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Further relationship between second-order cone and positive semidefinite matrix cone

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ABSTRACT
It is well known that second-order cone (SOC) programming can be regarded as a special case of positive semidefinite programming using the arrow matrix. This paper further studies the relationship between SOCs and positive semidefinite matrix cones. In particular, we explore the relationship to expressions regarding distance, projection, tangent cone, normal cone and the KKT system. Understanding these relationships will help us see the connection and difference between the SOC and its PSD reformulation more clearly.

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1. Introduction

The second-order cone (SOC) in $\mathbb{R}^n$, also called the Lorentz cone, is defined as

$$\mathcal{K}^n := \left\{(x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1} \,|\, x_1 \geq \|x_2\|\right\}, \tag{1}$$

where $\|\cdot\|$ denotes the Euclidean norm. If $n = 1$, $\mathcal{K}^n$ is the set of nonnegative reals $\mathbb{R}_+$. The positive semidefinite matrix cone (PSD cone), denoted by $\mathcal{S}^n_+$, is the collection of all symmetric positive semidefinite matrices in $\mathbb{R}^{n\times n}$, i.e.

$$\begin{aligned}\mathcal{S}^n_+ &:= \left\{X \in \mathbb{R}^{n\times n} \,|\, X \in \mathcal{S}^n \text{ and } X \succeq O\right\} \\ &:= \left\{X \in \mathbb{R}^{n\times n} \,|\, X = X^T \text{ and } v^T X v \geq 0 \;\; \forall v \in \mathbb{R}^n\right\}.\end{aligned}$$

It is well known that SOC and positive semidefinite matrix cone both belong to the category of symmetric cones,[1] which are unified under Euclidean Jordan algebra.

In [2], for each vector $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$, an arrow-shaped matrix L_x (alternatively called an arrow matrix and denoted by $\mathrm{Arw}(x)$) is defined as

$$L_x := \begin{bmatrix} x_1 & x_2^T \\ x_2 & x_1 I_{n-1} \end{bmatrix}. \tag{2}$$

It can be verified that there is a close relationship between the SOC and the PSD cone as below:

$$x \in \mathcal{K}^n \quad \Longleftrightarrow \quad L_x := \begin{bmatrix} x_1 & x_2^T \\ x_2 & x_1 I_{n-1} \end{bmatrix} \succeq O. \tag{3}$$

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Hence, a second-order cone program (SOCP) can be recast as a special semidefinite program (SDP). In the light of this, it seems that we just need to focus on SDP. Nevertheless, this reformulation has some disadvantages. For example, Ref. [3] indicates that

> Solving SOCPs via SDP is not a good idea, however. Interior-point methods that solve the SOCP directly have a much better worst-case complexity than an SDP method .... The difference between these numbers is significant if the dimensions of the second-order constraints are large.

This comment mainly concerns the algorithmic aspects; see [2,3] for more information.

In fact, 'reformulation' is usually the main idea behind many approaches to study various optimization problems and it is necessary to discuss the relationship between the primal problem and the transformed problem. For example, for complementarity problems (or variational inequality problems), we can reformulate these problems to work on a minimization optimization problem via merit functions (or gap functions). The properties of merit functions ensure the solution to complementarity problems is the same as the global optimal solution to the minimization problem. Nonetheless, finding a global optimal solution is very difficult. Thus, we turn to study the connection between the solution to complementarity problems and the stationary points of the transformed optimization problem. Similarly, for mathematical programming with complementarity constraints (MPCC), the ordinary KKT conditions do not hold because the standard constraint qualification fails to hold (due to the existence of complementarity constraints). One therefore considers to recast MPCC to other types of optimization problems with different approaches. These different approaches also ensure the solution set of MPCC is the same to that of the transformed optimization problems. But the KKT conditions for these transformed optimization problems are different, which are the source of various concepts of stationary points for MPCC, such as S-, M- and C-stationary points.

A similar question arises from SOCP and its SDP reformulation. In view of the above discussions, it could be interesting to study their relation from theoretical and numerical aspects. As mentioned above, Ref. [3] mainly deals with the SOCP and its SDP reformulation from the perspective of algorithm. The study on the relationship between SOCP and its corresponding SDP from theoretical aspect is rare. Sim and Zhao [4] discuss the relation between SOCP and its SDP counterpart from the perspective of duality theory. There are already some known relations between the SOC and the PSD cone; for instance,

(a) $x \in \operatorname{int} \mathcal{K}^n \iff L_x \in \operatorname{int} \mathcal{S}^n_+$;
(b) $x = 0 \iff L_x = 0$;
(c) $x \in \operatorname{bd} \mathcal{K}^n \setminus \{0\} \iff L_x \in \operatorname{bd} \mathcal{S}^n_+ \setminus \{O\}$.

Besides the interior, boundary point set, we know that for an optimization problem, some other topological structures, such as tangent cones, normal cones, projections and KKT systems, play very important roles. One may wonder whether there exists an analogous relationship between the SOC and the PSD cone. We will answer it in this paper. In particular, by comparing the expressions of distance, projection, tangent cone, normal cone and the KKT system between the SOC and the PSD cone, we will know more about the differences between SOCP and its SDP reformulation.

2. Preliminaries

In this section, we introduce some background materials that will be used in subsequent analysis. In the space of matrices, if we equip it with the trace inner product and the Frobenius norm

$$\langle X, Y\rangle_F := \operatorname{tr}(X^T Y), \quad \|X\|_F := \sqrt{\langle X, X\rangle_F},$$

then, for any $X \in \mathcal{S}^n$, its (repeated) eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ are real and it admits a spectral decomposition of the form:

$$X = P \operatorname{diag}[\lambda_1, \lambda_2, \ldots, \lambda_n]\, P^T \tag{4}$$

for some $P \in \mathcal{O}$. Here, $\mathcal{O}$ denotes the set of orthogonal $P \in \mathbb{R}^{n\times n}$, i.e. $P^T = P^{-1}$.

The above factorization (4) is the well-known spectral decomposition (eigenvalue decomposition) in matrix analysis.[5] There is a similar spectral decomposition associated with $\mathcal{K}^n$. To see this, we first introduce the so-called *Jordan product*. For any $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ and $y = (y_1, y_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$, their Jordan product [1] is defined by

$$x \circ y := \big(\langle x, y\rangle, y_1 x_2 + x_1 y_2\big).$$

Since the Jordan product, unlike scalar or matrix multiplication, is not associative, this is a main source on complication in the analysis of second-order cone complementarity problem (SOCCP). The identity element under this product is $e := (1, 0, \ldots, 0)^T \in \mathbb{R}^n$. It can be verified that the arrow matrix L_x is a linear mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$ given by $L_x y = x \circ y$. For each $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$, x admits a spectral decomposition [1,6–8] associated with $\mathcal{K}^n$ in the form of

$$x = \lambda_1(x) u_x^{(1)} + \lambda_2(x) u_x^{(2)}, \tag{5}$$

where $\lambda_1(x)$, $\lambda_2(x)$ and $u_x^{(1)}$, $u_x^{(2)}$ are the spectral values and the corresponding spectral vectors of x, respectively, given by

$$\lambda_i(x) := x_1 + (-1)^i \|x_2\| \quad \text{and} \quad u_x^{(i)} := \frac{1}{2}\begin{pmatrix} 1 \\ (-1)^i \bar{x}_2 \end{pmatrix}, \quad i = 1, 2, \tag{6}$$

with $\bar{x}_2 = x_2/\|x_2\|$ if $x_2 \neq 0$, and otherwise $\bar{x}_2$ being any vector in $\mathbb{R}^{n-1}$ with $\|\bar{x}_2\| = 1$. When $x_2 \neq 0$, the spectral decomposition is unique. The following lemma states the relation between the spectral decomposition of x and the eigenvalue decomposition of L_x.

Lemma 2.1: *Let $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ have the spectral decomposition given as in (5)–(6). Then, L_x has the eigenvalue decomposition:*

$$L_x = P \operatorname{diag}\big[\lambda_1(x), \lambda_2(x), x_1, \ldots, x_1\big] P^T$$

where

$$P = \Big[\sqrt{2} u_x^{(1)} \;\; \sqrt{2} u_x^{(2)} \;\; u_x^{(3)} \;\; \ldots \;\; u_x^{(n)}\Big] \in \mathbb{R}^{n\times n}$$

is an orthogonal matrix, and $u_x^{(i)}$ for $i = 3, \ldots, n$ have the form of $(0, \bar{u}_i)$ with $\bar{u}_3, \ldots, \bar{u}_n$ being any unit vectors in $\mathbb{R}^{n-1}$ that span the linear subspace orthogonal to x_2.

Proof: Please refer to [7–9]. □

From Lemma 2.1, it is not hard to calculate the inverse of L_x whenever it exists:

$$L_x^{-1} = \frac{1}{\det(x)} \begin{bmatrix} x_1 & -x_2^T \\ -x_2 & \dfrac{\det(x)}{x_1} I + \dfrac{1}{x_1} x_2 x_2^T \end{bmatrix} \tag{7}$$

where $\det(x) := x_1^2 - \|x_2\|^2$ denotes the determinant of x.

Throughout the whole paper, we use $\Pi_C(\cdot)$ to denote the projection mapping onto a closed and convex set C. In addition, for $\alpha \in \mathbb{R}$, $(\alpha)_+ := \max\{\alpha, 0\}$ and $(\alpha)_- := \min\{\alpha, 0\}$. Given a nonempty subset A in $\mathbb{R}^n$, we define $AA^T := \{uu^T \mid u \in A\}$ and $L_A := \{L_u \mid u \in A\}$, respectively. We denote Λ^n the set of all arrow-shaped matrices and Λ^n_+ the set of all positive semidefinite arrow matrices, i.e.

$$\Lambda^n := \{L_y \in \mathbb{R}^{n\times n} \mid y \in \mathbb{R}^n\} \quad \text{and} \quad \Lambda^n_+ := \{L_y \succeq O \mid y \in \mathbb{R}^n\}.$$

Lemma 2.2: *Let $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ have the spectral decomposition given as in (5)–(6). Then, the following hold:*

(a) $\Pi_{\mathcal{K}^n}(x) = (x_1 - \|x_2\|)_+ u_x^{(1)} + (x_1 + \|x_2\|)_+ u_x^{(2)}$,

(b) $\Pi_{\mathcal{S}^n_+}(L_x) = P \begin{bmatrix} (x_1 - \|x_2\|)_+ & 0 & 0 \\ 0 & (x_1 + \|x_2\|)_+ & 0 \\ 0 & 0 & (x_1)_+ I_{n-2} \end{bmatrix} P^T$ *where P is an orthogonal matrix of L_x.*

Proof: Please see [9,10] for a proof. □

3. Relation on distance and projection

In this section, we show the relation on distance and projection associated with the SOC and the PSD cone. We begin with some explanation for why we need to do so. First, let us consider the projection of x over $\mathcal{K}^n$. In the light of the relationship (3) between the SOC and the PSD cone, one may ask 'Can we obtain the expression of projection $\Pi_{\mathcal{K}^n}(x)$ by using $\Pi_{\mathcal{S}^n_+}(L_x)$, the projection of L_x over $\mathcal{S}^n_+$?'. In other words,

$$\text{Is} \quad \Pi_{\mathcal{K}^n}(x) = L^{-1}\left(\Pi_{\mathcal{S}^n_+}(L_x)\right) \quad \text{or} \quad \Pi_{\mathcal{S}^n_+}(L_x) = L\left(\Pi_{\mathcal{K}^n}(x)\right) \quad \text{right ?} \tag{8}$$

Here, the operator L, defined as $L(x) := L_x$, is a single-point mapping between $\mathbb{R}^n$ and $\mathcal{S}^n$, and L^{-1} is the inverse mapping of L, which can be achieved as in (7). To see this, take $x = (1, 2, 0) \in \mathbb{R}^3$; then, applying Lemma 2.1 yields

$$L_x = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Hence, by Lemma 2.2, we have

$$\Pi_{\mathcal{S}^3_+}(L_x) = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & \frac{3}{2} & 0 \\ \frac{3}{2} & \frac{3}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

which is not a form of the arrow matrix as shown in (2) because the diagonal entries are not equal. This means that we cannot seek a vector y such that $L_y = \Pi_{\mathcal{S}^n_+}(L_x)$. Note that

$$\Pi_{\mathcal{K}^n}(x) = (1+2)\frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} \\ \frac{3}{2} \\ 0 \end{pmatrix}$$

which gives

$$L\left(\Pi_{\mathcal{K}^n}(x)\right) = \begin{bmatrix} \frac{3}{2} & \frac{3}{2} & 0 \\ \frac{3}{2} & \frac{3}{2} & 0 \\ 0 & 0 & \frac{3}{2} \end{bmatrix}.$$

Hence, $\Pi_{\mathcal{K}^n}(x) \neq L^{-1}(\Pi_{\mathcal{S}^n_+}(L_x))$ and $\Pi_{\mathcal{S}^n_+}(L_x) \neq L(\Pi_{\mathcal{K}^n}(x))$. The distances $\mathrm{dist}(x, \mathcal{K}^n)$ and $\mathrm{dist}(L_x, \mathcal{S}^3_+)$ are also different since

$$\mathrm{dist}(x, \mathcal{K}^n) = \|x - \Pi_{\mathcal{K}^n}(x)\| = \left\| \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix} \right\| = \frac{\sqrt{2}}{2}$$

and

$$\mathrm{dist}(L_x, \mathcal{S}^n_+) = \|L_x - \Pi_{\mathcal{S}^n_+}(L_x)\| = \left\| \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\| = 1.$$

The failure of the above approach comes from the fact that the PSD cone is much larger, i.e. there exists a positive semidefinite matrix that is not arrow shaped. Consequently, we may ask whether (8) holds if we restrict the positive semidefinite matrices to arrow-shaped matrices. Still for $x = (1, 2, 0)$, by the expression given as in Theorem 3.1 below, we know that

$$\Pi_{\Lambda^n_+}(L_x) = \begin{bmatrix} \frac{7}{5} & \frac{7}{5} & 0 \\ \frac{7}{5} & \frac{7}{5} & 0 \\ 0 & 0 & \frac{7}{5} \end{bmatrix}$$

which implies $L^{-1}(\Pi_{\Lambda^n_+}(L_x)) = (\frac{7}{5}, \frac{7}{5}, 0)$. To sum up, $\Pi_{\mathcal{K}^n}(x) \neq L^{-1}(\Pi_{\Lambda^n_+}(L_x))$ and $\Pi_{\Lambda^n_+}(L_x) \neq L(\Pi_{\mathcal{K}^n}(x))$. All the above observations and discussions lead us to further explore some relationship, other than (3), between the SOC and the PSD cone.

Lemma 3.1: *The problem of finding the projection of L_x onto Λ^n_+:*

$$\begin{array}{ll} \min & \|L_x - L_y\|_F \\ \text{s.t.} & L_y \in \Lambda^n_+ \end{array} \tag{9}$$

is equivalent to the following optimization problem:

$$\begin{array}{ll} \min & \|L_{x-y}\|_F \\ \text{s.t.} & y \in \mathcal{K}^n. \end{array} \tag{10}$$

Precisely, L_y is an optimal solution to (9) if and only if y is an optimal solution to (10).

Proof: The result follows from the facts that $L_x - L_y = L_{x-y}$ and $L_y \in \Lambda^n_+ \iff y \in \mathcal{K}^n$. □

The result of Lemma 3.1 will help us find the expressions of the distance and projection of x onto $\mathcal{K}^n$, L_x to $\mathcal{S}^n_+$ and Λ^n_+. In particular, the distance of x onto $\mathcal{K}^n$ and L_x to $\mathcal{S}^n_+$ can be obtained using their expression of the projection given in Lemma 2.2.

Theorem 3.1: *Let $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ have the spectral decomposition given as in (5)–(6). Then, the following holds:*

(a) $\mathrm{dist}(x, \mathcal{K}^n) = \sqrt{\frac{1}{2}(x_1 - \|x_2\|)^2_- + \frac{1}{2}(x_1 + \|x_2\|)^2_-}$;

(b) $\mathrm{dist}(L_x, \mathcal{S}^n_+) = \sqrt{(x_1 - \|x_2\|)^2_- + (x_1 + \|x_2\|)^2_- + (n-2)(x_1)^2_-}$;

(c) $\Pi_{\Lambda^n_+}(L_x) = \begin{cases} L_x & \text{if } \; x_1 \geq \|x_2\|, \\ O & \text{if } \; x_1 \leq -\frac{2}{n}\|x_2\|, \\ \frac{1}{1+\frac{2}{n}}\left(x_1 + \frac{2}{n}\|x_2\|\right) \begin{bmatrix} 1 & \bar{x}_2^T \\ \bar{x}_2 & I_{n-1} \end{bmatrix} & \text{if } \; -\frac{2}{n}\|x_2\| < x_1 < \|x_2\|, \end{cases}$

(d) $\mathrm{dist}(L_x, \Lambda^n_+) = \sqrt{\frac{2n}{n+2}\left(x_1 - \|x_2\|\right)^2_- + \frac{n^2}{n+2}\left(x_1 + \frac{2}{n}\|x_2\|\right)^2_-}$.

Proof: (a) From Lemma 2.2, we know that $x = (x_1 - \|x_2\|)u_x^{(1)} + (x_1 + \|x_2\|)u_x^{(2)}$ and $\Pi_{\mathcal{K}^n}(x) = (x_1 - \|x_2\|)_+ u_x^{(1)} + (x_1 + \|x_2\|)_+ u_x^{(2)}$. Thus, it is clear to see that

$$\begin{aligned} \mathrm{dist}(x, \mathcal{K}^n) &= \|x - \Pi_{\mathcal{K}^n}(x)\| \\ &= \left\|(x_1 - \|x_2\|)_- u_x^{(1)} + (x_1 + \|x_2\|)_- u_x^{(2)}\right\| \\ &= \sqrt{\frac{1}{2}(x_1 - \|x_2\|)^2_- + \frac{1}{2}(x_1 + \|x_2\|)^2_-} \end{aligned}$$

where the last step is derived from $\|u_x^{(i)}\| = \sqrt{2}/2$ for $i = 1, 2$ and $\langle u_x^{(1)}, u_x^{(2)} \rangle = 0$.
(b) By Lemma 2.1 and Lemma 2.2(b),

$$L_x = P \begin{bmatrix} x_1 - \|x_2\| & 0 & 0 \\ 0 & x_1 + \|x_2\| & 0 \\ 0 & 0 & x_1 I_{n-2} \end{bmatrix} P^T$$

and

$$\Pi_{\mathcal{S}_+^n}(L_x) = P \begin{bmatrix} (x_1 - \|x_2\|)_+ & 0 & 0 \\ 0 & (x_1 + \|x_2\|)_+ & 0 \\ 0 & 0 & (x_1)_+ I_{n-2} \end{bmatrix} P^T.$$

Combining the above yields

$$\begin{aligned} \operatorname{dist}(L_x, \mathcal{S}_+^n) &= \left\| \begin{bmatrix} (x_1 - \|x_2\|)_- & 0 & 0 \\ 0 & (x_1 + \|x_2\|)_- & 0 \\ 0 & 0 & (x_1)_- I_{n-2} \end{bmatrix} \right\| \\ &= \sqrt{(x_1 - \|x_2\|)_-^2 + (x_1 + \|x_2\|)_-^2 + (n-2)(x_1)_-^2}. \end{aligned}$$

(c) To find $\Pi_{\Lambda_+^n}(L_x)$, we need to solve the optimization problem (9). From Lemma 3.1, it is equivalent to look into problem (10). Thus, we first compute

$$\begin{aligned} &\|L_{x-y}\|_F \\ &= \sqrt{(x_1 - y_1 - \|x_2 - y_2\|)^2 + (x_1 - y_1 + \|x_2 - y_2\|)^2 + (n-2)(x_1 - y_1)^2} \\ &= \sqrt{n(x_1 - y_1)^2 + 2\|x_2 - y_2\|^2} \\ &= \sqrt{n}\sqrt{(x_1 - y_1)^2 + \frac{2}{n}\|x_2 - y_2\|^2} \\ &= \sqrt{n}\sqrt{(x_1 - y_1)^2 + \left\|\sqrt{\frac{2}{n}}x_2 - \sqrt{\frac{2}{n}}y_2\right\|^2}. \end{aligned}$$

Now, we denote

$$y' := \left(y_1, \sqrt{\frac{2}{n}}y_2\right) = (y_1, \gamma y_2) = \Gamma y \quad \text{where} \quad \gamma := \sqrt{\frac{2}{n}} \text{ and } \Gamma := \begin{bmatrix} 1 & 0 \\ 0 & \gamma I \end{bmatrix}.$$

Then, $y_1 \geq \|y_2\|$ if and only if $y_1' \geq \frac{1}{\gamma}\|y_2'\|$; that is, $y \in \mathcal{K}^n$ if and only if $y' \in \mathcal{L}_\theta$ with $\cot\theta = \frac{1}{\gamma}$, where $\mathcal{L}_\theta := \{x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1} | x_1 \geq \|x_2\| \cot\theta\}$; see [11]. We therefore conclude that the problem (10) is indeed equivalent to the following optimization problem:

$$\begin{aligned} \min \quad & \sqrt{(x_1 - y_1')^2 + \left\|\sqrt{\tfrac{2}{n}}x_2 - y_2'\right\|^2} \\ \text{s.t.} \quad & y' \in \mathcal{L}_\theta. \end{aligned} \tag{11}$$

The optimal solution to the problem (11) is $\Pi_{\mathcal{L}_\theta}(x')$, the projection of $x' := (x_1, \gamma x_2) = \Gamma x$ onto $\mathcal{L}_\theta$, which according to [11, Theorems 3.1 and 3.2] is expressed by

$$\begin{aligned}
&\Pi_{\mathcal{L}_\theta}(x') \\
&= \frac{1}{1+\cot^2\theta}(x_1' - \|x_2'\|\cot\theta)_+ \begin{pmatrix} 1 \\ -\bar{x}_2'\cot\theta \end{pmatrix} + \frac{1}{1+\tan^2\theta}(x_1' + \|x_2'\|\tan\theta)_+ \begin{pmatrix} 1 \\ \bar{x}_2'\tan\theta \end{pmatrix} \\
&= \frac{\gamma^2}{1+\gamma^2}(x_1 - \|x_2\|)_+ \begin{pmatrix} 1 \\ -\frac{1}{\gamma}\bar{x}_2 \end{pmatrix} + \frac{1}{1+\gamma^2}(x_1 + \gamma^2\|x_2\|)_+ \begin{pmatrix} 1 \\ \gamma\bar{x}_2 \end{pmatrix}.
\end{aligned}$$

Hence, the optimal solution to (10) is

$$\begin{aligned}
y &= \Gamma^{-1}y' = \Gamma^{-1}\Pi_{\mathcal{L}_\theta}(x') = \Gamma^{-1}\Pi_{\mathcal{L}_\theta}(\Gamma x) \\
&= \begin{bmatrix} \frac{\gamma^2}{1+\gamma^2}(x_1 - \|x_2\|)_+ + \frac{1}{1+\gamma^2}(x_1 + \gamma^2\|x_2\|)_+ \\ \left(-\frac{1}{1+\gamma^2}(x_1 - \|x_2\|)_+ + \frac{1}{1+\gamma^2}(x_1 + \gamma^2\|x_2\|)_+\right)\bar{x}_2 \end{bmatrix} \\
&= \begin{cases} x & \text{if } \ x_1 \geq \|x_2\|, \\ 0 & \text{if } \ x_1 \leq -\frac{2}{n}\|x_2\|, \\ \frac{1}{1+\gamma^2}\left(x_1 + \gamma^2\|x_2\|\right)\begin{pmatrix} 1 \\ \bar{x}_2 \end{pmatrix} & \text{if } \ -\frac{2}{n}\|x_2\| < x_1 < \|x_2\|. \end{cases}
\end{aligned} \tag{12}$$

By Lemma 3.1, the optimal solution to (9) is L_y, i.e.

$$L_y = \Pi_{\Lambda_+^n}(L_x) = \begin{cases} L_x & \text{if } \ x_1 \geq \|x_2\|, \\ O & \text{if } \ x_1 \leq -\frac{2}{n}\|x_2\|, \\ \frac{1}{1+\frac{2}{n}}\left(x_1 + \frac{2}{n}\|x_2\|\right)\begin{bmatrix} 1 & \bar{x}_2^T \\ \bar{x}_2 & I_{n-1} \end{bmatrix} & \text{if } \ -\frac{2}{n}\|x_2\| < x_1 < \|x_2\|. \end{cases}$$

(d) In view of the expression (12), we can compute the distance $\mathrm{dist}(L_x, \Lambda_+^n)$ as follows.

$$\begin{aligned}
\mathrm{dist}(L_x, \Lambda_+^n) &= \|L_x - L_y\|_F = \|L_{x-y}\|_F \\
&= \Bigg(n\left[x_1 - \frac{\gamma^2}{1+\gamma^2}(x_1 - \|x_2\|)_+ - \frac{1}{1+\gamma^2}\left(x_1 + \gamma^2\|x_2\|\right)_+\right]^2 \\
&\quad + 2\left[\|x_2\| + \frac{1}{1+\gamma^2}(x_1 - \|x_2\|)_+ - \frac{1}{1+\gamma^2}\left(x_1 + \gamma^2\|x_2\|\right)_+\right]^2 \Bigg)^{\frac{1}{2}} \\
&= \Bigg(n\left[x_1 - \frac{2}{n+2}(x_1 - \|x_2\|)_+ - \frac{n}{n+2}\left(x_1 + \frac{2}{n}\|x_2\|\right)_+\right]^2 \\
&\quad + 2\left[\|x_2\| + \frac{n}{n+2}(x_1 - \|x_2\|)_+ - \frac{n}{n+2}\left(x_1 + \frac{2}{n}\|x_2\|\right)_+\right]^2 \Bigg)^{\frac{1}{2}} \\
&= \Bigg(n\left[\frac{2}{n+2}(x_1 - \|x_2\|)_- + \frac{n}{n+2}\left(x_1 + \frac{2}{n}\|x_2\|\right)_-\right]^2 \\
&\quad + 2\left[-\frac{n}{n+2}(x_1 - \|x_2\|)_- + \frac{n}{n+2}\left(x_1 + \frac{2}{n}\|x_2\|\right)_-\right]^2 \Bigg)^{\frac{1}{2}} \\
&= \sqrt{\frac{2n}{n+2}\left(x_1 - \|x_2\|\right)_-^2 + \frac{n^2}{n+2}\left(x_1 + \frac{2}{n}\|x_2\|\right)_-^2},
\end{aligned}$$

where the third equation comes from the facts that

$$x_1 = \frac{2}{n+2}(x_1 - \|x_2\|) + \frac{n}{n+2}\left(x_1 + \frac{2}{n}\|x_2\|\right)$$

and

$$\|x_2\| = -\frac{n}{n+2}(x_1 - \|x_2\|) + \frac{n}{n+2}\left(x_1 + \frac{2}{n}\|x_2\|\right).$$

$\square$

Theorem 3.2: *For any $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$,*

$$\mathrm{dist}(x, \mathcal{K}^n) \le \mathrm{dist}(L_x, \mathcal{S}^n_+) \le \mathrm{dist}(L_x, \Lambda^n_+).$$

In particular, for $n = 2$,

$$\mathrm{dist}(x, \mathcal{K}^2) = \frac{\sqrt{2}}{2}\,\mathrm{dist}(L_x, \mathcal{S}^2_+) \text{ and } \mathrm{dist}(L_x, \mathcal{S}^2_+) = \mathrm{dist}(L_x, \Lambda^2_+).$$

Proof: The first inequality follows from the formula of distance given as in Theorem 3.1; the second inequality comes from the fact that Λ^n_+ is a subset of $\mathcal{S}^n_+$, i.e. $\Lambda^n_+ \subset \mathcal{S}^n_+$.

For $n = 2$, by part(d) of Theorem 3.1, we have

$$\mathrm{dist}(L_x, \Lambda^2_+) = \sqrt{(x_1 - \|x_2\|)^2_- + (x_1 + \|x_2\|)^2_-}.$$

Combining this and Theorem 3.1(a)–(b) yields $\mathrm{dist}(x, \mathcal{K}^2) = \frac{\sqrt{2}}{2}\,\mathrm{dist}(L_x, \mathcal{S}^2_+)$ and $\mathrm{dist}(L_x, \mathcal{S}^2_+) = \mathrm{dist}(L_x, \Lambda^2_+)$. $\square$

Note that Λ^2_+ is strictly included in $\mathcal{S}^2_+$, i.e. $\Lambda^2_+ \subsetneqq \mathcal{S}^2_+$, because in the arrow matrix, the diagonal element is the same, but positive semidefinite matrix does not impose this requirement. Thus, $\mathrm{dist}(L_x, \Lambda^2_+) \le \mathrm{dist}(L_x, \mathcal{S}^2_+)$. In Theorem 3.2, we further show that the equality holds.

In view of Theorem 3.2, a natural question arises here: Are these distances equivalent? Recall that for two functions $g, h : \mathbb{R}^n \to \mathbb{R}$, we say that they are equivalent if there exist $\tau_1, \tau_2 > 0$ such that

$$\tau_1 g(x) \le h(x) \le \tau_2 g(x), \quad \forall x \in \mathbb{R}^n.$$

For instance, 1-norm and 2-norm are equivalent in this sense. To answer this question, we need the following lemma.

Lemma 3.2: *For $a, b \in \mathbb{R}$, the following inequality holds:*

$$\left(\frac{a+b}{2}\right)^2_- \le \frac{1}{2}\left(a^2_- + b^2_-\right).$$

Proof: We assume without loss of generality that $a \le b$. Then, we consider the following four cases to proceed the proof.

Case 1: For $a \ge 0$ and $b \ge 0$, we have

$$\left(\frac{a+b}{2}\right)^2_- = 0 = \frac{1}{2}\left(a^2_- + b^2_-\right).$$

Case 2: For $a \le 0$ and $b \le 0$, we have

$$\left(\frac{a+b}{2}\right)_-^2 = \left(\frac{a+b}{2}\right)^2 \le \frac{a^2+b^2}{2} = \frac{1}{2}\left(a_-^2 + b_-^2\right).$$

Case 3: For $a \le 0$, $b \ge 0$ and $a \le -b$, there implies $(a+b)/2 \le 0$. Then, we have

$$\left(\frac{a+b}{2}\right)_-^2 = \left(\frac{a+b}{2}\right)^2 = \frac{a^2+b^2+2ab}{4} \le \frac{a^2+b^2}{4} \le \frac{1}{2}a^2 = \frac{1}{2}\left(a_-^2 + b_-^2\right),$$

where the first inequality comes from the fact that $ab \le 0$ and the second inequality follows from the fact that $a^2 \ge b^2$ due to $a \le -b \le 0$.
Case 4: For $a \le 0$, $b \ge 0$ and $a \ge -b$, we have

$$\left(\frac{a+b}{2}\right)_-^2 = 0 \le \frac{1}{2}a^2 = \frac{1}{2}\left(a_-^2 + b_-^2\right).$$

□

Theorem 3.3: *The distances* $\mathrm{dist}(x, \mathcal{K}^n)$, $\mathrm{dist}(L_x, \mathcal{S}_+^n)$ *and* $\mathrm{dist}(L_x, \Lambda_+^n)$ *are all equivalent in the sense of*

$$\mathrm{dist}(x, \mathcal{K}^n) \le \mathrm{dist}(L_x, \mathcal{S}_+^n) \le \sqrt{n}\,\mathrm{dist}(x, \mathcal{K}^n) \tag{13}$$

and

$$\mathrm{dist}(L_x, \mathcal{S}_+^n) \le \mathrm{dist}(L_x, \Lambda_+^n) \le \sqrt{\frac{2n}{n+2}}\,\mathrm{dist}(L_x, \mathcal{S}_+^n). \tag{14}$$

Proof: (i) The key part to prove inequality (13) is to look into $\mathrm{dist}^2(L_x, \mathcal{S}_+^n)$, which are computed as below:

$$\begin{aligned}
&\mathrm{dist}^2(L_x, \mathcal{S}_+^n)\\
&\quad = (x_1 - \|x_2\|)_-^2 + (x_1 + \|x_2\|)_-^2 + (n-2)(x_1)_-^2\\
&\quad = (x_1 - \|x_2\|)_-^2 + (x_1 + \|x_2\|)_-^2 + (n-2)\left(\frac{(x_1 - \|x_2\|) + (x_1 + \|x_2\|)}{2}\right)_-^2\\
&\quad \le (x_1 - \|x_2\|)_-^2 + (x_1 + \|x_2\|)_-^2 + \frac{n-2}{2}\left((x_1 - \|x_2\|)_-^2 + (x_1 + \|x_2\|)_-^2\right)\\
&\quad = n\left(\frac{1}{2}(x_1 - \|x_2\|)_-^2 + \frac{1}{2}(x_1 + \|x_2\|)_-^2\right)\\
&\quad = n\,\mathrm{dist}^2(x, \mathcal{K}^n),
\end{aligned}$$

where the inequality is due to Lemma 3.2. Hence, we achieve

$$\mathrm{dist}(x, \mathcal{K}^n) \le \mathrm{dist}(L_x, \mathcal{S}_+^n) \le \sqrt{n}\,\mathrm{dist}(x, \mathcal{K}^n),$$

which indicates that the distance between x to $\mathcal{K}^n$ and L_x to $\mathcal{S}_+^n$ is equivalent.
(ii) It remains to show the equivalence between $\mathrm{dist}(L_x, \mathcal{S}_+^n)$ and $\mathrm{dist}(L_x, \Lambda_+^n)$. To proceed, we need to consider the following cases.
Case 1: For $x_1 \ge \|x_2\|$, $\mathrm{dist}(L_x, \mathcal{S}_+^n) = 0 = \mathrm{dist}(L_x, \Lambda_+^n)$.
Case 2: For $x_1 \le -\|x_2\|$, $\mathrm{dist}(L_x, \Lambda_+^n) = \sqrt{nx_1^2 + 2\|x_2\|^2} = \mathrm{dist}(L_x, \mathcal{S}_+^n)$.
Case 3: For $0 \le x_1 \le \|x_2\|$, $\mathrm{dist}(L_x, \Lambda_+^n) = \sqrt{\frac{2n}{n+2}}|x_1 - \|x_2\||$ and $\mathrm{dist}(L_x, \mathcal{S}_+^n) = |x_1 - \|x_2\||$.

Case 4: For $-\frac{2}{n}\|x_2\| \le x_1 \le 0$, $\mathrm{dist}^2(L_x, \Lambda_+^n) = \frac{2n}{n+2}(x_1 - \|x_2\|)^2$ and $\mathrm{dist}^2(L_x, \mathcal{S}_+^n) = (x_1 - \|x_2\|)^2 + (n-2)x_1^2$. Then,

$$\frac{2n}{n+2}\,\mathrm{dist}^2(L_x, \mathcal{S}_+^n) = \frac{2n}{n+2}\left(x_1 - \|x_2\|\right)^2 + \frac{2n}{n+2}(n-2)x_1^2 \ge \mathrm{dist}^2(L_x, \Lambda_+^n).$$

Case 5: For $-\|x_2\| \le x_1 \le -\frac{2}{n}\|x_2\|$,

$$\mathrm{dist}^2(L_x, \Lambda_+^n) = nx_1^2 + 2\|x_2\|^2 \ \text{ and } \ \mathrm{dist}^2(L_x, \mathcal{S}_+^n) = (x_1 - \|x_2\|)^2 + (n-2)x_1^2.$$

Note that

$$\begin{aligned} &\mathrm{dist}(L_x, \Lambda_+^n) \le \sqrt{\frac{2n}{n+2}}\,\mathrm{dist}(L_x, \mathcal{S}_+^n) \\ \Longleftrightarrow\ & nx_1^2 + 2\|x_2\|^2 \le \frac{2n}{n+2}\left[(x_1 - \|x_2\|)^2 + (n-2)x_1^2\right] \\ \Longleftrightarrow\ & 4\|x_2\|\left[nx_1 + \|x_2\|\right] \le n(n-4)x_1^2. \end{aligned}$$

Since $x_1 \le -\frac{2}{n}\|x_2\|$, it implies that

$$4\|x_2\|\left[nx_1 + \|x_2\|\right] \le -4\|x_2\|^2 \le 4\frac{n-4}{n}\|x_2\|^2 = n(n-4)\left(-\frac{2}{n}\|x_2\|\right)^2 \le n(n-4)x_1^2,$$

where the second inequality is due to the fact $\frac{n-4}{n} \ge -1$ for all $n \ge 2$. Hence,

$$\mathrm{dist}(L_x, \Lambda_+^n) \le \sqrt{\frac{2n}{n+2}}\,\mathrm{dist}(L_x, \mathcal{S}_+^n),$$

which is the desired result. □

The following example demonstrates that the inequalities (13) and (14) in Theorem 3.3 may be strict.

Example 3.1: Consider $x = (-1, 2, \underbrace{0, \ldots, 0}_{n-2})$ with $n \ge 4$. Then,

$$\mathrm{dist}(x, \mathcal{K}^n) < \mathrm{dist}(L_x, \mathcal{S}_+^n) < \sqrt{n}\,\mathrm{dist}(x, \mathcal{K}^n)$$

and

$$\mathrm{dist}(L_x, \mathcal{S}_+^n) < \mathrm{dist}(L_x, \Lambda_+^n) < \sqrt{\frac{2n}{n+2}}\,\mathrm{dist}(L_x, \mathcal{S}_+^n).$$

To see this, from Theorem 3.1, we know that

$$\mathrm{dist}(x, \mathcal{K}^n) = \sqrt{\frac{9}{2}}, \quad \mathrm{dist}(L_x, \mathcal{S}_+^n) = \sqrt{n+7}, \quad \mathrm{dist}(L_x, \Lambda_+^n) = \sqrt{n+8}. \tag{15}$$

Note that for $n \ge 4$, we have

$$\sqrt{\frac{9}{2}} < \sqrt{n+7} < \sqrt{\frac{9n}{2}},$$

and

$$\sqrt{n+7} < \sqrt{n+8} < \sqrt{\frac{2n}{n+2}}\sqrt{n+7},$$

which says

$$\operatorname{dist}(x,\mathcal{K}^n) < \operatorname{dist}(L_x,\mathcal{S}^n_+) < \sqrt{n}\operatorname{dist}(x,\mathcal{K}^n),$$

and

$$\operatorname{dist}(L_x,\mathcal{S}^n_+) < \operatorname{dist}(L_x,\Lambda^n_+) < \sqrt{\frac{2n}{n+2}}\operatorname{dist}(L_x,\mathcal{S}^n_+).$$

From this example, we see that the distance related to SOC is independent of n; nonetheless, if we treat it as semidefinite matrix, the distance is dependent on n; see (15).

4. Relation on tangent cone

As shown earlier, all the distances introduced in Section 2 are equivalent. This allows us to study the relation on tangent cone because the tangent cone can be achieved by distance function.[12] More specifically, for a convex set C, there is

$$T_C(x) := \{h \mid \operatorname{dist}(x+th, C) = o(t),\ t \geq 0\}.$$

In the light of this, this section is devoted to exploring the relation on tangent cones.

Theorem 4.1: *Let $x=(x_1,x_2)\in\mathbb{R}\times\mathbb{R}^{n-1}$ belong to $\mathcal{K}^n$, i.e. $x\in\mathcal{K}^n$. Then,*

(a) $$T_{\mathcal{K}^n}(x) = \begin{cases} \mathcal{K}^n & \text{if } x=0, \\ \mathbb{R}^n & \text{if } x\in\operatorname{int}\mathcal{K}^n, \\ \left\{(d_1,d_2)\in\mathbb{R}^n \,\middle|\, d_2^Tx_2 - x_1d_1 \leq 0\right\} & \text{if } x\in\operatorname{bd}\mathcal{K}^n\backslash\{0\}. \end{cases}$$

(b) $$T_{\mathcal{S}^n_+}(L_x) = \begin{cases} \mathcal{S}^n_+ & \text{if } x=0, \\ \mathcal{S}^n & \text{if } x\in\operatorname{int}\mathcal{K}^n, \\ \left\{H\in\mathcal{S}^n \mid (u_x^{(1)})^THu_x^{(1)} \geq 0\right\} & \text{if } x\in\operatorname{bd}\mathcal{K}^n\backslash\{0\}. \end{cases}$$

(c) $T_{\Lambda^n_+}(L_x) = \{L_h \mid h\in T_{\mathcal{K}^n}(x)\} = T_{\mathcal{S}^n_+}(L_x)\cap\Lambda^n$.

Proof: The formulae of $T_{\mathcal{K}^n}(x)$ and $T_{\mathcal{S}_+}(L_x)$ follow from the results given in [13,14]. To verify part(c), we know that

$$T_{\Lambda^n_+}(L_x) = \left\{H\in\mathcal{S}^n \mid L_x + t_nH_n \in \Lambda^n_+,\ t_n\to 0^+, H_n\to H\right\}.$$

Due to $t_nH_n\in\Lambda^n_+ - L_x$, H_n is also an arrow matrix. This means $H_n = L_{h_n}$ for some $h_n\in\mathbb{R}^n$. In addition, $H_n\to H$ implies $H = L_h$ for some h with $h_n\to h$. Thus, we obtain that $L_x + t_nH_n = L_{x+t_nh_n}\in\Lambda^n_+$ which is equivalent to saying $x+t_nh_n\in\mathcal{K}^n$, i.e. $h\in T_{\mathcal{K}^n}(x)$. Moreover, since $\Lambda^n_+ = \mathcal{S}^n_+\cap\Lambda^n$ and $\mathcal{S}^n_+$, Λ^n cannot be separated, it yields

$$T_{\Lambda^n_+}(L_x) = T_{\mathcal{S}^n_+}(L_x)\cap T_{\Lambda^n}(L_x) = T_{\mathcal{S}^n_+}(L_x)\cap\Lambda^n$$

by [15, Theorem 6.42], where the last step comes from the fact that Λ^n is a subspace. □

The relation between $T_{\mathcal{K}^n}(x)$ and $T_{\mathcal{S}^n_+}(L_x)$ can be also characterized using their expression.

Theorem 4.2: *Let $x=(x_1,x_2)\in\mathbb{R}\times\mathbb{R}^{n-1}$ belong to $\mathcal{K}^n$, i.e. $x\in\mathcal{K}^n$. Then,*

$$L_{T_{\mathcal{K}^n}(x)} = T_{\mathcal{S}^n_+}(L_x)\cap\Lambda^n. \tag{16}$$

Proof: We proceed the proof by discussing the following three cases.

Case 1: For $x\in\operatorname{int}\mathcal{K}^n$, we have $L_x\in\operatorname{int}\mathcal{S}^n_+$. Thus, $T_{\mathcal{K}^n}(x)=\mathbb{R}^n$ and $T_{\mathcal{S}^n_+}(L_x)=\mathcal{S}^n$. This implies

$$L_{T_{\mathcal{K}^n}(x)} = L_{\mathbb{R}^n} = \Lambda^n = \mathcal{S}^n\cap\Lambda^n = T_{\mathcal{S}^n_+}(L_x)\cap\Lambda^n.$$

Case 2: For $x = 0$, we have $T_{\mathcal{K}^n}(x) = \mathcal{K}^n$ and $T_{\mathcal{S}^n_+}(L_x) = \mathcal{S}^n_+$. Since $y \in \mathcal{K}^n$ if and only if $L_y \in \mathcal{S}^n_+$,

$$L_{T_{\mathcal{K}^n}(x)} = L_{\mathcal{K}^n} = \Lambda^n_+ = \mathcal{S}^n_+ \cap \Lambda^n = T_{\mathcal{S}^n_+}(L_x) \cap \Lambda^n.$$

Case 3: For $x \in \operatorname{bd} \mathcal{K}^n \backslash \{0\}$, take $d \in T_{\mathcal{K}^n}(x)$. Then,

$$(u_x^{(1)})^T L_d \, u_x^{(1)} = \frac{1}{4}\begin{pmatrix} 1 & -\bar{x}_2^T \end{pmatrix} \begin{bmatrix} d_1 & d_2^T \\ d_2 & d_1 I \end{bmatrix} \begin{pmatrix} 1 \\ -\bar{x}_2 \end{pmatrix} = \frac{1}{2}(d_1 - d_2^T \bar{x}_2) \geq 0,$$

where the inequality comes from $d \in T_{\mathcal{K}^n}(x)$. Hence, $L_d \in T_{\mathcal{S}^n_+}(L_x)$ by Theorem 4.1, i.e. $L_{T_{\mathcal{K}^n}(x)} \subset T_{\mathcal{S}^n_+}(L_x) \cap \Lambda^n$. The converse inclusion can be proved by a similar argument. □

The restriction to Λ^n in (16) is required, which is illustrated by the following example. Taking $x = (1, 1) \in \mathbb{R}^2$, we have

$$T_{\mathcal{K}^2}(x) = \{d = (d_1, d_2) \in \mathbb{R}^2 \mid -d_1 + d_2 \leq 0\}$$

and

$$T_{\mathcal{S}^2_+}(L_x) = \left\{H \in \mathcal{S}^2 \mid (u_x^{(1)})^T H u_x^{(1)} \geq 0\right\} = \left\{H \in \mathcal{S}^2 \mid H_{11} - 2H_{12} + H_{22} \geq 0\right\}.$$

Hence, $L_{T_{\mathcal{K}^n}(x)}$ does not equal $T_{\mathcal{S}^n_+}(L_x)$.

5. Relation on normal cone

In this section, we continue to explore the relation on normal cone between the SOC and its PSD reformulation. To this end, we first write out the expressions of $N_{\mathcal{K}^n}(x)$, $N_{\mathcal{S}^n_+}(L_x)$, and $N_{\Lambda^n_+}(L_x)$, respectively.

Theorem 5.1: *Let $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ belong to $\mathcal{K}^n$, i.e. $x \in \mathcal{K}^n$. Then,*

(a) $$N_{\mathcal{K}^n}(x) = \begin{cases} -\mathcal{K}^n & \text{if } x = 0, \\ \{0\} & \text{if } x \in \operatorname{int} \mathcal{K}^n, \\ \mathbb{R}_+(-x_1, x_2) & \text{if } x \in \operatorname{bd} \mathcal{K}^n \backslash \{0\}. \end{cases}$$

(b) $$N_{\mathcal{S}^n_+}(L_x) = \begin{cases} -\mathcal{S}^n_+ & \text{if } x = 0, \\ \{O\} & \text{if } x \in \operatorname{int} \mathcal{K}^n, \\ \left\{\alpha \begin{bmatrix} 1 & -\bar{x}_2^T \\ -\bar{x}_2 & \bar{x}_2 \bar{x}_2^T \end{bmatrix} \,\middle|\, \alpha \leq 0 \right\} & \text{if } x \in \operatorname{bd} \mathcal{K}^n \backslash \{0\}. \end{cases}$$

(c) $N_{\Lambda^n_+}(L_x) = N_{\mathcal{S}^n_+}(L_x) + (\Lambda^n)^\perp$, *where*

$$(\Lambda^n)^\perp = \left\{H \in \mathcal{S}^n \mid tr(H) = 0,\ H_{1,i} = 0, i = 2, \ldots, n\right\}.$$

Proof: Parts (a) and (b) follow from [13] and [14]. For Part (c), since $\Lambda^n_+ = \mathcal{S}^n_+ \cap \Lambda^n$, it follows from [15, Theorem 6.42] that

$$N_{\Lambda^n_+}(L_x) = N_{\mathcal{S}^n_+}(L_x) + N_{\Lambda^n}(L_x).$$

Because Λ^n is a subspace, we know that $N_{\Lambda^n}(L_x) = (\Lambda^n)^\perp$, where

$$(\Lambda^n)^\perp = \left\{H \in \mathcal{S}^n \mid \langle H, L_y \rangle = 0,\ \forall y \in \mathbb{R}^n\right\} = \left\{H \in \mathcal{S}^n \mid tr(H) = 0,\ H_{1,i} = 0, i = 2, \ldots, n\right\}.$$

□

The relation between $N_{\Lambda^n_+}(L_x)$ and $N_{\mathcal{S}^n_+}(L_x)$ is already described in Theorem 5.1. Next, we further describe the relation between $N_{\mathcal{K}^n}(x)$ and $N_{\mathcal{S}^n_+}(L_x)$.

Theorem 5.2: *Let $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ belong to $\mathcal{K}^n$, i.e. $x \in \mathcal{K}^n$. Then, for $x \in \operatorname{int} \mathcal{K}^n$ and $x \in \operatorname{bd} \mathcal{K}^n \backslash \{0\}$,*

$$N_{\mathcal{S}_+^n}(L_x) = -N_{\mathcal{K}^n}(x) N_{\mathcal{K}^n}(x)^T.$$

Proof: Case 1: For $x \in \operatorname{int} \mathcal{K}^n$, $N_{\mathcal{K}^n}(x) = \{0\}$ and $N_{\mathcal{S}_+^n}(L_x) = \{O\}$. The desired result holds in this case.

Case 2: For $x \in \operatorname{bd} \mathcal{K}^n \backslash \{0\}$, it follows from Theorem 5.1 that

$$\begin{aligned} N_{\mathcal{S}_+^n}(L_x) &= \left\{ \alpha \begin{bmatrix} 1 & -\bar{x}_2^T \\ -\bar{x}_2 & \bar{x}_2 \bar{x}_2^T \end{bmatrix} \middle| \alpha \le 0 \right\} \\ &= \left\{ \alpha \begin{pmatrix} 1 \\ -\bar{x}_2 \end{pmatrix} \left(1, -\bar{x}_2^T\right) \middle| \alpha \le 0 \right\}. \end{aligned} \tag{17}$$

Since $N_{\mathcal{K}^n}(x) = \{y \mid y = \beta \hat{x}, \beta \le 0\}$ with $\hat{x} := (x_1, -x_2)$,

$$-N_{\mathcal{K}^n}(x) N_{\mathcal{K}^n}(x)^T = \{-\beta^2 \hat{x} \hat{x}^T \mid \beta \le 0\} = \left\{ -(\beta x_1)^2 \begin{pmatrix} 1 \\ -\bar{x}_2 \end{pmatrix} \left(1, -\bar{x}_2^T\right) \middle| \beta \le 0 \right\}. \tag{18}$$

Comparing with (17) and (18) yields the desired result. $\square$

From Theorem 5.1(c), we know that $N_{\Lambda_+^n}(L_x) \supset N_{\mathcal{S}_+^n}(L_x)$ since $O \in (\Lambda^n)^\perp$. For $x = 0$, $N_{\mathcal{K}^n}(x) = -\mathcal{K}^n$ and $N_{\mathcal{S}_+^n}(L_x) = -\mathcal{S}_+^n$. In this case, $N_{\mathcal{S}_+^n}(L_x)$ and $-N_{\mathcal{K}^n}(x) N_{\mathcal{K}^n}(x)^T$ do not coincide, i.e. Theorem 5.2 fails when $x = 0$. Below, we give the algebraic expressions for $N_{\Lambda_+^n}(L_x)$ and $N_{\mathcal{S}_+^n}(L_x)$ as $n = 2$, from which we can see the difference between them more clearly.

Theorem 5.3: *For $n = 2$, the explicit expressions of $N_{\mathcal{S}_+^2}(L_x)$ and $N_{\Lambda_+^2}(L_x)$ are as below:*

$$N_{\mathcal{S}_+^2}(L_x) = \begin{cases} \left\{ \begin{bmatrix} a & b \\ b & c \end{bmatrix} \middle| a \le 0, c \le 0, ac \ge b^2 \right\} & \text{if } \ x = 0, \\ \{O\} & \text{if } \ x \in \operatorname{int} \mathcal{K}^2, \\ \left\{ \alpha \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \middle| \alpha \le 0 \right\} & \text{if } \ x \in \operatorname{bd} \mathcal{K}^2 \backslash \{0\},\ x_2 > 0, \\ \left\{ \alpha \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \middle| \alpha \le 0 \right\} & \text{if } \ x \in \operatorname{bd} \mathcal{K}^2 \backslash \{0\},\ x_2 < 0. \end{cases}$$

and

$$N_{\Lambda_+^2}(L_x) = \begin{cases} \left\{ \begin{bmatrix} a & b \\ b & c \end{bmatrix} \middle| a + c \le -2|b| \right\} & \text{if } \ x = 0, \\ \left\{ \begin{bmatrix} a & b \\ b & c \end{bmatrix} \middle| a + c = 0, b = 0 \right\} & \text{if } \ x \in \operatorname{int} \mathcal{K}^2, \\ \left\{ \begin{bmatrix} a & b \\ b & c \end{bmatrix} \middle| a + c + 2b = 0,\ b \ge 0 \right\} & \text{if } \ x \in \operatorname{bd} \mathcal{K}^2 \backslash \{0\},\ x_2 > 0, \\ \left\{ \begin{bmatrix} a & b \\ b & c \end{bmatrix} \middle| a + c - 2b = 0,\ b \le 0 \right\} & \text{if } \ x \in \operatorname{bd} \mathcal{K}^2 \backslash \{0\},\ x_2 < 0. \end{cases}$$

Proof: First, we claim that

$$(\Lambda_+^2)^\circ = \left\{ \begin{bmatrix} a & b \\ b & c \end{bmatrix} \middle| a + c \le -2|b| \right\}.$$

In fact

$$\begin{aligned} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \in (\Lambda_+^2)^\circ \quad & \iff \left\langle \begin{bmatrix} a & b \\ b & c \end{bmatrix}, \begin{bmatrix} x_1 & x_2 \\ x_2 & x_1 \end{bmatrix} \right\rangle \le 0, \ \ \forall x_1 \ge |x_2|, \\ & \iff (a+c)x_1 + 2bx_2 \le 0, \quad \forall x_1 \ge |x_2|. \end{aligned} \tag{19}$$

If we plug in $x_1 = |x_2| + \tau$ with $\tau \ge 0$, then (19) can be rewritten as

$$(a+c)|x_2| + 2bx_2 + (a+c)\tau \le 0, \ \ \forall x_2 \in \mathbb{R} \ \text{ and } \ \tau \ge 0,$$

i.e.

$$(a+c+2b)x_2 + (a+c)\tau \le 0, \ \ \forall x_2 \ge 0 \ \text{ and } \ \tau \ge 0 \tag{20}$$

and

$$(-a-c+2b)x_2 + (a+c)\tau \le 0, \ \ \forall x_2 \le 0 \ \text{ and } \ \tau \ge 0. \tag{21}$$

With the arbitrariness of $\tau \ge 0$, we have $a + c \le 0$. Likewise, we have $a + c + 2b \le 0$ by (20) and $-a - c + 2b \ge 0$ by (21). Thus, $a + c \le -2b$ and $a + c \le 2b$. In other words, we conclude that the inequality (19) implies

$$a + c \le \min\{-2b, 2b\} = -2|b|.$$

Conversely, if a, b, c satisfies $a + c \le -2|b|$, then for $x_1 \ge |x_2|$, we have

$$(a+c)x_1 + 2bx_2 \le -2|b|x_1 + 2bx_2 \le -2|b||x_2| + 2bx_2 \le 0.$$

Consequently, the inequality (19) holds.
Case 1: For $x = 0$, we have

$$N_{\mathcal{S}_+^2}(O) = -\mathcal{S}_+^2 = \left\{ \begin{bmatrix} a & b \\ b & c \end{bmatrix} \middle| \ a \le 0, c \le 0, ac \ge b^2 \right\}$$

$$N_{\Lambda_+^2}(O) = (\Lambda_+^2)^\circ = \left\{ \begin{bmatrix} a & b \\ b & c \end{bmatrix} \middle| \ a + c \le -2|b| \right\}.$$

Case 2: For $x \in \operatorname{int} \mathcal{K}^2$, we claim that

$$N_{\Lambda_+^2}(L_x) = \left\{ \begin{bmatrix} a & b \\ b & c \end{bmatrix} \middle| \ a + c = 0, b = 0 \right\}.$$

In fact, since Λ_+^2 is a cone, $H \in N_{\Lambda_+^2}(L_x)$ is equivalent to saying that $H \in (\Lambda_+^2)^\circ$ and

$$\left\langle \begin{bmatrix} a & b \\ b & c \end{bmatrix}, \begin{bmatrix} x_1 & x_2 \\ x_2 & x_1 \end{bmatrix} \right\rangle = 0,$$

i.e. $(a + c)x_1 + 2bx_2 = 0$. Note that

$$0 = (a+c)x_1 + 2bx_2 \le -2|b|x_1 + 2bx_2 \le -2|b|x_1 + 2|b||x_2| \le -2|b|x_1 + 2|b|x_1 = 0.$$

Due to $x_1 > |x_2|$, we obtain $b = 0$ and $a + c = 0$.
Case 3: For $x \in \operatorname{bd} \mathcal{K}^2 \backslash \{0\}$, i.e. $x_1 = |x_2| \ne 0$, by a similar argument, we obtain the following expression. If $x_2 > 0$, then

$$N_{\Lambda_+^2}(L_x) = \left\{ \begin{bmatrix} a & b \\ b & c \end{bmatrix} \middle| \ a + c + 2b = 0, b \ge 0 \right\}.$$

If $x_2 < 0$, then

$$N_{\Lambda^2_+}(L_x) = \left\{ \begin{bmatrix} a & b \\ b & c \end{bmatrix} \,\middle|\, a + c - 2b = 0, b \le 0 \right\}.$$

□

By adopting the above expressions of $N_{\mathcal{S}^2_+}(L_x)$ and $N_{\Lambda^2_+}(L_x)$, we can see the decomposition in Theorem 5.1(c) more clearly. For example, take $\begin{bmatrix} a & b \\ b & c \end{bmatrix} \in N_{\Lambda^2_+}(L_x)$.

If $x = 0$, then

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix} = \begin{bmatrix} \frac{a+c}{2} & b \\ b & \frac{a+c}{2} \end{bmatrix} + \begin{bmatrix} \frac{a-c}{2} & 0 \\ 0 & \frac{c-a}{2} \end{bmatrix},$$

where $\begin{bmatrix} \frac{a+c}{2} & b \\ b & \frac{a+c}{2} \end{bmatrix} \in -\mathcal{S}^2_+$ since $a + c \le -2|b|$ by Theorem 5.3 and $\begin{bmatrix} \frac{a-c}{2} & 0 \\ 0 & \frac{c-a}{2} \end{bmatrix} \in (\Lambda^2)^\perp$.

If $x_1 = |x_2|$ and $x_2 > 0$, then

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix} = \begin{bmatrix} -b & b \\ b & -b \end{bmatrix} + \begin{bmatrix} a+b & 0 \\ 0 & c+b \end{bmatrix},$$

where $\begin{bmatrix} -b & b \\ b & -b \end{bmatrix} \in -\mathcal{S}^2_+$ and $\begin{bmatrix} a+b & 0 \\ 0 & c+b \end{bmatrix} \in (\Lambda^2)^\perp$ since $b \ge 0$ and $a+b+c+b = a+c+2b = 0$ by Theorem 5.3.

If $x_1 = |x_2|$ and $x_2 < 0$, then

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix} = \begin{bmatrix} b & b \\ b & b \end{bmatrix} + \begin{bmatrix} a-b & 0 \\ 0 & c-b \end{bmatrix},$$

where $\begin{bmatrix} b & b \\ b & b \end{bmatrix} \in -\mathcal{S}^2_+$ and $\begin{bmatrix} a-b & 0 \\ 0 & c-b \end{bmatrix} \in (\Lambda^2)^\perp$.

If $x_1 > |x_2|$, then $\begin{bmatrix} a & b \\ b & c \end{bmatrix} \in (\Lambda^2)^\perp$ by Theorem 5.3.

Theorem 5.4: *Let $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ belong to $\mathcal{K}^n$, i.e. $x \in \mathcal{K}^n$. Then, the following statements hold:*

(a) *If $x = 0$ or $x \in \operatorname{int} \mathcal{K}^n$, then $y \in N_{\mathcal{K}^n}(x) \Longleftrightarrow L_y \in N_{\mathcal{S}^n_+}(L_x)$.*
(b) *If $x, y \in \operatorname{bd} \mathcal{K}^n \backslash \{0\}$, then $y \in N_{\mathcal{K}^n}(x) \Longleftrightarrow L_y \in N_{\mathcal{S}^n_+}(L_x)$ if and only if $n = 2$.*

Proof: Part (a). Notice that

$$y \in N_{\mathcal{K}^n}(x) \Longleftrightarrow x \in \mathcal{K}^n,\ -y \in \mathcal{K}^n,\ \langle x, -y \rangle = 0, \tag{22}$$

and

$$\begin{aligned} L_y \in N_{\mathcal{S}^n_+}(L_x) &\Longleftrightarrow L_x \in \mathcal{S}^n_+,\ -L_y \in \mathcal{S}^n_+,\ \langle L_x, -L_y \rangle = 0 \\ &\Longleftrightarrow L_x \in \mathcal{S}^n_+,\ L_{-y} \in \mathcal{S}^n_+,\ \langle L_x, L_{-y} \rangle = 0, \end{aligned} \tag{23}$$

where the second equivalence is due to the fact $L_{-y} = -L_y$.

It is easy to see that these two systems (22) and (23) are equivalent for the case of $x = 0$ and $x \in \operatorname{int} \mathcal{K}^n$, which corresponds to $L_x = 0$ and $L_x \in \operatorname{int} \mathcal{S}^n_+$.

Part (b). For $x \in \operatorname{bd} \mathcal{K}^n \backslash \{0\}$ or $L_x \in \operatorname{bd} \mathcal{S}^n_+ \backslash \{O\}$, note that

$$\langle x, y \rangle = x_1 y_1 + x_2^T y_2, \quad \langle L_x, L_{-y} \rangle = -n x_1 y_1 - 2 x_2^T y_2.$$

For $y \neq 0$, this implies $\langle x, y\rangle = 0 \iff \langle L_x, L_{-y}\rangle = 0$ if and only if $n = 2$. This completes the proof. $\square$

We point out that the relationship between $L_{T_{\mathcal{K}}^n(x)}$ and $T_{\mathcal{S}_+^n}(L_x)$ has been described in Theorem 4.2. Although the normal cone is the polar cone of the tangent cone for a given convex set, it fails to achieve the relationship between $L_{N_{\mathcal{K}^n}}(x)$ and $N_{\mathcal{S}_+^n}(L_x)$ by taking polar on both sides of (16) because the operator L is not invariant under polar operator. More precisely, for x, y,

$$\langle x, y\rangle \leq 0 \quad \not\Rightarrow \quad \langle L_x, L_y\rangle \leq 0.$$

In fact, if $\langle x, y\rangle \leq 0$, i.e. $x_1y_1 + x_2^T y_2 \leq 0$, whereas $\langle L_x, L_y\rangle = nx_1y_1 + 2x_2^T y_2$. It is clear that for $n \geq 3$,

$$x_1y_1 + x_2^T y_2 \leq 0 \not\Rightarrow nx_1y_1 + 2x_2^T y_2 \leq 0.$$

For example, taking $x = (2, 1, 1)$ and $y = (2, -3, -2)$ gives $\langle x, y\rangle = -1 < 0$ and $\langle L_x, L_y\rangle = 2 > 0$. All the above explain why we need a different approach to prove Theorem 5.4.

6. Relation on KKT systems

In this section, we turn our attention to the relation on KKT systems. First, we know that the following second-order cone programming problem (SOCP)

$$\begin{array}{ll} \min & f(x) \\ \text{s.t.} & g(x) \in \mathcal{K}^n \end{array} \tag{24}$$

can be rewritten as a positive semidefinite programming problem (SDP)

$$\begin{array}{ll} \min & f(x) \\ \text{s.t.} & L_{g(x)} \in \mathcal{S}_+^n, \end{array} \tag{25}$$

where $g : \mathbb{R}^n \to \mathbb{R}^n$ is expressed as $g = (g_1, g_2, \ldots g_n)$. The KKT systems of the above SOCP (24) and SDP (25) are, respectively, denoted by $K(x)$ and $K(L_x)$, which are expressed as

$$\begin{aligned} K(x) &:= \left\{ \lambda \in \mathbb{R}^n \,\middle|\, 0 = \nabla f(x) + \sum_{i=1}^n \lambda_i \nabla g_i(x),\ \lambda \in N_{\mathcal{K}^n}(g(x)) \right\}, \\ K(L_x) &:= \left\{ \Gamma \in \mathbb{R}^{n\times n} \,\middle|\, 0 = \nabla f(x) + \left(\sum_{i=1}^n \Gamma_{ii}\right) \nabla g_1(x) + 2\sum_{i=2}^n \Gamma_{1i}\nabla g_i(x),\ \Gamma \in N_{\mathcal{S}_+^n}(L_{g(x)}) \right\}. \end{aligned}$$

In order to describe the relation between $K(x)$ and $K(L_x)$, we define the following two mappings: given $x \in \mathbb{R}^n$ and $X \in \mathcal{S}^n$, we define

$$M(X) := \begin{pmatrix} \sum_{i=1}^n X_{ii} & 2X_{12} & \ldots & 2X_{1n} \end{pmatrix} \tag{26}$$

and

$$\widetilde{M}(x) := \left\{ \Gamma \in \mathcal{S}_-^n \,\middle|\, \sum_{i=1}^n \Gamma_{ii} = x_1,\ \Gamma_{1i} = \frac{1}{2}x_i,\ i = 2, \ldots, n \right\}. \tag{27}$$

Then, the relation between KKT system of the above two problems is given as below.

Theorem 6.1: *Let $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ belong to $\mathcal{K}^n$, i.e. $x \in \mathcal{K}^n$. Suppose that the mappings M and $\widetilde{M}$ are defined as in (26) and (27), respectively. Then, the following statements hold:*

(a) $\widetilde{M}(N_{\mathcal{K}^n}(x)) = N_{\mathcal{S}^n_+}(L_x)$ and $M(N_{\mathcal{S}^n_+}(L_x)) = N_{\mathcal{K}^n}(x)$;
(b) $\widetilde{M}(K(x)) = K(L_x)$ and $M(K(L_x)) = K(x)$.

Proof: (a) We first show

$$\widetilde{M}(N_{\mathcal{K}^n}(x)) \subseteq N_{\mathcal{S}^n_+}(L_x) \text{ and } M(N_{\mathcal{S}^n_+}(L_x)) \subseteq N_{\mathcal{K}^n}(x). \tag{28}$$

Let $y \in N_{\mathcal{K}^n}(x)$. Take $\Gamma \in \widetilde{M}(y)$. Then, $\Gamma \in \mathcal{S}^n_-$ satisfies $\sum_{i=1}^n \Gamma_{ii} = y_1$ and $\Gamma_{1i} = \frac{1}{2}y_i$ for $i = 2, \dots, n$. Hence,

$$\langle \Gamma, L_x \rangle = \sum_{i=1}^n x_1 \Gamma_{ii} + 2\sum_{i=2}^n x_i \Gamma_{1i} = \sum_{i=1}^n x_i y_i = x^T y = 0,$$

where the last step is due to $y \in N_{\mathcal{K}^n}(x)$. This says $\Gamma \in N_{\mathcal{S}^n_+}(L_x)$, i.e. $\widetilde{M}(N_{\mathcal{K}^n}(x)) \subseteq N_{\mathcal{S}^n_+}(L_x)$.

For the other part, taking $\Gamma \in N_{\mathcal{S}^n_+}(L_x)$, then $-\Gamma \in \mathcal{S}^n_+$, which implies $-M(\Gamma) = M(-\Gamma) \in \mathcal{K}^n$ by [4, Theorem 1], i.e. $M(\Gamma) \in -\mathcal{K}^n$. Note that

$$\langle M(\Gamma), x \rangle = \sum_{i=1}^n \Gamma_{ii} x_1 + \sum_{i=2}^n 2x_i \Gamma_{1i} = \langle \Gamma, L_x \rangle = 0,$$

where the last step is obtained by $\Gamma \in N_{\mathcal{S}^n_+}(L_x)$. In summary, we have $M(\Gamma) \in N_{\mathcal{K}^n}(x)$. This shows $M(N_{\mathcal{S}^n_+}(L_x)) \subseteq N_{\mathcal{K}^n}(x)$.

Conversely, Let $\Gamma \in N_{\mathcal{S}^n_+}(L_x)$. Then, $\Gamma \in \mathcal{S}^n_-$. It follows from (28) that $M(\Gamma) \in N_{\mathcal{K}^n}(x)$; hence, $\Gamma \in \widetilde{M}(M(\Gamma)) \subseteq \widetilde{M}(N_{\mathcal{K}^n}(x))$. For $y \in N_{\mathcal{K}^n}(x)$, we see $\widetilde{M}(y) \in N_{\mathcal{S}^n_+}(L_x)$ by (28). Thus, $y = M(\widetilde{M}(y)) \subseteq M(N_{\mathcal{S}^n_+}(L_x))$.
(b) It follows from part(a) immediately. □

Clearly, M is a singleton mapping, whereas $\widetilde{M}$ is a set-valued mapping. In particular, in the proof of Theorem 6.1, we need to choose an element in $\widetilde{M}$. Below, we present a way to pick an element in $\widetilde{M}$.

Theorem 6.2: *For $x \in \mathcal{K}^n$ and $y \in N_{\mathcal{K}^n}(x)$, let*

$$\widehat{\Gamma} := \begin{cases} \begin{bmatrix} \alpha y_1 & \frac{1}{2}y_2^T \\ \frac{1}{2}y_2 & \beta \frac{1}{y_1}\bar{y}_2\bar{y}_2^T \end{bmatrix} & \text{if } y \neq 0, \\ O & \text{if } y = 0, \end{cases}$$

where

$$\alpha = \frac{1}{2}\left(1 \pm \sqrt{1 - \frac{\|y_2\|^2}{y_1^2}}\right) \text{ and } \beta = \frac{1}{2}\left(1 \mp \sqrt{1 - \frac{\|y_2\|^2}{y_1^2}}\right) y_1^2.$$

Then, $\widehat{\Gamma} \in \widetilde{M}(y)$.

Proof: Note first that $\alpha, \beta \geq 0$ due to $y \in -\mathcal{K}^n$. Since the case in which $y = 0$ is trivial, it suffices to prove the case where $y \neq 0$. Consider the following two subcases.
Case 1: For $y_2 \neq 0$, by a simple calculation, we can reach

$$\alpha\beta = \frac{\|y_2\|^2}{4} \quad \text{and} \quad \beta = (1-\alpha)y_1^2.$$

Using this, we have

$$\sum_{i=1}^n \widehat{\Gamma}_{ii} = \alpha y_1 + \frac{\beta}{y_1} = y_1 \quad \text{and} \quad \widehat{\Gamma}_{1i} = \frac{1}{2}y_i, \ \forall i = 2, 3, \dots, n.$$

Then, it remains to verify $\widehat{\Gamma} \preceq O$, i.e.

$$(u \;\; v^T)\, \widehat{\Gamma} \begin{pmatrix} u \\ v \end{pmatrix} \leq 0 \quad \forall\, (u \;\; v^T) \in \mathbb{R}^n.$$

This can be seen by verifying the following:

$$\begin{aligned}
(u \;\; v^T) \begin{bmatrix} \alpha y_1 & \frac{1}{2} y_2^T \\ \frac{1}{2} y_2 & \beta \frac{1}{y_1} \bar{y}_2 \bar{y}_2^T \end{bmatrix} \begin{pmatrix} u \\ v \end{pmatrix} &= \alpha y_1 u^2 + y_2^T v u + \beta \frac{1}{y_1 \|y_2\|^2} (y_2^T v)^2 \\
&= -(\sqrt{-\alpha y_1} u)^2 + y_2^T v u - \left(\sqrt{-\frac{\beta}{y_1}} \frac{1}{\|y_2\|} y_2^T v \right)^2 \\
&= - \left(\sqrt{-\alpha y_1} u - \sqrt{-\frac{\beta}{y_1}} \frac{1}{\|y_2\|} y_2^T v \right)^2 \\
&\leq 0.
\end{aligned}$$

Case 2: For $y_2 = 0$, we have

$$\widehat{\Gamma} = \begin{bmatrix} \alpha y_1 & 0 \\ 0 & \beta \frac{1}{y_1} \bar{y}_2 \bar{y}_2^T \end{bmatrix}$$

where $\alpha = 0$ and $\beta = y_1^2$ or $\alpha = 1$ and $\beta = 0$. Then, it is clear to see

$$\alpha y_1 + \beta \frac{1}{y_1} = y_1,$$

which indicates

$$\sum_{i=1}^{n} \widehat{\Gamma}_{ii} = y_1 \quad \text{and} \quad \widehat{\Gamma}_{1i} = 0 = \frac{1}{2} y_i, \quad i = 2, 3, \ldots, n.$$

Moreover, in this case, we also have $\widehat{\Gamma} \preceq O$ because

$$\begin{aligned}
(u \;\; v^T)\, \widehat{\Gamma} \begin{pmatrix} u \\ v \end{pmatrix} &= (u \;\; v^T) \begin{bmatrix} \alpha y_1 & 0 \\ 0 & \beta \frac{1}{y_1} \bar{y}_2 \bar{y}_2^T \end{bmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \\
&= \alpha y_1 u^2 + \beta \frac{1}{y_1} \left(\bar{y}_2^T v \right)^2 \\
&\leq 0,
\end{aligned}$$

where the last step follows from $\alpha, \beta \geq 0$ and $y_1 < 0$. □

7. Conclusion

In this paper, we have explored the relation between the SOC and its PSD counterpart in terms of distances, projections, tangent cones, normal cones and the KKT systems. It is known that SOCP and SDP are closely related; for example, SOCP can be regarded as a special case of SDP, and SOCP relaxation provides a nice approach to SDP as mentioned in [16]. The results obtained in this paper help us understand the differences between the SOC and its PSD reformulation better.

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