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SOC-monotone and SOC-convex functions vs. matrix-monotone and matrix-convex functions[☆]

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ABSTRACT

The SOC-monotone function (respectively, SOC-convex function) is a scalar valued function that induces a map to preserve the monotone order (respectively, the convex order), when imposed on the spectral factorization of vectors associated with second-order cones (SOCs) in general Hilbert spaces. In this paper, we provide the sufficient and necessary characterizations for the two classes of functions, and particularly establish that the set of continuous SOC-monotone (respectively, SOC-convex) functions coincides with that of continuous matrix monotone (respectively, matrix convex) functions of order 2.

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1. Introduction

Let \mathbb{H} be a real Hilbert space of dimension $\dim(\mathbb{H}) \geq 3$ endowed with an inner product $\langle \cdot, \cdot \rangle$ and its induced norm $\| \cdot \|$. Fix a unit vector $e \in \mathbb{H}$ and denote by $\langle e \rangle^\perp$ the orthogonal complementary

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space of e , i.e., $\langle e \rangle^\perp = \{x \in \mathbb{H} \mid \langle x, e \rangle = 0\}$. Then each x can be written as

$$x = x_e + x_0e \text{ for some } x_e \in \langle e \rangle^\perp \text{ and } x_0 \in \mathbb{R}.$$

The second-order cone (SOC) in \mathbb{H} , also called the Lorentz cone, is a set defined by

$$K := \left\{ x \in \mathbb{H} \mid \langle x, e \rangle \geq \frac{1}{\sqrt{2}} \|x\| \right\} = \{x_e + x_0e \in \mathbb{H} \mid x_0 \geq \|x_e\|\}.$$

From [7, Section 2], we know that K is a pointed closed convex self-dual cone. Hence, \mathbb{H} becomes a partially ordered space via the relation \succeq_K . In the sequel, for any $x, y \in \mathbb{H}$, we always write $x \succeq_K y$ (respectively, $x \succ_K y$) when $x - y \in K$ (respectively, $x - y \in \text{int}K$); and denote \bar{x}_e by the vector $\frac{x_e}{\|x_e\|}$ if $x_e \neq 0$, and otherwise by any unit vector from $\langle e \rangle^\perp$.

Associated with the second-order cone K , each $x = x_e + x_0e \in \mathbb{H}$ can be decomposed as

$$x = \lambda_1(x)u_1(x) + \lambda_2(x)u_2(x), \tag{1}$$

where $\lambda_i(x) \in \mathbb{R}$ and $u_i(x) \in \mathbb{H}$ for $i = 1, 2$ are the spectral values and the associated spectral vectors of x , defined by

$$\lambda_i(x) = x_0 + (-1)^i \|x_e\|, \quad u_i(x) = \frac{1}{2}(e + (-1)^i \bar{x}_e). \tag{2}$$

Clearly, when $x_e \neq 0$, the spectral factorization of x is unique by definition.

Let $f : J \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a scalar valued function, where J is an interval (finite or infinite, closed or open) in \mathbb{R} . Let S be the set of all $x \in \mathbb{H}$ whose spectral values $\lambda_1(x)$ and $\lambda_2(x)$ belong to J . Unless otherwise stated, in this paper S is always taken in this way. By the spectral factorization of x in (1) and (2), it is natural to define $f^{\text{soc}} : S \subseteq \mathbb{H} \rightarrow \mathbb{H}$ by

$$f^{\text{soc}}(x) := f(\lambda_1(x))u_1(x) + f(\lambda_2(x))u_2(x), \quad \forall x \in S. \tag{3}$$

It is easy to see that the function f^{soc} is well defined whether $x_e = 0$ or not. For example, by taking $f(t) = t^2$, we have that $f^{\text{soc}}(x) = x^2 = x \circ x$, where “ \circ ” means the Jordan product and the detailed definition is see in the next section. Note that

$$\begin{aligned} (\lambda_1(x) - \lambda_1(y))^2 + (\lambda_2(x) - \lambda_2(y))^2 &= 2(\|x\|^2 + \|y\|^2 - 2x_0y_0 - 2\|x_e\|\|y_e\|) \\ &\leq 2(\|x\|^2 + \|y\|^2 - 2\langle x, y \rangle) = 2\|x - y\|^2. \end{aligned}$$

We may verify that the domain S of f^{soc} is open in \mathbb{H} if and only if J is open in \mathbb{R} . Also, S is always convex since, for any $x = x_e + x_0e, y = y_e + y_0e \in S$ and $\beta \in [0, 1]$,

$$\begin{aligned} \lambda_1[\beta x + (1 - \beta)y] &= (\beta x_0 + (1 - \beta)y_0) - \|\beta x_e + (1 - \beta)y_e\| \geq \min\{\lambda_1(x), \lambda_1(y)\}, \\ \lambda_2[\beta x + (1 - \beta)y] &= (\beta x_0 + (1 - \beta)y_0) + \|\beta x_e + (1 - \beta)y_e\| \leq \max\{\lambda_2(x), \lambda_2(y)\}, \end{aligned}$$

which implies that $\beta x + (1 - \beta)y \in S$. Thus, $f^{\text{soc}}(\beta x + (1 - \beta)y)$ is well defined.

In this paper we are interested in two classes of special scalar valued functions that induce the maps via (3) to preserve the monotone order and the convex order, respectively.

Definition 1.1. A function $f : J \rightarrow \mathbb{R}$ is said to be *SOC-monotone* if for any $x, y \in S$,

$$x \succeq_K y \implies f^{\text{soc}}(x) \succeq_K f^{\text{soc}}(y); \tag{4}$$

and f is said to be *SOC-convex* if, for any $x, y \in S$ and any $\beta \in [0, 1]$,

$$f^{\text{soc}}(\beta x + (1 - \beta)y) \preceq_K \beta f^{\text{soc}}(x) + (1 - \beta)f^{\text{soc}}(y). \tag{5}$$

From Definition 1.1 and Eq. (3), it is easy to see that the set of SOC-monotone and SOC-convex functions are closed under positive linear combinations and pointwise limits.

The concept of SOC-monotone (respectively, SOC-convex) functions above is a direct extension of those given by [5,6] to general Hilbert spaces, and is analogous to that of matrix monotone (respectively, matrix convex) functions and more general operator monotone (respectively, operator convex) functions; see, e.g., [17, 15, 14, 2, 11, 23]. Just as the importance of matrix monotone (respectively, matrix convex) functions to the solution of convex semidefinite programming [19, 4], SOC-monotone (respectively, SOC-convex) functions also play a crucial role in the design and analysis of algorithms for convex second-order cone programming [3, 22]. For matrix monotone and matrix convex functions, after the seminal work of Löwner [17] and Kraus [15], there have been systematic studies and perfect characterizations for them; see [8, 16, 4, 13, 12, 21, 20] and the references therein. However, the study on SOC-monotone and SOC-convex functions just begins with [5], and the characterizations for them are still imperfect. Particularly, it is not clear what is the relation between the SOC-monotone (respectively, SOC-convex) functions and the matrix monotone (respectively, matrix convex) functions.

In this work, we provide the sufficient and necessary characterizations for SOC-monotone and SOC-convex functions in the setting of Hilbert spaces, and show that the set of continuous SOC-monotone (SOC-convex) functions coincides with that of continuous matrix monotone (matrix convex) functions of order 2. Some of these results generalize those of [5, 6] (see Propositions 3.2 and 4.2), and some are new, which are difficult to achieve by using the techniques of [5, 6] (see, for example, Proposition 4.4). In addition, we also discuss the relations between SOC-monotone functions and SOC-convex functions, verify Conjecture 4.2 in [5] under a little stronger condition (see Proposition 6.2), and present a counterexample to show that Conjecture 4.1 in [5] generally does not hold. It is worthwhile to point out that the analysis in this paper depends only on the inner product of Hilbert spaces, whereas most of the results in [5, 6] are obtained with the help of matrix operations.

Throughout this paper, all differentiability means Fréchet differentiability. If $F: \mathbb{H} \rightarrow \mathbb{H}$ is (twice) differentiable at $x \in \mathbb{H}$, we denote by $F'(x)$ ($F''(x)$) the first-order F-derivative (the second-order F-derivative) of F at x . In addition, we use $C^n(J)$ and $C^\infty(J)$ to denote the set of n times and infinite times continuously differentiable real functions on J , respectively. When $f \in C^1(J)$, we denote by $f^{[1]}$ the function on $J \times J$ defined by

$$f^{[1]}(\lambda, \mu) := \begin{cases} \frac{f(\lambda)-f(\mu)}{\lambda-\mu} & \text{if } \lambda \neq \mu, \\ f'(\lambda) & \text{if } \lambda = \mu; \end{cases}$$

and when $f \in C^2(J)$, denote by $f^{[2]}$ the function on $J \times J \times J$ defined by

$$f^{[2]}(\tau_1, \tau_2, \tau_3) := \frac{f^{[1]}(\tau_1, \tau_2) - f^{[1]}(\tau_1, \tau_3)}{\tau_2 - \tau_3}$$

if τ_1, τ_2, τ_3 are distinct, and for other values of $\tau_1, \tau_2, \tau_3, f^{[2]}$ is defined by continuity; e.g.,

$$f^{[2]}(\tau_1, \tau_1, \tau_3) = \frac{f(\tau_3) - f(\tau_1) - f'(\tau_1)(\tau_3 - \tau_1)}{(\tau_3 - \tau_1)^2}, \quad f^{[2]}(\tau_1, \tau_1, \tau_1) = \frac{1}{2}f''(\tau_1).$$

For a linear operator \mathcal{L} from \mathbb{H} into \mathbb{H} , we write $\mathcal{L} \geq 0$ (respectively, $\mathcal{L} > 0$) to mean that \mathcal{L} is positive semidefinite (respectively, positive definite), i.e., $\langle h, \mathcal{L}h \rangle \geq 0$ for any $h \in \mathbb{H}$ (respectively, $\langle h, \mathcal{L}h \rangle > 0$ for any $0 \neq h \in \mathbb{H}$).

2. Preliminaries

This section recalls some background material and gives several lemmas that will be used in the subsequent sections. We start with the definition of Jordan product [9]. For any $x = x_e + x_0e, y = y_e + y_0e \in \mathbb{H}$, the Jordan product of x and y is defined as

$$x \circ y := (x_0y_e + y_0x_e) + \langle x, y \rangle e.$$

A simple computation can verify that for any $x, y, z \in \mathbb{H}$ and the unit vector e , (i) $e \circ e = e$ and $e \circ x = x$; (ii) $x \circ y = y \circ x$; (iii) $x \circ (x^2 \circ y) = x^2 \circ (x \circ y)$, where $x^2 = x \circ x$; (iv) $(x + y) \circ z = x \circ z + y \circ z$. For any $x \in \mathbb{H}$, define its determinant by

$$\det(x) := \lambda_1(x)\lambda_2(x) = x_0^2 - \|x_e\|^2.$$

Then each $x = x_e + x_0e$ with $\det(x) \neq 0$ is invertible with respect to the Jordan product, i.e., there is a unique $x^{-1} = (-x_e + x_0e)/\det(x)$ such that $x \circ x^{-1} = e$.

We next give several lemmas where Lemma 2.1 is used in Section 3 to characterize SOC-monotonicity, and Lemmas 2.2 and 2.3 are used in Section 4 to characterize SOC-convexity.

Lemma 2.1. Let $\mathbb{B} := \{z \in \langle e \rangle^\perp \mid \|z\| \leq 1\}$. Then, for any given $u \in \langle e \rangle^\perp$ with $\|u\| = 1$ and $\theta, \lambda \in \mathbb{R}$, the following results hold.

- (a) $\theta + \lambda \langle u, z \rangle \geq 0$ for any $z \in \mathbb{B}$ if and only if $\theta \geq |\lambda|$.
- (b) $\theta - \|\lambda z\|^2 \geq (\theta - \lambda^2) \langle u, z \rangle^2$ for any $z \in \mathbb{B}$ if and only if $\theta - \lambda^2 \geq 0$.

Proof. (a) Suppose that $\theta + \lambda \langle u, z \rangle \geq 0$ for any $z \in \mathbb{B}$. If $\lambda = 0$, then $\theta \geq |\lambda|$ clearly holds. If $\lambda \neq 0$, take $z = -\text{sign}(\lambda)u$. Since $\|u\| = 1$, we have $z \in \mathbb{B}$, and consequently, $\theta + \lambda \langle u, z \rangle \geq 0$ reduces to $\theta - |\lambda| \geq 0$. Conversely, if $\theta \geq |\lambda|$, then using the Cauchy–Schwartz inequality yields $\theta + \lambda \langle u, z \rangle \geq 0$ for any $z \in \mathbb{B}$.

(b) Suppose that $\theta - \|\lambda z\|^2 \geq (\theta - \lambda^2) \langle u, z \rangle^2$ for any $z \in \mathbb{B}$. Then we must have $\theta - \lambda^2 \geq 0$. If not, for those $z \in \mathbb{B}$ with $\|z\| = 1$ but $\langle u, z \rangle \neq \|u\| \|z\|$, it holds that

$$(\theta - \lambda^2) \langle u, z \rangle^2 > (\theta - \lambda^2) \|u\|^2 \|z\|^2 = \theta - \|\lambda z\|^2,$$

which contradicts the given assumption. Conversely, if $\theta - \lambda^2 \geq 0$, the Cauchy–Schwartz inequality implies that $(\theta - \lambda^2) \langle u, z \rangle^2 \leq \theta - \|\lambda z\|^2$ for any $z \in \mathbb{B}$. \square

Lemma 2.2. For any given $a, b, c \in \mathbb{R}$ and $x = x_e + x_0e$ with $x_e \neq 0$, the inequality

$$a \left[\|h_e\|^2 - \langle h_e, \bar{x}_e \rangle^2 \right] + b [h_0 + \langle \bar{x}_e, h_e \rangle]^2 + c [h_0 - \langle \bar{x}_e, h_e \rangle]^2 \geq 0 \tag{6}$$

holds for all $h = h_e + h_0e \in \mathbb{H}$ if and only if $a \geq 0$, $b \geq 0$ and $c \geq 0$.

Proof. Suppose that (6) holds for all $h = h_e + h_0e \in \mathbb{H}$. By letting $h_e = \bar{x}_e$, $h_0 = 1$ and $h_e = -\bar{x}_e$, $h_0 = 1$, respectively, we get $b \geq 0$ and $c \geq 0$ from (6). If $a \geq 0$ does not hold, then by taking $h_e = \sqrt{\frac{b+c+1}{|a|}} \frac{z_e}{\|z_e\|}$ with $\langle z_e, x_e \rangle = 0$ and $h_0 = 1$, (6) gives a contradiction $-1 \geq 0$. Conversely, if $a \geq 0$, $b \geq 0$ and $c \geq 0$, then (6) clearly holds for all $h \in \mathbb{H}$. \square

Lemma 2.3. Let $f \in C^2(J)$ and $u_e \in \langle e \rangle^\perp$ with $\|u_e\| = 1$. For any $h = h_e + h_0e \in \mathbb{H}$, define

$$\mu_1(h) := \frac{h_0 - \langle u_e, h_e \rangle}{\sqrt{2}}, \quad \mu_2(h) := \frac{h_0 + \langle u_e, h_e \rangle}{\sqrt{2}}, \quad \mu(h) := \sqrt{\|h_e\|^2 - \langle u_e, h_e \rangle^2}.$$

Then, for any given $a, d \in \mathbb{R}$ and $\lambda_1, \lambda_2 \in J$, the following inequality

$$\begin{aligned} & 4f''(\lambda_1)f''(\lambda_2)\mu_1(h)^2\mu_2(h)^2 + 2(a-d)f''(\lambda_2)\mu_2(h)^2\mu(h)^2 \\ & + 2(a+d)f''(\lambda_1)\mu_1(h)^2\mu(h)^2 + (a^2 - d^2)\mu(h)^4 \\ & - 2[(a-d)\mu_1(h) + (a+d)\mu_2(h)]^2\mu(h)^2 \geq 0 \end{aligned} \tag{7}$$

holds for all $h = h_e + h_0e \in \mathbb{H}$ if and only if

$$a^2 - d^2 \geq 0, \quad f''(\lambda_2)(a-d) \geq (a+d)^2 \text{ and } f''(\lambda_1)(a+d) \geq (a-d)^2. \tag{8}$$

Proof. Suppose that (7) holds for all $h = h_e + h_0e \in \mathbb{H}$. Taking $h_0 = 0$ and $h_e \neq 0$ with $\langle h_e, u_e \rangle = 0$, we have $\mu_1(h) = 0, \mu_2(h) = 0$ and $\mu(h) = \|h_e\| > 0$, and then (7) gives $a^2 - d^2 \geq 0$. Taking $h_e \neq 0$ such that $|\langle u_e, h_e \rangle| < \|h_e\|$ and $h_0 = \langle u_e, h_e \rangle \neq 0$, we have $\mu_1(h) = 0, \mu_2(h) = \sqrt{2}h_0$ and $\mu(h) > 0$, and then (7) reduces to the following inequality

$$4 \left[(a - d)f''(\lambda_2) - (a + d)^2 \right] h_0^2 + (a^2 - d^2)(\|h_e\|^2 - h_0^2) \geq 0.$$

This implies that $(a - d)f''(\lambda_2) - (a + d)^2 \geq 0$. If not, by letting h_0 be sufficiently close to $\|h_e\|$, the last inequality yields a contradiction. Similarly, taking h with $h_e \neq 0$ satisfying $|\langle u_e, h_e \rangle| < \|h_e\|$ and $h_0 = -\langle u_e, h_e \rangle$, we get $f''(\lambda_1)(a + d) \geq (a - d)^2$ from (7).

Next, suppose that (8) holds. Then, the inequalities $f''(\lambda_2)(a - d) \geq (a + d)^2$ and $f''(\lambda_1)(a + d) \geq (a - d)^2$ imply that the left-hand side of (7) is greater than

$$4f''(\lambda_1)f''(\lambda_2)\mu_1(h)^2\mu_2(h)^2 - 4(a^2 - d^2)\mu_1(h)\mu_2(h)\mu(h)^2 + (a^2 - d^2)\mu(h)^4,$$

which is obviously nonnegative if $\mu_1(h)\mu_2(h) \leq 0$. Now assume that $\mu_1(h)\mu_2(h) > 0$. If $a^2 - d^2 = 0$, then the last expression is clearly nonnegative, and if $a^2 - d^2 > 0$, then the last two inequalities in (8) imply that $f''(\lambda_1)f''(\lambda_2) \geq (a^2 - d^2) > 0$, and therefore,

$$\begin{aligned} & 4f''(\lambda_1)f''(\lambda_2)\mu_1(h)^2\mu_2(h)^2 - 4(a^2 - d^2)\mu_1(h)\mu_2(h)\mu(h)^2 + (a^2 - d^2)\mu(h)^4 \\ & \geq 4(a^2 - d^2)\mu_1(h)^2\mu_2(h)^2 - 4(a^2 - d^2)\mu_1(h)\mu_2(h)\mu(h)^2 + (a^2 - d^2)\mu(h)^4 \\ & = (a^2 - d^2) \left[2\mu_1(h)\mu_2(h) - \mu(h)^2 \right]^2 \geq 0. \end{aligned}$$

Thus, we prove that inequality (7) holds. The proof is complete. \square

To close this section, we introduce the regularization of a locally integrable real function. Let φ be a real function of class C^∞ with the following properties: $\varphi \geq 0, \varphi$ is even, the support $\text{supp } \varphi = [-1, 1]$, and $\int_{\mathbb{R}} \varphi = 1$. For each $\varepsilon > 0$, let $\varphi_\varepsilon(t) = \frac{1}{\varepsilon}\varphi(\frac{t}{\varepsilon})$. Then $\text{supp } \varphi_\varepsilon = [-\varepsilon, \varepsilon]$ and φ_ε has all the properties of φ listed above. If f is a locally integrable real function, we define its regularization of order ε as the function

$$f_\varepsilon(s) := \int f(s - t)\varphi_\varepsilon(t)dt = \int f(s - \varepsilon t)\varphi(t)dt. \tag{9}$$

Note that f_ε is a C^∞ function for each $\varepsilon > 0$, and $\lim_{\varepsilon \rightarrow 0} f_\varepsilon(x) = f(x)$ if f is continuous.

3. Characterizations of SOC-monotone functions

In this section we present some characterizations for SOC-monotone functions, by which the set of continuous SOC-monotone functions is shown to coincide with that of continuous matrix monotone functions of order 2. To this end, we need the following technical lemma.

Lemma 3.1. For any given $f : J \rightarrow \mathbb{R}$ with J open, let $f^{\text{SOC}} : S \rightarrow \mathbb{H}$ be defined by (3).

- (a) f^{SOC} is continuous on S if and only if f is continuous on J .
- (b) f^{SOC} is (continuously) differentiable on S iff f is (continuously) differentiable on J . Also, when f is differentiable on J , for any $x = x_e + x_0e \in S$ and $v = v_e + v_0e \in \mathbb{H}$,

$$(f^{\text{SOC}})'(x)v = \begin{cases} f'(x_0)v & \text{if } x_e = 0; \\ (b_1(x) - a_0(x))\langle \bar{x}_e, v_e \rangle \bar{x}_e + c_1(x)v_0\bar{x}_e \\ \quad + a_0(x)v_e + b_1(x)v_0e + c_1(x)\langle \bar{x}_e, v_e \rangle e & \text{if } x_e \neq 0, \end{cases} \tag{10}$$

where $a_0(x) = \frac{f(\lambda_2(x)) - f(\lambda_1(x))}{\lambda_2(x) - \lambda_1(x)}, b_1(x) = \frac{f'(\lambda_2(x)) + f'(\lambda_1(x))}{2}, c_1(x) = \frac{f'(\lambda_2(x)) - f'(\lambda_1(x))}{2}$.

(c) If f is differentiable on J , then for any given $x \in S$ and all $v \in \mathbb{H}$,

$$(f^{\text{SOC}})'(x)e = (f')^{\text{SOC}}(x) \text{ and } \langle e, (f^{\text{SOC}})'(x)v \rangle = \langle v, (f')^{\text{SOC}}(x) \rangle.$$

(d) If f' is nonnegative (respectively, positive) on J , then for each $x \in S$,

$$(f^{\text{SOC}})'(x) \geq 0 \text{ (respectively, } (f^{\text{SOC}})'(x) > 0 \text{)}.$$

Proof. (a) Suppose that f^{SOC} is continuous. Let Ω be the set composed of those $x = te$ with $t \in J$. Clearly, $\Omega \subseteq S$, and f^{SOC} is continuous on Ω . Noting that $f^{\text{SOC}}(x) = f(t)e$ for any $x \in \Omega$, it follows that f is continuous on J . Conversely, if f is continuous on J , then f^{SOC} is continuous at any $x = x_e + x_0e \in S$ with $x_e \neq 0$ since $\lambda_i(x)$ and $u_i(x)$ for $i = 1, 2$ are continuous at such points. Next, let $x = x_e + x_0e$ be an arbitrary element from S with $x_e = 0$, and we prove that f^{SOC} is continuous at x . Indeed, for any $z = z_e + z_0e \in S$ sufficiently close to x , it is not hard to verify that

$$\|f^{\text{SOC}}(z) - f^{\text{SOC}}(x)\| \leq \frac{|f(\lambda_2(z)) - f(x_0)|}{2} + \frac{|f(\lambda_1(z)) - f(x_0)|}{2} + \frac{|f(\lambda_2(z)) - f(\lambda_1(z))|}{2}.$$

Since f is continuous on J , and $\lambda_1(z), \lambda_2(z) \rightarrow x_0$ as $z \rightarrow x$, it follows that

$$f(\lambda_1(z)) \rightarrow f(x_0) \text{ and } f(\lambda_2(z)) \rightarrow f(x_0) \text{ as } z \rightarrow x.$$

The last two equations imply that f^{SOC} is continuous at x .

(b) When f^{SOC} is (continuously) differentiable, using the similar arguments as in part (a) can show that f is (continuously) differentiable. Next assume that f is differentiable. Fix any $x = x_e + x_0e \in S$. We first consider the case where $x_e \neq 0$. Since $\lambda_i(x)$ for $i = 1, 2$ and $\frac{x_e}{\|x_e\|}$ are continuously differentiable at such x , it follows that $f(\lambda_i(x))$ and $u_i(x)$ are differentiable and continuously differentiable, respectively, at x . Then f^{SOC} is differentiable at such x by the definition of f^{SOC} . Also, an elementary computation shows that

$$[\lambda_i(x)]'v = \langle v, e \rangle + (-1)^i \frac{\langle x_e, v - \langle v, e \rangle e \rangle}{\|x_e\|} = v_0 + (-1)^i \frac{\langle x_e, v_e \rangle}{\|x_e\|}, \tag{11}$$

$$\left(\frac{x_e}{\|x_e\|}\right)'v = \frac{v - \langle v, e \rangle e}{\|x_e\|} - \frac{\langle x_e, v - \langle v, e \rangle e \rangle x_e}{\|x_e\|^3} = \frac{v_e}{\|x_e\|} - \frac{\langle x_e, v_e \rangle x_e}{\|x_e\|^3} \tag{12}$$

for any $v = v_e + v_0e \in \mathbb{H}$, and consequently,

$$[f(\lambda_i(x))]'v = f'(\lambda_i(x)) \left[v_0 + (-1)^i \frac{\langle x_e, v_e \rangle}{\|x_e\|} \right],$$

$$[u_i(x)]'v = \frac{1}{2}(-1)^i \left[\frac{v_e}{\|x_e\|} - \frac{\langle x_e, v_e \rangle x_e}{\|x_e\|^3} \right].$$

Together with the definition of f^{SOC} , we calculate that $(f^{\text{SOC}})'(x)v$ is equal to

$$\begin{aligned} & \frac{f'(\lambda_1(x))}{2} \left[v_0 - \frac{\langle x_e, v_e \rangle}{\|x_e\|} \right] \left(e - \frac{x_e}{\|x_e\|} \right) - \frac{f(\lambda_1(x))}{2} \left[\frac{v_e}{\|x_e\|} - \frac{\langle x_e, v_e \rangle x_e}{\|x_e\|^3} \right] \\ & + \frac{f'(\lambda_2(x))}{2} \left[v_0 + \frac{\langle x_e, v_e \rangle}{\|x_e\|} \right] \left(e + \frac{x_e}{\|x_e\|} \right) + \frac{f(\lambda_2(x))}{2} \left[\frac{v_e}{\|x_e\|} - \frac{\langle x_e, v_e \rangle x_e}{\|x_e\|^3} \right] \\ & = b_1(x)v_0e + c_1(x) \langle \bar{x}_e, v_e \rangle e + c_1(x)v_0\bar{x}_e + b_1(x) \langle \bar{x}_e, v_e \rangle \bar{x}_e \\ & \quad + a_0(x)v_e - a_0(x) \langle \bar{x}_e, v_e \rangle \bar{x}_e, \end{aligned}$$

where $\lambda_2(x) - \lambda_1(x) = 2\|x_e\|$ is used for the last equality. Thus, we get (10) for $x_e \neq 0$. We next consider the case where $x_e = 0$. Under this case, for any $v = v_e + v_0e \in \mathbb{H}$,

$$\begin{aligned}
 f^{\text{SOC}}(x + v) - f^{\text{SOC}}(x) &= \frac{f(x_0 + v_0 - \|v_e\|)}{2} (e - \bar{v}_e) + \frac{f(x_0 + v_0 + \|v_e\|)}{2} (e + \bar{v}_e) - f(x_0)e \\
 &= \frac{f'(x_0)(v_0 - \|v_e\|)}{2} e + \frac{f'(x_0)(v_0 + \|v_e\|)}{2} e \\
 &\quad + \frac{f'(x_0)(v_0 + \|v_e\|)}{2} \bar{v}_e - \frac{f'(x_0)(v_0 - \|v_e\|)}{2} \bar{v}_e + o(\|v\|) \\
 &= f'(x_0)(v_0 e + \|v_e\| \bar{v}_e) + o(\|v\|),
 \end{aligned}$$

where $\bar{v}_e = \frac{v_e}{\|v_e\|}$ if $v_e \neq 0$, and otherwise \bar{v}_e is an arbitrary unit vector from $\langle e \rangle^\perp$. Hence,

$$\|f^{\text{SOC}}(x + v) - f^{\text{SOC}}(x) - f'(x_0)v\| = o(\|v\|).$$

This shows that f^{SOC} is differentiable at such x with $(f^{\text{SOC}})'(x)v = f'(x_0)v$.

Assume that f is continuously differentiable. From (10), it is easy to see that $(f^{\text{SOC}})'(x)$ is continuous at every x with $x_e \neq 0$. We next argue that $(f^{\text{SOC}})'(x)$ is continuous at every x with $x_e = 0$. Fix any $x = x_0e$ with $x_0 \in J$. For any $z = z_e + z_0e$ with $z_e \neq 0$, we have

$$\begin{aligned}
 \|(f^{\text{SOC}})'(z)v - (f^{\text{SOC}})'(x)v\| &\leq |b_1(z) - a_0(z)|\|v_e\| + |b_1(z) - f'(x_0)|\|v_0\| \\
 &\quad + |a_0(z) - f'(x_0)|\|v_e\| + |c_1(z)|(\|v_0\| + \|v_e\|).
 \end{aligned} \tag{13}$$

Since f is continuously differentiable on J and $\lambda_2(z) \rightarrow x_0$, $\lambda_1(z) \rightarrow x_0$ as $z \rightarrow x$, we have

$$a_0(z) \rightarrow f'(x_0), \quad b_1(z) \rightarrow f'(x_0) \text{ and } c_1(z) \rightarrow 0.$$

Together with Eq. (13), we obtain that $(f^{\text{SOC}})'(z) \rightarrow (f^{\text{SOC}})'(x)$ as $z \rightarrow x$.

(c) The result is direct by the definition of $(f')^{\text{SOC}}$ and a simple computation from (10).

(d) Suppose that $f'(t) \geq 0$ for all $t \in J$. Fix any $x = x_e + x_0e \in S$. If $x_e = 0$, the result is direct. It remains to consider the case $x_e \neq 0$. Since $f'(t) \geq 0$ for all $t \in J$, we have $b_1(x) \geq 0$, $b_1(x) - c_1(x) = f'(\lambda_1(x)) \geq 0$, $b_1(x) + c_1(x) = f'(\lambda_2(x)) \geq 0$ and $a_0(x) \geq 0$. From part (b) and the definitions of $b_1(x)$ and $c_1(x)$, it follows that for any $h = h_e + h_0e \in \mathbb{H}$,

$$\begin{aligned}
 \langle h, (f^{\text{SOC}})'(x)h \rangle &= (b_1(x) - a_0(x))\langle \bar{x}_e, h_e \rangle^2 + 2c_1(x)h_0\langle \bar{x}_e, h_e \rangle + b_1(x)h_0^2 + a_0(x)\|h_e\|^2 \\
 &= a_0(x) \left[\|h_e\|^2 - \langle \bar{x}_e, h_e \rangle^2 \right] + \frac{1}{2} (b_1(x) - c_1(x)) [h_0 - \langle \bar{x}_e, h_e \rangle]^2 \\
 &\quad + \frac{1}{2} (b_1(x) + c_1(x)) [h_0 + \langle \bar{x}_e, h_e \rangle]^2 \geq 0.
 \end{aligned}$$

This implies that the operator $(f^{\text{SOC}})'(x)$ is positive semidefinite. Particularly, if $f'(t) > 0$ for all $t \in J$, we have that $\langle h, (f^{\text{SOC}})'(x)h \rangle > 0$ for all $h \neq 0$. The proof is complete. \square

Lemma 3.1(d) shows that the differential operator $(f^{\text{SOC}})'(x)$ corresponding to a differentiable non-decreasing f is positive semidefinite. So, the differential operator $(f^{\text{SOC}})'(x)$ associated with a differentiable SOC-monotone function is also positive semidefinite.

Proposition 3.1. Assume that $f \in C^1(J)$ with J open. Then f is SOC-monotone if and only if $(f^{\text{SOC}})'(x)h \in K$ for any $x \in S$ and $h \in K$.

Proof. If f is SOC-monotone, then for any $x \in S$, $h \in K$ and $t > 0$, we have

$$f^{\text{SOC}}(x + th) - f^{\text{SOC}}(x) \succeq_K 0,$$

which, by the continuous differentiability of f^{SOC} and the closedness of K , implies that

$$(f^{\text{SOC}})'(x)h \succeq_K 0.$$

Conversely, for any $x, y \in S$ with $x \succeq_K y$, from the given assumption we have that

$$f^{\text{SOC}}(x) - f^{\text{SOC}}(y) = \int_0^1 (f^{\text{SOC}})'(x + t(x - y))(x - y)dt \in K.$$

This shows that $f^{\text{SOC}}(x) \succeq_K f^{\text{SOC}}(y)$, i.e., f is SOC-monotone. The proof is complete. \square

Proposition 3.1 shows that the differential operator $(f^{\text{SOC}})'(x)$ associated with a differentiable SOC-monotone function f leaves K invariant. If, in addition, the linear operator $(f^{\text{SOC}})'(x)$ is bijective, then $(f^{\text{SOC}})'(x)$ belongs to the automorphism group of K . Such linear operators are important to study the structure of the cone K (see [9]).

Corollary 3.1. Assume that $f \in C^1(J)$ with J open. If f is SOC-monotone, then

- (a) $(f')^{\text{SOC}}(x) \in K$ for any $x \in S$;
- (b) f^{SOC} is a monotone function, i.e., $\langle f^{\text{SOC}}(x) - f^{\text{SOC}}(y), x - y \rangle \geq 0$ for any $x, y \in S$.

Proof. Part (a) is direct by using Proposition 3.1 with $h = e$ and Lemma 3.1(c). By part (a), $f'(\tau) \geq 0$ for all $\tau \in J$. Together with Lemma 3.1(d), $(f^{\text{SOC}})'(x) \geq 0$ for any $x \in S$. Applying the integral mean-value theorem, it then follows that

$$\langle f^{\text{SOC}}(x) - f^{\text{SOC}}(y), x - y \rangle = \int_0^1 \langle x - y, (f^{\text{SOC}})'(y + t(x - y))(x - y) \rangle dt \geq 0.$$

This proves the desired result of part (b). The proof is complete. \square

Note that the converse of Corollary 3.1(a) is not correct. For example, for the function $f(t) = -t^{-2}$ ($t > 0$), it is clear that $(f')^{\text{SOC}}(x) \in K$ for any $x \in \text{int}K$, but it is not SOC-monotone by Example 5.1(ii). The following proposition provides another sufficient and necessary characterization for differentiable SOC-monotone functions.

Proposition 3.2. Let $f \in C^1(J)$ with J open. Then f is SOC-monotone if and only if

$$\begin{bmatrix} f^{[1]}(\tau_1, \tau_1) & f^{[1]}(\tau_1, \tau_2) \\ f^{[1]}(\tau_2, \tau_1) & f^{[1]}(\tau_2, \tau_2) \end{bmatrix} = \begin{bmatrix} f'(\tau_1) & \frac{f(\tau_2) - f(\tau_1)}{\tau_2 - \tau_1} \\ \frac{f(\tau_1) - f(\tau_2)}{\tau_1 - \tau_2} & f'(\tau_2) \end{bmatrix} \geq 0, \quad \forall \tau_1, \tau_2 \in J. \quad (14)$$

Proof. The equality is direct by the definition of $f^{[1]}$. It suffices to prove that f is SOC-monotone if and only if the inequality in (14) holds for any $\tau_1, \tau_2 \in J$. Assume that f is SOC-monotone. By Proposition 3.1, $(f^{\text{SOC}})'(x)h \in K$ for any $x \in S$ and $h \in K$. Fix any $x = x_e + x_0e \in S$. It suffices to consider the case where $x_e \neq 0$. Since $(f^{\text{SOC}})'(x)h \in K$ for any $h \in K$, we particularly have $(f^{\text{SOC}})'(x)(z + e) \in K$ for any $z \in \mathbb{B}$, where \mathbb{B} is the set defined in Lemma 2.1. From Lemma 3.1(b), it follows that

$$(f^{\text{SOC}})'(x)(z + e) = [(b_1(x) - a_0(x)) \langle \bar{x}_e, z \rangle + c_1(x)]\bar{x}_e + a_0(x)z + [b_1(x) + c_1(x)\langle \bar{x}_e, z \rangle]e.$$

This means that $(f^{\text{SOC}})'(x)(z + e) \in K$ for any $z \in \mathbb{B}$ if and only if

$$b_1(x) + c_1(x)\langle \bar{x}_e, z \rangle \geq 0, \quad (15)$$

$$[b_1(x) + c_1(x)\langle \bar{x}_e, z \rangle]^2 \geq \|[(b_1(x) - a_0(x)) \langle \bar{x}_e, z \rangle + c_1(x)]\bar{x}_e + a_0(x)z\|^2. \quad (16)$$

By Lemma 2.1(a), we know that (15) holds for any $z \in \mathbb{B}$ if and only if $b_1(x) \geq |c_1(x)|$. Since by a simple computation the inequality in (16) can be simplified as

$$b_1(x)^2 - c_1(x)^2 - a_0(x)^2 \|z\|^2 \geq [b_1(x)^2 - c_1(x)^2 - a_0(x)^2] \langle z, \bar{x}_e \rangle^2,$$

applying Lemma 2.1(b) yields that (16) holds for any $z \in \mathbb{B}$ if and only if

$$b_1(x)^2 - c_1(x)^2 - a_0(x)^2 \geq 0.$$

This shows that $(f^{\text{soc}})'(x)(z + e) \in K$ for any $z \in \mathbb{B}$ if and only if

$$b_1(x) \geq |c_1(x)| \text{ and } b_1(x)^2 - c_1(x)^2 - a_0(x)^2 \geq 0. \tag{17}$$

The first condition in (17) is equivalent to $b_1(x) \geq 0, b_1(x) - c_1(x) \geq 0$ and $b_1(x) + c_1(x) \geq 0$, which, by the expressions of $b_1(x)$ and $c_1(x)$ and the arbitrariness of x , is equivalent to $f'(\tau) \geq 0$ for all $\tau \in J$; whereas the second condition in (17) is equivalent to

$$f'(\tau_1)f'(\tau_2) - \left[\frac{f(\tau_2) - f(\tau_1)}{\tau_2 - \tau_1} \right]^2 \geq 0, \quad \forall \tau_1, \tau_2 \in J.$$

The two sides show that the inequality in (14) holds for all $\tau_1, \tau_2 \in J$.

Conversely, if the inequality in (14) holds for all $\tau_1, \tau_2 \in J$, then from the arguments above we have $(f^{\text{soc}})'(x)(z + e) \in K$ for any $x = x_e + x_0e \in S$ and $z \in \mathbb{B}$. This implies that $(f^{\text{soc}})'(x)h \in K$ for any $x \in S$ and $h \in K$. By Proposition 3.1, f is SOC-monotone. \square

Propositions 3.1 and 3.2 provide the characterizations for continuously differentiable SOC-monotone functions. When f does not belong to $C^1(J)$, one may check the SOC-monotonicity of f by combining the following proposition with Propositions 3.1 and 3.2.

Proposition 3.3. *Let $f : J \rightarrow \mathbb{R}$ be a continuous function on the open interval J , and f_ε be its regularization defined by (9). Then, f is SOC-monotone if and only if f_ε is SOC-monotone on J_ε for every sufficiently small $\varepsilon > 0$, where $J_\varepsilon := (a + \varepsilon, b - \varepsilon)$ for $J = (a, b)$.*

Proof. Throughout the proof, for every sufficiently small $\varepsilon > 0$, we let S_ε be the set of all $x \in \mathbb{H}$ whose spectral values $\lambda_1(x), \lambda_2(x)$ belong to J_ε . Assume that f_ε is SOC-monotone on J_ε for every sufficiently small $\varepsilon > 0$. Let x, y be arbitrary vectors from S with $x \succeq_K y$. Then, for any sufficiently small $\varepsilon > 0$, we have $x + \varepsilon e, y + \varepsilon e \in S_\varepsilon$ and $x + \varepsilon e \succeq_K y + \varepsilon e$. Using the SOC-monotonicity of f_ε on J_ε yields that $f_\varepsilon^{\text{soc}}(x + \varepsilon e) \succeq_K f_\varepsilon^{\text{soc}}(y + \varepsilon e)$. Taking the limit $\varepsilon \rightarrow 0$ and using the convergence of $f_\varepsilon^{\text{soc}}(x) \rightarrow f^{\text{soc}}(x)$ and the continuity of f^{soc} on S implied by Lemma 3.1(a), we readily obtain that $f^{\text{soc}}(x) \succeq_K f^{\text{soc}}(y)$. This shows that f is SOC-monotone.

Now assume that f is SOC-monotone. Let $\varepsilon > 0$ be an arbitrary sufficiently small real number. Fix any $x, y \in S_\varepsilon$ with $x \succeq_K y$. Then, for all $t \in [-1, 1]$, we have $x - t\varepsilon e, y - t\varepsilon e \in S$ and $x - t\varepsilon e \succeq_K y - t\varepsilon e$. Therefore, $f^{\text{soc}}(x - t\varepsilon e) \succeq_K f^{\text{soc}}(y - t\varepsilon e)$, which is equivalent to

$$\begin{aligned} & \frac{f(\lambda_1 - t\varepsilon) + f(\lambda_2 - t\varepsilon)}{2} - \frac{f(\mu_1 - t\varepsilon) + f(\mu_2 - t\varepsilon)}{2} \\ & \geq \left\| \frac{f(\lambda_1 - t\varepsilon) - f(\lambda_2 - t\varepsilon)}{2} \bar{x}_e - \frac{f(\mu_1 - t\varepsilon) - f(\mu_2 - t\varepsilon)}{2} \bar{y}_e \right\|. \end{aligned}$$

Together with the definition of f_ε , it then follows that

$$\begin{aligned} & \frac{f_\varepsilon(\lambda_1) + f_\varepsilon(\lambda_2)}{2} - \frac{f_\varepsilon(\mu_1) + f_\varepsilon(\mu_2)}{2} \\ & = \int \left[\frac{f(\lambda_1 - t\varepsilon) + f(\lambda_2 - t\varepsilon)}{2} - \frac{f(\mu_1 - t\varepsilon) + f(\mu_2 - t\varepsilon)}{2} \right] \varphi(t) dt \\ & \geq \int \left\| \frac{f(\lambda_1 - \varepsilon) - f(\lambda_2 - \varepsilon)}{2} \bar{x}_e - \frac{f(\mu_1 - \varepsilon) - f(\mu_2 - \varepsilon)}{2} \bar{y}_e \right\| \varphi(t) dt \end{aligned}$$

$$\begin{aligned} &\geq \left\| \int \left[\frac{f(\lambda_1 - \varepsilon) - f(\lambda_2 - \varepsilon)}{2} \bar{x}_\varepsilon - \frac{f(\mu_1 - \varepsilon) - f(\mu_2 - \varepsilon)}{2} \bar{y}_\varepsilon \right] \varphi(t) dt \right\| \\ &= \left\| \frac{f_\varepsilon(\lambda_1) - f_\varepsilon(\lambda_2)}{2} \bar{x}_\varepsilon - \frac{f_\varepsilon(\mu_1) - f_\varepsilon(\mu_2)}{2} \bar{y}_\varepsilon \right\|. \end{aligned}$$

By the definition of $f_\varepsilon^{\text{SOC}}$, this shows that $f_\varepsilon^{\text{SOC}}(x) \succeq_K f_\varepsilon^{\text{SOC}}(y)$, i.e., f_ε is SOC-monotone. \square

From Proposition 3.2 and [2, Theorem V. 3.4], $f \in C^1(J)$ is SOC-monotone if and only if it is matrix monotone of order 2. When the continuous f is not in the class $C^1(J)$, the result also holds due to Proposition 3.3 and the fact that f is matrix monotone of order n if and only if f_ε is matrix monotone of order n . Thus, we have the following main result.

Theorem 3.1. *The set of continuous SOC-monotone functions on the open interval J coincides with that of continuous matrix monotone functions of order 2 on J .*

Remark 3.1. Combining Theorem 3.1 with Löwner’s theorem [17] shows that if $f : J \rightarrow \mathbb{R}$ is a continuous SOC-monotone function on the open interval J , then $f \in C^1(J)$.

4. Characterizations of SOC-convex functions

This section is devoted itself to the characterizations of SOC-convex functions, and shows that the continuous f is SOC-convex if and only if it is matrix convex of order 2. First, for the first-order differentiable SOC-convex functions, we have the following characterizations.

Proposition 4.1. *Assume that $f \in C^1(J)$ with J open. Then, the following results hold.*

(a) f is SOC-convex if and only if for any $x, y \in S$,

$$f^{\text{SOC}}(y) - f^{\text{SOC}}(x) - (f^{\text{SOC}})'(x)(y - x) \succeq_K 0. \tag{18}$$

(b) If f is SOC-convex, then $(f')^{\text{SOC}}$ is a monotone function on S .

Proof. By following the arguments as in [1, Proposition B.3(a)], the proof of part (a) can be done easily, and we omit the details. From part (a), it follows that for any $x, y \in S$,

$$f^{\text{SOC}}(x) - f^{\text{SOC}}(y) - (f^{\text{SOC}})'(y)(x - y) \succeq_K 0,$$

$$f^{\text{SOC}}(y) - f^{\text{SOC}}(x) - (f^{\text{SOC}})'(x)(y - x) \succeq_K 0.$$

Adding the last two inequalities, we immediately obtain that

$$\left[(f^{\text{SOC}})'(y) - (f^{\text{SOC}})'(x) \right] (y - x) \succeq_K 0.$$

Using the self-duality of K and Lemma 3.1(c) then yields

$$0 \leq \left\langle e, \left[(f^{\text{SOC}})'(y) - (f^{\text{SOC}})'(x) \right] (y - x) \right\rangle = \left\langle y - x, (f')^{\text{SOC}}(y) - (f')^{\text{SOC}}(x) \right\rangle.$$

This shows that $(f')^{\text{SOC}}$ is monotone. The proof is complete. \square

To provide sufficient and necessary characterizations for twice differentiable SOC-convex functions, we need the following lemma that offers the second-order differential of f^{SOC} .

Lemma 4.1. For any given $f: J \rightarrow \mathbb{R}$ with J open, let $f^{\text{SOC}}: S \rightarrow \mathbb{H}$ be defined by (3).

(a) f^{SOC} is twice (continuously) differentiable on S if and only if f is twice (continuously) differentiable on J . Furthermore, when f is twice differentiable on J , for any given $x = x_e + x_0e \in S$ and $u = u_e + u_0e, v = v_e + v_0e \in \mathbb{H}$, we have that

$$(f^{\text{SOC}})''(x)(u, v) = f''(x_0)u_0v_0e + f''(x_0)(u_0v_e + v_0u_e) + f''(x_0)\langle u_e, v_e \rangle e$$

if $x_e = 0$; and otherwise

$$\begin{aligned} (f^{\text{SOC}})''(x)(u, v) &= (b_2(x) - a_1(x))u_0\langle \bar{x}_e, v_e \rangle \bar{x}_e + (c_2(x) - 3d(x))\langle \bar{x}_e, u_e \rangle \langle \bar{x}_e, v_e \rangle \bar{x}_e \\ &\quad + d(x)[\langle u_e, v_e \rangle \bar{x}_e + \langle \bar{x}_e, v_e \rangle u_e + \langle \bar{x}_e, u_e \rangle v_e] + c_2(x)u_0v_0\bar{x}_e \\ &\quad + (b_2(x) - a_1(x))\langle \bar{x}_e, u_e \rangle v_0\bar{x}_e + a_1(x)(v_0u_e + u_0v_e) \\ &\quad + b_2(x)u_0v_0e + c_2(x)[v_0\langle \bar{x}_e, u_e \rangle + u_0\langle \bar{x}_e, v_e \rangle]e \\ &\quad + a_1(x)\langle u_e, v_e \rangle e + (b_2(x) - a_1(x))\langle \bar{x}_e, u_e \rangle \langle \bar{x}_e, v_e \rangle e, \end{aligned} \tag{19}$$

where

$$\begin{aligned} c_2(x) &= \frac{f''(\lambda_2(x)) - f''(\lambda_1(x))}{2}, & b_2(x) &= \frac{f''(\lambda_2(x)) + f''(\lambda_1(x))}{2}, \\ a_1(x) &= \frac{f'(\lambda_2(x)) - f'(\lambda_1(x))}{\lambda_2(x) - \lambda_1(x)}, & d(x) &= \frac{b_1(x) - a_0(x)}{\|x_e\|}. \end{aligned}$$

(b) If f is twice differentiable on J , then for any given $x \in S$ and $u, v \in \mathbb{H}$,

$$\begin{aligned} (f^{\text{SOC}})''(x)(u, v) &= (f^{\text{SOC}})''(x)(v, u), \\ \langle u, (f^{\text{SOC}})''(x)(u, v) \rangle &= \langle v, (f^{\text{SOC}})''(x)(u, u) \rangle. \end{aligned}$$

Proof. (a) The first part is direct by the given conditions and Lemma 3.1(b), and we only need to derive the differential formula. Fix any $u = u_e + u_0e, v = v_e + v_0e \in \mathbb{H}$. We first consider the case where $x_e = 0$. Without loss of generality, assume that $u_e \neq 0$. For any sufficiently small $t > 0$, using Lemma 3.1(b) and $x + tu = (x_0 + tu_0) + tu_e$, we have that

$$\begin{aligned} (f^{\text{SOC}})'(x + tu)v &= [b_1(x + tu) - a_0(x + tu)]\langle \bar{u}_e, v_e \rangle \bar{u}_e + c_1(x + tu)v_0\bar{u}_e \\ &\quad + a_0(x + tu)v_e + b_1(x + tu)v_0e + c_1(x + tu)\langle \bar{u}_e, v_e \rangle e. \end{aligned}$$

In addition, from Lemma 3.1(b), we also have that $(f^{\text{SOC}})'(x)v = f'(x_0)v_0e + f'(x_0)v_e$. Using the definition of $b_1(x)$ and $a_0(x)$, and the differentiability of f' on J , it follows that

$$\begin{aligned} \lim_{t \rightarrow 0} \frac{b_1(x + tu)v_0e - f'(x_0)v_0e}{t} &= f''(x_0)u_0v_0e, \\ \lim_{t \rightarrow 0} \frac{a_0(x + tu)v_e - f'(x_0)v_e}{t} &= f''(x_0)u_0v_e, \\ \lim_{t \rightarrow 0} \frac{b_1(x + tu) - a_0(x + tu)}{t} &= 0, \\ \lim_{t \rightarrow 0} \frac{c_1(x + tu)}{t} &= f''(x_0)\|u_e\|. \end{aligned}$$

Using the above four limits, it is not hard to obtain that

$$\begin{aligned} (f^{\text{SOC}})''(x)(u, v) &= \lim_{t \rightarrow 0} \frac{(f^{\text{SOC}})'(x + tu)v - (f^{\text{SOC}})'(x)v}{t} \\ &= f''(x_0)u_0v_0e + f''(x_0)(u_0v_e + v_0u_e) + f''(x_0)\langle u_e, v_e \rangle e. \end{aligned}$$

We next consider the case where $x_e \neq 0$. From Lemma 3.1(b), it follows that

$$\begin{aligned} (f^{\text{SOC}})'(x)v &= (b_1(x) - a_0(x)) \langle \bar{x}_e, v_e \rangle \bar{x}_e + c_1(x)v_0\bar{x}_e \\ &\quad + a_0(x)v_e + b_1(x)v_0e + c_1(x) \langle \bar{x}_e, v_e \rangle e, \end{aligned}$$

which in turn implies that

$$\begin{aligned} (f^{\text{SOC}})''(x)(u, v) &= [(b_1(x) - a_0(x)) \langle \bar{x}_e, v_e \rangle \bar{x}_e]' u + [c_1(x)v_0\bar{x}_e]' u \\ &\quad + [a_0(x)v_e + b_1(x)v_0e]' u + [c_1(x) \langle \bar{x}_e, v_e \rangle e]' u. \end{aligned} \tag{20}$$

By the expressions of $a_0(x)$, $b_1(x)$ and $c_1(x)$ and equations (11) and (12), we calculate that

$$\begin{aligned} (b_1(x))'u &= \frac{f''(\lambda_2(x)) [u_0 + \langle \bar{x}_e, u_e \rangle]}{2} + \frac{f''(\lambda_1(x)) [u_0 - \langle \bar{x}_e, u_e \rangle]}{2} \\ &= b_2(x)u_0 + c_2(x)\langle \bar{x}_e, u_e \rangle, \\ (c_1(x))'u &= c_2(x)u_0 + b_2(x)\langle \bar{x}_e, u_e \rangle, \\ (a_0(x))'u &= \frac{f'(\lambda_2(x)) - f'(\lambda_1(x))}{\lambda_2(x) - \lambda_1(x)} u_0 + \frac{b_1(x) - a_0(x)}{\|x_e\|} \langle \bar{x}_e, u_e \rangle \\ &= a_1(x)u_0 + d(x)\langle \bar{x}_e, u_e \rangle, \\ (\langle \bar{x}_e, v_e \rangle)'u &= \left\langle \frac{1}{\|x_e\|} u_e - \frac{\langle \bar{x}_e, u_e \rangle}{\|x_e\|} \bar{x}_e, v_e \right\rangle. \end{aligned}$$

Using these equalities and noting that $a_1(x) = c_1(x)/\|x_e\|$, we obtain that

$$\begin{aligned} [(b_1(x) - a_0(x)) \langle \bar{x}_e, v_e \rangle \bar{x}_e]' u &= [(b_2(x) - a_1(x))u_0 + (c_2(x) - d(x))\langle \bar{x}_e, u_e \rangle] \langle \bar{x}_e, v_e \rangle \bar{x}_e \\ &\quad + (b_1(x) - a_0(x)) \left\langle \frac{1}{\|x_e\|} u_e - \frac{\langle \bar{x}_e, u_e \rangle}{\|x_e\|} \bar{x}_e, v_e \right\rangle \bar{x}_e \\ &\quad + (b_1(x) - a_0(x)) \langle \bar{x}_e, v_e \rangle \left[\frac{1}{\|x_e\|} u_e - \frac{\langle \bar{x}_e, u_e \rangle}{\|x_e\|} \bar{x}_e \right] \\ &= [(b_2(x) - a_1(x))u_0 + (c_2(x) - d(x))\langle \bar{x}_e, u_e \rangle] \langle \bar{x}_e, v_e \rangle \bar{x}_e \\ &\quad + d(x)\langle u_e, v_e \rangle \bar{x}_e - 2d(x)\langle \bar{x}_e, v_e \rangle \langle \bar{x}_e, u_e \rangle \bar{x}_e + d(x)\langle \bar{x}_e, v_e \rangle u_e; \\ [a_0(x)v_e + b_1(x)v_0e]' u &= [a_1(x)u_0 + d(x)\langle \bar{x}_e, u_e \rangle] v_e + [b_2(x)u_0 + c_2(x)\langle \bar{x}_e, u_e \rangle] v_0e; \\ [c_1(x)v_0\bar{x}_e]' u &= [c_2(x)u_0 + b_2(x)\langle \bar{x}_e, u_e \rangle] v_0\bar{x}_e + c_1(x)v_0 \frac{u_e - \langle \bar{x}_e, u_e \rangle \bar{x}_e}{\|x_e\|} \\ &= [c_2(x)u_0 + b_2(x)\langle \bar{x}_e, u_e \rangle] v_0\bar{x}_e + a_1(x)v_0 [u_e - \langle \bar{x}_e, u_e \rangle \bar{x}_e]; \end{aligned}$$

and

$$\begin{aligned} [c_1(x)\langle \bar{x}_e, v_e \rangle e]^T u &= [c_2(x)u_0 + b_2(x)\langle \bar{x}_e, u_e \rangle] \langle \bar{x}_e, v_e \rangle e + c_1(x) \left\langle \frac{u_e - \langle \bar{x}_e, u_e \rangle \bar{x}_e}{\|x_e\|}, v_e \right\rangle e \\ &= c_2(x)u_0 \langle \bar{x}_e, v_e \rangle e + (b_2(x) - a_1(x)) \langle \bar{x}_e, u_e \rangle \langle \bar{x}_e, v_e \rangle e + a_1(x) \langle u_e, v_e \rangle e. \end{aligned}$$

Adding the equalities above and using Eq. (20) yields the formula in (19).

(b) By the formula in part (a), a simple computation yields the desired result. \square

Proposition 4.2. Assume that $f \in C^2(J)$ with J open. Then, the following results hold.

(a) f is SOC-convex if and only if for any $x \in S$ and $h \in \mathbb{H}$, $(f^{\text{SOC}})''(x)(h, h) \in K$.

(b) f is SOC-convex if and only if f is convex and for any $\tau_1, \tau_2 \in J$,

$$\frac{f''(\tau_2) f(\tau_2) - f(\tau_1) - f'(\tau_1)(\tau_2 - \tau_1)}{2(\tau_2 - \tau_1)^2} \geq \left[\frac{f(\tau_1) - f(\tau_2) - f'(\tau_2)(\tau_1 - \tau_2)}{(\tau_2 - \tau_1)^2} \right]^2. \tag{21}$$

(c) f is SOC-convex if and only if f is convex and for any $\tau_1, \tau_2 \in J$,

$$\frac{1}{4} f''(\tau_1) f''(\tau_2) \geq \frac{f(\tau_2) - f(\tau_1) - f'(\tau_1)(\tau_2 - \tau_1)}{(\tau_2 - \tau_1)^2} \cdot \frac{f(\tau_1) - f(\tau_2) - f'(\tau_2)(\tau_1 - \tau_2)}{(\tau_2 - \tau_1)^2}. \tag{22}$$

(d) f is SOC-convex if and only if for any $\tau_1, \tau_2 \in J$ and $s = \tau_1, \tau_2$,

$$\begin{bmatrix} f^{[2]}(\tau_2, s, \tau_2) & f^{[2]}(\tau_2, s, \tau_1) \\ f^{[2]}(\tau_1, s, \tau_2) & f^{[2]}(\tau_1, s, \tau_1) \end{bmatrix} \succeq 0.$$

Proof. (a) Suppose that f is SOC-convex. Since f^{SOC} is twice continuously differentiable by Lemma 4.1(a), we have for any given $x \in S$, $h \in \mathbb{H}$ and sufficiently small $t > 0$,

$$f^{\text{SOC}}(x + th) = f^{\text{SOC}}(x) + t(f^{\text{SOC}})'(x)h + \frac{1}{2}t^2(f^{\text{SOC}})''(x)(h, h) + o(t^2).$$

Applying Proposition 4.1(a) yields that $\frac{1}{2}(f^{\text{SOC}})''(x)(h, h) + o(t^2)/t^2 \succeq_K 0$. Taking the limit $t \downarrow 0$, we obtain $(f^{\text{SOC}})''(x)(h, h) \in K$. Conversely, fix any $z \in K$ and $x, y \in S$. Applying the mean-value theorem for the twice continuously differentiable $(f^{\text{SOC}}(\cdot), z)$ at x , we have

$$\begin{aligned} (f^{\text{SOC}}(y), z) &= (f^{\text{SOC}}(x), z) + \langle (f^{\text{SOC}})'(x)(y - x), z \rangle \\ &\quad + \frac{1}{2} \langle (f^{\text{SOC}})''(x + t_1(y - x))(y - x, y - x), z \rangle \end{aligned}$$

for some $t_1 \in (0, 1)$. Since $x + t_1(y - x) \in S$, the given assumption implies that

$$(f^{\text{SOC}}(y) - f^{\text{SOC}}(x) - (f^{\text{SOC}})'(x)(y - x), z) \geq 0.$$

This, by the arbitrariness of z in K , implies that $f^{\text{SOC}}(y) - f^{\text{SOC}}(x) - (f^{\text{SOC}})'(x)(y - x) \succeq_K 0$. From Proposition 4.1(a), it then follows that f is SOC-convex.

(b) By part (a), it suffices to prove that $(f^{\text{SOC}})''(x)(h, h) \in K$ for any $x \in S$ and $h \in \mathbb{H}$ if and only if f is convex and (21) holds. Fix any $x = x_e + x_0e \in S$. By the continuity of $(f^{\text{SOC}})''(x)$, we may assume that $x_e \neq 0$. From Lemma 4.1(a), for any $h = h_e + h_0e \in \mathbb{H}$,

$$\begin{aligned} (f^{\text{SOC}})''(x)(h, h) &= \left[(c_2(x) - 3d(x)) \langle \bar{x}_e, h_e \rangle^2 + 2(b_2(x) - a_1(x))h_0 \langle \bar{x}_e, h_e \rangle \right] \bar{x}_e \\ &\quad + \left[c_2(x)h_0^2 + d(x)\|h_e\|^2 \right] \bar{x}_e + \left[2a_1(x)h_0 + 2d(x) \langle \bar{x}_e, h_e \rangle \right] h_e \end{aligned}$$

$$\begin{aligned}
 & + \left[2c_2(x)h_0 \langle \bar{x}_e, h_e \rangle + b_2(x)h_0^2 + a_1(x)\|h_e\|^2 \right] e \\
 & + (b_2(x) - a_1(x))\langle \bar{x}_e, h_e \rangle^2 e.
 \end{aligned}$$

Therefore, $(f^{\text{soc}})''(x)(h, h) \in K$ if and only if the following two inequalities hold:

$$b_2(x) \left(h_0^2 + \langle \bar{x}_e, h_e \rangle^2 \right) + 2c_2(x)h_0 \langle \bar{x}_e, h_e \rangle + a_1(x) \left(\|h_e\|^2 - \langle \bar{x}_e, h_e \rangle^2 \right) \geq 0 \tag{23}$$

and

$$\begin{aligned}
 & \left[b_2(x) \left(h_0^2 + \langle \bar{x}_e, h_e \rangle^2 \right) + 2c_2(x)h_0 \langle \bar{x}_e, h_e \rangle + a_1(x) \left(\|h_e\|^2 - \langle \bar{x}_e, h_e \rangle^2 \right) \right]^2 \\
 & \geq \left\| \left(c_2(x)h_0^2 + d(x)\|h_e\|^2 \right) \bar{x}_e + 2(b_2(x) - a_1(x))h_0 \langle \bar{x}_e, h_e \rangle \bar{x}_e \right. \\
 & \quad \left. + (c_2(x) - 3d(x)) \langle \bar{x}_e, h_e \rangle^2 \bar{x}_e + 2(a_1(x)h_0 + d(x)\langle \bar{x}_e, h_e \rangle) h_e \right\|^2.
 \end{aligned} \tag{24}$$

Observe that the left-hand side of (23) can be rewritten as

$$\frac{f''(\lambda_2(x))(h_0 + \langle \bar{x}_e, h_e \rangle)^2}{2} + \frac{f''(\lambda_1(x))(h_0 - \langle \bar{x}_e, h_e \rangle)^2}{2} + a_1(x)(\|h_e\|^2 - \langle \bar{x}_e, h_e \rangle^2).$$

From Lemma 2.2, it then follows that (23) holds for all $h = h_e + h_0e \in \mathbb{H}$ if and only if

$$f''(\lambda_1(x)) \geq 0, \quad f''(\lambda_2(x)) \geq 0 \quad \text{and} \quad a_1(x) \geq 0. \tag{25}$$

In addition, by the definition of $b_2(x)$, $c_2(x)$ and $a_1(x)$, the left-hand side of (24) equals

$$\left[f''(\lambda_2(x))\mu_2(h)^2 + f''(\lambda_1(x))\mu_1(h)^2 + a_1(x)\mu(h)^2 \right]^2, \tag{26}$$

where $\mu_1(h)$, $\mu_2(h)$ and $\mu(h)$ are defined as in Lemma 2.3 with u_e replaced by \bar{x}_e . In the following, we use μ_1 , μ_2 and μ to represent $\mu_1(h)$, $\mu_2(h)$ and $\mu(h)$ respectively. Note that the sum of the first three terms in $\|\cdot\|^2$ on the right-hand side of (24) equals

$$\begin{aligned}
 & \frac{1}{2} (c_2(x) + b_2(x) - a_1(x)) (h_0 + \langle \bar{x}_e, h_e \rangle)^2 \bar{x}_e \\
 & + \frac{1}{2} (c_2(x) - b_2(x) + a_1(x)) (h_0 - \langle \bar{x}_e, h_e \rangle)^2 \bar{x}_e \\
 & + d(x) \left(\|h_e\|^2 - \langle \bar{x}_e, h_e \rangle^2 \right) \bar{x}_e - 2d(x) \langle \bar{x}_e, h_e \rangle^2 \bar{x}_e \\
 & = f''(\lambda_2(x))\mu_2^2 \bar{x}_e - f''(\lambda_1(x))\mu_1^2 \bar{x}_e - (a_1(x) + d(x))\mu_2^2 \bar{x}_e \\
 & \quad + (a_1(x) - d(x))\mu_1^2 \bar{x}_e + 2d(x)\mu_2\mu_1 \bar{x}_e + d(x)\mu^2 \bar{x}_e \\
 & =: E(x, h)\bar{x}_e,
 \end{aligned}$$

where $(\mu_2 - \mu_1)^2 = 2\langle \bar{x}_e, h_e \rangle^2$ is used for the equality, while the last term is

$$\begin{aligned}
 & (a_1(x) - d(x)) (h_0 - \langle \bar{x}_e, h_e \rangle) h_e + (a_1(x) + d(x)) (h_0 + \langle \bar{x}_e, h_e \rangle) h_e \\
 & = \sqrt{2} (a_1(x) - d(x)) \mu_1 h_e + \sqrt{2} (a_1(x) + d(x)) \mu_2 h_e.
 \end{aligned}$$

Thus, we calculate that the right-hand side of (24) equals

$$\begin{aligned}
 & E(x, h)^2 + 2 \left[(a_1(x) - d(x))\mu_1 + (a_1(x) + d(x))\mu_2 \right]^2 \|h_e\|^2 \\
 & + 2\sqrt{2}E(x, h)[a_1(x) - d(x)]\mu_1 \langle \bar{x}_e, h_e \rangle + 2\sqrt{2}E(x, h)[a_1(x) + d(x)]\mu_2 \langle \bar{x}_e, h_e \rangle
 \end{aligned}$$

$$\begin{aligned}
 &= E(x, h)^2 + 2\left[(a_1(x) - d(x))\mu_1 + (a_1(x) + d(x))\mu_2\right]^2 \left[\mu^2 + \frac{(\mu_2 - \mu_1)^2}{2}\right] \\
 &\quad + 2E(x, h)(\mu_2 - \mu_1)\left[(a_1(x) - d(x))\mu_1 + (a_1(x) + d(x))\mu_2\right] \\
 &= \left[E(x, h) + (\mu_2 - \mu_1)\left[(a_1(x) - d(x))\mu_1 + (a_1(x) + d(x))\mu_2\right]\right]^2 \\
 &\quad + 2\left[(a_1(x) - d(x))\mu_1 + (a_1(x) + d(x))\mu_2\right]^2 \mu^2, \tag{27}
 \end{aligned}$$

where the expressions of μ_1 , μ_2 and μ are used for the first equality. Now substituting the expression of $E(x, h)$ into (27) yields that the right-hand side of (27) equals

$$\left[f''(\lambda_2(x))\mu_2^2 - f''(\lambda_1(x))\mu_1^2 + d(x)\mu^2\right]^2 + 2\left[(a_1(x) - d(x))\mu_1 + (a_1(x) + d(x))\mu_2\right]^2 \mu^2.$$

Together with Eq. (26), it follows that (24) is equivalent to

$$\begin{aligned}
 &4f''(\lambda_1(x))f''(\lambda_2(x))\mu_1^2\mu_2^2 + 2(a_1(x) - d(x))f''(\lambda_2(x))\mu_2^2\mu^2 \\
 &+ 2(a_1(x) + d(x))f''(\lambda_1(x))\mu_1^2\mu^2 + (a_1(x)^2 - d(x)^2)\mu^4 \\
 &- 2\left[(a_1(x) - d(x))\mu_1 + (a_1(x) + d(x))\mu_2\right]^2 \mu^2 \geq 0.
 \end{aligned}$$

By Lemma 2.3, this inequality holds for any $h = h_e + h_0e \in \mathbb{H}$ if and only if

$$\begin{aligned}
 a_1(x)^2 - d(x)^2 &\geq 0, \quad f''(\lambda_2(x))(a_1(x) - d(x)) \geq (a_1(x) + d(x))^2, \\
 f''(\lambda_1(x))(a_1(x) + d(x)) &\geq (a_1(x) - d(x))^2,
 \end{aligned}$$

which, by the expression of $a_1(x)$ and $d(x)$, are respectively equivalent to

$$\begin{aligned}
 &\frac{f(\lambda_2) - f(\lambda_1) - f'(\lambda_1)(\lambda_2 - \lambda_1)}{(\lambda_2 - \lambda_1)^2} \cdot \frac{f(\lambda_1) - f(\lambda_2) - f'(\lambda_2)(\lambda_1 - \lambda_2)}{(\lambda_2 - \lambda_1)^2} \geq 0, \\
 &\frac{f''(\lambda_2)}{2} \frac{f(\lambda_2) - f(\lambda_1) - f'(\lambda_1)(\lambda_2 - \lambda_1)}{(\lambda_2 - \lambda_1)^2} \geq \left[\frac{f(\lambda_1) - f(\lambda_2) - f'(\lambda_2)(\lambda_1 - \lambda_2)}{(\lambda_2 - \lambda_1)^2}\right]^2, \tag{28} \\
 &\frac{f''(\lambda_1)}{2} \frac{f(\lambda_1) - f(\lambda_2) - f'(\lambda_2)(\lambda_1 - \lambda_2)}{(\lambda_2 - \lambda_1)^2} \geq \left[\frac{f(\lambda_2) - f(\lambda_1) - f'(\lambda_1)(\lambda_2 - \lambda_1)}{(\lambda_2 - \lambda_1)^2}\right]^2,
 \end{aligned}$$

where $\lambda_1 = \lambda_1(x)$ and $\lambda_2 = \lambda_2(x)$. Summing up the discussions above, f is SOC-convex if and only if (25) and (28) hold. In view of the arbitrariness of x , we have that f is SOC-convex if and only if f is convex and (21) holds.

(c) It suffices to prove that (21) is equivalent to (22). Clearly, (21) implies (22). We next prove that (22) implies (21). Fixing any $\tau_2 \in J$, we consider $g(t) : J \rightarrow \mathbb{R}$ defined by

$$g(t) = \frac{f''(\tau_2)}{2} \left[f(\tau_2) - f(t) - f'(t)(\tau_2 - t) \right] - \frac{[f(t) - f(\tau_2) - f'(\tau_2)(t - \tau_2)]^2}{(t - \tau_2)^2}$$

if $t \neq \tau_2$, and otherwise $g(\tau_2) = 0$. From the proof of [12, Theorem 2.3], we know that (21) implies that $g(t)$ attains its global minimum at $t = \tau_2$. Consequently, (21) follows.

(d) The result is immediate by part (b) and the definition of $f^{[2]}$ given in Section 2. \square

By Proposition 4.2(b) and (c), using the same arguments as in [12] gives the following result.

Corollary 4.1. *Let $f \in C^4(J)$ with J open. Then, the following results hold.*

(a) f is SOC-convex if and only if for any $t \in J$,

$$\begin{bmatrix} \frac{f''(t)}{6} & \frac{f^{(3)}(t)}{24} \\ \frac{f^{(3)}(t)}{6} & \frac{f^{(4)}(t)}{24} \end{bmatrix} \succeq 0$$

where $f^{(3)}$ and $f^{(4)}$ denote the third and the fourth derivatives of f , respectively.

(b) If $f''(t) > 0$ for every $t \in J$, then f is SOC-convex if and only if there exists a positive concave function $c(t)$ on J such that $f''(t) = c(t)^{-3}$ for every $t \in J$.

Propositions 4.1 and 4.2 and Corollary 4.1 provide the characterizations for continuously differentiable SOC-convex functions, which extend the corresponding results of [6, Section 4]. When f is not continuously differentiable, the following proposition shows that one may check the SOC-convexity of f by checking that of its regularization f_ε . Since the proof can be done easily by following that of Proposition 3.3, we omit the details.

Proposition 4.3. Let $f : J \rightarrow \mathbb{R}$ be a continuous function on the open interval J , and f_ε be its regularization defined by (9). Then, f is SOC-convex if and only if f_ε is SOC-convex on J_ε for every sufficiently small $\varepsilon > 0$, where $J_\varepsilon := (a + \varepsilon, b - \varepsilon)$ for $J = (a, b)$.

By Corollary 4.1, [12, Theorem 2.3] and Proposition 4.3, we have the following result.

Theorem 4.1. The set of continuous SOC-convex functions on the open interval J coincides with that of continuous matrix convex functions of order 2 on J .

Remark 4.1. Combining Theorem 4.1 with Kraus' theorem [15] shows that if $f : J \rightarrow \mathbb{R}$ is a continuous SOC-convex function, then $f \in C^2(J)$.

To close this section, we establish another sufficient and necessary characterization for twice continuously differentiable SOC-convex functions f by the differential operator $(f^{\text{SOC}})'$.

Proposition 4.4. Let $f \in C^2(J)$ with J open. Then f is SOC-convex if and only if

$$x \succeq_K y \implies (f^{\text{SOC}})'(x) - (f^{\text{SOC}})'(y) \succeq 0, \quad \forall x, y \in S. \tag{29}$$

Proof. Suppose that f is SOC-convex. Fix any $x, y \in S$ with $x \succeq_K y$, and $h \in \mathbb{H}$. Since f^{SOC} is twice continuously differentiable by Lemma 4.1(a), applying the mean-value theorem for the twice continuously differentiable $\langle h, (f^{\text{SOC}})'(\cdot)h \rangle$ at y , we have

$$\begin{aligned} \langle h, [(f^{\text{SOC}})'(x) - (f^{\text{SOC}})'(y)]h \rangle &= \langle h, (f^{\text{SOC}})''(y + t_1(x - y))(x - y, h) \rangle \\ &= \langle x - y, (f^{\text{SOC}})''(y + t_1(x - y))(h, h) \rangle \end{aligned} \tag{30}$$

for some $t_1 \in (0, 1)$, where Lemma 4.1(b) is used for the second equality. Noting that $y + t_1(x - y) \in S$ and f is SOC-convex, from Proposition 4.2(a) we have

$$(f^{\text{SOC}})''(y + t_1(x - y))(h, h) \in K.$$

This, together with $x - y \in K$, yields that $\langle x - y, (f^{\text{SOC}})''(y + t_1(x - y))(h, h) \rangle \geq 0$. Then, from (30) and the arbitrariness of h , we have $(f^{\text{SOC}})'(x) - (f^{\text{SOC}})'(y) \succeq 0$.

Conversely, assume that the implication in (29) holds for any $x, y \in S$. For any fixed $u \in K$, clearly, $x + tu \succeq_K x$ for all $t > 0$. Consequently, for any $h \in \mathbb{H}$, we have

$$\langle h, [(f^{\text{SOC}})'(x + tu) - (f^{\text{SOC}})'(x)]h \rangle \geq 0.$$

Note that $(f^{\text{SOC}})'(x)$ is continuously differentiable. The last inequality implies that

$$0 \leq \langle h, (f^{\text{SOC}})''(x)(u, h) \rangle = \langle u, (f^{\text{SOC}})''(x)(h, h) \rangle.$$

By the self-duality of K and the arbitrariness of u in K , this means that $(f^{\text{SOC}})''(x)(h, h) \in K$. Together with Proposition 4.2(a), it follows that f is SOC-convex. \square

5. Examples and applications

This section gives some examples of SOC-monotone functions and SOC-convex functions, and then discusses their applications in establishing certain important inequalities. First, by Proposition 3.2, one may easily verify that the following functions are SOC-monotone.

- Example 5.1.** (i) $f(t) = t^r$ on $[0, +\infty)$ is SOC-monotone if and only if $0 \leq r \leq 1$.
 (ii) $f(t) = -t^{-r}$ on $(0, +\infty)$ is SOC-monotone if and only if $0 \leq r \leq 1$.
 (iii) $f(t) = \ln(t)$ on $(0, +\infty)$ is SOC-monotone.
 (iv) $f(t) = -\cot(t)$ on $(0, \pi)$ is SOC-monotone.
 (v) $f(t) = \frac{t}{c+t}$ on $(-\infty, c)$ and $(c, +\infty)$ are SOC-monotone, where $c \geq 0$ is a constant.
 (vi) $f(t) = \frac{t}{c-t}$ on $(-\infty, c)$ and $(c, +\infty)$ are SOC-monotone, where $c \geq 0$ is a constant.

By Corollary 4.1, it is easy to verify that the following functions are SOC-convex.

- Example 5.2.** (i) The function $f(t) = t^r$ with $r \geq 0$ is SOC-convex on $[0, +\infty)$ if and only if $r \in [1, 2]$; and the function $f(t) = t^{-r}$ with $r > 0$ is SOC-convex on $(0, +\infty)$ if and only if $r \in [0, 1]$. Particularly, $f(t) = t^2$ is SOC-convex on \mathbb{R} .
 (ii) The function $f(t) = t^r$ with $r \geq 0$ is SOC-concave if and only if $r \in [0, 1]$.
 (iii) The entropy function $f(t) = t \ln t$ ($t \geq 0$) is SOC-convex.
 (iv) The logarithmic function $f(t) = -\ln t$ ($t > 0$) is SOC-convex.
 (v) $f(t) = t/(t - \sigma)$ with $\sigma \geq 0$ on $(\sigma, +\infty)$ is SOC-convex.
 (vi) $f(t) = -t/(t + \sigma)$ with $\sigma \geq 0$ on $(-\sigma, +\infty)$ is SOC-convex.
 (vii) $f(t) = t^2/(1 - t)$ on $(-1, 1)$ is SOC-convex.

Next we illustrate the applications of the SOC-monotonicity and SOC-convexity of certain functions in establishing some important inequalities. For example, by the SOC-monotonicity of $-t^{-r}$ and t^r with $r \in [0, 1]$, one can get the order-reversing inequality and the Löwner–Heinz inequality, and by the SOC-monotonicity and SOC-concavity of $-t^{-1}$, one may obtain the general harmonic–arithmetic mean inequality.

Proposition 5.1. For any $x, y \in \mathbb{H}$ and $0 \leq r \leq 1$, the following inequalities hold:

- (a) $y^{-r} \succeq_K x^{-r}$ if $x \succeq_K y \succ_K 0$;
- (b) $x^r \succeq_K y^r$ if $x \succeq_K y \succeq_K 0$;
- (c) $[\beta x^{-1} + (1 - \beta)y^{-1}]^{-1} \preceq_K \beta x + (1 - \beta)y$ for any $x, y \succ_K 0$ and $\beta \in (0, 1)$.

From the second inequality of Proposition 5.1, we particularly have the following result which generalizes [10, Eq. (3.9)], and is often used when analyzing the properties of the generalized FB SOC complementarity function $\phi_p(x, y) := (|x|^p + |y|^p)^{1/p} - (x + y)$.

Corollary 5.1. For any $x, y \in \mathbb{H}$, let $z(x, y) := (|x|^p + |y|^p)^{1/p}$ for any $p > 1$. Then,

$$z(x, y) \succeq_K |x| \succeq_K x \text{ and } z(x, y) \succeq_K |y| \succeq_K y.$$

The SOC-convexity can also be used to establish some matrix inequalities. From (10) we see that, when \mathbb{H} reduces to the n -dimensional Euclidean space \mathbb{R}^n , the differential operator $(f^{\text{soc}})'(x)$ becomes the following $n \times n$ symmetric matrix:

$$\begin{bmatrix} b_1(x) & c_1(x)\bar{x}_e^T \\ c_1(x)\bar{x}_e & a_0(x)I + (b_1(x) - a_0(x))\bar{x}_e\bar{x}_e^T \end{bmatrix}$$

where $a_0(x)$, $b_1(x)$ and $c_1(x)$ are same as before, and I is an identity matrix. Thus, from Proposition 4.4, we have the following result which is hard to get by direct calculation.

Proposition 5.2. *If $f \in C^2(J)$ is SOC-convex on the open interval J , then for any $x, y \in S$ with $x \succeq_K y$,*

$$\begin{bmatrix} b_1(x) & c_1(x)\bar{x}_e^T \\ c_1(x)\bar{x}_e & a_0(x)I + (b_1(x) - a_0(x))\bar{x}_e\bar{x}_e^T \end{bmatrix} \geq \begin{bmatrix} b_1(y) & c_1(y)\bar{y}_e^T \\ c_1(y)\bar{y}_e & a_0(y)I + (b_1(y) - a_0(y))\bar{y}_e\bar{y}_e^T \end{bmatrix}.$$

Particularly, when $f(t) = t^2$ ($t \in \mathbb{R}$), this conclusion reduces to the following implication

$$x \succeq_K y \implies \begin{bmatrix} x_0 & \bar{x}_e^T \\ \bar{x}_e & x_0I \end{bmatrix} \geq \begin{bmatrix} y_0 & \bar{y}_e^T \\ \bar{y}_e & y_0I \end{bmatrix}.$$

In addition, with certain SOC-monotone and SOC-convex functions, one can easily establish some determinant inequalities.

Proposition 5.3. *For any $x, y \in K$ and any real number $p \geq 1$, it holds that*

$$\sqrt[p]{\det(x + y)} \geq 2^{\frac{2}{p}-2} \left(\sqrt[p]{\det(x)} + \sqrt[p]{\det(y)} \right).$$

Proof. Since $f(t) = t^{1/p}$ is SOC-concave on $[0, +\infty)$, we have $\left(\frac{x+y}{2}\right)^{1/p} \succeq_K \frac{x^{1/p} + y^{1/p}}{2}$, which, by the fact that $\det(x) \geq \det(y)$ whenever $x \succeq_K y \succeq_K 0$, implies that

$$2^{-\frac{2}{p}} \det \left(\sqrt[p]{x + y} \right) = \det \left(\sqrt[p]{\frac{x + y}{2}} \right) \geq \det \left(\frac{\sqrt[p]{x} + \sqrt[p]{y}}{2} \right) \geq \frac{\det(\sqrt[p]{x}) + \det(\sqrt[p]{y})}{4},$$

where $\det(x + y) \geq \det(x) + \det(y)$ for $x, y \in K$ is used for the last inequality. In addition, by the definition of $\det(x)$, it is clear that $\det(\sqrt[p]{x}) = \sqrt[p]{\det(x)}$. Thus, from the last equation, we obtain the desired inequality. The proof is complete. \square

6. Relations between SOC-monotone and SOC-convex

Comparing Example 5.1 with Example 5.2, we observe that there are some relations between SOC-monotone and SOC-convex functions. For example, $f(t) = t \ln t$ and $f(t) = -\ln t$ are SOC-convex on $(0, +\infty)$, and its derivative functions are SOC-monotone on $(0, +\infty)$. This is similar to the case for matrix convex and matrix monotone functions. However, it is worthwhile to point out that they can not inherit all relations between matrix convex and matrix monotone functions, since the class of continuous SOC-monotone (SOC-convex) functions coincides with the class of continuous matrix monotone (matrix convex) functions of order 2 only, and there exist gaps between matrix monotone (matrix convex) functions of different orders (see [13,20]). Then, a question occurs to us: which relations for matrix convex and monotone functions still hold for SOC-convex and SOC-monotone functions.

Lemma 6.1. Assume that $f : J \rightarrow \mathbb{R}$ is three times differentiable on the open interval J . Then, f is a non-constant SOC-monotone function if and only if f' is strictly positive and $(f')^{-1/2}$ is concave.

Proof. “ \Leftarrow ”. Clearly, f is a non-constant function. Also, by [12, Proposition 2.2], we have

$$\frac{f(t_2) - f(t_1)}{t_2 - t_1} \leq \sqrt{f'(t_2)f'(t_1)}, \quad \forall t_2, t_1 \in J. \tag{31}$$

This, by the strict positivity of f' and Proposition 3.2, shows that f is SOC-monotone. “ \Rightarrow ”. The result is direct by [8, Theorem III] and Theorem 3.1. \square

Using Lemma 6.1, we may verify that SOC-monotone and SOC-convex functions inherit the following relation of matrix monotone and matrix convex functions.

Proposition 6.1. If $f : J \rightarrow \mathbb{R}$ is a continuous SOC-monotone function, then the function $g(t) = \int_a^t f(s)ds$ with some $a \in J$ is SOC-convex.

Proof. It suffices to consider the case where f is a non-constant SOC-monotone function. Due to Proposition 3.3, we may assume $f \in C^3(J)$. By Lemma 6.1, $f'(t) > 0$ for all $t \in J$ and $(f')^{-1/2}$ is concave. Since $g \in C^4(J)$ and $g''(t) = f'(t) > 0$ for all $t \in J$, by Corollary 4.1 in order to prove that g is SOC-convex, we only need to argue

$$\frac{g''(t)g^{(4)}(t)}{48} \geq \frac{[g^{(3)}(t)]^2}{36} \iff \frac{f'(t)f^{(3)}(t)}{48} \geq \frac{[f''(t)]^2}{36}, \quad \forall t \in J. \tag{32}$$

Since $(f')^{-1/2}$ is concave, its second-order derivative is nonpositive. From this, we have

$$\frac{1}{32} (f''(t))^2 \leq \frac{1}{48} f'(t)f^{(3)}(t) \quad \forall t \in J,$$

which implies the inequality (32). The proof is complete. \square

Similar to matrix monotone and matrix convex functions, the converse of Proposition 6.1 does not hold. For example, $f(t) = t^2/(1 - t)$ on $(-1, 1)$ is SOC-convex by Example 5.2(vii), but its derivative $g'(t) = \frac{1}{(1-t)^2} - 1$ is not SOC-monotone by Proposition 3.2. As a consequence of Proposition 6.1 and Proposition 4.4, we have the following corollary.

Corollary 6.1. Let $f \in C^2(J)$. If f' is SOC-monotone, then f is SOC-convex. This is equivalent to saying that for any $x, y \in S$ with $x \succeq_K y$,

$$(f')^{\text{soc}}(x) \succeq_K (f')^{\text{soc}}(y) \implies (f^{\text{soc}})'(x) - (f^{\text{soc}})'(y) \succeq 0.$$

From [2, Theorem V. 2.5], we know that a continuous function f mapping $[0, +\infty)$ into itself is matrix monotone if and only if it is matrix concave. However, for such f we can not prove that f is SOC-concave when it is SOC-monotone, but f is SOC-concave under a little stronger condition than SOC-monotonicity, i.e., the matrix monotonicity of order 4.

Proposition 6.2. Let f be a continuous function mapping $[0, +\infty)$ into itself. Then,

- (a) f is SOC-monotone whenever f is SOC-concave;
- (b) f is SOC-concave whenever f is matrix monotone of order 4.

Proof. Using the same arguments as [5, Proposition 4.1], we can prove part (a). By [18, Theorem 2.1], if f is continuous and matrix monotone of order $2n$, then f is matrix concave of order n . This, together with Theorem 4.1, implies part (b). \square

Note that Proposition 6.2 verifies Conjecture 4.2 of [5] partially. From [2], we know that the functions in Example 5.1(i)–(iii) are all matrix monotone, and so they are SOC-concave by Proposition 6.2(b). In addition, using Proposition 6.2(b) and noting that $-t^{-1}$ ($t > 0$) is SOC-monotone and SOC-concave on $(0, +\infty)$, we readily have the following corollary.

Corollary 6.2. *Let f be a continuous function from $(0, +\infty)$ into itself. If f is matrix monotone of order 4, then the function $g(t) = \frac{1}{f(t)}$ is SOC-convex.*

Proposition 6.3. *Let f be a continuous real function on the interval $[0, \alpha)$. If f is SOC-convex, then the indefinite integral of $\frac{f(t)}{t}$ is also SOC-convex.*

Proof. The result follows directly by [21, Proposition 2.7] and Theorem 4.1. \square

For a continuous real function f on the interval $[0, \alpha)$, Theorem V. 2.9 of [2] states that the following two conditions are equivalent:

- (i) f is matrix convex and $f(0) \leq 0$;
- (ii) The function $g(t) = f(t)/t$ is matrix monotone on $(0, \alpha)$.

Now we cannot establish the equivalence between the two conditions for the SOC-monotone and SOC-convex functions though the given examples in Examples 5.1 and 5.2 imply that this seems to hold, and we will leave it as a future topic.

At the end of this section, let us take a look at Example 5.1(i)–(iii) and (v)–(vi). We find that these functions are continuous, nondecreasing and concave. Then, one naturally ask whether such functions are SOC-monotone or not. This is exactly Conjecture 4.1(a) of [5]. The following counterexample shows that Conjecture 4.1(a) of [5] does not hold generally.

Example 6.1. Consider $f(t) = \begin{cases} -t \ln t + t & t \in (0, 1), \\ 1 & t \in [1, +\infty). \end{cases}$ This function is continuously differentiable, nondecreasing and concave on $(0, +\infty)$. However, letting $t_1 = 0.1$ and $t_2 = 3$,

$$f'(t_1)f'(t_2) - \left(\frac{f(t_1) - f(t_2)}{t_1 - t_2}\right)^2 = -\left(\frac{-t_1 \ln t_1 + t_1 - 1}{t_1 - t_2}\right)^2 = -0.0533.$$

By Proposition 3.2, we know that the function f is not SOC-monotone.

7. Conclusions

We provided the necessary and sufficient characterizations for SOC-monotone and SOC-convex functions in the setting of a general Hilbert space, by which the set of continuous SOC-monotone (respectively, SOC-convex) functions is shown to coincide with that of continuous matrix monotone (respectively, matrix convex) functions of order 2. These results will be helpful to characterize the SC-monotone (i.e., symmetric cone monotone) and SC-convex functions, which are scalar valued functions that induce Löwner operators in Euclidean Jordan algebra to preserve the monotone order and convex order, respectively.

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