Comput Optim Appl
DOI 10.1007/s10589-016-9889-y

On merit functions for p-order cone complementarity problem

Xin-He Miao[1] · **Yu-Lin Chang**[2] · **Jein-Shan Chen**[2]

Received: 17 August 2016

Abstract Merit function approach is a popular method to deal with complementarity problems, in which the complementarity problem is recast as an unconstrained minimization via merit function or complementarity function. In this paper, for the complementarity problem associated with p-order cone, which is a type of nonsymmetric cone complementarity problem, we show the readers how to construct merit functions for solving p-order cone complementarity problem. In addition, we study the conditions under which the level sets of the corresponding merit functions are bounded, and we also assert that these merit functions provide an error bound for the p-order cone complementarity problem. These results build up a theoretical basis for the merit method for solving p-order cone complementarity problem.

Keywords p-order cone complementarity problem · Merit function · Error bound

---

X.-H. Miao: The author's work is supported by National Natural Science Foundation of China (No. 11471241).
J.-S. Chen: The author's work is supported by Ministry of Science and Technology, Taiwan.

✉ Jein-Shan Chen
jschen@math.ntnu.edu.tw

Xin-He Miao
xinhemiao@tju.edu.cn

Yu-Lin Chang
ylchang@math.ntnu.edu.tw

1 Department of Mathematics, Tianjin University, Tianjin 300072, China

2 Department of Mathematics, National Taiwan Normal University, Taipei 11677, Taiwan

Published online: 24 December 2016

1 Motivation and introduction

The general conic complementarity problem is to find an element $x \in \mathbb{R}^n$ such that

$$x \in \mathcal{K}, \quad F(x) \in \mathcal{K}^* \quad \text{and} \quad \langle x, F(x) \rangle = 0, \tag{1}$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product, $F : \mathbb{R}^n \to \mathbb{R}^n$ is a continuously differentiable mapping, $\mathcal{K}$ represents a closed convex cone, and $\mathcal{K}^*$ is the dual cone of $\mathcal{K}$ given by $\mathcal{K}^* := \{v \in \mathbb{R}^n \mid \langle v, x \rangle \geq 0, \ \forall x \in \mathcal{K}\}$. When $\mathcal{K}$ is a symmetric cone, the problem (1) is called the symmetric cone complementarity problem [9–11,18,20]. In particular, when $\mathcal{K}$ is the so-called second-order cone which is defined as

$$\mathcal{K}^n := \{(x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1} \mid \|x_2\| \leq x_1\},$$

the problem (1) reduces to the second-order cone complementarity problem [1–5,7,8]. In contrast to symmetric cone programming and symmetric cone complementarity problem, we are not familiar with their nonsymmetric counterparts. Referring the reader to [14,16,19,22] and the bibliographies therein, we observe that there is no unified way to handle nonsymmetric cone constraints, and the study on each item for such problems usually uses certain specific features of the nonsymmetric cones under consideration.

In this paper, we focus on a special nonsymmetric cone $\mathcal{K}$ for problem (1), i.e., p-order cone. Then, the problem (1) reduces to the p-order cone complementarity problem (POCCP for short). Indeed , the p-order cone [17,22] is a generalization of the second-order cone in $\mathbb{R}^n$, denoted by $\mathcal{K}_p$, and can be expressed as

$$\mathcal{K}_p := \left\{ x \in \mathbb{R}^n \,\middle|\, x_1 \geq \left(\sum_{i=2}^{n} |x_i|^p \right)^{\frac{1}{p}} \right\} \quad (p > 1).$$

If we write $x := (x_1, \bar{x}) \in \mathbb{R} \times \mathbb{R}^{n-1}$, the p-order cone $\mathcal{K}_p$ can be equivalently expressed as

$$\mathcal{K}_p = \left\{ x = (x_1, \bar{x}) \in \mathbb{R} \times \mathbb{R}^{n-1} \mid x_1 \geq \|\bar{x}\|_p \right\}, \quad (p > 1).$$

When $p = 2$, it is obvious that the p-order cone is exactly the second-order cone, which means the p-order cone complementarity problem is actually the second-order cone complementarity problem. Thus, the p-order cone complementarity problem (POCCP) can be viewed as the generalization of the second-order cone complementarity problem. As shown in [15,17], $\mathcal{K}_p$ is a convex cone and its dual cone is given by

$$\mathcal{K}_p^* = \left\{ y \in \mathbb{R}^n \,\middle|\, y_1 \geq \left(\sum_{i=2}^{n} |y_i|^q \right)^{\frac{1}{q}} \right\}$$

or equivalently

$$\mathcal{K}_p^* = \left\{ y = (y_1, \bar{y}) \in \mathbb{R} \times \mathbb{R}^{n-1} \mid y_1 \geq \|\bar{y}\|_q \right\} = \mathcal{K}_q,$$

where $q > 1$ and satisfies $\frac{1}{p} + \frac{1}{q} = 1$. In addition, the dual cone $\mathcal{K}_p^*$ is also a convex cone. For more details regarding p-order cone and its involved optimization problems, please refer to [15,17,22].

During the past decade, there had active research and various methods for complementarity problems, which include the interior-point methods, the smoothing Newton methods, the semismooth Newton methods, and the merit function methods, see [1–12,14,21] and references therein. As seen in the literature, almost all the attention was paid to symmetric cone complementarity problems, that is, nonlinear complementarity problem (NCP), positive semi-definite complementarity problem (SDCP), second-order cone complementarity problem (SOCCP). As mentioned earlier, there is no unified framework to deal with general nonsymmetric cone complementarity problems. Consequently, the study about nonsymmetric cone complementarity problem is very limited. Nonetheless, we believe that that merit function approach, in which the complementarity problem is recast as an unconstrained minimization via merit function or complementarity function, may be appropriately viewed as a unified way to deal with nonsymmetric cone complementarity problem. Indeed, the main difficulty lies on how to construct complementarity functions or merit functions for nonsymmetric cone complementarity problem. For circular cone setting, several successful ways were shown in [16]. Inspired by the work [16], we employ the similar ways to construct merit functions for solving p-order cone complementarity problem. For completeness, the idea is roughly described again as below.

Recall that for solving the problem (1), a popular approach is to reformulate it as an unconstrained smooth minimization problem or a system of nonsmooth equations. In this category of methods, it is important to adapt a merit function. A *merit function* for the p-order cone complementarity problem is a function $h : \mathbb{R}^n \to [0, +\infty)$, provided that

$$h(x) = 0 \quad \Longleftrightarrow \quad x \text{ solves the POCCP (1).}$$

Hence, solving the problem (1) is equivalent to handling the unconstrained minimization problem

$$\min_{x \in \mathbb{R}^n} h(x)$$

with the optimal value zero. Until now, for solving symmetric cone complementarity problem, a large number of merit functions have been proposed. Among them, one of the most popular merit functions is the natural residual (NR) merit function $\Psi_{\mathrm{NR}} : \mathbb{R}^n \to \mathbb{R}$, which is defined as

$$\Psi_{\mathrm{NR}}(x) := \frac{1}{2} \left\| \phi_{\mathrm{NR}}(x, F(x)) \right\|^2 = \frac{1}{2} \left\| x - (x - F(x))_+^{\mathcal{K}} \right\|^2,$$

where $(\cdot)_+^{\mathcal{K}}$ denotes the projection onto the symmetric cone $\mathcal{K}$. Then, we know that $\Psi_{\text{NR}}(x) = 0$ if and only if x is a solution to the symmetric cone complementarity problem. As remarked in [16], this function Ψ_{NR} (or ϕ_{NR}) can also serve as merit function (or complementarity function) for general conic complementarity problem. Hence, it is also applicable to p-order cone complementarity problem. Under this setting, for any $x \in \mathbb{R}^n$, we denote x_+ be the projection of x onto the p-order cone $\mathcal{K}_p$, and x_- be the projection of $-x$ onto the dual cone $\mathcal{K}_p^*$ of $\mathcal{K}_p$. By properties of projection onto the closed convex cone, it can be verified that $x = x_+ - x_-$. Moreover, the formula of projection of $x \in \mathbb{R}^n$ onto $\mathcal{K}_p$ is obtained in [17]. Besides the NR merit function Ψ_{NR}, are there any other types of merit functions for POCCP? In this paper, we answer this question by presenting other types of merit functions for the p-order cone complementarity problem. Moreover, we investigate the properties of these proposed merit functions, and study conditions under which these merit functions provide bounded level sets. Note that such properties will guarantee that the sequence generated by descent methods has at least one accumulation point, and build up a theoretical basis for designing the merit function method for solving p-order cone complementarity problem.

2 Preliminaries

In this section, we briefly review some basic concepts and background materials about the p-order cone, and define one type of product associated with p-order cone, which will be extensively used in subsequent analysis.

As mentioned, the p-order cone $\mathcal{K}_p$ is a pointed closed convex cone, and its dual cone denoted by $\mathcal{K}_p^*$ is given as

$$\mathcal{K}_p^* = \left\{ y \in \mathbb{R}^n \,\middle|\, y_1 \geq \left(\sum_{i=2}^{n} |y_i|^q \right)^{\frac{1}{q}} \right\}$$

or equivalently

$$\mathcal{K}_p^* = \left\{ y = (y_1, \bar{y}) \in \mathbb{R} \times \mathbb{R}^{n-1} \mid y_1 \geq \|\bar{y}\|_q \right\} = \mathcal{K}_q,$$

where $q > 1$ and satisfies $\frac{1}{p} + \frac{1}{q} = 1$. From the expression of the dual cone $\mathcal{K}_p^*$, it is easy to know that the dual cone $\mathcal{K}_p^*$ is also a closed convex cone. In addition, when $p \neq q$, we have $\mathcal{K}_p \neq \mathcal{K}_q = \mathcal{K}_p^*$, i.e., the p-order cone $\mathcal{K}_p$ is not a self-dual cone. That is to say, the p-order cone $\mathcal{K}_p$ is not a symmetric cone for $p \neq 2$.

It is well known that Jordan product plays a critical role in the study of symmetric cone programming or symmetric cone complementarity problems. However, there is no Jordan product for the setting of the p-order cone so far. Hence, we need to find one type of special product for the setting of the p-order cone, which is similar to the one for the setting of symmetric cone. To this end, for any $x = (x_1, \cdots, x_n)^T \in \mathbb{R}^n$ and $y = (y_1, \cdots, y_n)^T \in \mathbb{R}^n$, we define one type of product of x and y associated with p-order cone $\mathcal{K}_p$ as follows:

$$x \bullet y = \begin{bmatrix} \langle x, y \rangle \\ w \end{bmatrix} \quad \text{where} \quad w := (w_2, \cdots, w_n)^T \quad \text{with} \quad w_i = |x_1|^{\frac{p}{q}} |y_i| - |y_1| |x_i|^{\frac{p}{q}}. \tag{2}$$

From the above definition (2) of product, when $p = q = 2$, it is not hard to see that the product $x \bullet y$ is exactly the Jordan product in the setting of second-order cone. According to the product "$\bullet$" defined as in (2), we have the following equivalence.

Proposition 2.1 *For any $x = (x_1, \bar{x}) \in \mathbb{R} \times \mathbb{R}^{n-1}$ with $\bar{x} = (x_2, \cdots, x_n)^T \in \mathbb{R}^{n-1}$ and $y = (y_1, \bar{y}) \in \mathbb{R} \times \mathbb{R}^{n-1}$ with $\bar{y} = (y_2, \cdots, y_n)^T \in \mathbb{R}^{n-1}$, the following statements are equivalent:*

(a) $x \in \mathcal{K}_p$, $y \in \mathcal{K}_p^* = \mathcal{K}_q$ and $\langle x, y \rangle = 0$.
(b) $x \in \mathcal{K}_p$, $y \in \mathcal{K}_p^* = \mathcal{K}_q$ and $x \bullet y = 0$.

In each case, x and y satisfy the condition that there is $c \geq 0$ such that $|x_i|^p = c|y_i|^q$ or $|y_i|^q = c|x_i|^p$ for any $i = 2, \cdots, n$.

Proof (b) $\Rightarrow$ (a) From the definition of product $x \bullet y$ of x and y associated with $\mathcal{K}_p$, the implication is obvious.

(a) $\Rightarrow$ (b) When $\bar{x} = 0$ or $\bar{y} = 0$, from (a), we know $x \in \mathcal{K}_p$, $y \in \mathcal{K}_q$ and $\langle x, y \rangle = 0$. Then, it is easy to see that $x \bullet y = 0$. When $\bar{x} \neq 0$ and $\bar{y} \neq 0$, by $x \in \mathcal{K}_p$ and $y \in \mathcal{K}_q$, we have $x_1 \geq \|\bar{x}\|_p$ and $y_1 \geq \|\bar{y}\|_q$. Hence, it follows from $\langle x, y \rangle = 0$ that

$$\begin{aligned} 0 &= \langle x, y \rangle \\ &= x_1 y_1 + \langle \bar{x}, \bar{y} \rangle \\ &\geq \|\bar{x}\|_p \|\bar{y}\|_q - \|\bar{x}\|_p \|\bar{y}\|_q \\ &= 0. \end{aligned}$$

This implies that $x_1 = \|\bar{x}\|_p$, $y_1 = \|\bar{y}\|_q$ and $|x_i|^p = c|y_i|^q$ or $|y_i|^q = c|x_i|^p$ with some $c \geq 0$ for any $i = 2, \cdots, n$. Next, we only consider the case $|x_i|^p = c|y_i|^q$, and the same arguments apply for the case $|y_i|^q = c|x_i|^p$. Because $|x_i|^p = c|y_i|^q$, we have $|x_i|^{\frac{p}{q}} = c^{\frac{1}{q}} |y_i|$. This yields that, for any $i = 2, \cdots, n$,

$$\begin{aligned} |x_1|^{\frac{p}{q}} |y_i| - |y_1| |x_i|^{\frac{p}{q}} &= |x_1|^{\frac{p}{q}} |y_i| - |y_1| c^{\frac{1}{q}} |y_i| \\ &= \left(\sum_{k=2}^{n} |x_k|^p \right)^{\frac{1}{q}} |y_i| - \left(\sum_{k=2}^{n} |y_k|^q \right)^{\frac{1}{q}} c^{\frac{1}{q}} |y_i| \\ &= c^{\frac{1}{q}} \left(\sum_{k=2}^{n} |y_k|^q \right)^{\frac{1}{q}} |y_i| - c^{\frac{1}{q}} \left(\sum_{k=2}^{n} |y_k|^q \right)^{\frac{1}{q}} |y_i| \\ &= 0, \end{aligned}$$

where the second equality holds due to $x_1 = \|\bar{x}\|_p$, $y_1 = \|\bar{y}\|_q$. Then, it follows that $x \bullet y = 0$, and the proof is complete. $\square$

To close this section, we introduce some other concepts that will be needed in subsequent analysis. A function $F : \mathbb{R}^n \to \mathbb{R}^n$ is said to be *monotone* if, for any $x, y \in \mathbb{R}^n$, there holds

$$\langle x - y, F(x) - F(y)\rangle \geq 0;$$

and *strictly monotone* if, for any $x \neq y$, the above inequality holds strictly; and *strongly monotone* with modulus $\rho > 0$ if, for any $x, y \in \mathbb{R}^n$, the following inequality holds

$$\langle x - y, F(x) - F(y)\rangle \geq \rho \|x - y\|^2.$$

The following technical result is crucial for achieving the property of bounded level sets. Although the analysis technique is similar to [16, Lemma 4.1], we present the details for completeness.

Proposition 2.2 *Suppose that the POCCP has a strictly feasible point z, i.e.,* $z \in \text{int}(\mathcal{K}_p)$ *and* $F(z) \in \text{int}(\mathcal{K}_p^*)$ *and that F is a monotone function. Then, for any sequence* $\{x^k\}$ *satisfying*

$$\left\|x^k\right\| \to \infty, \quad \limsup_{k\to\infty} \left\|x_-^k\right\| < \infty \quad \textit{and} \quad \limsup_{k\to\infty} \left\|(-F(x^k))_+\right\| < \infty,$$

we have $\left\langle x^k, F(x^k)\right\rangle \to \infty$.

Proof Since F is monotone, for any $x^k \in \mathbb{R}^n$, we have

$$\left\langle x^k - z, F(x^k) - F(z)\right\rangle \geq 0,$$

which leads to

$$\left\langle x^k, F(x^k)\right\rangle + \langle z, F(z)\rangle \geq \left\langle x^k, F(z)\right\rangle + \left\langle z, F(x^k)\right\rangle. \tag{3}$$

From properties of projection, we write $x^k = x_+^k - x_-^k$ and $F(x^k) = (-F(x^k))_- - (-F(x^k))_+$. Then, it follows from (3) that

$$\begin{aligned}&\left\langle x^k, F(x^k)\right\rangle + \langle z, F(z)\rangle \\ &\quad \geq \left\langle x_+^k, F(z)\right\rangle - \left\langle x_-^k, F(z)\right\rangle + \left\langle z, (-F(x^k))_-\right\rangle - \left\langle z, (-F(x^k))_+\right\rangle. \end{aligned} \tag{4}$$

Now, we denote $x_+^k := \left([x_+^k]_1, \overline{x_+^k}^T\right)^T$ and $F(z) := \left([f(z)]_1, \overline{f(z)}^T\right)^T$. With these notations, we look into the first term on the right-hand side of (4):

$$
\begin{aligned}
\left\langle x_+^k, F(z)\right\rangle &= \left[x_+^k\right]_1 [f(z)]_1 + \left\langle \overline{x_+^k}, \overline{f(z)}\right\rangle \\
&\geq \left[x_+^k\right]_1 [f(z)]_1 - \left\|\overline{x_+^k}\right\|_p \cdot \left\|\overline{f(z)}\right\|_q \\
&\geq \left[x_+^k\right]_1 [f(z)]_1 - \left[x_+^k\right]_1 \left\|\overline{f(z)}\right\|_q \\
&= \left[x_+^k\right]_1 \left([f(z)]_1 - \left\|\overline{f(z)}\right\|_q\right) \\
&\geq 0.
\end{aligned} \tag{5}
$$

Note that $x^k = x_+^k - x_-^k$, which gives $\|x_+^k\| \geq \|x^k\| - \|x_-^k\|$. Using the assumptions on $\{x^k\}$, i.e., $\|x^k\| \to \infty$, and $\limsup_{k\to\infty} \|x_-^k\| < \infty$, we see that $\|x_+^k\| \to \infty$, and hence $[x_+^k]_1 \to \infty$. Because the POCCP has a strictly feasible point z, we know $[f(z)]_1 - \|\overline{f(z)}\|_q > 0$, which together with (5) implies that

$$
\langle x_+^k, F(z)\rangle \to \infty \quad \text{as } k \to \infty. \tag{6}
$$

Moreover, we also observe that

$$
\begin{aligned}
\limsup_{k\to\infty} \langle x_-^k, F(z)\rangle &\leq \limsup_{k\to\infty} \|x_-^k\| \|F(z)\| < \infty, \\
\limsup_{k\to\infty} \langle z, (-F(x^k))_+\rangle &\leq \limsup_{k\to\infty} \|z\| \|(-F(x^k))_+\| < \infty
\end{aligned}
$$

and $\langle z, (-F(x^k))_-\rangle \geq 0$. All of these together with (4) and (6) yield

$$
\left\langle x^k, F(x^k)\right\rangle \to \infty.
$$

Then, the proof is complete. □

3 Merit functions for POCCP

In this section, based on the product (2) of x and y associated with p-order cone in $\mathbb{R}^n$ and employing the same idea in [16], we propose several classes of merit functions for the p-order cone complementarity problem and investigate their favorable properties, respectively.

3.1 The first class of merit functions

In this subsection, we focus on the natural residual (NR) function $\phi_{\text{NR}} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, which is given by:

$$
\phi_{\text{NR}}(x, y) := x - (x - y)_+,
$$

where $(\cdot)_+$ denotes the projection function. We know that the NR function ϕ_{NR} is always an complementarity function for general conic complementarity problem, see

[16] or [8, Proposition 1.5.8]. In light of this, it is clear that the function $\Psi_{\rm NR}(x) = \frac{1}{2}\|\phi_{\rm NR}(x, F(x))\|^2$ serves a merit function for the POCCP.

Lemma 3.1 *Let $x, y \in \mathbb{R}^n$ and $\phi_{\rm NR}(x, y) = x - (x - y)_+$. For any closed convex cone $\mathcal{K}$, we have*

$$\|\phi_{\rm NR}(x, y)\| \geq \max\left\{\|x_-^{\mathcal{K}^*}\|, \|(-y)_+^{\mathcal{K}}\|\right\},$$

where $z_+^{\mathcal{K}}$ denotes the projection of z onto the closed convex cone $\mathcal{K}$, and $z_-^{\mathcal{K}^*}$ means the projection of $-z$ onto its dual cone $\mathcal{K}^*$.

Proof The proof is similar to [16, Lemma 3.2] because the cone therein can be replaced by any closed convex cone. Hence, we omit it here. □

In fact, Lu and Huang [12] considered a more general NR merit function, whose format is as bellow:

$$\Psi_\alpha(x) = \frac{1}{2}\|x - (x - \alpha F(x))_+\|^2, \qquad (\alpha > 0).$$

They also showed the property of error bound under the strong monotonicity and the global Lipschitz continuity of F.

Theorem 3.1 *[12, Theorem 3.3] Suppose that F is strongly monotone with modulus $\rho > 0$ and is Lipschitz continuous with constant $L > 0$, Then for any fixed $\alpha > 0$, the following inequality holds*

$$\frac{1}{2 + \alpha L}\sqrt{\Psi_\alpha(x)} \leq \|x - x^*\| \leq \frac{1 + \alpha L}{\alpha\rho}\sqrt{\Psi_\alpha(x)},$$

where x^* is the unique solution of the generally closed convex cone complementarity problems.

From Theorem 3.1, we know that the NR merit function $\Psi_{\rm NR}$ (i.e., $\alpha = 1$) provides an error bound for the POCCP. Unfortunately, when considering the boundedness of the level set for the NR function $\phi_{\rm NR}$, if under the same conditions used in Proposition 2.2, we cannot guarantee the boundedness of the level set for the function $\phi_{\rm NR}$. For example, as mentioned in [16], taking $F(x) = 1 - \dfrac{1}{x}$ and $x > 0$, it is easy to verify that the level set

$$\mathscr{L}_{\rm NR}(2) = \left\{x \in \mathbb{R}^n \mid \|\phi_{\rm NR}(x, F(x))\| \leq 2\right\}$$

is unbounded. Thus, a different condition is needed. In fact, in order to establish the boundedness of the level set for the natural residual function $\phi_{\rm NR}$ or the merit function Ψ_α, we need the following concept.

Definition 3.1 A mapping $F : \mathbb{R}^n \to \mathbb{R}^n$ is said to be strongly coercive if

$$\lim_{\|x\|\to\infty} \frac{\langle F(x), x - y\rangle}{\|x - y\|} = \infty$$

holds for all $y \in \mathbb{R}^n$.

Theorem 3.2 *Suppose that F is strongly coercive. Then, the level set*

$$\mathscr{L}_{\rm NR}(\gamma) = \left\{x \in \mathbb{R}^n \mid \left\|\phi_{\rm NR}(x, F(x))\right\| \leq \gamma\right\}$$

or

$$\mathscr{L}_{\Psi_\alpha}(\gamma) = \left\{x \in \mathbb{R}^n \mid \Psi_\alpha(x) \leq \gamma\right\}$$

is bounded for all $\gamma \geq 0$.

Proof The proof is similar to [16, Theorem 4.2]. Hence, we omit it. □

3.2 The second class of merit functions

For any $x \in \mathbb{R}^n$, we denote f_{LT} the LT (standing for Luo-Tseng) merit function associated with the p-order cone complementarity problem, whose mathematical formula is given as follows:

$$f_{LT}(x) := \varphi(\langle x, F(x)\rangle) + \frac{1}{2}\left[\|(x)_-\|^2 + \|(-F(x))_+\|^2\right], \tag{7}$$

where $\varphi : \mathbb{R} \to \mathbb{R}_+$ is an arbitrary smooth function satisfying

$$\varphi(t) = 0,\ \forall t \leq 0 \quad \text{and} \quad \varphi'(t) > 0,\ \forall t > 0.$$

It is easy to see that $\varphi(t) \geq 0$ for all $t \in \mathbb{R}$ from the above condition. This class of functions has been considered by Tseng [21]for the positive semidefinite complementarity problem, for the second-order cone complementarity problem by Chen [2], and for the general SCCP case by Pan and Chen [18], respectively. For the setting of general closed convex cone complementarity problems, the LT merit function has also been studied by Lu and Huang [12], with some favorable properties shown as below.

Property 3.1 ([12, Lemma 3.1 and Theorem 3.4]) *Let $f_{LT} : \mathbb{R}^n \to \mathbb{R}$ be given as in (7). Then, the following results hold.*

(a) For all $x \in \mathbb{R}^n$, we have $f_{LT}(x) \geq 0$; and $f_{LT}(x) = 0$ if and only if x solves the p-order cone complementarity problem.

(b) If $F(\cdot)$ is differentiable, then so is $f_{LT}(\cdot)$. Moreover,

$$\nabla f_{LT}(x) = \nabla\varphi(\langle x, F(x)\rangle)[F(x) + x\nabla F(x)] - x_- - \nabla F(x)(-F(x))_+$$

for all $x \in \mathbb{R}^n$.

Property 3.2 ([12, Theorem 3.6]) *Let f_{LT} be given as in (7). Suppose that $F : \mathbb{R}^n \to \mathbb{R}^n$ is a strongly monotone mapping and that the p-order cone complementarity problem has a solution x^*. Then, there exists a constant $\tau > 0$ such that*

$$\tau \|x - x^*\|^2 \leq \max\{0, \langle x, F(x)\rangle\} + \|x_-\| + \|(-F(x))_+\|, \quad \forall x \in \mathbb{R}^n.$$

Moreover,

$$\tau \|x - x^*\|^2 \leq \varphi^{-1}(f_{LT}(x)) + 2[f_{LT}(x)]^{\frac{1}{2}}, \quad \forall x \in \mathbb{R}^n.$$

Although the above properties were established in [12] for general closed convex cone setting, there is no study about bounded level set for f_{LT} therein. Hence, we hereby present the condition which ensures the boundedness of level sets for the LT merit function f_{LT} to solve the p-order cone complementarity problem.

Theorem 3.3 *Suppose that the p-order cone complementarity problem has a strictly feasible point and that F is monotone. Then, the level set*

$$\mathscr{L}_{f_{LT}}(\gamma) := \{x \in \mathbb{R}^n \mid f_{LT}(x) \leq \gamma\}$$

is bounded for all $\gamma \geq 0$.

Proof We prove this result by contradiction. Suppose there exists an unbounded sequence $\{x^k\} \subseteq \mathscr{L}_{f_{LT}}(\gamma)$ for some $\gamma \geq 0$. Then, the sequences $\{x^k_-\}$ and $\{(-F(x^k))_+\}$ must be bounded. If not, from the expression (7) of f_{LT} and the property $\varphi(t) \geq 0$ for all $t \in \mathbb{R}$, it follows that

$$f_{LT}(x^k) \geq \frac{1}{2}\left[\|x^k_-\|^2 + \|(-F(x^k))_+\|^2\right] \to \infty,$$

which contradicts $\{x^k\} \subseteq \mathscr{L}_{f_{LT}}(\gamma)$. Hence, we have

$$\limsup_{k\to\infty} \|x^k_-\| < \infty \quad \text{and} \quad \limsup_{k\to\infty} \|(-F(x^k))_+\| < \infty.$$

Then, applying Proposition 2.2 yields

$$\langle x^k, F(x^k)\rangle \to \infty.$$

Using the properties of the function φ again, we have $\varphi(\langle x^k, F(x^k)\rangle) \to \infty$, which leads to $f_{LT}(x^k) \to \infty$. It contradicts $\{x^k\} \subseteq \mathscr{L}_{f_{LT}}(\gamma)$. Thus, the level set $\mathscr{L}_{f_{LT}}(\gamma)$ is bounded for all $\gamma \geq 0$ and the proof is complete. □

3.3 The third class of merit functions

Motivated by the construction way of the merit function f_{LT}, we make a slight modification on the LT merit function f_{LT} associated with the p-order cone complementarity problem, which leads to the third class of merit functions. More specifically, we first look into the set $\Omega := \mathcal{K}_p \cap \mathcal{K}_p^*$. Indeed, the set Ω is characterized as follows:

$$\Omega := \mathcal{K}_p \cap \mathcal{K}_p^* = \begin{cases} \mathcal{K}_p & \text{for } 1 \le p \le 2, \\ \mathcal{K}_p^* = \mathcal{K}_q & \text{for } p \ge 2, \end{cases}$$

where q satisfies the condition $q \ge 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Moreover, it is easy to check that Ω is also a closed convex cone. In light of this closed convex cone Ω, another function is considered:

$$\widehat{f_{LT}}(x) := \frac{1}{2}\|(x \bullet F(x))_+^{\Omega}\|^2 + \frac{1}{2}\left[\|x_-\|^2 + \|(-F(x))_+\|^2\right], \tag{8}$$

where $(x \bullet y)_+^{\Omega}$ denotes the projection of $x \bullet y$ onto Ω. As shown in the following theorem, we see that the function $\widehat{f_{LT}}$ is also a type of merit functions for the p-order cone complementarity problem.

Theorem 3.4 *Let the function $\widehat{f_{LT}}$ be given as in (8). Then, for all $x \in \mathbb{R}^n$, we have*

$$\widehat{f_{LT}}(x) = 0 \iff x \in \mathcal{K}_p, \quad F(x) \in \mathcal{K}_p^* \text{ and } \langle x, F(x)\rangle = 0,$$

where $\mathcal{K}_p^*$ denotes the dual cone of $\mathcal{K}_p$, i.e., $\mathcal{K}_p^* = \mathcal{K}_q$ with $p, q \ge 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof From the definition of the function $\widehat{f_{LT}}$ given in (8), we have

$$\begin{aligned} \widehat{f_{LT}}(x) = 0 &\iff \|(x \bullet F(x))_+^{\Omega}\| = 0, \quad \|x_-\| = 0 \text{ and } \|(-F(x))_+\| = 0, \\ &\iff (x \bullet F(x))_+^{\Omega} = 0, \quad x_- = 0 \text{ and } (-F(x))_+ = 0, \\ &\iff x \bullet F(x) \in -\mathcal{K}_p \ \left(\text{or } x \bullet F(x) \in -\mathcal{K}_p^*\right), \quad x \in \mathcal{K}_p \\ &\qquad \text{and } F(x) \in \mathcal{K}_p^*, \\ &\iff -x \bullet F(x) \in \mathcal{K}_p \ \left(\text{or } -x \bullet F(x) \in \mathcal{K}_p^*\right), \quad x \in \mathcal{K}_p \\ &\qquad \text{and } F(x) \in \mathcal{K}_p^*, \\ &\iff x \in \mathcal{K}_p, \quad F(x) \in \mathcal{K}_p^* \\ &\qquad \text{and } \langle x, F(x)\rangle = 0, \end{aligned}$$

where the last equivalence holds due to the properties of $\mathcal{K}_p$ and $\mathcal{K}_p^*$. Thus, the proof is complete. □

In the following, we investigate the error bound property and the boundedness property of level sets of the merit function $\widehat{f_{LT}}$ for the p-order cone complementarity problem. In order to achieve these results, we need a novel lemma as below.

Lemma 3.2 *For any $x, y \in \mathbb{R}^n$, we have*

$$\langle x, y \rangle \leq \left\| (x \bullet y)_+^{\Omega} \right\|,$$

where the product $x \bullet y$ is defined as in (2).

Proof Given any $x, y \in \mathbb{R}^n$, let $x = (x_1, \cdots, x_n)^T$ and $y = (y_1, \cdots, y_n)^T$. Recall from the product (2), we know

$$x \bullet y = \begin{bmatrix} \langle x, y \rangle \\ w \end{bmatrix} \quad \text{where} \quad w := \left(|x_1|^{\frac{p}{q}} |y_i| - |y_1||x_i|^{\frac{p}{q}} \right)_{i=2}^{n}.$$

To proceed the arguments, we consider the following three cases.

Case 1 When $x \bullet y \in \Omega$, we have $(x \bullet y)_+^{\Omega} = x \bullet y$. Then, it is easy to verify that

$$\left\| (x \bullet y)_+^{\Omega} \right\| \geq \langle x, y \rangle.$$

Case 2 When $x \bullet y \in -\Omega^*$, where Ω^* denotes the dual cone of Ω, we have $(x \bullet y)_+^{\Omega} = 0$ and $\langle x, y \rangle \leq 0$. This clearly implies that

$$\left\| (x \bullet y)_+^{\Omega} \right\| \geq \langle x, y \rangle.$$

Case 3 When $x \bullet y \notin \Omega \cup (-\Omega^*)$, let $(x \bullet y)_+^{\Omega} := (v_1, \bar{v}^T)^T$. If $\langle x, y \rangle \leq 0$, then the result is obvious. Thus, we only need to look into the case of $\langle x, y \rangle > 0$. If $\Omega = \mathcal{K}_p$, by the property of projection onto the p-order cone, we have

$$(x \bullet y)_+^{\Omega} - x \bullet y = \begin{bmatrix} v_1 - \langle x, y \rangle \\ \bar{v} - w \end{bmatrix} \in \Omega^* = \mathcal{K}_p^* = \mathcal{K}_q.$$

From the definition of dual order cone $\mathcal{K}_p^*$ again, it follows that $v_1 - \langle x, y \rangle \geq 0$, i.e., $v_1 \geq \langle x, y \rangle$. Hence, this yields that

$$\left\| (x \bullet y)_+^{\Omega} \right\| \geq |v_1| \geq v_1 \geq \langle x, y \rangle.$$

With similar arguments, for the case of $\Omega = \mathcal{K}_p^* = \mathcal{K}_q$, we also obtain that

$$\langle x, y \rangle \leq \left\| (x \bullet y)_+^{\Omega} \right\|.$$

From all the above cases, we have shown that $\langle x, y \rangle \leq \left\| (x \bullet y)_+^{\Omega} \right\|$. Thus, the proof is complete. □

Theorem 3.5 *Let the function $\widehat{f_{LT}}$ be given as in (8). Suppose that $F : \mathbb{R}^n \to \mathbb{R}^n$ is strongly monotone mapping and that x^* is a solution to the p-order cone complementarity problem. Then, there exists a scalar $\tau > 0$ such that*

$$\tau \|x - x^*\|^2 \leq \left(2 + \sqrt{2}\right) \left[\widehat{f_{LT}}(x) \right]^{\frac{1}{2}}.$$

Proof Since the function F is strongly monotone and x^* is a solution to the p-order cone complementarity problem, there exists a scalar $\rho > 0$ such that, for any $x \in \mathbb{R}^n$,

$$\begin{aligned}
\rho \|x - x^*\|^2 &\le \langle F(x) - F(x^*), x - x^* \rangle \\
&= \langle F(x), x \rangle + \langle F(x^*), -x \rangle + \left\langle -F(x), x^* \right\rangle \\
&= \langle F(x), x \rangle + \langle F(x^*), x_- - x_+ \rangle + \left\langle (-F(x))_+ - (-F(x))_-, x^* \right\rangle \\
&\le \langle F(x), x \rangle + \left\langle F(x^*), x_- \right\rangle + \left\langle (-F(x))_+, x^* \right\rangle \\
&\le \left\| (x \bullet F(x))_+^{\Omega} \right\| + \|x_-\| \|F(x^*)\| + \left\| x^* \right\| \|(-F(x))_+\| \\
&\le \max \left\{1, \|F(x^*)\|, \|x^*\| \right\} \left(\left\| (x \bullet F(x))_+^{\Omega} \right\| + \|x_-\| + \|(-F(x))_+\| \right),
\end{aligned}$$

where the second inequality holds due to the properties of $\mathcal{K}_p$ and $\mathcal{K}_p^*$, and the third inequality follows from Lemma 3.2. Then, setting $\tau := \dfrac{\rho}{\max\{1, \|F(x^*)\|, \|x^*\|\}}$ yields

$$\tau \left\| x - x^* \right\|^2 \le \left\| (x \bullet F(x))_+^{\Omega} \right\| + \|x_-\| + \|(-F(x))_+\|.$$

Moreover, we observe that

$$\left\| (x \bullet F(x))_+^{\Omega} \right\| = \sqrt{2} \left(\frac{1}{2} \|(x \bullet F(x))_+^{\Omega}\|^2 \right)^{\frac{1}{2}} \le \sqrt{2} \left[\widehat{f_{LT}}(x) \right]^{\frac{1}{2}},$$

and

$$\|x_-\| + \|(-F(x))_+\| \le \sqrt{2} \left(\|x_-\|^2 + \|(-F(x))_+\|^2 \right)^{\frac{1}{2}} \le 2 \left[\widehat{f_{LT}}(x) \right]^{\frac{1}{2}}.$$

Putting all the above together gives

$$\tau \|x - x^*\|^2 \le (2 + \sqrt{2}) \left[\widehat{f_{LT}}(x) \right]^{\frac{1}{2}},$$

which is the desired result. □

Next, we study the boundedness of level sets of merit function $\widehat{f_{LT}}$.

Theorem 3.6 *Let the merit function $\widehat{f_{LT}}$ be given as in (8). Suppose that the p-order cone complementarity problem has a strictly feasible point and that F is monotone. Then, the level set*

$$\mathscr{L}_{\widehat{f_{LT}}}(\gamma) = \left\{ x \in \mathbb{R}^n \,\middle|\, \widehat{f_{LT}}(x) \le \gamma \right\}$$

is bounded for all $\gamma \ge 0$.

Proof Like the proof of Theorem 3.3, we prove this result by contradiction. Suppose there exists an unbounded sequence $\{x^k\} \subseteq \mathscr{L}_{\widehat{f_{LT}}}(\gamma)$ for some $\gamma \geq 0$. We claim that the sequences $\{x_-^k\}$ and $\{(-F(x^k))_+\}$ are bounded. If not, by the expression (8) of $\widehat{f_{LT}}$, we obtain

$$\widehat{f_{LT}}(x^k) \geq \frac{1}{2}\left[\|x_-^k\|^2 + \|(-F(x^k))_+\|^2\right] \to \infty,$$

which contradicts $\{x^k\} \subseteq \mathscr{L}_{\widehat{f_{LT}}}(\gamma)$. Therefore, it follows that

$$\limsup_{k\to\infty} \|x_-^k\| < \infty \quad \text{and} \quad \limsup_{k\to\infty} \|(-F(x^k))_+\| < \infty.$$

Then, applying Proposition 2.2 yields $\langle x^k, F(x^k)\rangle \to \infty$. This together with Lemma 3.2 implies

$$\left\|(x^k \bullet F(x^k))_+^{\Omega}\right\| \geq \left\langle x^k, F(x^k)\right\rangle \to \infty,$$

which leads to $\widehat{f_{LT}}(x^k) \to \infty$. This clearly contradicts $\{x^k\} \subseteq \mathscr{L}_{\widehat{f_{LT}}}(\gamma)$. Hence, the level set $\mathscr{L}_{\widehat{f_{LT}}}(\gamma)$ is bounded and the proof is complete. □

Remark 3.1 In fact, if the term $\left(x^k \bullet F(x^k)\right)_+^{\Omega}$ in the expression of $\widehat{f_{LT}}$ is replaced by $x^k \bullet F(x^k)$, all Theorem 3.4, Lemma 3.2, Theorems 3.5 and 3.6 still hold.

3.4 The fourth class of merit function

In this subsection, in light of the product $x \bullet y$ and the NR merit function Ψ_{NR}, we consider another merit function as below:

$$f_r(x) := \frac{1}{2}\left\|\phi_{\mathrm{NR}}(x, F(x))\right\|^2 + \frac{1}{2}\left\|(x \bullet F(x))_+^{\Omega}\right\|^2, \tag{9}$$

where $(x \bullet y)_+^{\Omega}$ denotes the projection of $x \bullet y$ onto Ω. As seen below, we verify that $f_r(x)$ is also a merit function for the p-order cone complementarity problem.

Theorem 3.7 *Let the function f_r be given as in (9). Then, for all $x \in \mathbb{R}^n$, we have*

$$f_r(x) = 0 \iff x \in \mathcal{K}_p, \quad F(x) \in \mathcal{K}_p^* \text{ and } \langle x, F(x)\rangle = 0,$$

where $\mathcal{K}_p^*$ denotes the dual cone of $\mathcal{K}_p$, i.e., $\mathcal{K}_p^* = \mathcal{K}_q$.

Proof In view of the definition of f_r given as in (9), we have

$$\begin{aligned} f_r(x) = 0 &\iff \left\|(x \bullet F(x))_+^{\Omega}\right\|^2 = 0 \text{ and } \Psi_{\mathrm{NR}}(x) = \frac{1}{2}\left\|\phi_{\mathrm{NR}}(x, F(x))\right\|^2 = 0, \\ &\iff x \in \mathcal{K}_p, \ F(x) \in \mathcal{K}_p^* \text{ and } \langle x, F(x)\rangle = 0, \end{aligned}$$

where the second equivalence holds because Ψ_{NR} is a merit function for p-order cone complementarity problem. Thus, the proof is complete. □

From Theorem 3.7, we see that if the squared exponent in the expression of f_r is deleted, i.e.,

$$\widetilde{f}_r(x) := \left\|\phi_{\text{NR}}(x, F(x))\right\| + \left\|(x \bullet F(x))_+^{\Omega}\right\|, \tag{10}$$

then $\widetilde{f}_r$ is also a merit function for the POCCP. In fact, for these two merit functions f_r and $\widetilde{f}_r$, there has no big differences between them in addition to the nature that f_r is better than $\widetilde{f}_r$. Next, we will establish the error bound properties for f_r and $\widetilde{f}_r$.

Theorem 3.8 *Let f_r and $\widetilde{f}_r$ be given as in (9) and (10), respectively. Suppose that $F : \mathbb{R}^n \to \mathbb{R}^n$ is strongly monotone mapping and that x^* is a solution to the p-order cone complementarity problem. Then, there exists a scalar $\tau > 0$ such that*

$$\tau\|x - x^*\|^2 \leq 3\sqrt{2}\big[f_r(x)\big]^{\frac{1}{2}} \quad \textit{and} \quad \tau\|x - x^*\|^2 \leq 2\widetilde{f}_r(x).$$

Proof From Lemma 3.1, we know

$$\|\phi_{\text{NR}}(x, F(x))\| \geq \max\{\|x_-\|, \|(-F(x))_+\|\}.$$

Then, following similar arguments as in Theorem 3.5, we have

$$\begin{aligned}
\tau\|x - x^*\|^2 &\leq \left\|(x \bullet F(x))_+^{\Omega}\right\| + \|x_-\| + \|(-F(x))_+\| \\
&\leq \sqrt{2}(f_r(x))^{\frac{1}{2}} + 2\|\phi_{\text{NR}}(x, F(x))\| \\
&= \sqrt{2}(f_r(x))^{\frac{1}{2}} + 2\sqrt{2}\left(\frac{1}{2}\|\phi_{\text{NR}}(x, F(x))\|^2\right)^{\frac{1}{2}} \\
&\leq \sqrt{2}(f_r(x))^{\frac{1}{2}} + 2\sqrt{2}(f_r(x))^{\frac{1}{2}} \\
&= 3\sqrt{2}\left[f_r(x)\right]^{\frac{1}{2}}
\end{aligned}$$

and

$$\begin{aligned}
\tau\|x - x^*\|^2 &\leq \left\|(x \bullet F(x))_+^{\Omega}\right\| + \|x_-\| + \|(-F(x))_+\| \\
&\leq \left\|(x \bullet F(x))_+^{\Omega}\right\| + 2\left\|\phi_{\text{NR}}(x, F(x))\right\| \\
&\leq 2\widetilde{f}_r(x),
\end{aligned}$$

where $\tau := \dfrac{\rho}{\max\{1, \|F(x^*)\|, \|x^*\|\}}$. Thus, the proof is complete. □

The following theorem presents the boundedness of the level sets of the functions $\widetilde{f}_r$ and f_r.

Theorem 3.9 *Let f_r and $\widetilde{f}_r$ be given as in (9) and (10), respectively. Suppose that that the p-order cone complementarity problem has a strictly feasible point and that F is monotone. Then, the level sets*

$$\mathscr{L}_{f_r}(\gamma) = \left\{x \in \mathbb{R}^n \mid f_r(x) \leq \gamma\right\}$$

and

$$\mathscr{L}_{\widetilde{f}_r}(\gamma) = \left\{x \in \mathbb{R}^n \,\middle|\, \widetilde{f}_r(x) \leq \gamma\right\}$$

are both bounded for all $\gamma \geq 0$.

Proof Here we only show the boundedness of the level sets of the function $\widetilde{f}_r$ for all $\gamma \geq 0$ because the same arguments can be easily applied to the case of f_r.
As the proof in Theorems 3.3 and 3.6, we prove this result by contradiction. Suppose there exists an unbounded sequence $\{x^k\} \subset \mathscr{L}_{\widetilde{f}_r}(\gamma)$ for some $\gamma \geq 0$. If $\|x_-^k\| \to \infty$ or $\left\|(-F(x^k))_+\right\| \to \infty$, by Lemma 3.1, we know

$$\widetilde{f}_r(x^k) \geq \left\|\phi_{\text{NR}}(x^k, F(x^k))\right\| \to \infty,$$

which contradicts $x^k \in \mathscr{L}_{\widetilde{f}_r}(\gamma)$. Hence, we have

$$\limsup_{k\to\infty} \left\|x_-^k\right\| < \infty \quad \text{and} \quad \limsup_{k\to\infty} \left\|(-F(x^k))_+\right\| < \infty.$$

Then, applying Proposition 2.2 yields $\langle x^k, F(x^k)\rangle \to \infty$. This together with Lemma 3.2 gives

$$\left\langle x^k, F(x^k)\right\rangle \leq \left\|(x^k \bullet F(x^k))_+^{\Omega}\right\| \to \infty,$$

which leads to $\widetilde{f}_r(x^k) \to \infty$. This is a contradiction because $\widetilde{f}_r(x^k) \leq \gamma$. Thus, the proof is complete. □

Remark 3.2 As Remark 3.1, if the term $\left(x^k \bullet F(x^k)\right)_+^{\Omega}$ in the expressions of f_r and $\widetilde{f}_r$ is replaced by $x^k \bullet F(x^k)$, all Theorems 3.7, 3.8, and 3.9 still hold.

3.5 The fifth class of merit functions

In this subsection, we introduce the implicit Lagrangian merit associated with the POCCOP. For any $x \in \mathbb{R}^n$ and $\alpha > 0$, the implicit Lagrangian merit function is given by

$$\begin{aligned} M_\alpha(x) := \langle x, F(x)\rangle + \frac{1}{2\alpha}\Big\{&\|(x - \alpha F(x))_+\|^2 - \|x\|^2 \\ &+ \|(\alpha x - F(x))_-\|^2 - \|F(x)\|^2\Big\}. \end{aligned} \tag{11}$$

This class of functions was first introduced by Mangasarian and Solodov [13] for solving nonlinear complementarity problems, and was extended by Kong et al. [11] to the setting of symmetric cone complementarity problems. Moreover, for the setting of general closed convex cone complementarity problems in Hilbert space, Lu and Huang [12] further investigated this merit function. Accordingly, the corresponding results in [12] can be applied to the the setting of POCCP. For completeness, as below, the error bound property of the merit function M_α for the POCCP is also presented.

Property 3.3 ([12, Theorem 3.9]) *Let M_α be given as in (11). Suppose that $F:\mathbb{R}^n\to\mathbb{R}^n$ is a strongly monotone mapping with modulus $\rho>0$ and is Lipschitz continuous with $L>0$. Assume that the p-order cone complementarity problem has a solution x^*. Then, for any fixed $\alpha>0$, the following inequality holds*

$$\frac{1}{(\alpha-1)(2+L)^2}\,M_\alpha(x)\le\|x-x^*\|\le\frac{\alpha(1+L)^2}{(\alpha-1)\rho^2}\,M_\alpha(x).$$

In the following theorem, we present the boundedness property of the level sets on the merit function M_α for solving the p-order cone complementarity problem.

Theorem 3.10 *Suppose that the p-order cone complementarity problem has a strictly feasible point and that F is monotone. Then, the level set*

$$\mathscr{L}_{M_\alpha}(\gamma):=\left\{x\in\mathbb{R}^n\mid M_\alpha(x)\le\gamma\right\}$$

is bounded for all $\gamma\ge 0$.

Proof First, we note that

$$\begin{aligned}
M_\alpha(x)&=\langle x,F(x)\rangle+\frac{1}{2\alpha}\Big\{\|(x-\alpha F(x))_+\|^2-\|x\|^2\\
&\quad+\|(\alpha x-F(x))_-\|^2-\|F(x)\|^2\Big\}\\
&=\langle x,F(x)\rangle+\frac{1}{\alpha}\langle x-\alpha F(x),(x-\alpha F(x))_+\rangle\\
&\quad-\frac{1}{2\alpha}\Big\{\|(x-\alpha F(x))_+\|^2+\|x\|^2\Big\}\\
&\quad+\left\langle F(x),\frac{1}{\alpha}[\alpha x-F(x)+(\alpha x-F(x))_-]-x\right\rangle\\
&\quad+\frac{1}{2\alpha}\Big\{\|\alpha x-F(x)+(\alpha x-F(x))_--\alpha x\|^2\Big\}\\
&=-\langle F(x),(x-\alpha F(x))_+-x\rangle-\frac{1}{2\alpha}\,\|(x-\alpha F(x))_+-x\|^2\\
&\quad+\left\langle F(x),\frac{1}{\alpha}(\alpha x-F(x))_+-x\right\rangle+\frac{\alpha}{2}\left\|\frac{1}{\alpha}(\alpha x-F(x))_+-x\right\|^2\\
&\ge\frac{1}{\alpha}\left[-\left\langle\alpha F(x),\frac{1}{\alpha}(\alpha x-F(x))_+-x\right\rangle-\frac{1}{2}\|\frac{1}{\alpha}(\alpha x-F(x))_+-x\|^2\right]
\end{aligned}$$

$$+\alpha\left[\left\langle\frac{1}{\alpha}F(x),\frac{1}{\alpha}(\alpha x-F(x))_+-x\right\rangle+\frac{\alpha}{2}\|\frac{1}{\alpha}(\alpha x-F(x))_+-x\|^2\right]$$
$$=\frac{\alpha^2-1}{2\alpha}\left\|\frac{1}{\alpha}(\alpha x-F(x))_+-x\right\|^2$$
$$=\frac{\alpha^2-1}{2\alpha}\left\|(x-\frac{1}{\alpha}F(x))_+-x\right\|^2.$$

From Theorem 3.2, we know that the level set $\mathscr{L}_{\Psi_\alpha}(\gamma)$ of the general NR merit function Ψ_α is bounded for all $\gamma \geq 0$. With this, it is easy to see that the level set $\mathscr{L}_{M_\alpha}(\gamma)$ is bounded for all $\gamma \geq 0$. Then, the proof is complete. □

4 Conclusion and future direction

Although the p-order cone complementarity problem belongs to nonsymmetric cone complementarity problem, for which there is no unified framework, we believe that the merit function approach may be an appropriate method that can be extended from symmetric cone complementarity problem to nonsymmetric cone complementarity problem. The key point to do such extension is constructing merit functions. Hence, in this paper, we present how to construct merit functions (by defining a new product) for the p-order cone complementarity problem. In addition, we have also shown under what conditions these merit functions have properties of error bounds and bounded level sets. These results provide a theoretical basis for designing the merit function method for solving the special nonsymmetric cone complementarity problem, i.e., p-order cone complementarity problem. We leave this topic as our future direction. At last, we point out that the main idea is employed from [16] (which is for circular cone setting) and most analysis techniques look similar to those used in [16]. Nonetheless, the product is novel which contributes to the literature by providing a new way to deal with such complementarity problem.

References

1. Chen, J.-S.: A new merit function and its related properties for the second-order cone complementarity problem. Pac. J. Optim. **2**, 167–179 (2006)
2. Chen, J.-S.: Two classes of merit functions for the second-order cone complementarity problem. Math. Methods Oper. Res. **64**, 495–519 (2006)
3. Chen, J.-S.: Conditions for error bounds and bounded level sets of some merit functions for the second-order cone complementarity problem. J. Optim. Theory Appl. **135**, 459–473 (2007)
4. Chen, X.D., Sun, D., Sun, J.: Complementarity functions and numerical experiments on some smoothing Newton methods for second-order cone complementarity problems. Comput. Optim. Appl. **25**, 39–56 (2003)
5. Chen, J.-S., Tseng, P.: An unconstrained smooth minimization reformulation of second-order cone complementarity problem. Math. Program. **104**, 293–327 (2005)
6. Chen, J.-S., Pan, S.-H.: A survey on SOC complementarity functions and solution methods for SOCPs and SOCCPs. Pac. J. Optim. **8**, 33–74 (2012)
7. Fukushima, M., Luo, Z.-Q., Tseng, P.: Smoothing functions for second- order-cone complimentarity problems. SIAM J. Optim. **12**, 436–460 (2002)

8. Facchinei, F., Pang, J.-S.: Finite-Dimensional Variational Inequalities and Complementarity Problems, vol. I. Springer, New York (2003)
9. Han, D.R.: On the coerciveness of some merit functions for complimentarity problems over symmetric cones. J. Math. Anal. Appl. **336**, 727–737 (2007)
10. Kong, L., Sun, J., Xiu, N.: A regularized smoothing Newton method for symmetric cone complementarity problems. SIAM J. Optim. **19**, 1028–1047 (2008)
11. Kong, L., Tuncel, T., Xiu, N.: Vector-valued implicit Lagrangian for symmetric cone complementarity problems. Asia-Pac. J. Oper. Res. **26**, 199–233 (2009)
12. Lu, N., Huang, Z.-H.: Three classes of merit functions for the complementarity problem over a closed convex cone. Optimization **62**, 545–560 (2013)
13. Mangasarian, O.L., Solodov, M.V.: Nonlinear complementarity as unconstrained and constrained minimization. Math Program **62**, 277–297 (1993)
14. Matsukawa, Y., Yoshise, A.: A primal barrier function Phase I algorithm for nonsymmetric conic optimization problems. Jpn. J. Ind. Appl. Math. **29**, 499–517 (2012)
15. Miao, X.-H., Chen, J.-S.: Characterizations of solution sets of cone-constrained convex programming problems. Optim. Lett. **9**, 1433–1445 (2015)
16. Miao, X.-H., Guo, S.-J., Qi, N., Chen, J.-S.: Constructions of complementarity functions and merit functions for circular cone complementarity problem. Comput. Optim. Appl. **63**, 495–522 (2016)
17. Miao, X.H., Qi, N., Chen, J.-S.: Projection formula and one type of spectral factorization associated with p-order cone, to appear in Journal of Nonlinear and Convex Analysis, (2016)
18. Pan, S.-H., Chen, J.-S.: A one-parametric class of merit functions for the symmetric cone complementarity problem. J. Math. Anal. Appl. **355**, 195–215 (2009)
19. Skajaa, A., Ye, Y.: A homogeneous interior-point algorithm for nonsymmetric convex conic optimization. Math. Program. **150**, 391–422 (2015)
20. Sun, D., Sun, J.: Löwner's operator and spectral functions in Euclidean Jordan algebras. Math. Oper. Res. **33**, 421–445 (2008)
21. Tseng, P.: Merit functions for semi-definite complementarity problems. Math. Program. **83**, 159–185 (1998)
22. Xue, G.-L., Ye, Y.-Y.: An efficient algorithm for minimizing a sum of p-norm. SIAM J. Optim. **10**, 551–579 (2000)