

NON-LERFNESS OF ARITHMETIC HYPERBOLIC MANIFOLD GROUPS AND MIXED 3-MANIFOLD GROUPS

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Abstract

We will show that for any noncompact arithmetic hyperbolic m -manifold with $m > 3$, and any compact arithmetic hyperbolic m -manifold with $m > 4$ that is not a 7-dimensional one defined by octonions, its fundamental group is not locally extended residually finite (LERF). The main ingredient in the proof is a study on abelian amalgamations of hyperbolic 3-manifold groups. We will also show that a compact orientable irreducible 3-manifold with empty or tori boundary supports a geometric structure if and only if its fundamental group is LERF.

1. Introduction

For a group G and a subgroup $H < G$, we say that H is *separable* in G if for any $g \in G \setminus H$ there exists a finite-index subgroup $G' < G$ such that $H < G'$ and $g \notin G'$. Here, G is called *LERF* (locally extended residually finite) or *subgroup separable* if all finitely generated subgroups of G are separable.

The LERFness of a group is a property closely related with low-dimensional topology, especially the virtual Haken conjecture (settled in [3]). In this paper, we are mostly interested in fundamental groups of some nice manifolds and graphs of groups constructed from these groups.

Among fundamental groups of low-dimensional manifolds, the following groups are known to be LERF: free groups (see [19]), surface groups (see [36]), Seifert manifold groups (see [36]), and hyperbolic 3-manifolds groups (see [3] and [43]); while the following groups are known to be non-LERF: groups of nontrivial graph manifolds (see [30]), and groups of fibered 3-manifolds whose monodromy is reducible and satisfies some further condition (see [24]).

In this paper, we give more examples of non-LERF groups arising from topology. These results imply that 3-manifolds with LERF fundamental groups support

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geometric structures, and it seems that hyperbolic manifolds with LERF fundamental groups have dimension at most 3.

One main result of this paper is about high-dimensional arithmetic hyperbolic manifolds (with dimension at least 4). Comparing to 3-dimensional case, there are much fewer examples of hyperbolic manifolds with dimension at least 4. Many examples of high-dimensional hyperbolic manifolds are constructed by arithmetic methods, and some other examples are constructed by doing cut-and-paste surgery on these arithmetic examples. So the following results suggest that having a non-LERF fundamental group is a general phenomenon in a high-dimensional hyperbolic world.

THEOREM 1.1

Let M^m be an arithmetic hyperbolic manifold with $m \geq 5$ which is not a 7-dimensional arithmetic hyperbolic manifold defined by octonions. Then its fundamental group is not LERF.

Moreover, if M is closed, then there exists a nonseparable subgroup isomorphic to a free product of surface groups and free groups. If M is not closed, there exists a nonseparable subgroup that is isomorphic to either a free subgroup, or a free product of surface groups and free groups.

Comparing with Theorem 1.1, it is shown in [6] that all geometrically finite subgroups of standard arithmetic hyperbolic manifold groups are separable. It will be easy to see that nonseparable subgroups constructed in the proof of Theorem 1.1 are not geometrically finite (see Remark 5.1).

Theorem 1.1 does not cover the case of arithmetic hyperbolic 4-manifolds. By using a slightly different method in Theorem 1.2, we show that noncompact arithmetic hyperbolic manifolds with dimension at least 4 have non-LERF fundamental groups. Of course, the only case in Theorem 1.2 that is not covered by Theorem 1.1 is the 4-dimensional case.

Note that in the more recent work [38], it is proved that all closed arithmetic hyperbolic 4-manifolds also have non-LERF fundamental groups. So, with possible exceptions in 7-dimensional arithmetic hyperbolic manifolds defined by octonions, all arithmetic hyperbolic manifolds with dimension at least 4 have non-LERF fundamental groups.

THEOREM 1.2

Let M^m be a noncompact arithmetic hyperbolic m -manifold with $m \geq 4$. Then $\pi_1(M)$ is not LERF.

Moreover, there exist a nonseparable subgroup isomorphic to a free group and another nonseparable subgroup isomorphic to a surface group.

Some examples of high-dimensional nonarithmetic hyperbolic manifolds are constructed in [2], [18], and [5]. These examples are constructed by cutting arithmetic hyperbolic manifolds along codimension-1 totally geodesic submanifolds and then pasting along isometric boundary components. Since all these nonarithmetic hyperbolic manifolds contain codimension-1 arithmetic hyperbolic submanifolds, Theorem 1.1 implies Theorem 5.2, which claims that all nonarithmetic examples in [18] and [5] (only 4-dimensional examples are constructed in [2]) with dimension at least 6 have non-LERF fundamental groups.

In Theorem 5.3, we also show that compact reflection hyperbolic manifolds with dimension ≥ 5 and noncompact reflection hyperbolic manifolds with dimension ≥ 4 have non-LERF fundamental groups.

Another main result in this paper concerns compact orientable irreducible 3-manifolds with empty or tori boundary. Thurston's geometrization conjecture (confirmed by Perelman) implies that any compact orientable irreducible 3-manifold M with empty or tori boundary has a minimal collection of incompressible tori, such that each component of its complement supports one of Thurston's eight geometries. If this set of incompressible tori is empty, then we say that M is a *geometric 3-manifold*.

The following theorem implies that a compact orientable irreducible 3-manifolds with empty or tori boundary is geometric if and only if its fundamental group is LERF. The author thinks that this result is very interesting, since it gives a surprising relation between geometric structures on 3-manifolds and LERFness of 3-manifold groups, and these two topics in 3-manifold topology have been very popular in the last two decades. This result also confirms Conjecture 1.5 in [24].

THEOREM 1.3

For a compact orientable irreducible 3-manifold M with empty or tori boundary, M supports one of Thurston's eight geometries if and only if $\pi_1(M)$ is LERF.

When $\pi_1(M)$ is not LERF, there exists a nonseparable subgroup isomorphic to a free group. If M is a closed mixed 3-manifold, then there also exists a nonseparable subgroup isomorphic to a surface group.

The proof of Theorem 1.3 is enlightened by the construction in Section 8 of [24]. To prove this theorem, the main case we need to deal with is that M is a union of two geometric 3-manifolds along one torus, with one of them being hyperbolic.

From a group theory point of view, the above group is a \mathbb{Z}^2 -amalgamation of two LERF groups. An even simpler case is a \mathbb{Z} -amalgamation of two hyperbolic 3-manifold groups, that is, the fundamental group of a union of two hyperbolic 3-manifolds along one essential circle.

There have been a lot of works that study LERFness of \mathbb{Z} -amalgamated groups $A *_\mathbb{Z} B$, with both A and B being LERF. For instance, the first such non-LERF example of $A *_\mathbb{Z} B$ was constructed in [34]. It has been shown that if both A and B are free groups (see [8]), or if A is free, B is LERF, and $\mathbb{Z} < A$ is a maximal cyclic subgroup (see [17]), or if both A and B are surface groups (see [27]), then $A *_\mathbb{Z} B$ is LERF.

Here, we give a family of non-LERF \mathbb{Z} -amalgamations of 3-manifold groups.

THEOREM 1.4

Let M_1, M_2 be two finite-volume hyperbolic 3-manifolds, and let $i_k : S^1 \rightarrow M_k$, $k = 1, 2$ be two π_1 -injective embedded circles. Then the fundamental group of

$$X = M_1 \cup_{S^1} M_2$$

is not LERF.

Moreover, if both M_1 and M_2 have cusps, then there exists a nonseparable subgroup isomorphic to a free group. If at least one of M_k is closed, then there exists a nonseparable subgroup isomorphic to a free product of surface groups and free groups.

Theorem 1.4 is the main ingredient to prove Theorem 1.1. We will use the fact that arithmetic hyperbolic manifolds have a lot of totally geodesic submanifolds of smaller dimension. If an arithmetic hyperbolic manifold has dimension at least 5, then there are two totally geodesic 3-dimensional submanifolds intersecting along a closed geodesic, which gives a picture addressed in Theorem 1.4.

In dimension 4, such a picture does not show up by dimension reason, so Theorem 1.4 does not help here. However, Theorem 1.3 implies that the double of any cusped hyperbolic 3-manifold has non-LERF fundamental group, and groups of all noncompact arithmetic hyperbolic manifolds with dimension at least 4 contain such doubled 3-manifold groups (see [25]). So Theorem 1.2 is a consequence of Theorem 1.3.

The organization of this paper is as follows. In Section 2, we review some background on group theory, 3-manifold topology, and arithmetic hyperbolic manifolds. In Section 3, we prove Theorem 1.3, which is enlightened by the construction in [24]. In Section 4, we prove Theorem 1.4, whose proof is similar to the proof of Theorem 1.3, with some modifications. In Section 5, we deduce Theorems 1.1 and 1.2 from Theorems 1.4 and 1.3, respectively. In Section 6, we ask some questions related to the results in this paper.

2. Preliminaries

In this section, we review some basic concepts in group theory, 3-manifold topology, and arithmetic hyperbolic manifolds.

2.1. Locally extended residually finite

In this subsection, we review basic concepts and properties on LERF groups.

Definition 2.1

Let G be a group, and let $H < G$ be a subgroup. We say that H is *separable* in G if, for any $g \in G \setminus H$, there exists a finite-index subgroup $G' < G$ such that $H < G'$ and $g \notin G'$.

An equivalent formulation is that H is separable in G if and only if H is a closed subset under the profinite topology of G .

Definition 2.2

A group G is *LERF* or *subgroup separable* if all finitely generated subgroups of G are separable in G .

A basic property on LERFness is that any subgroup of a LERF group is still LERF. This property is basic and well known, while the proof is very simple. However, since this property is crucial for us, we give a proof here.

LEMMA 2.3

Let G be a group, and let $\Gamma < G$ be a subgroup. For a further subgroup $H < \Gamma$, if H is separable in G , then H is separable in Γ .

In particular, if Γ is not LERF, then G is not LERF.

Proof

If we take an arbitrary element $\gamma \in \Gamma \setminus H$, then $\gamma \in G \setminus H$ holds. Since H is separable in G , there exists a finite-index subgroup $G' < G$ such that $H < G'$ and $\gamma \notin G'$. Then $\Gamma' = G' \cap \Gamma$ is a finite-index subgroup of Γ , with $H < G' \cap \Gamma = \Gamma'$ and $\gamma \notin G' \cap \Gamma = \Gamma'$. So H is also separable in Γ .

If Γ is not LERF, then it contains a finitely generated subgroup H which is not separable in Γ . Then the previous paragraph implies that H is not separable in G . So G is not LERF. \square

In this paper, the main method to prove that a group G is not LERF is to find a descending tower of subgroups of G until we get a subgroup which has a nice structure such that a topological argument can be applied to prove its non-LERFness.

2.2. Geometric decomposition of irreducible 3-manifolds

In this paper, we assume that all manifolds are connected and oriented and that all 3-manifolds are compact and have empty or tori boundary. For any noncompact finite-volume hyperbolic manifold M , we always truncate M by deleting a horocusp for each cusp end of M . Then we can consider M as a compact 3-manifold with tori boundary, and the boundary has an induced Euclidean structure.

Let M be an irreducible 3-manifold with empty or tori boundary. By the geometrization of 3-manifolds, which is achieved by Perelman and Thurston, exactly one of the following hold:

- M is geometric; that is, M supports one of the following eight geometries: \mathbb{E}^3 , \mathbb{S}^3 , $\mathbb{S}^2 \times \mathbb{E}^1$, $\mathbb{H}^2 \times \mathbb{E}^1$, Nil, Sol, $\widetilde{\text{PSL}}_2(\mathbb{R})$, and \mathbb{H}^3 .
- There is a nonempty minimal union $\mathcal{T}_M \subset M$ of disjoint essential tori and Klein bottles, unique up to isotopy, such that each component of $M \setminus \mathcal{T}_M$ is either Seifert-fibered or atoroidal. In the Seifert-fibered case, the interior supports both the $\mathbb{H}^2 \times \mathbb{E}^1$ -geometry and the $\widetilde{\text{PSL}}_2(\mathbb{R})$ -geometry; in the atoroidal case, the interior supports the \mathbb{H}^3 -geometry.

If M has nontrivial geometric decomposition (as in the second case), we say that M is a *nongeometric 3-manifold* and call components of $M \setminus \mathcal{T}_M$ *Seifert pieces* or *hyperbolic pieces*, according to their geometry. If all components of $M \setminus \mathcal{T}_M$ are Seifert pieces, then M is called a *graph manifold*. Otherwise, M contains a hyperbolic piece, and it is called a *mixed manifold*. Since we only consider virtual properties of 3-manifolds in this paper, we can pass to a double cover of the 3-manifold and assume that all components of \mathcal{T}_M are tori.

The geometric decomposition is very closely related to, but slightly different from, the more traditional Jaco–Shalen–Johannson (JSJ) decomposition. Since these two decompositions agree with each other if M has no decomposing Klein bottle (which can be achieved by a double cover), and we are studying virtual properties, we will not make much of a difference between them.

2.3. Fibered structures of 3-manifolds

In the construction of nonseparable subgroups in Theorem 1.3 and Theorem 1.4, all subgroups have a graph of group structures, and the vertex groups are fibered surface subgroups in geometric pieces. We will briefly review the theory of the Thurston norm and its relation with fibered structures on 3-manifolds.

If a 3-manifold M has a surface bundle over circle structure with $b_1(M) > 1$, then M has infinitely many different such structures (which works for all dimensions). These fibered structures of the 3-manifold M are organized by the Thurston norm on $H_2(M, \partial M; \mathbb{R})$ ($\cong H^1(M; \mathbb{R})$ by duality) defined in [39].

For any $\alpha \in H_2(M, \partial M; \mathbb{Z})$, its Thurston norm is defined by:

$$\|\alpha\| = \inf \{ |\chi(T_0)| \mid (T, \partial T) \subset (M, \partial M) \text{ represents } \alpha \},$$

where $T_0 \subset T$ excludes S^2 and D^2 components of T . In [39], it is shown that this norm can be extended to $H_2(M, \partial M; \mathbb{R})$ homogeneously and continuously, and the Thurston norm unit ball is a polyhedron whose faces are dual with elements in $H_1(M; \mathbb{Z})/Tor$. For a general 3-manifold, the Thurston norm is only a seminorm, while it is a genuine norm for finite-volume hyperbolic 3-manifolds.

For a top-dimensional open face F of the Thurston norm unit ball, let C be the open cone over F . In [39], Thurston showed that an integer point $\alpha \in H_2(M, \partial M; \mathbb{R})$ corresponds to a surface bundle structure of M if and only if α is contained in an open cone C as above, and all integer points in C correspond to surface bundle structures of M . In this case, C is called a *fibred cone*, and the corresponding face F is called a *fibred face*. Any point (which may not be an integer point) in a fibred cone we call a *fibred class*.

Thurston's theorem implies that the set of fibred classes of M is an open subset of $H_2(M, \partial M; \mathbb{R})$. In particular, for any fibred class $\alpha \in H_2(M, \partial M; \mathbb{R})$ and any $\beta \in H_2(M, \partial M; \mathbb{R})$, there exists $\epsilon > 0$, such that $\alpha + c\beta \in H_2(M, \partial M; \mathbb{R})$ is a fibred class for any $c \in (-\epsilon, \epsilon)$.

2.4. Virtual retractions of hyperbolic 3-manifold groups

In the proof of Theorem 1.3 and 1.4, we need to perturb a fibred class $\alpha \in H_2(M, \partial M; \mathbb{R})$ to get a new fibred class with some desired property. To make sure the desired perturbation exists, we need the virtual retract property of geometrically finite subgroups of hyperbolic 3-manifold groups.

Definition 2.4

For a group G and a subgroup $H < G$, we say that H is a *virtual retraction* of G if there exists a finite-index subgroup $G' < G$ and a homomorphism $\phi : G' \rightarrow H$, such that $H < G'$ and $\phi|_H = id_H$.

For a finite-volume hyperbolic 3-manifold M , the following dichotomy for a finitely generated infinite-index subgroup $H < \pi_1(M)$ holds:

- (1) H is a geometrically finite subgroup of $\pi_1(M)$ from the Kleinian group point of view. Equivalently, H is (relatively) quasiconvex in the (relative) hyperbolic group $\pi_1(M)$ from the geometric group theory point of view.
- (2) H is a geometrically infinite subgroup of $\pi_1(M)$. In this case, H is a virtual fibred surface subgroup of M .

Here, we do not give the definition of geometrically finite and geometrically infinite subgroups. Readers only need to know that if H is not a virtual fibred surface subgroup, then it is a geometrically finite subgroup. An introduction of geometrically

finite subgroups can be found in [7] and [26, Chapter VI]. The proof of the above dichotomy relies on the covering theorem (see [10], [41]) and the Tameness theorem (see [1], [9]) on open hyperbolic 3-manifolds.

In [11], it is shown that (relatively) quasiconvex subgroups of virtually compact special (relative) hyperbolic groups are virtual retractions. The celebrated virtual compact special theorem of Wise (see [43] for cusped case) and Agol (see [3] for closed case) implies that groups of finite-volume hyperbolic 3-manifolds are virtually compact special. These two results together give us the following theorem.

THEOREM 2.5

Let M be a finite-volume hyperbolic 3-manifold, and let $H < \pi_1(M)$ be a geometrically finite subgroup (i.e., H is not a virtual fibered surface subgroup). Then H is a virtual retraction of $\pi_1(M)$.

2.5. Arithmetic hyperbolic manifolds

In this subsection, we briefly review the definition of (standard) arithmetic hyperbolic manifolds. Most material can be found in [42, Chapter 6].

Recall that the hyperboloid model of \mathbb{H}^n is defined as the following. Equip \mathbb{R}^{n+1} with a bilinear form $B : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ with

$$B((x_1, \dots, x_n, x_{n+1}), (y_1, \dots, y_n, y_{n+1})) = x_1 y_1 + \dots + x_n y_n - x_{n+1} y_{n+1}.$$

Then the hyperbolic space \mathbb{H}^n is identified with

$$I^n = \{\vec{x} = (x_1, \dots, x_n, x_{n+1}) \mid B(\vec{x}, \vec{x}) = -1, x_{n+1} > 0\}.$$

The hyperbolic metric is given by the restriction of $B(\cdot, \cdot)$ on the tangent space of I^n .

The isometry group of \mathbb{H}^n consists of all linear transformations of \mathbb{R}^{n+1} that preserve $B(\cdot, \cdot)$ and fix I^n . Let $J = \text{diag}(1, \dots, 1, -1)$ be the $(n+1) \times (n+1)$ matrix defining the bilinear form $B(\cdot, \cdot)$. Then the isometry group of \mathbb{H}^n is given by

$$\text{Isom}(\mathbb{H}^n) \cong \text{PO}(n, 1; \mathbb{R}) = \{X \in \text{GL}(n+1, \mathbb{R}) \mid X^t J X = J\} / (X \sim -X).$$

The orientation-preserving isometry group of \mathbb{H}^n is given by

$$\text{Isom}_+(\mathbb{H}^n) \cong \text{SO}_0(n, 1; \mathbb{R}),$$

which is the component of

$$\text{SO}(n, 1; \mathbb{R}) = \{X \in \text{SL}(n+1, \mathbb{R}) \mid X^t J X = J\}$$

that contains the identity matrix.

Now we give the definition of *standard arithmetic hyperbolic manifolds*; they are also called *arithmetic hyperbolic manifolds of simplest type*.

Let $K \subset \mathbb{R}$ be a totally real number field, and let $\sigma_1 = id, \sigma_2, \dots, \sigma_k$ be all embeddings of K into \mathbb{R} . Let

$$f(x) = \sum_{i,j=1}^{n+1} a_{ij} x_i x_j, \quad a_{ij} = a_{ji} \in K$$

be a nondegenerate symmetric quadratic form defined over K with negative inertia index 1 (as a quadratic form over \mathbb{R}). We further suppose that, for any $l > 1$, the quadratic form

$$f^{\sigma_l}(x) = \sum_{i,j=1}^{n+1} \sigma_l(a_{ij}) x_i x_j$$

is positive definite. Then the information of K and f can be used to define an arithmetic hyperbolic group.

Let \mathcal{O}_K be the ring of algebraic integers in K , and let A be the $(n+1) \times (n+1)$ matrix defining f . Since the negative inertia index of A is 1, the *special orthogonal group of f* ,

$$\mathrm{SO}(f; \mathbb{R}) = \{X \in \mathrm{SL}(n+1, \mathbb{R}) \mid X^t A X = A\},$$

is conjugate to $\mathrm{SO}(n, 1; \mathbb{R})$ by a matrix P (satisfying $P^t A P = J$). Moreover, $\mathrm{SO}(f; \mathbb{R})$ has two components, and we let $\mathrm{SO}_0(f; \mathbb{R})$ be the component that contains the identity matrix.

Then we form the set of algebraic integer points

$$\mathrm{SO}(f; \mathcal{O}_K) = \{X \in \mathrm{SL}(n+1, \mathcal{O}_K) \mid X^t A X = A\}$$

in $\mathrm{SO}(f; \mathbb{R})$. The theory of arithmetic groups implies that

$$\mathrm{SO}_0(f; \mathcal{O}_K) = \mathrm{SO}(f; \mathcal{O}_K) \cap \mathrm{SO}_0(f; \mathbb{R})$$

is conjugate to a lattice of $\mathrm{Isom}_+(\mathbb{H}^n)$ (by the matrix P); that is, it has finite covolume. For simplicity, we abuse notation and still use $\mathrm{SO}_0(f; \mathcal{O}_K)$ to denote its P -conjugation in $\mathrm{SO}_0(n, 1; \mathbb{R}) \cong \mathrm{Isom}_+(\mathbb{H}^n)$.

Here, $\mathrm{SO}_0(f; \mathcal{O}_K) \subset \mathrm{Isom}_+(\mathbb{H}^n)$ is called *the arithmetic group* and is defined by number field K and quadratic form f , and $\mathbb{H}^n / \mathrm{SO}_0(f; \mathcal{O}_K)$ is a finite-volume hyperbolic arithmetic orbifold. A hyperbolic n -manifold (orbifold) M is called a *standard arithmetic hyperbolic manifold (orbifold)* if M is commensurable with $\mathbb{H}^n / \mathrm{SO}_0(f; \mathcal{O}_K)$ for some K and f .

The arithmetic orbifold $\mathbb{H}^n / \mathrm{SO}_0(f; \mathcal{O}_K)$ is noncompact if and only if $f(\vec{x}) = 0$ has a nontrivial solution in $\in K^{n+1}$, which happens only if $K = \mathbb{Q}$ (i.e., $\mathcal{O}_K = \mathbb{Z}$). When $n \geq 4$, $\mathbb{H}^n / \mathrm{SO}_0(f; \mathcal{O}_K)$ is noncompact if and only if $K = \mathbb{Q}$.

For this paper, the most important property of standard arithmetic hyperbolic manifolds is that they contain a lot of finite-volume hyperbolic 3-manifolds as totally geodesic submanifolds. This can be done by diagonalizing the matrix A (over K) and taking an indefinite 4×4 submatrix.

The above recipe using quadratic forms over number fields gives all even-dimensional arithmetic hyperbolic manifolds (orbifolds). In any odd dimension, there is another family of arithmetic hyperbolic manifolds (orbifolds) which are defined by (skew-Hermitian) quadratic forms over quaternion algebras. We do not give the definition of this family here; readers can find a detailed definition in [23].

This family of arithmetic hyperbolic manifolds defined over quaternions also has many finite-volume hyperbolic 3-manifolds that are totally geodesic submanifolds. This can be done by diagonalizing the quadratic form over quaternions and taking a 2×2 submatrix. Note that this fact is also used in [22].

In dimension 7, there is a third way to construct arithmetic hyperbolic manifolds by using octonions. Only sporadic examples exist, and the author does not know whether these manifolds have totally geodesic (or π_1 -injective) 3-dimensional submanifolds. All examples in this family are compact manifolds.

3. Non-LERFness of nongeometric 3-manifold groups

In this section, we prove that groups of nongeometric 3-manifolds are not LERF. The construction of nonseparable (surface) subgroups is enlightened by the construction in [24] (and also in [35]). The proof of nonseparability is essentially a computation of the spirality character defined in [24]. Here, we modify the construction in [24] and give an elementary proof of nonseparability without using the spirality character explicitly.

3.1. Finite semicovers of nongeometric 3-manifolds

We first review the notion of finite semicovers of nongeometric 3-manifolds, which was introduced in [32].

Definition 3.1

Let M be a nongeometric 3-manifold with tori or empty boundary. A *finite semicover* of M is a compact 3-manifold N and a local embedding $f : N \rightarrow M$, such that its restriction on each boundary component of N is a finite cover to a decomposition torus or a boundary component of M .

For a finite semicover $f : N \rightarrow M$, if M has no decomposing Klein bottle, the decomposition tori of N is exactly $f^{-1}(\mathcal{T}_M) \setminus \partial N$, and the restriction of f on each geometric piece of N is a finite cover of the corresponding geometric piece of M .

One important property of finite semicovers is given by the following lemma in [24].

LEMMA 3.2 ([24, Lemma 6.2])

If N is a connected finite semicover of a nongeometric 3-manifold M with empty or tori boundary, then N has an embedded lifting in a finite cover of M . In fact, the semicovering map $N \rightarrow M$ is π_1 -injective, and $\pi_1(N)$ is separable in $\pi_1(M)$.

Remark 3.3

In [24], this lemma is only stated in the case that M is a closed orientable irreducible nongeometric 3-manifold, but it also clearly holds for irreducible nongeometric 3-manifolds with nonempty boundary. This is because we can first take the double $D(M)$ of M , apply the closed manifold version of Lemma 3.2 to $N \rightarrow D(M)$, and apply Lemma 2.3 to get separability of $\pi_1(N)$ in $\pi_1(M)$.

3.2. Reduction to nongeometric 3-manifolds with very simple dual graph

To prove Theorem 1.3, we will reduce it to the case that the dual graph of M consists of two vertices and two edges (a bigon), and M has at least one hyperbolic piece.

Let M be an orientable irreducible nongeometric 3-manifold with tori or empty boundary. It is known that all graph manifolds have non-LERF fundamental groups (see [30]), so we can assume that M has at least one hyperbolic piece; that is, M is a mixed 3-manifold.

The dual graph of M is a graph with vertices corresponding to geometric pieces of M and edges corresponding to decomposition tori. The following lemma is the first step of our reduction of 3-manifolds, which reduces the non-LERFness of mixed 3-manifold groups to a very simple case: the dual graph of M has only two vertices and one edge.

LEMMA 3.4

Let M be a mixed 3-manifold. Then there exists a 3-manifold $N = N_1 \cup_T N_2$ such that the following hold:

- (1) N_1 is a cusped hyperbolic 3-manifold, and N_2 is a geometric 3-manifold.
- (2) $N_1 \cap N_2 = T$ is a single torus, and $N = N_1 \cup_T N_2$ is a fibered 3-manifold.
- (3) N is a finite semicover of M , so $\pi_1(N)$ is a subgroup of $\pi_1(M)$.

Proof

By [31], we take a finite cover of M such that it is a fibered 3-manifold; we still denote it by M .

We first suppose that M has at least two geometric pieces. Take any hyperbolic piece N_1 , and take another (distinct) geometric piece N_2 adjacent to N_1 . It is possible that $N_1 \cap N_2$ consists of more than one tori, and let T be one of them. We cut M along all decomposition tori in \mathcal{T}_M except T ; then the component containing N_1 and N_2 is the desired N , which is clearly a finite semicover of M .

The fibered structure on M induces a fibered structure on N , since fibered structures of 3-manifolds are compatible with geometric decomposition. It is easy to see that all other desired conditions hold for N .

It remains to consider the case that M has only one geometric piece, and we denote it by N_1 . Since the geometric decomposition of M is nontrivial, there is a decomposition torus T of M that is adjacent to N_1 on both sides. Then we take a double cover of M dual to T and reduce it to the previous case. \square

By Lemma 2.3, to prove non-LERFness of mixed 3-manifold groups, we only need to consider the case $M = M_1 \cup_T M_2$ as in Lemma 3.4 (we use M and M_i instead of N and N_i since we will do further constructions). The dual graph of M has two vertices and one edge, which is not our desired model for constructing nonseparable subgroups. Actually, we need a cycle in the dual graph of the 3-manifold. So we use the following lemma to pass it to a further finite semicover, such that its dual graph consists of two vertices and two edges (a bigon).

LEMMA 3.5

Let $M = M_1 \cup_T M_2$ be a 3-manifold satisfying the conclusion of Lemma 3.4. Then there exists a 3-manifold $N = N_1 \cup_{T \cup T'} N_2$ with nonempty boundary such that the following hold:

- (1) N_1 is a cusped hyperbolic 3-manifold, and N_2 is a geometric 3-manifold.
- (2) $N_1 \cap N_2 = T \cup T'$ is a union of two tori, and $N = N_1 \cup_{T \cup T'} N_2$ is a fibered 3-manifold.
- (3) The homomorphism $H_1(T \cup T'; \mathbb{Z}) \rightarrow H_1(N_1; \mathbb{Z})$ induced by inclusion is injective.
- (4) N is a finite semicover of M , so $\pi_1(N)$ is a subgroup of $\pi_1(M)$.
- (5) There exists a fibered surface S of N , which is a union of two subsurfaces $S = S_1 \cup_{c \cup c'} S_2$, such that $S_i = S \cap N_i$, $c = S \cap T$, and $c' = S \cap T'$. Here, both S and S' are connected, while both c and c' are one single circle.

Proof

Claim. *There exists a 3-manifold $N = N_1 \cup_{T \cup T'} N_2$ satisfying conditions (1)–(4).*

We first prove this claim.

We take a basepoint of M_1 on T . For $\mathbb{Z}^2 \cong \pi_1(T) < \pi_1(M_1) < \text{Isom}_+(\mathbb{H}^3)$, we take any hyperbolic element $g \in \pi_1(M_1)$ which maps the fixed point of $\pi_1(T)$ on S_∞^2 to a different point. By the Klein combination theorem (see [26, Section VII, Theorem A.13]), for large-enough positive integer k , the subgroup of $\pi_1(M_1)$ generated by $\pi_1(T)$ and $g^k \pi_1(T) g^{-k}$ is isomorphic to the free product of these two groups, that is, isomorphic to $\mathbb{Z}^2 * \mathbb{Z}^2$, and we denote it by H .

Since $H < \pi_1(M_1)$ is not a surface subgroup, it is geometrically finite. By Theorem 2.5, there exists a finite cover N_1 of M_1 , such that $H < \pi_1(N_1)$, and there exists a retraction homomorphism $\pi_1(N_1) \rightarrow H$. Since hyperbolic 3-manifolds have LERF fundamental groups (see [3], [43]), by passing to a further finite cover (still denoted by N_1), we can assume that $g^k \notin \pi_1(N_1)$ and that N_1 has at least three boundary components.

Since $g^k \notin \pi_1(N_1)$, any (embedded) arc γ in N_1 (starting from the lifted basepoint) corresponding to $g^k \in \pi_1(M)$ connects two different boundary components of N_1 , and we denote them by T_1 and T'_1 . Note that the restriction of covering map $N_1 \rightarrow M_1$ maps both T_1 and T'_1 to T by homeomorphisms. Then $H < \pi_1(N_1)$ corresponds to the fundamental group of the union of T_1 , T'_1 , and γ . Since $H = \pi_1(T_1 \cup T'_1 \cup \gamma)$ is a retraction of $\pi_1(N_1)$, $H_1(T_1 \cup T'_1 \cup \gamma; \mathbb{Z}) \cong H_1(T_1 \cup T'_1; \mathbb{Z})$ is a retraction of $H_1(N_1; \mathbb{Z})$. So condition (3) holds for N_1 .

If M_2 is a cusped hyperbolic 3-manifold, by doing the same construction as M_1 , we get a finite cover $N_2 \rightarrow M_2$ such that two boundary components T_2 and T'_2 of N_2 are mapped to T by homeomorphisms. By identifying T_1 and T'_1 with T_2 and T'_2 , respectively, we get a semifinite cover $N = N_1 \cup_{T \cup T'} N_2$ of M satisfying conditions (1)–(4). Here, we use T to denote the image of T_1 and T_2 , and we use T' to denote the image of T'_1 and T'_2 .

If M_2 is a Seifert-fibered space, then we first do the following preparation before doing the above construction for M_1 . Since M is a fibered 3-manifold, we have $M = S \times I / \phi$, where $\phi : S \rightarrow S$ is a reducible homeomorphism on a surface S . By taking some finite cyclic cover M' of M along S , we can assume that M' has two adjacent geometric pieces, such that one of them is a cusped hyperbolic 3-manifold, and another one is homeomorphic to $\Sigma \times S^1$ with $\chi(\Sigma) < 0$.

We take the union of these two adjacent pieces along a common torus and get our new $M = M_1 \cup_T M_2$ with $M_2 = \Sigma \times S^1$. Then we do the same construction for M_1 as above to get a finite cover N_1 . For M_2 , let c be the boundary component of Σ corresponding to the boundary component $T \subset \partial M_2$. Since $\chi(\Sigma) < 0$, there exists a

double cover $\Sigma' \rightarrow \Sigma$ such that there are two boundary components $c_2, c'_2 \subset \partial\Sigma'$ that are mapped to c by homeomorphisms.

Then $N_2 = \Sigma' \times S^1$ is a finite cover of M_2 . Let T_2 and T'_2 be the boundary components of N_2 corresponding to $c_2 \times S^1$ and $c'_2 \times S^1$, respectively; then they are both mapped to T by homeomorphisms. We paste N_1 and N_2 together to get the desired finite semicover $N = N_1 \cup_{T \cup T'} N_2$.

This finishes the proof of the claim.

Now $N = N_1 \cup_{T \cup T'} N_2$ satisfies conditions (1)–(4), so we need to work on condition (5).

Since M is a fibered 3-manifold, the semicover N has an induced fibered structure. The corresponding fibered surface S might be more complicated than what we want in condition (5), since $S \cap N_i$, $S \cap T$, and $S \cap T'$ may not be connected.

We write N as $N = S \times I/\phi$. Since N has nontrivial torus decomposition, $\phi : S \rightarrow S$ is a reducible self-homeomorphism of S . Let \mathcal{C} be the set of reduction circles such that $\phi| : S \setminus \mathcal{C} \rightarrow S \setminus \mathcal{C}$ is either pseudo-Anosov or periodic on each ϕ -component (by [40]).

We first suppose that there are two components S_1 and S_2 of $S \setminus \mathcal{C}$ such that $S_i \subset N_i$, and such that $S_1 \cap S_2$ contains two circles c and c' with $c \subset T$ and $c' \subset T'$. Take a positive integer k , such that ϕ^k preserves each component of both $S \setminus \mathcal{C}$ and \mathcal{C} . In this case, $N' = ((S_1 \cup_{c \cup c'} S_2) \times I)/\phi^k$ is a finite semicover of N . Let $\bar{N}_i = S_i \times I/\phi^k$, and let \bar{T} and \bar{T}' be the components of $\partial\bar{N}_1$ (also $\partial\bar{N}_2$) containing c and c' , respectively. Then it is easy to check that $\bar{N} = \bar{N}_1 \cup_{\bar{T} \cup \bar{T}'} \bar{N}_2$ satisfies all desired conditions.

If there are not two components of $S \setminus \mathcal{C}$ satisfying the above condition, we need to modify the fibered surface S . The new fibered surface is the Haken sum of S and a multiple of T_1 , and the detail is as follows.

We take a tubular neighborhood $N(T_1)$ of T_1 in N_1 and give it a coordinate by $N(T_1) = T_1 \times I = (S^1 \times I) \times S^1$ such that

$$S \cap N(T_1) = (\{a_1, a_2, \dots, a_k\} \times I) \times S^1,$$

with a_1, \dots, a_k following a cyclic order on S^1 . The fibered structure on $N(T_1)$ is given by a fibered structure of $S^1 \times I$ and then crossed with S^1 . For any integer j , we modify the fibered structure on $N(T_1)$ by modifying the fibered structure on $S^1 \times I$. For each fixed integer j , a new fibered structure on $S^1 \times I$ is given by a union of disjoint embedded arcs $I_i \subset S^1 \times I$, such that I_i connects $(a_i, 0)$ to $(a_{i+j}, 1)$ (modulo k), where $i = 1, 2, \dots, k$. This fibered structure on $N(T_1)$ can be pasted with the original fibered structure of $N \setminus N(T_1)$ to get a new fibered structure of N .

If we start from one component $S_1 \subset S \cap N_1$, then take any component $S_2 \subset S \cap N_2$ such that $S_1 \cap S_2 \cap T' \neq \emptyset$. Then $S_1 \cap T_1$ and $S_2 \cap T_2$ are two families of parallel

circles on T , but it is possible any two circles in these two families are not identified with each other. Then we apply the above modification of the fibered structure for a proper chosen j , such that the new fibered surface satisfies the assumption of the previous case. \square

Actually, condition (5) is not really necessary in the proof of Theorem 1.3, but it will make the immersed π_1 -injective surface constructed in Proposition 3.6 a simple shape.

3.3. Construction of nonseparable surface subgroups

In this subsection, we construct a π_1 -injective properly immersed subsurface in the 3-manifold $N = N_1 \cup_{T \cup T'} N_2$ constructed in Lemma 3.5 and then prove that this surface subgroup is not separable in $\pi_1(N)$.

The following proposition constructs a π_1 -injective properly immersed subsurface in N , which is our candidate of nonseparable surface subgroup. Readers may want to compare this construction with the construction in Section 8 of [24].

PROPOSITION 3.6

For the 3-manifold $N = N_1 \cup_{T \cup T'} N_2$ and fibered subsurface $S = S_1 \cup_{c \cup c'} S_2$ constructed in Lemma 3.5, there exists a connected π_1 -injective properly immersed surface $i : \Sigma \looparrowright N$ such that the following hold:

- (1) Σ is a union of connected subsurfaces as $\Sigma = (\Sigma_{1,1} \cup \Sigma_{1,2}) \cup (\bigcup_{k=1}^{2n} \Sigma_{2,k})$, with $i(\Sigma_{1,j}) \subset N_1$ and $i(\Sigma_{2,k}) \subset N_2$.
- (2) The restrictions of i on $\Sigma_{1,j}$ and $\Sigma_{2,k}$ are embeddings, and their images are fibered surfaces in N_1 and N_2 , respectively.
- (3) Each $\Sigma_{2,k}$ is a parallel copy of S_2 in N_2 , so $\Sigma_{2,k}$ intersects with both T and T' along exactly one circle.
- (4) $\Sigma_{1,1} \cap \Sigma_{2,1}$ consists of two circles s and s' , with $i(s) \subset T$ and $i(s') \subset T'$.
- (5) $\Sigma_{1,1} \cap T$ consists of A parallel copies of c , and $\Sigma_{1,1} \cap T'$ consists of B parallel copies of c' , with $A \neq B$.

Proof

When we cut N along $T \cup T'$ and cut S along $c \cup c'$, we use T_i and T'_i to denote the copies of T and T' in N_i , respectively, and use c_i and c'_i to denote the copies of c and c' in S_i , respectively.

Let $\alpha \in H^1(N; \mathbb{Z})$ be the fibered class dual to S , and let $\alpha_1 = \alpha|_{N_1}$. Then $\alpha_1|_{T_1}$ is dual to $c_1 \subset T_1$, and $\alpha_1|_{T'_1}$ is dual to $c'_1 \subset T'_1$.

Since $H_1(T_1 \cup T'_1; \mathbb{Z}) \rightarrow H_1(N_1; \mathbb{Z})$ is injective, there exists a direct summand $A < H_1(N_1; \mathbb{Z})$ such that $A \cong \mathbb{Z}^4$ and $H_1(T_1 \cup T'_1; \mathbb{Z}) < A$. Since $\mathbb{Z}^4 \cong H_1(T_1 \cup$

$T'_1; \mathbb{Z}) < A \cong \mathbb{Z}^4$ is a finite-index subgroup, there exists a homomorphism $\tau : A \rightarrow \mathbb{Z}$ such that $\tau|_{H_1(T_1; \mathbb{Z})}$ is equal to $l\alpha_1|_{T_1}$ for some $l \in \mathbb{Z}_+$, and $\tau|_{H_1(T'_1; \mathbb{Z})} = 0$.

Let $\phi : H_1(N_1; \mathbb{Z}) \rightarrow A$ be a retraction given by the direct sum structure. Then we get a cohomology class $\beta \in H^1(N_1; \mathbb{Z})$ defined by $\tau \circ \phi : H_1(N_1; \mathbb{Z}) \rightarrow \mathbb{Z}$. By the construction of τ , $\beta|_{T_1} = \tau \circ \phi|_{T_1} = \tau|_{T_1} = l\alpha_1|_{T_1}$ for some $l \in \mathbb{Z}_+$ and $\beta|_{T'_1} = 0$.

Since α_1 is a fibered class on N_1 , for large-enough $n \in \mathbb{Z}_+$, $\alpha_{1,1} = n\alpha_1 + \beta$ and $\alpha_{1,2} = n\alpha_1 - \beta$ are both fibered classes in $H^1(N_1; \mathbb{Z})$. Here, we can also assume that $n > l$ and $\gcd(n, l) = 1$.

Since $\alpha_{1,1}|_{T_1}$ is dual to $n + l$ copies of c_1 , $\alpha_{1,1}|_{T'_1}$ is dual to n copies of c'_1 , and $\gcd(n, l) = 1$, $\alpha_{1,1} \in H^1(N_1; \mathbb{Z})$ is a primitive class. Similarly, $\alpha_{1,2} \in H^1(N_1; \mathbb{Z})$ is also primitive.

Let $\Sigma_{1,1} \subset N_1$ be the connected fibered surface dual to $\alpha_{1,1} \in H^1(N_1; \mathbb{Z})$, and let $\Sigma_{1,2} \subset N_1$ be the connected fibered surface dual to $\alpha_{1,2}$. Then $\Sigma_{1,1} \cap T_1$ consists of $A = n + l$ copies of c_1 (as oriented curves), $\Sigma_{1,1} \cap T'_1$ consists of $B = n$ copies of c'_1 , $\Sigma_{1,2} \cap T_1$ consists of $n - l$ copies of c_1 , and $\Sigma_{1,2} \cap T'_1$ consists of n copies of c'_1 . So $(\Sigma_{1,1} \cup \Sigma_{1,2}) \cap T_1$ and $(\Sigma_{1,1} \cup \Sigma_{1,2}) \cap T'_1$ consist of $2n$ (oriented) copies of c_1 and c'_1 , respectively.

Note that both $S_2 \cap T_2$ and $S_2 \cap T'_2$ are exactly one (oriented) copy of c_2 and c'_2 , respectively. We take $2n$ copies of S_2 in N_2 , and denote them by $\Sigma_{2,k}$, with $k = 1, 2, \dots, 2n$. Then we identify parallel circles in $(\Sigma_{1,1} \cup \Sigma_{1,2}) \cap T_1$ with $(\bigcup_{k=1}^{2n} \Sigma_{2,k}) \cap T_2$ on $T = T_1 = T_2$ and identify parallel circles in $(\Sigma_{1,1} \cup \Sigma_{1,2}) \cap T'_1$ with $(\bigcup_{k=1}^{2n} \Sigma_{2,k}) \cap T'_2$ on $T' = T'_1 = T'_2$ to get an immersed surface Σ . In the identification process, we first identify one circle in $\Sigma_{1,1} \cap T_1$ with the circle in $\Sigma_{2,1} \cap T_2$ and identify one circle in $\Sigma_{1,1} \cap T'_1$ with the circle in $\Sigma_{2,1} \cap T'_2$. Then we identify the remaining circles arbitrarily. There are actually many ways to do the identification in the second step, since we can isotopy any Σ_{2,k_0} such that its intersection with T_2 slides over the other circles $\Sigma_{2,k} \cap T_2$, while the other surfaces in $\{\Sigma_{2,k}\}$ are fixed.

It is easy to see that $i : \Sigma \looparrowright N$ is a properly immersed surface, and it satisfies conditions (1)–(5) in the proposition by the construction.

Moreover, by conditions (3) and (5), there exists some Σ_{2,k_0} such that both $\Sigma_{1,1} \cap \Sigma_{2,k_0}$ and $\Sigma_{1,2} \cap \Sigma_{2,k_0}$ are not empty. So $\Sigma_{1,1}$ and $\Sigma_{1,2}$ lie in the same connected component of Σ . Then Σ must be connected, since each $\Sigma_{2,k}$ intersects with at least one of $\Sigma_{1,1}$ and $\Sigma_{1,2}$.

Now we show that i is π_1 -injective by using classical 3-manifold topology. Suppose there is a map $j : S^1 \rightarrow \Sigma$ which is not null-homotopic in Σ , but $i \circ j : S^1 \rightarrow N$ is null-homotopic in N .

We can assume that $i \circ j$ is transverse with the decomposition tori $T \cup T'$, and j minimizes the number of points in $(i \circ j)^{-1}(T \cup T') \subset S^1$ in the homotopy class

of j . This number is not zero; otherwise it contradicts the π_1 -injectivity of fibered surfaces.

Since $i \circ j$ is null-homotopic, it can be extended to a map $k : D^2 \rightarrow N$ such that $k|_{S^1} = i \circ j$. We can homotopy k relative to S^1 such that it is transverse with $T \cup T'$, and $k^{-1}(T \cup T')$ consists of disjoint simple arcs in D^2 .

Then there exists a subarc $\alpha \subset S^1$ and an arc component β in $k^{-1}(T \cup T') \subset D^2$, such that α and β share endpoints and there are no other components of $k^{-1}(T \cup T')$ lying in the subdisk $B \subset D^2$ bounded by $\alpha \cup \beta$. Without loss of generality, we suppose that $j(\alpha)$ lies in $\Sigma_{1,1} \subset N_1$, $k(\beta) \subset T$, and $k(B) \subset N_1$. Then it is easy to see that the k -images of two endpoints of α lie in the same component of $\Sigma_{1,1} \cap T$ by considering the algebraic intersection number between $\Sigma_{1,1}$ and $\alpha \cup \beta$. Moreover, $k|_{\beta} : \beta \rightarrow T$ is homotopic to a map into $\Sigma_{1,1} \cap T$, relative to the boundary of β .

Then it is routine to check that $j|_{\alpha} : \alpha \rightarrow \Sigma_{1,1}$ is homotopy to a map with image in $i^{-1}(T)$, relative to the boundary of α . After a further homotopy of j supporting on a neighborhood of α , we get another $j' : S^1 \rightarrow \Sigma$ which is homotopy to j and has a fewer number of points in $(i \circ j')^{-1}(T \cup T') \subset S^1$.

So we get a contradiction with the minimality of j , and $i : \Sigma \looparrowright N$ is π_1 -injective. \square

The following proposition proves the nonseparability of $i_*(\pi_1(\Sigma)) < \pi_1(N)$ constructed in Proposition 3.6. Essentially, the proof checks that the spirality character of $\Sigma \rightarrow N$ is nontrivial (defined in [24]), but we do not use the terminology of the spirality character here, since the picture is relatively simple and we can give a direct proof.

PROPOSITION 3.7

For the properly immersed subsurface $i : \Sigma \looparrowright N$ constructed in Proposition 3.6, $i_(\pi_1(\Sigma)) < \pi_1(N)$ is a nonseparable subgroup.*

Proof

Suppose that $i_*(\pi_1(\Sigma)) < \pi_1(N)$ is separable; we will get a contradiction.

Let \tilde{N} be the covering space of N corresponding to $i_*(\pi_1(\Sigma))$. Since each component of $\Sigma \cap i^{-1}(N_k)$ is a fibered surface in N_k for $k = 1, 2$, it is easy to see that \tilde{N} is homeomorphic to $\Sigma \times \mathbb{R}$. So $i : \Sigma \looparrowright N$ lifts to an embedding $\Sigma \hookrightarrow \tilde{N}$.

Since $i_*(\pi_1(\Sigma)) < \pi_1(N)$ is separable, by [36] there exists an intermediate finite cover $\hat{N} \rightarrow N$ of $\tilde{N} \rightarrow N$ such that $i : \Sigma \looparrowright N$ lifts to an embedding $\hat{i} : \Sigma \hookrightarrow \hat{N}$.

Since $i : \Sigma \looparrowright N$ is a proper immersion, $\hat{i} : \Sigma \hookrightarrow \hat{N}$ is also a proper embedding. So Σ defines a nontrivial cohomology class $\sigma \in H^1(\hat{N}; \mathbb{Z})$.

For each decomposition torus $\hat{T}_s \subset \hat{N}$, suppose $\Sigma \cap \hat{T}_s$ consists of k_s parallel circles. Let K be the least common multiple of all k_s . By taking the K -sheet cyclic cover of \hat{N} along Σ (corresponding to the kernel of $H_1(\hat{N}; \mathbb{Z}) \xrightarrow{\sigma} \mathbb{Z} \rightarrow \mathbb{Z}_K$), we get a further finite cover $\bar{N} \rightarrow N$. Then Σ embeds into \bar{N} , and it intersects with each decomposition torus of \bar{N} exactly once.

Let \bar{N}_1 and \bar{N}_2 be the geometric pieces of \bar{N} containing $\Sigma_{1,1}$ and $\Sigma_{2,1}$, respectively. Since $\Sigma_{1,1} \cap \Sigma_{2,1} = s \cup s'$, let \bar{T} and \bar{T}' be the decomposition tori in $\bar{N}_1 \cap \bar{N}_2$ containing s and s' , respectively. Then the finite cover $\bar{N} \rightarrow N$ induces finite covers:

$$\begin{aligned} \bar{N}_1 &\rightarrow N_1, & \bar{N}_2 &\rightarrow N_2, \\ \bar{T} &\rightarrow T, & \bar{T}' &\rightarrow T'. \end{aligned}$$

Since both $\bar{T} \rightarrow T$ and $\bar{T}' \rightarrow T'$ are induced by $\bar{N}_1 \rightarrow N_1$ and $\bar{N}_2 \rightarrow N_2$, we will get two relations between $\deg(\bar{T} \rightarrow T)$ and $\deg(\bar{T}' \rightarrow T')$ and then get a contradiction.

Since $\Sigma_{1,1}$ is an embedded fibered surface in both \bar{N}_1 and N_1 , \bar{N}_1 is a finite cyclic cover of N_1 along $\Sigma_{1,1}$. Similarly, \bar{N}_2 is a finite cyclic cover of N_2 along $\Sigma_{2,1}$.

Since $\Sigma_{1,1} \cap T$ consists of A parallel circles and $\Sigma_{1,1} \cap T'$ consists of B parallel circles, while $\Sigma_{1,1} \cap \bar{T}$ and $\Sigma_{1,1} \cap \bar{T}'$ are both only one circle, $\bar{N}_1 \rightarrow N_1$ is a cyclic cover whose degree is a multiple of $\text{lcm}(A, B)$, and

$$A \cdot \deg(\bar{T} \rightarrow T) = \deg(\bar{N}_1 \rightarrow N_1) = B \cdot \deg(\bar{T}' \rightarrow T'). \quad (1)$$

We also have that $\Sigma_{2,1}$ is an embedded fibered surface in both \bar{N}_2 and N_2 . Since $\Sigma_{2,1} \cap T$, $\Sigma_{2,1} \cap T'$, $\Sigma_{2,1} \cap \bar{T}$, and $\Sigma_{2,1} \cap \bar{T}'$ are all just one circle, and $\bar{N}_2 \rightarrow N_2$ is a finite cyclic cover, we have

$$\deg(\bar{T} \rightarrow T) = \deg(\bar{N}_2 \rightarrow N_2) = \deg(\bar{T}' \rightarrow T'). \quad (2)$$

Equations (1) and (2) imply that $A = B$, which contradicts with condition (5) in Proposition 3.6. So $i_*(\pi_1(\Sigma))$ must be a nonseparable subgroup of $\pi_1(N)$. \square

Remark 3.8

From the proof of Proposition 3.7, readers can see that the main ingredient for proving the nonseparability of $\pi_1(\Sigma)$ is the subsurface $\Sigma_{1,1} \cup_{s \cup s'} \Sigma_{2,1}$. However, the author cannot prove that $\pi_1(\Sigma_{1,1} \cup_{s \cup s'} \Sigma_{2,1})$ is nonseparable in $\pi_1(N)$ yet, although it seems quite plausible.

In the proof of Proposition 3.7, we do need the properness of the immersed subsurface $i : \Sigma \looparrowright N$ so that we can take the finite cyclic cover of \hat{N} along Σ to get \bar{N} and then get the contradiction. Actually, most of the proof can be translated to purely group theoretical language, except that the author does not know how to interpret ‘‘properly immersed subsurface’’ algebraically.

3.4. Proof of Theorem 1.3

Now we are ready to prove Theorem 1.3.

Proof

Suppose that M supports one of Thurston's eight geometries. Since the fundamental group is finite or virtually abelian, if M supports the S^3 - or $S^2 \times \mathbb{E}^1$ -geometry, then LERFness trivially holds. If M supports the \mathbb{E}^3 -, Nil-, $\mathbb{H}^2 \times \mathbb{E}^1$ -, or $\widetilde{\text{PSL}}_2(\mathbb{R})$ -geometry, then M is a Seifert manifold and LERFness is proved in [36]. If M supports the Sol-geometry, then M is virtually a torus bundle over circle, and a proof of LERFness can be found in [30]. If M is a hyperbolic 3-manifold, then LERFness is shown by the celebrated works of Wise (see [43] for the cusped case) and Agol (see [3] for the closed case).

Now we need to show that nongeometric 3-manifolds have non-LERF fundamental groups. We first suppose that M is a mixed 3-manifold; that is, M has a hyperbolic piece.

If M is not a closed manifold, then Lemmas 3.4 and 3.5 imply that M has a finite semicover $N = N_1 \cup_{T \cup T'} N_2$ satisfying the conditions in Lemma 3.5. In particular, $\pi_1(N)$ is a subgroup of $\pi_1(M)$. Then Proposition 3.6 constructs a nonclosed surface subgroup (free subgroup) $\pi_1(\Sigma) < \pi_1(N)$, and Proposition 3.7 shows that $\pi_1(\Sigma)$ is not separable in $\pi_1(N)$. Finally, Lemma 2.3 implies that $\pi_1(\Sigma)$ is not separable in $\pi_1(M)$, and thus $\pi_1(M)$ is not LERF.

If M is a closed mixed 3-manifold, then the above proof also shows the existence of a nonseparable free subgroup in $\pi_1(M)$. We need also to construct a nonseparable closed surface subgroup.

Let $N \rightarrow M$ be the finite semicover constructed in Lemma 3.5 (with $\partial N \neq \emptyset$), and let $\Sigma \looparrowright N$ be the π_1 -injective properly immersed surface constructed in Proposition 3.6. To make the geometric picture simpler, we apply Lemma 3.2 to construct a finite cover M' of M such that N lifts to an embedded submanifold of M' .

In this case, the induced map $\Sigma \looparrowright M'$ is an immersion but is not a proper immersion. So we cannot use the proof of Proposition 3.7 for this Σ . Now we extend Σ to a closed surface Σ' , with an immersion $j : \Sigma' \looparrowright M'$. Then we can apply the argument in the proof of Proposition 3.7 to prove the nonseparability of $\pi_1(\Sigma') < \pi_1(M')$.

The construction of $j : \Sigma' \looparrowright M'$ is actually done in Section 8 of [24], so we only give a sketch here.

Let the boundary components of Σ be s_1, \dots, s_m , with each s_i lying on a decomposition torus $T_i \subset M'$. By Theorem 4.11 of [13], there exists an essentially immersed subsurface $R_i \looparrowright M'$, such that ∂R_i consists of two components b_i and \bar{b}_i , while b_i and \bar{b}_i are mapped to a positive and a negative multiple of $s_i \subset T_i$, respectively, with

the same covering degree. Moreover, a neighborhood of ∂R_i in R_i is mapped to the side of T_i that is not N , and R_i intersects with $\mathcal{T}_{M'}$ minimally.

Then we take some finite cover of $\hat{\Sigma} \rightarrow \Sigma$ such that each boundary component of $\hat{\Sigma}$ that is mapped to s_i has covering degree $\deg(b_i \rightarrow s_i)$, and we take another copy of $\hat{\Sigma}$ with opposite orientation. Together with a proper number of copies of R_i , $i = 1, \dots, m$, they can be pasted together to get a π_1 -injective immersed closed subsurface $\Sigma' \looparrowright M'$. A similar argument as in Proposition 3.7 can be applied to $\hat{\Sigma} \subset \Sigma'$ to show that $\pi_1(\Sigma')$ is not separable in $\pi_1(M')$, and so it is not separable in $\pi_1(M)$.

If M is a graph manifold, it was already shown in [30] that $\pi_1(M)$ is not LERF. So we only sketch the construction of nonseparable surface subgroups.

The first step is to show that M has a finite semicover $N = S \times I/\phi$, where $S = S_1 \cup_{c \cup c'} S_2$ and ϕ is a composition of Dehn twists along c and c' . Then we perturb the fibered structures on both N_1 and N_2 (since Seifert-fibered spaces have less flexible fibered structures) to get a π_1 -injective properly immersed subsurface similar to what we get in Proposition 3.6. Then a similar argument as in Proposition 3.7 shows that this surface subgroup is not separable. Here, we do need to use the fact that two adjacent Seifert pieces in a graph manifold have incompatible regular fibers on their intersection torus.

However, in general, it does not seem easy to construct a nonseparable closed surface subgroup in a closed graph manifold. \square

Remark 3.9

In [35], the authors constructed a π_1 -injective properly immersed subsurface $\Sigma \looparrowright M$ for some graph manifold M . Then [29] proved that $\pi_1(\Sigma)$ is not contained in any finite-index subgroup of $\pi_1(M)$ (not engulfed). In the proof of [29], only the infinite plane property of the surfaces constructed in [35] is used. Since the surfaces we constructed in the proof of Theorem 1.3 also have the infinite plane property, for any mixed 3-manifold M , we can find a finite cover $M' \rightarrow M$ and a π_1 -injective properly immersed subsurface $\Sigma \looparrowright M'$ such that $\pi_1(\Sigma)$ is not contained in any finite-index subgroup of $\pi_1(M')$.

In [30], it is shown that all graph manifold groups contain

$$L = \langle x, y, r, s \mid r x r^{-1} = x, r y r^{-1} = y, s x s^{-1} = x \rangle$$

as a subgroup. Then the non-LERFness of L implies the non-LERFness of all graph manifold groups. It is easy to see that some mixed manifolds (e.g., the double of any cusped hyperbolic 3-manifold) do not contain L as a subgroup in their fundamental groups. So L is not the source of the non-LERFness of these groups.

Since any free product of LERF groups is still LERF, we have the following direct corollary of Theorem 1.3.

COROLLARY 3.10

Let M be a compact orientable 3-manifold with empty or tori boundary. Then $\pi_1(M)$ is LERF if and only if all prime factors of M support one of Thurston's eight geometries.

Knot complements in S^3 also form a classical family of interesting 3-manifolds, and each knot is a torus knot, a hyperbolic knot, or a satellite knot. We have the following corollary for knot complements.

COROLLARY 3.11

Let M be the complement of a knot $K \subset S^3$. Then $\pi_1(M)$ is LERF if and only if K is either a torus knot or a hyperbolic knot.

4. Union of two hyperbolic 3-manifolds along a circle

In this section, we will give the proof of Theorem 1.4. The proof is very similar to the proof of Theorem 1.3. For some lemmas and propositions in this section, we will only give a sketch of the proof; we point out necessary modifications of the corresponding proofs in Section 3.

In the proof of the non-LERFness of $\pi_1(M_1 \cup_{S^1} M_2)$, we actually only use machinery on hyperbolic 3-manifolds for M_1 (the crucial ingredient is the virtual retract property of its geometrically finite subgroups), and do not have much requirement for M_2 . So we will have some more general results on the non-LERFness of \mathbb{Z} -amalgamated groups in Section 4.2.

4.1. Non-LERFness of $\pi_1(M_1 \cup_{S^1} M_2)$ for hyperbolic 3-manifolds M_1 and M_2

Suppose that M_1 and M_2 are two finite volume hyperbolic 3-manifolds (possibly with cusps), and let $i_k : S^1 \rightarrow M_k$, $k = 1, 2$ be two essential circles. Here, we can assume that both i_k are embeddings into $\text{int}(M_k)$, and we denote the image of i_k by γ_k . It is possible that the element in $\pi_1(M_k)$ corresponding to γ_k is a parabolic element or a nonprimitive element. For simplicity, the readers can think of γ_k as a simple closed geodesic in M_k most of the time.

Let $X = M_1 \cup_{\gamma} M_2$ be the space obtained by identifying γ_1 and γ_2 by a homeomorphism; then we need to show that $\pi_1(X)$ is not LERF. For a standard graph of space, the edge space should be $S^1 \times I$. Here, we directly paste M_1 and M_2 together along the circles, which makes the picture simpler. We also give orientations on γ_1 and γ_2 such that the pasting preserves orientations on these two circles.

For any point in X , either it has a neighborhood homeomorphic to B^3 (the open unit ball in \mathbb{R}^3) or B_+^3 (the points in B^3 with nonnegative z -coordinate), or it has a neighborhood homeomorphic to a union of two B^3 's along $I_z = B^3 \cap (z\text{-axis})$; that is, $B^3 \cup_{I_z} B^3$.

We first give a name for the spaces that locally look like B^3 , B_+^3 , or $B^3 \cup_I B^3$.

Definition 4.1

A compact Hausdorff space X is called a *singular 3-manifold* if, for any point $x \in X$, either it has a neighborhood homeomorphic to B^3 or B_+^3 , or it has a neighborhood homeomorphic to $B^3 \cup_{I_z} B^3$ with $x \in I_z$. We call points in the first class *regular points*, and we call points in the second class *singular points*.

We can think of a singular 3-manifold X as a union of finitely many 3-manifolds along disjoint simple closed curves, and we call each of these 3-manifolds a *3-manifold piece of X* .

In the proof of Theorem 1.3, the concept of a finite semicover played an important role, so we need to define a corresponding concept for singular 3-manifolds. Here, the set of singular points in singular 3-manifolds corresponds to the set of decomposition tori in 3-manifolds.

Definition 4.2

Let Y, Z be two singular 3-manifolds. A map $i : Y \rightarrow Z$ is called a *singular finite semicover* if, for any point $y \in Y$, one of the following holds:

- (1) i maps a neighborhood of y to a neighborhood of $i(y)$ by homeomorphism.
- (2) y is a regular point and $i(y)$ is a singular point, such that i maps a B^3 neighborhood of y to one of the B^3 's in a $B^3 \cup_{I_z} B^3$ neighborhood of $i(y)$ by homeomorphism.

Under a singular finite semicover, all singular points are mapped to singular points, and all regular points not lying in a finite union of simple closed curves in Y are mapped to regular points. It maps each 3-manifold piece of Y to a 3-manifold piece of Z by a finite cover.

It is easy to see that a singular finite semicover $i : Y \rightarrow Z$ induces an injective homomorphism on fundamental groups. The author also believes that a singular finite semicover gives a separable subgroup $\pi_1(Y) < \pi_1(Z)$, but we do not need this result here.

The following lemma corresponds to Lemma 3.4.

LEMMA 4.3

Let $X = M_1 \cup_\gamma M_2$ be a union of two finite volume hyperbolic 3-manifolds along an essential circle. There then exists a singular 3-manifold $Y = N_1 \cup_c N_2$ such that the following hold:

- (1) Y is a union of two hyperbolic 3-manifolds N_1 and N_2 , where N_k is a finite cover of M_k ($k = 1, 2$), and the set of singular points is one oriented circle.
- (2) Each N_k is a fibered 3-manifold with a fixed fibered surface S_k such that the algebraic intersection number $[S_k] \cap [c] = 1$ for $k = 1, 2$.
- (3) Y is a singular finite semicover of X , so $\pi_1(Y)$ is a subgroup of $\pi_1(X)$.

Proof

By Agol's virtual fibering theorem and virtual infinite Betti number theorem (see [3]), there exists a finite cover M'_1 of M_1 such that M'_1 is a fibered 3-manifold and $b_1(M'_1) > 1$. Let $\gamma'_1 \subset M'_1$ be one oriented elevation (one component of the preimage) of $\gamma_1 \subset M_1$. If γ'_1 is null-homologous in M'_1 , we apply Theorem 2.5 to find a further finite cover M''_1 such that γ'_1 lifts to a non-null-homologous curve in M''_1 .

Since the fibered cone is an open set in $H^1(M''_1; \mathbb{R})$, there exists a fibered surface S_1 in M''_1 which has positive intersection number with γ'_1 . So we have $[S_1] \cap [\gamma'_1] = a_1 \in \mathbb{Z}_+$ and $\deg(\gamma'_1 \rightarrow \gamma_1) = b_1$.

By the same construction, we get a finite cover $M''_2 \rightarrow M_2$ with a fibered surface S_2 such that, for some oriented elevation γ'_2 of γ_2 , $[S_2] \cap [\gamma'_2] = a_2 \in \mathbb{Z}_+$ and $\deg(\gamma'_2 \rightarrow \gamma_2) = b_2$.

Let N_1 be the $a_1 b_2$ -sheet cyclic cover of M''_1 along S_1 , and let c_1 be one elevation of γ'_1 . Then $[S_1] \cap [c_1] = 1$ and $\deg(c_1 \rightarrow \gamma_1) = b_1 b_2$. Similarly, let N_2 be the $a_2 b_1$ -sheet cyclic cover of M''_2 along S_2 , and let c_2 be one elevation of γ'_2 . Then $[S_2] \cap [c_2] = 1$ and $\deg(c_2 \rightarrow \gamma_2) = b_1 b_2$.

Since $c_1 \rightarrow \gamma_1$ and $c_2 \rightarrow \gamma_2$ have the same degree, we can identify c_1 and c_2 (as oriented curves) to get the desired singular finite semicover $Y = N_1 \cup_c N_2$. \square

Remark 4.4

Actually, we may get a result as strong as Lemma 3.4; that is, $Y = N_1 \cup_c N_2$ is an $S_1 \vee S_2$ bundle over S^1 . However, we did not state Lemma 4.3 in this way. One reason is that we need to homotopy the curve c_k in N_k to get this fibered structure. Moreover, the closed curve c_k may not (virtually) be a closed orbit of the pseudo-Anosov suspension flow of a (virtual) fibered structure of N_k . So it is not a natural object, from the dynamical point of view. Nevertheless, $S_1 \vee S_2$ is a homotopy fiber of $Y = N_1 \cup_c N_2$, from a homotopy point of view.

For simplicity, we still use $X = M_1 \cup_\gamma M_2$ to denote the singular 3-manifold obtained in Lemma 4.3. Then we have the following lemma corresponding to Lemma 3.5.

LEMMA 4.5

For the singular 3-manifold $X = M_1 \cup_\gamma M_2$ constructed in Lemma 4.3, there exists a singular 3-manifold $Y = N_1 \cup_{c \cup c'} N_2$ such that the following hold:

- (1) Y is a union of two hyperbolic 3-manifolds N_1 and N_2 , where each N_k is a finite cover of M_k ($k = 1, 2$), and the set of singular points consists of two oriented circles.
- (2) The homomorphism $H_1(c \cup c'; \mathbb{Z}) \rightarrow H_1(N_1; \mathbb{Z})$ induced by inclusion is injective.
- (3) For each N_k ($k = 1, 2$), there exists a fibered surface $S'_k \subset N_k$, such that $[S'_k] \cap [c] = [S'_k] \cap [c'] = 1$ holds for the algebraic intersection number.
- (4) Y is a singular finite semicover of X , so $\pi_1(Y)$ is a subgroup of $\pi_1(X)$.

Proof

Let γ_i be the oriented copy of γ in M_i .

By a similar argument as in the proof of Lemma 3.5, and using the virtual retract property of a $\mathbb{Z} * \mathbb{Z} = \langle \pi_1(\gamma), g^n \pi_1(\gamma) g^{-n} \rangle$ subgroup in $\pi_1(M_1)$, we can find a finite cover N_1 of M_1 and two distinct homeomorphic liftings c_1 and c'_1 of $\gamma_1 \subset M_1$, such that $H_1(c_1 \cup c'_1; \mathbb{Z}) \rightarrow H_1(N_1; \mathbb{Z})$ is injective.

In the conclusion of Lemma 4.3, we fixed a fibered surface S_1 in M_1 whose algebraic intersection number with γ_1 is 1. For an elevated fibered surface $S'_1 \subset N_1$, the algebraic intersection numbers of S'_1 with c_1 and c'_1 are both equal to 1.

By doing a similar construction for M_2 (actually, a simpler construction works since we do not require condition (2) for N_2), we get a finite cover N_2 of M_2 , with two homeomorphic liftings c_2 and c'_2 of γ_2 , and a fibered surface S'_2 of N_2 with $[S'_2] \cap [c_2] = [S'_2] \cap [c'_2] = 1$.

Then we paste N_1 and N_2 together by identifying c_1 with c_2 (denoted by c) and identifying c'_1 and c'_2 (denoted by c') to get the desired singular finite semicover Y . \square

For singular 3-manifolds, we need a definition in this singular world that corresponds to immersed surfaces in 3-manifolds.

We first define singular surfaces, which play the same role as surfaces in 3-manifolds.

Definition 4.6

A compact Hausdorff space K is called a *singular surface* if, for any point $k \in K$, either it has a neighborhood homeomorphic to B^2 or B_+^2 , or it has a neighborhood homeomorphic to $B^2 \vee B^2$, with k lying in the intersection of two disks. We call the points in the first class *regular points*, and we call points in the second class *singular points*.

We can think of a singular surface K as a union of finitely many compact surfaces, pasting along finitely many points in their interior. We call each of these surfaces a *surface piece of K* .

Now we define singular immersions from singular surfaces to singular 3-manifolds.

Definition 4.7

Let $i : K \rightarrow X$ be a map from a singular surface to a singular 3-manifold. We say that i is a *singular immersion* if the following conditions hold:

- (1) i maps the singular set of K to the singular set of X .
- (2) The restriction of i on each surface piece of K is a proper immersion from the surface to a 3-manifold piece of X .
- (3) For any singular point $k \in K$, there exist a $B^2 \vee B^2$ neighborhood of k and a $B^3 \cup_{I_z} B^3$ neighborhood of $i(k)$, such that i maps the two B^2 s to distinct B^3 s in $B^3 \cup_{I_z} B^3$, and each B^2 is mapped to the intersection of B^3 with the xy -plane by homeomorphism.

Note that Definition 4.7 is not a good candidate for “proper singular immersion.” Under Definition 4.7, some regular point of K can be mapped to a singular point of X . If we consider the corresponding manifold picture, this corresponds to the case that a boundary component of a surface is mapped to a JSJ torus of a 3-manifold, which is not proper. In the following proposition, we construct a singular immersion that gets rid of this picture in the algebraic topology sense, and the readers may compare it with Proposition 3.6.

PROPOSITION 4.8

For the singular 3-manifold $Y = N_1 \cup_{c \cup c'} N_2$ and fibered surfaces $S'_k \subset N_k$ constructed in Lemma 4.5, there exists a connected singular surface K and a π_1 -injective singular immersion $i : K \hookrightarrow Y$ such that the following hold:

- (1) K is a union of oriented connected subsurfaces as $K = (\Sigma_{1,1} \cup \Sigma_{1,2}) \cup (\bigcup_{k=1}^{2n} \Sigma_{2,k})$, with $i(\Sigma_{1,j}) \subset N_1$ and $i(\Sigma_{2,k}) \subset N_2$.

- (2) *There are $4n$ singular points in K . Each singular point lies in $\Sigma_{1,j} \cap \Sigma_{2,k}$ for some $j \in \{1, 2\}$, $k \in \{1, 2, \dots, 2n\}$, and each $\Sigma_{2,k}$ contains exactly two singular points.*
- (3) *The restrictions of i on the $\Sigma_{1,j}$'s and $\Sigma_{2,k}$'s are all embeddings, and their images are fibered surfaces of N_1 and N_2 , respectively.*
- (4) *Each $\Sigma_{2,k}$ is a copy of S'_2 in N_2 , and the two singular points in $\Sigma_{2,k}$ are mapped to the intersection of $\Sigma_{2,k}$ with c and c' , respectively.*
- (5) *$\Sigma_{1,1} \cap \Sigma_{2,1}$ consists of two singular points p and p' , with $i(p) \in c$ and $i(p') \in c'$.*
- (6) *We have the algebraic intersection numbers $[\Sigma_{1,1}] \cap [c_1] = A$ and $[\Sigma_{1,1}] \cap [c'_1] = B$, with $A \neq B$, and $[\Sigma_{1,2}] \cap [c_1] = 2n - A$ and $[\Sigma_{1,2}] \cap [c'_1] = 2n - B$. Here, c_1 and c'_1 are the oriented copies of c and c' in N_1 , respectively.*
- (7) *The set $\{\text{singular points in } \Sigma_{1,1}\} \cap i^{-1}(c)$ has cardinality A . Suppose this set is $\{a_1, \dots, a_A\}$; then $\Sigma_{1,1}$ has positive local intersection number with c_1 at each a_l . The same statement holds for $\{\text{singular points in } \Sigma_{1,1}\} \cap i^{-1}(c')$ (with A replaced by B), $\{\text{singular points in } \Sigma_{1,2}\} \cap i^{-1}(c)$, and $\{\text{singular points in } \Sigma_{1,2}\} \cap i^{-1}(c')$ (with A replaced by $2n - A$ and $2n - B$, respectively).*
- (8) *For each $l \in \{1, \dots, A\}$, take the embedded oriented subarc of c from $i(a_1)$ to $i(a_l)$. Then slightly move it along the positive direction of c_1 to get an oriented arc ρ_l with endpoints away from $\Sigma_{1,1}$. Then the algebraic intersection number between $\Sigma_{1,1}$ and ρ_l is equal to $l - 1$. Similar statements also hold for $\{\text{singular points in } \Sigma_{1,1}\} \cap i^{-1}(c')$, $\{\text{singular points in } \Sigma_{1,2}\} \cap i^{-1}(c)$, and $\{\text{singular points in } \Sigma_{1,2}\} \cap i^{-1}(c')$.*

This proposition looks more complicated than Proposition 3.6, and we give some remarks here.

Remark 4.9

The conditions (1)–(6) in Proposition 4.8 correspond to the conditions in Proposition 3.6, and conditions (7) and (8) in Proposition 4.8 correspond to the “properness” of this singular immersion. Although we do not assume

$$i^{-1}\{\text{singular points in } Y\} = \{\text{singular points in } K\},$$

conditions (6) and (7) imply that the total algebraic intersection number between $\Sigma_{1,j}$ and c_1 at the points in $(i^{-1}(c) \cap \Sigma_{1,j}) \setminus \{\text{singular points in } \Sigma_{1,j}\}$ is zero, and it also holds for c' . So it is a weak and algebraic version of $i^{-1}(c \cup c') \cap \Sigma_{1,j} = \{\text{singular points in } \Sigma_{1,j}\}$.

Here, we use the algebraic intersection number instead of the geometric intersection number (or number of components in the intersection), as in Proposition 3.6. For

a fibered surface and a closed orbit of the suspension flow (or a boundary component of the 3-manifold), the algebraic intersection number is always equal to the geometric intersection number (or the number of components in the intersection). However, the circles c_1 and c'_1 in N_1 may not be (virtually) closed orbits of the suspension flow, even up to homotopy. Although we can homotopy c_1 and c'_1 such that their algebraic intersection numbers with one fibered surface are equal to the corresponding geometric intersection numbers, here there are two fibered surfaces $\Sigma_{1,1}$ and $\Sigma_{1,2}$ in N_1 , and we may not be able to do so simultaneously for both $\Sigma_{1,1}$ and $\Sigma_{1,2}$.

Proof

By the same argument as in the proof of Proposition 3.6, we can construct two fibered surfaces $\Sigma_{1,1}$ and $\Sigma_{1,2}$ in N_1 such that condition (6) holds. Take $2n$ copies of S'_2 in N_2 , and denote them by $\Sigma_{2,1}, \dots, \Sigma_{2,2n}$.

First, suppose we choose any A , B , $2n - A$, and $2n - B$ points in $\Sigma_{1,1} \cap c_1$, $\Sigma_{1,1} \cap c'_1$, $\Sigma_{1,2} \cap c_1$, and $\Sigma_{1,2} \cap c'_1$, respectively, such that the corresponding surfaces and curves have positive local intersection numbers at these points. If we identify these points with $(\bigcup_{k=1}^{2n} \Sigma_{2,k}) \cap (c_2 \cup c'_2)$ in an arbitrary way, then we get a singular surface K and a singular immersion satisfying conditions (1)–(4), (6), and (7).

So we need to choose these points carefully so that condition (8) holds, and then do the correct pasting such that condition (5) holds.

The choice of these four families of points follows the same process, so we only consider $\Sigma_{1,1} \cap c_1$. Although the algebraic intersection number between $\Sigma_{1,1}$ and c_1 is A , there might be more geometric intersection points. So we assume that there are $A + 2m$ intersection points in $\Sigma_{1,1} \cap c_1$. Take any positive intersection point a'_1 in $\Sigma_{1,1} \cap c_1$. By following the orientation of c_1 , we denote the other points of $\Sigma_{1,1} \cap c_1$ by a'_2, \dots, a'_{A+2m} . For any $l \in \{1, \dots, A + 2m\}$, take the embedded oriented subarc in c_1 from a'_1 to a'_l , then move it slightly along the positive direction of c_1 , and denote it by ρ'_l . Whenever we move from a'_l to a'_{l+1} , the algebraic intersection number $[\Sigma_{1,1}] \cap [\rho'_l]$ differs from $[\Sigma_{1,1}] \cap [\rho'_{l+1}]$ by 1 or -1 , depending on whether $\Sigma_{1,1}$ intersects with c_1 positively or negatively at a'_{l+1} . Since $[\Sigma_{1,1}] \cap [\rho'_1] = 0$ and $[\Sigma_{1,1}] \cap [\rho'_{A+2m}] = A - 1$, it is easy to find A points in $\{a'_1, a'_2, \dots, a'_{A+2m}\}$ (with $a_1 = a'_1$), such that they are all positive intersection points and satisfy condition (8).

Then we can paste the $2n$ points in $(\Sigma_{1,1} \cup \Sigma_{1,2}) \cap c_1$ (and $(\Sigma_{1,1} \cup \Sigma_{1,2}) \cap c'_1$) chosen above with the $2n$ points in $(\bigcup_{k=1}^{2n} \Sigma_{2,k}) \cap c_2$ (and $(\bigcup_{k=1}^{2n} \Sigma_{2,k}) \cap c'_2$) to get a connected singular immersed surface $i : K \looparrowright Y$. By doing isotopy of $\Sigma_{2,1}$ in N_2 , we can make sure the pasting satisfies condition (5).

The π_1 -injectivity of $i : K \looparrowright Y$ follows from the same π_1 -injectivity argument in Lemma 3.6. Note that we do need condition (8) here. \square

Then we show that the above π_1 -injective singular immersion gives a nonseparable subgroup in $\pi_1(Y)$. This proof is similar to the proof of Proposition 3.7.

PROPOSITION 4.10

For the singular immersion $i : K \looparrowright Y$ constructed in Proposition 4.8, $i_*(\pi_1(K)) < \pi_1(Y)$ is a nonseparable subgroup.

Proof

Suppose that $i_*(\pi_1(K)) < \pi_1(Y)$ is separable; then we will get a contradiction.

Since each surface piece of K is mapped to a fibered surface in the corresponding 3-manifold piece of Y , the covering space \tilde{Y} of Y corresponding to $\pi_1(K)$ is homeomorphic to a union of $\Sigma_{1,j} \times \mathbb{R}$ (with $j = 1, 2$) and $\Sigma_{2,k} \times \mathbb{R}$ (with $k = 1, \dots, 2n$) by pasting along the preimage of c_i and c'_i (with $i = 1, 2$). In particular, $i : K \looparrowright Y$ lifts to an embedding in \tilde{Y} . By the separability of $i_*(\pi_1(K))$, [36] implies that there exists an intermediate finite cover $p : \hat{Y} \rightarrow Y$ of $\tilde{Y} \rightarrow Y$ such that $i : K \looparrowright Y$ lifts to an embedding $\hat{i} : K \hookrightarrow \hat{Y}$.

In Proposition 3.7, we took a finite cyclic cover of \hat{N} along Σ . It can be done either geometrically, that is, by taking finitely many copies of $\hat{N} \setminus \Sigma$ and pasting them together, or algebraically, that is, by taking a finite cyclic cover dual to the cohomology class defined by Σ . Here, we will follow the algebraic process.

Now we show that $K \subset \hat{Y}$ defines a cohomology class $\kappa \in H^1(Y; \mathbb{Z})$ by using duality, that is, by taking the algebraic intersection number.

Since K intersects with all 3-manifold pieces of \tilde{Y} , it also intersects with all 3-manifold pieces of \hat{Y} . For each 3-manifold piece \hat{N}_s of \hat{Y} , $K \cap \hat{N}_s$ is a properly embedded oriented surface in \hat{N}_s , so it defines a cohomology class $\kappa_s \in H^1(\hat{N}_s; \mathbb{Z})$.

For each component \hat{c} of $p^{-1}(c \cup c')$, suppose that it is adjacent to \hat{N}_1 and \hat{N}_2 . Then we need to show that $\kappa_1|_{\hat{c}} = \kappa_2|_{\hat{c}}$, that is, that the algebraic intersection numbers $[\hat{N}_1 \cap K] \cap [\hat{c}]$ and $[\hat{N}_2 \cap K] \cap [\hat{c}]$ are equal to each other. Here, \hat{N}_1 and \hat{N}_2 are finite covers of N_1 and N_2 , respectively.

Since $\Sigma_{1,1}$ and $\Sigma_{1,2}$ are different fibered surfaces in N_1 , only one of them lies in \hat{N}_1 . Without loss of generality, we suppose that $K \cap \hat{N}_1 = \Sigma_{1,1}$, and \hat{c} is a component of $p^{-1}(c)$. All other cases follow from the same argument.

Since $\Sigma_{1,1}$ is a fibered surface in both \hat{N}_1 and N_1 , we have that $\hat{N}_1 \rightarrow N_1$ is a finite cyclic cover dual to $\Sigma_{1,1}$, and we let the covering degree be D . Recall that $[\Sigma_{1,1}] \cap [c_1] = A$. Then $p^{-1}(c) \cap \hat{N}_1$ has $\gcd(A, D)$ many components (\hat{c} is one of them), and each of them has algebraic intersection number $\frac{A}{\gcd(A, D)}$ with $\Sigma_{1,1}$.

So we have that $\langle \kappa_1, \hat{c} \rangle = \frac{A}{\gcd(A, D)}$, and we need to show $\langle \kappa_2, \hat{c} \rangle = \frac{A}{\gcd(A, D)}$.

We first show that, for $\hat{i}|_{\Sigma_{1,1}} : \Sigma_{1,1} \rightarrow \hat{N}_1$, there are exactly $\frac{A}{\gcd(A, D)}$ points in $\{a_1, \dots, a_A\}$ that are mapped to \hat{c} . For two points $a_s, a_t \in \{a_1, \dots, a_A\}$, if $\hat{i}(a_s)$ and

$\hat{i}(a_t)$ lie in the same component of $p^{-1}(c) \cap \hat{N}_1$, then there is an oriented subarc τ of $p^{-1}(c)$ from $\hat{i}(a_s)$ to $\hat{i}(a_t)$. Take an oriented path γ in $\Sigma_{1,1}$ from a_s to a_t . Then $\tau \cdot \hat{i}(\gamma^{-1})$ is a loop in \hat{N}_1 and it projects to a loop δ in N_1 .

Since $\tau \cdot \hat{i}(\gamma^{-1})$ is a loop in \hat{N}_1 , the algebraic intersection number of $\Sigma_{1,1}$ with δ is a multiple of D . On the other hand, δ consists of the projection of τ and $\hat{i}(\gamma^{-1})$ in N_1 , which helps us to compute the algebraic intersection number by another way. Since γ lies in $\Sigma_{1,1}$, its projection in N_1 has algebraic intersection number 0 with $\Sigma_{1,1}$. Since $p \circ \tau$ is a path on c_1 with initial point a_s and terminal point a_t , by condition (8) in Proposition 4.8, the algebraic intersection number between $\Sigma_{1,1}$ and $p \circ \tau$ equals $nA + (t - s)$ for some $n \in \mathbb{Z}$. (Actually, we need to slightly push τ and $\hat{i}(\gamma^{-1})$ along the positive direction of the corresponding component of $p^{-1}(c)$, such that their endpoints are away from $\Sigma_{1,1}$.)

From these two computations of the algebraic intersection number between $\Sigma_{1,1}$ and δ , we get that $mD = nA + (t - s)$ holds for some integers m and n . So $t - s$ is a multiple of $\gcd(A, D)$. It implies that, for each component of $p^{-1}(c) \cap \hat{N}_1$, there are exactly $\frac{A}{\gcd(A, D)}$ points in $\{a_1, \dots, a_A\}$ mapped to it. In particular, it holds for \hat{c} .

There are exactly $\frac{A}{\gcd(A, D)}$ points in $\{a_1, \dots, a_A\} \cap \hat{c}$. Each of them lies in a fibered surface in $K \cap \hat{N}_2$, and the algebraic intersection number between the fibered surface and \hat{c} is 1. So we have $\langle \kappa_2, \hat{c} \rangle = \frac{A}{\gcd(A, D)} = \langle \kappa_1, \hat{c} \rangle$.

By a Mayer–Vietoris sequence argument, we get that $\hat{i} : K \hookrightarrow \hat{Y}$ defines a cohomology class $\kappa \in H^1(Y; \mathbb{Z})$ by taking the algebraic intersection number of any 1-cycle in Y with K .

As in Proposition 3.7, we take a finite cover of \hat{Y} dual to κ to get a further finite cover $q : \bar{Y} \rightarrow Y$ such that K embeds in \bar{Y} . We can further require that each component of $q^{-1}(c \cup c')$ intersects with exactly two surface pieces in K , with algebraic intersection number 1. Let \bar{N}_1 and \bar{N}_2 be the 3-manifold pieces of \bar{Y} containing $\Sigma_{1,1}$ and $\Sigma_{2,1}$, respectively, and let $\bar{c}, \bar{c}' \subset \bar{N}_1 \cap \bar{N}_2$ be the singular circles containing the two points in $\Sigma_{1,1} \cap \Sigma_{1,2}$. Then we can compute the relation between $\deg(\bar{c} \rightarrow c)$ and $\deg(\bar{c}' \rightarrow c')$ from both $\deg(\bar{N}_1 \rightarrow N_1)$ and $\deg(\bar{N}_2 \rightarrow N_2)$, and get a contradiction as in Proposition 3.7. \square

Now we are ready to prove Theorem 1.4.

Proof

For a singular 3-manifold $X = M_1 \cup_\gamma M_2$, Lemmas 4.3 and 4.5 imply that there exists a singular finite semicover $Y = N_1 \cup_{c \cup c'} N_2$ of X such that the conditions in Lemma 4.5 hold.

By Proposition 4.8, there exists a π_1 -injective singular immersion $i : K \looparrowright Y$ satisfying the conditions in Proposition 4.8. Then Proposition 4.10 implies that

$i_*(\pi_1(K))$ is not separable in $\pi_1(Y)$. Since $\pi_1(Y)$ is a subgroup of $\pi_1(X)$, Lemma 2.3 implies that $i_*(\pi_1(K))$ is not separable in $\pi_1(X)$.

If both M_1 and M_2 are 3-manifolds with boundary, then K is a union of surfaces with boundary along finitely many points, making $\pi_1(K)$ a free group. If at least one of M_1 and M_2 is a closed 3-manifold, then K is a union of closed surfaces and (possibly empty set of) bounded surfaces along finitely many points, so $\pi_1(K)$ is a free product of free groups and surface groups. \square

The following direct corollary of Theorem 1.4 implies that any Higman–Neumann–Neumann (HNN) extension of a hyperbolic 3-manifold group along cyclic subgroups is not LERF.

The readers may compare this corollary with the result in [28], which gives a sufficient and necessary condition for an HNN extension of a free group along cyclic subgroups being LERF. Note that Niblo’s condition holds for a generic pair of cyclic subgroups in a free group.

COROLLARY 4.11

Let M be a finite volume hyperbolic 3-manifold, and let $A, B < \pi_1(M)$ be two infinite cyclic subgroups with an isomorphism $\phi : A \rightarrow B$. Then the HNN extension

$$\pi_1(M) *_{A'=B} = \langle \pi_1(M), t \mid tat^{-1} = \phi(a), \forall a \in A \rangle$$

is not LERF.

Proof

Let $\pi_1(M)_{A'=B} \rightarrow \mathbb{Z}_2$ be the homomorphism which kills all elements in $\pi_1(M)$ and maps t to $\bar{1} \in \mathbb{Z}_2$. Then the kernel H is an index-2 subgroup of $\pi_1(M) *_{A'=B}$.

The subgroup H has a graph of group structure such that the graph consists of two vertices and two edges connecting these two vertices (a bigon). Both vertex groups are isomorphic to $\pi_1(M)$, and both edge groups are infinite cyclic. So H contains a subgroup that is a \mathbb{Z} -amalgamation of two copies of $\pi_1(M)$. Then Theorem 1.4 and Lemma 2.3 imply that $\pi_1(M)_{A'=B}$ is not LERF. \square

4.2. More general cases

Actually, the proof of Theorem 1.4 only uses the machinery on hyperbolic 3-manifolds for M_1 (virtual retract property of geometrically finite subgroups), and M_2 only needs to satisfy some mild conditions. So we have the following generalization of Theorem 1.4.

THEOREM 4.12

Let M_1 be a finite-volume hyperbolic 3-manifold, and let M_2 be a compact fibered

manifold over the circle; that is, $M_2 = N \times I / \phi$ for some orientation-preserving self-homeomorphism $\phi : N \rightarrow N$ on a compact n -manifold N ($n > 0$). We also suppose that $\pi_1(N)$ has some nontrivial finite quotient.

Let $S^1 \rightarrow M_1$ be an essential circle in M_1 , and let $S^1 \rightarrow M_2$ be an essential circle that has nonzero algebraic intersection number with N . Then the \mathbb{Z} -amalgamation

$$\pi_1(M_1 \cup_{S^1} M_2)$$

is not LERF.

We give a sketch of the proof which parallels the proof of Theorem 1.4. It is not hard to see that the proof works even if we only assume that N is a CW complex and M_2 is a mapping torus of N (with $\pi_1(N)$ still in need of having a nontrivial finite quotient), but we prefer to state only the result for the manifold case.

Proof

At first, we can find a singular finite semicover $M'_1 \cup_\gamma M'_2$ such that similar conditions in Lemma 4.3 hold. For M_1 , we still use the virtual fibering theorem, the virtual infinite Betti number theorem, and the virtual retract property to find a fibered structure in some finite cover of M_1 , such that conditions (1) and (2) in Lemma 4.3 hold. For M_2 , it already has a fibered structure, and we may only need to take a finite cyclic cover of M_2 along N .

Then we can find a further singular semicover $N_1 \cup_{c \cup c'} N_2$ such that similar conditions in Lemma 4.5 hold. For M'_1 , we still use the virtual retract property and LERFness to get a finite cover N' satisfying conditions (2) and (3) in Lemma 4.5. For M'_2 , we use the fact that $\pi_1(N)$ admits a nontrivial finite quotient to find a finite cover N_2 of M'_2 , such that the preimage of γ in N_2 contains at least two components, and they are mapped to γ by homeomorphisms.

In the construction of the π_1 -injective immersed singular object (not a singular surface if $n \neq 2$) in Proposition 4.8, we only perturb fibered structures in N_1 , do all nontrivial works over there, and always use the original fibered structure of N_2 . So the same construction gives a π_1 -injective immersed singular object in $N_1 \cup_{c \cup c'} N_2$, which satisfies the conditions in Proposition 4.8.

The proof of Proposition 4.10 does not use any 3-manifold topology. It uses only the fiber bundle over circle structures and counts covering degrees. So the same proof shows that the above π_1 -injective immersed singular object gives a nonseparable subgroup in $\pi_1(N_1 \cup_{c \cup c'} N_2)$, which is also nonseparable in $\pi_1(M_1 \cup_{S^1} M_2)$.

The nonseparable subgroup constructed above is a free product of surface groups, finite-index subgroups of $\pi_1(N)$, and free groups. \square

Since the perturbation of fiber bundle over circle structures works in any dimension, we have the following further corollary.

COROLLARY 4.13

Let M_1 be a finite-volume hyperbolic 3-manifold, and let M_2 be a compact manifold with a fiber bundle over circle structure and $b_1(M_2) \geq 2$.

Let $S^1 \rightarrow M_1$ be an essential circle in M_1 , and let $S^1 \rightarrow M_2$ be a circle in M_2 with nonzero image in $H_1(M_2; \mathbb{Q})$. Then the \mathbb{Z} -amalgamation

$$\pi_1(M_1 \cup_{S^1} M_2)$$

is not LERF.

Proof

At first, $b_1(M_2) \geq 2$ implies that, for any fiber bundle over circle structure $M_2 = N \times I/\phi$, $b_1(N) \geq 1$ holds. So for any such N , $\pi_1(N)$ has a nontrivial finite quotient.

We take any fibered structure of M_2 and write M_2 as $M_2 = N \times I/\phi$. By perturbing the fibered structure on M_2 , we can assume that $[N]$ has a nonzero algebraic intersection number with $[S^1] \in H_1(M_2; \mathbb{Z})$. So we are in the situation of Theorem 4.12, and $\pi_1(M_1 \cup_{S^1} M_2)$ is not LERF. \square

5. Non-LERFness of arithmetic hyperbolic manifold groups

In this section, we give the proof of Theorems 1.1 and 1.2, as well as some further results on non-LERFness of high-dimensional nonarithmetic hyperbolic manifold groups. These results imply that most known examples of high-dimensional hyperbolic manifolds have non-LERF fundamental groups.

For all proofs in this section, to prove that a group is not LERF, we only need to show that it contains a subgroup isomorphic to one of the non-LERF groups in Theorems 1.3 or 1.4.

We start with proving Theorem 1.2, which claims that all noncompact arithmetic hyperbolic manifolds with dimension at least 4 have non-LERF fundamental groups.

Proof

We start with the case that M is a noncompact standard arithmetic hyperbolic manifold.

We first show that M contains a (immersed) noncompact, totally geodesic 3-dimensional submanifold N . This is well known for experts, but the author did not find a reference on it, so we give a short proof here.

Since M is noncompact, it is defined by \mathbb{Q} and a nondegenerate quadratic form $f : \mathbb{Q}^{m+1} \rightarrow \mathbb{Q}$ with negative inertia index 1. Let the symmetric bilinear form defining f be denoted by $B(\cdot, \cdot)$.

Since M is not compact, f represents 0 nontrivially in \mathbb{Q}^{m+1} , and thus there exists $\vec{w} \neq \vec{0} \in \mathbb{Q}^{m+1}$ such that $B(\vec{w}, \vec{w}) = f(\vec{w}) = 0$. Since f is nondegenerate, there exists $\vec{v} \neq 0 \in \mathbb{Q}^{m+1}$ such that $B(\vec{v}, \vec{w}) \neq 0$. Let $V = \text{span}_{\mathbb{Q}}(\vec{v}, \vec{w})$. Then it is easy to check that $V^\perp \cap V = \{\vec{0}\}$, and the restriction of $B(\cdot, \cdot)$ on V^\perp is positive definite.

Let $(\vec{v}_1, \dots, \vec{v}_{m-1})$ be a \mathbb{Q} -basis of V^\perp such that $B(\vec{v}_i, \vec{v}_j) = \delta_{ij}$ for $i, j \in \{1, \dots, m-1\}$. Then $W = \text{span}_{\mathbb{Q}}(\vec{v}_1, \vec{v}_2, \vec{v}, \vec{w})$ is a 4-dimensional subspace of \mathbb{Q}^{m+1} , such that the restriction of f on W has negative inertia index 1, and f represents 0 nontrivially in W .

So W and $f|_W$ define a (immersed) noncompact totally geodesic 3-dimensional suborbifold in $\mathbb{H}^m/\text{SO}_0(f, \mathbb{Z})$, which gives a (immersed) noncompact, totally geodesic 3-dimensional submanifold N^3 in M .

Now we are ready to prove the theorem. Here, we consider M and N^3 as compact manifolds by truncating their horocusps.

Each boundary component of M has a naturally induced Euclidean structure, so it is finitely covered by T^{m-1} , and each boundary component of N is homeomorphic to T^2 . We first take two copies of N . For each T^2 component of ∂N , take a long-enough immersed $T^2 \times I$ in the corresponding boundary component of M , which is finitely covered by $T^{m-1} = (T^2 \times S^1) \times T^{m-4}$, such that the T^2 factor of $T^2 \times I$ is identified with $(T^2 \times *) \times * \subset (T^2 \times S^1) \times T^{m-4}$, and the I factor wraps around the S^1 factor. This construction is the same with the Freedman tubing construction in dimension 3 (see [16]). In [25], it is shown that as long as the I factor wraps around S^1 for sufficiently many times, this immersed $N \cup (\partial N \times I) \cup N$ is π_1 -injective, and so $\pi_1(N \cup (\partial N \times I) \cup N) < \pi_1(M)$.

Topologically, $N \cup (\partial N \times I) \cup N$ is just the double of N along ∂N . Since the double of N is a closed mixed 3-manifold with nontrivial geometric decomposition, Theorem 1.3 implies that $\pi_1(N \cup (\partial N \times I) \cup N)$ is not LERF. Then Lemma 2.3 implies that $\pi_1(M)$ is not LERF.

Moreover, Theorem 1.3 implies the existence of nonseparable free subgroups and nonseparable surface subgroups in $\pi_1(M)$.

If M is a noncompact arithmetic hyperbolic manifold defined by quaternions, then it also contains noncompact 3-dimensional totally geodesic submanifolds, by doing the same process as above for quadratic forms over quaternions. So the above proof also works in the quaternion case.

Since 7-dimensional arithmetic hyperbolic manifolds defined by octonions are all compact, the proof is done. \square

Then we give the proof of Theorem 1.1, which claims that all arithmetic hyperbolic manifolds with dimension at least 5 which are not those sporadic examples in

dimension 7 have non-LERF fundamental groups. In this proof, we use two totally geodesic 3-dimensional submanifolds instead of just using one such submanifold as in the proof of Theorem 1.2.

Proof

We first suppose that M^m is a standard arithmetic hyperbolic manifold, with $m \geq 5$.

By the definition of standard arithmetic hyperbolic manifolds, there exists a totally real number field K , and a nondegenerate quadratic form $f : K^{m+1} \rightarrow K$ defined over K , such that the negative inertial index of f is 1 and f^σ is positive definite for all nonidentity embeddings $\sigma : K \rightarrow \mathbb{R}$. Moreover, $\pi_1(M)$ is commensurable with $\mathrm{SO}_0(f; \mathcal{O}_K)$. So to prove that $\pi_1(M)$ is not LERF, we need only to show $\mathrm{SO}_0(f; \mathcal{O}_K)$ is not LERF.

We first diagonalize the quadratic form f such that the symmetric matrix corresponding to f is $A = \mathrm{diag}(k_1, \dots, k_m, k_{m+1})$, with $k_1, \dots, k_m > 0$ and $k_{m+1} < 0$.

First suppose that there exists $i \in \{1, \dots, m\}$ such that $-\frac{k_i}{k_{m+1}}$ is not a square in K , and we can assume $i = 1$. Then f has two quadratic subforms defined by $\mathrm{diag}(k_1, k_2, k_3, k_{m+1})$ and $\mathrm{diag}(k_1, k_4, k_5, k_{m+1})$, respectively. These two subforms satisfy the conditions for defining arithmetic groups in $\mathrm{Isom}_+(\mathbb{H}^3)$, and we denote these two subforms by f_1 and f_2 .

Then $\mathrm{SO}_0(f_1; \mathcal{O}_K)$ and $\mathrm{SO}_0(f_2; \mathcal{O}_K)$ are both subgroups of $\mathrm{SO}_0(f; \mathcal{O}_K)$. Each of them fix a 3-dimensional totally geodesic plane in \mathbb{H}^m , and these two planes perpendicularly intersect with each other along a 1-dimensional bi-infinite geodesic. (Here, we do use that $m \geq 5$.) We denote these two 3-dimensional planes by P_1 and P_2 with $P_1 \cap P_2 = L$. Then $M_i = P_i / \mathrm{SO}_0(f_i; \mathcal{O}_K)$ is a hyperbolic 3-orbifold for each $i = 1, 2$. Moreover, it is easy to see that $\mathrm{SO}_0(f_1; \mathcal{O}_K) \cap \mathrm{SO}_0(f_2; \mathcal{O}_K) = \mathrm{SO}_0(f_3; \mathcal{O}_K)$, where f_3 is defined by $\mathrm{diag}(k_1, k_{m+1})$ and $\mathrm{SO}_0(f_3; \mathcal{O}_K)$ fixes L . The condition that $-\frac{k_1}{k_{m+1}}$ is not a square in K implies that f_3 only represents 0 trivially in K^2 , and so $\mathrm{SO}_0(f_3; \mathcal{O}_K) \cong \mathbb{Z}$.

By a routine argument in hyperbolic geometry and using the LERFness of hyperbolic 3-manifold groups (see, e.g., [6, Lemma 7.1]), there exist torsion-free finite-index subgroups $\Lambda_i < \mathrm{SO}_0(f_i; \mathcal{O}_K)$ with $\mathrm{SO}_0(f_3; \mathcal{O}_K) < \Lambda_i$ for $i = 1, 2$, and the subgroup of $\mathrm{SO}_0(f; \mathcal{O}_K)$ generated by Λ_1 and Λ_2 is isomorphic to $\Lambda_1 *_\mathbb{Z} \Lambda_2$.

So $\mathrm{SO}_0(f; \mathcal{O}_K)$ contains a subgroup $\Lambda_1 *_\mathbb{Z} \Lambda_2$, which is the fundamental group of $M_1 \cup_\gamma M_2$ for two hyperbolic 3-manifolds M_1 and M_2 . By Theorem 1.4, $\mathrm{SO}_0(f; \mathcal{O}_K)$ is not LERF and $\pi_1(M)$ is not LERF.

If M is closed, then both M_1 and M_2 are closed, and the nonseparable subgroup can be chosen to be a free product of closed surface groups and free groups. If M has cusps, then the nonseparable subgroup might be a free group.

If $-\frac{k_i}{k_{m+1}}$ is a square in K for all $i \in \{1, \dots, m\}$, then the quadratic form f is equivalent to the diagonal form $\text{diag}(\underbrace{1, \dots, 1}_m, -1)$ and $K = \mathbb{Q}$. It is easy to check that f is also equivalent to the diagonal form $\text{diag}(2, 2, \underbrace{1, \dots, 1}_{m-2}, -1)$, and we reduce to the previous case.

If M^m is an arithmetic hyperbolic manifold defined by a quadratic form over quaternions, then we can also find two totally geodesic 3-dimensional submanifolds intersecting along one circle. This can be done by diagonalizing the (skew-Hermitian) matrix with quaternion entries and taking two 2×2 submatrices with one common entry, which contributes to the negative inertia index. Then the same proof as above also works in this case. \square

Remark 5.1

Actually, the nonseparable subgroups constructed in this proof are not geometrically finite, so it is consistent with the result in [6] (geometrically finite subgroups of standard arithmetic hyperbolic manifold groups are separable). For a cocompact lattice $\Lambda < \text{Isom}_+(\mathbb{H}^n)$ and a finitely generated subgroup $H < \Gamma$, H is geometrically finite if and only if H is a quasiconvexity subgroup of Λ (see [37]), which is equivalent to saying that the inclusion $H \hookrightarrow \Lambda$ is a quasi-isometric embedding ([4]).

In our construction, the nonseparable subgroup $H < \pi_1(M^m)$ is a free product $H \cong H_1 * H_2$, where H_1 is a fibered surface subgroup of a (immersed) 3-dimensional totally geodesic submanifold M_1 of M^m . So, since $H_1 \hookrightarrow H_1 * H_2 \cong H$ and $\pi_1(M_1) \hookrightarrow \pi_1(M^m)$ are both quasi-isometric embeddings, if $H \hookrightarrow \pi_1(M^m)$ is a quasi-isometric embedding, then $H_1 \hookrightarrow \pi_1(M_1)$ must also be a quasi-isometric embedding. However, this is impossible since fibered surface subgroups of hyperbolic 3-manifold groups have exponential distortion:

$$\begin{array}{ccc} H_1 & \rightarrow & H \cong H_1 * H_2 \\ \downarrow & & \downarrow \\ \pi_1(M_1) & \rightarrow & \pi_1(M^m) \end{array}$$

In [2], [18], and [5], the authors did cut-and-paste surgery on standard arithmetic hyperbolic manifolds along codimension-1 totally geodesic arithmetic submanifolds and constructed many nonarithmetic hyperbolic manifolds.

In [18], the authors took two noncommensurable standard arithmetic hyperbolic m -manifolds, cut them along isometric codimension-1 totally geodesic submanifolds, and then glued them together in another way. This process is called ‘‘interbreeding.’’ In [2] and [5], the authors cut one standard arithmetic hyperbolic m -manifold along two isometric codimension-1 totally geodesic submanifolds and then glued it back in

a different way. This process is called “inbreeding,” which was first carried out in [2] for the 4-dimensional case and then generalized to higher dimensions in [5].

Since all manifolds constructed in [2], [18], and [5] contain codimension-1 totally geodesic arithmetic submanifolds, we have the following direct corollary of Theorem 1.1.

THEOREM 5.2

If M^m is a nonarithmetic hyperbolic m -manifold constructed in [18] or [5], with $m \geq 6$, then $\pi_1(M)$ is not LERF.

Moreover, if M is closed, then there exists a nonseparable subgroup isomorphic to a free product of surface groups and free groups. If M is not closed, then there exists a nonseparable subgroup isomorphic to either a free subgroup or a free product of surface groups and free groups.

Another geometric way to construct hyperbolic m -manifolds is the reflection group method. Suppose P is a finite-volume polyhedron in \mathbb{H}^m such that any two codimension-1 faces that intersect with each other have dihedral angle $\frac{\pi}{n}$ with integer $n \geq 2$. Then the group generated by reflections along codimension-1 faces of P is a discrete subgroup of $\text{Isom}(\mathbb{H}^m)$ with finite covolume.

For any torsion-free finite-index subgroup of a reflection group consisting of orientation-preserving isometries, the quotient of \mathbb{H}^m is a finite-volume hyperbolic m -manifold M . Here, M is a closed manifold if and only if P is compact. The hyperbolic manifolds constructed by this method are not necessarily arithmetic, and it is known that there exist closed nonarithmetic reflection hyperbolic manifolds with dimension ≤ 5 and noncompact nonarithmetic reflection hyperbolic manifolds with dimension ≤ 10 (see [42, Chapter 6.3.2]).

When $m \geq 5$, it is easy to see that M still contains two totally geodesic 3-dimensional submanifolds intersecting along a closed geodesic. To get such a picture, we take two totally geodesic 3-dimensional planes in \mathbb{H}^m that contain two 3-dimensional faces of P and intersect with each other along one edge of P . Then their images in M are two immersed totally geodesic 3-dimensional submanifolds. Similarly, for any $m \geq 4$, noncompact reflection hyperbolic m -manifolds also have noncompact totally geodesic 3-dimensional submanifolds.

So we get the following theorem for finite-volume hyperbolic manifolds that arise from reflection groups. The proof is exactly the same as the proof of Theorems 1.1 and 1.2.

THEOREM 5.3

Let M be either a closed hyperbolic m -manifold, such that $m \geq 5$, or a noncompact

finite-volume hyperbolic m -manifold with $m \geq 4$. If $\pi_1(M)$ is commensurable with the reflection group of some finite-volume polyhedron in \mathbb{H}^m , then $\pi_1(M)$ is not LERF.

Moreover, if M is closed, then there exists a nonseparable subgroup isomorphic to a free product of surface groups and free groups. If M is noncompact, then there exists a nonseparable subgroup isomorphic to a free group and another nonseparable subgroup isomorphic to a surface subgroup.

By the dimension reason, there is no π_1 -injective $M_1 \cup_\gamma M_2$ submanifold in a 4-dimensional (arithmetic) hyperbolic manifold, so Theorem 1.4 does not give us any non-LERF fundamental group in dimension 4. Actually, the non-LERFness of 4-dimensional closed arithmetic hyperbolic manifold groups is proved in the author's more recent work [38].

6. Further questions

In this section, we raise a few questions related to the results in this paper.

(1) In Remark 3.9, we get that, for any mixed 3-manifold M , there exists a finite cover M' of M and a π_1 -injective properly immersed subsurface $\Sigma \looparrowright M'$, such that $\pi_1(\Sigma)$ is not contained in any finite-index subgroup of $\pi_1(M')$. We may ask whether taking this finite cover is necessary.

Question 6.1

For any mixed 3-manifold M , does there exist a π_1 -injective (properly) immersed subsurface $\Sigma \looparrowright M$ such that $\pi_1(\Sigma)$ is not contained in any finite-index subgroup of $\pi_1(M)$?

(2) None of the results in this paper cover the case of compact (arithmetic) hyperbolic 4-manifolds, since they contain neither $M_1 \cup_\gamma M_2$ as a singular submanifold nor a \mathbb{Z}^2 subgroup (or a mixed 3-manifold group as its subgroup).

One possible approach for compact (arithmetic) hyperbolic 4-manifolds is to study the group of $M_1 \cup_S M_2$, with M_1 and M_2 being compact arithmetic hyperbolic 3-manifolds and with S being a hyperbolic surface embedded in both M_1 and M_2 . In this case, the edge group is a closed surface group, which is much more complicated than \mathbb{Z} or \mathbb{Z}^2 . The method in this paper seems to not work directly in this case. Even if it works (under some clever modification), the nonseparable (finitely generated) subgroup constructed by this method would be infinitely presented.

This question is actually solved in the author's more recent work [38], which is built on the constructions in this paper.

(3) Given the non-LERFness results of high-dimensional (arithmetic) hyperbolic manifolds in this paper, maybe it is not too ambitious to ask the following question about general high-dimensional hyperbolic manifolds.

Question 6.2

Do all finite-volume hyperbolic manifolds with dimension at least 4 have non-LERF fundamental groups?

The main difficulty is that we do not have many examples of finite-volume high-dimensional hyperbolic manifolds. To the best of the author's knowledge, the main methods for constructing high-dimensional hyperbolic manifolds (with dimension at least 4) are the arithmetic method, the interbreeding and inbreeding method, and the reflection group method. In this paper, it is shown in Theorems 1.1, 1.2, 5.2, and 5.3 that these three constructions give non-LERF fundamental groups in dimensions at least 5 (not 7-dimensional sporadic examples), at least 6, and at least 5, respectively. Besides these methods, there are other constructions of high-dimensional hyperbolic manifolds that invoke some specific right-angled hyperbolic polytopes (see, e.g., [12], [14], [15], [21], [33]). The hyperbolic manifolds obtained by these constructions also contain many totally geodesic 3-dimensional submanifolds. If the dimension is at least 5, then Theorem 1.1 implies that these manifolds have non-LERF fundamental groups, and [38] confirms the non-LERFness when the dimension equals 4.

However, it is difficult to understand a general high-dimensional hyperbolic manifold if we do not assume that it lies in one of the above families. The author does not know whether a general high-dimensional hyperbolic manifold group contains 3-manifold subgroups. Maybe a generalization of [20] (which shows that each closed hyperbolic 3-manifold admits a π_1 -injective immersed almost totally geodesic closed subsurface) can do this job, but it seems to be very difficult.

(4) The author expects that the method in this paper can be used to prove that more groups are not LERF. However, since the author does not have very broad knowledge in group theory, we only consider groups of finite-volume hyperbolic manifolds in this paper, which is one of the author's favorite family of groups.

The author also expects that the method in this paper can be translated to a purely algebraic proof instead of a geometric one. Actually, most parts of the proof are essentially algebraic, except for one point. In Propositions 3.6 and 3.7 (also Propositions 4.8 and 4.10), although the essential part that gives the nonseparability is $\Sigma_{1,1} \cup \Sigma_{2,1}$, we still need to take a bigger (singular) surface so that it defines a nontrivial 1-dimensional cohomology class in some finite cover. Then we take a finite cyclic cover dual to this cohomology class and get a contradiction. Although it seems that this process can be done algebraically, the author does not know how to work it out.

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