

Kodaira dimension of algebraic fiber spaces over abelian varieties

Junyan Cao¹ · Mihai Păun²

Received: 2 May 2015 / Accepted: 19 May 2016
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Abstract In this article we provide a proof of the Iitaka C_{nm} conjecture for algebraic fiber spaces over tori.

1 Introduction

Let $p : X \rightarrow Y$ be an algebraic fiber space, i.e. X and Y are non-singular projective manifolds and p is surjective with connected fibers. An important problem in birational geometry is the *Iitaka conjecture*, stating that

$$\kappa(X) \geq \kappa(Y) + \kappa(X/Y) \quad (1.1)$$

where $\kappa(X)$ is the Kodaira dimension of X , and $\kappa(X/Y)$ is the Kodaira dimension of a general fiber of p .

In this article our goal is to show that *the log-version of the inequality (1.1) holds true, provided that the base Y is an abelian variety*; this generalizes

✉ Junyan Cao
junyan.cao@imj-prg.fr
Mihai Păun
paun@kias.re.kr

¹ Institut de Mathématiques de Jussieu, Université Paris 6, 4, Place Jussieu, 75252 Paris, France

² Korea Institute for Advanced Study, 85 Hoegiro, Dongdaemun-gu, Seoul 130-722, South Korea

the result obtained by Kawamata in 1982, cf. [25]. More precisely, our main theorem states as follows.

Theorem 1.1 *Let $p : X \rightarrow A$ be an algebraic fiber space, where A is an Abelian variety. Let Δ be an effective \mathbb{Q} -divisor such that the pair (X, Δ) is klt, and let F be a generic fiber of p . Then*

$$\kappa(K_X + \Delta) \geq \kappa(K_F + \Delta_F), \tag{1.2}$$

where $\Delta_F = \Delta|_F$.

We will give next a few hints about the proof, so as to situate our work in the impressive body of papers dedicated to this problem, cf. [6, 7, 16–18, 21, 22, 24, 28–30, 35, 37, 39, 40] among many others. The key ingredient of their proof is the positivity of direct image sheaves $p_*(mK_{X/Y} + m\Delta)$ (notable exceptions to this statement are the works of Hacon–Chen cf. [18] and Birkar–Chen [7], respectively), where $m \in \mathbb{N}$ such that $m\Delta$ is a line bundle. Up to a certain point, the arguments presented here follow this *main stream*.

Roughly speaking, we only have to deal with the following two extreme cases: either the determinant of the direct image $p_*(mK_{X/Y} + m\Delta)$ is big, or it is topologically trivial. This is obtained as consequence of Theorem 3.3 in Sect. 3.

In the first case we show that the relative canonical bundle $K_{X/Y} + \Delta$ is greater than the p -inverse image of an ample divisor, up to a divisor whose image has codimension at least two in Y . This will allow us to extend pluricanonical sections from the fibers of p , and therefore proves Theorem 1.1 in the case under discussion. The main result involved in this part of our proof is Theorem 3.4, which is nothing but a generalization of Viehweg’s *weak semi-stability* results, cf. [37, 40].

If the determinant of the direct image is flat, then we consider the direct image sheaf

$$p_*(mK_{X/Y} + m\Delta),$$

where m is as above. General results show the existence of a subset $\Sigma_0 \subset Y$ whose codimension is at least two, such that the restriction

$$\mathcal{E} := p_*(mK_{X/Y} + m\Delta)|_{Y \setminus \Sigma_0}$$

corresponds to a vector bundle. Now, if the vector bundle \mathcal{E} is e.g. trivial, then it is a simple matter to extend the pluricanonical sections $mK_{X/Y} + m\Delta|_{X_y}$ defined on a general enough fiber X_y . Of course, a priori we cannot expect this to happen; nevertheless, we show that there exists an open set $\Sigma \subset Y$ whose codimension is at least two, such that

$$\mathcal{E}|_{Y \setminus \Sigma}$$

is a Hermitian flat vector bundle when endowed with a canonical L^2 metric (the so-called “Narasimhan-Simha”). This relies on the metric properties of the sheaves $p_*(mK_{X/Y} + m\Delta)$ (for which we refer to [34]), combined with a result of Raufi concerning the existence of the curvature current corresponding to a singular Hermitian metric (see the results in [36]).

We remark that the classification of flat vector bundles on elliptic curves used in [25] is replaced in this paper by the following considerations. The bundle \mathcal{E} considered above gives a unitary representation ρ of rank r of $\pi_1(A \setminus \Sigma)$. As we have already mentioned, the analytic set Σ has codimension at least 2, so the fundamental group of its complement in A equals $\pi_1(A)$, which is a free abelian group. If the image of ρ is finite, then an appropriate power of any section of the restriction $(mK_{X/Y} + m\Delta)|_{X_y}$ extends to $X \setminus p^{-1}(\Sigma)$. Moreover, the resulting section has finite L^2 norm, so it extends across $p^{-1}(\Sigma)$. Next, a unitary representation of any free abelian group splits as a direct sum of representations of rank one. If the image of the representation ρ is infinite, then one of the factors of the splitting will have the same property. In this case we conclude by using another crucial result due Campana-Peternell cf. [12, Theorem 3.1], as well as the generalization in [11].

Our article is organized as follows. In Sect. 2 we will collect some basic facts concerning the construction of metrics on relative pluricanonical bundles and their direct images, as well as a few results concerning singular Hermitian metrics on vector bundles. The main result we establish in Sect. 3 is Theorem 3.4; the techniques needed to prove it refine the arguments of E. Viehweg and H. Tsuji, among many others. The proof of the inequality (1.2) is completed in Sect. 4. As a complement to the techniques and results we obtain in this article, we establish in Sect. 5 a version of (1.2) for an arbitrary algebraic fiber space $p : X \rightarrow Y$ for which the line bundle $\det p_*(mK_{X/Y} + m\Delta)$ is topologically trivial.

2 Positivity of direct images: a few techniques and results

The positivity results for direct images of twisted pluricanonical relative bundles are part of the main tools in our proof. In this section we recall the construction of the Bergman metric, and some of its properties; we will equally collect a few results taken from [2, 4, 34]. Even if our “language” is mostly analytic, a large part of the results here have counterparts/versions in algebraic geometry, cf. [35] and the references therein.

2.1 The relative Bergman metric

Let X and Y be two projective manifolds, which are assumed to be non-singular. Let $p : X \rightarrow Y$ be a surjective map, and let $(L, h_L) \rightarrow X$ be a line

bundle endowed with a Hermitian metric h_L . We make the convention that unless explicitly mentioned otherwise, the metric in this article are allowed to be singular. As part of the set-up, we assume that we have

$$\Theta_{h_L}(L) \geq 0 \tag{2.1}$$

in the sense of currents on X . By definition, this means the following: let $\Omega \subset X$ be any trivialization subset for L , such that the restriction $h_L|_{\Omega}$ corresponds to the metric $|\cdot|^2 e^{-\varphi_L}$. Then (2.1) requires that φ_L is psh, so that we have $\sqrt{-1}\partial\bar{\partial}\varphi_L \geq 0$ in the sense of currents.

In this context we recall the construction of the Bergman metric $e^{-\varphi_{X/Y}}$ on the bundle $K_{X/Y} + L$; we refer to [2] for further details.

Let Y_0 be a Zariski open subset of Y such that p is smooth over Y_0 , and for every $y \in Y_0$, the fiber X_y satisfies $h^0(X_y, K_{X/Y} \otimes L \otimes \mathcal{I}(h_L|_{X_y})) = \text{rank } p_*(K_{X/Y} \otimes L \otimes \mathcal{I}(h_L))$. Let X^0 be the p -inverse image of Y_0 and let $x_0 \in X^0$ be an arbitrary point; let z^1, \dots, z^{n+m} be local coordinates centered at x_0 , and let t^1, \dots, t^m be a coordinate centered at $y_0 := p(x_0)$. We can assume that $z^{n+j} = p^*(t_j)$ for every j . We consider as well a trivialization of L near x_0 . With this choice of local coordinates, we have a local trivialization of the tangent bundles of X and Y respectively, and hence of the (twisted) relative canonical bundle.

The local weight of the metric $e^{-\varphi_{X/Y}}$ with respect to this is given by the equality

$$e^{\varphi_{X/Y}(x_0)} = \sup_{\|u\|_{y_0} \leq 1} |F_u(x_0)|^2 \tag{2.2}$$

where the notations are as follows: u is a section of $K_{X_{y_0}} + L|_{X_{y_0}}$, and F_u is the coefficient of $dz^1 \wedge \dots \wedge dz^{n+m}$ in the local expression of $u \wedge p^* dt$. The norm which appears in the definition (2.2) is obtained by the fiber integral

$$\|u\|_{y_0}^2 := \int_{X_{y_0}} |u|^2 e^{-\varphi_L}. \tag{2.3}$$

An equivalent way of defining (2.2) is via an orthonormal basis, say u_1, \dots, u_k of sections of $K_{X_{y_0}} + L|_{X_{y_0}}$. Then we see that

$$e^{\varphi_{X/Y}(x_0)} = \sum_{j=1}^N |F_j(x_0)|^2 \tag{2.4}$$

where F_j are the functions corresponding to u_j .

The Bergman metric $h_{X/Y} = e^{-\varphi_{X/Y}}$ can also be introduced in an intrinsic manner as follows. Let ξ be a vector in the fiber over x_0 of the line bundle $-(K_{X/Y} + L)_{x_0}$. Then we have

$$|\xi|^2 = \sup_{\|u\|_{y_0} \leq 1} |\langle \xi, u_{x_0} \rangle|^2. \tag{2.5}$$

This defines a metric on the dual bundle, which induces $h_{X/Y}$ on $K_{X/Y} + L$. As we see from (2.4), the restriction of the metric $e^{\varphi_{X/Y}}$ to the fiber X_{y_0} coincides with the metric induced by any orthonormal basis of the space of holomorphic sections of $K_{X_{y_0}} + L|_{X_{y_0}}$. Hence the variation from one fiber to another is in general a C^∞ operation, since the said orthonormalization process is involved. Thus it is a remarkable fact that this metric has positive curvature in the sense of currents on X .

Theorem 2.1 [2, Theorem 0.1] *The curvature of the metric $h_{X/Y}$ on the twisted relative canonical bundle $K_{X/Y} + L|_{X^0}$ is positive in the sense of currents. Moreover, the local weights $\varphi_{X/Y}$ are uniformly bounded from above on X^0 , so they admit a unique extension as psh functions.*

Remark 2.2 The fact that the uniform boundness of $\varphi_{X/Y}$ on X_0 implies that it admits a unique extension is a standard result in pluripotential theory (cf. [27], p. 52 and the references therein, as well as [20], pp. 43–44) which we briefly recall now. Let X be a complex manifold (not necessarily compact) and let Z be a complex subvariety in X . Let φ be a psh function defined on $X \setminus Z$. The following assertions hold true.

- (i) If $\text{codim}_X(Z) \geq 2$, then φ admits a unique extension as a psh function on X .
- (ii) If $\text{codim}_X(Z) \geq 1$ and φ is uniformly bounded from above on $X \setminus Z$, then φ admits a unique extension as a psh function on X .

We will use these two properties frequently in the article.

The definition (2.2), although not intrinsically formulated, is explicit enough so as to imply the following statement. Let $p : X \rightarrow Y$ be a dominant map, such that X is Kähler; we denote by Δ the analytic set corresponding to the critical values of p . We assume that Δ is a snc divisor of Y , and we also assume that the p -inverse image of Δ equals

$$\sum_{i \in I} e_i W_i \tag{2.6}$$

where e_i are positive integers, and W_i are reduced hypersurfaces of X .

The next statement can be seen as a metric version of the corresponding results due to Kawamata in [26] and Campana in [9], respectively.

Theorem 2.3 *Let $\Theta_{X/Y}$ be the curvature current corresponding to the Bergman metric (2.2). Then we have*

$$\Theta_{X/Y} \geq [\Sigma_p] := \sum_{i \in I_h} (e_i - 1)[W_i] \tag{2.7}$$

in the sense of currents on X where I_h is the set of indexes $i \in I$ such that $p(W_i)$ is a divisor of Y . In particular, the current $\Theta_{X/Y}$ is singular along the multiple fibers of the map p .

Proof Let $x_0 \in W_1$ be a non-singular point of one of the sets appearing in (2.6).

We consider a coordinate set Ω containing the point x_0 , and we fix the coordinates (z_1, \dots, z_{n+m}) on Ω , such that $W_1 \cap \Omega = (z_{n+1} = 0)$. The local structure of the map p is as follows

$$(z_1, \dots, z_{n+m}) \rightarrow (z_{n+1}^{e_1}, z_{n+2}, \dots, z_{n+m}), \tag{2.8}$$

so that we assume implicitly that $p(W_1)$ is given locally by $t_1 = 0$.

The intersection $p^{-1}(t) \cap \Omega$ of the fibers of p with the coordinate set Ω can be identified with an open set in \mathbb{C}^n . This allows us to bound the absolute value of the quantity which computes the Bergman metric locally at x_0 , as we see next.

Let $t \in Y \setminus (t_1 = 0)$ be a point near $p(x_0)$, and let u be a section of the $K_{X_t} + L|_{X_t}$ as in (2.3). If $\|u\|_{X_t}^2 = 1$, then by the construction of F_u , we have

$$\int_{X_t \cap \Omega} \frac{|F_u|^2}{|z_{n+1}|^{2e_1-2}} d\lambda \leq \|u\|_{X_t}^2 = 1,$$

where $d\lambda$ is the Lebesgue measure with respect to z_1, \dots, z_n . Combining this with (2.2), we have thus

$$\varphi_{X/Y}(z) \leq (e_1 - 1) \log |z_{n+1}|^2 + O(1), \tag{2.9}$$

where the quantity $O(1)$ in (2.9) is uniform with respect to $z \in \Omega \setminus W_1$. This shows that the Lelong number of $\varphi_{X/Y}$ at any generic point of W_1 is greater than $e_1 - 1$. As a consequence, we have

$$\Theta_{X/Y} \geq (e_1 - 1)[W_1],$$

and the proof is finished. □

The construction of the metric $h_{X/Y}$ has a perfect pluricanonical analogue, as we recall next. Let u be a section of the bundle $mK_{X_y} + L$, where $m \geq 1$ is a positive integer. Then we define

$$\|u\|_y^{\frac{2}{m}} := \int_{X_y} |u|^{\frac{2}{m}} e^{-\frac{1}{m}\varphi L}, \tag{2.10}$$

and the definition (2.5) generalizes immediately, as follows. Let ξ be a vector in the fiber over x of the dual bundle $-(mK_{X/Y} + L)_x$. Then we have

$$|\xi|^2 = \sup_{\|u\|_y \leq 1} |\langle \xi, u_x \rangle|^2. \tag{2.11}$$

We denote the resulting metric by $h_{X/Y}^{(m)}$.

We recall next the analogue of Theorem 2.1, as follows.

Theorem 2.4 [4, Theorem 0.1] *The curvature of the metric $h_{X/Y}^{(m)}$ on the twisted relative pluricanonical bundle $mK_{X/Y} + L|_{X^0}$ is positive in the sense of currents. Moreover, the local weights $\varphi_{X/Y}$ are uniformly bounded from above on X^0 , so they admit a unique extension as psh functions.*

Remark 2.5 If the map p verifies the hypothesis of Theorem 2.3, then we infer that

$$\Theta_{h_{X/Y}^{(m)}}(mK_{X/Y} + L) \geq m[\Sigma_p]. \tag{2.12}$$

The proof is the same as in Theorem 2.3: if the local structure of the map p is as in (2.8), then the $L^{2/m}$ normalization bound for the sections involved in the computation of the metric $h_{X/Y}^{(m)}$ imply that the local pointwise norm of these sections is bounded. The weights of the metric $h_{X/Y}^{(m)}$ are given by the wedge product with $dt^{\otimes m}$, so the conclusion follows.

2.2 Singular metrics on vector bundles and direct image sheaves

We recall first the definition of singular Hermitian metrics on vector bundles cf. [2, 33, 35]. Let $E \rightarrow X$ be a holomorphic vector bundle of rank r on a complex manifold X . We denote by

$$H_r := \{A = (a_{i\bar{j}})\}$$

the set of $r \times r$, semi-positive definite Hermitian matrices. Let \bar{H}_r be the space of semi-positive, possibly unbounded Hermitian forms on \mathbb{C}^r . A singular

Hermitian metric h on E is given locally by a measurable map with values in \bar{H}_r such that

$$0 < \det h < +\infty$$

almost everywhere. In the definition above, a matrix valued function $h = (h_{i\bar{j}})$ is measurable provided that all entries $h_{i\bar{j}}$ are measurable. This notion is somehow too general; in particular, it is impossible to define a curvature current corresponding to it, as soon as the rank of E is at least two, see [36] for a clear example illustrating this.

Nevertheless, as observed in [2], one can still formulate the notion of negativity/positivity in the sense of Griffiths: the bundle (E, h_E) is *negatively curved* if

$$x \rightarrow \log |u|_{h_E, x}^2 \tag{2.13}$$

is psh, for any choice of a holomorphic local section u of E . The bundle (E, h_E) is *positively curved* if (E^*, h_E^*) is negative.

It is important to notice the following consequence of the Griffiths negativity assumption for a singular Hermitian vector bundle (E, h_E) . Let ξ be a local holomorphic section of E defined on a coordinate open set $U \subset X$. Since the function

$$|\xi|^2 = \sum_{\alpha, \beta} \xi_\alpha \bar{\xi}_\beta h_{\alpha\bar{\beta}}$$

is psh, in particular it follows that it is unambiguously defined at each point of U . We infer that the same is true for the coefficients $(h_{\alpha\bar{\beta}})$. Moreover, the function $|\xi|^2$ is bounded from above on any relatively compact $U' \Subset U$ so it follows that we have

$$\sup_{z \in U'} |h_{\alpha\bar{\beta}}(z)| \leq C. \tag{2.14}$$

The following result is a particular case of [36]; it gives a sufficient criteria in order to define the notion of *curvature current* associated to (E, h_E) which fits perfectly to what we will need later on in the paper.

Theorem 2.6 [36, Theorem 1.6] *Let (E, h_E) be a positively curved singular Hermitian vector bundle of rank r . We assume that the induced metric $\det h_E$ on the determinant $\Lambda^r E$ of E is non-singular. Then the coefficients of the Chern connection form $\theta_E := h_E^{-1} \partial h_E$ belong to L^2_{loc} . As a consequence, the curvature current $\Theta_{h_E}(E)$ is well defined and it is moreover positive in the sense of Griffiths. Moreover, it can be written locally as $\bar{\partial}\theta_E$.*

We provide here a few explanations about the statement 2.6. The fact that $\Theta_{h_E}(E)$ is well defined as matrix-valued $(1, 1)$ -current means that locally on some coordinate set U centered at some point $x \in X$ we have

$$\Theta_{h_E}(E)|_U = \sum_{j,k,\alpha,\beta} \mu_{j\bar{k}\alpha\bar{\beta}} dz^j \wedge d\bar{z}^k e_\alpha \otimes e_\beta^* \tag{2.15}$$

where $\mu_{j\bar{k}\alpha\bar{\beta}}$ are measures on U (rather than smooth functions as in the classical case), $(e_\alpha)_{\alpha=1,\dots,r}$ is a local holomorphic frame of E and $(z^i)_{i=1,\dots,n}$ are local coordinates. The positivity in the sense of Griffiths we are referring to in Theorem 2.6 means that for any local holomorphic vector field $\sum v^j \frac{\partial}{\partial z^j}$ and for any local holomorphic section $\sum \xi^\alpha e_\alpha$, the measure

$$\sum \mu_{j\bar{k}\alpha\bar{\beta}} v^j \bar{v}^k \xi^\alpha \bar{\xi}^\beta \tag{2.16}$$

is (real and) positive on U . The positivity of the measure (2.16) is obtained by Raufi in [36] by using an approximation procedure: he shows that under the hypothesis of Theorem 2.6, locally near each point of X there exists a sequence of non-singular metrics $h_{E,k}$ such that $(E, h_{E,k})$ is Griffiths-positively curved (in the usual sense), and that the corresponding curvature form is converging to $\bar{\partial}\theta_E$.

We will apply next this result in the context of direct images of twisted (pluri)canonical bundles. The set-up is the same as in the previous subsection; let $Y_1 \subset Y$ be the intersection of the set of regular values of p with the maximal subset of Y on which the direct image sheaf $p_*(K_{X/Y} + L)$ is locally free.

The fiberwise canonical L^2 -metrics

$$g_{1,y}(u, u) := \int_{X_y} |u|^2 e^{-\varphi_L} \leq +\infty \tag{2.17}$$

for $u \in H^0(X_y, K_{X_y} + L_y)$ induces a singular Hermitian metric $g_{X/Y}$ on the bundle $p_*(K_{X/Y} + L)|_{Y_1}$ whose curvature is positive (cf. [40]). The following result gives an important precision concerning this framework.

Theorem 2.7 [34, Theorem 3.3.5] *We suppose that the natural inclusion*

$$p_*((K_{X/Y} + L) \otimes \mathcal{I}(h_L)) \subset p_*(K_{X/Y} + L) \tag{2.18}$$

is generically isomorphic. Then the canonical L^2 -metric $g_{X/Y}$ on the direct image $p_(K_{X/Y} + L)|_{Y_1}$ has positive curvature, and it extends as a singular Hermitian metric $\tilde{g}_{X/Y}$ on the torsion free sheaf $p_*(K_{X/Y} + L)$ with positive curvature. We say that $\tilde{g}_{X/Y}$ is the L^2 metric on $p_*(K_{X/Y} + L)$ with respect to h_L .*

Since $p_*(K_{X/Y} + L)$ is torsion free, it is locally free outside a set of codimension at least two, say $\Sigma \subset Y$. Theorem 2.7 shows in particular that the restriction $(p_*(K_{X/Y} + L)|_{Y \setminus \Sigma}, \tilde{g}_{X/Y})$ is a positively curved singular Hermitian vector bundle.

We consider next the line bundle $\det p_*(K_{X/Y} + L)$; by this notation we mean the top exterior power of the direct image $p_*(K_{X/Y} + L)$. Then we have the following consequence of the previous results.

Corollary 2.8 *We assume that the hypothesis of Theorem 2.7 are fulfilled, and that $p_*(K_{X/Y} + L)$ is non-trivial. Then the determinant line bundle*

$$\det p_*(K_{X/Y} + L)$$

admits a singular hermitian metric whose curvature current Θ is positive. Moreover, we have the following statements.

- (a) *If Θ is a non-singular (1,1) form on some open subset $\Omega \subset Y \setminus \Sigma$, then the curvature current of $p_*(K_{X/Y} + L)|_{\Omega}$ is well-defined.*
- (b) *If Θ vanishes on an open subset $\Omega' \subset Y \setminus \Sigma$, then so does the full curvature tensor corresponding to $p_*(K_{X/Y} + L)$. In this case the relative metric $\tilde{g}_{X/Y}|_{\Omega'}$ is smooth.*

Proof The metric $\tilde{g}_{X/Y}$ on the direct image induces a metric on the determinant bundle $\det p_*(K_{X/Y} + L)|_{Y \setminus \Sigma}$ whose curvature is positive (and it actually equals the trace of the curvature of the direct image in the complement of an algebraic set). It is well-known that psh functions extend across sets of codimension at least two (cf. Remark 2.2), hence the the first part of the corollary follows.

The statement (a) is a direct consequence of Theorem 2.6, because the metric induced on the determinant bundle on Ω is *smooth*, by standard regularity results. As for part (b), we use Theorem 2.6 again, and it implies that the restriction of the curvature current corresponding to $p_*(K_{X/Y} + L)|_{\Omega'}$ is well-defined. We establish its vanishing next; as we will see, it is a consequence of the positivity in the sense of Griffiths of the curvature of $p_*(K_{X/Y} + L)$, combined with the fact that its trace Θ is equal to zero. We remark at this point that is really important to have at our disposal the curvature current as given by Theorem 2.6, and not only the positivity in the sense of (2.13).

A by-product of the proof of Theorem 2.6 (cf. [36, Remark 4.1]) is the fact that the curvature current Θ of $\det E$ is simply the trace of the matrix-valued current $\Theta_{h_E}(E)$. By using the notations (2.15) at the beginning of this section, this is equivalent to the fact that

$$\sum_{j,k} \sum_{\alpha} \mu_{j\bar{k}\alpha\bar{\alpha}} dz^j \wedge dz^{\bar{k}} = 0. \tag{2.19}$$

Since $\Theta_{h_E}(E)$ is assumed to be positive in the sense of Griffiths, we infer that the current

$$\sum_{j,k} \mu_{j\bar{k}\alpha\bar{\alpha}} dz^j \wedge d\bar{z}^k \tag{2.20}$$

is positive for each index α . When combined with (2.19), this implies that

$$\mu_{j\bar{k}\alpha\bar{\alpha}} \equiv 0 \tag{2.21}$$

for each j, k, α . But then we are done, since the positivity of $\Theta_{h_E}(E)$ together with (2.21) shows that for each pair of indexes α, β we have

$$\operatorname{Re} \left(\xi^\alpha \bar{\xi}^\beta \sum_{j,k} \mu_{j\bar{k}\alpha\bar{\beta}} v^j \bar{v}^k \right) \geq 0 \tag{2.22}$$

[notations as in (2.16)] which in turn implies that $\mu_{j\bar{k}\alpha\bar{\beta}} \equiv 0$ for any j, k, α, β . The current $\Theta_{h_E}(E)$ is therefore identically zero.

The regularity statement is verified as follows. In the first place we already know that the coefficients of h are bounded, where h stands for the local expression of the metric $\tilde{g}_{X/Y}$. This follows thanks to relation (2.14) which implies that the absolute value of the coefficients of the dual metric h^* is bounded from above, combined with the fact that the determinant $\det h$ is smooth.

Since $\bar{\partial}$ of the connection form (=curvature current) is equal to zero, it follows that the connection is smooth. Locally near a point of Ω' we therefore have

$$\partial h = h \cdot \Psi \tag{2.23}$$

where Ψ is smooth. The relation (2.23) holds in the sense of distributions; by applying the $\bar{\partial}$ operator to it, we see that h satisfies an elliptic equation. In conclusion, it is smooth. □

3 Some technical results

We recall here a result due to E. Viehweg which has been widely used in the previous works concerning the Iitaka conjecture...

Proposition 3.1 [39, Lemma 7.3] *Let $p : X \rightarrow Y$ be a surjective map between two non-singular, projective manifolds. Then there exists a commutative diagram*

$$\begin{array}{ccc}
 X' & \xrightarrow{\pi_X} & X \\
 p' \downarrow & & \downarrow p \\
 Y' & \xrightarrow{\pi_Y} & Y
 \end{array}$$

such that X' and Y' are smooth, the morphisms π_X, π_Y are birational, and moreover, each hypersurface $W \subset X'$ such that $\text{codim}_{Y'} p'(W) \geq 2$ is π_X -contractible, i.e., $\text{codim}_X \pi_X(W) \geq 2$.

As we see in [39, Lemma 7.3], the statement above is a quick consequence of Hironaka’s flattening theorem.

Remark 3.2 Let Δ be an effective klt \mathbb{Q} -divisor on X . Then we have

$$\pi_X^*(K_X + \Delta) + E' = K_{X'} + \Delta'$$

where E' is effective and π_X -exceptional, Δ' is klt. As a consequence, there exists a set $Z \subset Y$ such that $\text{codim}_Y Z \geq 2$ and such that we have an isometry

$$p'_*(mK_{X'/Y'} + m\Delta')|_{Y' \setminus Z'} \rightarrow \pi_Y^* p_*(mK_{X/Y} + m\Delta)|_{Y \setminus Z}, \tag{3.1}$$

where $Z' := \pi_Y^{-1}(Z)$. Indeed, this is simply a consequence of the definition of the canonical L^2 metrics $\tilde{g}_{X'/Y'}$ and $\tilde{g}_{X/Y}$ on $p'_*(mK_{X'/Y'} + m\Delta')$ and $p_*(mK_{X/Y} + m\Delta)$ respectively (cf. the constructions in the beginning of Sect. 4), together with the properties of the maps/manifolds in Proposition 3.1.

Moreover, by considering a further modification, we can assume that the singular locus Σ of the fibration p' is normal crossing and $p'^{-1}(\Sigma)$ is also normal crossing.

We will recall now two results whose combination will reveal the strategy of our proof.

Theorem 3.3 [8, Chapter 3.3], [13] *Let $T = \mathbb{C}^n / \Gamma$ be a complex torus of dimension m , and let $\alpha \in H^{1,1}(T, \mathbb{Z})$ be a pseudo-effective non trivial class. If α is not ample then there exists a submersion*

$$\pi : T \rightarrow S \tag{3.2}$$

*to an abelian variety S of dimension smaller than m so that we have $\alpha = \pi^*c_1(H)$ for some ample line bundle H on S . Moreover, after passing to some finite étale cover, the fiber of π is also a torus.*

Sketch of the proof Since the pseudo-effective class on the torus can be represented by a closed constant semipositive $(1, 1)$ -form, there exists a holomorphic line bundle L on T with a smooth hermitian metric h such that

$c_1(L) = \alpha \in H^{1,1}(T, \mathbb{Z})$ and $i\Theta_h(L) \geq 0$. Since L is not ample, by applying [8, Chapter 3.3], L is semiample and defines a nontrivial fibration to a subvariety of the dual torus of T . Combining with the fact that $\kappa(T) = 0$, after a finite étale cover, the image of the fibration should be an abelian variety. The theorem is thus proved. \square

The following statement originates in the seminal work of E. Viehweg, cf. Chapter 6 of [40] as well as Proposition 4.5 in [41]; the generalization presented below is stated in the article by Tsuji [37, Sect. 2.6]. We will nevertheless provide a complete treatment here, for the sake of completeness. Also, we stress that in the next theorem the base Y is not necessarily the modification of an abelian variety.

Theorem 3.4 *Let $f : X \rightarrow Y$ be a fibration between two projective manifolds. Let L be a \mathbb{Q} -line bundle on X endowed with metric h_L whose corresponding curvature current is semi-positive definite and such that $e^{-\varphi_L}$ is L^1 -integrable on X , where φ_L is the potential of h_L . Let $m \in \mathbb{N}$ be a positive integer such that mL is a line bundle, and let $\Sigma \subset Y$ be the singular locus of f . We assume that Σ is snc and $f^{-1}(\Sigma)$ is a normal crossing divisor. Then there exist a constant $\varepsilon_0 > 0$ and an effective \mathbb{Q} -divisor F in X satisfying $\text{codim}_Y f_*(F) \geq 2$, such that*

$$c_1(K_{X/Y} + F + L) \geq \varepsilon_0 \cdot f^*c_1(\det f_*(mK_{X/Y} + mL)). \tag{3.3}$$

Proof Let $Y_0 \subset Y$ be the maximal Zariski open set such that $f|_{f^{-1}(Y_0)}$ is flat and such that the direct image $f_*(mK_{X/Y} + mL)$ is locally free when restricted to Y_0 . Then $\text{codim}_Y(Y \setminus Y_0) \geq 2$, as is well-known (given that X and Y are non-singular).

By hypothesis, the inverse image of the discriminant of f can be written as

$$f^*\Sigma = \sum W_i + \sum a_i V_i, \tag{3.4}$$

where $\sum W_i + \sum V_i$ are snc and $a_i \geq 2$. Set $W := \sum W_i$ and $V := \sum V_i$.

Next, we see that there exists a natural morphism

$$\det f_*(mK_{X/Y} + mL) \rightarrow \bigotimes^r f_*(mK_{X/Y} + mL) \text{ on } Y_0, \tag{3.5}$$

where r is the rank of $f_*(mK_{X/Y})$. In order to give a useful interpretation of (3.5), we consider the fibered product

$$X^r := X \times_Y X \times_Y \cdots \times_Y X \tag{3.6}$$

corresponding to the map p (as always, the convention here is that X^r is the component of the fibered product (3.6) which maps surjectively onto Y).

Let $f^r : X^r \rightarrow Y$ be the natural induced fibration, and let $\text{pr}_i : X^r \rightarrow X$ be the projection on the i -th factor. Set $X_0^r := (f^r)^{-1}(Y_0)$ and $L_r := \bigotimes_{i=1}^r \text{pr}_i^*(L)$. According to the definition in [23, Definition 5.13], the map f is a flat Cohen-Macaulay fibration over Y_0 . Moreover, the results established in [23, Corollary 5.24] show that we have the crucial equality

$$\omega_{X^r/Y} = \bigotimes_{i=1}^r \text{pr}_i^*(K_{X/Y}) \quad \text{on } X_0^r. \tag{3.7}$$

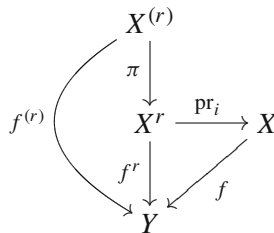
Combining (3.7) with [23, Lemma 3.17], we infer that

$$\bigotimes_{i=1}^r f_{\star}(mK_{X/Y} + mL) \simeq (f^r)_{\star}((\omega_{X^r/Y} \otimes L_r)^{\otimes m}) \quad \text{on } Y_0.$$

As a consequence, by (3.5) we infer that we have

$$H^0(X_0^r, (\omega_{X^r/Y} \otimes L_r)^{\otimes m} \otimes (f^{r\star} \det f_{\star}(mK_{X/Y} + mL))^*) \neq 0. \tag{3.8}$$

Let $\pi : X^{(r)} \rightarrow X^r$ be a desingularization of X^r which is an isomorphism at non-singular points of X^r , and let $f^{(r)} := f^r \circ \pi$ be the map induced by f^r . Set $X_0^{(r)} := \pi^{-1}(X_0^r)$.



By hypothesis W is snc, so the variety X_0^r is normal at each point of the subset $W_0^r := W \times_{Y_0} \cdots \times_{Y_0} W$; moreover, it is Gorenstein with at most rational singularities. This is yet another consequence of [23, Lemma 3.13, Theorem 5.12].

We consider the canonical bundle $K_{X^{(r)}}$ of the manifold $X^{(r)}$; we will compare next its π -direct image with ω_{X^r} . To this end we recall that there exists a non-zero morphism

$$\pi_{\star} \mathcal{O}(K_{X^{(r)}}) \rightarrow \omega_{X^r} \tag{3.9}$$

on X^r , cf. [23, 3.20], which is moreover an isomorphism on the locus where X_0^r is normal and has at most rational singularities.

The map (3.9) and the identity (3.7) show in particular the existence of a meromorphic section of the bundle

$$K_{X^{(r)}/Y}^{-1} \otimes \pi^* \left(\bigotimes_{i=1}^r \text{pr}_i^*(K_{X/Y}) \right) \tag{3.10}$$

whose zeroes and poles are contained in $X^{(r)} \setminus X_0^r$, together with the complement of the locus where X_0^r is normal and has at most rational singularities.

As a consequence, there exists a couple of effective divisors E_1, E_2 on $X^{(r)}$ such that we have

$$K_{X^{(r)}/Y} + E_1 = \pi^* \omega_{X^r/Y} + E_2 \quad \text{on } X_0^{(r)} \tag{3.11}$$

and moreover, each component Λ of the support of E_i belong to one of the following category.

- (a) The $f^{(r)}$ -image of the divisor Λ is contained in $Y \setminus Y_0$, i.e. a set of codimension at least two.
- (b) The codimension of the projection of Λ by some maps $\text{pr}_i \circ \pi$ is at least 2, or it is equal to one of the V_l .

These properties will be important for the rest of our proof.

When combined with (3.8) and [32, III, Lemma 5.10], the equality (3.11) shows that the bundle

$$mK_{X^{(r)}/Y} + m\pi^*L_r + mE_1 - f^{(r)*} \det f_* (mK_{X/Y} + mL) + E_3 \tag{3.12}$$

is effective, where E_3 is an effective divisor on $X^{(r)}$ which projects in codimension two, i.e. we have $(\pi \circ f^{(r)})_*(E_3) \subset (Y \setminus Y_0)$.

Let $\epsilon > 0$ small enough (which depends on m and L_r) such that if we define

$$\Delta = \epsilon(mK_{X^{(r)}/Y} + m\pi^*L_r + mE_1 + E_3 - f^{(r)*} \det f_* (mK_{X/Y} + mL))$$

then the pair $(X_y^{(r)}, \Delta + \pi^*L_r|_{X_y^{(r)}})$ is klt for any $y \in Y$ in the complement of a set of measure zero. Here, ‘‘klt’’ means that $e^{-2 \ln |\Delta| - \sum_{i=1}^r (\pi \circ \text{pr}_i)^* \varphi_L}$ is L^1 -integrable on the fiber $X_y^{(r)}$. We set $\tilde{\Delta} := \Delta + \pi^*L_r$.

By the results in [4], there exists a very ample line bundle $A_Y \rightarrow Y$ such that for any point $y \in Y$ as above the restriction map

$$\begin{aligned} &H^0\left(X^{(r)}, f^{(r)*}A_Y + kK_{X^{(r)}/Y} + k\tilde{\Delta}\right) \\ &\rightarrow H^0\left(X_y^{(r)}, f^{(r)*}A_Y + kK_{X^{(r)}/Y} + k\tilde{\Delta}|_{X_y^{(r)}}\right) \end{aligned} \tag{3.13}$$

is surjective for every k sufficiently divisible.

To simplify the notations we set $D_k = A_Y - k\epsilon \det f_*(mK_{X/Y} + mL)$ and we have

$$\begin{aligned} &f^{(r)*}A_Y + kK_{X^{(r)}/Y} + k\tilde{\Delta} \\ &= f^{(r)*}D_k + k(1 + \epsilon m)(K_{X^{(r)}/Y} + \pi^*L_r) + \epsilon kmE_1 + \epsilon kE_3. \end{aligned} \tag{3.14}$$

Therefore the map (3.13) becomes

$$\begin{aligned} &H^0(X^{(r)}, f^{(r)*}D_k + k(1 + \epsilon m)(K_{X^{(r)}/Y} + \pi^*L_r) + \epsilon kmE_1 + \epsilon kE_3) \\ &\rightarrow H^0(X_y^{(r)}, kK_{X^{(r)}/Y} + k\tilde{\Delta}). \end{aligned} \tag{3.15}$$

As a consequence, we have the following crucial extension property.

Claim 3.5 *There exists a constant $C > 0$ independent of k such that for any section u of the bundle $k(1 + \epsilon m)(K_{X/Y} + L)|_{X_y}$, there exists a section*

$$U \in H^0(X, rf^*D_k + rk(1 + \epsilon m)(K_{X/Y} + L) + Ck[V] + kF) \tag{3.16}$$

whose restriction to the fiber X_y is equal to $u^{\otimes r}$, where F is an effective divisor on X (independent of k) such that $\text{codim}_Y f_*(F) \geq 2$.

We admit this statement for the time being, and we finish next the proof of Theorem 3.4. By Claim 3.5 the bundle

$$rf^*D_k + rk(1 + \epsilon m)(K_{X/Y} + L) + Ck[V] + kF \geq 0$$

is effective, which is equivalent to say that

$$\begin{aligned} &rf^*A_Y + rk(1 + \epsilon m)(K_{X/Y} + L) + Ck[V] \\ &\quad - rk\epsilon f^* \det f_*(mK_{X/Y} + mL) + kF \end{aligned}$$

is effective as well. Thanks to Theorem 2.3, we have

$$K_{X/Y} + L \geq \epsilon_0[V_h]$$

for some $\varepsilon_0 > 0$, where V_h is the divisor corresponding to the components of V projecting in codimension one. Therefore we have

$$rf^*A_Y + \left(rk(1 + \varepsilon m) + \frac{Ck}{\varepsilon_0} \right) (K_{X/Y} + L) - rk\varepsilon f^* \det f_*(mK_{X/Y} + mL) + k\tilde{F} \geq 0$$

for some effective divisor \tilde{F} satisfying $\text{codim}_Y f_*(\tilde{F}) \geq 2$. Theorem 3.4 is thus proved by letting $k \rightarrow \infty$. □

We establish next the claim.

Proof The point $y \in Y$ is supposed to be generic, so we have the equality

$$X_y^{(r)} = X_y \times \cdots \times X_y \tag{3.17}$$

where the number of the factors in the product (3.17) is r . Since X_y is a smooth fiber, we have

$$k(K_{X^{(r)}/Y} + \tilde{\Delta}) + f^{(r)*}A_Y|_{X_y^{(r)}} = k(1 + \varepsilon m)(K_{X^{(r)}/Y} + \pi^*L_r)|_{X_y^{(r)}}$$

by construction. Thus the section u we are given by hypothesis defines a section $u^{(r)}$ of the bundle

$$k(K_{X^{(r)}/Y} + \tilde{\Delta}) + f^{(r)*}A_Y|_{X_y^{(r)}} \tag{3.18}$$

by considering the tensor product of the pr_i -inverse images of u .

The property (3.15) and the relation (3.14) show that there exists a section $U^{(r)}$ of the bundle

$$f^{(r)*}D_k + k(1 + \varepsilon m)(K_{X^{(r)}/Y} + \pi^*L_r) + \varepsilon kmE_1 + \varepsilon kE_3, \tag{3.19}$$

extending $u^{(r)}$ and we show next that the “restriction to the diagonal” of $U^{(r)}$ satisfies all the properties required by the claim.

We recall that by properties (a) and (b), for any component Λ of the divisor $E_i|_{X_0^{(r)}}$ there exists a projection $\text{pr}_i \circ \pi$ for which the image of Λ is contained in one of the V_l . In particular, we have

$$E_1 + E_2 \leq C \sum_{i,l} (\text{pr}_i \circ \pi)^* V_l \tag{3.20}$$

on the open set of $X_0^{(r)}$ whose complement projects in codimension greater than 2 via $\text{pr}_i \circ \pi$.

We consider next a covering of $X^{(r)}$ with coordinate open sets Ω_α on which all our global bundles become trivial. Let s_{E_i} be the canonical section corresponding to E_i . The expression

$$\frac{s_{E_1}^k}{s_{E_2}^{k+\epsilon km}} \cdot U^{(r)} \tag{3.21}$$

can be seen as a meromorphic section of the bundle

$$k(1 + \epsilon m)\pi^* \left(\bigotimes_{i=1}^r \text{pr}_i^*(K_{X/Y} + L) \right) + f^{(r)*}D_k$$

[via (3.11)] which is holomorphic on $f^{(r)}$ -inverse image of the intersection of Y_0 with the set of regular values of p . It corresponds to a collection of local meromorphic functions $(U_\alpha^{(r)})$ for which the pole order is given by (3.21).

Let $Y \setminus \Sigma$ be the set of regular values of f ; then the fiber $X_{y_0}^{(r)}$ is equal to $X_{y_0} \times \dots \times X_{y_0}$ for any $y_0 \in Y \setminus \Sigma$. We consider the collection of holomorphic functions

$$s_{V,\alpha}^{ck} U_\alpha^{(r)}(x, \dots, x, f(x)) \tag{3.22}$$

which are defined on the diagonal subset of the product $X_{y_0} \times \dots \times X_{y_0}$ intersected with $\Omega_\alpha \cap (f^{(r)})^{-1}(Y \setminus \Sigma)$. Here $s_{V,\alpha}$ denote the local equations corresponding to the divisor V , and c is a constant which can be easily computed from (3.21) [we use the bound (3.20)]. The local holomorphic functions (3.22) glue together as a section U of

$$rk(1 + \epsilon m)(K_{X/Y} + L) + rf^*D_k + ckV \tag{3.23}$$

except that it is only defined on some open set $X_0 \subset X$ whose codimension in $X \setminus V$ is at least two. However, the relations (3.20) and (3.21) show that U is bounded near the generic point of the support of $V \cap f^{-1}(Y_0)$, so it extends to $f^{-1}(Y_0)$ and another application of [32, III, Lemma 5.10] will end the proof, as follows.

Indeed, the lemma in question shows that there exists a divisor F on X , such that $f_*(F) \subset Y \setminus Y_0$ and such that

$$\begin{aligned} & f_*(rk(1 + \epsilon m)(K_{X/Y} + L) + ckV)^{**} \\ &= f_*(rk(1 + \epsilon m)(K_{X/Y} + L) + ckV + kF) \end{aligned} \tag{3.24}$$

where we denote by \mathcal{F}^{**} the double dual of \mathcal{F} . The sheaves we are dealing with are torsion free, and in this case (3.24) combined with the projection formula

show that we have

$$\begin{aligned}
 & f_* (rk(1 + \epsilon m)(K_{X/Y} + L) + rf^*D_k + ckV)^{**} \\
 &= f_* (rk(1 + \epsilon m)(K_{X/Y} + L) + rf^*D_k + ckV + kF) \quad (3.25)
 \end{aligned}$$

(we are using here that $(\mathcal{F} \otimes L)^{**} = \mathcal{F}^{**} \otimes L$ for any locally free L and torsion free \mathcal{F} , respectively). The claim is therefore established. \square

Remark 3.6 The proof just finished shows that given any line bundle P such that the sheaf

$$\left(\bigotimes^r p_*(mK_{X/Y} + mL) \right) \otimes P^{-1}$$

has a global (non-identically zero) section, then $c_1(K_{X/Y} + L + F) \geq \epsilon_0 c_1(P)$. However, the bundle $\det p_*(mK_{X/Y} + mL)$ seems to be “the best” for what we have to do next.

The following result is classical, cf. [18,25]: it shows that in order to prove (1.2) it would be enough to establish the inequality $\kappa(K_X + \Delta) \geq \min\{1, \kappa(K_F + \Delta_F)\}$.

Proposition 3.7 *Let $p : X \rightarrow A$ be a fibration from a projective manifold to a simple Abelian variety A (i.e. there is no strict subtorus in A) and let Δ be a klt \mathbb{Q} -divisor on X . Let F be a generic fiber of p . If $\kappa(X + \Delta) \geq 1$, then*

$$\kappa(K_X + \Delta) \geq \kappa(K_F + \Delta_F),$$

where $K_F + \Delta_F = (K_X + \Delta)|_F$.

Proof We use here an approach which goes back to Y. Kawamata, [25, p. 62]. Modulo desingularization, we can assume that the Iitaka fibration of $K_X + \Delta$ is a morphism $\varphi : X \rightarrow W$.

$$\begin{array}{ccc}
 X & \xrightarrow{\varphi} & W \\
 & \searrow p & \\
 & & A
 \end{array}$$

Let G be the generic fiber of φ and set $\Delta_G := \Delta|_G$. Then $K_G + \Delta_G = (K_X + \Delta)|_G$ and we have

$$\kappa(K_G + \Delta_G) = 0. \tag{3.26}$$

Let $p : G \rightarrow p(G)$ be the restriction of p on G . We will analyze next among three cases which may occur.

Case 1: We assume that $p(G) = A$; then we argue as follows. Let $\tilde{p} : G \rightarrow \tilde{A}$ be the Stein factorisation of $p : G \rightarrow A$:

$$\begin{array}{ccc}
 G & \xrightarrow{p} & A \\
 & \searrow \tilde{p} & \nearrow s \\
 & & \tilde{A}
 \end{array}$$

After some desingularization \tilde{p} , we can assume that \tilde{A} is smooth. There are two subcases:

Subcase 1 The ramification locus of $s : \tilde{A} \rightarrow A$ is of codimension 1 in A . Let $[E]$ be the divisor corresponding to the ramification locus. Since A is a simple torus, $[E]$ is an ample divisor on A . Therefore $K_{\tilde{A}}$ is big. In this case, it is well known that $\kappa(K_G + \Delta_G) \geq \dim \tilde{A} \geq 1$. We get a contradiction with (3.26).

Subcase 2 The ramification locus of $s : \tilde{A} \rightarrow A$ is of codimension at least 2 in A . As the Stein factorisation extend uniquely over closed analytic subsets of codimension ≥ 2 , \tilde{A} is thus an abelian variety.

Let $t \in \tilde{A}$ be a generic point. Let G_t be the fiber of \tilde{p} over t and set $\Delta_{G_t} := \Delta|_{G_t}$. By induction, (3.26) implies

$$\kappa(K_{G_t} + \Delta_{G_t}) = 0. \tag{3.27}$$

We next estimate the dimension of G . Let F be the fiber of $p : X \rightarrow A$ over $s(t) \in A$. Then F is a generic fiber. By restricting φ on F , we obtain a morphism

$$\varphi_t : F \rightarrow V$$

where V is a subvariety of W . Let $\tilde{V} \rightarrow V$ be the Stein factorisation of φ_t .

$$\begin{array}{ccc}
 F & \xrightarrow{\varphi_t} & V \\
 & \searrow \tilde{\varphi}_t & \nearrow \\
 & & \tilde{V}
 \end{array}$$

Since G is generic, we infer that the fiber of $\tilde{p} : G \rightarrow \tilde{A}$ over t coincides with a generic fiber of $\tilde{\varphi}_t$. Combining this with (3.27), then [38, Theorem 5.11] implies that

$$\kappa(K_F + \Delta_F) \leq \dim \tilde{V} = \dim F - \dim G_t.$$

Therefore we have

$$\dim G_t \leq \dim F - \kappa(K_F + \Delta_F)$$

and thus we infer that

$$\begin{aligned} \dim G &= \dim G_t + \dim \tilde{A} \leq \dim F - \kappa(K_F + \Delta_F) + \dim A \\ &= \dim X - \kappa(K_F + \Delta_F). \end{aligned}$$

Finally, by construction of the Iitaka fibration, $\dim G = \dim X - \kappa(K_X + \Delta)$; we obtain the inequality

$$\dim X - \kappa(K_X + \Delta) \leq \dim X - \kappa(K_F + \Delta_F),$$

and in conclusion $\kappa(K_X + \Delta) \geq \kappa(K_F + \Delta_F)$.

Case 2: We assume that the image $p(G)$ has dimension zero. Since G is connected, $p(G)$ is a point in A . This means that we can define a map $W \rightarrow A$, which can be assumed to be regular by blowing up W . We have thus the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & W \\ & \searrow p & \swarrow q \\ & & A \end{array}$$

Set $t := p(G)$. Let F be the fiber of p over t . Then F is a generic fiber of p and G is a generic fiber of

$$\varphi : F \rightarrow \varphi(F),$$

and by [38, Theorem 5.11] we infer that

$$\kappa(K_F + \Delta_F) \leq \kappa(K_G + \Delta_G) + \dim \varphi(F) = \dim \varphi(F).$$

Note that $\varphi(F)$ is the fiber of q over $t \in A$. We have $\dim W = \dim \varphi(F) + \dim A$. Therefore $\dim W \geq \kappa(K_F + \Delta_F) + \dim A$. Combining this with the fact that φ is the Iitaka fibration, we have thus

$$\kappa(K_X + \Delta) = \dim W \geq \kappa(K_F + \Delta_F) + \dim A,$$

and we are done.

Case 3: The remaining case $p(G)$ is a proper subvariety of A . Since we are assuming that A is simple, by [38, Corollary 10.10], any desingularization of $p(G)$ is of general type. In this case, it is well known that $\kappa(K_G + \Delta_G) \geq \dim p(G) \geq 1$. We get a contradiction. \square

4 Proof of the main theorem

After the preparations in the previous sections, we will present here the arguments for our main result. To start with, we apply Proposition 3.1 for the map $p : X \rightarrow A$. We will use the notation

$$p' : X' \rightarrow A' \tag{4.1}$$

for the resulting fiber space, and we keep in mind that the projective manifold A' is birational to an abelian variety.

$$\begin{array}{ccc} X' & \xrightarrow{\pi_X} & X \\ p' \downarrow & & \downarrow p \\ A' & \xrightarrow{\pi_A} & A \end{array}$$

The gain is that we can allow ourselves any effective “error divisor” in X' whose p' -projection in A' has codimension at least two.

We will choose next a metric on the relative canonical bundle of p' which reflects the properties of the canonical algebra of a generic fiber of p . By hypothesis, we know that $\kappa(K_{X_a} + \Delta_a) \geq 0$ for any generic $a \in A$, where $\Delta_a := \Delta|_{X_a}$. The important result [5] states that the canonical ring of the pair (X_a, Δ_a) is finitely generated. Let $m \gg 0$ be a large enough positive integer, so that the singularities of the metric of $K_{X_a} + \Delta_a$ induced by the linear system $mK_{X_a} + m\Delta_a$ are *minimal* (i.e. equivalent to the singularities given by the generators of the canonical algebra). Of course, the same integer m will work for any $a \in A'$ generic. As we have already recalled in Sect. 2, we can construct the m -Bergman kernel metric $h_{X'/A'}$ on the bundle $K_{X'/A'} + \Delta'$ by using the fiberwise liner systems $mK_{X_a} + m\Delta_a$.

Our proof relies on the positivity and regularity properties of the direct image sheaf

$$F'_m := p'_*(mK_{X'/A'} + m\Delta') \tag{4.2}$$

of the relative pluricanonical bundle of p' .

We introduce the bundle

$$L'_m := (m - 1)K_{X'/A'} + m\Delta' \tag{4.3}$$

and we endow it with the corresponding power of the relative Bergman metric $(m - 1)h_{X'/A'} + h_{\Delta'}$, where $h_{\Delta'}$ is the canonical singular metric on Δ' ; then we have

$$p'_\star \left((K_{X'/A'} + L'_m) \otimes \mathcal{I} \left(h_{X'/A'}^{\otimes(m-1)} \cdot h_{\Delta'} \right) \right) \subset F'_m. \tag{4.4}$$

It turns out that (4.4) is generically isomorphic, as we verify next.

Let $a \in A'$ be a generic point and let $u \in H^0(X'_a, mK_{X'/A'} + m\Delta')$ be a holomorphic section. Thanks to the explicit construction of $h_{X'/A'}$, we infer that we have

$$\log |u|^2 \leq (m - 1)\varphi_{X'/A'}|_{X'_a} + O(1) \tag{4.5}$$

(by a slight abuse of notation). Combining this with the fact that (X', Δ') is klt, we obtain $u \in H^0(X'_a, (K_{X'/A'} + L'_m) \otimes \mathcal{I}(h_{X'/A'}^{\otimes(m-1)} \cdot h_{\Delta'}))$. Therefore (4.4) is generically isomorphic. We refer to [3, Corollary A.2.4] for more details.

Thanks to Theorem 2.7, the torsion free sheaf F'_m can be endowed with the canonical L^2 metric $\tilde{g}_{X'/A'}$ with respect to the metric on L'_m constructed above; it has positive curvature. Moreover, we see that the Hermitian line bundle $(K_{X'/A'} + L'_m, h_{X'/A'}^{\otimes(m-1)} \cdot h_{\Delta'})$ satisfies the hypothesis of Theorem 2.7. We can therefore apply Corollary 2.8 to $(F'_m, \tilde{g}_{X'/A'})$ and thus the determinant line bundle $\det F'_m$ has a positive curvature current denoted by $\Xi \geq 0$ if endowed with the metric induced by $\tilde{g}_{X'/A'}$.

Set $F_m := p_*(mK_{X/A} + m\Delta)$. By using the same construction as above, we have a canonical L^2 metric $\tilde{g}_{X/A}$ on F_m . Thanks to Remark 3.2, we have an isometry

$$(F'_m, \tilde{g}_{X'/A'})|_{A' \setminus Z'} \rightarrow \pi_A^*(F_m, \tilde{g}_{X/A})|_{A \setminus Z} \tag{4.6}$$

where $\text{codim}_A Z \geq 2$ and $Z' := \pi_A^{-1}(Z)$.

Theorems 3.3 and 3.4 show clearly how we will proceed for the rest of our proof. Roughly speaking, if $(\pi_A)_*\Xi$ is neither ample nor trivial, then we are done by induction, cf. Theorem 3.3. If not, then we will analyze the remaining two extreme cases in the following subsections.

4.1 The direct image of the curvature current of the determinant is non-zero

By the construction at the beginning of this section, we have a commutative diagram satisfying the properties in Proposition 3.1 and Remark 3.2:

$$\begin{array}{ccc}
 X' & \xrightarrow{\pi_X} & X \\
 p' \downarrow & & \downarrow p \\
 A' & \xrightarrow{\pi_A} & A
 \end{array}$$

If the curvature current corresponding to the determinant of F'_m is non-identically zero on $A' \setminus Z'$, i.e.

$$\pi_{A\star} \Xi \neq 0, \tag{4.7}$$

then Ξ is automatically a big class in $H^{1,1}(A', \mathbb{Z})$ (cf. Theorem 3.3) if A is a simple torus¹.

As a consequence of Theorem 3.4 we obtain the following result.

Corollary 4.1 *If the class $\{\Xi\}$ is big, then for any generic point $a \in A'$ and for any $k \gg 1$ sufficiently divisible we can find an effective divisor E in X' satisfying $\text{codim}_{A'} p'_\star(E) \geq 2$, such that the restriction map*

$$H^0(X', kK_{X'/A'} + k\Delta' + E) \rightarrow H^0(X'_a, kK_{X'/A'} + k\Delta')$$

is surjective.

Proof Thanks to Remark 3.2, we can assume that the singular locus of p' is normal crossing and we can thus apply Theorem 3.4 to the fibration p' . Let A_Y be a very ample divisor on A' . Since $\det p'_\star(mK_{X'/A'} + m\Delta') = \Xi$ is big on A' , by Theorem 3.4 we can find a parameter $m_1 \in \mathbb{N}$ large enough, and an effective divisor E satisfying $\text{codim}_{A'} p'_\star(E) \geq 2$, such that

$$m_1 K_{X'/A'} + m_1 \Delta' + E = (p')^\star A_Y + \Delta_1. \tag{4.8}$$

for some pseudo-effective divisor Δ_1 on X' .

By [4], the restriction map

$$\begin{aligned}
 &H^0(X', kK_{X'/A'} + k\Delta' + (p')^\star A_Y + \Delta_1) \\
 &\rightarrow H^0(X'_a, kK_{X'/A'} + k\Delta' + (p')^\star A_Y + \Delta_1)
 \end{aligned} \tag{4.9}$$

¹ We can add this assumption by the argument in the beginning of Theorem 4.4.

is surjective, for any $k \geq m_1$ divisible enough. Indeed, this can be seen as follows: Δ_1 is pseudo-effective, so its first Chern class contains a closed positive current T . If we choose $a \in A'$ very general, and $m_1 \gg 0$, then the restriction of the current $\frac{1}{k}T$ to X'_y is well-defined, and the corresponding multiplier ideal sheaf $\mathcal{I}(\Delta' + \frac{1}{k}T)$ is trivial for any $k \geq m_1$. We endow the \mathbb{Q} -bundle

$$K_{X'/A'} + \Delta' + \frac{1}{k}\Delta_1$$

with the k -Bergman metric, and then we write

$$\begin{aligned} &kK_{X'/A'} + k\Delta' + (p')^*A_Y + \Delta_1 \\ &= K_{X'} + \Delta' + \frac{1}{k}\Delta_1 + (k-1)\left(K_{X'/A'} + \Delta' + \frac{1}{k}\Delta_1\right) + (p')^*A_Y \end{aligned}$$

so that the surjectivity of the map (4.9) follows by the usual extension results.

The relation (4.8) shows that we have

$$kK_{X'/A'} + k\Delta' + (p')^*A_Y + \Delta_1 = (m_1 + k)(K_{X'/A'} + \Delta') + E,$$

we therefore infer

$$H^0(X', (m_1 + k)(K_{X'/A'} + \Delta') + E) \twoheadrightarrow H^0(X'_a, (m_1 + k)(K_{X'/A'} + \Delta') + E)$$

and this last vector space is equal to

$$H^0(X'_a, (m_1 + k)(K_{X'/A'} + \Delta')),$$

as a consequence fact that $p'_*(E) \subsetneq A'$. The corollary is proved. □

In particular we have $\kappa(K_X + \Delta) \geq \kappa(K_{X_a} + \Delta_a)$ by Proposition 3.1.

4.2 The curvature current of the determinant is zero

In this subsection we assume that $p : X \rightarrow A$ is a fibration from a projective manifold to an abelian variety, and we have

$$\pi_{A,\star}\Xi = 0, \tag{4.10}$$

where Ξ is the determinant of F'_m as in (4.7).

Set $F_m := p_*(mK_{X/A} + m\Delta)$ and let $\tilde{g}_{X/A}$ be the corresponding canonical L^2 metric on F_m constructed in the beginning of Sect. 4. By (4.6), it follows

that the curvature of $\det(F_m)$ is equal to zero on $A \setminus \Sigma$, i.e. in the complement of a set of codimension at least two. However, it is known that (cf. for example [20, Chapter III, Corollary 2.11]) the support of a closed positive $(1, 1)$ -current cannot be contained in such a small set, unless the said current is identically zero. We therefore have

$$\Theta(\det(F_m), \det \tilde{g}_{X/A}) = 0, \tag{4.11}$$

and it follows that there exists a subset of A still denoted by Σ such that the next properties hold true.

- (a) The codimension of Σ in A is at least two, i.e. $\text{codim}_A \Sigma \geq 2$.
- (b) The restriction of the direct image sheaf $F_m|_{A \setminus \Sigma}$ is a vector bundle.
- (c) The canonical L^2 metric $\tilde{g}_{X/A}$ is non-singular on $A \setminus \Sigma$, and the couple $(F_m|_{A \setminus \Sigma}, \tilde{g}_{X/A})$ is a Hermitian flat vector bundle.

This is a consequence of Corollary 2.8, combined with (4.11). In what follows, we will write

$$(\mathcal{E}, h) := (F_m, \tilde{g}_{X/A})|_{A \setminus \Sigma} \tag{4.12}$$

in order to simplify the notation. Thus $\mathcal{E} \rightarrow A \setminus \Sigma$ is a vector bundle endowed with a non-singular metric h whose associated curvature is equal to zero. Then the parallel transport induces a representation

$$\rho : \pi_1(A \setminus \Sigma) \rightarrow \mathbb{U}(r), \tag{4.13}$$

where r is the rank of \mathcal{E} and where $\mathbb{U}(r)$ is the unitary group of degree r . The next proposition is a consequence of the fact that $\pi_1(A)$ is commutative.

Proposition 4.2 *Let $\rho : \pi_1(A) \rightarrow \mathbb{U}(r)$ be the representation (4.13). The following assertions hold true.*

- (i) *If the image of ρ is infinite, then there exists a non-zero section of the bundle $mK_X + m\Delta + L$, where L is a non-torsion topologically trivial line bundle on X .*
- (ii) *If the image of ρ is finite, we have $\kappa(K_X + \Delta) \geq \kappa(K_F + \Delta_F)$.*

Proof Since $\pi_1(A) = \pi_1(A \setminus \Sigma)$ is abelian, ρ can be decomposed as the direct sum of r representations $\{\rho_i\}_{i=1}^r$

$$\rho_i : \pi_1(A \setminus \Sigma) \rightarrow \mathbb{U}(1).$$

- (i) If the image of ρ is infinite, there exists at least one index, say $i = 1$, such that the image of the corresponding representation ρ_1 is infinite. Then ρ_1

corresponds to a topologically trivial *non-torsion* line bundle L_1 . We therefore infer the existence of a section

$$s \in H^0(p^{-1}(A \setminus \Sigma), mK_X + m\Delta - L_1) \tag{4.14}$$

such that $|s|_{\tilde{g}_{X/A}}(y) = 1$ for every generic point $y \in A$.

It follows that the L^2 -norm of s with respect to any smooth metric on the bundle $mK_X + m\Delta - L_1$ is finite. Indeed, this is a consequence of the fact that the weights of the metric $h_{X/Y}^{(m)}$ are bounded from above, cf. Theorem 2.4. Then s can be extended as an element in $H^0(X, mK_X + m\Delta - L_1)$ and the claim (i) of our proposition is established.

(ii) If the image of ρ is finite, set $l := |\rho(\pi_1(A))|$. For every $\tau \in H^0(F, mK_F + m\Delta)$, the parallel transport of $\tau^{\otimes l}$ induces an element in

$$s \in H^0(p^{-1}(A \setminus \Sigma), mlK_X + ml\Delta),$$

We invoke the same argument as above (namely the finiteness of the L^2 norm of s) to show that it extends to X . It defines therefore a non-zero element in $H^0(X, mlK_X + ml\Delta)$, and the proposition is proved. \square

The following result due to Campana-Peternell [12, Theorem 3.1] together with its generalization in [11] will allow us to conclude.

Theorem 4.3 [12, Theorem 3.1] [11, Theorem 0.1] *Let X be a projective complex manifold, and let Δ be an effective \mathbb{Q} -divisor on X , such that the pair (X, Δ) is log canonical. Let $L \in \text{Pic}^0(X)$ be a topologically trivial line bundle. Then:*

- (a) *For any positive integer $m \geq 1$ we have $\kappa(K_X + \Delta) \geq \kappa(mK_X + m\Delta + L)$.*
- (b) *Let $l \in \mathbb{N}$ be a positive integer. Then there exists infinitely many $d \in \mathbb{N}$ for which we have*

$$h^0(X, l(K_X + \Delta) + dL) \geq h^0(X, l(K_X + \Delta) + L).$$

- (c) *If $\kappa(K_X + \Delta) = \kappa(mK_X + m\Delta + L) = 0$, then L is a torsion bundle.*

In [11] the point (b) is not explicitly stated, so we will provide here a complete treatment for the convenience of the readers. The argument we invoke in what follows is borrowed from [12, Proposition 3.2, (4)].

Proof We remark that the point (a) is a direct consequence of [11, Theorem 0.1]. In order to establish (b), we first observe that for each q and l , the set

$$V_{q,l} = \{\lambda \in \text{Pic}^0(X) : h^0(X, l(K_X + \Delta) + \lambda) \geq q\}$$

is a finite union of torsion translates of complex subtori of $\text{Pic}^0(X)$, cf. [11, Sect. 2]. In particular, if $L \in V_{q,l}$, then there exists a torsion line bundle ρ and a subtorus $T \subset \text{Pic}^0(X)$ such that L belongs to the component $\rho + T$ of $V_{q,l}$. If we denote by s the order of ρ , then we see that $d \cdot L \in V_{q,l}$ provided that $d := p \cdot s + 1$, where p is any positive integer. The point (b) is thus proved.

To prove the point (c), we suppose by contradiction that there exists a non-torsion bundle $L \in \text{Pic}^0(X)$ such that $\kappa(K_X + \Delta) = \kappa(mK_X + m\Delta + L) = 0$. After passing to some multiple of m and L , we can assume for simplicity that

$$h^0(X, mK_X + m\Delta + L) = 1.$$

As $V_{1,m}$ is a finite union of torsion translates of complex subtori of $\text{Pic}^0(X)$, we can find a torsion bundle $L_{\text{tor}} \in \text{Pic}^0(X)$ and a non-trivial bundle $F \in T'$, (where T' is a subtorus of $\text{Pic}^0(X)$), such that $L = L_{\text{tor}} + F$ and $L_{\text{tor}} + T' \subset V_{1,m}$. As a consequence, for every $t \in \mathbb{R}$, we can find three non-trivial sections

$$\begin{aligned} s_t &\in H^0(X, mK_X + m\Delta + L_{\text{tor}} + tF), \\ s_{-t} &\in H^0(X, mK_X + m\Delta + L_{\text{tor}} - tF) \end{aligned} \tag{4.15}$$

as well as $s_0 \in H^0(X, mK_X + m\Delta + L_{\text{tor}})$. When $|t|$ is small enough, $s_t \cdot s_{-t}$ and $s_0 \cdot s_0$ are two linearly independent elements in $H^0(X, 2mK_X + 2m\Delta + 2L_{\text{tor}})$. Then $\kappa(K_X + \Delta) \geq 1$ and we get a contradiction. \square

We are now ready to prove our main theorem.

Theorem 4.4 *Let $p : X \rightarrow A$ be a fibration from a projective manifold to an Abelian variety. Let Δ be an effective klt \mathbb{Q} -divisor on X and let F be a generic fiber of p . Then*

$$\kappa(K_X + \Delta) \geq \kappa(K_F + \Delta_F), \tag{4.16}$$

where $\Delta_F = \Delta|_F$.

Proof Without loss of generality, we can assume that A is a simple torus, i.e., there is no nontrivial subtorus in A . In fact, since A is Abelian, if there is a nontrivial subtorus A_1 , by Poincaré’s reductibility theorem (cf. [19, Theorem 8.1]), after a finite smooth cover, $A = A_1 \times A_2$, where A_2 is another subtorus of A . Then (4.16) can be proved by induction.

We follow the notations in the beginning of Sect. 4. In particular, we have the commutative diagram

$$\begin{array}{ccc}
 X' & \xrightarrow{\pi_X} & X \\
 p' \downarrow & & \downarrow p \\
 A' & \xrightarrow{\pi_A} & A
 \end{array}$$

and the positive curvature current Ξ on A' .

If $(\pi_A)_*(\Xi) \neq 0$, we can use the results in the case 4.1. In particular, as A is simple, Theorem 3.3 implies that Ξ is big on A' . By using Corollary 4.1 and Proposition 3.1, we obtain $\kappa(K_X + \Delta) \geq \kappa(K_F + \Delta_F)$.

If $(\pi_A)_*(\Xi) = 0$, we are in the case 4.2. If the image of the representation ρ in Proposition 4.2 is finite, then Proposition 4.2 implies (4.16). If not, using Proposition 4.2 again, there exists $L \in \text{Pic}^0(A)$ which is not a torsion point, such that

$$\kappa(mK_{X/A} + m\Delta + p^*L) \geq 0. \tag{4.17}$$

Combined this with Theorem 4.3, we have

$$\kappa(K_X + \Delta) \geq 1. \tag{4.18}$$

Indeed, we necessarily have $\kappa(K_X + \Delta) \geq 0$, and we cannot have equality as it would contradict (4.17) and the point (b) of Theorem 4.3. Thanks to (4.18) and Proposition 3.7, we obtain (4.16) and the theorem is proved. \square

5 Further results and remarks

Throughout the current section, the notations we will be using are the same as in Sect. 4. The following result is a direct consequence of the arguments used in paragraph 4.1, so we simply state it without any comment about the proof.

Theorem 5.1 *Let $p : X \rightarrow Y$ be an algebraic fiber space, and let Δ be an effective \mathbb{Q} -divisor on X such that (X, Δ) is klt. We assume that for some positive m divisible enough we have $\det F_m$ is big. Then we infer that*

$$\kappa(K_X + \Delta) \geq \kappa(K_F + \Delta_F) + \kappa(Y), \tag{5.1}$$

where $\Delta_F = \Delta|_F$.

Developing further the ideas in the proof of our main result here, we obtain the following statement in which the flatness of $\det F_m$ is shown to have stronger consequences.

Theorem 5.2 *Let $p : X \rightarrow Y$ be a fibration between two projective manifolds and let Δ be an effective \mathbb{Q} -divisor, such that (X, Δ) is klt. If $\det F_m$ is topologically trivial, then $(F_m, \tilde{g}_{X/Y})$ is a hermitian flat vector bundle on Y .*

Proof We use the same argument as in the beginning of Sect. 4.2, and infer the existence of a subset of $\Sigma \subset Y$ such that the properties (a)–(c) cf. 4.2, right after (4.11), hold true.

As we will see next, our result is a consequence of the correspondence between the unitary representations of the fundamental group of Y and the Hermitian flat vector bundles on Y , together with the fact that $\text{codim}_Y \Sigma \geq 2$.

Let (U_α) be a finite cover of Y with contractible coordinate sets. For each index α the set $U_\alpha \setminus \Sigma$ is equally contractible –this is a consequence of (a) we are referring above. The parallel translation with respect to the flat metric $\tilde{g}_{X/Y}|_{Y \setminus \Sigma}$ gives a holomorphic trivialization

$$\theta_\alpha : F_m|_{U_\alpha \setminus \Sigma} \rightarrow (U_\alpha \setminus \Sigma) \times \mathbb{C}^r \tag{5.2}$$

which is moreover an *isometry* provided that the right hand side in (5.2) is endowed with the Euclidean metric. Therefore the transition functions, say $g_{\alpha\beta}$, corresponding to the trivializations θ_α are matrices in $\mathbb{U}(r)$.

Let (\mathcal{E}, h_E) be the Hermitian vector bundle of rank r on Y given locally by $(U_\alpha \times \mathbb{C}^r)_\alpha$, together with the transition functions $(g_{\alpha\beta})_{\alpha,\beta}$ (so that the metric h_E corresponds to the flat metric on $U_\alpha \times \mathbb{C}^r$). This construction gives an isometry

$$i : F_m|_{Y \setminus \Sigma} \rightarrow \mathcal{E}|_{Y \setminus \Sigma}. \tag{5.3}$$

By Hartogs’ theorem, i extends to an injection of sheaves

$$F_m \rightarrow \mathcal{E} \tag{5.4}$$

on Y which we still denote by i . We show next that in fact the map (5.4) is an isomorphism.

To this end let U be a coordinate topologically open subset of Y and $u \in H^0(U, \mathcal{E})$ be a holomorphic section whose norm at each point of U is equal to one, i.e.

$$|u|_{h_E}(y) = 1 \tag{5.5}$$

for each $y \in U$. By (5.3), the restriction of u to $U \setminus \Sigma$ belongs to the image of i , so that we have

$$u|_{U \setminus \Sigma} = i(v_0) \tag{5.6}$$

for a $v_0 \in H^0(U \setminus \Sigma, F_m)$. The main point next is that i is an isometry, so we infer that

$$|v_0|_{\bar{g}_{X/Y}}(y) = 1 \tag{5.7}$$

for any $y \in U \setminus \Sigma$.

The section v_0 corresponds to an element in $H^0(p^{-1}(U \setminus \Sigma), mK_{X/Y} + m\Delta)$ whose pointwise norm with respect to the canonical L^2 metric is equal to one. Therefore, we are in the same situation as in Proposition 4.2, so v_0 extends across the inverse image of Σ . In conclusion, we have constructed a local section v of $F_m|_U$ such that $i(v) = u$ and the theorem is proved. \square

A rather immediate application of Theorems 3.4 and 5.2, is the next result, which already appears in [28] in a slightly less general form.

Lemma 5.3 *Let $p : X \rightarrow Y$ be an algebraic fiber space, and let Δ be an effective \mathbb{Q} -divisor, such that (X, Δ) is klt. If there exists an integer $m \geq 2$ such that*

$$c_1(p_*(mK_{X/Y} + m\Delta)) = 0 \in H^{1,1}(Y, \mathbb{R}),$$

then $F_{m_1} := p_(m_1K_{X/Y} + m_1\Delta)$ is hermitian flat for any $m_1 \in \mathbb{N}$ such that F_{m_1} is non zero.*

Proof Let m_1 be a positive integer as above. The idea of the proof is very simple: the positivity carried by the determinant of F_{m_1} can be injected into F_m via Theorem 3.4. Since we already know that the first Chern class of F_m is equal to zero, the conclusion follows. We give next the details.

As we have already seen, the determinant $L_1 := \det F_{m_1}$ of the sheaf F_{m_1} is a pseudo-effective line bundle. There exists thus a metric h_L on L_1 such that

$$\Theta_{h_L}(L_1) \geq 0$$

in the sense of currents on Y . By applying Theorem 3.4 (with the \mathbb{Q} -bundle “ L ” corresponding to Δ) we infer the existence of an effective divisor T on X such that the codimension of the projection is large, i.e. $\text{codim}_Y(p_*(T)) \geq 2$ and such that $c_1(K_{X/Y} + \Delta + T - \epsilon_1 p^*L_1)$ is pseudo-effective on X for some positive $\epsilon_1 > 0$.

As a consequence, we can equip the bundle $(m - 1)(K_{X/Y} + \Delta) + T$ with a metric h such that

$$\Theta_h((m - 1)(K_{X/Y} + \Delta) + T) \geq \epsilon_2 p^* \Theta_{h_L}(L_1) \tag{5.8}$$

on X , where $\epsilon_2 > 0$ is a positive real number, and such that

$$\begin{aligned} H^0(X_y, (m(K_{X/Y} + \Delta) + T) \otimes \mathcal{I}(h|_{X_y})) \\ = H^0(X_y, m(K_{X/Y} + \Delta) + T|_{X_y}) \end{aligned} \tag{5.9}$$

for generic $y \in Y$.

Let $h_{X/Y}$ be the metric on $p_*(mK_{X/Y} + m\Delta + T)$ induced by h as in Theorem 2.7.

By Lemma 5.4 below, the relation (5.8) gives the following lower bound for the curvature of the determinant: we have

$$\Theta_{\det h_{X/Y}} \det p_*(mK_{X/Y} + m\Delta + T) \geq r\epsilon_2 \Theta_{h_L}(L_1), \tag{5.10}$$

on Y , where r is the rank of the direct image $p_*(mK_{X/Y} + m\Delta + T)$.

Since $\text{codim}_Y(p_*(T)) \geq 2$, we have

$$\det p_*(mK_{X/Y} + m\Delta + T) = \det p_*(mK_{X/Y} + m\Delta) = 0 \in H^{1,1}(Y, \mathbb{R}).$$

Combining this with (5.10), we know that the current $\Theta_{h_L}(L_1)$ is identically zero, and the lemma is proved. \square

The following statement was used during the proof of Lemma 5.3.

Lemma 5.4 *Let $p : X \rightarrow Y$ be an algebraic fiber space. Let (G, h_G) and (L, h_L) be Hermitian line bundles on X and Y , respectively. The metrics h_G and h_L are allowed to be singular and we assume that the following requirements hold true.*

(i) *We have $\Theta_{h_L}(L) \geq 0$ and*

$$\Theta_{h_G}(G) \geq \epsilon_0 p^* \Theta_{h_L}(L) \tag{5.11}$$

on X , where ϵ_0 is a positive real number.

(ii) *The direct image $p_*(K_{X/Y} + G)$ is non-zero, and we have*

$$p_*((K_{X/Y} + G) \otimes \mathcal{I}(h_G))_y = p_*(K_{X/Y} + G)_y \tag{5.12}$$

for any generic $y \in Y$.

Let $h_{X/Y}$ be the L^2 metric on the direct image sheaf $p_(K_{X/Y} + G)$ induced by h_G . Then we have*

$$\Theta_{\det h_{X/Y}}(\det p_*(K_{X/Y} + G)) \geq r\epsilon_0 \Theta_{h_L}(L) \tag{5.13}$$

on Y .

Proof Our arguments are based on the main results in [34] combined with very basic properties of psh functions, as follows.

Let $y_0 \in Y$ be an arbitrary point, and let $\Omega_0 \subset Y$ be an open subset of Y centered at y_0 . Let $\Omega \subset p^{-1}(\Omega_0)$ be an open subset of X contained in the inverse image of Ω_0 . We assume that the restriction of h_G to Ω is given by the weight $e^{-\varphi_G}$, and that the restriction of h_L to Ω_0 is given by $e^{-\varphi_L}$. By hypothesis we have

$$\sqrt{-1}\partial\bar{\partial}\varphi_L \geq 0, \quad \sqrt{-1}\partial\bar{\partial}\varphi_G \geq \varepsilon_0 p^* \sqrt{-1}\partial\bar{\partial}\varphi_L \tag{5.14}$$

on Ω . In particular, we can write the local weight of h_G

$$\varphi_G = (\varphi_G - \varepsilon_0\varphi_L \circ p) + \varepsilon_0\varphi_L \circ p \tag{5.15}$$

as sum of two psh functions (actually the difference $\varphi_G - \varepsilon_0\varphi_L \circ p$ is only psh up to modification in the complement of a measure zero set, but this is not relevant for what we have to do next).

We remark that the local expressions $\varphi_G - \varepsilon_0\varphi_L \circ p$ glue together as a global metric on $G|_{p^{-1}(\Omega_0)}$; the resulting object is denoted by

$$h_{0G} := e^{-\varepsilon_0\varphi_L \circ p} h_G \tag{5.16}$$

and we have $\Theta_{h_{0G}}(G|_{p^{-1}(\Omega_0)}) \geq 0$. By [34], the restriction

$$p_*(K_{X/Y} + G)|_{\Omega_0} \tag{5.17}$$

is positively curved when endowed with the L^2 metric induced by h_{0G} . Here the important fact is that the hypothesis (5.12) implies a similar relation for h_{0G} .

Let ξ be a local holomorphic section of the dual of the bundle (5.17). The positivity of the curvature together with the expression (5.16) show that we have

$$\sqrt{-1}\partial\bar{\partial} \log |\xi|_{h_{X/Y}}^2 \geq \varepsilon_0 \sqrt{-1}\partial\bar{\partial}\varphi_L \tag{5.18}$$

and this implies (5.13). □

We recall next the following deep result due to Zuo [44, Corollary 1] (see also [10, Theorem 1]); it will play a crucial role in our next statement.

Theorem 5.5 [44, Corollary 1] *Let Y be a compact Kähler manifold, and let ρ be a finite dimensional representation of $\pi_1(Y)$, whose Zariski closure is a reductive algebraic group. If $\kappa(Y) = 0$, the ρ splits as a direct sum of representations of rank one, modulo an étale cover of Y .*

As a consequence of the considerations in this paper together with Theorem 5.5, we obtain the following main theorem of this section.

Theorem 5.6 *Let $p : X \rightarrow Y$ be an algebraic fiber space, and let Δ be an effective \mathbb{Q} -divisor on X such that (X, Δ) is klt. We assume that for some $m \geq 2$ the line bundle $\det F_m$ is topologically trivial. Then we have*

$$\kappa(K_X + \Delta) \geq \kappa(K_F + \Delta_F) + \kappa(Y), \tag{5.19}$$

where $\Delta_F = \Delta|_F$.

Proof Since $\det F_m = 0$ we infer that F_m is hermitian flat vector bundle of rank r over Y , by Theorem 5.2. We obtain a representation

$$\rho : \pi_1(Y) \rightarrow \mathbb{U}(r)$$

induced by the parallel transport corresponding to $(F_m, \tilde{g}_{X/Y})$.

The unitary group $\mathbb{U}(r)$ is compact, therefore the Zariski closure of the image of ρ is automatically reductive.

We are therefore lead to distinguish among three cases, corresponding to the Kodaira dimension of Y .

- *Case 1: The Kodaira dimension of Y is equal to $-\infty$, i.e. $\kappa(Y) = -\infty$* In this case, we have immediately (5.19).
- *Case 2: The Kodaira dimension of Y is equal to zero, i.e. $\kappa(Y) = 0$* Thanks to Theorem 5.5, we infer that ρ splits into a direct sum of 1-dimensional representations, after a finite étale cover $p_Y : Y' \rightarrow Y$. We denote by X' the fibered product of X and Y' over Y , $p_X : X' \rightarrow X$ the natural projection and $a : Y' \rightarrow \text{Alb } Y'$ the Albanese map of Y' .

$$\begin{array}{ccc} X' & \xrightarrow{p_X} & X \\ p' \downarrow & & \downarrow p \\ Y' & \xrightarrow{p_Y} & Y \\ a \downarrow & & \\ \text{Alb } Y' & & \end{array}$$

We have thus

$$p'_*(mK_{X'/Y'} + m\Delta') = p_Y^*F_m = \bigoplus_{j=1}^r L_j, \tag{5.20}$$

where r is the rank of F_m , $\Delta' = p_X^*\Delta$ and $L_j \in \text{Pic}^0(Y')$.

Thanks to Lemma 5.3, we can assume that m is large and sufficiently divisible. Then $\kappa(Y) = 0$ implies that $a_*(mK_{Y'})$ is a trivial line bundle. Since L_1, \dots, L_r are pulled back from $\text{Alb } Y'$, we have $L_j = a^*\tilde{L}_j$ for some $\tilde{L}_j \in \text{Pic}^0(\text{Alb } Y')$. We have thus

$$(a \circ p')_*(mK_{X'} + m\Delta') \simeq a_*(mK_{Y'}) \otimes \left(\bigoplus_{j=1}^r \tilde{L}_j \right) \simeq \bigoplus_{j=1}^r \tilde{L}_j. \tag{5.21}$$

Set $V_{1,m} := \{L \in \text{Pic}^0(\text{Alb } Y') \mid h^0(mK_{X'} + m\Delta' + L) \geq 1\}$. Then (5.21) implies that

$$V_{1,m} = \{-\tilde{L}_1, \dots, -\tilde{L}_r\}. \tag{5.22}$$

In this case it follows from Theorem 4.3 (b) that $V_{1,m}$ is a finite union of torsion line bundles (because the corresponding subtori much have dimension zero). Then the bundles $\tilde{L}_1, \dots, \tilde{L}_r$ have finite order. Therefore L_1, \dots, L_r have finite order in $\text{Pic}^0(Y')$

By [5], we infer that

$$\bigotimes^q F_m \rightarrow F_{qm} = p_*(qmK_{X/Y} + qm\Delta)$$

is surjective for every $q \in \mathbb{N}$. Combining this with (5.20), we obtain the surjective morphism

$$\bigotimes^q \left(\bigoplus_{j=1}^r L_j \right) \rightarrow p_Y^* F_{qm} \quad \text{for every } q \in \mathbb{N}.$$

As L_1, \dots, L_r have finite order, by taking q divisible enough we have

$$h^0(X', qmK_{X'} + qm\Delta') \geq C_1 \text{rank } F_{qm} \geq C_2 q^{\kappa(K_F + \Delta_F)}$$

for some $C_1, C_2 > 0$ independent of q ; we remark that $p_X : X' \rightarrow X$ depends on m but is independent of q . Then

$$\kappa(X', K_{X'} + \Delta') \geq \kappa(K_F + \Delta_F).$$

Since the Iitaka dimension is not affected by the finite étale cover, we have thus

$$\kappa(X, K_X + \Delta) \geq \kappa(K_F + \Delta_F) + \kappa(Y).$$

• *Case 3: The Kodaira dimension of Y is greater than one, i.e. $\kappa(Y) \geq 1$* In this case we use the Iitaka map corresponding to the canonical bundle of Y .

Let $m_0 \gg 0$ be a large enough positive integer, such that the Iitaka map

$$q_0 : Y \dashrightarrow Z$$

of K_Y is given by the linear system $|m_0 K_Y|$. Here Z is a smooth manifold, of dimension equal to $\kappa(Y)$.

There exist a modification of $\pi_Y : Y' \rightarrow Y$ such that the induced map $q := q_0 \circ \pi_Y$ is an algebraic fiber space

$$q : Y' \rightarrow Z \tag{5.23}$$

such that $\kappa(Y'_z) = 0$, where $Y'_z := q^{-1}(z)$ is a general fiber of q . The map π_Y is such that the base points of the bundle $\pi_Y^*(m_0 K_Y)$ are divisorial, i.e. there exists an effective divisor D on Y' such that

$$\pi_Y^*(m_0 K_Y) = D + L \tag{5.24}$$

where L is a line bundle generated by its global sections. Actually, the map q in (5.23) is associated to the base-point free linear system $|L|$. Therefore, for any point $z \in Z$ which is not a critical value of q , the fiber Y'_z is given by the equations

$$\sigma_1 = 0, \dots, \sigma_d = 0 \tag{5.25}$$

where $d = \kappa(Y)$ and the σ_j above are sections of L .

Let X' be a desingularization of the fibered product $X \times_Y Y'$. We consider the natural projection $p' : X' \rightarrow Y'$; we can assume that the exceptional divisor of the map

$$X' \rightarrow X \times_Y Y' \tag{5.26}$$

is contained in the p' -inverse image of $Y' \setminus Y_0$, where by definition Y_0 is the Zariski open subset of Y' such that $\pi_Y|_{Y_0}$ is biholomorphic. Let

$$X'_z := (p')^{-1}(Y'_z) \tag{5.27}$$

be the inverse image of Y'_z ; it is a non-singular manifold, provided that z is general enough. We have

$$K_{X'/Y'} + \Delta_{X'} + E_Y = \pi_X^*(K_{X/Y} + \Delta_X) + E_X \tag{5.28}$$

where E_X is π_X -exceptional, $p'(E_Y)$ is π_Y -exceptional and $(X', \Delta_{X'})$ is klt. We will denote by $F'_m := p'_*(mK_{X'/Y'} + m\Delta_{X'})$ the direct image corresponding to p' and $\tilde{g}_{X'/Y'}$ the corresponding metric on F'_m .

$$\begin{array}{ccc} X' & \xrightarrow{\pi_X} & X \\ p' \downarrow & & \downarrow p \\ Y' & \xrightarrow{\pi_Y} & Y \\ q \downarrow & & \\ Z & & \end{array}$$

The arguments we will use for the rest of our proof are a bit more complicated than what they should be, because it is not clear that the bundle F'_m is flat.

In any case, we have the natural isometry

$$i : \pi_Y^* F_m \rightarrow F'_m \quad \text{on } Y_0. \tag{5.29}$$

The inverse image bundle $\pi_Y^* F_m$ is Hermitian flat on Y' . As i is an isometry, by the argument at the end of the proof of Theorem 5.2, given any flat local section u of $\pi_Y^* F_m$ the image

$$i(u|_{Y_0}) \tag{5.30}$$

extends across $Y' \setminus Y_0$ as a section of F'_m . In this way we obtain a morphism

$$i : \pi_Y^* F_m \rightarrow F'_m \tag{5.31}$$

defined globally on Y' .

Theorem 5.6 will be established by using classical L^2 -extension theorem and the following three lemmas; our goal is to show that we have

$$\kappa(K_X + \Delta) \geq \kappa(K_F + \Delta_F) + \kappa(Y). \tag{5.32}$$

The first step of the remaining part of our proof is as follows.

Lemma 5.7 *Let $z \in Z$ be a general point. Then we have*

$$\kappa(Y'_z, K_{Y'_z} + S|_{Y'_z}) = 0 \tag{5.33}$$

for any effective divisor S on Y' whose support is contained in the exceptional loci of π_Y .

Proof Let

$$K_{Y'} = \pi_Y^* K_Y + \Lambda \tag{5.34}$$

where Λ is the exceptional divisor of π_Y . The properties of the Iitaka map show in particular that we have

$$h^0(Y'_z, lK_{Y'_z}) = h^0(Y'_z, l(\pi_Y^* K_Y + \Lambda)|_{Y'_z}) = 1 \tag{5.35}$$

for every l sufficiently divisible.

Fix u_0 a non-zero section of some multiple of K_Y . The support of S is contained in $\text{Supp}(\Lambda)$, and therefore it would be enough to show that we have

$$h^0\left(Y'_z, r_1 K_{Y'_z} + r_2 \Lambda|_{Y'_z}\right) = 1 \tag{5.36}$$

for any $r_1, r_2 \in \mathbb{N}$ sufficiently divisible. This is however immediate, because the sections in (5.36) can be injected in the space $H^0(Y'_z, (r_1 + r_2)K_{Y'_z})$ by multiplication with an appropriate power of u_0 . Thanks to (5.35), $h^0(Y'_z, (r_1 + r_2)K_{Y'_z}) = 1$ and so the proof is finished. \square

The next result establish a very useful map between the sections of the bundles $m(K_{X'/Y'} + \Delta_{X'})|_{X'_z}$ and $m\pi_X^*(K_{X/Y} + \Delta_X)|_{X'_z}$, respectively.

Lemma 5.8 *For every $L \in \text{Pic}^0(Y'_z)$, the morphism (5.31) induces an isomorphism*

$$H^0(Y'_z, \pi_Y^* F_m \otimes L) \rightarrow H^0(Y'_z, F'_m \otimes L). \tag{5.37}$$

In particular, we have the natural isomorphism

$$H^0(Y'_z, \pi_Y^* F_m) \rightarrow H^0(Y'_z, F'_m) \tag{5.38}$$

and

$$\kappa(K_{X'_z} + \Delta_{X'}|_{X'_z}) \geq \kappa(K_F + \Delta_F). \tag{5.39}$$

Proof As F_m is hermitian flat and $\kappa(Y'_z) = 0$, by Theorem 5.5, we have

$$\pi_X^* F_m|_{Y'_z} = \bigoplus_{i=1}^r L_i, \tag{5.40}$$

modulo an étale cover of Y'_z , where $L_i \in \text{Pic}^0(Y'_z)$ and r is the rank of F_m . In what follows, we still denote by Y'_z the resulting manifold, in order to keep

the notation as clean as possible –we observe that it is enough to establish the lemma for the pull-back of u .

Let $u \in H^0(Y'_z, F'_m \otimes L)$. Thanks to (5.29), u induces a section

$$\widehat{u} := i^{-1}(u) \otimes (\sigma_T|_{Y'_z}) \in H^0(Y'_z, \pi_X^* F_m \otimes T \otimes L),$$

for some effective divisor T supported in the exceptional loci of π_Y , where σ_T is the canonical section of T . Let s_i be the components of section \widehat{u} according to (5.40), and let $v_0 \in H^0(Y'_z, m_1 K_{Y'_z})$ for some $m_1 \in \mathbb{N}$ large enough. Then we have

$$v_0 \otimes s_i \in H^0(Y'_z, m_1 K_{Y'_z} + T + L_i + L). \tag{5.41}$$

By combining Lemma 5.7 with Theorem 4.3, we infer that $L_i + L$ is a torsion point in $\text{Pic}^0(Y'_z)$ as soon as the corresponding section s_i is non-zero. Moreover, we have already a section $v_0 \otimes (\sigma_T|_{Y'_z}) \in H^0(Y'_z, m_1 K_{Y'_z} + T)$. Using again Lemma 5.7, we infer that

$$\text{div}(s_i) = [\sigma_T|_{Y'_z}].$$

Therefore

$$\text{div}(\widehat{u}) = \text{div}\left(i^{-1}(u) \otimes (\sigma_T|_{Y'_z})\right) = [\sigma_T|_{Y'_z}].$$

As a consequence, $\frac{\widehat{u}}{\sigma_T} \in H^0(Y'_z, \pi_X^* F_m \otimes L)$ and (5.37) is proved.

It remains to prove (5.39). Set

$$V_{1,m} := \left\{ L \in \text{Pic}^0(Y'_z) \mid h^0(X'_z, mK_{X'_z} + m\Delta_{X'} + q^*L) \geq 1 \right\}.$$

Then (5.37) and (5.40) imply that

$$V_{1,m} = \{-L_1, \dots, -L_r\}. \tag{5.42}$$

Indeed, let $L \in \text{Pic}^0(Y'_z)$ be a line bundle in the set $V_{1,m}$. The equality (5.37) shows that some of the bundles $L - L_j$ is effective; this establishes (5.42).

By using the same argument as in the case 2 of the proof of Theorem 5.6, we deduce that L_1, \dots, L_r have finite order. The inequality (5.39) is therefore proved. □

The last technical statement which we need is the following.

Lemma 5.9 *Let V be a section of the bundle $m\pi_X^*(K_{X/Y} + \Delta_X)|_{X'_z}$. Then we have*

$$\sup_{X'_z} |V|^2 e^{-\varphi_{X/Y}} < C < +\infty \tag{5.43}$$

for some constant $C > 0$. Here $e^{-\varphi_{X/Y}}$ is the pull-back of the corresponding relative Bergman metric on X .

Proof We consider a flat local frame for π^*F_m ; by this we mean that we have local sections $\sigma_1, \dots, \sigma_r$ of π^*F_m at a point $y_0 \in Y'_z$ such that the quantity

$$\int_{X_y} |\sigma_j^0|^2 e^{-\varphi_{\Delta_X} - (m-1)\varphi_{X/Y}^{(m)}} \tag{5.44}$$

is a (non-identically zero) constant with respect to y in the complement of an analytic subset. In the notation above, σ_j^0 is such that $\sigma_j^0 \wedge dp = \sigma_j$. By definition, the weight of the Bergman metric satisfies the pointwise estimate

$$\varphi_{X/Y}(x) \geq \log \frac{|\sigma_j|^2}{\int_{X_y} |\sigma_j^0|^2 e^{-\varphi_{\Delta_X} - (m-1)\varphi_{X/Y}^{(m)}}} \tag{5.45}$$

where $y = p(x)$ and combined with (5.44), we obtain

$$|\sigma_j|^2 e^{-\varphi_{X/Y}} \leq \int_{X_y} |\sigma_j^0|^2 e^{-\varphi_{\Delta_X} - (m-1)\varphi_{X/Y}^{(m)}} \leq C. \tag{5.46}$$

The result follows, since we can express the section V in terms of the flat generators σ_j . □

We have now at our disposal all the ingredients needed to finish the proof of Theorem 5.6. Let u be a section of the bundle $m\pi_X^*(K_{X/Y} + \Delta_X)|_{X'_z}$, and let τ_0 be a section of the bundle m_0K_Y , where we assume that m and m_0 are greater than 2. Thanks to (5.31), the tensor product

$$\hat{u} := u \otimes (p \circ \pi_X)^*\tau_0 \tag{5.47}$$

induces a section of the bundle

$$K_{X'} + \Delta_{X'} + (m - 1)\pi_X^*(K_{X/Y} + \Delta_X) + (m_0 - 1)(\pi_Y \circ p')^*K_Y|_{X'_z}. \tag{5.48}$$

We endow the bundle $\pi_X^*(K_{X/Y} + \Delta_X)$ with the m^{th} root of the metric used in Lemma 5.9, and the bundle $\pi_Y^*K_Y$ with the metric given by the decomposition

(5.24). By Ohsawa-Takegoshi extension theorem, there exists a section U of the bundle

$$K_{X'} + \Delta_{X'} + (m - 1)\pi_X^*(K_{X/Y} + \Delta_X) + (m_0 - 1)(\pi_Y \circ p')^*K_Y \tag{5.49}$$

whose restriction to the fiber X'_z is precisely \widehat{u} in (5.47). Indeed, we are using here the fact that X'_z is given by the inverse image of the (5.25), together with Lemma 5.9.

Let (u_j) be a basis of the space of global sections of the bundle $m\pi_X^*(K_{X/Y} + \Delta_X)|_{X'_z}$ and let (U_j) be the sections of (5.49) obtained by the procedure explained above. If we denote by (τ_k) a basis of the bundle $(m - m_0)K_Y$, then the family of sections

$$U_j \otimes (\pi_Y \circ p')^*\tau_k$$

are linearly independent. Moreover, they induce holomorphic sections of the bundle $K_{X'} + \Delta_{X'} + (m - 1)p_X^*(K_X + \Delta_X)$. Combining this with (5.38) and (5.39), we get

$$\kappa(K_{X'} + \Delta_{X'} + E_Y) \geq \kappa(K_F + \Delta_F) + \kappa(Y).$$

Notice that E_Y is the exceptional loci of π_X , the proof of Theorem 5.6 is thus finished. □

Remark 5.10 An interesting question related to the proof just finished is the following. We assume that the bundle $\det F_m$ is topologically trivial; it follows that F_m itself is flat. If we consider a birational map $\pi_Y : Y' \rightarrow Y$ and $p' : X' \rightarrow Y'$ as in (5.26), does it follows that F'_m is flat as well? This is clearly the case if π_Y is finite instead of birational.

Recently, the first named author of the present article has obtained the following result.

Theorem 5.11 [14] *Let $p : X \rightarrow Y$ be an algebraic fiber space, where Y is a projective variety of dimension at most two. Let Δ be an effective \mathbb{Q} -divisor such that the pair (X, Δ) is klt, and let F be a generic fiber of p . Then*

$$\kappa(K_X + \Delta) \geq \kappa(K_F + \Delta_F) + \kappa(Y), \tag{5.50}$$

where $\Delta_F = \Delta|_F$.

This settles the Iitaka conjecture in the case of a two-dimensional base. The proof in [14] uses the techniques developed in the present article, together with some results in the theory of orbifolds of Calabi-Yau type.

Acknowledgments We owe a debt of gratitude to Bo Berndtsson, Sébastien Boucksom, Frédéric Campana, Philippe Eyssidieux, Christopher Hacon, Andreas Höring, Zhi Jiang, Yujiro Kawamata, Mihnea Popa, Hossein Raufi, Christian Schnell and Shigeharu Takayama for sharing generously with us their results and intuitions on the topics analyzed here. It is our pleasure to acknowledge the partial support we have benefited from the ANR project “MACK” during the preparation of the present article. Last but not least, we would like to thank the anonymous referee for constructive criticism and excellent suggestions who helped us to improve substantially the quality of this work.

References

- Berndtsson, B.: Curvature of vector bundles associated to holomorphic fibrations. *Ann. Math.* **169**, 531–560 (2009)
- Berndtsson, B., Păun, M.: Bergman kernels and the pseudoeffectivity of relative canonical bundles. *Duke Math. J.* **145**(2), 343–378 (2008)
- Berndtsson, B., Păun, M.: Bergman kernels and subadjunction. [arXiv:1002.4145v1](https://arxiv.org/abs/1002.4145v1)
- Berndtsson, B., Păun, M.: Quantitative extensions of pluricanonical forms and closed positive currents. *Nagoya Math. J.* **205**, 25–65 (2012). [arXiv:1002.4145](https://arxiv.org/abs/1002.4145)
- Birkar, C., Cascini, P., Hacon, C.D., McKernan, J.: Existence of minimal models for varieties of general type. *J. Am. Math. Soc.* **23**, 405–468 (2010)
- Birkar, C.: The Iitaka conjecture C_{nm} in dimension six. *Compos. Math.* **145**, 1442–1446 (2009)
- Birkar, C., Chen, J.A.: Varieties fibred over abelian varieties with fibres of log general type. *Adv. Math.* **270**, 206–222 (2015)
- Birkenhake, C., Lange, H.: *Complex Abelian Varieties*. Grundlehren der Mathematischen Wissenschaften, vol. 302, 2nd edn, pp. xii+635. Springer, Berlin (2004)
- Campana, F.: Orbifolds, special varieties and classification theory. *Ann. Inst. Fourier* **54**, 499–665 (2004)
- Campana, F., Claudon, B., Eyssidieux, P.: Représentations linéaires des groupes kahlériens: factorisations et conjecture de Shafarevich linéaire. *Compos. Math.* **151**(2), 351–376 (2015)
- Campana, F., Koziarz, V., Păun, M.: Numerical character of the effectivity of adjoint line bundles. *Annales de l’Institut Fourier*. Tome **62**, 107–119 (2012)
- Campana, F., Peternell, T., Matei, T.: Geometric stability of the cotangent bundle and the universal cover of a projective manifold. *Bull. Soc. Math. France* (2011)
- Cao, J.: On the approximation of Kähler manifolds by algebraic varieties. *Math. Ann.* **363**(1–2), 393–422 (2015)
- Cao, J.: Kodaira dimension of algebraic fiber spaces over surfaces. [arXiv:1511.07048](https://arxiv.org/abs/1511.07048)
- Catanese, F., Dettweiler, M.: Answer to a question by Fujita on variation of hodge structures (2014). [arXiv:1311.3232](https://arxiv.org/abs/1311.3232)
- Chen, J.A., Hacon, C.D.: On algebraic fiber spaces over varieties of maximal Albanese dimension. *Duke Math. J.* **111**(1), 159–175 (2002)
- Chen, J.A., Hacon, C.D.: On Ueno’s conjecture K. *Math. Ann.* **345**(2), 287–296 (2009)
- Chen, J.A., Hacon, C.D.: Kodaira dimension of irregular varieties. *Invent. Math.* **186**(3), 480–500 (2011)
- Debarre, O.: *Tores et variétés abéliennes complexes*. Cours Spécialisés, vol. 6, pp. vi+125. EDP Sciences, Les Ulis, Paris (1999)
- Demailly, J.-P.: *Complex analytic and differential geometry*. Université de Grenoble (2007). <https://www-fourier.ujf-grenoble.fr/~demailly/documents.html>
- Fujino, O.: On maximal Albanese dimensional varieties. *Proc. Jpn. Acad. Ser. A Math. Sci.* **89**(8), 92–95 (2013)

22. Hacon, C.D.: A derived category approach to generic vanishing. *J. Reine Angew. Math.* **575**, 173–187 (2004)
23. Höring, A.: Positivity of direct image sheaves—a geometric point of view. *L'Enseignement Math.* **56**(1), 87–142 (2010)
24. Kawamata, Y.: Characterization of Abelian varieties. *Compos. Math.* **43**(2), 253–276 (1981)
25. Kawamata, Y.: Kodaira dimension of algebraic fiber spaces over curves. *Invent. Math.* **66**, 57–71 (1982)
26. Kawamata, Y.: Subadjunction of log canonical divisors, II. *Am. J. Math.* **120**, 893–899 (1998)
27. Klimek, M.: Pluripotential theory. London Mathematical Society Monographs. Oxford Science Publications, Oxford (1991)
28. Kollár, J.: Subadditivity of the Kodaira dimension: fibers of general type. In: *Algebraic Geometry, Sendai, 1985*. *Adv. Stud. Pure Math.*, vol. 10, pp. 361–398. North-Holland, Amsterdam (1987)
29. Kovács, S., Patakfalvi, Z.: Projectivity of the moduli space of stable log-varieties and subadditivity of log-Kodaira dimension (2015). [arXiv:1503.02952](https://arxiv.org/abs/1503.02952)
30. Lai, C.-J.: Varieties fibered by good minimal models. *Math. Ann.* **350**(3), 533–547 (2011)
31. Mourougane, Ch., Takayama, S.: Hodge metrics and the curvature of higher direct images. *Ann. Sci. Éc. Norm. Supér.* **41**(6), 905–924 (2008)
32. Nakayama, N.: Zariski-decomposition and abundance *MSJ Memoirs*, vol. 14. Mathematical Society of Japan, Tokyo (2004)
33. Păun, M.: Singular Hermitian metrics and positivity of direct images of pluricanonical bundles (2016). [arXiv:1606.00174](https://arxiv.org/abs/1606.00174)
34. Păun, M., Takayama, S.: Positivity of twisted relative pluricanonical bundles and their direct images (2014). [arXiv:1409.5504](https://arxiv.org/abs/1409.5504)
35. Popa, M., Schnell, Ch.: On direct images of pluricanonical bundles. *Algebra Number Theory* **8**(9), 2273–2295 (2014). [arXiv:1405.6125](https://arxiv.org/abs/1405.6125)
36. Raufi, H.: Singular hermitian metrics on holomorphic vector bundles. *Arkiv för Matematik* **53**(2), 359–382 (2015)
37. Tsuji, H.: Global generation of the direct images of relative pluricanonical systems (2010). [arXiv:1012.0884](https://arxiv.org/abs/1012.0884)
38. Ueno, K.: Classification theory of algebraic varieties and compact complex spaces. Notes written in collaboration with P. Cherenack. *Lecture Notes in Mathematics*, vol. 439, pp. xix+278. Springer, Berlin-New York (1975)
39. Viehweg, E.: Weak positivity and the additivity of the Kodaira dimension for certain fibrespaces. *Proc. Algebraic Varieties and Analytic Varieties, 1981*. *Adv. Studies in Math.* vol. 1, pp. 329–353. Kinokunya-North-Holland Publ., Tokyo (1983)
40. Viehweg, E.: Quasi-projective moduli for polarized manifolds. *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*, vol. 30. Springer, Berlin (1995)
41. Viehweg, E.: Positivity of direct image sheaves and applications to families of higher dimensional manifolds *School on Vanishing Theorems and Effective Results in Algebraic Geometry (Trieste, 2000)*, ICTP Lect. Notes, vol. 6, pp. 249–284. Abdus Salam Int. Cent. Theoret. Phys., Trieste (2001)
42. Voisin, C.: Théorie de Hodge et géométrie algébrique complexe. *Cours Spécialisés*, vol. 10. Société Mathématique de France, Paris (2002)
43. Yang, J.-H.: Holomorphic vector bundles over complex tori. *J. Korean Math. Soc.* **26**(1), 117–142 (1989)
44. Zuo, K.: Kodaira dimension and Chern hyperbolicity of the Shafarevich maps for representations of π_1 of compact Kahler manifolds. *J. Reine Angew. Math.* **472**, 139–156 (1996)