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A Robust Proof of the Instability of Naked Singularities of a Scalar Field in Spherical Symmetry

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Abstract: Published in 1999, Christodoulou proved that the naked singularities of a self-gravitating scalar field are not stable in spherical symmetry and therefore the cosmic censorship conjecture is true in this context. The original proof is by contradiction and sharp estimates are obtained strictly depending on spherical symmetry. In this paper, appropriate a priori estimates for the solution are obtained. These estimates are more relaxed but sufficient for giving another robust argument in proving the instability, in particular not by contradiction. In a companion paper, we are able to prove certain instability theorems of the spherically symmetric naked singularities of a scalar field under gravitational perturbations without symmetries. The argument given in this paper plays a central role.

1. Introduction

In the paper [4], Christodoulou proved both the *weak cosmic censorship conjecture* and the *strong cosmic censorship conjecture* for spherically symmetric solutions of the Einstein equations coupled with a massless scalar field. The coupled system reads

$$\mathbf{Ric}_{\alpha\beta} - \frac{1}{2}\mathbf{R}g_{\alpha\beta} = 2\mathbf{T}_{\alpha\beta}$$

where

$$\mathbf{T}_{\alpha\beta} = \nabla_\alpha\phi\nabla_\beta\phi - \frac{1}{2}g_{\alpha\beta}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi,$$

which we call the Einstein-scalar field equations. The proof, which is by contradiction, contains sharp estimates which may not be easily obtained beyond spherical symmetry. In this paper, we will provide a robust proof which is not by contradiction, and contains only relaxed estimates. The main advantage of this proof is that it has the potential to be extended beyond spherical symmetry.

1.1. The work of Christodoulou. Consider the characteristic initial value problem of the Einstein-scalar field equations in spherical symmetry. The initial data is given on a null cone C_o issuing from a fixed point o of the symmetry group $SO(3)$, and consists of a function $\alpha_0 = \frac{\partial}{\partial r}(r\phi)\big|_{C_o}$ defined on $[0, +\infty)$, where r is area radius of the orbit spherical sections of C_o, and ϕ is the scalar field function. Then what was exactly proved by Christodoulou is the following theorem.

Theorem 1 (Christodoulou, Theorem 4.1 in [4]). *Let $\mathcal{E}$ be the complement of the collection of functions $\alpha_0 \in BV$ whose maximal future development is either complete, or possesses a complete future null infinity and a strictly spacelike singular future boundary. Then there exist two functions f_1, $f_2 \in BV$, such that $\alpha_0+\lambda_1 f_1+\lambda_2 f_2 \notin \mathcal{E}$ and has non-complete maximal future development for all λ_1, λ_2 with $\lambda_1 \neq 0$ or $\lambda_2 \neq 0$. Moreover, if $\alpha_0 + \lambda_1 f_1 + \lambda_2 f_2 \equiv \alpha_0' + \lambda_1' f_1' + \lambda_2' f_2'$, then $\alpha_0 \equiv \alpha_0'$, $f_1 \equiv f_1'$, $f_2 \equiv f_2'$ and $\lambda_1 = \lambda_1'$, $\lambda_2 = \lambda_2'$. We may therefore say that the exceptional set $\mathcal{E}$ is of codimension at least* 2.

Remark 1. In the proof of the above theorem in [4], f_2 is shown to be absolutely continuous, and therefore the conclusion that the exceptional set is of codimension at least 1, still positive but not 2, holds for α_0 being absolutely continuous, which of course is more regular than being of bounded variation.

The proof can roughly be divided into three steps. Consider an arbitrary initial data $\alpha_0 \in BV$. The first step, which was shown in [3], is that the maximal future development of such data is complete, unless there exists a singular endpoint e of the central timelike geodesic Γ from o. A sharp criterion of the appearance of e was also found: $\frac{2m}{r} \nrightarrow 0$ when approaching e from the past, where m is the mass function. The second step is to understand how an apparent horizon forms. We have the following collapse theorem also by Christodoulou.

Theorem 2 (Christodoulou, [2]). *Consider the spherically symmetric solution of the Einstein-scalar field equations with initial data given on a null cone C_o. Let S_1 and S_2 be two spherical sections with area radii r_1, r_2 and mass contents m_1, m_2, and S_2 is in the exterior to S_1. Denote*

$$\delta = \frac{r_2}{r_1} - 1.$$

Then there exist two universal positive constants c_0, c_1 such that if $\delta \leq c_0$ and

$$2(m_2 - m_1) > c_1 r_2 \delta \log\left(\frac{1}{\delta}\right),$$

then the incoming null cone through S_2 intersects the apparent horizon and enters the trapped region, the region of trapped surfaces. Moreover, there exists an event horizon and the trapped region terminates at a spacelike singular future boundary.

Inspired by this theorem, one can then consider the future of $\underline{C}_e \cup C_o$, where $\underline{C}_e$ is the boundary of the causal past of e and intersects C_o at $s = s_e$ and C_o here refers to the part $s \geq s_e$. Then the last step, what was really proved in [4] is that, allowing a perturbation on α_0, there exists a sequence $p_n \in \Gamma$ where $p_n \to e$ such that the null cone C_{p_n} issuing from p_n satisfies the assumptions of Theorem 2 at two spheres $S_{1,n}$ and $S_{2,n}$ on C_{p_n}, between which the distance tends to zero. The corresponding spacetimes have an event horizon and therefore possess a complete future null infinity, which verifies the *weak cosmic censorship conjecture*. Moreover, the distance of $S_{1,n}$ and $S_{2,n}$ tending to zero implies that the apparent horizon issues from e and therefore the future boundary of the maximal development is spacelike and singular. This verifies the *strong cosmic censorship conjecture*.

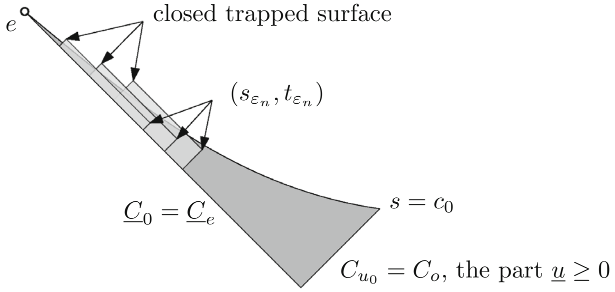

Fig. 1. The original proof by Christodoulou. The closed trapped surfaces on the top of the light grey regions are predicted by Theorem 2

1.2. The content of this paper. In this paper, we are going to give a new argument of this last step. This argument is more robust and can potentially be extended beyond spherical symmetry, which has been studied in our companion paper [6]. The present article can then be understood as an introduction to our new methods applied on the original spherically symmetric model studied by Christodoulou, so we are not going to prove any essentially new results in this paper.

We would like to give a statement of what we prove exactly in this paper. First, we introduce a double null coordinate $(\underline{u}, u)$ of the quotient spacetime relative to the singular endpoint e of Γ as follows. Let $\underline{u}$ and u be optical functions, i.e., their level sets, which we denote by $\underline{C}_{\underline{u}}$ and C_u, are incoming and outgoing null cones respectively. We take $\underline{u} = 0$, $u = -r$ on $\underline{C}_e$, and take $u = u_0$ and $\underline{u}$ increasing towards the future on C_o where $-u_0 = r_0$ is the area radius of the sphere $\underline{C}_e \cap C_o$. Using this notation, we will write $\underline{C}_e = \underline{C}_0$, $C_o = C_{u_0}$. In terms of the double null coordinate $(\underline{u}, u)$ relative to e, what we will reprove can be stated as follows.

Theorem 3. *Let $\mathcal{E}$ be the complement of the collection of functions $\alpha_0 \in BV$ whose maximal future development is either complete or has the property, that if e is the singular endpoint of Γ and $(\underline{u}, u)$ is the double null coordinate relative to e, then there exist two sequences $\delta_n \to 0^+$ and $u_n \to 0^-$ such that*

$$2(m - m_n) > \frac{c_1 r(r - r_n)}{r_n} \log \frac{r_n}{r - r_n}, \ \text{with} \ \frac{r - r_n}{r_n} \leq c_0 \tag{1.1}$$

where m and r take values at $(\underline{u}, u) = (\delta_n, u_n)$, $m_n = m(0, u_n)$, $r_n = |u_n|$ and c_0, c_1 are the constants given in Theorem 2. Then if $\alpha_0 \in \mathcal{E}$, there exist two functions $f_1, f_2 \in BV$, such that $\alpha_0 + \lambda_1 f_1 + \lambda_2 f_2 \notin \mathcal{E}$ and has non-complete maximal future development for all λ_1, λ_2 with $\lambda_1 \neq 0$ or $\lambda_2 \neq 0$. Moreover, if $\alpha_0 + \lambda_1 f_1 + \lambda_2 f_2 \equiv \alpha_0' + \lambda_1' f_1' + \lambda_2' f_2'$, then $\alpha_0 \equiv \alpha_0'$, $f_1 \equiv f_1'$, $f_2 \equiv f_2'$ and $\lambda_1 = \lambda_1'$, $\lambda_2 = \lambda_2'$.

We discuss some differences between the original proof of Christodoulou and the proof in this paper in the rest of the introduction. The proof of Christodoulou is roughly depicted in Fig. 1. Instead of using the double null coordinate, Christodoulou worked in a dimensionless coordinate (s, t) relative to the singular endpoint e defined as follows:

$$u = u_0 \mathrm{e}^{-t}, \ -2r = u_0 \mathrm{e}^{s-t}.$$

The proof should be done by deriving suitable estimates and Christodoulou started by assuming that the conclusion of Theorem 3 is not true for a generic class of the initial data, this is to say, given any $\varepsilon > 0$, the opposite of (1.1), i.e.,

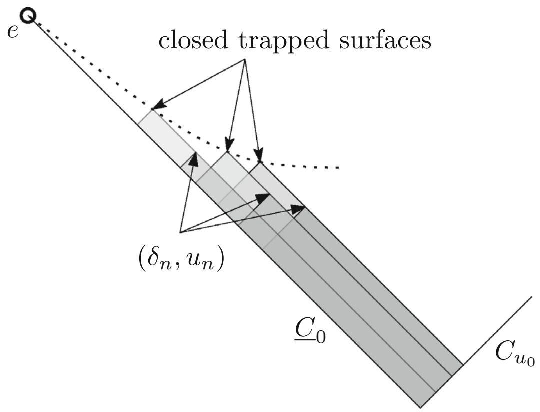

Fig. 2. The proof in this paper. Similar to Fig. 1, the closed trapped surfaces on the top of the light grey regions are predicted by Theorem 2

$$2(m(s,t) - m(0,t)) \leq c_1 r(s,t) s \log\left(\frac{1}{s}\right) \tag{1.2}$$

holds in the region $\{0 \leq s \leq c_0\} \cap \{0 \leq \underline{u} \leq \varepsilon\}$, the intersection of the dark grey region in Fig. 1 and the part of the spacetime before the incoming null cone $\underline{u} = \varepsilon$. For the spherically symmetric Einstein-scalar field system, the mass m governs the whole system and has good monotonicity properties. Therefore Christodoulou was able to estimate all related geometric quantities in a sharp way in terms of $m(s,t) - m(0,t)$, which is bounded from (1.2), without knowing any a priori bounds of the solution. In particular, he was able to compare the value of $r\frac{\partial}{\partial \underline{u}}\phi$[1] in the region $\{0 \leq s \leq c_0\} \cap \{0 \leq \underline{u} \leq \varepsilon\}$ to its values on $\underline{C}_0 = \underline{C}_e$ and on $C_{u_0} = C_o$. By carefully choosing a sequence (s_n, t_n) approaching the singularity, i.e., $s_n \to 0^+$ and $t_n \to +\infty$, the estimates in particular give a lower bound of $r\frac{\partial}{\partial \underline{u}}\phi$ and hence a lower bound of $m - m_n$ at (s_n, t_n), which is roughly speaking an L^2 integral of $r\frac{\partial}{\partial \underline{u}}\phi$ relative to $\underline{u}$. Christodoulou found that such a lower bound on $m - m_n$ contradicts to (1.2) at (s_n, t_n) for sufficiently large n. Therefore, (1.1) should hold for some particular $(s_\varepsilon, t_\varepsilon) \in \{0 \leq s \leq c_0\} \cap \{0 \leq \underline{u} \leq \varepsilon\}$. And since the above argument works for any $\varepsilon > 0$, it was concluded that (1.1) should hold for a sequence of $(s_{\varepsilon_n}, t_{\varepsilon_n})$ and this is what one wanted to prove.

However, in order to extend this result beyond spherical symmetry, much more things need to be concerned. First of all, we need to derive suitable a priori estimates in order to establish the *existence* of the solution. Second, we may not benefit from the assumptions like (1.2) which is from proof by contradiction because the mass can no longer govern the whole system without symmetries. In addition, the estimates derived by Christodoulou are so sharp that it is not easy to extend them beyond spherical symmetry.

The proof given in this paper, which is depicted in Fig. 2, is along the following line. Different from the dimensionless coordinate (s, t), we work in the double null coordinate $(\underline{u}, u)$, in which many new techniques pioneered by Christodoulou in the work [5] were developed recently. In the double null coordinate, we are able to derive a priori L^∞ bounds of the geometric quantities, including $\frac{\partial}{\partial \underline{u}}\phi$, $\frac{\partial}{\partial u}\phi$, and the derivatives of r and

[1] Strictly speaking, Christodoulou estimated the quantity $r\frac{\partial}{\partial r}\phi$.

Ω defined by $-2\Omega^2 = g\left(\frac{\partial}{\partial \underline{u}}, \frac{\partial}{\partial u}\right)$. These a priori estimates are proved by a bootstrap argument to hold in all rectangular regions, the dark grey regions in Fig. 2, in the form $0 \le \underline{u} \le \delta$, $u_0 \le u \le u_1$ where $\delta |u_1|^{-1}$ should be sufficiently small. Then we carefully choose two sequences $\delta_n \to 0^+$ and $u_n \to 0^-$ according to the condition (1.1)[2] and our main new observation is that the quantity $\delta_n |u_n|^{-1}$ is small enough to guarantee that the a priori estimates are satisfied in $0 \le \underline{u} \le \delta_n$, $u_0 \le u \le u_n$ for sufficiently large n. Moreover, these a priori estimates, which are in principle more relaxed than the estimates derived by Christodoulou, are still enough to give a suitable lower bound of $r \frac{\partial}{\partial \underline{u}} \phi$ for $0 \le \underline{u} \le \delta_n$, $u = u_n$, and hence a lower bound of $m - m_n$ which implies the condition (1.1) for a generic class of the initial data. In particular, we do not need to argue by contradiction since we already have L^∞ a priori bounds which are much stronger than the original assumption (1.2) of Christodoulou. One of the advantages of not arguing by contradiction is that we are able to identify the exact locations where the condition (1.1) holds.

The a priori estimates we derive are robust and analogues of them may hold when no symmetries are imposed, and the new argument presented in this paper can possibly be extended. The generalization of these estimates without symmetries will also be used in proving the *existence* of the solution. In our companion paper [6], we consider the characteristic initial value problem of the Einstein-scalar field equations, with the initial data given on two null cones intersecting at a sphere. The incoming null cone is assumed to be spherically symmetric and singular at its vertex, in the sense that $\frac{2m}{r} \not\to 0$ when approaching it from the past. No symmetries are imposed on the outgoing null cone. Then we will show that the argument presented in this paper can be directly generalized and we can also prove certain instability theorems. We suggest the readers refer to [6] for the precise statements. Finally, we should also mention that the estimates derived in this paper, and also those in [6], are inspired by and share many common features with those in the work of An-Luk [1] where they worked with the spacetime region deep near the vertex which is regular. Readers may also refer to [6] for some discussions about this.

2. Double Null Coordinates and Equations

2.1. Double null coordinate. The spherically symmetric spacetime can be studied through its 2-dimensional quotient spacetime manifold with boundary Γ, the fixed point set of the $SO(3)$ action, being a timelike geodesic, which we call the central line. We use a double null coordinate $(\underline{u}, u)$, where $\underline{u}, u$ are optical functions, which means that their level sets $\underline{C}_{\underline{u}}$ and C_u are incoming and outgoing null cones invariant under the $SO(3)$ action respectively. In the quotient spacetime, $\underline{C}_{\underline{u}}$ and C_u are then incoming and outgoing null lines respectively. We then denote

$$L = \frac{\partial}{\partial \underline{u}}, \ \underline{L} = \frac{\partial}{\partial u},$$

and define the lapse function Ω by

$$-2\Omega^2 = g(L, \underline{L}).$$

[2] We remark that the the choice of two sequences δ_n and u_n, is similar to the choice of (s_n, t_n) in Christodoulou's proof.

Then the metric has the form

$$-2\Omega^2(\mathrm{d}\underline{u} \otimes \mathrm{d}u + \mathrm{d}u \otimes \mathrm{d}\underline{u}) + r^2 \mathrm{d}\sigma_{\mathbb{S}^2}$$

where the area radius function $r = r(\underline{u}, u)$ is defined by

$$\mathrm{Area}(S_{\underline{u},u}) = 4\pi r^2,$$

and $\mathrm{d}\sigma_{\mathbb{S}^2}$ is the standard metric of the unit sphere.

2.2. Equations. From the form of the metric, the unknowns of the Einstein-scalar field equations are r, Ω and the scalar field function ϕ. What we really concern are their derivatives. We define the null expansions relative to the normalized pair of null vectors $\Omega^{-2}L$, $\underline{L}$ and the mass function m by

$$h = \Omega^{-2}Dr, \ \underline{h} = \underline{D}r, \ m = \frac{r}{2}(1 + h\underline{h}),$$

where D and $\underline{D}$ are the restrictions on the orbit spheres of the Lie derivatives along L and $\underline{L}$. When applying on functions, D and $\underline{D}$ are simply the ordinary derivatives. We then define the D derivative of the lapse Ω

$$\omega = D \log \Omega,$$

while its $\underline{D}$ derivative is not needed in this paper. Finally, we also need the derivatives of the scalar field function ϕ:

$$L\phi = \frac{\partial}{\partial \underline{u}}\phi, \ \underline{L}\phi = \frac{\partial}{\partial u}\phi.$$

We then list below all the equations satisfied by the above quantities and needed in this paper. First of all, we have the following five null structure equations:[3]

$$Dh = - r\Omega^{-2}(L\phi)^2, \tag{2.1}$$

$$\underline{D}(\Omega^2 h) = - \frac{\Omega^2(1 + h\underline{h})}{r}, \tag{2.2}$$

$$D\underline{h} = - \frac{\Omega^2(1 + h\underline{h})}{r}, \tag{2.3}$$

$$\underline{D}(\Omega^{-2}\underline{h}) = - r\Omega^{-2}(\underline{L}\phi)^2, \tag{2.4}$$

$$\underline{D}\omega = \frac{\Omega^2(1 + h\underline{h})}{r^2} - L\phi\underline{L}\phi. \tag{2.5}$$

The following two equations, which are equivalent, are the wave equation satisfied by ϕ:

$$\underline{D}(rL\phi) = - \Omega^2 h\underline{L}\phi, \tag{2.6}$$

$$D(r\underline{L}\phi) = - \underline{h}L\phi. \tag{2.7}$$

[3] Readers may refer to [2] for the derivations of these equations, though the notations have some slight differences. These equations can also be directly written down from the general null structure equations without symmetries, which can be found in the authors companion paper [6] mentioned above. The derivations of these equations in vacuum can be found in Christodoulou's work [5] on the formation of black holes.

Finally, we have the following equation about the mass function m:

$$Dm = -\frac{1}{2}\underline{h}\Omega^{-2}(rL\phi)^2, \tag{2.8}$$

$$\underline{D}m = -\frac{1}{2}h(r\underline{L}\phi)^2. \tag{2.9}$$

3. A Priori Bounds for the Solution

We begin the proof of Theorem 3. Recall that we start from an arbitrary initial data $\alpha_0 \in BV$ and the central line has a singular endpoint e, approaching which $\frac{2m}{r} \nrightarrow 0$. The double null coordinate $(\underline{u}, u)$ is chosen such that $\underline{u} = 0$, $u = -r$ on the boundary of the causal past of e, and $u = u_0 = -r_0$ and $\underline{u}$ increases towards the future on the initial null cone C_o where r_0 is the area radius of $\underline{C}_e \cap C_o$.

3.1. Geometry on $\underline{C}_0$. First of all we would like derive some identities on $\underline{C}_0$, the boundary of the causal past of e. We denote the restrictions on $\underline{C}_0$ of some geometric quantities, which are considered as functions of u:

$$\psi = \psi(u) = r\underline{L}\phi\Big|_{\underline{C}_0}, \ \varphi = \varphi(u) = rL\phi\Big|_{\underline{C}_0},$$

$$\Omega_0 = \Omega_0(u) = \Omega\Big|_{\underline{C}_0}, h_0 = h_0(u) = h\big|_{\underline{C}_0}.$$

From $u = -r$ on $\underline{C}_0$, we must have $\underline{h}\big|_{\underline{C}_0} \equiv -1$. Substituting this into (2.4), we find

$$\frac{\partial}{\partial u}\log\Omega_0 = -\frac{1}{2}\frac{\psi^2}{|u|}, \text{ and hence } -\log\frac{\Omega_0^2(u)}{\Omega_0^2(u_0)} = \int_{u_0}^{u}\frac{\psi^2(u')}{|u'|}\mathrm{d}u'. \tag{3.1}$$

From (2.2), we have

$$\frac{\partial}{\partial u}(\Omega_0^2 h_0) = -\frac{\Omega_0^2(1-h_0)}{|u|}, \tag{3.2}$$

$$\text{and hence } -\log\frac{\Omega_0^2(u)h_0(u)}{\Omega_0^2(u_0)h_0(u_0)} = \int_{u_0}^{u}\frac{1}{|u'|}\left(\frac{1}{h_0(u')}-1\right)\mathrm{d}u'. \tag{3.3}$$

Because $m\big|_{\underline{C}_0} \geq 0$[4] and the apparent horizon, where $m = 2r$, does not intersects $\underline{C}_0$,[5] and hence none of the spherical sections of $\underline{C}_0$ are closed trapped surfaces, then $2m < r$ on $\underline{C}_0$. From the definition of the mass m and that $\underline{h}\big|_{\underline{C}_0} \equiv -1$ on $\underline{C}_0$, we have $1 - h_0 = \frac{2m}{r}\big|_{\underline{C}_0}$ which lies in $[0, 1)$ and hence $0 < h_0 \leq 1$. Then we are going to prove an important lemma.

[4] See for example Proposition 4.1 in [2]. Roughly speaking, the non-negativity of the mass m follows from that it is monotonically increasing along future outgoing null cones and it vanishes on the central line.

[5] Intuitively thinking, if the apparent horizon intersects $\underline{C}_0$, then we have nothing to prove because we already have a black hole and the singularity is not naked. In fact, it was proved by Christodoulou in the complete version of the collapse theorem in [3] that the apparent horizon contains no incoming null pieces and intersects the singular boundary. This would contradict to the assumption that $\underline{C}_0$ is the boundary of the past of the singular endpoint of the central line.

Lemma 1. *Both $\Omega_0^2 h_0$ and Ω_0 are monotonically decreasing and converge to* 0 *as* $u \to 0^-$.

Proof. The monotonicity follows from the fact that the integrands in (3.1) and (3.3) are positive. From Lemma 2 in [4] (3.3) tends to infinity as $u \to 0^-$. We rewrite this proof using the notations in this paper. Indeed, on $\underline{C}_0$, it holds $\frac{2m}{r}\big|_{\underline{C}_0} = 1 - h_0$, then using the fact that $m(0, u)$ is decreasing which follows from (2.9), we have

$$\int_{3u}^{u} \frac{1}{|u'|}\left(\frac{1}{h_0(u')} - 1\right) \mathrm{d}u' = \int_{3u}^{u} \frac{1}{|u'|} \frac{\frac{2m(0,u')}{|u'|}}{1 - \frac{2m(0,u')}{|u'|}} \mathrm{d}u'$$
$$\geq \int_{3u}^{u} \frac{1}{|u'|} \frac{\frac{|u|}{|u'|}\frac{2m(0,u)}{|u|}}{1 - \frac{|u|}{|u'|}\frac{2m(0,u)}{|u|}} \mathrm{d}u' = \log\left(\frac{1 - \frac{1}{3}\frac{2m(0,u)}{|u|}}{1 - \frac{2m(0,u)}{|u|}}\right).$$

If (3.3) is bounded for all $u \in [u_0, 0)$, then the first integral above should tend to zero when $u \to 0^-$. However, this implies that $\frac{2m}{r} \to 0$ from the above inequality, a contradiction. Therefore $\Omega_0^2 h_0 \to 0$ as $u \to 0^-$. If $\Omega_0 \nrightarrow 0$, then Ω_0 has a positive lower bound because it is decreasing. Therefore $\Omega^2 h_0 \to 0$ implies that $h_0 \to 0$. Substitute this to (3.2), we find as $u \to 0^-$,

$$\Omega_0^2(u_0)h_0(u_0) - \Omega_0^2(u)h_0(u) = \int_{u_0}^{u} \frac{\Omega_0^2(1 - h_0)}{|u'|} \mathrm{d}u' \to +\infty$$

and hence $\Omega_0^2(u)h_0(u) \to -\infty$, a contradiction. We conclude that $\Omega_0 \to 0$ as $u \to 0^-$ and the proof is completed. □

Remark 2. The proof in the rest of the paper then depends only on the fact that $\Omega_0 \to 0$ monotonically. The infiniteness of (3.3) depends strictly on the monotonicity of mass m along $\underline{C}_0$. We do not expect a robust argument of this proof since the criteria $\frac{2m}{r} \nrightarrow 0$ may not make sense beyond spherical symmetry. A robust version of this lemma, which is beyond reach right now, should include another suitable criterion outside spherical symmetry, which is still an active area of research.

3.2. The a priori estimates. We are going to derive the a priori estimates for the geometric quantities. We fix a small constant $\delta > 0$, a constant $u_1 \in (u_0, 0)$ and denote

$$\mathscr{F} = \mathscr{F}(u_0, u_1) = \max\{1, \sup_{u_0 \leq u \leq u_1} |\varphi(u)|\},$$

and

$$\mathcal{A} = \mathcal{A}(\delta, u_0, u_1) = \max\{1, \sup_{0 \leq \underline{u} \leq \delta} \mathscr{F}^{-1}(|rL\phi(\underline{u}, u_0)| + |u_0||\omega(\underline{u}, u_0)|)\}.$$

Here $\mathscr{F}^{-1}$ is understood as the reciprocal of the quantity $\mathscr{F}$ but not the inverse of the function. We remark that the product $\mathscr{F}\mathcal{A}$ contains informations from both C_{u_0} and $\underline{C}_0$ and hence it is the expected bound of $rL\phi$ and the whole system.[6] Without loss of generality, we also assume that $\Omega(u_0) \leq 1$. By the monotonicity of Ω_0, we have $\Omega_0(u) \leq 1$ for all $u \in [u_0, 0)$. Then we are going to prove

[6] In our companion paper [6], $\mathscr{F}$ can be arbitrarily chosen and we establish the existence theorem in a general form. Then we can apply the existence theorem in different situations by choosing different $\mathscr{F}$.

Theorem 4. *There exists a universal large constant* $C_0 \geq 1$ *such that the following statement is true. Suppose that*

$$\mathcal{A} < +\infty,$$

and for some $C \geq C_0$ *we have*

$$C^2\delta|u_1|^{-1}\mathscr{F}\mathscr{W}^{\frac{1}{2}}\mathcal{A} \leq 1, \text{ where } \mathscr{W} = \mathscr{W}(u_0, u_1) = \max\left\{1, \left|\log\frac{\Omega_0(u_1)}{\Omega_0(u_0)}\right|\right\}. \tag{3.4}$$

Then we have the following estimates for $0 \leq \underline{u} \leq \delta$, $u_0 \leq u \leq u_1$*:*[7]

$$\frac{1}{2}\Omega_0(u) \leq \Omega \leq 2\Omega_0(u),\ \frac{1}{2}|u| \leq r \leq 2|u|, \tag{3.5}$$

$$|rL\phi| \lesssim \mathscr{F}\mathcal{A}, \tag{3.6}$$

$$|r\underline{L}\phi - \psi| \lesssim \delta|u|^{-1}\mathscr{F}\mathcal{A}, \tag{3.7}$$

$$|h - h_0| \lesssim \delta|u|^{-1}\Omega_0^{-2}\mathscr{F}^2\mathcal{A}^2, \tag{3.8}$$

$$|\underline{h} + 1| \lesssim \delta|u|^{-1}\mathscr{F}\mathcal{A}, \tag{3.9}$$

$$|u||\omega| \lesssim \mathscr{F}\mathscr{W}^{\frac{1}{2}}\mathcal{A}. \tag{3.10}$$

Moreover, we have the improved estimate

$$|rL\phi(\underline{u}, u) - \varphi(u)| \lesssim |rL\phi(\underline{u}, u_0) - \varphi(u_0)| + \delta|u|^{-1}\mathscr{F}^2\mathscr{W}^{\frac{1}{2}}\mathcal{A}^2. \tag{3.11}$$

Proof. We begin the proof by introducing the set $\mathcal{U}_{\delta,u_1}$ of $u_* \in [u_0, u_1]$ such that the following *bootstrap assumptions* hold for $0 \leq \underline{u} \leq \delta$, $u_0 \leq u \leq u_*$:

$$|rL\phi| \lesssim C^{\frac{1}{4}}\mathscr{F}\mathcal{A}, \tag{3.12}$$

$$|u||\omega| \lesssim C^{\frac{1}{4}}\mathscr{F}\mathscr{W}^{\frac{1}{2}}\mathcal{A}. \tag{3.13}$$

$\mathcal{U}_{\delta,u_1}$ is non-empty since $u_0 \in \mathcal{U}_{\delta,u_1}$ by the definition of $\mathscr{F}$ and $\mathcal{A}$, and closed in $[u_0, u_1]$ by the continuity of $L\phi$, ω relative to u.[8] Then we choose any $u_* \in \mathcal{U}_{\delta,u_1}$ and hope to derive estimates for $0 \leq \underline{u} \leq \delta$, $u_0 \leq u \leq u_*$ where (3.12) and (3.13) hold. To this end, we must introduce another set $\underline{\mathcal{U}}_{\delta,u_*}$ of $\underline{u}_* \in [0, \delta]$ such that the following *secondary bootstrap assumptions* hold for $0 \leq \underline{u} \leq \underline{u}_*$, $0 \leq u \leq u_*$:

$$|r\underline{L}\phi - \psi| \lesssim C^{\frac{1}{4}}\delta|u|^{-1}\mathscr{F}\mathcal{A}, \tag{3.14}$$

$$|h - h_0| \lesssim C^{\frac{1}{2}}\delta|u|^{-1}\Omega_0^{-2}\mathscr{F}^2\mathcal{A}^2, \tag{3.15}$$

$$|\underline{h} + 1| \lesssim C^{\frac{1}{4}}\delta|u|^{-1}\mathscr{F}\mathcal{A}. \tag{3.16}$$

For any $u_* \in \mathcal{U}_{\delta,u_1}$, the set $\underline{\mathcal{U}}_{\delta,u_*}$ is still non-empty since $0 \in \underline{\mathcal{U}}_{\delta,u_*}$ and closed by the continuity $\underline{L}\phi, h, \underline{h}$ relative to $\underline{u}$. We then choose $\underline{u}_* \in \underline{\mathcal{U}}_{\delta,u_*}$ and begin to derive estimates for $0 \leq \underline{u} \leq \underline{u}_*$, $u_0 \leq u \leq u_*$, where (3.12)–(3.16) hold.

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[7] The notation $A \lesssim B$ means $A \leq cB$ for some universal constant c.

[8] This is because we are dealing with a solution of bounded variation, then the quantities $L\phi, \underline{L}\phi, h, \underline{h}, \omega$ are bounded and Ω, r are continuous. Moreover, from the Eqs. (2.1)–(2.7), if $r > 0$, then $L\phi, h, \underline{h}, \omega$ are continuous relative to u and $\underline{L}\phi, h, \underline{h}$ are continuous relative to $\underline{u}$.

Before closing the bootstrap argument, the estimates we obtain below only hold for $0 \le \underline{u} \le \underline{u}_*$, $u_0 \le u \le u_*$. From (3.13), we have

$$|\log \Omega - \log \Omega_0| \le \int_0^{\underline{u}_*} |\omega| \mathrm{d}\underline{u} \lesssim C^{\frac{1}{4}} \delta |u|^{-1} \mathscr{F} \mathscr{W}^{\frac{1}{2}} \mathcal{A} \lesssim C^{-1}.$$

The last inequality is because of (3.4) and $|u| \ge |u_1|$.[9] By choosing C_0 (and hence C) sufficiently large, we have

$$|\log \Omega - \log \Omega_0| \le \log 2$$

and therefore (3.5) holds for Ω. Moreover, we have

$$|\Omega - \Omega_0| \le \int_0^{\underline{u}_*} |\Omega\omega| \,\mathrm{d}\underline{u} \lesssim C^{\frac{1}{4}} \delta |u|^{-1} \Omega_0 \mathscr{F} \mathscr{W}^{\frac{1}{2}} \mathcal{A}. \tag{3.17}$$

For r, we note that, from (3.15),

$$|\Omega^2 h| \lesssim \Omega_0^2 \left(h_0 + C^{\frac{1}{2}} \delta |u|^{-1} \Omega_0^{-2} \mathscr{F}^2 \mathcal{A}^2 \right) \lesssim \mathscr{F} \mathcal{A}. \tag{3.18}$$

The second inequality holds because $\Omega_0 \le 1 \le \mathscr{F} \mathcal{A}$ by definition. We then use the equation $Dr = \Omega^2 h$ to obtain

$$|r - |u|| \le \int_0^{\underline{u}_*} |\Omega^2 h| \mathrm{d}\underline{u} \lesssim \delta \mathscr{F} \mathcal{A}. \tag{3.19}$$

We then deduce that $|r - |u|| \lesssim C^{-1}|u|$ and (3.5) holds for r if C_0 is sufficiently large.[10]

For $L\phi$, we consider the Eq. (2.6). We write

$$\frac{\partial}{\partial u}(rL\phi - \varphi) = -\left(\Omega^2 h r^{-1} (r\underline{L}\phi) - \Omega_0^2 h_0 |u|^{-1} \psi \right). \tag{3.20}$$

Using (3.14), (3.15), (3.17), (3.18) and (3.19), the right hand side can be estimated by

$$\begin{aligned}
&|\Omega^2 h r^{-1} (r\underline{L}\phi) - \Omega_0^2 h_0 |u|^{-1} \psi| \\
&\quad \lesssim |\Omega^2 - \Omega_0^2||h_0|u|^{-1}\psi| + |\Omega^2||h - h_0|||u|^{-1}\psi| \\
&\qquad + |\Omega^2 h||r^{-1} - |u|^{-1}||\psi| + |\Omega^2 h r^{-1}||r\underline{L}\phi - \psi| \\
&\quad \lesssim C^{\frac{1}{4}} \delta |u|^{-1} \Omega_0^2 \mathscr{F} \mathscr{W}^{\frac{1}{2}} \mathcal{A} \cdot |u|^{-1}|\psi| + C^{\frac{1}{2}} \delta |u|^{-1} \mathscr{F}^2 \mathcal{A}^2 \cdot |u|^{-1}|\psi| \\
&\qquad + \mathscr{F}\mathcal{A} \cdot \delta |u|^{-2} \mathscr{F}\mathcal{A} \cdot |\psi| + |u|^{-1} \mathscr{F}\mathcal{A} \cdot C^{\frac{1}{4}} \delta |u|^{-1} \mathscr{F}\mathcal{A} \\
&\quad \lesssim C^{\frac{1}{2}} \delta |u|^{-1} \mathscr{F}^2 \mathcal{A}^2 \cdot |u|^{-1} \left(1 + |\psi| + \Omega_0^2 \mathscr{W}^{\frac{1}{2}} |\psi| \right).
\end{aligned}$$

[9] Because (3.4) is used very frequently in a similar manner, we will not point it out again when we use (3.4) in the rest of the paper. Because $\mathscr{W} \ge 1$, (3.4) will also be used in the form $C^2 \delta |u_1|^{-1} \mathscr{F} \mathcal{A} \le 1$.

[10] Similar to (3.4), the estimates (3.5) are used frequently and we will not point this out in the argument.

For the first two terms above, we use (3.1) to compute

$$\begin{aligned}&\int_{u_0}^{u} C^{\frac{1}{2}}\delta|u'|^{-1}\mathscr{F}^2\mathcal{A}^2\cdot|u'|^{-1}\,(1+|\psi|)\,\mathrm{d}u'\\&\quad\lesssim C^{\frac{1}{2}}\delta|u|^{-1}\mathscr{F}^2\mathcal{A}^2+C^{\frac{1}{2}}\delta\mathscr{F}^2\mathcal{A}^2\left(\int_{u_0}^{u}\frac{|\psi|^2}{|u'|}\mathrm{d}u'\right)^{\frac{1}{2}}\left(\int_{u_0}^{u}\frac{1}{|u'|^3}\mathrm{d}u'\right)^{\frac{1}{2}}\\&\quad\lesssim C^{\frac{1}{2}}\delta|u|^{-1}\mathscr{F}^2\mathscr{W}^{\frac{1}{2}}\mathcal{A}^2.\end{aligned}$$

For the last term, we note that from (3.1), $\frac{|\psi|^2}{|u|}=\partial_u(-\log\Omega_0^2)$, and then $\int_{u_0}^{u}\frac{\Omega_0^2|\psi|^2}{|u'|}\mathrm{d}u'\leq\Omega_0^2(u_0)\leq 1$. We use this fact to derive that

$$\begin{aligned}&\int_{u_0}^{u} C^{\frac{1}{2}}\delta|u'|^{-1}\mathscr{F}^2\mathcal{A}^2\cdot|u'|^{-1}\Omega_0^2\mathscr{W}^{\frac{1}{2}}|\psi|\mathrm{d}u'\\&\lesssim C^{\frac{1}{2}}\delta\mathscr{F}^2\mathscr{W}^{\frac{1}{2}}\mathcal{A}^2\left(\int_{u_0}^{u}\frac{\Omega_0^2|\psi|^2}{|u'|}\mathrm{d}u'\right)^{\frac{1}{2}}\left(\int_{u_0}^{u}\frac{1}{|u'|^3}\mathrm{d}u'\right)^{\frac{1}{2}}\lesssim C^{\frac{1}{2}}\delta|u|^{-1}\mathscr{F}^2\mathscr{W}^{\frac{1}{2}}\mathcal{A}^2.\end{aligned}$$

Combining the above two estimates, we have

$$\int_{u_0}^{u}|\Omega^2hr^{-1}(r\underline{L}\phi)-\Omega_0^2h_0|u'|^{-1}\psi|\mathrm{d}u'\lesssim C^{\frac{1}{2}}\delta|u|^{-1}\mathscr{F}^2\mathscr{W}^{\frac{1}{2}}\mathcal{A}^2. \tag{3.21}$$

Integrating the Eq. (3.20) and using this estimate, we then have

$$|rL\phi-\varphi|\lesssim|rL\phi-\varphi|_{|C_{u_0}}+C^{\frac{1}{2}}\delta|u|^{-1}\mathscr{F}^2\mathscr{W}^{\frac{1}{2}}\mathcal{A}^2.$$

By the definition of $\mathscr{F}$ and $\mathcal{A}$, $|rL\phi-\varphi|_{|C_{u_0}}\leq|rL\phi|_{|C_{u_0}}+|\varphi(u_0)|\lesssim\mathscr{F}\mathcal{A}$. Then the estimate (3.6) follows by (3.4).

For $\underline{L}\phi$, we note that, from (3.16)

$$|\underline{h}|\lesssim 1+C^{\frac{1}{4}}\delta|u|^{-1}\mathscr{F}\mathcal{A}\lesssim 1+C^{-1}\lesssim 1. \tag{3.22}$$

We then simply integrate the Eq. (2.7) and obtain, by (3.5), (3.6) and (3.22) we have proved above,

$$|r\underline{L}\phi-\psi|\lesssim\int_0^{\underline{u}_*}|\underline{h}L\phi|\mathrm{d}\underline{u}\lesssim\delta|u|^{-1}\mathscr{F}\mathcal{A}.$$

For h and $\underline{h}$, we use the Eqs. (2.1) and (2.3). From (2.1), using (3.5) and (3.6), we have

$$|h-h_0|\lesssim\int_0^{\underline{u}_*}|r\Omega^{-2}(L\phi)^2|\mathrm{d}\underline{u}\lesssim\delta\Omega_0^{-2}|u|^{-1}\mathscr{F}^2\mathcal{A}^2,$$

which is the desired estimate (3.8). From (2.3), using (3.5), (3.18) and (3.22), we have

$$|\underline{h}+1|\lesssim\int_0^{\underline{u}_*}\left|\frac{\Omega^2(1+h\underline{h})}{r}\right|\mathrm{d}\underline{u}\lesssim\delta(1+\mathscr{F}\mathcal{A})\cdot|u|^{-1}\lesssim\delta|u|^{-1}\mathscr{F}\mathcal{A},$$

which is the desired estimate (3.9).

For ω, we use the Eq. (2.5). The right hand side of (2.5) can be estimated by, using (3.5), (3.6), (3.7), (3.18) and (3.22),

$$\left|\frac{\Omega^2(1+h\underline{h})}{r^2} - L\phi\underline{L}\phi\right| \lesssim |u|^{-2}\mathscr{F}\mathcal{A} + |u|^{-2}\mathscr{F}\mathcal{A}(|\psi| + \delta|u|^{-1}\mathscr{F}\mathcal{A}).$$

Integrating (2.5) and using (3.1), we then have

$$|\omega| \lesssim |u|^{-1}\mathscr{F}\mathscr{W}^{\frac{1}{2}}\mathcal{A},$$

which is the desired estimate (3.10).

We have completed the estimates in the region $0 \le \underline{u} \le \underline{u}_*$, $u_0 \le u \le u_*$ and conclude that the estimates (3.6)–(3.10) hold under the *bootstrap assumptions* (3.12)–(3.16) if C_0 is sufficiently large. We begin to close the bootstrap argument. We find that the estimates (3.6)–(3.10) improve the *bootstrap assumptions* (3.12)–(3.16) if C_0 is sufficiently large. In particular, the *secondary bootstrap assumptions* (3.14), (3.15) and (3.16) are improved. By the continuity of $\underline{L}\phi, h, \underline{h}$ relative to $\underline{u}$, these *secondary bootstrap assumptions* hold for $0 \le \underline{u} \le \underline{u}_* + \varepsilon$, $u_0 \le u \le u_*$ for some $\varepsilon > 0$ if $\underline{u}_* < \delta$. This shows that $\underline{\mathcal{U}}_{\delta,u_*}$ is open in $[0,\delta]$. Then $\underline{\mathcal{U}}_{\delta,u_*} = [0,\delta]$ for any $u_* \in \mathcal{U}_{\delta,u_1}$, and in particular, the *secondary bootstrap assumptions* (3.14), (3.15) and (3.16) hold for $0 \le \underline{u} \le \delta$, $u_0 \le u \le u_*$ and we have closed the secondary bootstrap argument. By the same argument as in deriving the above estimates (3.6)–(3.10), with $\underline{u}_*$ replaced by δ, we conclude that the estimates (3.6)–(3.10) hold for $0 \le \underline{u} \le \delta$, $u_0 \le u \le u_*$. In particular, (3.6) and (3.10) improve the *bootstrap assumptions* (3.12) and (3.13). By the continuity of $L\phi$ and ω relative to u, $\mathcal{U}_{\delta,u_1}$ is open in $[u_0, u_1]$ and then $\mathcal{U}_{\delta,u_1} = [u_0, u_1]$. In particular, the *bootstrap assumptions* (3.12) and (3.13) hold for $0 \le \underline{u} \le \delta$, $u_0 \le u \le u_1$ and we have closed the whole bootstrap argument. By the same argument as in deriving the above estimates (3.6)–(3.10) again and the secondary bootstrap argument, we conclude that (3.6)–(3.10) hold for $0 \le \underline{u} \le \delta$, $u_0 \le u \le u_1$. At last, using these estimates, we can find that (3.21) holds without C, and then (3.11) also holds. This completes the proof. □

4. Instability Theorems

We then turn to the proof of the instability theorems. We divide the proof in two cases according to the behavior of $\varphi(u)$ as $u \to 0^-$. The first case is the following.

Theorem 5. *If $\varphi(u)$ is unbounded as $u \to 0^-$, then there exist two sequences $\delta_n \to 0^+$ and $u_n \to 0^-$ such that (1.1) holds for $\underline{u} = \delta_n$, $u = u_n$.*

Proof. Because $\varphi(u)$ is unbounded, we can find a sequence $u_n \to 0^-$ such that

$$|\varphi_n| = \sup_{u_0 \le u \le u_n} |\varphi(u)| \to \infty \text{ as } n \to \infty$$

where $\varphi_n = \varphi(u_n)$. Define δ_n in terms of $u_n = -r_n$ by

$$\varphi_n^2 = 2^8 c_1 \Omega_n^4 \log \frac{r_n}{4\Omega_n^2\delta_n} \tag{4.1}$$

where $\Omega_n = \Omega_0(u_n)$ and c_1 is the constant in Theorem 2. It is obvious that $\delta_n \to 0^+$ because $\Omega_n \to 0$. We are going to prove such δ_n, u_n are the two sequences we need.

We hope to apply Theorem 4 for $\delta = \delta_n$, $u_1 = u_n$. We find that

$$\mathscr{F}_n = \mathscr{F}(u_0, u_n) = \max\{1, \sup_{u_0 \le u \le u_1} |\varphi(u)|\} = |\varphi_n|$$

if n is sufficiently large, and the corresponding $\mathcal{A}_n$ is bounded since the initial data is of bounded variation, and hence bounded. So we compute, for each n,

$$C^2 \delta_n |u_n|^{-1} |\varphi_n| \mathscr{W}_n^{\frac{1}{2}} = \frac{1}{4} C^2 \Omega_n^{-2} \exp\left(-\frac{\varphi_n^2}{2^8 c_1 \Omega_n^4}\right) |\varphi_n| \mathscr{W}_n^{\frac{1}{2}}$$

where $\mathscr{W}_n = \mathscr{W}(u_0, u_n) = \max\left\{1, \left|\log \frac{\Omega_n}{\Omega_0(u_0)}\right|\right\}$. We can see the right hand side tends to zero and therefore (3.4) holds for $\delta = \delta_n$, $u_1 = u_n$ for sufficiently large n depending on C and the initial bound of $L\phi$ on C_{u_0}. As a consequence, we have the following estimates for a sufficiently large $C \ge C_0$ and $\underline{u} \in [0, \delta_n]$, $u = u_n$:

- $|\underline{h} + 1| \le C^{-1}$, which implies $-\underline{h} \ge \dfrac{1}{2}$.
- $\Omega^{-2} \ge \dfrac{1}{4}\Omega_n^{-2}$, $1 \ge \dfrac{r}{2r_n}$.
- $|rL\phi - \varphi_n| \le c|rL\phi(\underline{u}, u_0) - \varphi(u_0)| + cC^{-1}|\varphi_n|$ for some c depending on the initial bound of $L\phi$ on C_{u_0} which follows from (3.11) and implies that $|rL\phi| > \dfrac{1}{2}|\varphi_n|$ for n sufficiently large.
- $\Omega_n^2 \delta_n \ge \Omega_n^2 h_n \delta_n = \displaystyle\int_0^{\delta_n} \Omega_n^2 h_n \mathrm{d}\underline{u} \ge \frac{1}{4}\int_0^{\delta_n} \Omega^2 h \mathrm{d}\underline{u} = \frac{1}{4}(r - r_n)$, where we use $h_n = h(0, u_n) \ge h$ because of $Dh \le 0$ from Eq. (2.1).

From (2.8), (4.1) and the above all estimates, we have, for n sufficiently large,

$$\begin{aligned}
m - m_n &= \frac{1}{2}\int_0^{\delta_n} (-\underline{h})\Omega^{-2}(rL\phi)^2 \mathrm{d}\underline{u} \\
&> \frac{1}{2^6}\delta_n \Omega_n^{-2}\varphi_n^2 \\
&= \frac{1}{2^6}\delta_n \Omega_n^{-2} \cdot 2^8 c_1 \Omega_n^4 \log \frac{r_n}{4\Omega_n^2 \delta_n} \\
&\ge \frac{c_1 r}{2r_n} \cdot 4\Omega_n^2 \delta_n \log \frac{r_n}{4\Omega_n^2\delta_n} \\
&\ge \frac{c_1 r}{2r_n}(r - r_n) \log \frac{r_n}{r - r_n}
\end{aligned}$$

which is the inequality in (1.1). For the last inequality, note that the function $x \log \frac{r_n}{x}$ is monotonically increasing for $(0, \frac{r_n}{\mathrm{e}}]$ and

$$\frac{4\Omega_n^2 \delta_n}{r_n} = \exp\left(-\frac{\varphi_n^2}{2^8 c_1 \Omega_n^4}\right)$$

which will be smaller than e^{-1} for sufficiently large n since the right hand side tends to zero. Then $4\Omega_n^2\delta_n \le \frac{r_n}{\mathrm{e}}$ and the last inequality holds. Finally, $\frac{r - r_n}{r_n} \le c_0$ follows also from

$$\frac{r-r_n}{r_n} \le \frac{4\Omega_n^2\delta_n}{r_n} = \exp\left(-\frac{\varphi_n^2}{2^8 c_1 \Omega_n^4}\right) \le c_0$$

for sufficiently large n. The proof is then completed. □

The second case is the following.

Theorem 6. *Suppose that $\varphi(u)$ is bounded by some $\Phi \ge 1$, and there exists some $\gamma \in (0,4)$ such that*

$$\limsup_{u\to 0^-} \Omega_0^{\gamma-4}(u) f(u;\gamma) > 1 \tag{4.2}$$

where the function f is defined by

$$f(u;\gamma) = \frac{1}{\delta(u;\gamma)} \int_0^{\delta(u;\gamma)} |rL\phi(\underline{u},u_0) + (\varphi(u) - \varphi(u_0))|^2 \mathrm{d}\underline{u}$$

and $\delta(u;\gamma)$ is defined in terms of u by

$$\Omega_0^{4-\gamma}(u) = 2^8 c_1 \Omega_0^4(u) \log \frac{|u|}{4\Omega_0^2(u)\delta(u;\gamma)}. \tag{4.3}$$

Then the conclusion of Theorem 5 also holds.

Proof. From (4.2), there exists a sequence $u_n \to 0^-$ such that

$$f(u_n;\gamma) > \Omega_0^{4-\gamma}(u_n). \tag{4.4}$$

From (4.3), we have $\delta_n = \delta(u_n;\gamma) \to 0^+$ and

$$\mathscr{F}_n = \mathscr{F}(u_0,u_n) = \max\{1, \sup_{u_0\le u\le u_n} |\varphi(u)|\} \le \Phi.$$

The corresponding $\mathcal{A}_n$ is also bounded. Then

$$C^2\delta_n|u_n|^{-1}\mathscr{F}_n\mathscr{W}_n \le C^2\frac{1}{4}\Omega_n^{-2}\exp\left(-\frac{1}{2^8 c_1\Omega_n^{\gamma}}\right)\Phi\mathscr{W}_n.$$

The right hand side tends to zero and therefore (3.4) holds for $\delta = \delta_n$, $u = u_n$ for n sufficiently large. Applying Theorem 4, the following estimates hold for $0 \le \underline{u} \le \delta_n$, $u = u_n$ if C_0 and n is sufficiently large, similar to those listed in the proof of Theorem 5:

$$-\underline{h} \ge \frac{1}{2},\ \Omega^{-2} \ge \frac{1}{4}\Omega_n^{-2},\ 1 \ge \frac{r}{2r_n},\ \Omega_n^2\delta_n \ge \frac{1}{4}(r-r_n). \tag{4.5}$$

However, the estimate we need for $L\phi$ is slightly different. We go back to Eq. (3.20). Integrating it from u_0 to u_n leads to

$$\begin{aligned}&(rL\phi(\underline{u},u_n) - \varphi_n) - (rL\phi(\underline{u},u_0) - \varphi(u_0))\\ &= \int_{u_0}^{u_n} -\left(\Omega^2 h r^{-1}(r\underline{L}\phi) - \Omega_0^2 h_0 |u'|^{-1}\psi\right)\mathrm{d}u'.\end{aligned}$$

The right hand side can be estimated by (3.21), with C dropped since we have the estimates (3.6)–(3.10) instead of the *bootstrap assumptions* (3.12)–(3.16). Then we have, for n sufficiently large,

$$|rL\phi(\underline{u}, u_n)| \geq |rL\phi(\underline{u}, u_0) + (\varphi_n - \varphi(u_0))| - c\delta_n |u_n|^{-1}\Phi^2 \mathscr{W}_n^{\frac{1}{2}}$$

for some constant c depending on the initial bound of $L\phi$ on C_{u_0}. Now from (4.3) again, we have

$$c\delta_n |u_n|^{-1}\Phi^2 \mathscr{W}_n^{\frac{1}{2}} = \frac{1}{4}\Omega_n^{-2}\exp\left(-\frac{1}{2^8 c_1 \Omega_n^{\gamma}}\right)\mathscr{W}_n^{\frac{1}{2}} \cdot c\Phi^2$$

and the rate of the right hand side tending to zero is faster than any positive power of Ω_n, then in particular it is less than $\sqrt{\frac{1}{4}\Omega_n^{4-\gamma}}$ if n is sufficiently large. From (4.3) and (4.4) we arrive at the lower bound

$$\begin{aligned}\int_0^{\delta_n} |rL\phi(\underline{u}, u_n)|^2 \mathrm{d}\underline{u} &> \frac{1}{2}\delta_n f(u_n;\gamma) - \frac{1}{4}\delta_n \Omega_n^{4-\gamma} \geq \frac{1}{4}\delta_n\Omega_n^{4-\gamma}\\ &= 2^6 c_1 \delta_n \Omega_n^4 \log\frac{r_n}{4\Omega_n^2\delta_n}.\end{aligned}$$

Therefore, by the above estimate, the estimates (4.5), and that $\frac{4\Omega_n^2\delta_n}{r_n} = \exp\left(-\frac{1}{2^8 c_2 \Omega_n^{\gamma}}\right)$ tends to zero, we have $\frac{r-r_n}{r_n} \leq c_0$ and

$$\begin{aligned}m - m_n &= \frac{1}{2}\int_0^{\delta_n}(-\underline{h})\Omega^{-2}(rL\phi)^2 \mathrm{d}\underline{u}\\ &\geq \frac{1}{2^4}\Omega_n^{-2}\int_0^{\delta_n}(rL\phi)^2\mathrm{d}\underline{u}\\ &> \frac{1}{2^4}\Omega_n^{-2}\cdot 2^6 c_1\delta_n\Omega_n^4\log\frac{r_n}{4\Omega_n^2\delta_n}\\ &\geq \frac{c_1 r}{2r_n}(r - r_n)\log\frac{r_n}{r-r_n}\end{aligned}$$

for sufficiently large n as in the proof of Theorem 5. The proof of Theorem 6 is then completed. □

Remark 3. It is worth mentioning that in Christodoulou's original proof, when $\varphi(u)$ is bounded but does not tend to zero, the conclusions of Theorem 5 holds without any additional conditions like (4.2). Indeed, the condition (4.2) is slightly different from that in Christodoulou's proof and we can see when $\varphi(u)$ is bounded but does not tend to zero, (4.2) holds identically because $rL\phi(\underline{u}, u_0)$ is of bounded variations and hence can be made right-continuous.

The remaining part of the proof of Theorem 3 is then similar to that in the last section in [4]. We still present the proof here for the sake of completeness.

Proof of Theorem 3. We fix the coordinate $\underline{u} = r - r_0$ on $C_o = C_{u_0}$. Then

$$\alpha_0 = \frac{\partial}{\partial r}(r\phi) = rL\phi\big|_{C_o} + \phi\big|_{C_o}.$$

We denote $\theta_0 = \theta_0(r) = rL\phi\big|_{\underline{u}=r-r_0, u=u_0}$. As in [4], α_0 being of bounded variation is equivalent to θ_0 being bounded variation and $\frac{|\theta_0|}{r} \in L^1(0, +\infty)$. Therefore we consider instead θ_0 in such a space. Suppose that $\theta_0 \in \mathcal{E}$, then there exists a singular endpoint e on Γ and we have a double null coordinate $(\underline{u}, u)$ relative to e and in particular, $\underline{C}_e \cap C_o$ has area radius r_0. Moreover, we must have $\varphi(u)$ is bounded and

$$\limsup_{u \to 0^-} \Omega_0^{\gamma-4}(u) f(u;\gamma) \leq 1 \tag{4.6}$$

for all $\gamma \in (0,4)$. This is because by Theorems 5 and 6, if $\varphi(u)$ is not bounded, or (4.6), the opposite of (4.2), does not hold for some $\gamma \in (0,4)$, then there exist two sequences $\delta_n \to 0^+$ and $u_n \to 0^-$ such that (1.1) holds, and hence $\theta_0 \notin \mathcal{E}$ by the definition of $\mathcal{E}$ in Theorem 3. We then define $f_1 = f_1(r)$ such that it vanishes on $[0, r_0)$ and near infinity, and is absolutely continuous on $[r_0, +\infty)$ with $f_1(r_0) = 1$. We also define $f_2 = f_2(r)$ to be absolutely continuous on $[0, +\infty)$ such that it vanishes on $[0, r_0]$ and near infinity, and

$$f_2(r) = \sqrt{\frac{\mathrm{d}}{\mathrm{d}r}\left[(r-r_0)\Omega_0^2(u(r-r_0;2))\right]}, \; r \in [r_0, r_0+1]$$

where $u(\delta; 2)$ is the inverse function of $\delta(u; 2)$ defined by (4.3) where $\gamma = 2$. More specifically, $u(r - r_0; 2)$ in the definition of $f_2(r)$ above is defined through

$$1 = 2^8 c_1 \Omega_0^2(u(r-r_0;2)) \log \frac{|u(r-r_0;2)|}{4\Omega_0^2(u(r-r_0;2))(r-r_0)}.$$

Recalling that $f_1(r_0) = 1$ and it is right-continuous at r_0, we must have $f_1(r) > \frac{1}{2}$ in $[r_0, r_0 + \varepsilon)$ for some $\varepsilon > 0$ and then for all $\gamma \in (0,4)$ and $\lambda_1 \neq 0$,

$$\lim_{u \to 0^-} \frac{\Omega_0^{\gamma-4}(u)}{\delta(u;\gamma)} \int_0^{\delta(u;\gamma)} \lambda_1^2 f_1^2(\underline{u} + r_0)\mathrm{d}\underline{u} \geq \lim_{u \to 0^-} \frac{\lambda_1^2 \Omega_0^{\gamma-4}(u)}{4} = +\infty. \tag{4.7}$$

On the other hand, we define $u_{(\gamma)}$ through $\delta(u;\gamma) = \delta(u_{(\gamma)}; 2)$. From (4.3), or equivalently,

$$\delta(u;\gamma) = |u|(4\Omega_0^2(u))^{-1} \exp\left(-\frac{1}{2^8 c_1 \Omega_0^\gamma(u)}\right).$$

When u increases, or equivalently $|u|$ decreases since $u < 0$, $\Omega_0(u)$ decreases and hence $(4\Omega_0^2(u))^{-1}\exp\left(-\frac{1}{2^8 c_1 \Omega_0^\gamma(u)}\right)$ decreases when $\Omega_0(u)$ is sufficiently small, and then $\delta(u;\gamma)$ decreases. On the other hand, when γ increases, $\delta(u;\gamma)$ also decreases. Then if $\gamma \in (0,2)$, we must have $|u| < |u_{(\gamma)}|$ and therefore $\Omega_0(u_{(\gamma)}) > \Omega_0(u)$. We then have

$$\begin{aligned}&\lim_{u \to 0^-} \frac{\Omega_0^{\gamma-4}(u)}{\delta(u;\gamma)} \int_0^{\delta(u;\gamma)} \lambda_2^2 f_2^2(\underline{u} + r_0)\mathrm{d}\underline{u} \\ &\quad = \lim_{u \to 0^-} \lambda_2^2 \Omega_0^{\gamma-4}(u)\Omega_0^2(u_{(\gamma)}) \geq \lambda_2^2 \lim_{u \to 0^-} \Omega_0^{\gamma-2}(u) = +\infty.\end{aligned} \tag{4.8}$$

If $\gamma \in (2,4)$, we have $\Omega_0(u_{(\gamma)}) < \Omega_0(u)$ and

$$\begin{aligned}&\lim_{u \to 0^-} \frac{\Omega_0^{\gamma-4}(u)}{\delta(u;\gamma)} \int_0^{\delta(u;\gamma)} \lambda_2^2 f_2^2(\underline{u} + r_0)\mathrm{d}\underline{u} \\ &\quad = \lim_{u \to 0^-} \lambda_2^2 \Omega_0^{\gamma-4}(u)\Omega_0^2(u_{(\gamma)}) \leq \lambda_2^2 \lim_{u \to 0^-} \Omega_0^{\gamma-2}(u) = 0.\end{aligned} \tag{4.9}$$

We then compute, when $\lambda_1 \neq 0$, for $\gamma \in (2, 4)$, from (4.6), (4.7), (4.8) and (4.9),

$$\begin{aligned}
&\limsup_{u\to 0^-} \frac{\Omega_0^{\gamma-4}(u)}{\delta(u;\gamma)} \int_0^{\delta(u;\gamma)} |rL\phi(\underline{u}, u_0) + \lambda_1 f_1(\underline{u}+r_0) + \lambda_2 f_2(\underline{u}+r_0) + (\varphi(u) - \varphi(u_0))|^2 \mathrm{d}\underline{u} \\
&\geq \liminf_{u\to 0^-} \frac{\Omega_0^{\gamma-4}(u)}{2\delta(u;\gamma)} \int_0^{\delta(u;\gamma)} |\lambda_1 f_1(\underline{u}+r_0)|^2 \mathrm{d}\underline{u} \\
&\quad - \limsup_{u\to 0^-} \frac{\Omega_0^{\gamma-4}(u)}{\delta(u;\gamma)} \int_0^{\delta(u;\gamma)} |\lambda_2 f_2(\underline{u}+r_0)|^2 \mathrm{d}\underline{u} \\
&\quad - \limsup_{u\to 0^-} \frac{\Omega_0^{\gamma-4}(u)}{\delta(u;\gamma)} \int_0^{\delta(u;\gamma)} |rL\phi(\underline{u}, u_0) + (\varphi(u) - \varphi(u_0))|^2 \mathrm{d}\underline{u} \\
&= +\infty.
\end{aligned}$$

When $\lambda_1 = 0$, $\lambda_2 \neq 0$, we compute, for $\gamma \in (0, 2)$, from (4.6) and (4.8),

$$\begin{aligned}
&\limsup_{u\to 0^-} \frac{\Omega_0^{\gamma-4}(u)}{\delta(u;\gamma)} \int_0^{\delta(u;\gamma)} |rL\phi(\underline{u}, u_0) + \lambda_1 f_1(\underline{u}+r_0) + \lambda_2 f_2(\underline{u}+r_0) + (\varphi(u) - \varphi(u_0))|^2 \mathrm{d}\underline{u} \\
&\geq \liminf_{u\to 0^-} \frac{\Omega_0^{\gamma-4}(u)}{2\delta(u;\gamma)} \int_0^{\delta(u;\gamma)} |\lambda_2 f_2(\underline{u}+r_0)|^2 \mathrm{d}\underline{u} \\
&\quad - \limsup_{u\to 0^-} \frac{\Omega_0^{\gamma-4}(u)}{\delta(u;\gamma)} \int_0^{\delta(u;\gamma)} |rL\phi(\underline{u}, u_0) + (\varphi(u) - \varphi(u_0))|^2 \mathrm{d}\underline{u} \\
&= +\infty.
\end{aligned}$$

This proves that $\theta_0 + \lambda_1 f_1 + \lambda_2 f_2 \notin \mathcal{E}$ for all λ_1, λ_2 with $\lambda_1 \neq 0$ or $\lambda_2 \neq 0$. Now suppose that $\theta, \theta' \in \mathcal{E}$ and

$$\theta_{\lambda_1,\lambda_2} := \theta_0 + \lambda_1 f_1 + \lambda_2 f_2 \equiv \theta'_{\lambda'_1,\lambda'_2} := \theta'_0 + \lambda'_1 f'_1 + \lambda'_2 f'_2.$$

Assume that e' is the singular endpoint of Γ in the maximal future development of θ'_0 (and hence of $\theta'_{\lambda'_1,\lambda'_2}$) and $\underline{C}_{e'} \cap C_o$ has area radius r'_0. We then have $e = e'$ and $r_0 = r'_0$. Because f_i, f'_i vanish on $[0, r_0)$, we will have $\theta(r) \equiv \theta'(r)$ for $r \in [0, r_0)$ and hence the double coordinate $(\underline{u}, u)$, the functions $\varphi(u)$ and $\Omega_0(u)$, are the same. Therefore $f_1 \equiv f'_1$, $f_2 \equiv f'_2$. We then write

$$\theta_{\lambda_1-\lambda'_1,\lambda_2-\lambda'_2} \equiv \theta'_0.$$

From the above argument, when $\lambda_1 \neq \lambda'_1$ or $\lambda_2 \neq \lambda'_2$, $\theta_{\lambda_1-\lambda'_1,\lambda_2-\lambda'_2} \notin \mathcal{E}$ but $\theta'_0 \in \mathcal{E}$. Therefore we must have $\lambda_1 = \lambda'_1$ and $\lambda_2 = \lambda'_2$. Finally, we conclude that $\theta \equiv \theta'$ and the proof is completed. □

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References

1. An, X., Luk, J.: Trapped surfaces in vacuum arising dynamically from mild incoming radiation. Adv. Theor. Math. Phys. **21**, 1–120 (2017)
2. Christodoulou, D.: The formation of black holes and singularities in spherically symmetric gravitational collapse. Commun. Pure Appl. Math. **44**(3), 339–373 (1991)
3. Christodoulou, D.: Bounded variation solutions of the spherically symmetric Einstein-scalar field equations. Commun. Pure Appl. Math. **46**(8), 1131–1220 (1993)
4. Christodoulou, D.: The instability of naked singularities in the gravitational collapse of a scalar field. Ann. Math. **149**, 183–217 (1999)
5. Christodoulou, D.: The Formation of Black Holes in General Relativity, Monographs in Mathematics, European Mathematical Soc. (2009)
6. Li, J., Liu, J.: Instability of spherical naked singularities of a scalar field under gravitational perturbations, arXiv:1710.02422

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