The No Trade Principle and the Characterization of Compact Beliefs[*†]

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This Version: November 22, 2015

Running title: No Trade and Compact Beliefs

[*]This article is a substantially revised and extended version of our previous working paper entitled, "The No Trade Principle in General Environments." We thank an associate editor and two anonymous referees for offering useful comments and advice. We also thank Xiao Luo and Paul Milgrom for stimulating questions, Harrison Cheng, Tai-Wei Hu and Atsushi Kajii for general comments.

[†]This research is supported in partial by the Taiwan MOST grants 102-2115-M-110-002-MY2 and 104-2115-M-110-009-MY2.

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Abstract

We establish the no trade principle, i.e., the no trade theorem and its converse, for any dual pair of bet and extended belief spaces, defined on a given measurable space. A key condition is that, except perhaps one of the agents, everyone else has (weak*) compact sets of beliefs. We find out that in most of the models of uncertainty adopted in the economic literature, roughly speaking, the epistemic statement that an agent has a compact set of beliefs is equivalent to the economic statement that he has an open cone of positive bets. This improves our understanding of what compactness actually means within an economic context.

JEL classification: C70; D82; D84

Keywords: No trade; Common priors; Dual pairs; Compact beliefs; Dual cones; Separation theorem

1. Introduction

The (interim) no trade principle asserts that risk-neutral agents are not prepared to trade for purely informational reasons, if and only if a common prior exists. This is an important result because it characterizes an epistemic term (i.e., common prior) in terms of an economic term (i.e., no trade). It gives the widely used common prior assumption an economic meaning.

The purpose of this article is twofold. Firstly, we incorporate the existing literature on the study of the no trade principle into a unified framework, using the concept of dual pairs. [1] By extending existing results, we aim to have a comprehensive understanding of the principle. Secondly, we provide a first attempt to characterize the epistemic term, "compact beliefs" [2], in terms of an economic term. This gives it an economic meaning.

We achieve our first goal by proving Theorem 2.2.[3] The result is the no trade principle established for *any* dual pair $\langle X, X^* \rangle$, where X is a space of bets and X^* is a space of (signed) charges (also called finitely additive measures), both defined on a measurable space $(\Omega, \mathscr{B})$. Here, $(\Omega, \mathscr{B})$ represents the uncertainty environment, in which Ω and $\mathscr{B}$ are the sets of all possible *states*, and *events*, respectively. A *belief* is a probability measure in X^*. [4] Roughly speaking, the principle holds as long as each agent has a convex and compact set of beliefs.

We achieve our second goal by showing that the assumption of compact beliefs in the no trade principle is not merely technical. We explain as follows. When an agent has a set K of beliefs, we say that a bet is *positive* for him if it can lead to positive expected gains with respect to all his beliefs in K. Let C_K be the dual cone, which collects all the positive bets of the agent. We show that, in fact, for several broad classes of dual pairs, a closed set K of beliefs is compact if and only if its dual cone C_K is open (Theorems 4.3, 4.7 and Corollary 4.4). That C_K is open describes a condition on making economic choices. This

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[1] We refer to Aliprantis and Border (2007) for all the mathematical terms used in this article.

[2] For the sake of simplicity, we say that an agent has compact/closed beliefs if the set of all his possible beliefs is compact/closed.

[3] Theorem 5.2 is technically equivalent to Theorem 2.2 in our framework. We regard these two theorems as a single result.

[4] For the sake of convenience, we call X^* the *extended belief space*.

condition says that whenever a bet is positive for an agent, any bet that is close enough to it is also positive for him.

As pointed out in the opening paragraph, the economic content of the no trade principle is widely understood. It appears that, in some sense, the main contribution of Theorem 2.2 is just a technical one. However, the duality argument we use not only provides insightful reasons why the no trade principle is true, but also improves our understanding of the roles played by various assumptions, in particular, compactness assumptions.

Our setting is clearly a generalization of the one in Ng (2003), but we do not follow its duality argument. We explain the distinction in the next two paragraphs.

Ng (2003) has already observed that a major reason for the validity of the no trade principle is that any (non-empty) closed and convex set of beliefs has the bipolar property (see Lemma 4.1 and the discussion below). The no trade principle then becomes an immediate consequence because of three features that come along with the model in Ng (2003): (1) any closed set of beliefs is compact; (2) any compact set of beliefs has an open dual cone; (3) after being normalized, every positive continuous linear functional on the bet space can be identified with a unique Borel probability measure (see Remarks 6.5, 6.6, 6.8 and 6.9). While the bipolar property holds, for any dual pair, none of these three features is true in general. The duality argument used in Ng (2003) may have obscured our understanding of the no trade principle.

We have explained that the approach in Ng (2003) does not work in general. Nonetheless, we have not arrived at the end of reasoning yet, because presumably, we can regard the no trade principle as a generalization of the Separation Theorem. It is quite natural to assume directly that at least some agents have compact beliefs. Moreover, there should be some duality argument that is independent of the topology of the bet space, avoiding the use of features (2) and (3). It turns out that Lemma 2.1(b) provides such an argument. This lemma is a new economic observation that explains why the no trade principle generally holds for any dual pair. It establishes an aggregate duality relationship for any finitely many agents with convex and compact beliefs. It says that a bet is positive with respect to the set of all common beliefs if and only if it is a sum of some individual positive bets. Based on this observation, we can obtain the no trade principle right away just by using a simple compactness argument and the Separation Theorem. This line of reasoning does not depend on any particular dual pair.

We see from Theorem 2.2 that the two fundamental assumptions that we impose on each agent's set of beliefs are convexity and compactness. One can justify the convexity assumption from the interim viewpoint because by Bayes' rule, any set of prior beliefs is non-empty and convex. Compact sets of beliefs are always closed (because the weak* topology is always Hausdorff). Lemma 4.1 shows that for any closed and convex set K of beliefs we have the bipolar property, i.e., we can recover all the beliefs from its dual cone C_K of positive bets. However, the bipolar property is, in general, not sufficient for proving the no trade principle, because it only establishes individual duality relationship between beliefs and bets. Thus, compactness naturally comes into play.

Compactness assumptions are pervasive in the economic literature, indicating their importance. We make two remarks below.

First of all, Feinberg (2000) has demonstrated the failure of the (interim) no trade principle for the measurable space $(\mathbb{N}, 2^{\mathbb{N}})$,[5] by giving an example in which there is no common prior and yet there is no trade involving bounded bets. The paper goes on to argue that the compactness assumption put on the state space cannot be discarded since in the example, the state space $\mathbb{N}$ is non-compact. The reasoning is quite weak. In fact, in the setting of the paper, conditions[6] that ensure the validity of the no trade principle would imply that every agent has compact prior beliefs. Theorem 5.2 shows that we can adopt the weaker assumption of compact prior beliefs, and extend the no trade principle to a setting with arbitrary dual pair arising from a given measurable space $(\Omega, \mathscr{B})$. Moreover, we can easily find examples in which the underlying state space Ω is non-compact, or even has no topology imposed (see Section 3). Consequently, the compactness of the state space is not a key to the validity of the no trade principle. What really matters is the assumption of compact

[5]We let $\mathbb{N}$ be the set of all natural numbers, and if necessary, be endowed with the discrete topology.

[6]In Feinberg (2000), the state space is compact Hausdorff, and posterior beliefs vary continuously across states. These are precisely the two conditions which imply that each agent has compact prior beliefs.

prior beliefs. As a verification, we can actually show that in the counterexample of Feinberg (2000), no agent has compact prior beliefs (see Section 5.2 for details).

Secondly, we remark that the assumption of compact beliefs has an immediate implication. It implies that there is no common belief for all the agents if and only if there is no common belief for some finite subset of agents[7]. This is a major step in proving that the no trade principle holds for any number of agents, finite or infinite.

In the next few paragraphs, we shall discuss about the main contribution of Theorems 4.3 and 4.7, and Corollary 4.4. We emphasize that the question of giving an economic characterization of compact beliefs is new, and our results show that such a characterization is possible for quite general uncertainty environment.

It is customary to model an uncertainty environment as a measurable space $(\Omega, \mathscr{B})$. In fact, the Savage's theory of choice (regarding the integral representation of preference relations) and many of its extensions (see Fishburn (1970) for details) are built upon such a general environment. Moreover, this basic model has been widely adopted in various areas of economic theory, such as the theory of general equilibrium with uncertainty. Some important results in economic theory are based on the assumption that Ω is a topological space.[8] For example, the construction of the universal space, which consists of (coherent) hierarchies of beliefs as elements, when there is incomplete information, can be worked out if Ω is a compact Hausdorff space (see Mertens and Zamir (1985)), or if Ω is a Polish space (see Brandenburger and Dekel (1993)).

If one considers Ω as purely set-theoretic, then a natural candidate of bet space is the space of all $\mathscr{B}$-measurable functions on Ω. Because of integrability requirement, if the belief space is rich enough (see Remark 6.4), then every bet is bounded. It is well known that $\langle B(\Omega, \mathscr{B}), ba(\mathscr{B})\rangle$[9] is a dual pair, and it turns out that our characterization holds for it.

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[7] Ng (2003) has got the same result as a consequence of, rather than as a reason for, the no trade principle.

[8] In this case, it makes no sense if $\mathscr{B}$ has no connection with the topology on Ω, and so we naturally let $\mathscr{B}$ be the σ-algebra of all Borel subsets of Ω.

[9] In order to save space and not to repeat the introduction of notation, we refer to Section 3 for detailed description of all the dual pairs discussed.

Suppose that there is a topology on Ω such that $\mathscr{B}$ is exactly the σ-algebra of all Borel subsets of Ω. This happens, for example, when $\mathscr{B} = 2^{\Omega}$. Then in addition to $\langle B(\Omega, \mathscr{B}), ba(\mathscr{B})\rangle$, there are some other modeling choices of dual pairs for which our characterization result holds. In this article, we give three such examples (see Corollary 4.4 and Theorem 4.7). The first one is $\langle C(\Omega), rca(\mathscr{B})\rangle$, where Ω is compact Hausdorff. The second one is $\langle C(\Omega), rca_c(\mathscr{B})\rangle$, where Ω is a metric space. The third one is $\left\langle C^b(\Omega), rca(\mathscr{B})\right\rangle$, where Ω is a complete metric space. Note that in each of these models, bets are continuous and beliefs are regular, so that the dual pair depends on the topological structure of Ω. Note also that the aforementioned examples cover most of the models of uncertainty we have encountered in the economic literature.

For the dual pair $\langle C(\Omega), rca_c(\mathscr{B})\rangle$, where Ω is a metric space, Theorem 4.7 also shows that a closed set of beliefs is compact if and only if it has a uniform compact support. This means that an agent with compact beliefs acts as if he has a subjective state space which is compact. A similar result holds for the dual pair $\left\langle C^b(\Omega), rca(\mathscr{B})\right\rangle$, where Ω is a complete metric space. In this case, a closed set of beliefs is compact if and only if it is uniformly tight, and this means an agent acts as if he has a subjective state space that is "approximately" compact. Technically speaking, these two results nicely connect any compact set of beliefs to the topology of Ω. As we shall see in Section 5.2, verifying whether a set of beliefs has a uniform compact support, or whether it is uniformly tight, is much easier than verifying directly whether it is (weak*) compact.

We regard this article as a sequel to Ng (2003), which has already given a literature review on the study of the no trade principle. Hence, we just briefly unify the existing literature using the dual pair framework. Morris (1994) and Samet (1998) have studied the case of finite state space. In our words, the underlying measurable space is $(\Omega, 2^{\Omega})$, where Ω is a finite set of n elements, and the dual pair considered is $\langle \mathbb{R}^n, \mathbb{R}^n\rangle$. Feinberg (2000) and Ng (2003) have extended to the case when Ω is compact Hausdorff, and the dual pair considered is $\langle C(\Omega), rca(\mathscr{B})\rangle$. In this article, Theorem 2.2 is an extension of the no trade principle as we can relax the compactness assumption of Ω without any additional condition (see Examples 3.2 and 3.3).

Billot et al. (2000) is not a paper about the no trade principle, but its Theorem 2 is a mathematical result that one can apply, after some generalization, to obtain the principle. We do not follow this approach because of two reasons. Firstly, it is hard to give an economic interpretation of the translation that appears in Theorem 2 of Billot et al. (2000). Secondly, Billot et al. (2000) only explains its theorem from a geometric viewpoint, while our approach offers an economic insight, i.e., Lemma 2.1(b), which plays a significant role in the explanation of the no trade principle.

This article provides a pioneer work in the economic characterization of compact beliefs. It is worthwhile to point out that the proof of Theorem 4.7 is based on two classical results, the Alaoglu-Bourbaki Theorem in functional analysis, as well as the Prokhorov Theorem in probability theory (see Prokhorov (1956)).

The rest of this article proceeds as follows. In Section 2, we set up a dual pair framework, and present a general form of the no trade principle. In Section 3, we give some modeling choices of dual pairs. In Section 4, we investigate the economic characterization of compact beliefs. In Section 5, we illustrate how we can apply our main results, to obtain the interim no trade principle, and to give a detailed analysis of the interesting case when the underlying measurable space is $(\mathbb{N}, 2^{\mathbb{N}})$. We make some concluding remarks in Section 6, and put all the detailed proofs in the Appendix.

2. Dual Pair and the No Trade Principle

In this section, we present a general form of the no trade principle using the concept of dual pairs.

Let $(\Omega, \mathscr{B})$ be a measurable space, where Ω is a non-empty set and $\mathscr{B}$ is a σ-algebra of subsets of Ω. Let X be a vector space of real-valued functions on Ω and X^* be a vector space of signed charges on $\mathscr{B}$. We assume that $\langle X, X^* \rangle$ is a dual pair with a (non-degenerate) bilinear form $\langle \cdot, \cdot \rangle$ defined on $X \times X^*$ via integration. That is, for each $f \in X$ and $\mu \in X^*$, we have $\langle f, \mu \rangle = \int_\Omega f \, d\mu$. Let P be the set of all probability measures on $\mathscr{B}$. It will be convenient to write $P^* = P \cap X^*$. As described in Section 1, we interpret X as the *bet space* of all available bets and P^* as the *belief space* of all admissible beliefs.

Without confusion, we denote by c the function on Ω that takes the constant value $c \in \mathbb{R}$ for all $\omega \in \Omega$. In order that the equation $\int_\Omega d\mu = 1$ makes sense for all $\mu \in P^*$, we require that $1 \in X$, i.e., X contains all constant functions on Ω. We shall always equip X^* with the weak* topology, i.e., the $\sigma(X^*, X)$-topology. It is well known that X^* is a locally convex (Hausdorff) space and its topological dual is exactly X. Clearly, P^* is a convex subset of X^* satisfying $\langle 1, \mu \rangle = 1$ for all $\mu \in P^*$.

Let K be a non-empty subset of X^*. We associate K with its (strict) *dual cone*

$$C_K = \{f \in X : \langle f, \mu \rangle > 0 \quad \text{for all } \mu \in K\},$$

which is clearly convex. If $K \subseteq P^*$, then $1 \in C_K$, and so $C_K \neq \varnothing$. If an agent has a set K of beliefs in P^*, then C_K is the set of all his *positive bets*, i.e., C_K collects any bet that can lead to positive expected gains with respect to all his beliefs in K.

Let I denote a non-empty index set of risk-neutral agents. We define

$$\sum_I C_{K_i} = \left\{ \sum_{i \in J} f_i : J \text{ is a finite subset of } I \text{ and } f_i \in C_{K_i} \right\},$$

which equals $\operatorname{conv}\left(\bigcup_{i\in I} C_{K_i}\right)$, i.e., the convex hull of $\bigcup_{i\in I} C_{K_i}$. Define *a trade* to be a function $\mathbf{f} : I \longrightarrow X$ such that for some finite subset J of I, we have $\sum_{i\in J} \mathbf{f}_i = 0$.

Lemma 2.1. *Suppose that for each $i \in I$, we have $K_i \subseteq P^*$ and $\bigcap_{i\in I} K_i \neq \varnothing$. Then,*

(a) $0 \notin \sum_I C_{K_i}$;

(b) (Aggregate duality)

$$C_{\bigcap_{i\in I} K_i} = \sum_I C_{K_i}$$

if I is finite and all K_i's are weak compact and convex.*

We are now ready to state the no trade principle.

Theorem 2.2 (No Trade Principle)**.** *For each $i \in I$, let K_i be non-empty, convex and weak* closed in P^*. Suppose that all K_i's, possibly with one exception, are weak* compact. Then, $\bigcap_{i\in I} K_i \neq \varnothing$ if and only if $0 \notin \sum_I C_{K_i}$.*

Lemma 2.1(a) is just the standard No Trade Theorem (with no private information). Lemma 2.1(b) provides an insightful argument why the no trade principle is generally true, as we have already explained its economic message in Section 1.

Theorem 2.2 is the no trade principle, established for *any* dual pair $\langle X, X^* \rangle$. Its economic content is well understood. If it is common knowledge that all agents choose positive bets, then there is a common belief if and only if they are not prepared to trade.

3. Examples of Dual Pairs

In this section, we are going to present three general examples of dual pairs, which are useful in related areas of economic research. If X is a locally convex (Hausdorff) space and X^* is its topological dual, then $\langle X, X^* \rangle$ forms a dual pair with the bilinear form $\langle x, x^* \rangle = x^*(x)$ for all $(x, x^*) \in X \times X^*$. This is the way we get our examples.

Example 3.1 ($\langle X, X^* \rangle = \langle B(\Omega, \mathscr{B}), ba(\mathscr{B}) \rangle$). We let $X = B(\Omega, \mathscr{B})$ be the space of all bounded $\mathscr{B}$-measurable functions on Ω. It is well known that $B(\Omega, \mathscr{B})$ equipped with the sup norm is a Banach space. Its topological dual is $X^* = ba(\mathscr{B})$, which consists of all signed charges on $\mathscr{B}$ with bounded variations (see for example, Section 14.1 of Aliprantis and Border (2007)). □

In the next two examples, we shall assume that Ω is a Tychonoff space, i.e., a completely regular Hausdorff space.[10] In this case, we always take $\mathscr{B}$ to be the σ-algebra of all the Borel subsets of Ω. We introduce below some standard notations and definitions before the presentation of our examples.

Let $C(\Omega)$ denote the space of all the continuous real-valued functions on Ω, and $C^b(\Omega)$ the subspace of all the bounded functions in $C(\Omega)$. The space $rca(\mathscr{B})$ consists of all the regular signed measures on $\mathscr{B}$ with bounded variations. We recall that the *support* of a (regular) measure μ, denoted by $\operatorname{supp} \mu$, is the complement of the union of all open sets $V \subset \Omega$ such that $\mu(V) = 0$. A (regular) signed measure has a compact support if its total variation

[10]Indeed, the state space Ω, modeled as a Tychonoff space, provides the most general topological set up for dual pair of continuous functions and regular measures, as indicated in (Gillman and Jerison, 1976, Theorem 3.6).

measure has a compact support. The space $rca_c(\mathscr{B})$ consists of all the elements in $rca(\mathscr{B})$ with compact supports.

Example 3.2 ($\langle X, X^* \rangle = \langle C(\Omega), rca_c(\mathscr{B}) \rangle$)**.** We endow $X = C(\Omega)$ with the compact-open topology (also called the topology of uniform convergence on compact sets). Then X is a locally convex (Hausdorff) space and its topology is generated by the following family of semi-norms:

$$\|f\|_{\Omega_0} = \sup\{|f(\omega)| : \omega \in \Omega_0\},$$

where Ω_0 runs through all the non-empty compact subsets of Ω. The topological dual of X is $X^* = rca_c(\mathscr{B})$ (see Jarchow (1981), Section 7.6.5). □

Example 3.3 ($\langle X, X^* \rangle = \left\langle C^b(\Omega), rca(\mathscr{B}) \right\rangle$)**.** A real-valued function f on Ω is said to be *vanishing at infinity* if for all $\epsilon > 0$, the set $\{\omega \in \Omega : |f(\omega)| \geq \epsilon\}$ is compact. Let $M_0(\Omega)$ be the set of all the bounded real-valued functions on Ω vanishing at infinity. Each $s \in M_0(\Omega)$ defines a semi-norm p_s on $C^b(\Omega)$ as follows.

$$p_s(f) = \sup_{t \in \Omega} |f(t)s(t)|.$$

This family of semi-norms generates a locally convex (Hausdorff) topology on $C^b(\Omega)$, called the *strict topology*. The topological dual of X is $X^* = rca(\mathscr{B})$ (see Jarchow (1981), Section 7.6.3). □

We remark that when Ω is compact Hausdorff, the two preceding examples coincide.

4. Individual Duality and Characterization of Compact Beliefs

Motivated by Theorem 2.2 and its proof, we shall study how the assumptions imposed on a given set K of beliefs are related to the duality relationship between K and its dual cone C_K. In particular, our main purpose in this section is to characterize all the weak* compact subsets of P^*. We start with the following observation.

Lemma 4.1 (Individual Duality)**.** *Let $\varnothing \neq K \subseteq P^*$. If K is convex and weak* closed in P^* (i.e., $K = \overline{K} \cap P^*$), then it satisfies the following* bipolar property:

$$K = \{\mu \in P^* : \langle f, \mu \rangle > 0 \quad \text{for all } f \in C_K\}. \tag{4.1}$$

The converse holds whenever C_K is weakly open.

The above lemma shows that any convex and relatively weak* closed subset K of P^* can be exactly described by its dual cone C_K, and thus establishes a duality relationship between beliefs and bets. Such relationship, however, is only at the individual level. To obtain an aggregate duality relationship between beliefs and bets, we can further assume that a set K of beliefs is weak* compact as in Lemma 2.1(b). Another way is to further assume that its dual cone C_K is open (in an appropriate topology) and follow the argument in Ng (2003). The following lemma shows a tight connection between these two approaches.

We recall that a topology τ on X is *consistent* with the dual pair $\langle X, X^* \rangle$ if (X, τ) is a locally convex Hausdorff space with topological dual exactly equal to X^*. This definition is not void since the weak topology on X, i.e., the $\sigma(X, X^*)$ topology, is always, consistent with $\langle X, X^* \rangle$.

Lemma 4.2. *Let X be endowed with a topology consistent with the dual pair $\langle X, X^* \rangle$. Suppose that $\varnothing \neq K \subseteq P^*$. If C_K is open, then K is relatively weak* compact (in X^*).*

We now examine the converse of Lemma 4.2 for each of the three dual pairs described in Section 3.

Theorem 4.3. *Consider the dual pair $\langle X, X^* \rangle = \langle B(\Omega, \mathscr{B}), ba(\mathscr{B}) \rangle)$. Let $\varnothing \neq K \subseteq P^*$, and K be weak* closed in $ba(\mathscr{B})$. Then K is weak* compact if and only if C_K is open in the sup norm topology.*

Using exactly the same arguments as in the proof of Theorem 4.3, we obtain the following corollary.

Corollary 4.4. *Consider the dual pair $\langle X, X^* \rangle = \langle C(\Omega), rca(\mathscr{B}) \rangle$, where Ω is a compact Hausdorff space. Let $\varnothing \neq K \subseteq P^*$, and K is weak* closed in $rca(\mathscr{B})$. Then K is weak* compact if and only if C_K is open in the sup norm topology.*

We can generalize the dual pair $\langle C(\Omega), rca(\mathscr{B}) \rangle$ where Ω is compact Hausdorff in two ways by considering the two dual pairs, $\langle C(\Omega), rca_c(\mathscr{B}) \rangle$ and $\left\langle C^b(\Omega), rca(\mathscr{B}) \right\rangle$ where Ω is

Tychonoff. We would like to introduce two concepts that are useful in the presentation of our results.

A subset K of $P \cap rca_c(\mathscr{B})$ is said to have *uniform compact support* if

$$S_K = \overline{\bigcup \{\operatorname{supp} \mu : \mu \in K\}}$$

is compact. A subset K of $P \cap rca(\mathscr{B})$ is said to be *uniformly tight* (also simply called *tight*) if for each $\epsilon > 0$, there is a compact subset Ω_0 of Ω such that for every $\mu \in K$, we have $\mu(\Omega \setminus \Omega_0) < \epsilon$.

Lemma 4.5. *Let Ω be a Tychonoff space. Consider the dual pair $\langle C(\Omega), rca_c(\mathscr{B})\rangle$ (respectively, $\left\langle C^b(\Omega), rca(\mathscr{B})\right\rangle$). Let $\varnothing \neq K \subseteq P^*$.*

(a) K is relatively weak compact in P^* if K has a uniform compact support (respectively, K is uniformly tight).*

(b) K has a uniform compact support (respectively, K is uniformly tight) if either (i) C_K is open, or (ii) Ω is a metric space (respectively, a complete metric space) and K is relatively weak compact in P^*.*

Lemma 4.6. *Consider the dual pair $\langle C(\Omega), rca_c(\mathscr{B})\rangle$ (respectively, $\left\langle C^b(\Omega), rca(\mathscr{B})\right\rangle$). Let $K \subseteq P^*$ be non-empty and weak* closed in P^*. Then, K has a uniform compact support (respectively, K is uniformly tight) if and only if C_K is open in the compact-open topology (respectively, strict topology).*

Combining Lemmas 4.5 and 4.6, we obtain the following characterization result.

Theorem 4.7. *Consider the dual pair*

$$\langle X, X^*\rangle = \langle C(\Omega), rca_c(\mathscr{B})\rangle \quad (\textit{respectively, } \left\langle C^b(\Omega), rca(\mathscr{B})\right\rangle).$$

Let Ω be a metric space (respectively, a complete metric space), and let K be a relatively weak closed nonempty subset of P^*. Then, the following statements are equivalent.*

(a) K is weak compact.*

(b) K has a uniform compact support (respectively, K is uniformly tight).

(c) C_K is open.

5. Applications

In this section, we shall state the interim no trade principle as a direct consequence of Theorem 2.2, and take a close look at it for the case when $(\Omega, \mathscr{B}) = \big(\mathbb{N}, 2^{\mathbb{N}}\big)$.

5.1. **Interim No Trade Principle.** We can enrich our original model by adding an interim stage, at which each agent $i \in I$ has obtained private information about the state of the world. He has also formed *posterior beliefs* over Ω according to Bayes' rule.

Formally, we define a *type space* to be a tuple $\{((\Omega, \mathscr{B}), (\mathscr{B}_i, t_i)_{i\in I})\}$ where for each $i \in I$, we assume that $\mathscr{B}_i$ is a sub-σ-algebra of $\mathscr{B}$, and that $t_i : \mathscr{B} \times \Omega \to [0, 1]$ is a *type*, i.e.,

- (a)**:** (posterior beliefs) $t_i(\cdot \,|\, \omega) \in P^*$ for each $\omega \in \Omega$;
- (b)**:** $t_i(E \,|\, \cdot)$ is a $\mathscr{B}_i$-measurable function for each $E \in \mathscr{B}$;
- (c)**:** for each $B \in \mathscr{B}_i$, we have $t_i(B \,|\, \omega) = \begin{cases} 1, & \text{if } \omega \in B; \\ 0, & \text{if } \omega \notin B. \end{cases}$

Let $T_i = \{t_i(\cdot \,|\, \omega) \in P^* : \omega \in \Omega\}$, which is the set of all the posterior beliefs of agent i. For the sake of convenience, we also call T_i the *type* of agent i. Following Harsanyi (1968), a *prior* (belief) for agent i is an element $\mu \in P^*$ such that the following Bayes' rule holds:

$$\mu(E \cap B) = \int_B t_i(E \,|\, \omega) d\mu(\omega) \quad \text{for every } E \in \mathscr{B} \text{ and } B \in \mathscr{B}_i\,. \tag{5.1}$$

Let $K(T_i)$ be the set of all the prior beliefs of agent i. Given any dual pair $\langle X, X^* \rangle$ defined on $(\Omega, \mathscr{B})$, by using exactly the same argument as in Ng (2003), we can easily obtain the following properties of $K(T_i)$.

Proposition 5.1. *For each $i \in I$,*

(a) conv $(T_i) \subseteq K(T_i) \subseteq \overline{\text{conv}}$ (T_i)*;*
(b) (Dynamic Consistency) $C_{K(T_i)} = C_{T_i}$.

From the Bayes' rule and Proposition 5.1(a), $K(T_i)$ is always a non-empty convex subset of P^*. Combining this fact with Proposition 5.1(b) and Theorem 2.2, we obtain the interim no trade principle, which is true for any dual pair defined on $(\Omega, \mathscr{B})$.

Theorem 5.2 (Interim No Trade Principle)**.** *Suppose that $K(T_i)$ is weak* closed in P^* for each $i \in I$, and all $K(T_i)$'s are weak* compact (with possibly one exception). Then, $\bigcap_{i\in I} K(T_i) \neq \varnothing$ if and only if $0 \notin \sum_I C_{T_i}$.*

5.2. **The Measurable Space** $(\mathbb{N}, 2^{\mathbb{N}})$**.** Let $(\Omega, \mathscr{B}) = \big(\mathbb{N}, 2^{\mathbb{N}}\big)$. Consider any type space $\big\{((\mathbb{N}, 2^{\mathbb{N}}), (\mathscr{B}_i, t_i)_{i\in I})\big\}$. We have the following general observation.

General Claim 1: $K(T) = \sigma$-conv (T) provided that T is a type for some agent, where

$$\sigma\text{-conv } (T) = \left\{ \sum_{k=1}^{\infty} \alpha_k \mu_k \in P^* : \alpha_k \geq 0, \mu_k \in T \text{ for each } k \text{ and } \sum_{k=1}^{\infty} \alpha_k = 1 \right\}.$$

Proof. Observe that T is countable. Since $T \subseteq K(T)$, from (5.1), we have σ-conv $T \subseteq K(T)$.

Now, suppose that T is a type for agent i, and that $\mu \in K(T)$. Let $\mathbb{B}$ be the partition of $\mathbb{N}$ generated from $\mathscr{B}_i$. Each $B \in \mathbb{B}$ is associated with exactly one element $t_B \in T$. For any $E \in 2^{\mathbb{N}}$, by countable additivity, we have

$$\mu(E) = \sum_{B\in\mathbb{B}} \mu(E \cap B) = \sum_{B\in\mathbb{B},\ \mu(B)>0} \mu(B) \cdot \frac{\mu(E\cap B)}{\mu(B)} = \sum_{B\in\mathbb{B},\ \mu(B)>0} \mu(B)\, t_B(E).$$

Since $\sum_{\mu(B)>0} \mu(B) = 1$, we have $\mu \in$ σ-conv T. □

As usual, let $\ell_\infty(\mathbb{N})$ be the set of all the bounded real-valued functions on $\mathbb{N}$. We shall also regard $\mathbb{N}$ as a topological space endowed with the discrete topology. Then, it is not difficult to see that

$$B(\mathbb{N}, 2^{\mathbb{N}}) = C^b(\mathbb{N}) = \ell_\infty(\mathbb{N}), \text{ and } C(\mathbb{N}) = \mathbb{R}^{\mathbb{N}}.$$

Since $\mathbb{N}$ is a Polish space, $rca(2^{\mathbb{N}}) = ca(2^{\mathbb{N}})$, which is the space of all signed measures on $2^{\mathbb{N}}$ with bounded variations. Similarly, $rca_c(2^{\mathbb{N}}) = ca_c(2^{\mathbb{N}})$, which consists of all the elements in $ca(2^{\mathbb{N}})$ with compact supports.

General Claim 2: The next statement is true for each of the three dual pairs, $\big\langle \ell_\infty(\mathbb{N}), ba(2^{\mathbb{N}})\big\rangle$, $\big\langle \ell_\infty(\mathbb{N}), ca(2^{\mathbb{N}})\big\rangle$ and $\big\langle \mathbb{R}^{\mathbb{N}}, ca_c(2^{\mathbb{N}})\big\rangle$. If T is a type for some agent, then $K(T)$ is weak* compact if and only if T is a finite set.

Proof. In any topological vector space, the convex hull of any finite set is compact. It follows that no matter which dual pair we consider, if T is finite, then $K(T) = \text{conv}\ (T)$ is weak* compact.

We now prove the other implication. Note that a subset of $\mathbb{N}$ is compact if and only if it is finite.

We first consider the dual pair $\langle \mathbb{R}^{\mathbb{N}}, ca_c(2^{\mathbb{N}}) \rangle$. By Theorem 4.7, if $K(T)$ is weak* compact, then it has a uniform compact support. This implies that T is a finite set.

We next consider the dual pair $\langle \ell_\infty(\mathbb{N}), ca(2^{\mathbb{N}}) \rangle$. If T is an infinite set, then for any finite set $A \subset \mathbb{N}$, there is always a $\mu \in T$ such that $\mu(\mathbb{N} \setminus A) = 1$, and thus $K(T)$ cannot be uniformly tight. By Theorem 4.7, we know that $K(T)$ is not weak* compact. As a result, if $K(T)$ is weak* compact, then T is a finite set.

Finally, for the dual pair $\langle \ell_\infty(\mathbb{N}), ba(2^{\mathbb{N}}) \rangle$, observe that $P^* = P \cap\ ba(2^{\mathbb{N}}) = P \cap\ ca(2^{\mathbb{N}})$. Therefore, $K(T)$, as a subset of P^*, is weak* compact with respect to $\langle \ell_\infty(\mathbb{N}), ba(2^{\mathbb{N}}) \rangle$, if and only if it is weak* compact with respect to $\langle \ell_\infty(\mathbb{N}), ca(2^{\mathbb{N}}) \rangle$, if and only if it is finite. □

Feinberg (2000) has also given an analysis for the measurable space $(\mathbb{N}, 2^{\mathbb{N}})$ and presented an example in which the interim no trade principle may fail to hold. The bet space and belief space considered in the example are respectively $X = \ell_\infty$ and P. Hence, we can choose X^* to be either $ba(2^{\mathbb{N}})$, or $ca(2^{\mathbb{N}})$, so that $\langle X, X^* \rangle$ forms a dual pair. Since in the example, every agent has an infinitely partitioned information structure, our result shows that no agent can have compact prior beliefs. Notice that if we choose $X^* = ba(2^{\mathbb{N}})$, we just need to treat $\mathbb{N}$ as a set with no topological structure at all.

For the dual pair $\langle \mathbb{R}^{\mathbb{N}}, ca_c(2^{\mathbb{N}}) \rangle$, the following example shows that when agents do not have compact prior beliefs, the interim no trade principle may not hold.

Example 5.3. Consider the dual pair $\langle \mathbb{R}^{\mathbb{N}}, ca_c(2^{\mathbb{N}}) \rangle$. For each $n \in \mathbb{N}$, let δ_n be the point mass at n. Define the following two sequences: for $n \in \mathbb{N}$,

$$\mu_1 = \delta_1, \quad \text{and} \quad \mu_{n+1} = \frac{2}{3}\,\delta_{2n} + \frac{1}{3}\,\delta_{2n+1};$$
$$\nu_n = \frac{2}{3}\,\delta_{2n-1} + \frac{1}{3}\,\delta_{2n}.$$

Suppose that Alice's type is $\{\mu_n : n \in \mathbb{N}\}$ and Bob's type is $\{\nu_n : n \in \mathbb{N}\}$. It is easy to verify that there is no common prior, and yet there is no trade. $\square$

6. Concluding Remarks

We have shown that the no trade principle holds for any dual pair that we can reasonably defined on a given measurable space. Furthermore, for several broad classes of dual pairs, provided that every agent has a closed beliefs, the condition that an agent has compact beliefs is equivalent to the condition that his set of positive bets is open. These two results are important because they express "epistemic terms" in terms of "economic terms". They improve our understanding of two standard assumptions, i.e., the common prior and compact belief assumptions, in an economic context.

We would like to end this article with a few remarks.

Remark 6.1 (The choice of X^*)**.** For the convenience of presentation, we have taken a dual pair $\langle X, X^* \rangle$ as given, and assumed that $P^* = P \cap X^*$. If we take a bet space X and a convex belief space P^* as given, then the choice of X^* so that $P^* = P \cap X^*$ and $\langle X, X^* \rangle$ forms a dual pair may not be unique (see Section 5.2 for an example). Nonetheless, whether or not a subset of P^* is weak* compact, weak* closed in P^* or convex is irrespective of the choice of X^*. This means that if Theorem 2.2 holds for any particular choice of X^*, it holds for every choice of X^*.

Remark 6.2 (More examples of dual pairs)**.** Based on the fact that $\langle B(\Omega, \mathscr{B}), ba(\mathscr{B}) \rangle$ is a dual pair, we can actually discover more examples of dual pairs.

Let $ca(\mathscr{B})$ be the space of all the signed measures on $\mathscr{B}$ with bounded variations. It is evident that $\langle B(\Omega, \mathscr{B}), ca(\mathscr{B}) \rangle$ forms a dual pair.

Let $M(\Omega, \mathscr{B})$ be the space of all the $\mathscr{B}$-measurable functions on Ω. Let $ba_f(\mathscr{B})$ be the space of all the elements in $ba(\mathscr{B})$ with finite supports. We can similarly define the space $ca_f(\mathscr{B})$. Then both $\langle M(\Omega, \mathscr{B})), ba_f(\mathscr{B}) \rangle$ and $\langle M(\Omega, \mathscr{B})), ca_f(\mathscr{B}) \rangle$ are dual pairs.

Unlike the dual pair $\langle B(\Omega, \mathscr{B}), ba(\mathscr{B}) \rangle$, we are not aware of any "nice" consistent topology on the bet space for any of the above three examples.

Remark 6.3 (Discontinuous bets)**.** Consider the measurable space $([0,1],\mathcal{B})$, where $\mathcal{B}$ is the σ-algebra of all the Lebesgue measurable subsets of $[0,1]$. Let λ denote the Lebesgue measure. Let $L^1([0,1])$ (respectively, $L^\infty([0,1])$) be the Banach space of all the Lebesgue integrable (respectively, essentially bounded) functions on $[0,1]$. It is well known that $\langle X, X^*\rangle = \langle L^1([0,1]), L^\infty([0,1])\rangle$ forms a dual pair. Each element g in $X^* = L^\infty([0,1])$ can be identified as a unique measure μ_g defined by

$$\mu_g(A) = \int_A g\, d\lambda, \quad \forall A \in \mathcal{B}.$$

It is clear that every continuous bet is in the bet space $X = L^1([0,1])$, while there exist discontinuous bets in X as well.

Remark 6.4 (Boundedness)**.** Suppose that $\langle X, X^*\rangle$ is a dual pair defined on $(\Omega, \mathscr{B})$, and X^* contains all the countable convex combinations of point masses on Ω. Then all functions in X are bounded. Otherwise, we can find a function $f \in X$ and a sequence $\{\omega_n\}$ in Ω such that $|f(\omega_n)| \geq 2^n$ for each $n \in \mathbb{N}$. By defining $\mu = \sum_{i=1}^{\infty} \frac{1}{2^n}\delta_{\omega_n}$, we see that f is not integrable with respect to μ, a contradiction.

Remark 6.5 (P^* may not be weak* closed)**.** Consider the dual pair $\left\langle \ell_\infty, ba(2^{\mathbb{N}})\right\rangle$. Let $T = \{\delta_n : n \in \mathbb{N}\}$ be the set of all the point masses on $\mathbb{N}$. By Claims 1 and 2 in Section 5.2, $P^* = \sigma$-conv T is not weak* compact. On the other hand, the norm-closed unit ball in $ba(2^{\mathbb{N}})$ is weak* compact by the Banach-Alaoglu Theorem, and it contains P^* as a subset. It follows that P^* cannot be weak* closed. In fact, any weak* limit point of T can be identified with a 0-1 measure defined from a free ultrafilter on $\mathbb{N}$, and so cannot be countably additive.

If we consider $\langle X, X^*\rangle = \langle C(\Omega), rca_c(\mathscr{B})\rangle$, or $\left\langle C^b(\Omega), rca(\mathscr{B})\right\rangle$, where Ω is a Tychonoff space, then P^* is weak* closed. To see this, note that for either dual pair, we have

$$P^* = \{\mu \in X^* : \langle 1, \mu\rangle = 1 \text{ and } \langle f, \mu\rangle \geq 0, \forall f \geq 0 \text{ in } X\}.$$

Remark 6.6 (Compactness of P^*)**.** If Ω is a complete metric space, then from Theorem 4.7 and the previous remark, P^* is weak* compact with respect to the dual pair $\left\langle C^b(\Omega), rca(\mathscr{B})\right\rangle$ if and only if P^* is uniformly tight, or equivalently, Ω is compact. We can also obtain a similar result for the dual pair $\langle C(\Omega), rca_c(\mathscr{B})\rangle$.

Corollary 6.7. *Consider the dual pair $\langle C(\Omega), rca_c(\mathscr{B})\rangle$ (respectively, $\big\langle C^b(\Omega), rca(\mathscr{B})\big\rangle$). If Ω is a metric space (respectively, a complete metric space), then P^* is weak* compact if and only if Ω is compact.*

Remark 6.8 (Compact but not uniformly tight)**.** Consider the dual pair $\big\langle C^b(\Omega), rca(\mathscr{B})\big\rangle$, where Ω is a Tychonoff space. In general, it is not true that every weak* compact subset of P^* is uniformly tight (see Section 4 in Topsøe (1974)). Thus, according to Lemma 4.6, the dual cone of a weak* compact subset of P^* may not be (strictly) open in $C^b(\Omega)$.

For the dual pair $\langle C(\Omega), rca_c(\mathscr{B})\rangle$, we are not aware of any example in the existing literature, in which a weak* compact subset of P^* may not have a uniform compact support.

Remark 6.9 (Beliefs as linear functionals)**.** In Theorem 2.2, we can actually replace P^* with any non-empty convex subset Q of X^* satisfying $\int_\Omega d\mu = 1$ for every $\mu \in Q$.

We note that in Ng (2003), beliefs are treated as (normalized) positive and continuous linear functionals on the bet space, rather than as probability measures on $\mathscr{B}$. This approach is weaker than ours. For the dual pair $\langle X, X^*\rangle$, define

$$Q^* = \left\{ \mu \in X^* : \int_\Omega d\mu = 1 \text{ and } \int_\Omega f\, d\mu \geq 0, \forall f \geq 0 \text{ in } X \right\}.$$

Using the argument in Ng (2003), we can only obtain the no trade principle under the assumption that beliefs are elements of Q^*. This approach creates some difficulty if not every element in Q^* is countably additive. We explain as follows.

Consider the dual pair $\langle B(\Omega, \mathscr{B}), ba(\mathscr{B})\rangle$. It is evident that Q^* is the set of all the probability charges on $\mathscr{B}$ (i.e., any $\mu \in ba(\mathscr{B})$ satisfying $\mu \geq 0$ and $\int_\Omega d\mu = 1$). In other words, beliefs are finitely additive, but may not be countably additive. Observe that Theorem 5.2 does not hold when the belief space is Q^* because in general, the law of iterated expectations holds for probability measures, but not for probability charges.

Remark 6.10 (Topology on X)**.** It is obvious that if the characterization result in Theorems 4.3, 4.7 or Corollary 4.4 holds for a certain topology on X, then it holds for any weaker topology on X. In particular, Theorem 4.3 and Corollary 4.4 are true with any consistent topology on X because the norm topology is the strongest consistent topology, i.e., the Mackey topology, on X.

Remark 6.11 (Generalizations)**.** As shown by Morris (1994) and Ng (2003), one can easily extend Theorem 2.2 (and hence Theorem 5.2) to the case in which all agents are risk-averse by standard convex programming arguments.

We now discuss about other possible generalizations. Recall that Theorem 2.2 actually consists of two parts: the no trade theorem with no private information (i.e., Lemma 2.1(a)) and its converse.

Since Lemma 2.1(a) follows just from definitions, the interim no trade theorem holds trivially for dynamically consistent agents. An interesting question is: how far can we deviate from the framework of expected utility so that the interim no trade theorem still holds? Halevy (2004) has provided some answer to this question when the number of agents is finite and the state space is finite.

Note that Theorem 2.2 itself has already taken care of the case when agents have multiple prior beliefs. The question of whether the interim no trade principle holds in this case is reduced to the question of finding conditions on a given updating rule so that all agents are dynamically consistent. The paper, Kajii and Ui (2009), has provided some nice examples along this line of thought. However, as the converse of Lemma 2.1(a) relies heavily on the dual pair structure of the bet and (extended) belief spaces, there is little room left for an extension of Theorem 2.2 to the case when agents have general preferences.

Remark 6.12 (More on $(\mathbb{N}, 2^{\mathbb{N}})$)**.** Recently, there are some papers (for example, Hellman (2014)) which attempt to establish the no trade principle for the measurable space $(\mathbb{N}, 2^{\mathbb{N}})$. The approach is not a standard one (i.e., there is no decision-theoretic foundation) as these papers assume that beliefs may be improper. In other words, an agent may assign infinite probability to the state space. In this case, the bet space X cannot contain any constant bet. It does not seem that there is a natural duality between beliefs and bets in the sense of Lemma 4.1. And as expected, it turns out that their results are not entirely positive.

7. Appendix: proofs

In many of our proofs, we shall use the Separation Theorem (see Theorem 3.4, Rudin (1991), p. 59) and the Fundamental Theorem of Duality (see Theorem 5.91, Aliprantis and

Border (2007), p. 212). We shall also use the following standard fact from general topology (see Section 3.11(a), Gillman and Jerison (1976), p.42).

Proposition 7.1. *Let Ω be a completely regular space. Let A and B be two disjoint non-empty subsets of Ω. If A is compact and B is closed, then there exists a continuous function $f : \Omega \longrightarrow [0,1]$ such that $f|_A \equiv 1$ and $f|_B \equiv 0$.*

7.1. **Proof of Lemma 2.1.** (a) If $\bigcap_{i\in I} K_i \neq \varnothing$, then each $K_i \neq \varnothing$. It follows just from the definition of each C_{K_i} that

$$\begin{aligned}\varnothing \neq \bigcap_{i\in I} K_i \subseteq \bigcap_{i\in I} \{\mu \in P^* : \langle f, \mu\rangle > 0, \quad \forall\, f \in C_{K_i}\} \\ = \left\{\mu \in P^* : \langle f, \mu\rangle > 0, \quad \forall\, f \in \bigcup_{i\in I} C_{K_i}\right\} \\ = \left\{\mu \in P^* : \langle f, \mu\rangle > 0, \quad \forall\, f \in \sum_I C_{K_i}\right\}.\end{aligned}$$

It is clear that $0 \notin \sum_I C_{K_i}$. □

(b) It suffices to consider the case when $I = \{1, 2\}$.

For any $f \in (C_{K_1} \cup C_{K_2})$ and $\mu \in (K_1 \cap K_2)$, we have $\langle f, \mu\rangle > 0$. Hence,

$$\sum_I C_{K_i} = \text{conv } (C_{K_1} \cup C_{K_2}) \subseteq C_{K_1 \cap K_2}\,.$$

If $f \in C_{K_1\cap K_2}$ but $f \notin (C_{K_1} \cup C_{K_2})$, then for $i = 1, 2$, we can find $\alpha_i \in K_i$ such that $\langle f, \alpha_i\rangle \leq 0$. Moreover, for any $\mu \in (K_1 \cap K_2)$, we have $\langle f, \mu\rangle > 0$. It follows from the convexity of K_i and the linearity of f on X^* that $\langle f, \beta_i\rangle = 0$ for some $\beta_i \in K_i$. Define

$$H = \{\mu \in X^* : \langle f, \mu\rangle = 0\} \text{ and } K_i(f) = K_i \cap H \text{ for } i = 1, 2.$$

Then $K_1(f)$ and $K_2(f)$ are disjoint, non-empty, convex and weak* compact subsets of the locally convex subspace H of X^*. Since any continuous linear functional on H can be extended to be an element of X (see Theorem 5.87, Aliprantis and Border (2007), p. 210), by the Separation Theorem and a translation argument, we can find $g_i \in C_{K_i(f)}$ such that $\langle g_1 + g_2, \mu\rangle = 0$ for all $\mu \in H$. Now, fix $i = 1$ or 2 and consider the weak* closed and convex set

$$H_i = \{\mu \in H : \langle g_i, \mu \rangle = 0\},$$

which is disjoint from the weak* compact convex set K_i. By the Separation Theorem, there is an $h_i \in X$ and $c_i \in \mathbb{R}$ such that

$$\inf\{\langle h_i, \mu \rangle : \mu \in K_i\} > c_i > \sup\{\langle h_i, \mu \rangle : \mu \in H_i\}.$$

It is obvious that $\langle h_i, \mu \rangle = 0$ for all $\mu \in H_i$. Indeed, since H_i is a vector space, if there is an $\nu \in H_i$ such that $\langle h_i, \nu \rangle = d_i \neq 0$, then $\left\langle h_i, \left(\frac{c_i}{d_i}\right)\nu \right\rangle = c_i$, a contradiction. Consequently, $h_i \in C_{K_i}$ and by the Fundamental Theorem of Duality, we can find $\lambda_i \in \mathbb{R}$ such that $h_i = \lambda_i\, g_i$ on H.

For any $\mu \in K_i(f)$, we have $\langle h_i, \mu \rangle > 0$ and $\langle g_i, \mu \rangle > 0$ so that $\lambda_i > 0$. Let $f_i = \frac{h_i}{\lambda_i} \in C_{K_1}$. Then, $\langle f_1 + f_2, \mu \rangle = 0$ for all $\mu \in H$. Using the Fundamental Theorem of Duality, we can find $\lambda \in \mathbb{R}$ such that $f_1 + f_2 = \lambda\, f$. By letting both sides of the last equation act on any $\mu \in K_1 \cap K_2$, we get $\lambda > 0$. This implies that $f \in (C_{K_1} + C_{K_2})$.

As a result, $C_{K_1 \cap K_2} = \sum_I C_{K_i}$. □

7.2. **Proof of Theorem 2.2.** We need to prove that under our assumptions, the converse of Lemma 2.1(a) is also true.

Suppose that $\bigcap_{i\in I} K_i = \varnothing$. Then there are at least two agents. We may assume that K_1 is weak* compact. By the finite intersection property, we may assume that there are $K_1, \ldots, K_m, K_{m+1}$ from K_i's with $\bigcap_{i=1}^m K_i \neq \varnothing$ but $\bigcap_{i=1}^{m+1} K_i = \varnothing$. We may also assume that K_{m+1} is weak* closed in P^*, while K_i is weak* compact for every $i = 1, 2, \ldots, m$. Since $\overline{K_{m+1}} \cap (\bigcap_{i=1}^m K_i) = \varnothing$. By the Separation Theorem and a translation argument, there is an $f \in X$ such that $-f \in C_{\overline{K_{m+1}}} \subseteq C_{K_{m+1}}$, and $f \in C_{\bigcap_{i=1}^m K_i} = \sum_{i=1}^m C_{K_i}$ (by Lemma 2.1(b)). This implies that $0 \in \sum_{i=1}^{m+1} C_{K_i} \subseteq \sum_I C_{K_i}$. □

7.3. **Proof of Lemma 4.1.** Suppose that $K \subseteq P^*$ is non-empty, convex and weak* closed in P^*. Consider the convex set

$$D = \{\mu \in P^* : \langle f, \mu \rangle > 0 \quad \text{for all } f \in C_K\}.$$

By definition, $K \subseteq D$. Suppose that $\nu \in P^*$ but $\nu \notin K$. Then $\nu \notin \overline{K}$. By the Separation Theorem, there is an $f \in X$ and a constant c such that

$$\langle f, \mu \rangle > c > \langle f, \nu \rangle \quad \forall\, \mu \in \overline{K},$$
$$\text{i.e.,} \quad \langle f - c, \mu \rangle > 0 > \langle f - c, \nu \rangle \quad \forall\, \mu \in \overline{K}.$$

The first inequality shows that $f - c \in C_{\overline{K}} \subseteq C_K$. The second inequality then shows that $\nu \notin D$. It follows that $K = D$.

Conversely, we assume C_K is weakly open, and K satisfies the bipolar property. It suffices to prove that D is weak* closed in P^*. Let μ be a weak* limit point of D in P^*. Fix any $f \in C_K$. Since C_K is weakly open, we can find $\mu_1, \ldots, \mu_n \in X^*$ such that we have $g \in C_K$ whenever $g \in X$ and $-1 < \int_\Omega (f - g)\, d\mu_i < 1$ for all i. Take $\epsilon > 0$ so that $-1 < \epsilon \int_\Omega d\mu_i < 1$ for all i. This implies that $f - \epsilon \in C_K$. By the definition of C_K and the fact that μ is a weak* limit point of D in P^*, we have $\int_\Omega f\, d\mu \geq \epsilon > 0$. We conclude that $K = D$ is weak* closed in P^*. □

7.4. **Proof of Lemma 4.2.** Given $\varnothing \neq K \subseteq P^*$, define $\overline{C}_K = \{f \in X : \langle f, \mu \rangle \geq 0, \forall\, \mu \in K\}$ and $D = \left\{f \in X : f + 1 \in \overline{C}_K\right\}$. By definition,

$$\begin{aligned} K &\subseteq \left\{\mu \in P^* : \langle f, \mu \rangle \geq 0, \quad \forall\, f \in \overline{C}_K\right\} \\ &= \{\mu \in P^* : \langle f, \mu \rangle \geq -1, \quad \forall\, f \in D\}. \end{aligned}$$

Since C_K is open and $1 \in C_K$, by translation, 0 is an interior point of D. Therefore, there is a balanced neighborhood U of 0 such that $U \subseteq D$. Now, the absolute polar of U is

$$\begin{aligned} U^\circ &= \{\mu \in X^* : |\langle f, \mu \rangle| \leq 1, \quad \forall\, f \in U\} \\ &= \{\mu \in X^* : \langle f, \mu \rangle \geq -1, \quad \forall\, f \in U\}, \end{aligned}$$

which is weak* compact by the Alaoglu-Bourbaki Theorem (see Theorem 5.105, Aliprantis and Border (2007), p.218). Since $U \subseteq D$ and $K \subseteq \{\mu \in P^* : \langle f, \mu \rangle \geq -1,\ \forall\, f \in D\}$, we have $K \subseteq U^\circ$. As a result, the weak* closed subset $\overline{K}$ of U° is weak* compact. □

7.5. **Proof of Theorem 4.3.** In view of Lemma 4.2, we only need to show that if K is weak* compact, then C_K is open. The key is that $X = B(\Omega, \mathscr{B})$, equipped with the sup norm, is a Banach space. Fix any $f \in C_K$. Since K is weak* compact, we may let $2\epsilon = \min_{\nu \in K} \int_\Omega f \, d\nu > 0$. For any $\mu \in K$ and for any $g \in X$ such that $||g - f|| < \epsilon$, we have

$$\int_\Omega g \, d\mu \geq \int_\Omega f \, d\mu - \int_\Omega |g - f| \, d\mu \geq 2\epsilon - \epsilon = \epsilon > 0.$$

It follows that $g \in C_K$ and hence C_K is open. □

7.6. **Proof of Lemma 4.5.** We note that for the dual pair $\langle C^b(\Omega), rca(\mathscr{B}) \rangle$, the statement in part (a) is just Prokhorov's Theorem (see Theorem 12.6.3, p. 268 in Jarchow (1981)).

7.6.1. *Proof of (a).* We are left with the dual pair $\langle C(\Omega), rca_c(\mathscr{B}) \rangle$.

Let K have a uniform compact support S_K in Ω. Let $\mathscr{D} = \{B \cap S_K : B \in \mathscr{B}\}$ be the collection of all the Borel subsets of S_K. Let $P(\mathscr{D})(\subseteq rca(\mathscr{D}) = rca_c(\mathscr{D}))$ be the set of all the regular probability measures on $\mathscr{D}$. Let $\{\mu_\lambda\} \subseteq K$ be a net in P^*. This induces a net $\{\nu_\lambda\} \subseteq P(\mathscr{D})$ defined by $\nu_\lambda(D) = \mu_\lambda(D)$ for all $D \in \mathscr{D}$. Now, $P(\mathscr{D})$ is weak* compact with respect to the dual pair $\langle C(S_K), rca(\mathscr{D}) \rangle$. We may assume that it has a weak* limit $\nu \in P(\mathscr{D})$.

Define μ on $\mathscr{B}$ by $\mu(B) = \nu(B \cap S_K)$. It is obvious that $\mu \in P^*$ since $\operatorname{supp} \mu$ is contained in the compact set S_K. We claim that μ is a weak* limit of $\{\mu_\lambda\}$ with respect to the dual pair $\langle C(\Omega), rca_c(\mathscr{B}) \rangle$. Indeed, by taking any $f \in C(\Omega)$, we have

$$\int_\Omega f \, d\mu_\lambda = \int_{S_K} f|_{S_K} \, d\nu_\lambda \longrightarrow \int_{S_K} f|_{S_K} \, d\nu = \int_\Omega f \, d\mu \, .$$

It follows that K is relatively weak* compact in P^*. □

7.6.2. *Proof the first part of (b).* Assume that condition $(b)(i)$ is satisfied.

(Case 1) We first consider the dual pair $\langle C(\Omega), rca_c(\mathscr{B}) \rangle$. Let $\varnothing \neq K \subseteq P^*$. Suppose that C_K is open in $C(\Omega)$. Since $1 \in C_K$, there exists a compact subset Ω_0 of Ω and a $\delta > 0$ such that $\{f \in C(\Omega) : \|f - 1\|_{\Omega_0} < \delta\} \subseteq C_K$.

Claim: $\bigcup_{\mu \in K} \operatorname{supp} \mu \subseteq \Omega_0$ and thus $S_K \subseteq \Omega_0$ is compact (since Ω_0 is a compact set and by definition, S_K is closed).

Suppose not, then there exists an $\omega_0 \in \operatorname{supp}\mu_0$, for some $\mu_0 \in K$ but $\omega_0 \notin \Omega_0$. Since Ω is completely regular, there exists a continuous function $g : \Omega \to [0,1]$ such that $g(\omega_0) = 1$ and $g|_{\Omega_0} \equiv 0$. Fix an ϵ satisfying $0 < \epsilon < 1$. By the continuity of g, there exists an open set $V \subseteq \Omega$ such that $\omega_0 \in V$ and for every $\omega \in V$, $g(\omega) \geq 1 - \epsilon$. Let $S = \operatorname{supp}\mu_0$. Since $\omega_0 \in (V \cap S) \neq \varnothing$, we have $\mu_0(V \cap S) > 0$ (otherwise $\mu_0(V) = \mu_0(V \cap S) + \mu_0(V \cap S^c) = 0 + 0 = 0$ and so $V \subseteq S^c$, which implies that $V \cap S = \varnothing$, a contradiction). It follows that

$$\int_\Omega g\, d\mu_0 \geq \int_V g\, d\mu_0 \geq (1-\epsilon)\mu_0(V \cap S) > 0.$$

For any $\lambda > 0$, observe that $\|(1 - \lambda g) - 1\|_{\Omega_0} = 0 < \delta$. Hence, $(1 - \lambda g) \in C_K$ for all $\lambda > 0$. Since $\mu_0 \in K$, we have $\int_\Omega (1 - \lambda g) d\mu_0 > 0$, or $\lambda \int_\Omega g\, d\mu_0 < 1$ for all $\lambda > 0$, which is impossible. This shows that S_K is compact and the proof is complete. □

(Case 2) We next consider the dual pair $\langle C^b(\Omega), rca(\mathscr{B})\rangle$. Let $\varnothing \neq K \subseteq P^*$. Suppose that C_K is open in $C^b(\Omega)$. Since $1 \in C_K$, there exist a $\delta > 0$ and a function $s \in M_0(\Omega)$ such that

$$\left\{ f \in C^b(\Omega) : p_s(f - 1) = \sup_{t \in \Omega} |(f-1)(t)s(t)| < \delta \right\} \subseteq C_K.$$

By definition, s vanishes at infinity. Thus, for each ϵ satisfying $0 < \epsilon < \delta$, there exists a compact set $\Omega_0 \subseteq \Omega$ such that

$$|s(t)| < \frac{\epsilon^2}{2}\,, \quad \forall\, t \notin \Omega_0.$$

We <u>claim</u> that $\mu(\Omega \setminus \Omega_0) < \epsilon$ for all $\mu \in K$ and thus K is uniformly tight.

Suppose not, then there exists some $\mu_0 \in K$ such that $\mu_0(\Omega \setminus \Omega_0) \geq \epsilon$. Let $S = \operatorname{supp}\mu_0$ and $\omega_0 \in S \cap \Omega_0^c$. Since μ_0 is regular, there exists a compact set $\Omega_1 \subseteq (S \cap \Omega_0^c)$ such that $\mu_0(\Omega_1) \geq \frac{\epsilon}{2}$ and $\omega_0 \in \Omega_1$. By Proposition 7.1, there exists a continuous function $g : \Omega \to [0,1]$ such that $g|_{\Omega_1} \equiv 1$ and $g|_{\Omega_0} \equiv 0$. Thus, $\int_\Omega g\, d\mu_0 \geq \int_{\Omega_1} g\, d\mu_0 = \mu_0(\Omega_1) \geq \frac{\epsilon}{2}$. Now,

$$p_s\left(1 - \frac{2g}{\epsilon} - 1\right) = p_s\left(-\frac{2g}{\epsilon}\right) = \frac{2}{\epsilon} \sup_{t \in \Omega} |g(t)s(t)| \leq \frac{2}{\epsilon} \cdot \frac{\epsilon^2}{2} = \epsilon < \delta$$

and therefore, $\left(1 - \frac{2g}{\epsilon}\right) \in C_K$. However, $\mu_0 \in K$, implying that

$$1 > \frac{2}{\epsilon} \int_\Omega g\, d\mu_0 \geq \frac{2}{\epsilon} \cdot \frac{\epsilon}{2} = 1,$$

a contradiction. This proves that K is uniformly tight. □

7.6.3. *Proof of the second part of (b).* Assume that condition $(b)(ii)$ is satisfied. Let (Ω, d) be a metric space. For $x \in \Omega$ and $r > 0$, we denote $B(x, r) = \{\omega \in \Omega : d(x, \omega) < r\}$.

(Case 1) Consider the dual pair $\langle C(\Omega), rca_c(\mathscr{B})\rangle$. Let K be a non-empty and relatively weak* compact set in P^*.

Suppose on the contrary that S_K is not a compact subset of Ω. Then, S_K contains a countably infinite subset $A = \{x_n : n \in \mathbb{N}\}$ that has no limit points. Let $F = \bigcup_{\mu \in K} \operatorname{supp} \mu$. Note that $S_K = \overline{F}$. For each $n \in \mathbb{N}$, pick an $\omega_n \in \left(F \cap B(x_n, \frac{1}{n})\right)$ and let $B = \{\omega_n : n \in \mathbb{N}\} \subseteq F$. Obviously, A and B have the same set of limit points. This implies that B has no limit points. In particular, B is a discrete subset of Ω. Therefore, there is a sequence $\{V_n\}$ of pairwise disjoint open sets such that $\omega_n \in V_n \subseteq B\left(\omega_n, \frac{1}{n}\right)$ for each $n \in \mathbb{N}$.

Now, for each $n \in \mathbb{N}$, pick $\mu_n \in K$ such that $\omega_n \in \operatorname{supp} \mu_n$. Then, $M = \{\mu_n : n \in \mathbb{N}\}$ must be an infinite subset of K (otherwise, $\bigcup_{\mu \in M} \operatorname{supp} \mu$ would be compact, and so B would have a limit point, a contradiction). But by assumption, K is relatively weak* compact in P^*. Consequently, M has a weak* limit point $\mu_0 \in \overline{K} \cap P^*$. By definition, $\operatorname{supp} \mu_0$ is compact and so $B \cap (\operatorname{supp} \mu_0)$ is finite. Replacing B by $B \setminus (\operatorname{supp} \mu_0)$, we may assume that $B \cap (\operatorname{supp} \mu_0) = \varnothing$. Thus, we may further assume that $(\bigcup_n V_n) \cap (\operatorname{supp} \mu_0) = \varnothing$ (simply take disjoint open sets V and W such that the closed set $B \subseteq V$ and the compact set $\operatorname{supp} \mu_0 \subseteq W$, and replace each V_n by $V \cap V_n$).

Since μ_n is regular, there is a compact set $\Omega_n \subseteq V_n \cap (\operatorname{supp} \mu_n)$ such that $\omega_n \in \Omega_n$ and $\mu_n(\Omega_n) = \epsilon_n > 0$. By Proposition 7.1, there exists a continuous function $f_n : \Omega \longrightarrow \left[0, \frac{1}{\epsilon_n}\right]$ satisfying $f_n|_{\Omega_n} \equiv \frac{1}{\epsilon_n}$ and $f_n|_{V_n^c} \equiv 0$.

The function $f = \sum_{n=1}^{\infty} f_n$ is well-defined because $f(\omega) \neq 0$ only when $\omega \in V_n$ for exactly one $n \in \mathbb{N}$. Clearly, $f \geq 0$. We claim that $f \in C(\Omega)$. To see this, let $\omega \in \Omega$. We separate the proof into two cases.

We first suppose that $f(\omega) > 0$. Then, for some m, we have $\omega \in V_m$ an $f|_{V_m} = f_m|_{V_m}$. Hence, f is continuous at ω.

We next suppose that $f(\omega) = 0$. If f is not continuous at ω, then there is an $\epsilon > 0$ and an infinite set $C = \{y_m\}$ such that $y_m \in B\left(\omega, \frac{1}{m}\right)$ and $f(y_m) \geq \epsilon$ for each m. But this implies that $C \subseteq \bigcup_{n=1}^{\infty} V_n$. Moreover, $V_n \cap C$ is finite for each n. In fact, if $V_r \cap C$ is infinite for some r, then we can find an infinite set $\{z_k\} \subseteq V_r \cap C$ satisfying $f(\omega) \geq f_r(\omega) = \lim_k f_r(z_k) \geq \epsilon > 0$, a contradiction. As a result, we can define inductively two strictly increasing sequences, $\{m_k\}$ and $\{n_k\}$, in $\mathbb{N}$ such that $y_{m_k} \in V_{n_k} \subseteq B\left(\omega_{n_k}, \frac{1}{n_k}\right)$. Hence, we have

$$d\left(\omega, \omega_{n_k}\right) \leq d\left(\omega, y_{m_k}\right) + d\left(y_{m_k}, \omega_{n_k}\right) < \frac{1}{m_k} + \frac{1}{n_k} \longrightarrow 0 \quad \text{as } k \longrightarrow \infty.$$

It follows that ω is a limit point of B, a contradiction. Consequently, f is continuous at ω.

To complete our proof, observe that for all $n \in \mathbb{N}$,

$$\int_{\Omega} f \, d\mu_n \geq \int_{\Omega_n} f_n \, d\mu_n = \frac{\mu_n(\Omega_n)}{\epsilon_n} = 1.$$

However, since $(\bigcup_n V_n) \cap (\operatorname{supp} \mu_0) = \varnothing$, we have $\int_{\Omega} f \, d\mu_0 = 0$, which contradicts to the fact that μ_0 is a weak* limit point of M. We conclude that K has a uniform compact support.□

(Case 2) We now consider the dual pair $\left\langle C^b(\Omega), rca(\mathscr{B})\right\rangle$. It is well known that the assertion holds if Ω is a Polish space (see Theorem 15.22, p. 519 in Aliprantis and Border (2007)). By an argument similar to the proof in Section 7.6.1, it suffices to show that if (Ω, d) is a complete metric space and K is a relatively weak* compact set in P^*, then there is a Polish subspace Ω_∞ of Ω such that $\mu(\Omega_\infty) = 1$ for every $\mu \in K$.

Observe that any subset of a uniformly tight set is also uniformly tight. Therefore, we may assume that K is weak* compact.

All elements in P^* are regular. Thus, for each $\mu \in K$ and each $n \in \mathbb{N}$, there exists a non-empty compact set Ω_n^μ such that $1 - \frac{1}{2n} < \mu(\Omega_n^\mu) \leq 1$. Define the closed set

$$F_n^\mu = \left\{\omega \in \Omega : d(\omega, \Omega_n^\mu) \geq \frac{1}{n}\right\}.$$

We have $\Omega_n^\mu \cap F_n^\mu = \varnothing$. If $F_n^\mu \neq \varnothing$, then by Proposition 7.1, there is a continuous function $f_n^\mu : \Omega \longrightarrow [0,1]$ such that $f_n^\mu|_{\Omega_n^\mu} \equiv 1$ and $f_n^\mu|_{F_n^\mu} \equiv 0$. If $F_n^\mu = \varnothing$, then we simply take $f_n^\mu \equiv 1$. For every $n \in \mathbb{N}$ and $\mu \in K$, define the following open set in K:

$$U_n^\mu = \left\{\nu \in K : 1 - \frac{1}{n} < \int_{\Omega} f_n^\mu \, d\nu \leq 1\right\}.$$

Obviously, $\mu \in U_n^\mu$ for every $\mu \in K$. In other words, $\{U_n^\mu : \mu \in K\}$ is an open cover of the weak* compact set K. Hence, there is a finite subset M_n of K such that $K = \bigcup_{\mu \in M_n} U_n^\mu$.

Let $f_n = \max\{f_n^\mu : \mu \in M_n\}$. Define

$$\Omega_n = \bigcup_{\mu \in M_n} \Omega_n^\mu \quad \text{and} \quad F_n = \left\{\omega \in \Omega : d\left(\omega, \Omega_n\right) \geq \frac{1}{n}\right\}.$$

Then, $f_n : \Omega \longrightarrow [0,1]$ is continuous. Clearly, $f_n|_{\Omega_n} \equiv 1$ and $f_n|_{F_n} \equiv 0$. Moreover, for each $\mu \in K$, we have:

$$1 - \frac{1}{n} < \int_\Omega f_n \, d\mu \leq 1.$$

It follows that for each $\mu \in K$, we have

$$\mu(F_n) = 1 - \int_{\Omega \setminus F_n} d\mu \leq 1 - \int_\Omega f_n \, d\mu < \frac{1}{n}. \tag{7.1}$$

Observe that every compact metric space is Polish. Then, $\Omega_\infty = \overline{\bigcup_{n=1}^\infty \Omega_n}$ is clearly a Polish subspace of Ω. We now complete our proof by showing that $\mu\left(\Omega_\infty\right) = 1$ for every $\mu \in K$. To see this, let $\mu \in K$ and Ω_0 be any compact subset of Ω such that $\Omega_0 \cap \Omega_\infty = \varnothing$. Choose $m \in \mathbb{N}$ so that $d\left(\Omega_0, \Omega_\infty\right) = \min_{\omega \in \Omega_0} d(\omega, \Omega_\infty) \geq \frac{1}{m}$. For any $n \geq m$, we have $d(\Omega_0, \Omega_n) \geq d(\Omega_0, \Omega_\infty) \geq \frac{1}{n}$, and so $\Omega_0 \subseteq F_n$. From (7.1), $\mu(\Omega_0) < \frac{1}{n}$ for all $n \geq m$. This implies that $\mu(\Omega_0) = 0$. By the regularity of μ, we must have $\mu(\Omega_\infty) = 1$. □

7.7. **Proof of Lemma 4.6.** Because of Lemma 4.5$(b)(i)$, we only need to prove the "only if" part, and in view of Lemma 4.5(a), we can directly assume that K is weak* compact.

(Case 1) Consider the dual pair $\langle C(\Omega), rca_c(\mathscr{B})\rangle$.

Let $f \in C_K$. Since K is weak* compact with a uniform compact support S_K, we may let $\epsilon = \min_{\bar\mu \in K} \int_\Omega f \; d\bar\mu = \min_{\bar\mu \in K} \int_{S_K} f \; d\bar\mu > 0$. For any $\mu \in K$ and $g \in C(\Omega)$ such that $\|g - f\|_{S_K} < \epsilon$, we have $\int_\Omega g \; d\mu = \int_{S_K} g \; d\mu \geq \int_{S_K} f \; d\mu - \int_{S_K} |g - f| d\mu > \epsilon - \epsilon = 0$. Thus, $g \in C_K$. As a result, C_K is open.

(Case 2) Consider the dual pair $\left\langle C^b(\Omega), rca(\mathscr{B})\right\rangle$.

Fix $f \in C_K$. Since K is weak* compact, we may let $2\epsilon = \min_{\bar\mu \in K} \int_\Omega f \; d\bar\mu > 0$. Since K is uniformly tight, there is an increasing sequence of compact sets, $\Omega_1 \subseteq \Omega_2 \subseteq \cdots$, such that for each $n \in \mathbb{N}$ and $\mu \in K$, we have $\mu\left(\Omega \setminus \Omega_n\right) > \frac{1}{n2^n}$. Note that for any Borel subset E of Ω and $\mu \in K$, we have $\mu(E) = \lim_{n \to \infty} \mu(E \cap \Omega_n)$.

Choose an $n \in \mathbb{N}$ so that $\epsilon > \frac{1}{n}$. Define a function $s \in M_0(\Omega)$ by

$$s(t) = \begin{cases} 1 & \text{if } t \in \Omega_n, \\ \frac{1}{m} & \text{if } t \in \Omega_{m+1} \setminus \Omega_m \text{ for } m \geq n, \\ 0 & \text{if } t \notin \bigcup_{m=1}^{\infty} \Omega_m. \end{cases}$$

Take any $g \in C^b(\Omega)$ such that $p_s(g - f) = \sup_{t \in \Omega} |(g - f)(t)s(t)| < \epsilon$. If $\mu \in K$, then

$$\begin{aligned} \int_\Omega g \, d\mu &\geq \int_\Omega f \, d\mu - \int_\Omega |g - f| \, d\mu \\ &\geq 2\epsilon - \int_{\Omega_n} |g - f| \, d\mu - \int_{\Omega \setminus \Omega_n} |g - f| \, d\mu \\ &> 2\epsilon - \epsilon - \sum_{m \geq n} \int_{\Omega_{m+1} \setminus \Omega_m} |g - f| \, d\mu \\ &> \epsilon - \epsilon \sum_{m \geq n} m\mu(\Omega_{m+1} \setminus \Omega_m) \quad \text{(since } p_s(g - f) < \epsilon) \\ &\geq \epsilon - \epsilon \sum_{m \geq n} m\mu(\Omega \setminus \Omega_m) \\ &> \epsilon - \epsilon \sum_{m \geq n} \frac{1}{2^m} \\ &\geq \epsilon - \epsilon = 0. \end{aligned}$$

Thus, $g \in C_K$. As a result, C_K is open. □

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