

OPERATOR IDEALS ARISING FROM GENERATING SEQUENCES

NGAI-CHING WONG

Dedicated to Professor Kar-Ping Shum in honor of his seventieth birthday

ABSTRACT. In this note, we will discuss how to relate an operator ideal on Banach spaces to the sequential structures it defines. Concrete examples of ideals of compact, weakly compact, completely continuous, Banach-Saks and weakly Banach-Saks operators will be demonstrated.

1. INTRODUCTION

Let $T : E \rightarrow F$ be a linear operator between Banach spaces. Let U_E, U_F be the closed unit balls of E, F , respectively. Note that closed unit balls serve simultaneously the basic model for open sets and bounded sets in Banach spaces. The usual way to describe T is to state either the bornological property, via TU_E , or the topological property, via $T^{-1}U_F$, of T . An other way classifying T is through the sequential structures T preserves. Here are some well-known examples.

- Examples 1.1.**
- (1) T is *bounded* (i.e., TU_E is a bounded subset of F)
 - $\Leftrightarrow T$ is *continuous* (i.e., $T^{-1}U_F$ is a 0-neighborhood of E in the norm topology)
 - $\Leftrightarrow T$ is *sequentially bounded* (i.e., T sends bounded sequences to bounded sequences);
 - (2) T is *of finite rank* (i.e., TU_E spans a finite dimensional subspace of F)
 - $\Leftrightarrow T$ is *weak-norm continuous* (i.e., $T^{-1}U_F$ is a 0-neighborhood of E in the weak topology)
 - $\Leftrightarrow T$ sends bounded sequences to sequences spanning finite dimensional subspaces of F ;

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- (3) T is *compact* (i.e., TU_E is totally bounded in F)
 $\Leftrightarrow T$ is continuous in the topology of uniform convergence on norm compact subsets of E' (i.e., $T^{-1}U_F \supseteq K^\circ$, the polar of a norm compact subset K of the dual space E' of E)
 $\Leftrightarrow T$ is *sequentially compact* (i.e., T sends bounded sequences to sequences with norm convergent subsequences); and
- (4) T is *weakly compact* (i.e., TU_E is relatively weakly compact in F)
 $\Leftrightarrow T$ is continuous in the topology of uniform convergence on weakly compact subsets of E' (i.e., $T^{-1}U_F \supseteq K^\circ$, the polar of a weakly compact subset K of E')
 $\Leftrightarrow T$ is *sequentially weakly compact* (i.e., T sends bounded sequences to sequences with weakly convergent subsequences).

In [18, 17, 15], we investigate the duality of the topological and bornological characters of operators demonstrated in the above examples, and applied them in the study of operator ideal theory in the sense of Pietsch [6]. In [16], we use these concepts in classifying locally convex spaces. They are also used by other authors in operator algebra theory in [13, 2].

After giving a brief account of the equivalence among the notions of operator ideals, generating topologies, and generating bornologies developed in [18, 17, 15] in Section 2, we shall explore into the other option of using sequential structures in Section 3. The theory is initiated by Stephani [11, 12]. We provide a variance here. In particular, we will show that the notions of operator ideals, generating bornologies and generating sequences are equivalent in the context of Banach spaces. However, we note that the sequential methodology does not seem to be appropriate in the context of locally convex spaces, as successfully as in [18, 16]. Some examples of popular operator ideals are treated in our new ways as demonstrations in Section 4.

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2. THE TRIANGLE OF OPERATORS, TOPOLOGIES AND BORNLOGIES

Let X be a vector space over $\mathbb{K} = \mathbb{R}$ or \mathbb{C} . Following Hogbe-Nlend [4], by a *vector bornology* on X we mean a family \mathcal{B} of subsets of X satisfies the following conditions:

- (VB₁) $X = \cup \mathcal{B}$;
- (VB₂) if $B \in \mathcal{B}$ and $A \subseteq B$ then $A \in \mathcal{B}$;
- (VB₃) $B_1 + B_2 \in \mathcal{B}$ whenever $B_1, B_2 \in \mathcal{B}$;
- (VB₄) $\lambda B \in \mathcal{B}$ whenever $\lambda \in \mathbb{K}$ and $B \in \mathcal{B}$;
- (VB₅) the circle hull $\text{ch } B$ of any B in \mathcal{B} belongs to \mathcal{B} .

Elements in \mathcal{B} are called \mathcal{B} -bounded sets in X . A vector bornology \mathcal{B} on X is called a *convex bornology* if $\Gamma B \in \mathcal{B}$ for all B in \mathcal{B} , where ΓB is the absolutely convex hull of B . The pair (X, \mathcal{B}) is called a *convex bornological space* which is denoted by $X^{\mathcal{B}}$.

Let (X, \mathcal{T}) be a locally convex space. For any subset B of X , the σ -disked hull of B is defined to be

$$\Gamma_{\sigma} B = \{ \sum_{n=1}^{\infty} \lambda_n b_n : \sum_{n=1}^{\infty} |\lambda_n| \leq 1, \lambda_n \in \mathbb{K}, b_n \in B, n = 1, 2, \dots \}.$$

A set B in X is said to be σ -disked if $B = \Gamma_{\sigma} B$. An absolutely convex, bounded subset B of X is said to be *infracomplete* (or a *Banach disk*) if the normed space $(X(B), r_B)$ is complete, where

$$X(B) = \bigcup_{n \geq 1} nB \quad \text{and} \quad r_B(x) := \inf \{ \lambda > 0 : x \in \lambda B \}$$

is the gauge of B defined on $X(B)$.

Lemma 2.1 ([18]). *Let (X, \mathcal{P}) be a locally convex space and B an absolutely convex bounded subset of X .*

- (a) *If B is σ -disked then B is infracomplete.*
- (b) *If B is infracomplete and closed then B is σ -disked.*

Let X, Y be locally convex spaces. Denote by $\sigma(X, X')$ the weak topology of X with respect to its dual space X' , while $\mathcal{P}_{\text{ori}}(X)$ is the original topology of X . We employ the notion $\mathcal{M}_{\text{fin}}(Y)$ for the *finite dimensional bornology* of Y which has a basis consisting of

all convex hulls of finite sets. On the other hand, $\mathcal{M}_{\text{von}}(Y)$ is used for the *von Neumann bornology* of Y which consists of all $\mathcal{P}_{\text{ori}}(Y)$ -bounded subsets of Y . Ordering of topologies and bornologies are induced by set-theoretical inclusion, as usual. Moreover, we write briefly $X_{\mathcal{P}}$ for a vector space X equipped with a locally convex topology \mathcal{P} and $Y^{\mathcal{M}}$ for a vector space Y equipped with a convex vector bornology \mathcal{M} .

Let X, Y be locally convex spaces. We denote by $\mathfrak{L}(X, Y)$ and $L^{\times}(X, Y)$ the collection of all linear operators from X into Y which are continuous and (locally) bounded (i.e., sending bounded sets to bounded sets), respectively.

Definition 2.2 (see [16]). Let \mathcal{C} be a subcategory of locally convex spaces.

- (1) (“**Operators**”) A family $\mathfrak{A} = \{\mathfrak{A}(X, Y) : X, Y \in \mathcal{C}\}$ of algebras of operators associated to each pair of spaces X and Y in \mathcal{C} is called an *operator ideal* if
 - OI₁**: $\mathfrak{A}(X, Y)$ is a nonzero vector subspace of $\mathfrak{L}(X, Y)$ for all X, Y in \mathcal{C} ; and
 - OI₂**: $RTS \in \mathfrak{A}(X_0, Y_0)$ whenever $R \in \mathfrak{L}(Y, Y_0)$, $T \in \mathfrak{A}(X, Y)$ and $S \in \mathfrak{L}(X_0, X)$ for any X_0, X, Y and Y_0 in \mathcal{C} .
- (2) (“**Topologies**”) A family $\mathcal{P} = \{\mathcal{P}(X) : X \in \mathcal{C}\}$ of locally convex topologies associated to each space X in \mathcal{C} is called a *generating topology* if
 - GT₁**: $\sigma(X, X') \subseteq \mathcal{P}(X) \subseteq \mathcal{P}_{\text{ori}}(X)$ for all X in \mathcal{C} ; and
 - GT₂**: $\mathfrak{L}(X, Y) \subseteq \mathfrak{L}(X_{\mathcal{P}}, Y_{\mathcal{P}})$ for all X and Y in \mathcal{C} .
- (3) (“**Bornologies**”) A family $\mathcal{M} = \{\mathcal{M}(Y) : Y \in \mathcal{C}\}$ of convex vector bornologies associated to each space Y in \mathcal{C} is called a *generating bornology* if
 - GB₁**: $\mathcal{M}_{\text{fin}}(Y) \subseteq \mathcal{M}(Y) \subseteq \mathcal{M}_{\text{von}}(Y)$ for all Y in \mathcal{C} ; and
 - GB₂**: $\mathfrak{L}(X, Y) \subseteq L^{\times}(X^{\mathcal{M}}, Y^{\mathcal{M}})$ for all X and Y in \mathcal{C} .

Definition 2.3. Let \mathfrak{A} be an operator ideal, \mathcal{P} a generating topology and \mathcal{M} a generating bornology on \mathcal{C} .

- (1) (“**Operators**” \rightarrow “**Topologies**”) For each X_0 in \mathcal{C} , the \mathfrak{A} -*topology* of X_0 , denoted by $\mathcal{T}(\mathfrak{A})(X_0)$, is the projective topology of X_0 with respect to the family

$$\{T \in \mathfrak{A}(X_0, Y) : Y \in \mathcal{C}\}.$$

In other words, a seminorm p of X_0 is $\mathcal{T}(\mathfrak{A})(X_0)$ -continuous if and only if there is a T in $\mathfrak{A}(X_0, Y)$ for some Y in \mathcal{C} and a continuous seminorm q of Y such that

$$p(x) \leq q(Tx), \quad \forall x \in X_0.$$

In this case, we call p an \mathfrak{A} -*seminorm* of X_0 .

- (2) (“**Operators**” \rightarrow “**Bornologies**”) For each Y_0 in \mathcal{C} , the \mathfrak{A} -*bornology* of Y_0 , denoted by $\mathcal{B}(\mathfrak{A})(Y_0)$, is the inductive bornology of Y_0 with respect to the family

$$\{T \in \mathfrak{A}(X, Y_0) : X \in \mathcal{C}\}.$$

In other words, a subset B of Y_0 is $\mathcal{B}(\mathfrak{A})(Y_0)$ -bounded if and only if there is a T in $\mathfrak{A}(X, Y_0)$ for some X in \mathcal{C} and a topologically bounded subset A of X such that

$$B \subseteq TA.$$

In this case, we call B an \mathfrak{A} -*bounded subset* of Y_0 .

- (3) (“**Topologies**” \rightarrow “**Operators**”) For X, Y in \mathcal{C} , let

$$\mathcal{O}(\mathcal{P})(X, Y) = \mathfrak{L}(X_{\mathcal{P}}, Y)$$

be the vector space of all continuous operators from X into Y which is still continuous with respect to the $\mathcal{P}(X)$ -topology.

- (4) (“**Bornologies**” \rightarrow “**Operators**”) For X, Y in \mathcal{C} , let

$$\mathcal{O}(\mathcal{M})(X, Y) = \mathfrak{L}(X, Y) \cap L^\times(X, Y^{\mathcal{M}})$$

be the vector space of all continuous operators from X into Y which send bounded sets to $\mathcal{M}(Y)$ -bounded sets.

- (5) (“**Topologies**” \leftrightarrow “**Bornologies**”) For X, Y in \mathcal{C} , the $\mathcal{P}^\circ(Y)$ -*bornology* of Y (resp. $\mathcal{M}^\circ(X)$ -*topology* of X) is defined to be the bornology (resp. topology) polar to $\mathcal{P}(X)$ (resp. $\mathcal{M}(Y)$). More precisely,

- a bounded subset A of Y is $\mathcal{P}^\circ(Y)$ -bounded if and only if its polar A° is a $\mathcal{P}(Y'_\beta)$ -neighborhood of zero; and
- a neighborhood V of zero of X is a $\mathcal{M}^\circ(X)$ -neighborhood of zero if and only if V° is $\mathcal{M}(X'_\beta)$ -bounded.

Here are two examples: the ideals \mathfrak{K} of compact operators and \mathfrak{P} of absolutely summing operators (see e.g. [6]), the generating systems \mathcal{P}_{pc} of precompact topologies (see

e.g. [7]) and \mathcal{P}_{pn} of prenuclear topologies (see e.g. [8, p. 90]), and the generating systems \mathcal{M}_{pc} of precompact bornologies and \mathcal{M}_{pn} of prenuclear bornologies (see e.g. [5]), respectively.

Definition 2.4. An operator ideal \mathfrak{A} on Banach spaces is said to be

- (1) *injective* if $S \in \mathfrak{A}(E, F_0)$ infers $T \in \mathfrak{A}(E, F)$, whenever $T \in \mathfrak{L}(E, F)$ and $\|Tx\| \leq \|Sx\|, \forall x \in E$;
- (2) *surjective* if $S \in \mathfrak{A}(E_0, F)$ infers $T \in \mathfrak{A}(E, F)$, whenever $T \in \mathfrak{L}(E, F)$ and $TU_E \subseteq SU_{E_0}$.

The *injective hull* $\mathfrak{A}^{\text{inj}}$ and the *surjective hull* $\mathfrak{A}^{\text{sur}}$, of \mathfrak{A} is the intersection of all injective and surjective operator ideals containing \mathfrak{A} , respectively. For example, $\mathfrak{K} = \mathfrak{K}^{\text{inj}} = \mathfrak{K}^{\text{sur}}$ and $\mathfrak{P} = \mathfrak{P}^{\text{inj}} \subsetneq \mathfrak{P}^{\text{sur}}$ (see, e.g., [6]).

In the following, we see that the notions of operator ideals, generating topologies, and generating bornologies are equivalent (see also [13] for the operator algebra version).

Theorem 2.5 ([9, 10, 18, 16]). *Let \mathfrak{A} be an operator ideal, \mathcal{P} a generating topology and \mathcal{M} a generating bornology on Banach spaces. We have*

- (1) $\mathcal{T}(\mathfrak{A}) = \{\mathcal{T}(\mathfrak{A})(X) : X \in \mathcal{C}\}$ is a generating topology.
- (2) $\mathcal{B}(\mathfrak{A}) = \{\mathcal{B}(\mathfrak{A})(Y) : Y \in \mathcal{C}\}$ is a generating bornology.
- (3) $\mathcal{O}(\mathcal{P}) = \{\mathcal{O}(\mathcal{P})(X, Y) : X, Y \in \mathcal{C}\}$ is an operator ideal.
- (4) $\mathcal{O}(\mathcal{M}) = \{\mathcal{O}(\mathcal{M})(X, Y) : X, Y \in \mathcal{C}\}$ is an operator ideal.
- (5) $\mathcal{P}^\circ = \{\mathcal{P}^\circ(Y) : Y \in \mathcal{C}\}$ is a generating bornology.
- (6) $\mathcal{M}^\circ = \{\mathcal{M}^\circ(Y) : Y \in \mathcal{C}\}$ is a generating topology.
- (7) $\mathcal{O}(\mathcal{T}(\mathfrak{A})) = \mathfrak{A}^{\text{inj}}$.
- (8) $\mathcal{O}(\mathcal{B}(\mathfrak{A})) = \mathfrak{A}^{\text{sur}}$.
- (9) $\mathcal{T}(\mathcal{O}(\mathcal{P})) = \mathcal{P}$.
- (10) $\mathcal{B}(\mathcal{O}(\mathcal{M})) = \mathcal{M}$.

For generating topologies \mathcal{P} and \mathcal{P}_1 , and generating bornologies \mathcal{M} and \mathcal{M}_1 on Banach spaces, we can also associate to them the operator ideals with components

$$\begin{aligned}\mathcal{O}(\mathcal{P}/\mathcal{P}_1)(X, Y) &= \mathfrak{L}(X_{\mathcal{P}}, Y_{\mathcal{P}_1}), \\ \mathcal{O}(\mathcal{M}/\mathcal{M}_1)(X, Y) &= L^\times(X^{\mathcal{M}_1}, Y^{\mathcal{M}}) \cap \mathfrak{L}(X, Y) \\ \mathcal{O}(\mathcal{P}/\mathcal{M})(X, Y) &= \mathfrak{L}(X_{\mathcal{P}}, Y^{\mathcal{M}}).\end{aligned}$$

These give rise to

$$\begin{aligned}\mathcal{O}(\mathcal{P}/\mathcal{P}_1) &= \mathcal{O}(\mathcal{P}_1)^{-1} \circ \mathcal{O}(\mathcal{P}) \quad [11], \\ \mathcal{O}(\mathcal{M}/\mathcal{M}_1) &= \mathcal{O}(\mathcal{M}) \circ \mathcal{O}(\mathcal{M}_1)^{-1} \quad [12], \\ \mathcal{O}(\mathcal{P}/\mathcal{M}) &= \mathcal{O}(\mathcal{M}) \circ \mathcal{O}(\mathcal{P}) = \mathcal{O}(\mathcal{M}) \circ \mathcal{O}(\mathcal{P}) \quad [18].\end{aligned}$$

Here, the product $\mathfrak{A} \circ \mathfrak{B}$ consists of operators of the form ST with $S \in \mathfrak{A}$ and $T \in \mathfrak{B}$, while the quotient $\mathfrak{A} \circ \mathfrak{B}^{-1}$ consists of those S such that $ST \in \mathfrak{A}$ whenever $T \in \mathfrak{B}$. Readers are referred to Pietsch's classic [6] for information regarding quotients and products of operator ideals.

3. GENERATING SEQUENCES

The following is based on a version in [11].

Definition 3.1. A family $\Phi(E)$ of bounded sequences in a Banach space E is called a *vector sequential structure* if

- $\{ax_n + by_n\} \in \Phi(E)$ whenever $\{x_n\}, \{y_n\} \in \Phi(E)$ and a, b are scalars;
- every subsequence of a sequence in $\Phi(E)$ belongs to $\Phi(E)$.

Clearly, the biggest vector sequential structure on a Banach space E is the family $\Phi_b(E)$ of all bounded sequences in E . On the other hand, we let $\Phi_{\text{fin}}(E)$ to be the vector sequential structure on E of all bounded sequences $\{x_n\}$ with finite dimensional ranges, i.e., there is a finite dimensional subspace of F containing all x_n 's.

Given two Banach spaces E, F with vector sequential structures Φ, Ψ , we let $\mathfrak{L}(E^\Phi, F^\Psi)$ be the vector space of all continuous linear operators sending a bounded sequence in Φ to a bounded sequence in F with a subsequence in Ψ .

Definition 3.2. A family $\Phi := \{\Phi(F) : F \text{ is a Banach space}\}$ of (bounded) vector sequential structures on each Banach space is called a *generating sequential structure* if

$$\mathbf{GS}_1: \Phi_{\text{fin}}(F) \subseteq \Phi(F) \subseteq \Phi_b(F) \text{ for every Banach space } F.$$

$$\mathbf{GS}_2: \mathfrak{L}(E, F) \subseteq \mathfrak{L}(E^\Phi, F^\Phi) \text{ for all Banach spaces } E, F.$$

Definition 3.3. Let \mathfrak{A} be an operator ideal on Banach spaces. Call a bounded sequence $\{y_n\}$ in a Banach space F an \mathfrak{A} -sequence if there is a bounded sequence $\{x_n\}$ in a Banach space E and a continuous linear operator T in $\mathfrak{A}(E, F)$ such that $Tx_n = y_n, n = 1, 2, \dots$. Denote by $\mathcal{S}(\mathfrak{A})(F)$ the family of all \mathfrak{A} -sequences in F .

Lemma 3.4. *Let \mathfrak{A} be an operator ideal on Banach spaces. Then*

$$\mathcal{S}(\mathfrak{A}) := \{\mathcal{S}(\mathfrak{A})(F) : F \text{ is a Banach space}\}$$

is a generating sequential structure.

Proof. It is easy to see that all $\mathcal{S}(\mathfrak{A})(F)$ is a bounded vector sequential structure. Since the ideal \mathfrak{F} of continuous linear operators of finite rank is the smallest operator ideal, while the ideal \mathfrak{L} of all continuous linear operators is the biggest, we see that (\mathbf{GS}_1) is satisfied. On the other hand, (\mathbf{GS}_2) follows from (\mathbf{OI}_2) directly. \square

We now work on the converse of Lemma 3.4. Let Φ be a generating sequential structure. Denote by $\mathcal{M}_{\text{base}}^\Phi(E)$ the family of all bounded sets M in a Banach space E such that every sequence $\{x_n\}$ in M has a subsequence $\{x_{n_k}\}$ in $\Phi(E)$.

Lemma 3.5. *The family $\mathcal{M}^\Phi(E)$ of all (bounded) sets with circle hull in $\mathcal{M}_{\text{base}}^\Phi(E)$ in a Banach space E forms a generating bornology \mathcal{M}^Φ .*

Proof. Let $M, N \in \mathcal{M}^\Phi(E)$, let λ be a scalar, and let $W \subseteq M$. It is easy to see that the circle hull $\text{ch } M, \lambda M$ and W are all in $\mathcal{M}^\Phi(E)$. For the sum $M + N$, we first notice that both $\text{ch } M$ and $\text{ch } N$ belong to $\mathcal{M}_{\text{base}}^\Phi(E)$. Now

$$M + N \subseteq \text{ch } M + \text{ch } N \in \mathcal{M}_{\text{base}}^\Phi(E)$$

and

$$\text{ch}(M + N) \subseteq \text{ch } M + \text{ch } N$$

give $M + N \in \mathcal{M}^\Phi(E)$. All these together say that $\mathcal{M}^\Phi(E)$ is a vector bornology on the Banach space E .

The condition (GB₁) follows directly from (GS₁). For (GB₂), let $T \in \mathfrak{L}(E, F)$ and $M \in \mathcal{M}^\Phi(E)$. Then for every sequence $\{Tx_n\}$ in TM , we will have a subsequence of $\{x_n\}$ in $\Phi(E)$. By (GS₂), we have a further subsequence $\{Tx_{n_k}\}$ belonging to $\mathcal{M}^\Phi(F)$. \square

Definition 3.6. A generating sequential structure Φ is said to be *normal* if for every $\{x_n\}$ in $\Phi(E)$ and every bounded scalar sequence (λ_n) in ℓ^∞ of norm not greater than 1, we have $\{\lambda_n x_n\} \in \Phi(E)$.

It is clear that $\mathcal{S}(\mathfrak{A})$ is normal for any operator ideal \mathfrak{A} .

Lemma 3.7. *For every normal generating sequential structure Φ , we have*

$$\mathcal{M}_{\text{base}}^\Phi = \mathcal{M}^\Phi.$$

Proof. It suffices to show that every M in $\mathcal{M}_{\text{base}}^\Phi(E)$ has its circle hull $\text{ch } M$ in $\mathcal{M}_{\text{base}}^\Phi(E)$ for each Banach space E , i.e., $\mathcal{M}_{\text{base}}^\Phi$ is itself circled. To this end, let $\{x_n\}$ be from $\text{ch } M$. Then there is a (λ_n) from ℓ^∞ of norm not greater than 1 such that

$$x_n = \lambda_n y_n, \quad n = 1, 2, \dots$$

for some $\{y_n\}$ in M . Now, there is a subsequence $\{y_{n_k}\}$ in $\Phi(E)$ ensuring that $\{x_{n_k}\} = \{\lambda_{n_k} y_{n_k}\} \in \Phi(E)$ for Φ being normal. \square

Definition 3.8. Let Φ be a generating sequential structure on Banach spaces. Let E, F be Banach spaces. Denote by $\mathcal{O}(\Phi)(E, F)$ the family of all continuous linear operators T in $\mathfrak{L}(E, F)$ sending every bounded sequence $\{x_n\}$ in E to a bounded sequence in F with a subsequence $\{Tx_n\}$ in $\Phi(F)$. In other words, $\mathcal{O}(\Phi)(E, F) = \mathfrak{L}(E^{\Phi_b}, F^\Phi)$.

Theorem 3.9. *Let Φ be a generating sequential structure on Banach spaces. Then $\mathcal{O}(\Phi)$ is a surjective operator ideal. More precisely,*

$$\mathcal{O}(\Phi) = \mathcal{O}(\mathcal{M}^\Phi).$$

Proof. The ideal property of $\mathcal{O}(\Phi)$ is trivial. For the surjectivity of $\mathcal{O}(\Phi)$, we let $S \in \mathcal{O}(\Phi)(E_0, F)$ and $T \in \mathfrak{L}(E, F)$ such that $TU_E \subseteq SU_{E_0}$. Here, U_E, U_{E_0} are the closed

unit balls of E, E_0 , respectively. We need to show that $T \in \mathcal{O}(\Phi)(E, F)$. In fact, for any bounded sequence $\{x_n\}$ in E , we have

$$Tx_n = Sy_n, \quad n = 1, 2, \dots,$$

with some bounded sequence $\{y_n\}$ in F . Now, the fact $\{Sy_n\}$ has a subsequence $\{Sy_{n_k}\} \in \Phi(F)$ implies that $T \in \mathcal{O}(\Phi)(E, F)$, as asserted.

Finally, we check the representation. The inclusion $\mathcal{O}(\Phi) \subseteq \mathcal{O}(\mathcal{M}^\Phi)$ is an immediate consequence of the definition of \mathcal{M}^Φ . Conversely, let $T \in \mathcal{O}(\mathcal{M}^\Phi)(E, F)$, i.e., $TU_E \in \mathcal{M}^\Phi(F)$. Let $\{x_n\}$ be a bounded sequence in E . We can assume $\|x_n\| \leq 1$ for all n . Then $\{Tx_n\} \subseteq TU_E$ and hence possesses a subsequence $\{Tx_{n_k}\} \in \Phi(F)$. In other words, $T \in \mathcal{O}(\Phi)(E, F)$. \square

Corollary 3.10. *If a generating sequential structure Φ is normal then*

$$\mathcal{S}(\mathcal{O}(\Phi)) = \Phi.$$

Conversely, if \mathfrak{A} is an operator ideal on Banach spaces then

$$\mathcal{O}(\mathcal{S}(\mathfrak{A})) = \mathfrak{A}^{\text{sur}}.$$

Theorem 3.11. *Let Φ, Ψ be two generating sequential structures. Let*

$$\mathcal{O}(\Phi/\Psi)(E, F) := \mathfrak{L}(E^\Psi, F^\Phi),$$

i.e., $T \in \mathcal{O}(\Phi/\Psi)(E, F)$ if and only if the continuous linear operator T sends each bounded sequence $\{x_n\}$ in $\Psi(E)$ to a sequence in F with a subsequence $\{Tx_{n_k}\}$ in $\Phi(F)$. Then $\mathcal{O}(\Phi/\Psi)(E, F)$ is an operator ideal on Banach spaces. Moreover, if Ψ is normal, then

$$\mathcal{O}(\Phi/\Psi) = \mathcal{O}(\mathcal{M}^\Phi/\mathcal{M}^\Psi) = \mathcal{O}(\mathcal{M}^\Phi) \circ \mathcal{O}(\mathcal{M}^\Psi)^{-1}.$$

Proof. Conditions (OI_1) and (OI_2) follow from Conditions (GS_1) and (GS_2) , respectively.

Assume Ψ is normal. Let $T \in \mathcal{O}(\Phi/\Psi)(E, F)$ and $M \in \mathcal{M}^\Psi(E)$ be circled. Suppose, on contrary, $TM \notin \mathcal{M}^\Phi(F)$. By definition, there is a sequence $\{Tx_n\}$ in TM having no subsequence $\{Tx_{n_k}\}$ in $\Phi(F)$. But since $\{x_n\} \subseteq M$, it has a subsequence $\{x_{n_k}\}$ in $\Psi(E)$. And thus $\{Tx_{n_k}\}$ has a further subsequence in $\Phi(F)$, a contradiction.

Conversely, let $T \in \mathcal{O}(\mathcal{M}^\Phi/\mathcal{M}^\Psi)(E, F)$. It is easy to see that the range of any sequence $\{x_n\}$ in $\Psi(E)$ is a bounded set in $\mathcal{M}^\psi(E)$, by noting that Ψ is normal. The range of the sequence $\{Tx_n\}$ is bounded in $\mathcal{M}^\Phi(F)$. Therefore, there is a subsequence $\{Tx_{n_k}\}$ of $\{Tx_n\}$ belongs to $\Phi(F)$. This ensures $T \in \mathcal{O}(\Phi/\Psi)(E, F)$. \square

Definition 3.12. Given two generating sequential structures Φ and Ψ . We say that $\Psi(E)$ is *coordinated to* $\Phi(E)$ if the following conditions holds. Suppose $\{x_n\} \in \Psi(E)$. The bounded sequence $\{x_n\} \notin \Phi(E)$ if and only if $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ which has no further subsequence belonging to $\Phi(E)$. We write, in this case, $\Phi(E) \ll \Psi(E)$. If $\Phi(E) \ll \Psi(E)$ for all Banach spaces E then we write $\Phi \ll \Psi$.

Theorem 3.13. *Assume Φ, Ψ be two generating sequential structures such that $\Phi(F) \ll \Psi(F)$ for some Banach space F . Then $\mathcal{O}(\Phi/\Psi)(E, F)$ consists of exactly those continuous linear operators sending bounded sequences in $\Psi(E)$ to bounded sequences in $\Phi(F)$ for any Banach space E .*

Proof. Assume $T \in \mathcal{O}(\Phi/\Psi)(E, F)$ and $\{x_n\} \in \Psi(E)$. Now, suppose $\{Tx_n\} \notin \Phi(F)$. Since $\{Tx_n\} \in \Psi(F)$ by (GS₂), we can verify the condition in Definition 3.12. Therefore, we might have a subsequence $\{Tx_{n_k}\}$ having no further subsequence in $\Phi(F)$. However, by the facts that $\{x_{n_k}\} \in \Psi(E)$ and $T \in \mathcal{O}(\Phi/\Psi)(E, F)$, we arrive at the asserted contradiction. \square

Corollary 3.14. *Let Φ, Ψ be two generating sequential structures such that $\Phi \ll \Psi$ and Ψ is normal. Then, the operator ideal $\mathcal{O}(\mathcal{M}^\Phi, \mathcal{M}^\Psi)$ consists of exactly those operators sending sequences in Ψ to sequences in Φ .*

4. EXAMPLES

We begin with some useful generating bornologies.

Examples 4.1. A set W in a Banach space F is called

- (1) *nuclear* (see, e.g., [11]) if

$$W \subseteq \left\{ y \in F : y = \sum_{n=1}^{\infty} \lambda_n y_n, |\lambda_n| \leq 1 \right\}$$

for an absolutely summable sequence $\{y_n\}$ in F ;

(2) *unconditionally summable* (see, e.g., [11]) if

$$W \subseteq \{y \in F : y = \sum_{n=1}^{\infty} \lambda_n y_n \text{ in norm, } |\lambda_n| \leq 1\}$$

for an unconditionally summable series $\sum_{n=1}^{\infty} y_n$ in F ;

(3) *weakly unconditionally summable* (see, e.g., [11]) if

$$W \subseteq \{y \in F : y = \sum_{n=1}^{\infty} \lambda_n y_n \text{ weakly, } |\lambda_n| \leq 1\}$$

for a weakly unconditionally summable series $\sum_{n=1}^{\infty} y_n$ in F ;

(4) *limited* ([1]) if

$$\lim_{n \rightarrow \infty} \sup_{a \in W} |y'_n(a)| = 0$$

for any $\sigma(F', F)$ -null sequence $\{y'_n\}$ in F' .

The above defines generating bornologies \mathcal{M}_ν , \mathcal{M}_{uc} , \mathcal{M}_{wuc} , and \mathcal{M}_{lim} , respectively.

Another way to obtain generating bornologies is through generating sequential structures.

Examples 4.2. A bounded sequence $\{x_n\}$ in a Banach space F is called

(1) *δ_0 -fundamental* ([11]) if

$$t_n = x_{n+1} - x_n, \quad n = 1, 2, \dots,$$

forms a weakly unconditionally summable series;

(2) *limited* ([1]) if

$$\langle x_n, x'_n \rangle \rightarrow 0 \text{ as } n \rightarrow \infty$$

for any $\sigma(F', F)$ -null sequence $\{x'_n\}$ in F' ;

(3) *Banach-Saks* if

$$\lim_{n \rightarrow \infty} \frac{x_1 + x_2 + \dots + x_n}{n} = x_0$$

for some x_0 in F in norm.

Examples 4.3. The following are all generating sequential structures.

- (1) Φ_{δ_0} of all δ_0 -fundamental sequences.
- (2) Φ_{lim} of all limited sequences.
- (3) Φ_{BS} of all Banach-Saks sequences.

- (4) Φ_c of all convergent sequences.
- (5) Φ_{wc} of all weakly convergence sequences.
- (6) Φ_{wCa} of all weakly Cauchy sequences.

Examples 4.4 ([11]). The following generating bornologies can be induced by the corresponding generating sequential structures as in Lemma 3.5.

- (1) The δ_0 -compact bornology \mathcal{M}_{δ_0} is defined by Φ_{δ_0} .
- (2) The compact bornology \mathcal{M}_c is defined by Φ_c .
- (3) The weakly compact bornology \mathcal{M}_{wc} is defined by Φ_{wc} .
- (4) The Rosenthal compact bornology \mathcal{M}_{wCa} is defined by Φ_{wCa} .
- (5) The Banach-Saks bornology \mathcal{M}_{BS} is defined by Φ_{BS} .

Proposition 4.5. *The generating bornology induced by Φ_{lim} is \mathcal{M}_{lim} .*

Proof. First, we notice that Φ_{lim} is normal. Hence, $\mathcal{M}_{\text{base}}^{\Phi_{\text{lim}}} = \mathcal{M}^{\Phi_{\text{lim}}}$ by Lemma 3.7. Let $W \in \mathcal{M}_{\text{lim}}(F)$. Then, by definition,

$$\lim_{n \rightarrow \infty} \sup_{a \in W} |\langle a, y'_n \rangle| = 0$$

for any $\sigma(F', F)$ -null sequence $\{y'_n\}$ in F' and any sequence $\{y_n\}$ in W . Hence, $W \in \mathcal{M}^{\Phi_{\text{lim}}}(F)$.

Conversely, if W is a bounded set in F such that every sequence $\{y_n\}$ in W has a limited subsequence $\{y_{n_k}\}$, we need to check that $W \in \mathcal{M}_{\text{lim}}$. Assume, on contrary, that there were some $\sigma(F', F)$ -null sequence $\{y'_n\}$ in F' and some sequence $\{a_n\}$ in W such that

$$\lim_{n \rightarrow \infty} |\langle a_{n_k}, y'_{n_k} \rangle| = 0.$$

This is a contradiction and thus W is a member of $\mathcal{M}_{\text{lim}}(F)$. □

Proposition 4.6. *The generating bornologies \mathcal{M}_c , \mathcal{M}_{wc} , \mathcal{M}_{wCa} , \mathcal{M}_ν , \mathcal{M}_{uc} , \mathcal{M}_{wuc} and \mathcal{M}_{lim} are all σ -disked.*

Proof. For the case of \mathcal{M}_{lim} , see [1]. For all others, see [11]. □

We are now ready to give a number of examples.

Examples 4.7. Using Definition 2.3 and Theorem 3.9, we can obtain the following operator ideals on Banach spaces through the associated generating bornologies.

- (1) The ideal $\mathfrak{N}^{\text{sur}}$ of *co-nuclear operators* for the nuclear bornology \mathcal{M}_ν .
- (2) The ideal $\mathfrak{L}_{\text{lim}}$ of *limited operators* for the limited bornology \mathcal{M}_{lim} .
- (3) The ideal \mathfrak{BS} of *Banach-Saks operators* for the Banach-Saks bornology \mathcal{M}_{BS} .
- (4) The ideal \mathfrak{K} of *compact operators* for the compact bornology \mathcal{M}_c .
- (5) The ideal \mathfrak{W} of *weakly compact operators* for the weakly compact bornology \mathcal{M}_{wc} .
- (6) The ideal \mathfrak{R} of *Rosenthal compact operators* for the Rosenthal compact bornology \mathcal{M}_{wCa} .
- (7) The ideal \mathfrak{D}_0 of δ_0 -*compact operators* for the δ_0 -compact bornology \mathcal{M}_{δ_0} .

For a proof of (a), see [6, p. 112]. For (b), see [3]. For (c), see Theorem 3.9. For others, see [11].

Concerning Theorem 3.13, we have

- Examples 4.8.**
- (1) $\Phi_c \ll \Phi_{wc}$.
 - (2) $\Phi_c \ll \Phi_{wCa}$.
 - (3) $\Phi_{wc} \ll \Phi_{wCa}$.
 - (4) $\Phi_c \ll \Phi_{\delta_0}$.
 - (5) $\Phi_c \ll \Phi_{\text{lim}}$.

The first four can be found in [11]. For the last one, we observe the fact that each limited sequence is weakly Cauchy (see [1]) and (c).

Examples 4.9. The following operator ideals have desirable representations.

- (1) The ideal \mathfrak{U} of unconditionally summing operators

$$\begin{aligned} \mathfrak{U} &= \mathcal{O}(\mathcal{M}_{uc}/\mathcal{M}_{wuc}) = \mathcal{O}(\mathcal{M}_{uc}) \circ \mathcal{O}(\mathcal{M}_{wuc})^{-1} \\ &= \mathcal{O}(\mathcal{M}_c/\mathcal{M}_{\delta_0}) = \mathcal{O}(\mathcal{M}_c) \circ \mathcal{O}(\mathcal{M}_{\delta_0})^{-1} = \mathfrak{K} \circ \mathfrak{D}_0^{-1}. \end{aligned}$$

- (2) The ideal \mathfrak{V} of completely continuous operators

$$\begin{aligned} \mathfrak{V} &= \mathcal{O}(\Phi_c/\Phi_{wc}) = \mathcal{O}(\mathcal{M}_c/\mathcal{M}_{wc}) = \mathcal{O}(\mathcal{M}_c) \circ \mathcal{O}(\mathcal{M}_{wc})^{-1} = \mathfrak{K} \circ \mathfrak{W}^{-1} \\ &= \mathcal{O}(\Phi_c/\Phi_{wCa}) = \mathcal{O}(\mathcal{M}_c/\mathcal{M}_{wCa}) = \mathcal{O}(\mathcal{M}_c) \circ \mathcal{O}(\mathcal{M}_{wCa})^{-1} = \mathfrak{K} \circ \mathfrak{R}^{-1}. \end{aligned}$$

(3) The ideal $w\mathfrak{S}\mathfrak{C}$ of weakly sequentially complete operators

$$s\mathfrak{S}\mathfrak{C} = \mathcal{O}(\Phi_{wc}/\Phi_{wCa}) = \mathcal{O}(\mathcal{M}_{wc}/\mathcal{M}_{wCa}) = \mathcal{O}(\mathcal{M}_{wc}) \circ \mathcal{O}(\mathcal{M}_{wCa})^{-1} = \mathfrak{W} \circ \mathfrak{K}^{-1}.$$

(4) The ideal $\mathfrak{G}\mathfrak{P}$ of Gelfand-Phillips operators

$$\mathfrak{G}\mathfrak{P} = \mathcal{O}(\Phi_c/\Phi_{\text{lim}}) = \mathcal{O}(\mathcal{M}_c/\mathcal{M}_{\text{lim}}) = \mathcal{O}(\mathcal{M}_c) \circ \mathcal{O}(\mathcal{M}_{\text{lim}})^{-1} = \mathfrak{K} \circ \mathfrak{L}_{\text{lim}}^{-1}.$$

(5) The ideal $w\mathfrak{B}\mathfrak{S}$ of weakly Banach-Saks operators

$$w\mathfrak{B}\mathfrak{S} = \mathcal{O}(\Phi_{\text{BS}}/\Phi_{wc}) = \mathcal{O}(\mathcal{M}_{\text{BS}}/\mathcal{M}_{wc}) = \mathcal{O}(\mathcal{M}_{\text{BS}}) \circ \mathcal{O}(\mathcal{M}_{wc})^{-1} = \mathfrak{B}\mathfrak{S} \circ \mathfrak{W}^{-1}.$$

See [11] for a proof of (a), (b) and (c). See [3] for (d). For (e), we simply recall that a T in $\mathfrak{L}(E, F)$ is called *weakly Banach-Saks* if T sends each weakly convergent sequence to a sequence possessing a Banach-Saks subsequence, by definition.

Proposition 4.10 ([11]). *Let \mathfrak{A} be a surjective operator ideal such that \mathfrak{A} is idempotent, i.e., $\mathfrak{A}^2 = \mathfrak{A}$. Then for any operator ideal $\mathfrak{B} \supseteq \mathfrak{A}$, there exists an operator ideal \mathfrak{C} such that*

$$\mathfrak{A} = \mathfrak{C} \circ \mathfrak{B}^{\text{sur}},$$

and \mathfrak{C} can be chosen to be $\mathfrak{A} \circ (\mathfrak{B}^{\text{sur}})^{-1}$.

Proof. We present a proof here for completeness. Denote

$$\mathcal{M}_{\mathfrak{A}} := \mathcal{M}(\mathfrak{A}), \quad \mathcal{M}_{\mathfrak{B}} := \mathcal{M}(\mathfrak{B})$$

and set

$$\mathfrak{C} := \mathcal{O}(\mathcal{M}_{\mathfrak{A}}/\mathcal{M}_{\mathfrak{B}}).$$

Now the facts $\mathfrak{A} \subseteq \mathfrak{B}$ and

$$\mathfrak{A} \subseteq \mathfrak{C} = \mathcal{O}(\mathcal{M}_{\mathfrak{A}}) \circ \mathcal{O}(\mathcal{M}_{\mathfrak{B}})^{-1} = \mathfrak{A} \circ (\mathfrak{B}^{\text{sur}})^{-1}$$

implies that

$$\mathfrak{A} = \mathfrak{A}^2 \subseteq \mathfrak{C} \circ \mathfrak{B} \subseteq \mathfrak{C} \circ \mathfrak{B}^{\text{sur}}.$$

On the other hand, if $T \in \mathfrak{C} \circ \mathfrak{B}^{\text{sur}}(E, F)$ then $T = RS$ for some $R \in \mathfrak{C}(G, F) = L^\times(G^{\mathcal{M}_{\mathfrak{B}}}, F^{\mathcal{M}_{\mathfrak{A}}}) \cap \mathfrak{L}(G, F)$ and $S \in \mathfrak{B}^{\text{sur}}(E, G) = L^\times(E, G^{\mathcal{M}_{\mathfrak{B}}}) \cap \mathfrak{L}(E, G)$ with some Banach space G . Hence,

$$T \in L^\times(E, F^{\mathcal{M}_{\mathfrak{A}}}) \cap \mathfrak{L}(E, F) = \mathfrak{A}(E, F).$$

□

Examples 4.11 ([11]). (1) Since the surjective ideal \mathfrak{K} of compact operators is idempotent and contained in the surjective ideal \mathfrak{W} , \mathfrak{R} , and \mathfrak{D}_0 of weakly compact operators, Rosenthal compact operators, and δ_0 -compact operators, respectively, we have

$$(a) \mathfrak{K} = \mathfrak{W} \circ \mathfrak{W} \text{ since } \mathfrak{W} = \mathcal{O}(\mathcal{M}_c/\mathcal{M}_{wc}) = \mathfrak{K} \circ \mathfrak{W}^{-1};$$

$$(b) \mathfrak{K} = \mathfrak{W} \circ \mathfrak{R} \text{ since } \mathfrak{W} = \mathcal{O}(\mathcal{M}_c/\mathcal{M}_{wcA}) = \mathfrak{K} \circ \mathfrak{R}^{-1};$$

$$(c) \mathfrak{K} = \mathfrak{U} \circ \mathfrak{D}_0 \text{ since } \mathfrak{U} = \mathcal{O}(\mathcal{M}_c/\mathcal{M}_{\delta_0}) = \mathfrak{K} \circ \mathfrak{D}_0^{-1}.$$

(2) Since the ideal \mathfrak{W} of weakly compact operators is idempotent and contained in the surjective ideal \mathfrak{R} of Rosenthal compact operators, we have

$$\mathfrak{W} = w\mathfrak{S}\mathfrak{C} \circ \mathfrak{R},$$

where $w\mathfrak{S}\mathfrak{C} = \mathcal{O}(\mathcal{M}_{wc}/\mathcal{M}_{wCa}) = \mathfrak{W} \circ \mathfrak{R}^{-1}$ is the ideal of weakly sequentially complete operators.

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DEPARTMENT OF APPLIED MATHEMATICS, NATIONAL SUN YAT-SEN UNIVERSITY, AND NATIONAL CENTER FOR THEORETICAL SCIENCES, KAOHSIUNG, 80424, TAIWAN, R.O.C.

E-mail address: wong@math.nsysu.edu.tw