (3) on the interval (τ, T) we mean the function $u \in L^2(\tau, T; V) \cap L^p(Q_{\tau,T})$ satisfying the equation in (3) in the dual space $L^2(\tau, T; V') + L^{p'}(Q_{\tau,T})$, where p' is the conjugate exponent of p, i.e. $\frac{1}{p} + \frac{1}{p'} = 1$.

We have the following theorem.

Theorem 2.1 *Suppose that $u_0 \in H$, $h \in L^2_{loc}(\mathbb{R}, H)$ and f satisfies* (F1)–(F3). *Then, problem* (3) *has a unique weak solution $u \in C([\tau, T]; H) \cap L^p(Q_{\tau,T}) \cap L^2(\tau, T; V)$.*

Proof The existence statement can be proved as in [11, Theorem 2.1 and Lemma 2.2], while the uniqueness was proved in [11, Theorem 3.1]. □

For more information about the regularity of weak solutions, see [18, 19].

We give now the definition of weak solution for problem (1). Assume that $u_0 \in H$ and $\phi \in L^2_H$.

Definition 2.1 The function $u \in L^2(-\rho, T; H) \cap L^2(\tau, T; V) \cap L^p(Q_{\tau,T})$ is said to be a weak solution to problem (1) on the interval $(-\rho, T)$ if and only if $u(x, \tau) = u_0(x)$ a.e. in Ω, $u(\tau + .) = \phi$ in L^2_H, and u satisfies the equation in (1) in the dual space $L^2(\tau, T; V') + L^{p'}(Q_{\tau,T})$.

The following statement is the main result in this section.

Theorem 2.2 *Suppose that* $u_0 \in H$, $\phi \in L^2_H$, $h \in L^2_{loc}(\mathbb{R}, H)$, f *satisfies* (F1)–(F3) *and* g *satisfies* (G1). *If* $\ell < \lambda_1$ *then the problem* (1) *has a unique weak solution* u *on* $(-\rho, T)$ *such that* $u \in C([\tau, T]; H) \cap L^2(-\rho, T; H) \cap L^2(\tau, T; V) \cap L^p(Q_{\tau,T})$.

Proof Uniqueness. Suppose that u and v are weak solutions to (1). Denoting $w = u - v$, we have

$$\frac{1}{2}\frac{d}{dt}\|w(t)\|_H^2 + \|A^{\frac{1}{2}}w(t)\|_H^2 + \int_\Omega (f(u) - f(v))(u - v) = \int_\Omega (g(u_t) - g(v_t))w.$$

Then, using (F3), (G1) and the Cauchy inequality, one has

$$\frac{d}{dt}\|w\|_H^2 \leqslant (2\ell + 1)\|w\|_H^2 + C_g\|u_t - v_t\|_{L^2_H}.$$

Integrating the last inequality over (τ, t), $t > \tau$, we obtain

$$\|w(t)\|_H^2 \leqslant (2\ell + 1)\int_\tau^t \|w(s)\|_H^2 ds + C_g \int_\tau^t ds \int_{-\rho}^0 \|w(s + z)\|_H^2 dz. \tag{4}$$

Noting that $w(r) = 0$ for $r \in (\tau - \rho, \tau)$, one has

$$\begin{aligned}\int_\tau^t ds \int_{-\rho}^0 \|w(s + z)\|_H^2 dz &= \int_\tau^t ds \int_{s-\rho}^s \|w(r)\|_H^2 dr \\ &\leqslant \int_\tau^t ds \int_{\tau-\rho}^t \|w(r)\|_H^2 dr \\ &= \int_\tau^t ds \int_\tau^t \|w(r)\|_H^2 dr \\ &\leqslant (T - \tau)\int_\tau^t \|w(s)\|_H^2 ds. \end{aligned} \tag{5}$$

Combining this with (4), we arrive at

$$\|w(t)\|_H^2 \leqslant (2\ell + 1 + C_g(T - \tau))\int_\tau^t \|w(s)\|_H^2 ds.$$

Then the Gronwall inequality ensures the uniqueness as desired.

We now construct a sequence of functions $\{u^n\}$ as follows. Let u^1 be the solution of the problem

$$\begin{aligned}&\frac{\partial u^1}{\partial t} + Au^1 = h(x, t) - f(u^1), \quad x \in \Omega, t \in (\tau, T), \\ &u^1(x, \tau) = u_0(x), \quad x \in \Omega,\end{aligned} \tag{6}$$

such that

$$u^1(x, \tau + \theta) = \phi(x, \theta), \quad x \in \Omega,\ \theta \in (-\rho, 0) \text{ for } \phi \in L^2_H.$$

By Theorem 2.1, problem (6) has a unique weak solution. For $n \geqslant 2$, we consider the problem

$$\begin{aligned}&\frac{\partial u^n}{\partial t} + Au^n = h(x,t) + g(u_t^{n-1}) - f(u^n), \quad x \in \Omega, t \in (\tau, T),\\ &u^n(x,\tau) = u_0(x), \qquad u^n(x,\tau+\theta) = \phi(x,\theta), \quad x \in \Omega,\ \theta \in (-\rho, 0).\end{aligned} \tag{7}$$

Since $g(u_t^{n-1}) \in H$, using Theorem 2.1 again, we have the existence of u^n. It remains to prove that $\{u^n\}$ is a Cauchy sequence in $L^2(\tau-\rho, T; H)$ and converges to the weak solution of (1).

Putting

$$w^n = u^{n+1} - u^n, \quad \text{for } n \geqslant 2,$$

we observe that

$$\begin{aligned}&\|w^n(t)\|_H^2 + 2\int_\tau^t \|A^{\frac{1}{2}} w^n(s)\|_H^2 ds + 2\int_\tau^t ds \int_\Omega (f(u^{n+1}) - f(u^n))(u^{n+1} - u^n)\\ &= 2\int_\tau^t ds \int_\Omega (g(u_s^n) - g(u_s^{n-1}))w^n,\end{aligned}$$

for $t > \tau$. It follows from (F3), the Cauchy inequality, and (G1) that

$$\begin{aligned}&\|w^n(t)\|_H^2 + 2\int_\tau^t \|A^{\frac{1}{2}} w^n(s)\|_H^2 ds\\ &\quad \leqslant (2\ell + \epsilon)\int_\tau^t \|w^n(s)\|_H^2 ds + C\int_\tau^t \|w^{n-1}(s)\|_{L_H^2}^2 ds,\end{aligned}$$

where $C = C(\epsilon, C_g) > 0$.

Choosing $\epsilon > 0$ such that $\lambda_1 > \ell + \frac{\epsilon}{2}$ and using the fact that $\|A^{\frac{1}{2}} u\|_H^2 \geqslant \lambda_1 \|u\|_H^2$, for all $u \in V$, we get

$$\|w^n(t)\|_H^2 + (2\lambda_1 - 2\ell - \epsilon)\int_\tau^t \|w^n(s)\|_H^2 ds \leqslant C\int_\tau^t \|w^{n-1}(s)\|_{L_H^2}^2 ds.$$

Taking (5) into account, we have

$$\|w^n(t)\|_H^2 \leqslant (T-\tau)C\int_\tau^t \|w^{n-1}(s)\|_H^2 ds. \tag{8}$$

We see that, for $r \in (\tau, t]$,

$$\begin{aligned}\|w^n(r)\|_H^2 &\leqslant (T-\tau)C\int_\tau^r \|w^{n-1}(s)\|_H^2 ds\\ &\leqslant (T-\tau)C\int_\tau^t \|w^{n-1}(s)\|_H^2 ds.\end{aligned}$$

Therefore,

$$\begin{aligned}\sup_{\tau\leqslant r\leqslant t}\|w^n(r)\|_H^2 &\leqslant (T-\tau)C\int_\tau^t \|w^{n-1}(s)\|_H^2 ds\\ &\leqslant (T-\tau)C\int_\tau^t \sup_{\tau\leqslant r\leqslant s}\|w^{n-1}(r)\|_H^2 ds.\end{aligned}$$

Denoting

$$\eta^n(t)=\sup_{\tau\leqslant r\leqslant t}\|w^n(r)\|_H^2,\quad \text{for each } n\geqslant 2,$$

it follows that $\eta^n(\cdot)$ is an increasing function on $[\tau, T]$ and

$$\eta^n(t)\leqslant (T-\tau)C\int_\tau^t \eta^{n-1}(s)ds.$$

By iterative estimates, we have

$$\eta^n(t)\leqslant \frac{(T-\tau)^n C^{n-1}}{(n-1)!}\eta^1(T),\quad n\geqslant 2,\ t\in(\tau,T]. \tag{9}$$

Using the last inequality, one easily concludes that $\{u^n\}$ is a Cauchy sequence in $C(\tau, T; H)$ and $L^2(\tau, T; H)$. It is evident that $\{u^n\}$ is a Cauchy sequence in $L^2([\tau-\rho, T], H)$ also.

We are in the position to show that $\{u^n\}$ converges to the solution of (1). We have proved that there exists a function $u\in L^2(\tau-\rho, T; H)$ such that

$$u^n\to u,\quad \text{strongly in } C([\tau,T];H), \tag{10}$$

$$u^n\to u,\quad \text{strongly in } L^2(\tau-\rho,T;H), \tag{11}$$

$$u^n\to u,\quad \text{a.e. in } Q_{\tau,T}. \tag{12}$$

On the other hand, in light of (7),

$$\begin{aligned}&\|u^n(t)\|_H^2+2\int_\tau^t\|A^{\frac12}u^n(s)\|_H^2ds+2\int_\tau^t ds\int_\Omega f(u^n)u^n\\ &=\|u^n(\tau)\|_H^2+2\int_\tau^t(h+g(u_t^{n-1}),u^n)_H ds\\ &\leqslant \|u_0\|_H^2+\int_\tau^t(\|h(.,s)\|_H^2+2\|u^n(s)\|_H^2+\|g(u_s^{n-1})\|_H^2)ds\\ &\leqslant \|u_0\|_H^2+\int_\tau^t(\|h(.,s)\|_H^2+2\|u^n(s)\|_H^2+C_g\|u_s^{n-1}\|_{L_H^2}^2)ds\\ &\leqslant \|u_0\|_H^2+C_g(T-\tau)\|\phi\|_{L_H^2}^2\\ &\quad+\int_\tau^t(\|h(.,s)\|_H^2+2\|u^n(s)\|_H^2+C_g(T-\tau)\|u^{n-1}(s)\|_H^2)ds.\end{aligned} \tag{13}$$

Here, we have used the following estimate:

$$\begin{aligned}\|u_s^{n-1}\|_{L_H^2}^2 &= \int_{-\rho}^{0} \|u^{n-1}(s+z)\|_H^2 dz = \int_{s-\rho}^{s} \|u^{n-1}(r)\|_H^2 dr \\ &\leqslant \int_{\tau-\rho}^{s} \|u^{n-1}(r)\|_H^2 dr \\ &= \int_{\tau-\rho}^{\tau} \|u^{n-1}(r)\|_H^2 dr + \int_{\tau}^{s} \|u^{n-1}(r)\|_H^2 dr \\ &= \|\phi\|_{L_H^2}^2 + \int_{\tau}^{s} \|u^{n-1}(r)\|_H^2 dr. \end{aligned} \tag{14}$$

In view of the fact that $\{u^n\}$ is a bounded sequence in $L^2(\tau, T; H)$, from (F2) and (13) we deduce that

- $\{u^n\}$ is bounded in $L^2(\tau, T; V)$.
- $\{u^n\}$ is bounded in $L^p(Q_{\tau,T})$.

These allow us to state that

- $\{Au^n\}$ is bounded in $L^2(\tau, T; V')$.
- $\{f(u^n)\}$ is bounded in $L^{p'}(Q_{\tau,T})$ in view of (F1).

Rewriting the equation in (7) as

$$\frac{du^n}{dt} = h + g(u_t^{n-1}) - Au^n - f(u^n), \tag{15}$$

one can see that the sequence $\{\frac{du^n}{dt}\}$ is bounded in $L^2(\tau, T; V') + L^{p'}(Q_{\tau,T})$. Consequently, we obtain

$$u^n \rightharpoonup u, \quad \text{in } L^2(\tau, T; V), \tag{16}$$

$$f(u^n) \rightharpoonup \chi, \quad \text{in } L^{p'}(Q_{\tau,T}), \tag{17}$$

$$\frac{du^n}{dt} \rightharpoonup \frac{du}{dt}, \quad \text{in } L^2(\tau, T; V') + L^{p'}(Q_{\tau,T}), \tag{18}$$

$$g(u_t^{n-1}) \to g(u_t), \quad \text{in } L^2(\tau, T; H) \text{ by (G1) and (11).} \tag{19}$$

Finally, using (12) and (17) it follows that $\chi = f(u)$ (see [20]) and we have the weak solution of (1) by passing to the limit of (15) in $L^2(\tau, T; V') + L^{p'}(Q_{\tau,T})$. At last, (10) ensures that $u \in C([\tau, T]; H)$ and therefore the initial condition is meaningful. The proof completes. □

3 Existence of Pullback Attractors

The definition of pullback attractor was proposed to deal with nonautonomous evolution equations, since one observes that the initial time is just as important as the final time and the trajectories of a dynamical system may be unbounded as the time goes to infinity. For more detail about the discussion of this notion, see [15]. Let us now recall some related definitions and results.

Let X be a complete metric space (which in our case is $H \times L^2_H$) and let $B_X(a, r)$ be the ball in X centered at a with radius r. Instead of semigroup, we will use a two-parameters process on X, denoted by $U(t, \tau)$, which has the following properties:

- $U(\tau, \tau) = \mathrm{Id}$.
- $U(t, \tau)U(\tau, r) = U(t, r)$, for all $t \geqslant \tau \geqslant r$.

Definition 3.1 Let U be a process on a complete metric space X. A family of compact sets $\{\mathcal{A}(t)\}_{t\in\mathbb{R}}$ is said to be an X-pullback attractor for U if

(i) $U(t, \tau)\mathcal{A}(\tau) = \mathcal{A}(t)$ for all $t \geqslant \tau$,
(ii) $\lim_{s\to+\infty} \mathrm{dist}(U(t, t-s)B, \mathcal{A}(t)) = 0$, for all bounded set B of X.

In this definition, $\mathrm{dist}(B_1, B_2)$ is the Hausdorff semidistance between two subsets $B_1, B_2 \subset X$; i.e.,

$$\mathrm{dist}(B_1, B_2) = \sup_{b_1\in B_1} \inf_{b_2\in B_2} d(b_1, b_2).$$

The pullback attracting property (ii) considers the state of the system at time t when the initial time $t - s$ goes to $-\infty$.

Definition 3.2 A family of sets $\{\mathcal{B}(t)\}_{t\in\mathbb{R}}$ is said to be X-pullback absorbing with respect to the process $U(t, \tau)$ if, for any bounded subset B of X and any $t \in \mathbb{R}$, there exists $\tau(t, B) \leqslant t$ such that $U(t, \tau)B \subset \mathcal{B}(t)$ for all $\tau \leqslant \tau(t, B)$.

The following theorem [15, 21] shows the sufficient conditions for the existence of X-pullback attractor.

Theorem 3.1 *Let $U(t, \tau)$ be a continuous process on X. If there exists a compact X-pullback absorbing $\{\mathcal{B}(t)\}_{t\in\mathbb{R}}$ with respect to the process $U(t, \tau)$, then there exists a X-pullback attractor $\{\mathcal{A}(t)\}_{t\in\mathbb{R}}$, and $\mathcal{A}(t) \subset \mathcal{B}(t)$ for all $t \in \mathbb{R}$. Furthermore,*

$$\mathcal{A}(t) = \overline{\bigcup_{\substack{B\subset X\\ \text{bounded}}} \Lambda_B(t)},$$

where

$$\Lambda_B(t) = \bigcap_{n\in\mathbb{N}} \overline{\bigcup_{s\geqslant n} U(t, t-s)B}.$$

We will apply this result to our work to study the existence of a pullback attractor for the process generated by (1). A solution u of (1) with initial data (u_0, ϕ) will be denoted by $u = u(t, \tau, u_0, \phi)$. We define the process with respect to (1) as a mapping on $H \times L^2_H$,

$$U(t, \tau)(u_0, \phi) = (u(t, \tau, u_0, \phi), u_t(., \tau, u_0, \phi)). \tag{20}$$

This is a natural way, since the phase space is designated at the step of setting for (1). However, there is more than a way to define the process for our problem as we change the phase space. For more details, see [15].

Lemma 3.1 *Suppose that the assumptions of the Theorem 2.1 take place. Let* $u = u(t,\tau,u_0,\phi)$ *and* $v = v(t,\tau,v_0,\psi)$ *be the weak solutions of* (1). *Then, there exists a number* $M > 0$ *such that*

$$\|u(t)-v(t)\|_H^2 \leqslant (\|u_0-v_0\|_H^2 + M\|\phi-\psi\|_{L_H^2}^2)e^{M(t-\tau)},$$

for all $t \geqslant \tau$.

Proof It follows from (1) that

$$\frac{1}{2}\frac{d}{dt}\|u(t)-v(t)\|_H^2 + \|A^{\frac{1}{2}}(u(t)-v(t))\|_H^2 + \int_\Omega (f(u(t))-f(v(t)))(u(t)-v(t))$$
$$= \int_\Omega (g(u_t)-g(v_t))(u(t)-v(t)).$$

Using (F3), (G1) and the Cauchy inequality, with ϵ chosen such that $\lambda_1 - \ell - \frac{1}{2}\epsilon > 0$, we have

$$\begin{aligned}\frac{d}{dt}\|u(t)-v(t)\|_H^2 &+ 2\lambda_1\|u(t)-v(t)\|_H^2 - 2\ell\|u(t)-v(t)\|_H^2\\ &\leqslant C_\epsilon C_g\|u_t-v_t\|_{L_H^2}^2 + \epsilon\|u(t)-v(t)\|_H^2.\end{aligned}$$

Then,

$$\frac{d}{dt}\|u(t)-v(t)\|_H^2 \leqslant C\int_{-\rho}^0 \|u(t+s)-v(t+s)\|_H^2 ds,$$

where $C = C_\epsilon C_g$. Integrating from τ to t, we obtain

$$\begin{aligned}\|u(t)-v(t)\|_H^2 &\leqslant \|u_0-v_0\|_H^2 + C\int_\tau^t ds\int_{-\rho}^0 \|u(s+z)-v(s+z)\|_H^2 dz\\ &= \|u_0-v_0\|_H^2 + C\int_{-\rho}^0 dz\int_\tau^t \|u(s+z)-v(s+z)\|_H^2 ds\\ &\leqslant \|u_0-v_0\|_H^2 + C\int_{-\rho}^0 dz\int_{\tau-\rho}^t \|u(s)-v(s)\|_H^2 ds\\ &= \|u_0-v_0\|_H^2 + C\rho\|\phi-\psi\|_{L_H^2}^2 + C\rho\int_\tau^t \|u(s)-v(s)\|_H^2 ds.\end{aligned}$$

Hence, the Gronwall inequality gives

$$\|u(t)-v(t)\|_H^2 \leqslant (\|u_0-v_0\|_H^2 + C\rho\|\phi-\psi\|_{L_H^2}^2)e^{C\rho(t-\tau)},$$

for any $t \geqslant \tau$. □

The aim of the next lemma is to construct an $H \times L^2_H$-pullback absorbing for the process $U(t,\tau)$.

Lemma 3.2 *Under the assumptions of Theorem 2.2 and assuming that g satisfies* (G2), *the solution of* (1) *satisfies*

$$\|u(t)\|_H^2 \leqslant e^{-\alpha(t-\tau)}\|u_0\|_H^2 + l_1\rho e^{-\alpha(t-\tau-\rho)}\|\phi\|_{L^2_H} + \int_\tau^t e^{-\alpha(t-s)}\|h(s)\|_H^2 ds + l_2(1-e^{-\alpha(t-\tau)}),$$

where α, l_1, l_2 are positive constants.

Proof From (1), we get

$$\frac{d}{dt}\|u(t)\|_H^2 + 2\lambda_1\|u(t)\|_H^2 + 2C_0\|u(t)\|^p_{L^p(\Omega)} \leqslant 2C_1|\Omega| + 2(h(t)+g(u_t),u(t))_H.$$

In view of (G2) and the Cauchy inequality, we have

$$\begin{aligned}&\frac{d}{dt}\|u(t)\|_H^2 + 2\lambda_1\|u(t)\|_H^2 + 2C_0\|u(t)\|^p_{L^p(\Omega)}\\ &\quad\leqslant 2C_1|\Omega| + 2M_1 + (C_\epsilon+1)\|u(t)\|_H^2 + \epsilon\|u_t\|^2_{L^2_H} + \|h(t)\|_H^2,\end{aligned}$$

where ϵ is chosen later. Since $p > 2$, it follows from the Young inequality that there is a number $C_p > 0$ such that

$$(C_\epsilon+1)\|u\|_H^2 \leqslant 2C_0\|u\|^p_{L^p(\Omega)} + C_p.$$

Therefore,

$$\begin{aligned}&\frac{d}{dt}\|u(t)\|_H^2 + 2\lambda_1\|u(t)\|_H^2\\ &\quad\leqslant 2C_1|\Omega| + 2M_1 + C_p + \epsilon\int_{-\rho}^0 \|u(t+s)\|_H^2 ds + \|h(t)\|_H^2.\end{aligned} \tag{21}$$

Now, with $\alpha > 0$, we have

$$\begin{aligned}\frac{d}{dt}(e^{\alpha t}\|u(t)\|_H^2) &= \alpha e^{\alpha t}\|u(t)\|_H^2 + e^{\alpha t}\frac{d}{dt}\|u(t)\|_H^2\\ &\leqslant e^{\alpha t}\Bigg[(\alpha-2\lambda_1)\|u(t)\|_H^2\\ &\qquad + \|h(t)\|_H^2 + 2C_1|\Omega| + 2M_1 + C_p + \epsilon\int_{-\rho}^0\|u(t+s)\|_H^2 ds\Bigg].\end{aligned}$$

Integrating on $[\tau,t]$ for $t > \tau$ yields

$$e^{\alpha t}\|u(t)\|_H^2 \leqslant e^{\alpha\tau}\|u(\tau)\|_H^2 + (\alpha-2\lambda_1)\int_\tau^t e^{\alpha s}\|u(s)\|_H^2 ds + \int_\tau^t e^{\alpha s}\|h(.,s)\|_H^2 ds$$

$$
\begin{aligned}
&+\frac{2C_1|\Omega|+2M_1+C_p}{\alpha}(e^{\alpha t}-e^{\alpha\tau})\\
&+\epsilon\int_{-\rho}^{0}dz\int_{\tau}^{t}e^{\alpha s}\|u(s+z)\|_H^2ds. \qquad (22)
\end{aligned}
$$

The last term in (22) can be estimated as

$$
\begin{aligned}
\int_{-\rho}^{0}dz\int_{\tau}^{t}e^{\alpha s}\|u(s+z)\|_H^2ds &= \int_{-\rho}^{0}dz\int_{\tau}^{t}e^{-\alpha z}e^{\alpha(s+z)}\|u(s+z)\|_H^2ds\\
&\leqslant \rho e^{\alpha\rho}\int_{\tau-\rho}^{t}e^{\alpha s}\|u(s)\|_H^2ds\\
&\leqslant \rho e^{\alpha\rho}\left[e^{\alpha\tau}\int_{\tau-\rho}^{\tau}\|u(s)\|_H^2ds+\int_{\tau}^{t}e^{\alpha s}\|u(s)\|^2ds\right].
\end{aligned}
$$

Thus,

$$
\begin{aligned}
e^{\alpha t}\|u(t)\|_H^2 \leqslant{}& e^{\alpha\tau}\|u(\tau)\|_H^2+(\alpha+\epsilon\rho e^{\alpha\rho}-2\lambda_1)\int_{\tau}^{t}e^{\alpha s}\|u(s)\|_H^2ds\\
&+\epsilon\rho e^{\alpha(\tau+\rho)}\|\phi\|_{L_H^2}+\int_{\tau}^{t}e^{\alpha s}\|h(s)\|_H^2ds\\
&+\frac{2C_1|\Omega|+2M_1+C_p}{\alpha}(e^{\alpha t}-e^{\alpha\tau}). \qquad (23)
\end{aligned}
$$

Now, choosing ϵ and α such that

$$\alpha+\epsilon\rho e^{\alpha\rho}\leqslant 2\lambda_1,$$

we obtain that

$$
\begin{aligned}
\|u(t)\|_H^2 \leqslant{}& e^{-\alpha(t-\tau)}\|u_0\|_H^2+\epsilon\rho e^{-\alpha(t-\tau-\rho)}\|\phi\|_{L_H^2}\\
&+\int_{\tau}^{t}e^{-\alpha(t-s)}\|h(s)\|_H^2ds+\frac{2C_1|\Omega|+2M_1+C_p}{\alpha}(1-e^{-\alpha(t-\tau)}). \qquad (24)
\end{aligned}
$$

We just have the conclusion of the lemma. □

As a consequence, we have the following result.

Theorem 3.2 *Suppose that f satisfies* (F1)–(F3) *and g satisfies* (G1)–(G2). *If $\ell<\lambda_1$ and h has the following property*:

$$\int_{-\infty}^{t}\|h(s)\|_H^2ds<+\infty, \quad \textit{for each } t\in\mathbb{R},$$

then the process $U(t,\tau)$ associated with (1) *has an $H\times L_H^2$-pullback absorbing.*

Proof By Lemma 3.2, we have

$$\|u(t+\theta,\tau)\|_H^2 \leqslant e^{-\alpha(t+\theta-\tau)}\|u_0\|_H^2 + l_1\rho e^{-\alpha(t+\theta-\tau-\rho)}\|\phi\|_{L^2_H} + \int_{-\infty}^{t+\theta}\|h(s)\|_H^2 ds + l_2(1-e^{-\alpha(t+\theta-\tau)}),$$

for $t \geqslant \tau+\rho$ and for all $\theta \in (-\rho, 0)$.

One can see that there exists $\hat{\tau} = \hat{\tau}(t, u_0, \phi)$ such that, for all $\tau \leqslant \hat{\tau}$, the following inequality holds:

$$e^{-\alpha(t-\tau)}\|u_0\|_H^2 + l_1\rho e^{-\alpha(t-\tau-\rho)}\|\phi\|_{L^2_H} \leqslant 1.$$

Therefore,

$$\|u(t,\tau)\|_H^2 \leqslant \int_{-\infty}^{t}\|h(s)\|_H^2 ds + l_2 + 1, \tag{25}$$

$$\|u_t(\theta,\tau)\|_H^2 \leqslant \int_{-\infty}^{t}\|h(s)\|_H^2 ds + l_2 + 1 \tag{26}$$

for all $\tau \leqslant \hat{\tau} - \rho$. Hence, it is obvious that

$$\|U(t,\tau)(u_0,\phi)\|_{H\times L^2_H} = \|u(t)\|_H + \|u_t\|_{L^2_H} \leqslant (1+\sqrt{\rho})R(t), \tag{27}$$

where

$$R(t) = \left(\int_{-\infty}^{t}\|h(s)\|_H^2 ds + l_2 + 1\right)^{\frac{1}{2}}. \tag{28}$$

Then, for any bounded set $D \subset H \times L^2_H$, one easily deduces that

$$U(t,\tau)D \subset \mathcal{B}_H(t) = B_{H\times L^2_H}(0, (1+\sqrt{\rho})R(t)), \quad \text{for all } \tau \leqslant \hat{\tau}(t,D) - \rho.$$

Thus, the process $U(t,\tau)$ has an $H \times L^2_H$-pullback absorbing. □

Denote

$$F(u) = \int_0^u f(r)dr. \tag{29}$$

In order to prove some further properties of the process $U(t,\tau)$, we need the following condition.

(F4) There exist a positive constants C_2, C_3, C_4 such that

$$C_2|u|^p - C_4 \leqslant F(u) \leqslant C_3|u|^p + C_4.$$

Theorem 3.3 *Under the assumptions of Theorem* 3.2 *and condition* (F4), *the process* $U(t,\tau)$ *associated with* (1) *has a* $V \times L^2_V$*-pullback absorbing.*

Proof Let $u(t) = U(t,\tau)(u_0,\phi)$. We first need an estimate for $\int_\zeta^{\zeta+1}(\|u(s)\|_V^2 + F(u(s)))ds$. One can proceed as in the proof of Lemma 3.2 to get that

$$\frac{d}{dt}\|u(t)\|_H^2 + 2\|u(t)\|_V^2 + 2C_0\|u(t)\|_{L^p(\Omega)}^p \\ \leqslant 2C_1|\Omega| + 2M_1 + (1+M_0^2)\|u(t)\|_H^2 + \|u_t\|_{L_H^2}^2 + \|h(t)\|_H^2.$$

Since $p > 2$, it follows from the Young inequality that there is a number $C_p > 0$ such that

$$(1+M_0^2)\|u\|_H^2 \leqslant C_0\|u\|_{L^p(\Omega)}^p + C_p.$$

Therefore,

$$\frac{d}{dt}\|u(t)\|_H^2 + 2\|u(t)\|_V^2 + C_0\|u(s)\|_{L^p(\Omega)}^p \\ \leqslant 2C_1|\Omega| + 2M_1 + C_p + \int_{-\rho}^0 \|u(t+s)\|_H^2 ds + \|h(t)\|_H^2. \tag{30}$$

Now, for a given time $t > \tau$ and for all ζ such that $\tau < \zeta \leqslant t$, we have

$$\|u(\zeta+1)\|_H^2 - \|u(\zeta)\|_H^2 + 2\int_\zeta^{\zeta+1}\left(\|u(s)\|_V^2 + \frac{C_0}{2}\|u(s)\|_{L^p(\Omega)}^p\right)ds \\ \leqslant 2C_1|\Omega| + 2M_1 + C_p + \int_\zeta^{\zeta+1} ds\int_{-\rho}^0 \|u(s+z)\|_H^2 dz + \int_{-\infty}^{t+1}\|h(s)\|_H^2 ds.$$

This implies that

$$\int_\zeta^{\zeta+1}\left(\|u(s)\|_V^2 + \frac{C_0}{2}\|u(s)\|_{L^p(\Omega)}^p\right)ds \\ \leqslant C_1|\Omega| + M_1 + \frac{C_p}{2} + \frac{1}{2}\|u(\zeta)\|_H^2 + \frac{\rho}{2}\int_{\zeta-\rho}^{\zeta+1}\|u(s)\|_H^2 ds + \frac{1}{2}\int_{-\infty}^{t+1}\|h(s)\|_H^2 ds.$$

Invoking (25) and (28), we have

$$\int_\zeta^{\zeta+1}\left(\|u(s)\|_V^2 + \frac{C_0}{2}\|u(s)\|_{L^p(\Omega)}^p\right)ds \\ \leqslant C_1|\Omega| + M_1 + \frac{C_p}{2} + \frac{1}{2}R^2(\zeta) + \frac{1}{2}\rho(\rho+1)R^2(\zeta+1) + \frac{1}{2}\int_{-\infty}^{t+1}\|h(s)\|_H^2 ds, \tag{31}$$

for all $\zeta \leqslant t$ and $\tau \leqslant \hat{\tau} - \rho$. In view of (F4), we can write

$$\int_\zeta^{\zeta+1}\left(\|u(s)\|_V^2 + \int_\Omega F(u(s))\right)ds \leqslant \bar{R}^2(t), \tag{32}$$

where

$$\min\left(1, \frac{C_0}{2C_3}\right)\bar{R}^2(t) = C_1|\Omega| + M_1 + \frac{C_p}{2} + \frac{C_0C_4}{2C_3}|\Omega| + \frac{1}{2}R^2(t) + \frac{1}{2}\rho(\rho+1)R^2(t+1) + \frac{1}{2}\int_{-\infty}^{t+1}\|h(s)\|_H^2 ds.$$

The next step is to get an estimate for $\frac{d}{d\zeta}(\|u(\zeta)\|^2 + \int_\Omega F(u(s)))$. Multiplying the equation in (1) by $\dot{u} = \frac{du}{d\zeta}$, using (G2) and the Cauchy inequality, we have

$$\|\dot{u}(\zeta)\|_H^2 + \frac{d}{d\zeta}\left(\|u(\zeta)\|_V^2 + \int_\Omega F(u(\zeta))\right) = (h + g(u_\zeta), \dot{u}(\zeta)) \leqslant \|\dot{u}(\zeta)\|_H^2 + M_1 + \frac{M_0^2}{2}\|u_\zeta\|_{L_H^2}^2 + \frac{1}{2}\|h(\zeta)\|_H^2.$$

Taking (26) into account, one gets

$$\frac{d}{d\zeta}\left(\|u(\zeta)\|_V^2 + \int_\Omega F(u(\zeta))\right) \leqslant M_1 + \frac{\rho M_0^2}{2}R^2(t) + \frac{1}{2}\|h(\zeta)\|_H^2 \tag{33}$$

for all $\zeta \leqslant t$ and $\tau \leqslant \hat{\tau} - \rho$. On the other hand, it is evident that

$$\int_\zeta^{\zeta+1}\|h(s)\|_H^2 ds \leqslant \int_{-\infty}^{t+1}\|h(s)\|_H^2 ds < +\infty. \tag{34}$$

Putting (32)–(34) into the uniform Gronwall inequality, we deduce that there exists $\hat{R} = \hat{R}(t, u_0, \phi, h) > 0$ such that

$$\|u(\zeta)\|_V^2 + \int_\Omega F(u(\zeta)) \leqslant \hat{R}^2(t),$$

for all $\zeta \leqslant t$ and $\tau \leqslant \hat{\tau} - \rho$.

Using (F4) again, we obtain

$$\|u(t)\|_V^2 + C_2\|u(t)\|_{L^p(\Omega)}^p \leqslant \hat{R}^2(t) + C_4|\Omega|, \tag{35}$$

$$\|u_t(\theta)\|_V^2 + C_2\|u_t(\theta)\|_{L^p(\Omega)}^p \leqslant \hat{R}^2(t) + C_4|\Omega|, \tag{36}$$

$$\|u_t\|_{L_V^2}^2 \leqslant \rho\hat{R}^2(t) + \rho C_4|\Omega|, \tag{37}$$

for all $\tau \leqslant \hat{\tau} - \rho$ and $\theta \in (-\rho, 0)$.

We conclude that, for any bounded set $D \subset V \times L_V^2$, we have

$$U(t,\tau)D \subset \mathcal{B}_V(t) = B_{V\times L_V^2}\left(0, (1+\sqrt{\rho})\sqrt{\hat{R}^2(t) + C_4|\Omega|}\right), \quad \text{for all } \tau \leqslant \hat{\tau}(t, D) - \rho.$$

Thus, the process $U(t,\tau)$ has a $V \times L_V^2$-pullback absorbing. □

Remark 3.1 In our case, the external force h is not supposed to be translation bounded, i.e.

$$\|h\|^2_{L^2_b(H)} := \sup_{t\in\mathbb{R}} \int_t^{t+1} \|h(s)\|^2_H ds < +\infty,$$

since we do not need the uniform boundedness of the trajectories of (1) under the analysis of the pullback attractor. However, this class of external forces can be used for our problem. Indeed, if h is translation bounded, then for $\alpha > 0$ we have

$$\begin{aligned}\int_\tau^t e^{-\alpha(t-s)}\|h(s)\|^2_H ds &\leqslant \sum_{k=0}^{\infty}\int_{t-k-1}^{t-k} e^{-\alpha(t-s)}\|h(s)\|^2_H ds\\ &\leqslant \sum_{k=0}^{\infty} e^{-\alpha k}\int_{t-k-1}^{t-k} \|h(s)\|^2_H ds\\ &\leqslant \frac{1}{1-e^{-\alpha}}\|h\|^2_{L^2_b(H)}.\end{aligned}$$

Thus, using Lemma 3.2, we obtain the same results as in Theorems 3.2 and 3.3. It should be mentioned that there is no relations between our class of external forces and the set of translation bounded functions.

Remark 3.2 By a priori estimates, we can prove the existence of a C_H-pullback absorbing and a C_V-pullback absorbing w.r.t. the process $U(t,\tau)$, if we change the phase space from $H \times L^2_H$ to C_H. In this case, we have a simple definition for $U(t,\tau)$ as follows:

$$U(t,\tau)(\phi) = u_t(.,\tau,\phi(0),\phi),$$

where $u \in C([-\rho,T];H) \cap L^2(\tau,T;V) \cap L^p(Q_{\tau,T})$ is the solution of (1) on the interval $[-\rho,T]$, for any $T \in \mathbb{R}$, $T > \tau$. The readers are referred to [15] for this approach.

We now can state the main theorem of this section.

Theorem 3.4 *Under the assumptions of Theorem* 3.3, *the process* $U(t,\tau)$ *associated with* (1) *has an* $H \times L^2_H$*-pullback attractor.*

Proof By Lemma 3.1, we see that $U(t,\tau)$ is continuous mapping on $H \times L^2_H$. In order to apply Theorem 3.1, we prove that, there exists a compact $H \times L^2_H$-pullback absorbing with respect to the process $U(t,\tau)$. From Theorem 3.3, $U(t,\tau)$ has a $V \times L^2_V$-pullback absorbing $\{\mathcal{B}_V(t)\}$. Let

$$\mathcal{B}(t) = \bigcup_{\tau\leqslant\hat{\tau}(t,\mathcal{B}_V)-\rho} U(t,\tau)\mathcal{B}_V(t).$$

It is easy to see that, $\{\mathcal{B}(t)\}$ is a $V \times L^2_V$-pullback absorbing of $U(t,\tau)$. We now show that $\mathcal{B}(t)$ is precompact in $H \times L^2_H$. Let Π_1 and Π_2 be projectors on $H \times L^2_H$, i.e.

$\Pi_1 : (u_0, \phi) \mapsto u_0$, $\Pi_2 : (u_0, \phi) \mapsto \phi$. One observes that $\Pi_1\mathcal{B}(t)$ is bounded in V and then it is precompact in H. It remains to prove that $\Pi_2\mathcal{B}(t)$ is precompact in L^2_H.

Let $u_t \in \Pi_2\mathcal{B}(t)$. For a given $t > \tau + \rho$, (36) ensures that $u(t+\theta)$, $\theta \in (-\rho, 0)$, belong to a bounded set in $V \cap L^p(\Omega)$. Denoting $\Theta = [t-\rho, t]$ and $Q_\Theta = \Omega \times \Theta$, it follows that u belong to a bounded set in $L^2(\Theta; V \cap L^p(\Omega))$. We rewrite an equation in (1) as one in the dual space $L^2(\Theta; V') + L^{p'}(Q_\Theta)$,

$$\dot{u}(\zeta) = h(\zeta) + g(u_\zeta) - Au(\zeta) - f(u(\zeta)). \tag{38}$$

At first, for any $v \in L^2(\Theta; V) \cap L^p(Q_\Theta)$, we have

$$\begin{aligned} |\langle h, v\rangle| &\leqslant \|h\|^2_{L^2(Q_\Theta)}\|v\|^2_{L^2(Q_\Theta)} \\ &\leqslant C\|h\|^2_{L^2(Q_\Theta)}\|v\|^2_{L^2(\Theta;V)}. \end{aligned} \tag{39}$$

Using (G2), one gets

$$\begin{aligned} |\langle g(u_\zeta), v\rangle| &\leqslant M_0 \int_{t-\rho}^{t} \|u_\zeta\|_{L^2_H}\|v(\zeta)\|_H d\zeta + \rho M_1 \\ &\leqslant CM_0\left(\int_{t-\rho}^{t} d\zeta \int_{-\rho}^{0} \|u(\zeta+z)\|^2_H dz\right)^{\frac{1}{2}} \|v\|_{L^2(\Theta;V)} + \rho M_1 \\ &\leqslant \rho CM_0\left(\int_{t-2\rho}^{t} \|u(\zeta)\|^2_H d\zeta\right)^{\frac{1}{2}} \|v\|_{L^2(\Theta;V)} + \rho M_1. \end{aligned} \tag{40}$$

It is obvious that

$$|\langle Au, v\rangle| \leqslant \|u\|_{L^2(\Theta;V)}\|v\|_{L^2(\Theta;V)}. \tag{41}$$

At last, taking (F1) into account, we have

$$\begin{aligned} |\langle f(u), v\rangle| &\leqslant \left(\int_{Q_\Theta} |f(u)|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}} \left(\int_{Q_\Theta} |v|^p\right)^{\frac{1}{p}} \\ &\leqslant C_f\|u\|^{p-1}_{L^p(Q_\Theta)}\|v\|_{L^p(Q_\Theta)} + C. \end{aligned} \tag{42}$$

Combining (39)–(42), we obtain that $\dot{u}$ belongs to a bounded set in $L^2(\Theta; V') + L^{p'}(Q_\Theta) \subset L^{p'}(\Theta; V' + L^{p'}(\Omega))$. Using the compactness lemma in [20], we conclude that u belongs to a compact set in $L^2(\Theta; H)$; or equivalently, $\{u_t \in \Pi_2\mathcal{B}(t)\}$ is precompact in $L^2([-\rho, 0]; H)$. The proof is complete. ☐

4 Further Remarks

Let us discuss some special cases of the operator A. A typical example for A is that $A = -\Delta$ and $D(A) = H^2(\Omega) \cap H^1_0(\Omega)$. In this case, we have $V = D(A^{\frac{1}{2}}) = H^1_0(\Omega)$. Now, we introduce two examples in which A is a degenerate elliptic operator.

Grushin Type Operator Let

$$A = -G_k := -\Delta_x - |x|^{2k}\Delta_y = -\sum_{i=1}^{N_1}\frac{\partial^2}{\partial x_i^2} - |x|^{2k}\sum_{i=1}^{N_2}\frac{\partial^2}{\partial y_i^2},$$

where $k \geqslant 0$. In this case, Ω is a smooth bounded domain in $\mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$. This operator was first introduced in [22] and one knows that it is not elliptic if $k > 0$ and Ω intersects the hyperplane $\{x = 0\}$.

In [23], to study the boundary-value problem, the authors use the natural energy space $S_0^1(\Omega)$ defined as the completion of $C_0^1(\bar{\Omega})$ in the norm

$$\|u\|_{S_0^1(\Omega)} = \left(\int_\Omega \big(|\nabla_x u|^2 + |x|^{2k}|\nabla_y u|^2\big)dxdy\right)^{1/2}.$$

We have the continuous embedding

$$S_0^1(\Omega) \subset L^p(\Omega), \quad \text{for } 2 \leqslant p \leqslant 2_k^* = \frac{2N(k)}{N(k)-2},$$

where

$$N(k) = N_1 + (k+1)N_2.$$

Moreover, this embedding is compact if $2 \leqslant p < 2_k^*$ and 2_k^* is the so-called critical exponent for the embedding. In view of the compact embedding $S_0^1(\Omega) \subset L^2(\Omega)$, we see that G_k is positively definite and has a compact resolvent (for more details, see [23]). Thus, it can be used for our problem with $V = S_0^1(\Omega)$.

Caldiroli-Musina Type Operator In this example, we are interested in the case $A = -\mathrm{div}(\sigma(x)\nabla u)$. The degeneracy of A is considered in the sense that the measurable, nonnegative diffusion coefficient $\sigma(x)$ is allowed to have at most a finite number of (essential) zeroes at some points. In [24], where a semilinear degenerate elliptic problem was studied, the authors assume that the function $\sigma : \Omega \to \mathbb{R}$ satisfies the following assumption for some $\alpha \in (0, 2)$:

$$(\mathcal{H}_\alpha) \quad \sigma \in L^1_{loc}(\Omega) \quad \text{and} \quad \liminf_{x\to z} |x-z|^{-\alpha}\sigma(x) > 0, \quad \text{for every } z \in \overline{\Omega}.$$

By $\mathcal{D}_0^{1,2}(\Omega, \sigma)$, we denote the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|v\| = \left(\int_\Omega \sigma(x)|\nabla v|^2\right)^{\frac{1}{2}}.$$

According to [24], if σ satisfies $(\mathcal{H}_\alpha)$, then the following assertions hold:

(i) $\mathcal{D}_0^{1,2}(\Omega, \sigma) \subset L^{2_\alpha^*}(\Omega)$ continuously.
(ii) $\mathcal{D}_0^{1,2}(\Omega, \sigma) \subset L^r(\Omega)$ compactly, if $r \in [1, 2_\alpha^*)$, where $2_\alpha^* = \frac{2N}{N-2+\alpha}$.

In particular, we have a compact embedding $\mathcal{D}_0^{1,2}(\Omega) \subset L^2(\Omega)$. This leads to the fact that A is positive definite and has a compact resolvent. We can put it into (1) and the energy space $\mathcal{D}_0^{1,2}(\Omega)$ plays the role of V.

Some Classes of Nonlinearity As a final demonstration, we consider some particular cases of nonlinearity $F(u, u_t)$ in our problem. In the case $g(u_t) = 0$, we have the reaction-diffusion problem, which was studied by many authors (see [11, 18–20] and references therein).

The particular case $f(u) = |u|^{p-1}u,\ p > 2$ satisfies (F1)–(F4) obviously.

Let us mention that one can deal with the more general case, namely,

$$f(u) = |u|^{p-1}u + k(u)$$

provided that:

- $k(u) \leqslant C_k(|u|^{q-1} + 1),\ C_k > 0,\ q < p.$
- $(k(u) - k(v))(u - v) \geqslant -\ell|u - v|^2.$
- $k_0|u|^q - k_1 \leqslant K(u) \leqslant k_0|u|^q - k_2$, where $K(u) = \int_0^u k(s)ds$ and k_0, k_1, k_2 are positive numbers.

As a special case of the retarded term $g(u_t)$, we recall the work of Rezounenko and Wu [17], in which a so-called *state-dependent* selective delay term was introduced,

$$g(u_t) = \int_{-\rho}^{0} \left\{ \int_{\Omega} b(u(x + \theta, y))k(x - y)dy \right\} \chi(\theta, u_t)d\theta.$$

We can give some restrictions on g, which are similar to those in [17]; precisely:

- $b : \mathbb{R} \to \mathbb{R}$ has the Lipschitz property and $|b(w)| \leqslant b_0|w| + b_1, b_0, b_1 > 0.$
- $k : \Omega \to \mathbb{R}$ is bounded.
- $\chi : [-\rho, 0] \times L^2_H \to \mathbb{R}$ is Lipschitz with respect to the second coordinate and

$$\|\chi(., v)\|_{L^2(-\rho,0)} \leqslant C_\chi \quad \text{for some } C_\chi > 0 \text{ and for all } v \in H.$$

By these assumptions, one can proceed as in [17] to prove that

$$|(g(\xi), \eta)_H| \leqslant M_0(\|\xi\|_{L^2_H} + 1)\|\eta\|_H,$$

for some $M_0 > 0$ and for all $(\xi, \eta) \in L^2_H \times H$. Actually, the last inequality can replace (G2), since we employ (G2) in the situation that

$$|(g(\xi), \eta)_H| \leqslant \epsilon\|\xi\|^2_{L^2_H} + C_\epsilon(\|\eta\|^2_H + 1).$$

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