The Isotropic Semicircle Law and Deformation of Wigner Matrices

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May 23, 2012

We analyse the spectrum of additive finite-rank deformations of $N \times N$ Wigner matrices H. The spectrum of the deformed matrix undergoes a transition, associated with the creation or annihilation of an outlier, when an eigenvalue d_i of the deformation crosses a critical value ± 1. This transition happens on the scale $|d_i| - 1 \sim N^{-1/3}$. We allow the eigenvalues d_i of the deformation to depend on N under the condition $\big||d_i| - 1\big| \geqslant (\log N)^{C \log \log N} N^{-1/3}$. We make no assumptions on the eigenvectors of the deformation. In the limit $N \to \infty$, we identify the law of the outliers and prove that the non-outliers close to the spectral edge have a universal distribution coinciding with that of the extremal eigenvalues of a Gaussian matrix ensemble.

A key ingredient in our proof is the *isotropic local semicircle law*, which establishes optimal high-probability bounds on the quantity $\langle \mathbf{v}, \big((H-z)^{-1} - m(z)\mathbb{1}\big)\mathbf{w}\rangle$, where $m(z)$ is the Stieltjes transform of Wigner's semicircle law and $\mathbf{v}, \mathbf{w}$ are arbitrary deterministic vectors.

AMS Subject Classification (2010): 15B52, 60B20, 82B44

Keywords: Random matrix, universality, deformation, outliers

*Partially supported by NSF grant DMS-0757425
†Partially supported by NSF grant DMS-1001655

1. Introduction

Random matrices were introduced by Wigner [35] in the 1950s to model the excitation spectra of large atomic nuclei, and have since been the subject of intense mathematical investigation. In this paper we study Wigner matrices – random matrices whose entries are independent up to symmetry constraints – that have been deformed by a finite-rank perturbation. By Weyl's eigenvalue interlacing inequalities, such a deformation does not influence the global statistics of the eigenvalues. Thus, the empirical eigenvalue densities of deformed and undeformed Wigner matrices have the same large-scale asymptotics, and are governed by Wigner's famous semicircle law. However, the behaviour of individual eigenvalues may change dramatically under a deformation. In particular, deformed Wigner matrices may exhibit *outliers*, eigenvalues located away from the bulk spectrum. Such models were first investigated by Füredi and Komlós [29]. Subsequently, much progress [5–7, 11–13, 28, 32] has been made in the analysis of the spectrum of such deformed matrix models. See e.g. [32] for a review of recent developments. Analogous deformations of covariance matrices, so-called *spiked population models*, as well as generalizations thereof, were studied in [1, 2, 4].

In a seminal work [3], Baik, Ben Arous, and Péché investigated the spectrum of deformed (spiked) complex Gaussian sample covariance matrices. They established a phase transition, sometimes referred to as the *BBP transition*, in the distribution of the extremal eigenvalues. In [31], Péché proved a similar result for additive deformations of GUE (the Gaussian Unitary Ensemble). Subsequently, the results of [3] and [31] were extended to the other Gaussian ensembles, such as GOE (the Gaussian Orthogonal Ensemble), by Bloemendal and Virág [9, 10]. We sketch the results of [3, 9, 10, 31] in the case of additive deformations of GUE. For simplicity, we consider rank-one deformations, although the results of [3, 9, 10, 31] cover arbitrary rank-k deformations. Thus, let H be an $N \times N$ GUE matrix, normalized so that its entries have variance N^{-1}. Let $\widetilde{H}(d) := H + d\mathbf{v}\mathbf{v}^*$, where $\mathbf{v}$ is a normalized vector and d is independent of N. If $d > 1$ then the spectrum of $\widetilde{H}(d)$ consists of a *bulk spectrum* asymptotically contained in $[-2, 2]$, and an *outlier*, located at $d + d^{-1}$ and having a normal law with variance of order N^{-1}. If $d < 1$ then there is no such outlier, and the statistics of the extremal eigenvalues of $\widetilde{H}(d)$ coincide with those of H. Thus, as d increases from $1 - \varepsilon$ to $1 + \varepsilon$ for some small $\varepsilon > 0$, the largest eigenvalue of $\widetilde{H}(d)$ detaches itself from the bulk spectrum and becomes an outlier.

The phase transition takes place on the scale $d = 1 + wN^{-1/3}$ where w is of order one. This may be heuristically understood as follows. The largest eigenvalues of H are known to fluctuate on the scale $N^{-2/3}$ around 2. The critical scale for d, i.e. the scale on which the outlier is separated from 2 by a gap of order $N^{-2/3}$, is therefore $d = 1 + wN^{-1/3}$ (since in that case $d + d^{-1} = 2 + w^2N^{-2/3} + O(w^3N^{-1})$). In [3, 9, 10, 31], the authors established the weak convergence as $N \to \infty$

$$N^{2/3}\Big(\lambda_N\big(\widetilde{H}(1 + wN^{-1/3})\big) - 2\Big) \implies \Lambda_w\,,$$

where $\lambda_N(A)$ denotes the largest eigenvalue of A. Moreover, the asymptotics in w of the law Λ_w was analysed in [3, 8–10, 31]: as $w \to +\infty$, the law Λ_w converges to a Gaussian; as $w \to -\infty$, the law Λ_w converges to the Tracy-Widom-β distribution (where $\beta = 1$ for GOE and $\beta = 2$ for GUE). As mentioned above, the results of [3, 9, 10, 31] also apply to rank-k deformations, where the picture is similar; each eigenvalue $d_i \in [-1, 1]^c$ gives rise to an outlier located around $d_i + d_i^{-1}$, while eigenvalues $d_i \in (-1, 1)$ do not change the statistics of the extremal eigenvalues of $\widetilde{H}$.

The proofs of [3, 31] use an asymptotic analysis of Fredholm determinants, while those of [9, 10] use an explicit tridiagonal representation of H; both of these approaches rely heavily on the Gaussian nature of H. In order to study the phase transition for non-Gaussian matrix ensembles, and in particular address the question of spectral universality, a different approach is needed. Interestingly, it was observed in [11–13] that the distribution of the outliers is not universal, and may depend on the geometry of the eigenvectors of A. The non-universality of the outliers was further investigated in [32].

In the present paper we take H to be a real symmetric or complex Hermitian Wigner matrix, and A to be a rank-k deterministic matrix whose symmetry class (real symmetric or complex Hermitian) coincides with that of H. We make the following assumptions on the perturbation A.

(A1) The eigenvalues $d_1, \dots, d_k$ of A may depend on N; they satisfy $\big||d_i| - 1\big| \geqslant (\log N)^{C \log \log N} N^{-1/3}$, i.e., on the scale of the phase transition, the eigenvalues of A are separated from the transition points by at least a logarithmic factor.

(A2) The eigenvectors of A are arbitrary orthonormal vectors.

Our main results on the spectrum of $H + A$ may be informally summarized as follows.

(R1) The non-outliers "stick" to eigenvalues of the undeformed matrix H (Theorem 2.7). In particular, the extremal bulk eigenvalues of $H + A$ are universal.

(R2) We identify the distribution of the outliers of $H + A$ (Theorem 2.14).

A key ingredient in our proof is a generalization of the *local semicircle law*. The study of the local semicircle law was initiated in [21, 22]; it provides a key step towards establishing universality for Wigner matrices [17, 23, 26, 27, 33, 34]. The strongest versions of the local semicircle law, proved in [15, 16, 26], give precise estimates on the local eigenvalue density, down to scales containing N^ε eigenvalues. In fact, as formulated in [26], the local semicircle law gives optimal high-probability estimates on the quantity

$$G_{ij}(z) - \delta_{ij} m(z) \,, \tag{1.1}$$

where $m(z)$ denotes the Stieltjes transform of Wigner's semicircle law and $G(z) = (H - z)^{-1}$ is the resolvent of H. Starting from such estimates on (1.1), the two following facts are established in [26].

(i) The eigenvalue density is governed by Wigner's semicircle law down to scales containing N^ε eigenvalues.

(ii) *Eigenvalue rigidity*: optimal high-probability bounds on the eigenvalue locations.

Another key ingredient in the proof of universality of random matrices is the *Green function comparison method* introduced in [27]. It uses a Lindeberg replacement strategy, which previously appeared in the context of random matrix theory in [14, 33, 34]. A fundamental input in the Green function comparison method is a precise control on the matrix entries of G, which is provided by the local semicircle law. The Green function comparison method has subsequently been applied to proving the spectral universality of adjacency matrices of random graphs [15, 16] as well as the universality of eigenvectors of Wigner matrices [30].

In this paper, we extend the local semicircle law to the *isotropic local semicircle law*, which gives optimal high-probability estimates on the quantity

$$\big\langle \mathbf{v} \,, \big(G(z) - m(z)\mathbb{1}\big)\mathbf{w} \big\rangle \,, \tag{1.2}$$

where $\mathbf{v}$ and $\mathbf{w}$ are arbitrary deterministic vectors. Note that (1.1) is a special case obtained from (1.2) by setting $\mathbf{v} = \mathbf{e}_i$ and $\mathbf{w} = \mathbf{e}_j$, where $\mathbf{e}_i$ denotes i-th standard basis vector of $\mathbb{C}^N$.

1.1. Outline and sketch of proofs. In Section 2, we introduce basic definitions and state our results. In a first part, we state the isotropic semicircle law (Theorem 2.2) and some important corollaries, such as the isotropic delocalization estimate (Theorem 2.5). The second part of Section 2 is devoted to the spectra of deformed Wigner matrices. Our main results are deviation estimates on the eigenvalue locations (Theorem 2.7) and the distribution of the outliers (Theorem 2.14). In subsequent remarks we discuss some special cases of interest, in particular making the link to the previous results of [11–13, 32].

The remainder of this paper is devoted to proofs. As it turns out, the proof of the isotropic local semicircle law is considerably simpler if the third moments of the matrix entries of H vanish. This case is dealt with in Section 3. The proof is based on the Green function comparison method and the local semicircle law of [26]. In Section 4, we give the additional arguments needed to extend the isotropic local semicircle law to arbitrary matrix entries. We remark that the Green function comparison method has been traditionally [16, 27, 30] used to obtain limiting distributions of smooth, bounded, observables that depend on the resolvent G. In this paper we use it in a novel setting: to obtain high-probability bounds on a fluctuating error.

In Section 5 we use the isotropic semicircle law to obtain an improved estimate outside of the classical spectrum $[-2, 2]$, and prove the isotropic delocalization result which yields optimal high-probability bounds on projections of the eigenvectors of H onto arbitrary deterministic vectors.

Section 6 is devoted to the proof of deviation estimates for the eigenvalues of $H + A$. Our starting point for locating the eigenvalues is a simple identity from linear algebra (Lemma 6.1) already used in the works [5–7, 32]. Similar identities were also used in [1, 2, 4] for deformed covariance matrices. Using such identities, the study of the eigenvalue distribution of the deformed ensemble can be reduced to the study of the resolvent. In our case, this study of the resolvent is considerably more involved because we allow very general perturbations and also identify the distribution of non-outliers. In order to illustrate our method, we first consider the rank-one case in Theorem 6.3. The general rank-k case is based on a bootstrap argument – in which the eigenvalues $\mathbf{d} = (d_1, \dots, d_k)$ of A are varied – which may be summarized in the following three steps.

(i) For arbitrary $\mathbf{d}$, we establish a “permissible region” $\Gamma(\mathbf{d}) \subset \mathbb{R}$ whose complement cannot contain eigenvalues of $H + A$. The region $\Gamma(\mathbf{d})$ consists essentially of small neighbourhoods of the extremal eigenvalues of H as well as of small neighbourhoods of the classical outlier locations $d_i + d_i^{-1}$ for i satisfying $|d_i| > 1$.

(ii) We fix $\mathbf{d}$ to be *independent* of N. In this simple case, we prove that each permissible neighbourhood of a classical outlier location $d_i + d_i^{-1}$ contains exactly one eigenvalue of $H + A$. Moreover, we prove that the non-outliers of $H + A$ stick to eigenvalues of H.

(iii) In order to allow arbitrary N-dependent $\mathbf{d}$’s, we construct a continuous path $(\mathbf{d}(t))_{t\in[0,1]}$ that takes an N-independent initial configuration $\mathbf{d}(0)$ to the desired N-dependent configuration $\mathbf{d} \equiv \mathbf{d}(1)$. Using (i), (ii), and the continuity of the eigenvalues of $H + A(t)$ as functions of t, we infer that the conclusions of (ii) remain valid for all $\mathbf{d}(t)$ where $t \in [0, 1]$, and in particular for $\mathbf{d}(1)$. (Here $A(t)$ denotes the perturbation with eigenvalues $\mathbf{d}(t)$.)

Finally, Section 7 contains the proof of Theorem 2.14, the distribution of the outliers. The proof consists of four main steps.

(i) We reduce the problem of identifying the distribution of an outlier to that of analysing the distribution of random variables of the form $\langle \mathbf{v}, G(\theta)\mathbf{v}\rangle$, where $\theta := d + d^{-1}$ and d is an eigenvalue of A with associated eigenvector $\mathbf{v}$. The argument is based on a precise control of the derivative of $G(z)$ and second-order perturbation theory.

(ii) We consider the case where H is Gaussian. Using the unitary invariance of the law of H, we prove that $\langle \mathbf{v}, G(\theta)\mathbf{v}\rangle$, when appropriately rescaled, converges to a normal random variable.

The remainder of the proof consists in analysing the difference between the general Wigner case and the Gaussian case. Ultimately, we shall apply the Green function comparison method to expressions of the form $\langle \mathbf{v}, G(\theta)\mathbf{v}\rangle$ (Step (iv) below). However, this method is only applicable if $\|\mathbf{v}\|_\infty$ is sufficiently small (in fact, our result shows that the Green function comparison method must fail if $\|\mathbf{v}\|_\infty$ is not small). We therefore have to perform a two-step comparison.

(iii) Let H be the Wigner matrix we are interested in. We introduce a cutoff ε_N (equal to φ^{-D} in the notation of Section 7.3). We define $\widehat{H}$ as the Wigner matrix obtained from H by replacing the (i,j)-th entry of H with a Gaussian whenever $|v_i| \leqslant \varepsilon_N$ and $|v_j| \leqslant \varepsilon_N$. We choose ε_N large enough that most entries of $\widehat{H}$ are Gaussian. We shall compare H with a Gaussian matrix V via the intermediate matrix $\widehat{H}$. In this step, (iii), we compare $\widehat{H}$ with V.

Our proof relies on a block expansion of $\widehat{H}$, which expresses the distribution of the difference

$$\big\langle \mathbf{v}, (\widehat{H}-\theta)^{-1}\mathbf{v}\big\rangle - \big\langle \mathbf{v}, (V-\theta)^{-1}\mathbf{v}\big\rangle$$

in terms of a sum of independent random variables ($\Gamma_1, \dots, \Gamma_6$ in the notation of Section 7.3) whose laws may be explicitly computed.

(iv) In the final step, we use the Green function comparison method to analyse the difference

$$\big\langle \mathbf{v}, (H-\theta)^{-1}\mathbf{v}\big\rangle - \big\langle \mathbf{v}, (\widehat{H}-\theta)^{-1}\mathbf{v}\big\rangle\,.$$

By definition of $\widehat{H}$, whenever the entry (i,j) of H differs from that of $\widehat{H}$, we have $|v_i| \leqslant \varepsilon_N$ and $|v_j| \leqslant \varepsilon_N$. As a consequence, as it turns out, the Green function comparison method is applicable. Of special note in this comparison argument is a shift in the mean of the outlier (arising from the second term on the right-hand side of (7.50)), depending on the third moments of the entries of H.

Acknowledgements. We are grateful to Alex Bloemendal, Paul Bourgade, László Erdős, and Horng-Tzer Yau for helpful comments.

2. Results

2.1. The setup. Let $H^\omega \equiv H = (h_{ij})$ be an $N \times N$ matrix; here ω denotes the running element in probability space, which we shall almost always drop from the notation. We assume that the upper-triangular entries $(h_{ij} : i \leqslant j)$ are independent complex-valued random variables. The remaining entries of H are given by imposing $H = H^*$. Here H^* denotes the Hermitian conjugate of H. We assume that all entries are centred, $\mathbb{E}h_{ij} = 0$. In addition, we assume that one of the two following conditions holds.

(i) *Real symmetric Wigner matrix*: $h_{ij} \in \mathbb{R}$ for all i,j and

$$\mathbb{E}h_{ii}^2 \;=\; \frac{2}{N}\,, \qquad \mathbb{E}h_{ij}^2 \;=\; \frac{1}{N} \qquad (i \neq j)\,.$$

(ii) *Complex Hermitian Wigner matrix*:

$$\mathbb{E}h_{ii}^2 \;=\; \frac{1}{N}\,, \qquad \mathbb{E}|h_{ij}|^2 \;=\; \frac{1}{N}\,, \qquad \mathbb{E}h_{ij}^2 \;=\; 0 \qquad (i \neq j)\,.$$

We use the abbreviation GOE/GUE to mean GOE if H is a real symmetric Wigner matrix with Gaussian entries and GUE if H is a complex Hermitian Wigner matrix with Gaussian entries. We assume that the entries of H have uniformly subexponential decay, i.e. that there exists a constant $\vartheta > 0$ such that

$$\mathbb{P}\big(\sqrt{N}|h_{ij}| \geqslant x\big) \;\leqslant\; \vartheta^{-1}\exp(-x^{\vartheta}) \tag{2.1}$$

for all i, j. Note that we do not assume the entries of H to be identically distributed.

The following quantities will appear throughout this paper. We choose a fixed but arbitrary constant $\Sigma \geqslant 3$. We define the logarithmic control parameter

$$\varphi_N \;\equiv\; \varphi \;:=\; (\log N)^{\log\log N}\,. \tag{2.2}$$

The parameter ζ will play the role of a fixed positive constant, which simultaneously dictates the power of φ in large deviations estimates and characterizes the decay of probability of exceptional events, according to the following definition.

Definition 2.1 (High probability events). *Let $\zeta > 0$. We say that an N-dependent event Ξ holds with* ζ-high probability *if there is some constant C such that*

$$\mathbb{P}(\Xi^c) \;\leqslant\; N^C \exp(-\varphi^{\zeta}) \tag{2.3}$$

for large enough N.

Introduce the spectral parameter

$$z \;=\; E + \mathrm{i}\eta\,,$$

which will be used as the argument of Stieltjes transforms and resolvents. In the following we shall often use the notation $E = \operatorname{Re} z$ and $\eta = \operatorname{Im} z$ without further comment. Let

$$\varrho(\xi) \;:=\; \frac{1}{2\pi}\sqrt{[4-\xi^2]_+} \qquad (\xi \in \mathbb{R})$$

denote the density of the local semicircle law, and

$$m(z) \;:=\; \int \frac{\varrho(\xi)}{\xi - z}\,\mathrm{d}\xi \qquad (z \notin [-2,2]) \tag{2.4}$$

its Stieltjes transform. To avoid confusion, we remark that the Stieltjes transform m was denoted by m_{sc} in the papers [15–27], in which m had a different meaning from (2.4). It is well known that the Stieltjes transform m satisfies the identity

$$m(z) + \frac{1}{m(z)} + z \;=\; 0\,. \tag{2.5}$$

For $\eta > 0$ we define the resolvent of H through

$$G(z) \;:=\; (H-z)^{-1}\,.$$

We use the notation $\mathbf{v} = (v_i)_{i=1}^N \in \mathbb{C}^N$ for the components of a vector. We introduce the standard scalar product $\langle \mathbf{v}, \mathbf{w} \rangle := \sum_i \overline{v}_i w_i$, which induces the Euclidean norm $\|\mathbf{v}\| := \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$. By definition, $\mathbf{v}$ is normalized if $\|\mathbf{v}\| = 1$.

We denote by C a generic positive large constant, whose value may change from one expression to the next. If this constant depends on some parameters α, we indicate this by writing C_α. Finally, for two positive quantities A_N and B_N we use the notation $A_N \asymp B_N$ to mean $C^{-1} A_N \leqslant B_N \leqslant C A_N$ for some positive constant C.

2.2. The isotropic local semicircle law. For $\zeta > 0$ let

$$\mathbf{S}(\zeta) \;:=\; \big\{ z \in \mathbb{C} : |E| \leqslant \Sigma \,,\, \varphi^\zeta N^{-1} \leqslant \eta \leqslant \Sigma \big\} \,. \tag{2.6}$$

For $z \in \mathbf{S}(\zeta)$ define the control parameter

$$\Psi(z) \;:=\; \sqrt{\frac{\operatorname{Im} m(z)}{N\eta}} + \frac{1}{N\eta} \,.$$

Our first main result is on the convergence of $G(z)$ to $m(z)\mathbb{1}$.

Theorem 2.2 (Isotropic local semicircle law). *Fix $\zeta > 0$. Then there exists a constant C_ζ such that*

$$\big| \langle \mathbf{v}, G(z)\mathbf{w} \rangle - m(z) \langle \mathbf{v}, \mathbf{w} \rangle \big| \;\leqslant\; \varphi^{C_\zeta} \Psi(z) \|\mathbf{v}\| \|\mathbf{w}\| \tag{2.7}$$

holds with ζ-high probability for all deterministic $\mathbf{v}, \mathbf{w} \in \mathbb{C}^N$ under either of the two following conditions.

- **A.** *The spectral parameter $z \in \mathbf{S}(C_\zeta)$ is arbitrary, and the third moments of the entries of H vanish in the sense that*

$$\mathbb{E} h_{ij}^3 \;=\; \mathbb{E} h_{ij}^2 \overline{h}_{ij} \;=\; 0 \qquad (i, j = 1, \dots, N) \,. \tag{2.8}$$

- **B.** *The spectral parameter $z \in \mathbf{S}(C_\zeta)$ satisfies*

$$\Psi(z)^3 \;\leqslant\; \varphi^{-C_0} N^{-1/2} \tag{2.9}$$

for some large enough constant C_0 depending on ζ.

Away from the asymptotic spectrum $[-2, 2]$, Theorem 2.2 can be strengthened as follows.

Theorem 2.3 (Isotropic local semicircle law outside of the spectrum). *Fix $\zeta > 0$ and $\Sigma \geqslant 3$. Then there exist constants C_1 and C_ζ such that for any*

$$E \;\in\; \big[-\Sigma, -2 - \varphi^{C_1} N^{-2/3} \big] \cup \big[2 + \varphi^{C_1} N^{-2/3}, \Sigma \big] \,,$$

any $\eta \in (0, \Sigma]$, and any deterministic $\mathbf{v}, \mathbf{w} \in \mathbb{C}^N$ we have

$$\big| \langle \mathbf{v}, G(z)\mathbf{w} \rangle - m(z) \langle \mathbf{v}, \mathbf{w} \rangle \big| \;\leqslant\; \varphi^{C_\zeta} \sqrt{\frac{\operatorname{Im} m(z)}{N\eta}} \, \|\mathbf{v}\| \|\mathbf{w}\| \,. \tag{2.10}$$

with ζ-high probability.

Remark 2.4. Using a simple lattice argument combined with the Lipschitz continuity of $z \mapsto G(z)$, one can easily strengthen the statement (2.7) of Theorem 2.2 to a simultaneous high probability statement for all z, as in (3.16) below. For more details, see e.g. Corollary 3.19 in [15].

Similarly, mimicking the proof of Lemma 7.2 below, we find

$$\sup\Big\{|\partial_z \langle \mathbf{v}\,, G(z)\mathbf{w}\rangle| : 2 + \varphi^{C_1} N^{-2/3} \leqslant |E| \leqslant \Sigma\,,\, 0 < |\eta| \leqslant \Sigma\Big\} \;\leqslant\; N \tag{2.11}$$

with ζ-high probability, from which we infer that the statement (2.10) of Theorem 2.3 holds with ζ-high probability simultaneously for all $z = E + \mathrm{i}\eta$ satisfying the conditions in (2.11).

For an $N \times N$ matrix A we denote by $\lambda_1(A) \leqslant \lambda_2(A) \leqslant \cdots \leqslant \lambda_N(A)$ the nondecreasing sequence of eigenvalues of A. Moreover, we denote by $\sigma(A)$ the spectrum of A. It is convenient to abbreviate the (random) eigenvalues of H by

$$\lambda_\alpha \;:=\; \lambda_\alpha(H)\,.$$

Denote by $\mathbf{u}^{(1)}, \mathbf{u}^{(2)}, \dots, \mathbf{u}^{(N)} \in \mathbb{C}^N$ the normalized eigenvectors of H associated with the eigenvalues $\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_N$. Our next result provides a bound on $\langle \mathbf{u}^{(\alpha)}\,, \mathbf{v}\rangle$ for arbitrary deterministic $\mathbf{v}$.

Theorem 2.5 (Isotropic delocalization). *Fix $\zeta > 0$. Then there is a constant C_ζ such that the following holds for any deterministic and normalized $\mathbf{v} \in \mathbb{C}^N$.*

(i) For any integers a and b satisfying $1 \leqslant a < b \leqslant N/2$ and

$$b - a \;\geqslant\; 2\varphi^{C_0}\Big(b^{1/3}N^{-1/6} + (ab)^{1/3}N^{-1/3}\Big) \tag{2.12}$$

we have

$$\frac{1}{b-a}\sum_{\alpha=a}^{b} |\langle \mathbf{u}^{(\alpha)}\,, \mathbf{v}\rangle|^2 \;\leqslant\; \varphi^{C_\zeta} N^{-1} \tag{2.13}$$

with ζ-high probability. Here C_0 is the constant from Theorem 2.2. By symmetry, a similar result holds for the eigenvectors $\alpha \geqslant N/2$.

(ii) If the third moments of the entries of H vanish in the sense of (2.8), then we have the stronger statement

$$\sup_\alpha |\langle \mathbf{u}^{(\alpha)}\,, \mathbf{v}\rangle|^2 \;\leqslant\; \varphi^{C_\zeta} N^{-1} \tag{2.14}$$

with ζ-high probability.

Remark 2.6. Theorem 2.5 implies that the coefficients of the eigenvectors of H are strongly oscillating. In order to see this, let $\alpha = 1, \dots, N$. If the third moments of the entries of H do not vanish, we require that $\alpha \notin [\varphi^{-4C_0}N^{1/2}, N - \varphi^{-4C_0}N^{1/2}]$. Then choosing $\mathbf{v} = N^{-1/2}(1, \dots, 1)$ and $\mathbf{v} = \mathbf{e}_i$ for $i = 1, \dots, N$ in Theorem 2.5 yields

$$\Bigg|\sum_{i=1}^N u_i^{(\alpha)}\Bigg| \;\leqslant\; \varphi^{C_\zeta}\,, \qquad \max_{1 \leqslant i \leqslant N} |u_i^{(\alpha)}| \;\leqslant\; \varphi^{C_\zeta} N^{-1/2} \tag{2.15}$$

with ζ-high probability. The second inequality implies

$$\sum_{i=1}^N |u_i^{(\alpha)}| \;\geqslant\; \varphi^{-C_\zeta} N^{1/2} \sum_{i=1}^N |u_i^{(\alpha)}|^2 \;=\; \varphi^{-C_\zeta} N^{1/2}$$

with ζ-high probability. Compare this with the first inequality of (2.15).

This behaviour is not surprising. In the GOE/GUE case, it is well known that each eigenvector $\mathbf{u}^{(\alpha)}$ is uniformly distributed on the unit sphere, so that its entries asymptotically behave like i.i.d. Gaussians.

2.3. Finite-rank deformation of Wigner matrices. Let $k \in \mathbb{N}$ be fixed, V be a deterministic $N \times k$ matrix satisfying $V^*V = \mathbb{1}$, and $d_1, \dots, d_k \in \mathbb{R} \setminus \{0\}$ be deterministic. We allow $d_1 \equiv d_1(N), \dots, d_k \equiv d_k(N)$ to depend on N. We also use the notation $V = [\mathbf{v}^{(1)}, \dots, \mathbf{v}^{(k)}]$, where $\mathbf{v}^{(1)}, \dots, \mathbf{v}^{(k)} \in \mathbb{C}^N$ are orthonormal. Define the rank-k perturbation

$$VDV^* \;=\; \sum_{i=1}^k d_i \mathbf{v}^{(i)} (\mathbf{v}^{(i)})^*\,, \qquad D \;=\; \operatorname{diag}(d_1, \dots, d_k)\,.$$

We shall study the spectrum of the deformed matrix

$$\widetilde{H} \;:=\; H + VDV^*\,.$$

We abbreviate the eigenvalues of $\widetilde{H}$ by

$$\mu_\alpha \;:=\; \lambda_\alpha(\widetilde{H})\,.$$

In order to state our results, we order the eigenvalues of D, i.e. we assume that $d_1 \leqslant \dots \leqslant d_k$. Define the numbers

$$k^{\pm} \;:=\; \#\{i : \pm d_i > 1\}\,.$$

As we shall see, k^- is the number of outliers to the left of the bulk and k^+ the number of outliers to the right of the bulk. We shall always assume that k^- and k^+ are independent of N.

Let

$$O \;:=\; \bigl\{i \in \{1, \dots, k\} : |d_i| > 1\bigr\} \;=\; \{1, \dots, k^-, k - k^+ + 1, \dots, k\} \tag{2.16}$$

denote the $k^- + k^+$ indices associated with the outliers. For $i \in O$ abbreviate the associated eigenvalue index by

$$\alpha(i) \;:=\; \begin{cases} N - k + i & \text{if } i \geqslant k - k^+ + 1 \\ i & \text{if } i \leqslant k^-\,. \end{cases} \tag{2.17}$$

Finally, for $d \in \mathbb{R} \setminus (-1, 1)$ we define

$$\theta(d) \;:=\; d + \frac{1}{d}\,. \tag{2.18}$$

Theorem 2.7 (Locations of the deformed eigenvalues). *Fix $\zeta > 0$, $K > 0$, $k \in \mathbb{N}$, and $0 < \mathfrak{b} < 1/3$. Then there exist positive constants C_2 and C_3 such that the following holds.*

Choose a sequence $\psi \equiv \psi_N$ satisfying $1 \leqslant \psi \leqslant N^{\mathfrak{b}}$. Suppose that

$$|d_i| \;\leqslant\; \Sigma - 1\,, \qquad \bigl||d_i| - 1\bigr| \;\geqslant\; \varphi^{C_2} \psi N^{-1/3} \tag{2.19}$$

for all $i = 1, \dots, k$. Then for $i \in O$ we have

$$\bigl|\mu_{\alpha(i)} - \theta(d_i)\bigr| \;\leqslant\; \varphi^{C_3} N^{-1/2} (|d_i| - 1)^{1/2} \tag{2.20}$$

with ζ-high probability. Moreover,

$$|\mu_\alpha - \lambda_{\alpha - k^-}| \;\leqslant\; \psi^{-1} N^{-2/3} \qquad \textit{for} \qquad k^- + 1 \;\leqslant\; \alpha \;\leqslant\; \varphi^K\,, \tag{2.21a}$$

$$|\mu_\alpha - \lambda_{\alpha + k^+}| \;\leqslant\; \psi^{-1} N^{-2/3} \qquad \textit{for} \qquad N - \varphi^K \;\leqslant\; \alpha \;\leqslant\; N - k^+\,, \tag{2.21b}$$

with ζ-high probability.

REMARK 2.8. In [12], Capitaine, Donati-Martin, and Féral proved that $\mu_{\alpha(i)} \to \theta(d_i)$ almost surely for all $i \in O$, under the assumptions that (i) D does not depend on N and (ii) the law of the entries of H is symmetric and satisfies a Poincaré inequality. Subsequently, the assumption (ii) was relaxed by Pizzo, Renfrew, and Soshnikov [32]. In fact, in [32] the authors proved, assuming (i), that the sequence $\sqrt{N}(\mu_{\alpha(i)} - \theta(d_i))$ is bounded in probability for all $i \in O$.

In [5, 6], Benaych-Georges, Guionnet, and Maïda considered deformations of Wigner matrices by finite-rank random matrices whose eigenvalues are independent of N and whose eigenvectors are either independent copies of a random vector with i.i.d. centred components satisfying a log-Sobolev inequality or are obtained by Gram-Schmidt orthonormalization of such independent copies. For these random perturbation models, they established eigenvalue sticking estimates similar to (2.21).

REMARK 2.9. Provided one is only interested in the locations of the outliers, i.e. (2.20), one can set $\psi = 1$ in Theorem 2.7.

We shall refer to the eigenvalues in (2.20), i.e. $\mu_1, \dots \mu_{k^-}, \mu_{N-k^++1}, \dots, \mu_N$, as the *outliers*, and to the eigenvalues in (2.21), i.e. $\mu_{k^-+1}, \dots, \mu_{\varphi^K}, \mu_{N-\varphi^K}, \dots, \mu_{N-k^+}$, as the *extremal bulk eigenvalues*.

REMARK 2.10. The phase transition associated with d_i happens on the scale $d_i = 1 + a_i N^{-1/3}$ where a_i is of order one. The condition (2.19) is optimal (up to powers of φ) in the sense that the power of N in (2.19) cannot be reduced. Indeed, in [3, 9, 10, 31] it is established that, for rank-one[1] deformations of GOE/GUE with $d = 1 + aN^{-1/3}$ and a of order one, μ_N fluctuates on the scale $N^{-2/3}$ and its distribution differs from that of λ_N. Hence in that case (2.21) cannot hold for $\psi \gg 1$. See also Remark 2.13 below for a more detailed discussion of the qualitative behaviour of eigenvalues of $\widetilde{H}$ as d_i crosses a transition point.

Note that the location $\theta(d_i)$ of the outlier associated with $d_i = 1+a_i N^{-1/3}$ satisfies $\theta(d_i) = 2+N^{-2/3}a_i^2 + O(a_i^3 N^{-1})$. In comparison, the largest eigenvalue of H fluctuates on a scale $N^{-2/3}$ around 2.

REMARK 2.11. An immediate corollary of Theorem 2.7 is the universality of the extremal bulk eigenvalues of $\widetilde{H}$. In other words, under the assumption $||d_i| - 1| \geqslant \varphi^{C_2+1} N^{-1/3}$ for all i, the statistics of the extremal bulk eigenvalues of $\widetilde{H}$ coincide with those of GOE/GUE.

Indeed, choosing $\psi = \varphi$ in Theorem 6.3 and invoking the edge universality for the Wigner matrix H proved in Theorem 1.1 of [30] (for similar results, see also [16, 26]), we find for all $\ell \in \mathbb{N}$ and all bounded and continuous f that

$$\lim_{N\to\infty}\Big[\mathbb{E} f\Big(N^{2/3}(\mu_{k^-+1}+2), \dots, N^{2/3}(\mu_{k^-+\ell}+2)\Big) - \mathbb{E}^G f\big(N^{2/3}(\lambda_1+2), \dots, N^{2/3}(\lambda_\ell+2)\big)\Big] \;=\; 0\,,$$

where $\mathbb{E}^G$ denotes expectation with respect to the $N \times N$ GOE/GUE matrices. A similar result holds at the other end of the spectrum.

REMARK 2.12. Theorem 2.7 was formulated for deterministic perturbations. However, it extends trivially to the case where V is random, independent of H, with arbitrary law satisfying $V^* V = \mathbb{1}$.

REMARK 2.13. The parameter ψ describes how strongly the extremal bulk eigenvalues of $\widetilde{H}$ stick to extremal eigenvalues of H. If d_i is within distance $CN^{-1/3}$ of a transition point ± 1, one does not expect the eigenvalues of $\widetilde{H}$ to stick to the eigenvalues of H. For very weak sticking on the scale $N^{-2/3}\varphi^{-1}$, corresponding to $\psi = \varphi$,

[1]For simplicity of presentation, we consider rank-one deformations, although the results of [3, 9, 10, 31] hold for rank-k deformations.

the eigenvalues d_i have to satisfy $\big||d_i|-1\big| \geqslant \varphi^{C_2+1}N^{-1/3}$. In particular, we may allow outliers at a distance $\varphi^{2C_2+2}N^{-2/3}$ from the spectral edge.

On the other hand, in order to obtain strong sticking on the scale $N^{-1+\varepsilon}$, corresponding to $\psi = N^{1/3-\varepsilon}$, the eigenvalues d_i have to satisfy $\big||d_i|-1\big| \geqslant \varphi^{C_2}N^{-\varepsilon}$. Now the outliers have to lie at a distance of at least $N^{2C_2-2\varepsilon}$ from the spectral edge.

Thus, Theorem 2.7 gives a clear picture of what happens to the extremal bulk eigenvalues as d_i passes a transition point ± 1. For definiteness, consider the case where d_i is varied from $1-c$ to $1+c$ for some small $c>0$, and all other eigenvalues of D are kept constant. Consider an extremal bulk eigenvalue near $+2$, say μ_α. By Theorem 2.7, for $d_i \leqslant 1-\varphi^{C_2+1}N^{-1/3}$, μ_α sticks to λ_β where $\beta := \alpha + k^+$. As d_i approaches 1, the eigenvalue μ_α progressively detaches itself from λ_β. Theorem 2.7 allows one to follow this behaviour down to $|d_i - 1| = \varphi^{C_2+1}N^{-1/3}$. Below this scale, as d_i passes 1, the eigenvalue μ_α "jumps" from from the vicinity of λ_β to the vicinity of $\lambda_{\beta+1}$. This jump happens in the range $d_i \in [1-\varphi^{C_2+1}N^{-1/3}, 1+\varphi^{C_2+1}N^{-1/3}]$. After the jump, i.e. for $d_i \geqslant 1+\varphi^{C_2+1}N^{-1/3}$, the eigenvalue μ_α sticks to $\lambda_{\beta+1}$ instead of λ_β, provided that $\beta < N$. If $\beta = N$, then μ_α escapes from the bulk spectrum and becomes an outlier. This jump happens simultaneously for all extremal bulk eigenvalues near $+2$, and is accompanied by the creation of an outlier. This may be expressed as $(k^0, k^+) \mapsto (k^0 - 1, k^+ + 1)$. Meanwhile, the extremal bulk eigenvalues on the other side of the spectrum, i.e. near -2, remain unaffected by the transition, and continue sticking to the same eigenvalues of H they stuck to before the transition.

Next, we identify the distribution of the outliers. We introduce the customary symmetry index β, by definition equal to 1 if H is real symmetric and 2 if H is complex Hermitian. In order to state our result, we define the moment matrices $M^{(3)} = (M^{(3)}_{ij})$ and $M^{(4)} = (M^{(4)}_{ij})$ of H through

$$M^{(3)}_{ij} \;:=\; N^{3/2}\mathbb{E}\big(|h_{ij}|^2 h_{ij}\big)\,, \qquad M^{(4)}_{ij} \;:=\; N^2\mathbb{E}|h_{ij}|^4\,.$$

By definition of H, the matrices $M^{(3)}$ and $M^{(4)}$ are Hermitian. Moreover, by (2.1) they have uniformly bounded entries. For $\mathbf{v} = (v_i) \in \mathbb{C}^N$ define

$$\begin{aligned} Q(\mathbf{v}) \;&:=\; \frac{1}{2\sqrt{N}}\sum_{i,j}\overline{v}_i M^{(3)}_{ij}\big(|v_i|^2 + |v_j|^2\big)v_j\,, \\ R(\mathbf{v}) \;&:=\; \frac{1}{N}\sum_{i,j}\big(M^{(4)}_{ij} - 4 + \beta\big)|v_j|^4\,, \\ S(\mathbf{v}) \;&:=\; \frac{1}{N}\sum_{i,j}\overline{v}_i M^{(3)}_{ij} v_j\,. \end{aligned} \tag{2.22}$$

The functions Q, R, and S are bounded on the unit ball in $\mathbb{C}^N$, uniformly in N.

Theorem 2.14 (Distribution of the outliers). *There is a constant C_2 such that the following holds. Suppose that*

$$|d_i| \;\leqslant\; \Sigma - 1\,, \qquad \big||d_i| - 1\big| \;\geqslant\; \varphi^{C_2}N^{-1/3} \tag{2.23}$$

for all $i = 1, \dots, k$. Suppose moreover that for all $i \in O$ we have

$$\min_{j \neq i}|d_i - d_j| \;\geqslant\; \varphi^{C_2}N^{-1/2}(|d_i| - 1)^{-1/2}\,. \tag{2.24}$$

For $i \in O$ define the random variable

$$\Pi_i \;:=\; (|d_i|+1)(|d_i|-1)^{1/2}\left(\frac{N^{1/2}\langle \mathbf{v}^{(i)}\,,H\mathbf{v}^{(i)}\rangle}{d_i^2}+\frac{S(\mathbf{v}^{(i)})}{d_i^4}\right)$$

and Υ_i, a random variable independent of Π_i with law

$$\Upsilon_i \;\overset{d}{=}\; \mathcal{N}\left(0\,,\,\frac{2(|d_i|+1)}{\beta d_i^4}+(|d_i|+1)^2(|d_i|-1)\left(\frac{4Q(\mathbf{v}^{(i)})}{d_i^5}+\frac{R(\mathbf{v}^{(i)})}{d_i^6}\right)\right).$$

Then we have, for all $i \in O$ and all bounded and continuous f,

$$\lim_{N\to\infty}\Big[\mathbb{E}f\Big(N^{1/2}(|d_i|-1)^{-1/2}\big(\mu_{\alpha(i)}-\theta(d_i)\big)\Big)-\mathbb{E}f(\Pi_i+\Upsilon_i)\Big] \;=\; 0\,. \tag{2.25}$$

Note that, by a standard approximation argument, (2.25) also holds for $f(x) = \mathbf{1}(x \leqslant a)$ where $a \in \mathbb{R}$; hence the convergence (2.25) may also be stated in terms of distribution functions.

Remark 2.15. In [11], Capitaine, Donati-Martin, and Féral identified the law of the outliers of deformed Wigner matrices subject to the following conditions: (i) D is independent of N but may have degenerate eigenvalues; (ii) the law of the matrix entries of H is symmetric and satisfies a Poincaré inequality; (iii) the eigenvectors of the deformation belong to one of two classes, corresponding roughly to either partially delocalized eigenvectors or strongly localized eigenvectors. Subsequently, the assumption (ii) was relaxed by Pizzo, Renfrew, and Soshnikov in [32]. (But assumption (iii) imposes that $S(\mathbf{v}^{(i)}) = Q(\mathbf{v}^{(i)}) = 0$ still holds for the results of [32].)

Remark 2.16. The condition (2.24) has the following interpretation. Let $i \in O$ and assume for definiteness that $d_i > 1$. If j is not associated with an outlier on the right-hand side of the bulk, i.e. if $d_j < 1$, then $d_i - d_j$ is bounded from below by the right-hand side of (2.24), as follows from (2.23). Hence the condition (2.24) is only needed to ensure that the outliers are not to close too each other; in fact, this condition is optimal (up to the factor φ^{C_2}) in guaranteeing that the distributions of the outliers have essentially no overlap. Indeed, by Theorem 2.7 we know that $\mu_{\alpha(i)}$ lies with ζ-high probability in an interval of length $2\varphi^{C_3}N^{-1/2}(d_i-1)^{1/2}$ centred around $\theta(d_i)$. Moreover, differentiating (2.18) yields

$$\theta(d_j)-\theta(d_i) \;\asymp\; (d_i-1)(d_j-d_i)\,.$$

Imposing the condition $|\theta(d_j)-\theta(d_i)| \geqslant \varphi^{C_3}N^{-1/2}(d_i-1)^{1/2}$ leads to (2.24) (with C_2 increased if necessary so that $C_2 \geqslant C_3$). In fact, in [3,31,32] it was proved (for D independent of N) that the distribution associated with degenerate outliers is not Gaussian.

The following remarks discuss some special cases of interest. In order to simplify notations, we set $k = 1$ and write $d \equiv d_1$, $\mathbf{v} \equiv \mathbf{v}^{(1)}$, $\Pi \equiv \Pi_1$, and $\Upsilon \equiv \Upsilon_1$.

Remark 2.17. In the GOE/GUE case, we have $M^{(3)} = 0$ and $M^{(4)}_{ij} = (4-\beta)+\delta_{ij}(17-8\beta)$. Thus we get that $Q(\mathbf{v}) = S(\mathbf{v}) = 0$ and $R(\mathbf{v}) = O(N^{-1})$. Since $N^{1/2}\langle\mathbf{v}\,,H\mathbf{v}\rangle$ is a centred Gaussian with variance $2\beta^{-1}$, we therefore find that $\Pi + \Upsilon$ has asymptotically[2] the distribution of a centred Gaussian with variance

$$\frac{2(|d|+1)^2(|d|-1)}{\beta d^4}+\frac{2(|d|+1)}{\beta d^4} \;=\; \frac{2(|d|+1)}{\beta d^2}\,.$$

[2] See Section 7.2 for precise definitions and more details.

Remark 2.18. If $\varphi^{C_2} N^{-1/3} \leqslant \big||d|-1\big| = o(1)$ then $\Pi + \Upsilon$ converges weakly to a centred Gaussian with variance $4\beta^{-1}$. As an outlier approaches the bulk spectrum, the dependence of its distribution on the details of H and $\mathbf{v}$ is washed out. Therefore, unlike outliers located at a distance of order one from the bulk spectrum, outliers close to ± 2 exhibit universality. Moreover, as an outlier approaches the bulk, its variance shrinks from N^{-1} (for $d-1 \asymp 1$) to $N^{-4/3}$ (for $d-1 \asymp N^{-1/3}$).

Remark 2.19. If $\max_i |v_i| \to 0$ as $N \to \infty$, we find that $Q(\mathbf{v}) \to 0$ and $R(\mathbf{v}) \to 0$ as $N \to \infty$. Moreover, the Central Limit Theorem implies in this case that $N^{1/2}\langle \mathbf{v}, H\mathbf{v}\rangle$ converges in distribution to a centred Gaussian with variance $2\beta^{-1}$. Therefore $\Pi + \Upsilon$ has asymptotically the distribution of

$$\mathcal{N}\biggl(\frac{(|d|+1)(|d|-1)^{1/2} S(\mathbf{v})}{d^4}\,,\,\frac{2(|d|+1)}{\beta d^2}\biggr)\,.$$

Thus, the only difference to the GOE/GUE case is a shift caused by the nonvanishing third moments of H. For example, if $M_{ij}^{(3)} = m^{(3)} \in \mathbb{R}$ is independent of i and j, and $\mathbf{v} = N^{-1/2}(1, \dots, 1)$, we find $S(\mathbf{v}) = m^{(3)} + O(N^{-1})$.

Remark 2.20. Typically, $R(\mathbf{v})$ is nonzero if $\mathbf{v}$ has entries which do not converge to zero. An example for which $Q(\mathbf{v})$ is nonzero is $M_{ij}^{(3)} = m^{(3)} \in \mathbb{R}$ independent of N and $\mathbf{v} = (2^{-1/2}, (2N-2)^{-1/2}, \dots, (2N-2)^{-1/2})$, in which case we have $Q(\mathbf{v}) = 2^{-3/2} m^{(3)} + O(N^{-1/2})$.

Remark 2.21. Consider now the case where $\max_i |v_i|$ does not tend to zero as $N \to \infty$. For definiteness, let $\mathbf{v} = (\mathbf{u}, \mathbf{w})$, where the dimension of $\mathbf{u}$ is constant and $\max_i |w_i| \to 0$ as $N \to \infty$. By the Central Limit Theorem and a short variance calculation, $N^{1/2}\langle \mathbf{v}, H\mathbf{v}\rangle$ has asymptotically the same distribution as $N^{1/2}\langle \mathbf{u}, H\mathbf{u}\rangle + 2\beta^{-1}(1-\|\mathbf{u}\|^2)(1+2\|\mathbf{u}\|^2)Z$, where Z is a standard normal random variable independent of H.

Let us take for example $\mathbf{v} = (1, 0, \dots, 0)$. Then $\Pi + \Upsilon$ has asymptotically the same distribution as $\Pi' + \Upsilon'$, where

$$\Pi' \;:=\; (|d|+1)(|d|-1)^{1/2} d^{-2} N^{1/2} h_{11}\,,$$

and Υ' is a centred Gaussian, independent of Π', with variance

$$\frac{2(|d|+1)}{\beta d^4} + \frac{(|d|+1)^2(|d|-1)}{N d^6} \sum_i \bigl(N^2 \mathbb{E}|h_{1i}|^4 - 4 + \beta\bigr)\,.$$

3. Proof of Theorem 2.2, Case A

In this section we prove Theorem 2.2 in the case **A**, i.e. where the first three moments of the entries of H coincide with those of GOE/GUE.

We start by introducing the following notations we shall use throughout the rest of the paper. For an $N \times N$ matrix A and $\mathbf{v}, \mathbf{w} \in \mathbb{C}^N$ we abbreviate

$$A_{\mathbf{v}\mathbf{w}} \;:=\; \langle \mathbf{v}, A\mathbf{w}\rangle\,.$$

We also write

$$A_{\mathbf{v}\mathbf{e}_i} \;\equiv\; A_{\mathbf{v}i}\,, \qquad A_{\mathbf{e}_i\mathbf{v}} \;\equiv\; A_{i\mathbf{v}}\,, \qquad A_{\mathbf{e}_i\mathbf{e}_j} \;\equiv\; A_{ij}\,,$$

where $\mathbf{e}_i \in \mathbb{C}^N$ denotes the i-th standard basis vector.

For definiteness, we consider the case where H is a complex Hermitian Wigner matrix; the proof for real symmetric Wigner matrices is the same. By Markov's inequality, in order to prove Theorem 2.2 it suffices to prove the following result.

Proposition 3.1. *Assume* (2.8) *and let* $\zeta > 0$ *be fixed. Then there exists a constant* C_ζ *such that, for all* $n \leqslant \varphi^\zeta$*, all deterministic* $\mathbf{v}, \mathbf{w} \in \mathbb{C}^N$*, and all* $z \in \mathbf{S}(C_\zeta)$*,*

$$\mathbb{E}\big|G_{\mathbf{v}\mathbf{w}}(z) - \langle \mathbf{v}, \mathbf{w}\rangle m(z)\big|^n \;\leqslant\; \big(\varphi^{C_\zeta}\Psi(z)\|\mathbf{v}\|\|\mathbf{w}\|\big)^n . \tag{3.1}$$

The rest of this section is devoted to the proof of Proposition 3.1.

3.1. Preliminaries. We start with a few basic tools. For $E \in \mathbb{R}$ define

$$\kappa_E \;:=\; \big||E| - 2\big| , \tag{3.2}$$

the distance from E to the spectral edges ± 2. In the following we use the notations

$$z \;=\; E + \mathrm{i}\eta , \qquad \kappa \;\equiv\; \kappa_E$$

without further comment. The following lemma collects some useful properties of m, the Stieltjes transform of the semicircle law.

Lemma 3.2. *For* $|z| \leqslant 2\Sigma$ *we have*

$$|m(z)| \;\asymp\; 1 , \qquad |1 - m(z)^2| \;\asymp\; \sqrt{\kappa + \eta} . \tag{3.3}$$

Moreover,

$$\operatorname{Im} m(z) \;\asymp\; \begin{cases} \sqrt{\kappa+\eta} & \text{if } |E| \leqslant 2 \\ \frac{\eta}{\sqrt{\kappa+\eta}} & \text{if } |E| \geqslant 2 . \end{cases}$$

(Here the implicit constants depend on Σ*.)*

Proof. The proof is an elementary calculation; see Lemma 4.2 in [27]. □

In addition to Ψ, we shall make use of a larger control parameter Φ, defined as

$$\Phi(z) \;:=\; \operatorname{Im} m(z) + \frac{1}{N\eta} , \qquad \Psi(z) \;=\; \sqrt{\frac{\operatorname{Im} m(z)}{N\eta}} + \frac{1}{N\eta} \;\asymp\; \sqrt{\frac{\Phi(z)}{N\eta}} . \tag{3.4}$$

From Lemma 3.2 we find, for any z satisfying $|z| \leqslant 2\Sigma$,

$$N^{-1/2} \;\lesssim\; \sqrt{\frac{\operatorname{Im} m(z)}{N\eta}} \;\lesssim\; \Psi(z) \;\lesssim\; \Phi(z) , \tag{3.5}$$

where $A_N \lesssim B_N$ means $A_N \leqslant C B_N$ for some constant C.

We shall often need to consider minors of H, which are the content of the following definition.

Definition 3.3 (Minors). *For $\mathbb{T} \subset \{1, \dots, N\}$ we define $H^{(\mathbb{T})}$ by*

$$(H^{(\mathbb{T})})_{ij} \;:=\; \mathbf{1}(i \notin \mathbb{T})\mathbf{1}(j \notin \mathbb{T})h_{ij}\,.$$

Moreover, we define the resolvent of $H^{(\mathbb{T})}$ through

$$G^{(\mathbb{T})}_{ij}(z) \;:=\; \mathbf{1}(i \notin \mathbb{T})\mathbf{1}(j \notin \mathbb{T})(H^{(\mathbb{T})} - z)^{-1}_{ij}\,.$$

We also set

$$\sum_i^{(\mathbb{T})} \;:=\; \sum_{i : i \notin \mathbb{T}}\,.$$

When $\mathbb{T} = \{a\}$, we abbreviate $(\{a\})$ by (a) in the above definitions; similarly, we write (ab) instead of $(\{a, b\})$.

We shall also need the following resolvent identities, proved in Lemma 4.2 of [25] and Lemma 6.10 of [16].

Lemma 3.4 (Resolvent identities). *For any i, j, k we have*

$$G_{ij} \;=\; G^{(k)}_{ij} + \frac{G_{ik}G_{kj}}{G_{kk}}\,. \tag{3.6}$$

Moreover, for $i \neq j$ we have

$$G_{ij} \;=\; -G_{ii} \sum_k^{(i)} h_{ik} G^{(i)}_{kj} \;=\; -G_{jj} \sum_k^{(j)} G^{(j)}_{ik} h_{kj}\,. \tag{3.7}$$

These identities also hold for minors $H^{(\mathbb{T})}$.

It is an immediate consequence of (3.6) that

$$G_{\mathbf{v}\mathbf{w}} \;=\; G^{(k)}_{\mathbf{v}\mathbf{w}} + \frac{G_{\mathbf{v}k}G_{k\mathbf{w}}}{G_{kk}}\,. \tag{3.8}$$

Moreover, we introduce the notations

$$\mathcal{G}_{\mathbf{v}i} \;:=\; -\sum_k^{(i)} G^{(i)}_{\mathbf{v}k} h_{ki}\,, \qquad \mathcal{G}_{i\mathbf{v}} \;:=\; -\sum_k^{(i)} h_{ik} G^{(i)}_{k\mathbf{v}}\,, \tag{3.9}$$

so that

$$G_{\mathbf{v}i} \;=\; G_{ii}\big(\overline{v}_i + \mathcal{G}_{\mathbf{v}i}\big)\,, \qquad G_{i\mathbf{v}} \;=\; G_{ii}\big(v_i + \mathcal{G}_{i\mathbf{v}}\big) \tag{3.10}$$

by (3.7).

Next, we record some basic large deviations estimates.

Lemma 3.5 (Large deviations estimates). *Let $a_1, \dots, a_N, b_1, \dots, b_M$ be independent random variables with zero mean and unit variance. Assume that there is a constant $\vartheta > 0$ such that*

$$\begin{aligned} \mathbb{P}(|a_i| \geqslant x) \;&\leqslant\; \vartheta^{-1} \exp(-x^{\vartheta}) \quad (i = 1, \dots, N)\,, \\ \mathbb{P}(|b_i| \geqslant x) \;&\leqslant\; \vartheta^{-1} \exp(-x^{\vartheta}) \quad (i = 1, \dots, M)\,. \end{aligned} \tag{3.11}$$

Then there exists a constant $\rho \equiv \rho(\vartheta) > 1$ such that, for any $\zeta > 0$ and any deterministic complex numbers A_i and B_{ij}, we have with ζ-high probability

$$\left|\sum_{i=1}^N A_i a_i\right| \;\leqslant\; \varphi^{\rho\zeta}\left(\sum_{i=1}^N |A_i|^2\right)^{1/2}, \tag{3.12}$$

$$\left|\sum_i A_i |a_i|^2 - \sum_i A_i\right| \;\leqslant\; \varphi^{\rho\zeta}\left(\sum_i |A_i|^2\right)^{1/2}, \tag{3.13}$$

$$\left|\sum_{i \neq j} \overline{a}_i B_{ij} a_j\right| \;\leqslant\; \varphi^{\rho\zeta}\left(\sum_{i\neq j} |B_{ij}|^2\right)^{1/2}, \tag{3.14}$$

$$\left|\sum_{i,j} a_i B_{ij} b_j\right| \;\leqslant\; \varphi^{\rho\zeta}\left(\sum_{i,j} |B_{ij}|^2\right)^{1/2}. \tag{3.15}$$

PROOF. The estimates (3.12) – (3.14) we proved in Appendix B of [25]. The estimate (3.15) follows easily from (3.12) in two steps. Defining $A_i := \sum_j B_{ij} b_j$, (3.12) yields $|A_i| \leqslant \varphi^{\rho\zeta}\big(\sum_j |B_{ij}|^2\big)^{1/2}$ with ζ-high probability. Since the families $\{A_i\}$ and $\{a_i\}$ are independent, (3.15) follows by using (3.12) again. □

Finally, we quote the following results which are proved in Theorems 2.1 and 2.2 of [26]. (Recall that we use the notation m for the quantity denoted by m_{sc} in [26].)

THEOREM 3.6 (LOCAL SEMICIRCLE LAW). *Fix $\zeta > 0$. Then there exists a constant C_ζ such that the event*

$$\bigcap_{z \in \mathbf{S}(C_\zeta)} \left\{ \max_{1 \leqslant i,j \leqslant N} \big|G_{ij}(z) - \delta_{ij} m(z)\big| \leqslant \varphi^{C_\zeta} \Psi(z) \right\} \tag{3.16}$$

holds with ζ-high probability.

Denote by $\gamma_1 \leqslant \gamma_2 \leqslant \cdots \leqslant \gamma_N$ the classical locations of the eigenvalues of H, defined through

$$N \int_{-\infty}^{\gamma_\alpha} \varrho(x)\,\mathrm{d}x \;=\; \alpha \qquad (1 \leqslant \alpha \leqslant N)\,. \tag{3.17}$$

THEOREM 3.7 (RIGIDITY OF EIGENVALUES). *Fix $\zeta > 0$. Then there exists a constant C_ζ such that*

$$|\lambda_\alpha - \gamma_\alpha| \;\leqslant\; \varphi^{C_\zeta}\big(\min\{\alpha, N+1-\alpha\}\big)^{-1/3} N^{-2/3}$$

for all $\alpha = 1, \dots, N$ with ζ-high probability.

3.2. Estimate of $G_{\mathbf{v}i}$. After these preparations, we may prove the key tool behind the proof of Proposition 3.1. It will be used as input in the Green function comparison method, throughout Sections 3.3, 3.4, and 4. Let us sketch its importance in the Green function comparison method. Anticipating the notation from the proof of Lemma 3.9, we shall have to estimate quantities of the form

$$(S - R)_{\mathbf{v}\mathbf{v}} \;=\; \big(-N^{-1/2} RVR + N^{-1} RVRVR + \cdots\big)_{\mathbf{v}\mathbf{v}}\,,$$

where the right-hand side is a resolvent expansion of the left-hand side. The first matrix product on the right-hand side may be written as

$$(RVR)_{\mathbf{v}\mathbf{v}} \;=\; R_{\mathbf{v}a}V_{ab}R_{b\mathbf{v}} + R_{\mathbf{v}b}V_{ba}R_{a\mathbf{v}}$$

(again anticipating the notation from the proof of Lemma 3.9). Lemma 3.8 will be used to estimate the resolvent entries of the form $R_{\mathbf{v}a}$ in such error estimates. These resolvent entries arise whenever the Green function comparison method is applied to the component $(\cdot)_{\mathbf{v}\mathbf{v}}$ of a resolvent.

LEMMA 3.8. *For any $\zeta > 0$ there exists a constant C_ζ such that*

$$|\mathcal{G}_{\mathbf{v}i}(z)| + |\mathcal{G}_{i\mathbf{v}}(z)| + |G_{\mathbf{v}i}(z)| + |G_{i\mathbf{v}}(z)| \;\leqslant\; \varphi^{C_\zeta}\sqrt{\frac{\operatorname{Im} G_{\mathbf{v}\mathbf{v}}(z)}{N\eta}} + C|v_i| \tag{3.18}$$

holds with ζ-high probability for all $z \in \mathbf{S}(C_\zeta)$.

PROOF. Since the families $(h_{ki})_k$ and $(G^{(i)}_{\mathbf{v}k})_k$ are independent, (3.9), (3.12), and (2.1) yield

$$|\mathcal{G}_{\mathbf{v}i}| \;\leqslant\; \varphi^{C_\zeta}\biggl(\frac{1}{N}\sum_k^{(i)}\bigl|G^{(i)}_{\mathbf{v}k}\bigr|^2\biggr)^{1/2}$$

with ζ-high probability for some constant C_ζ. By spectral decomposition one easily finds that

$$\frac{1}{N}\sum_k^{(i)}\bigl|G^{(i)}_{\mathbf{v}k}\bigr|^2 = \frac{1}{N\eta}\operatorname{Im} G^{(i)}_{\mathbf{v}\mathbf{v}}\,.$$

From (3.3) and (3.16) we find that

$$|G_{ii}| \;\leqslant\; C \tag{3.19}$$

with ζ-high probability provided that $\eta > \varphi^{C_\zeta}$ for some large enough C_ζ. Setting

$$X \;:=\; |\mathcal{G}_{\mathbf{v}i}| + |\mathcal{G}_{i\mathbf{v}}|\,,$$

we therefore conclude, using first (3.8) and then (3.10), that

$$X \;\leqslant\; \varphi^{C_\zeta}\biggl(\frac{\operatorname{Im} G_{\mathbf{v}\mathbf{v}} + |G_{ii}|\bigl|G_{\mathbf{v}i}/G_{ii}\bigr|\bigl|G_{i\mathbf{v}}/G_{ii}\bigr|}{N\eta}\biggr)^{1/2} \;\leqslant\; \varphi^{C_\zeta}\sqrt{\frac{\operatorname{Im} G_{\mathbf{v}\mathbf{v}}}{N\eta}} + \varphi^{C_\zeta}\frac{X}{\sqrt{N\eta}} + \varphi^{C_\zeta}\frac{|v_i|}{\sqrt{N\eta}}$$

with ζ-high probability. Thus we find for $\eta \geqslant \varphi^{2C_\zeta}N^{-1}$

$$X \;\leqslant\; \varphi^{C_\zeta}\sqrt{\frac{\operatorname{Im} G_{\mathbf{v}\mathbf{v}}}{N\eta}} + |v_i|$$

with ζ-high probability, and the claim for $|\mathcal{G}_{\mathbf{v}i}| + |\mathcal{G}_{i\mathbf{v}}|$ follows. The claim for $|G_{\mathbf{v}i}| + |G_{i\mathbf{v}}|$ follows using (3.10) and (3.19). □

3.3. Estimate of $\operatorname{Im} G_{\mathbf{vv}}$**.** The first step in the proof of Proposition 3.1 is the following estimate of $\operatorname{Im} G_{\mathbf{vv}}$. Note that $\operatorname{Im} G_{\mathbf{vv}}$ is a nonnegative quantity, as may be easily seen by spectral decomposition of G.

LEMMA 3.9. *Let $\zeta > 0$ be fixed. Then there exists a constant C_ζ such that, for all $n \leqslant \varphi^\zeta$, all deterministic and normalized $\mathbf{v} \in \mathbb{C}^N$, and all $z \in \mathbf{S}(C_\zeta)$, we have*

$$\mathbb{E}\big(\operatorname{Im} G_{\mathbf{vv}}(z)\big)^n \;\leqslant\; \big(\varphi^{C_\zeta}\Phi(z)\big)^n\,. \tag{3.20}$$

PROOF. We shall prove (3.20) using Green function comparison to GOE/GUE. First we claim that (3.20) holds if H is a GOE/GUE matrix. Indeed, in that case, using unitary invariance, (3.5), and (3.16), we find for $z \in \mathbf{S}(C_\zeta)$ that

$$\mathbb{E}\big(\operatorname{Im} G_{\mathbf{vv}}(z)\big)^n \;=\; \mathbb{E}\big(\operatorname{Im} G_{11}(z)\big)^n \;\leqslant\; \big(\varphi^{C_\zeta}\Phi(z)\big)^n + N^n N^C \exp(-\varphi^{2\zeta})\,,$$

where in the last inequality we used the rough bound $|G_{11}(z)| \leqslant \eta^{-1} \leqslant N$. Thus (3.20) for GOE/GUE follows from (3.5) and the estimate

$$N^{Cn}\exp(-\varphi^{2\zeta}) \;\leqslant\; C\,,$$

valid for $n \leqslant \varphi^\zeta$.

From now on we work on the product space generated by the Wigner matrix $H = (N^{-1/2}W_{ij})_{i,j}$ and the GOE/GUE matrix $(N^{-1/2}V_{ij})_{i,j}$. We fix a bijective ordering map on the index set of the independent matrix elements,

$$\phi : \{(i,j) : 1 \leqslant i \leqslant j \leqslant N\} \;\to\; \Big\{1, \dots, \gamma_{\max}\Big\} \qquad \text{where} \quad \gamma_{\max} \;:=\; \frac{N(N+1)}{2}\,, \tag{3.21}$$

and denote by $H_\gamma = (h_{ij}^\gamma)$, $\gamma = 0, \dots, \gamma_{\max}$, the Wigner matrix whose upper-triangular entries are defined by

$$h_{ij}^\gamma \;:=\; \begin{cases} N^{-1/2}W_{ij} & \text{if } \phi(i,j) \leqslant \gamma \\ N^{-1/2}V_{ij} & \text{otherwise}\,. \end{cases}$$

In particular, H_0 is a GOE/GUE matrix and $H_{\gamma_{\max}} = H$.

Let $E^{(ij)}$ denote the matrix whose matrix elements are given by $E^{(ij)}_{kl} := \delta_{ik}\delta_{jl}$. Fix $\gamma \geqslant 1$ and let (a,b) be determined by $\phi(a,b) = \gamma$. We shall compare $H_{\gamma-1}$ with H_γ for each γ and then sum up the differences. Note that the matrices $H_{\gamma-1}$ and H_γ differ only in the entries (a,b) and (b,a), and they can be written as

$$H_{\gamma-1} \;=\; Q + N^{-1/2}V \qquad \text{where} \quad V \;:=\; V_{ab}E^{(ab)} + \mathbf{1}(a \neq b)V_{ba}E^{(ba)}\,, \tag{3.22}$$

and

$$H_\gamma \;=\; Q + N^{-1/2}W \qquad \text{where} \quad W \;:=\; W_{ab}E^{(ab)} + \mathbf{1}(a \neq b)W_{ba}E^{(ba)}\,;$$

here the matrix Q satisfies $Q_{ab} = Q_{ba} = 0$.

Next, we introduce the Green functions

$$R \;:=\; \frac{1}{Q-z}\,, \qquad S \;:=\; \frac{1}{H_{\gamma-1}-z}\,, \qquad T \;:=\; \frac{1}{H_\gamma - z}\,, \tag{3.23}$$

which are well-defined for $\eta > 0$ since Q and H_γ are self-adjoint. Using the notation $G^\gamma := (H_\gamma - z)^{-1}$, we have the telescopic sum

$$\mathbb{E}\big(\operatorname{Im} G^{\gamma_{\max}}_{\mathbf{vv}}\big)^n - \mathbb{E}\big(\operatorname{Im} G^0_{\mathbf{vv}}\big)^n \;=\; \sum_{\gamma=1}^{\gamma_{\max}} \Big(\mathbb{E}\big(\operatorname{Im} G^\gamma_{\mathbf{vv}}\big)^n - \mathbb{E}\big(\operatorname{Im} G^{\gamma-1}_{\mathbf{vv}}\big)^n\Big)\,. \tag{3.24}$$

For any $K \in \mathbb{N}$ we have the resolvent expansions

$$S \;=\; \sum_{k=0}^{K-1} N^{-k/2}(-RV)^k R + N^{-K/2}(-RV)^K S \;=\; \sum_{k=0}^{K-1} N^{-k/2} R(-VR)^k + N^{-K/2} S(-VR)^K \tag{3.25}$$

and

$$R \;=\; \sum_{k=0}^{K-1} N^{-k/2}(SV)^k S + N^{-K/2}(SV)^K R \;=\; \sum_{k=0}^{K-1} N^{-k/2} S(VS)^k + N^{-K/2} R(VS)^K\,. \tag{3.26}$$

Now we choose $K = 10$ in (3.26). Applying Theorem 3.6 to the Wigner matrix S, using the rough bound $\|R\| \leqslant \eta^{-1} \leqslant N$ to estimate the rest term in (3.26), and recalling (2.1), we find

$$\big|R_{ij} - \delta_{ij} m\big| \;\leqslant\; \big|S_{ij} - \delta_{ij} m\big| + \varphi^{C_\zeta} N^{-1/2} \;\leqslant\; \varphi^{C_\zeta} \Psi \tag{3.27}$$

with 2ζ-high probability. Here we also used (3.5). Throughout the proof we shall tacitly make use of the bound $|R_{ij}| \leqslant C$ with 2ζ-high probability, as follows from (3.27).

Next, setting $K = 1$ in (3.25), recalling (2.1), and using Lemma 3.8, we find

$$|S_{\mathbf{v}a} - R_{\mathbf{v}a}| \;\leqslant\; N^{-1/2}\varphi^{C_\zeta}\Big(|S_{\mathbf{v}a} R_{ba}| + |S_{\mathbf{v}b} R_{aa}|\Big) \;\leqslant\; N^{-1/2}\varphi^{C_\zeta}\left(\sqrt{\frac{\operatorname{Im} S_{\mathbf{v}\mathbf{v}}}{N\eta}} + |v_a| + |v_b|\right) \tag{3.28}$$

with 2ζ-high probability. Now (3.28), (3.5), and Lemma 3.8 yield

$$|R_{\mathbf{v}a}| \;\leqslant\; \varphi^{C_\zeta}\sqrt{\frac{\operatorname{Im} S_{\mathbf{v}\mathbf{v}}}{N\eta}} + C|v_a| + \varphi^{C_\zeta} N^{-1/2} \;\leqslant\; \varphi^{C_\zeta}\sqrt{\frac{\operatorname{Im} S_{\mathbf{v}\mathbf{v}}}{N\eta}} + \varphi^{C_\zeta}\Psi + C|v_a| \tag{3.29}$$

with 2ζ-high probability. The same bound holds for $R_{a\mathbf{v}}$. Similarly, choosing $K = 1$ in (3.25) yields, using (3.29), that

$$|S_{\mathbf{v}\mathbf{v}} - R_{\mathbf{v}\mathbf{v}}| \;\leqslant\; N^{-1/2}\varphi^{C_\zeta}\big(|S_{\mathbf{v}a} R_{b\mathbf{v}}| + |S_{\mathbf{v}b} R_{a\mathbf{v}}|\big) \;\leqslant\; N^{-1/2}\varphi^{C_\zeta}\left(\frac{\operatorname{Im} S_{\mathbf{v}\mathbf{v}}}{N\eta} + |v_a|^2 + |v_b|^2\right) \tag{3.30}$$

with 2ζ-high probability.

After these preparations, we may start to estimate

$$\big(\operatorname{Im} S_{\mathbf{v}\mathbf{v}}\big)^n - \big(\operatorname{Im} R_{\mathbf{v}\mathbf{v}}\big)^n \;=\; \sum_{m=1}^{n} A_m \,(\operatorname{Im} R_{\mathbf{v}\mathbf{v}})^{n-m}\,,$$

where we defined

$$A_m \;:=\; \binom{n}{m}\big(\operatorname{Im} S_{\mathbf{v}\mathbf{v}} - \operatorname{Im} R_{\mathbf{v}\mathbf{v}}\big)^m\,.$$

We choose $K = 4$ in (3.25) and introduce the notation $S - R = \sum_{k=1}^{4} Y_k$, whereby Y_k has k factors V. We write

$$A_m \;=\; \sum_{k=m}^{4m} A_{m,k} \qquad \text{where} \quad A_{m,k} \;:=\; \binom{n}{m} \sum_{k_1,\dots,k_m=1}^{4} \mathbf{1}\big(k_1 + \cdots + k_m = k\big) \prod_{i=1}^{m} \operatorname{Im}(Y_{k_i})_{\mathbf{v}\mathbf{v}}\,. \tag{3.31}$$

Thus we have

$$\mathbb{E}\big(\operatorname{Im} S_{\mathbf{v}\mathbf{v}}\big)^n - \mathbb{E}\big(\operatorname{Im} R_{\mathbf{v}\mathbf{v}}\big)^n \;=\; \mathcal{A} + \sum_{m=1}^{n} \sum_{k=\max\{4,m\}}^{4m} \mathbb{E} A_{m,k} \big(\operatorname{Im} R_{\mathbf{v}\mathbf{v}}\big)^{n-m}\,, \tag{3.32}$$

where $\mathcal{A}$ depends on the randomness only through Q and the first three moments of V_{ab}.

We shall prove that

$$\sum_{m=1}^{n} \sum_{k=4}^{4m} \mathbb{E}\,|A_{m,k}| \big(\operatorname{Im} R_{\mathbf{v}\mathbf{v}}\big)^{n-m} \;\leqslant\; \frac{\mathcal{E}_{ab}}{\log N} \Big(\mathbb{E}\big(\operatorname{Im} S_{\mathbf{v}\mathbf{v}}\big)^n + (\varphi^{C_\zeta}\Phi)^n \Big)\,, \tag{3.33}$$

where we defined

$$\mathcal{E}_{ab} \;:=\; \sum_{\sigma,\tau=0}^{2} N^{-2+\sigma/2+\tau/2} |v_a|^\sigma |v_b|^\tau\,. \tag{3.34}$$

For future use, we note that the proof of (3.33) does not require the vanishing of the third moments of H as in (2.8). Before proving (3.33), we show how it implies (3.20). Let us abbreviate $X_\gamma := \mathbb{E}\big(\operatorname{Im} G^\gamma_{\mathbf{v}\mathbf{v}}\big)^n$ and $\mathcal{E}_\gamma := (\log N)^{-1} \mathcal{E}_{\phi^{-1}(\gamma)}$. Note that, since $\operatorname{Im} G^\gamma_{\mathbf{v}\mathbf{v}} \geqslant 0$, we have $X_\gamma \geqslant 0$ for all γ. Repeating the derivation of (3.32) for T instead of S, using that the first three moments of V_{ab} and W_{ab} are the same, and using the estimate (3.33) and its analogue with S replaced by T, we find

$$X_\gamma - X_{\gamma-1} \;\leqslant\; \mathcal{E}_\gamma \Big(X_\gamma + X_{\gamma-1} + \big(\varphi^{C_\zeta}\Phi\big)^n \Big)\,.$$

Abbreviating $r_\gamma := (1-\mathcal{E}_\gamma)^{-1}(1+\mathcal{E}_\gamma) \geqslant 1$ we therefore find

$$X_\gamma \;\leqslant\; r_\gamma\, X_{\gamma-1} + r_\gamma\, \mathcal{E}_\gamma \big(\varphi^{C_\zeta}\Phi\big)^n\,.$$

Since (3.20) holds for GOE/GUE, we have the initial estimate $X_0 \leqslant \big(\varphi^{C_\zeta}\Phi\big)^n$. Iteration therefore yields

$$X_\gamma \;\leqslant\; \Bigg(\prod_{j=1}^{\gamma} r_\gamma \Bigg) \Bigg(1 + \sum_{j=1}^{\gamma} \mathcal{E}_\gamma \Bigg) \big(\varphi^{C_\zeta}\Phi\big)^n\,.$$

Next, we observe that $\sum_\gamma \mathcal{E}_\gamma \leqslant 1$. Since $0 \leqslant \mathcal{E}_\gamma \leqslant 1/2$, we find $\prod_\gamma r_\gamma \leqslant C$. This implies

$$\mathbb{E}\big(\operatorname{Im} G_{\mathbf{v}\mathbf{v}}\big)^n \;=\; X_{\gamma_{\max}} \;\leqslant\; \big(\varphi^{C_\zeta}\Phi\big)^n\,,$$

which is (3.20).

What remains is to prove (3.33). Recall that in (3.31), $(Y_k)_{\mathbf{v}\mathbf{v}} = N^{-k/2}\big[(-RV)^k R\big]_{\mathbf{v}\mathbf{v}}$ if $k < 4$ and $(Y_4)_{\mathbf{v}\mathbf{v}} = N^{-2}\big[(-RV)^k S\big]_{\mathbf{v}\mathbf{v}}$. For each Y_{k_i} in (3.31), we write out the matrix multiplication in terms of matrix elements of S, R, and V. Then we multiply everything out. We classify the resulting terms using two additional parameters $s,t \geqslant 0$. Here s is the total number of matrix elements $R_{\mathbf{v}a}$, $R_{a\mathbf{v}}$, $S_{\mathbf{v}a}$, and $S_{a\mathbf{v}}$; t is defined similarly with a replaced by b. If $a=b$, we use the symmetric convention $s=t$.

We have the conditions

$$s+t \;=\; 2m\,, \qquad k \;\geqslant\; \max\{s,t\}\,. \tag{3.35}$$

The first one is immediate. The second one is clearly true if $a=b$. In order to prove it in the case $a \neq b$, assume for definiteness that $s \geqslant t$. Then each factor $R_{\mathbf{v}a}$, $R_{a\mathbf{v}}$, $S_{\mathbf{v}a}$, and $S_{a\mathbf{v}}$ is associated with a unique factor V_{ab} or V_{ba} (the one standing next to it in the matrix product); this proves the second condition of (3.35). Thus we have the decomposition

$$A_{m,k} \;=\; \sum_{s,t=0}^{k} \mathbf{1}(s+t=2m) A_{m,k,s,t}\,, \tag{3.36}$$

in self-explanatory notation.

Using Lemma 3.8 and (3.29), we get the bound

$$\begin{aligned}
&\Big(|R_{\mathbf{v}a}|+|R_{a\mathbf{v}}|+|S_{\mathbf{v}a}|+|S_{a\mathbf{v}}|\Big)^s \Big(|R_{\mathbf{v}b}|+|R_{b\mathbf{v}}|+|S_{\mathbf{v}b}|+|S_{b\mathbf{v}}|\Big)^t \\
&\quad\leqslant\; \left(\varphi^{C_\zeta}\sqrt{\frac{\operatorname{Im} S_{\mathbf{v}\mathbf{v}}}{N\eta}}+\varphi^{C_\zeta}\Psi+C|v_a|\right)^s \left(\varphi^{C_\zeta}\sqrt{\frac{\operatorname{Im} S_{\mathbf{v}\mathbf{v}}}{N\eta}}+\varphi^{C_\zeta}\Psi+C|v_b|\right)^t \\
&\quad\leqslant\; \left(\varphi^{C_\zeta}\frac{\operatorname{Im} S_{\mathbf{v}\mathbf{v}}}{N\eta}+\varphi^{C_\zeta}\Psi^2\right)^m + \left(\varphi^{C_\zeta}\frac{\operatorname{Im} S_{\mathbf{v}\mathbf{v}}}{N\eta}+\varphi^{C_\zeta}\Psi^2\right)^{m-s/2} (C|v_a|)^s \\
&\qquad+ \left(\varphi^{C_\zeta}\frac{\operatorname{Im} S_{\mathbf{v}\mathbf{v}}}{N\eta}+\varphi^{C_\zeta}\Psi^2\right)^{m-t/2} (C|v_b|)^t + (C|v_a|)^s (C|v_b|)^t \\
&\quad\leqslant\; \varphi^{-Dm}\left(\varphi^{C_{\zeta,D}}\left(\frac{\operatorname{Im} S_{\mathbf{v}\mathbf{v}}}{N\eta}+\Psi^2+N^{-1/2}\right)\right)^m \Big(1+N^{s/4}|v_a|^s+N^{t/4}|v_b|^t+N^{s/4+t/4}|v_a|^s|v_b|^t\Big)
\end{aligned} \tag{3.37}$$

with 2ζ-high probability, where in the second step we used Lemma 3.10 below and $s+t\leqslant \varphi^\zeta$, and in the third step the inequality $x^{m-a}y^a \leqslant (x+y)^m$. Here $D>0$ is some constant to be chosen later, and $C_{\zeta,D}$ denotes a constant depending on ζ and D. For the following it will be convenient to abbreviate

$$\mathcal{F}_{ab}(s,t) \;:=\; 1+N^{s/4}|v_a|^s+N^{t/4}|v_b|^t+N^{s/4+t/4}|v_a|^s|v_b|^t\,.$$

Using (3.4), (3.5), and Lemma 3.10 below, we find that there is a constant $C_{\zeta,D}$ such that for $z\in\mathbf{S}(C_{\zeta,D})$ we have

$$\begin{aligned}
&\Big(|R_{\mathbf{v}a}|+|R_{a\mathbf{v}}|+|S_{\mathbf{v}a}|+|S_{a\mathbf{v}}|\Big)^s \Big(|R_{\mathbf{v}b}|+|R_{b\mathbf{v}}|+|S_{\mathbf{v}b}|+|S_{b\mathbf{v}}|\Big)^t \\
&\qquad\qquad\leqslant\; \varphi^{-Dm}\Big(\big(\operatorname{Im} S_{\mathbf{v}\mathbf{v}}\big)^m + \big(\varphi^{C_{\zeta,D}}\Phi\big)^m\Big)\mathcal{F}_{ab}(s,t)
\end{aligned} \tag{3.38}$$

with 2ζ-high probability.

Next, we observe that (3.30) and (3.5) imply

$$\operatorname{Im} R_{\mathbf{v}\mathbf{v}} \;\leqslant\; \big(1+\varphi^{C_\zeta}N^{-1/2}\big)\operatorname{Im} S_{\mathbf{v}\mathbf{v}} + \varphi^{C_\zeta}\Phi \tag{3.39}$$

with 2ζ-high probability. Recall that, be definition, $A_{m,k,s,t}$ contains k factors V, s factors in the set $\{R_{\mathbf{v}a}, R_{a\mathbf{v}}, S_{\mathbf{v}a}, S_{a\mathbf{v}}\}$, and t factors in the set $\{R_{\mathbf{v}b}, R_{b\mathbf{v}}, S_{\mathbf{v}b}, S_{b\mathbf{v}}\}$. Therefore the definitions (3.31) and (3.36), as well as the estimates (2.1), (3.38), and (3.39), yield

$$\begin{aligned}
&|A_{m,k,s,t}|(\operatorname{Im} R_{\mathbf{v}\mathbf{v}})^{n-m} \\
&\leqslant\; (4n)^m\varphi^{kC_\zeta}N^{-k/2}\varphi^{-Dm}\Big(\big(\operatorname{Im} S_{\mathbf{v}\mathbf{v}}\big)^m+\big(\varphi^{C_{\zeta,D}}\Phi\big)^m\Big)\,\mathcal{F}_{ab}(s,t)\,\Big(\big(1+\varphi^{C_\zeta}N^{-1/2}\big)\operatorname{Im} S_{\mathbf{v}\mathbf{v}}+\varphi^{C_\zeta}\Phi\Big)^{n-m} \\
&\leqslant\; \varphi^{(C_\zeta-D)m}N^{-k/2}\Big((\operatorname{Im} S_{\mathbf{v}\mathbf{v}})^n+\big(\varphi^{C_{\zeta,D}}\Phi\big)^n\Big)\mathcal{F}_{ab}(s,t)
\end{aligned} \tag{3.40}$$

with 2ζ-high probability, where we used that $k \leqslant 4m$, that $n \leqslant \varphi^\zeta$, $\binom{n}{m} \leqslant n^m$, and Lemma 3.10 below. Denote by Ξ the event on which the estimate (3.40) holds; thus, $\mathbb{P}(\Xi^c) \leqslant N^C \exp(-\varphi^{2\zeta})$. Using (2.1) and the deterministic bound $\|R\| + \|S\| \leqslant N$, it is easy to see that on Ξ^c we have the rough estimate

$$\mathbb{E}|A_{m,k,s,t}|(\operatorname{Im} R_{\mathbf{v}\mathbf{v}})^{n-m}\mathbf{1}(\Xi^c) \;\leqslant\; N^n\Big(\mathbb{E}|A_{m,k,s,t}|^2\Big)^{1/2}\mathbb{P}(\Xi^c)^{1/2} \;\leqslant\; (N\varphi^\zeta)^{C\varphi^\zeta}\exp(-c\varphi^{2\zeta}) \;\leqslant\; \Phi^n N^{-10n}$$

for all $n \leqslant \varphi^\zeta$ and N large enough. Therefore choosing $D \equiv D_\zeta$ large enough we get from (3.40)

$$\mathbb{E}|A_{m,k,s,t}|(\operatorname{Im} R_{\mathbf{v}\mathbf{v}})^{n-m} \;\leqslant\; \varphi^{-m}\Big((\operatorname{Im} S_{\mathbf{v}\mathbf{v}})^n + \big(\varphi^{C_\zeta}\Phi\big)^n\Big)N^{-k/2}\,\mathcal{F}_{ab}(s,t)\,.$$

Therefore (3.33) follows using (3.35) if we can prove that

$$N^{-\max\{4,s,t\}/2}\Big(1 + N^{s/4}|v_a|^s + N^{t/4}|v_b|^t + N^{s/4+t/4}|v_a|^s|v_b|^t\Big) \;\leqslant\; C\mathcal{E}_{ab} \;=\; C\sum_{\sigma,\tau=0}^{2} N^{-2+\sigma/2+\tau/2}|v_a|^\sigma|v_b|^\tau\,. \tag{3.41}$$

for all s,t. We check that all terms on the left-hand side of (3.41) are bounded, for all $s,t \geqslant 0$, by the right-hand side of (3.41). The first term is trivial: $N^{-\max\{4,s,t\}/2} \leqslant N^{-2}$. The second term is bounded by

$$N^{-\max\{4,s,t\}/2}N^{s/4}|v_a|^s \;\leqslant\; N^{-2} + N^{-2+1/4}|v_a| + N^{-2+1}|v_a|^2\,.$$

The third term is bounded similarly. Finally, the last term is bounded by

$$N^{-\max\{4,s,t\}/2}N^{s/4+t/4}|v_a|^s|v_b|^t \;\leqslant\; E + N^{-2+1/2}|v_a||v_b| + N^{-2+1+1/4}(|v_a|^2|v_b| + |v_a||v_b|^2) + |v_a|^2|v_b|^2\,,$$

where E denotes a quantity bounded by the three previous terms. This completes the proof of (3.41), and hence of (3.33). □

What remains is to prove the following elementary result.

Lemma 3.10. *For $x,y \geqslant 0$ and $m \in \mathbb{N}$ we have*

$$(x+y)^m \;\leqslant\; Cx^m + (my)^m\,.$$

Proof. By convexity of the function $x \mapsto x^m$ we have, for any $\lambda \in (0,1)$,

$$(x+y)^m \;=\; \bigg((1-\lambda)\frac{x}{1-\lambda} + \lambda\frac{y}{\lambda}\bigg)^m \;\leqslant\; \frac{1}{(1-\lambda)^m}x^m + \frac{1}{\lambda^m}y^m\,.$$

Choosing $\lambda = 1/m$ yields the claim. □

3.4. Estimate of $G_{\mathbf{vv}} - m$. We now conclude the proof of Proposition 3.1. By polarization and linearity, it is enough to prove the following result.

LEMMA 3.11. *Let $\zeta > 0$ be fixed. Then there exists a constant C_ζ such that, for all $n \leqslant \varphi^\zeta$, all deterministic and normalized $\mathbf{v} \in \mathbb{C}^N$, and all $z \in \mathbf{S}(C_\zeta)$, we have*

$$\mathbb{E}\big|G_{\mathbf{vv}}(z) - m(z)\big|^n \;\leqslant\; \big(\varphi^{C_\zeta}\Psi(z)\big)^n . \tag{3.42}$$

PROOF. The proof is very similar to that of Lemma 3.9, whose notation we take over without further comment. In order to avoid dealing with complex numbers, we estimate the real and imaginary parts of $G_{\mathbf{vv}} - m$ separately. We give the argument for the real part; the imaginary part is dealt with in the same way. Throughout the following n denotes an even number less than φ^ζ.

For the GOE/GUE matrix H_0 we get from Theorem 3.6, as in the proof of Lemma 3.9, that

$$\mathbb{E}\big(\operatorname{Re} G^0_{\mathbf{vv}} - \operatorname{Re} m\big)^n \;\leqslant\; \big(\varphi^{C_\zeta}\Psi\big)^n . \tag{3.43}$$

In order to perform the comparison step, we write, similarly to (3.32),

$$\mathbb{E}\big(\operatorname{Re} S_{\mathbf{vv}} - \operatorname{Re} m\big)^n - \mathbb{E}\big(\operatorname{Re} R_{\mathbf{vv}} - \operatorname{Re} m\big)^n \;=\; \mathcal{B} + \sum_{m=1}^{n} \sum_{k=\max\{4,m\}}^{4m} \mathbb{E} B_{m,k}\big(\operatorname{Re} R_{\mathbf{vv}} - \operatorname{Re} m\big)^{n-m} ,$$

where $\mathcal{B}$ depends on the randomness only through Q and the first three moments of V_{ab}, and

$$B_{m,k} \;:=\; \binom{n}{m} \sum_{k_1,\dots,k_m=1}^{4} \mathbf{1}\big(k_1 + \cdots + k_m = k\big) \prod_{i=1}^{m} \operatorname{Re}(Y_{k_i})_{\mathbf{vv}} .$$

Similarly to (3.33), we shall prove that

$$\sum_{m=1}^{n}\sum_{k=4}^{4m} \mathbb{E}\left(|B_{m,k}|\big|\operatorname{Re} R_{\mathbf{vv}} - \operatorname{Re} m\big|^{n-m}\right) \;\leqslant\; \frac{\mathcal{E}_{ab}}{\log N}\,\mathbb{E}\left[\big(\operatorname{Re} S_{\mathbf{vv}} - \operatorname{Re} m\big)^n + \left(\varphi^{C_\zeta}\frac{\operatorname{Im} S_{\mathbf{vv}}}{N\eta}\right)^n + (\varphi^{C_\zeta}\Psi)^n\right]. \tag{3.44}$$

Using Lemma 3.9, (3.4), and (3.5) we find that the right-hand side of (3.44) is bounded by

$$\frac{\mathcal{E}_{ab}}{\log N}\,\mathbb{E}\left[\big(\operatorname{Re} S_{\mathbf{vv}} - \operatorname{Re} m\big)^n + (\varphi^{C_\zeta}\Psi)^n\right].$$

Therefore (3.43) and (3.44) yield (3.42), exactly as in the paragraph following (3.34).

What remains therefore is to prove (3.44). Using (3.37), (3.5), and Lemma 3.10 we get, for arbitrary $D > 0$,

$$\begin{aligned}\Big(|R_{\mathbf{v}a}| + |R_{a\mathbf{v}}| + |S_{\mathbf{v}a}| + |S_{a\mathbf{v}}|\Big)^s \Big(|R_{\mathbf{v}b}| + |R_{b\mathbf{v}}| + |S_{\mathbf{v}b}| + |S_{b\mathbf{v}}|\Big)^t \qquad\qquad& \\ \leqslant\; \varphi^{-Dm}\left(\left(\varphi^{C_{\zeta,D}}\frac{\operatorname{Im} S_{\mathbf{vv}}}{N\eta}\right)^m + \big(\varphi^{C_{\zeta,D}}\Psi\big)^m\right)\mathcal{F}_{ab}(s,t)&\end{aligned} \tag{3.45}$$

with 2ζ-high probability. Therefore we get, similarly to (3.40),

$$|B_{m,k,s,t}|\big|\mathrm{Re}\,R_{\mathbf{vv}}-\mathrm{Re}\,m\big|^{n-m}$$

$$\leqslant\;\varphi^{(C_\zeta-D)m}N^{-k/2}\Bigg(\big(\mathrm{Re}\,S_{\mathbf{vv}}-\mathrm{Re}\,m\big)^n+\bigg(\varphi^{C_{\zeta,D}}\frac{\mathrm{Im}\,S_{\mathbf{vv}}}{N\eta}\bigg)^n+\big(\varphi^{C_{\zeta,D}}\Psi\big)^n\Bigg)\mathcal{F}_{ab}(s,t)$$

with 2ζ-high probability, where we used (3.30), $N^{-1/2}\leqslant\Psi$, and Lemma 3.10. Choosing $D>0$ large enough and recalling (3.41) yields (3.44). (We omit the details of the analysis on the low-probability event, which are similar to those following (3.40).) This concludes the proof of Lemma 3.11. □

4. Proof of Theorem 2.2, Case **B**

In this section we prove Theorem 2.2 in the case **B**, i.e. we impose no condition on the third moments of the entries of H, and $\Psi(z)$ satisfies (2.9). By Markov's inequality, it suffices to prove the following result.

Proposition 4.1. *Fix $\zeta>0$. Then there are constants C_0 and C_ζ, both depending on ζ, such that the following holds. Assume that $z\in\mathbf{S}(C_\zeta)$ satisfies* (2.9) *with constant C_0. Then we have, for all $n\leqslant\varphi^\zeta$ and all deterministic $\mathbf{v},\mathbf{w}\in\mathbb{C}^N$, that*

$$\mathbb{E}\big|G_{\mathbf{vw}}(z)-\langle\mathbf{v},\mathbf{w}\rangle m(z)\big|^n\;\leqslant\;\big(\varphi^{C_\zeta}\Psi(z)\|\mathbf{v}\|\|\mathbf{w}\|\big)^n\,. \tag{4.1}$$

The rest of this section is devoted to the proof of Proposition 4.1. We take over the notation of Section 3, which we use throughout this section without further comment.

4.1. Estimate of $\mathrm{Im}\,G_{\mathbf{vv}}$.

In this section we derive an apriori bound on $\mathrm{Im}\,G_{\mathbf{vv}}$ by proving the following result.

Lemma 4.2. *Fix $\zeta>0$. Then there are large enough constants C_0 and C_ζ, both depending on ζ, such that the following holds. Assume that $z\in\mathbf{S}(C_\zeta)$ satisfies* (2.9) *with constant C_0. Then we have, for all $n\leqslant\varphi^\zeta$ and all deterministic and normalized $\mathbf{v}\in\mathbb{C}^N$, that*

$$\mathbb{E}\big(\mathrm{Im}\,G_{\mathbf{vv}}(z)\big)^n\;\leqslant\;\big(\varphi^{C_\zeta}\Phi(z)\big)^n\,. \tag{4.2}$$

The following (trivial) observation will be needed in the next section: The constant C_0 may be increased at will without changing C_ζ in (4.2).

The main technical estimate behind the proof of Lemma 4.2 is the following lemma. Recall the setup (3.21) of the Green function comparison, and in particular the definitions (3.23).

Lemma 4.3. *Fix $\zeta>0$. Then there are constants C_0 and C_1, both depending on ζ, such that if* (2.9) *holds with constant C_0 then we have the following. For any a,b we have*

$$\left|\sum_{m=1}^{n}\sum_{k=\max\{3,m\}}^{4m}\mathbb{E}A_{m,k}(\mathrm{Im}\,R_{\mathbf{vv}})^{n-m}\right|\;\leqslant\;\frac{C}{\log N}\bigg(\widetilde{\mathcal{E}}_{ab}+N^{-3/2}\frac{\varphi^{C_1}}{N\eta}\bigg)\Big(\mathbb{E}\big(\mathrm{Im}\,S_{\mathbf{vv}}\big)^n+(\varphi^{C_1}\Phi)^n\Big)\,, \tag{4.3}$$

where

$$\widetilde{\mathcal{E}}_{ab}\;:=\;\mathcal{E}_{ab}+\delta_{ab}\big(|v_a|^2+N^{-3/2}\big)\;=\;\sum_{\sigma,\tau=0}^{2}N^{-2+\sigma/2+\tau/2}|v_a|^\sigma|v_b|^\tau+\delta_{ab}\big(|v_a|^2+N^{-3/2}\big)\,.$$

Moreover, if

$$|v_a| + |v_b| \;\leqslant\; N^{-1/4}\sqrt{\frac{\varphi^{C_1}}{N\eta}} \tag{4.4}$$

then we have the stronger bound

$$\left|\sum_{m=1}^{n}\sum_{k=\max\{3,m\}}^{4m}\mathbb{E}A_{m,k}(\operatorname{Im}R_{\mathbf{vv}})^{n-m}\right| \;\leqslant\; \frac{C}{\log N}\,\widetilde{\mathcal{E}}_{ab}\left(\mathbb{E}\big(\operatorname{Im}S_{\mathbf{vv}}\big)^n + (\varphi^{C_1}\Phi)^n\right). \tag{4.5}$$

Before proving Lemma 4.3, we use it to complete the proof of Lemma 4.2.

Proof of Lemma 4.2. Let $B \subset \{1,\dots,N\}^2$ denote the subset

$$B \;:=\; \left\{(a,b)\,:\,|v_a| + |v_b| > N^{-1/4}\sqrt{\frac{\varphi^{C_1}}{N\eta}}\right\}.$$

Since $\|\mathbf{v}\| = 1$, the number of indices a such that $|v_a| \geqslant \varepsilon$ is bounded by ε^{-2}. Therefore

$$|B| \;\leqslant\; N^{3/2}\left(\frac{\varphi^{C_1}}{N\eta}\right)^{-1}.$$

Therefore we have

$$\sum_{(a,b)\in B}\frac{C}{\log N}\left(\widetilde{\mathcal{E}}_{ab} + N^{-3/2}\frac{\varphi^{C_1}}{N\eta}\right) + \sum_{(a,b)\in B^c}\frac{C}{\log N}\,\widetilde{\mathcal{E}}_{ab} \;\leqslant\; \frac{C}{\log N}\,.$$

Now (4.2) follows from (4.3) and (4.5), by repeating the argument after (3.34). □

Before proving Lemma 4.3, we record the following lower bound on η.

Lemma 4.4. *Let $C_0 > 0$. If (2.9) holds then*

$$\eta \;\geqslant\; \varphi^{C_0/3}N^{-5/6}\,. \tag{4.6}$$

Proof. The claim follows immediately from $(N\eta)^{-1} \leqslant \Psi \leqslant \varphi^{-C_0/3}N^{-1/6}$. □

Proof of Lemma 4.3. Note that the proof of (3.33) did not use the assumption (2.8). In particular, all statements in the proof of Lemma 3.9 after (3.35) remain true in the case **B**. By (3.33), it is enough to prove

$$\left|\mathbb{E}A_{m,3}(\operatorname{Im}R_{\mathbf{vv}})^{n-m}\right| \;\leqslant\; \frac{1}{\log N}\left(\widetilde{\mathcal{E}}_{ab} + N^{-3/2}\frac{\varphi^{C_\zeta}}{N\eta}\right)\left(\mathbb{E}\big(\operatorname{Im}S_{\mathbf{vv}}\big)^n + (\varphi^{C_\zeta}\Phi)^n\right) \tag{4.7}$$

for $m = 1,2,3$ as well as, assuming (4.4),

$$\left|\mathbb{E}A_{m,3}(\operatorname{Im}R_{\mathbf{vv}})^{n-m}\right| \;\leqslant\; \frac{1}{\log N}\,\widetilde{\mathcal{E}}_{ab}\left(\mathbb{E}\big(\operatorname{Im}S_{\mathbf{vv}}\big)^n + (\varphi^{C_\zeta}\Phi)^n\right) \tag{4.8}$$

for $m = 1,2,3$. In order to prove (4.7) and (4.8), we distinguish four cases depending on m and whether $a = b$. Recall from (3.35) that

$$s + t \;=\; 2m\,, \qquad s \;\leqslant\; 3\,, \qquad t \;\leqslant\; 3\,. \tag{4.9}$$

Case (i): $a = b$ **and** $m \leqslant 3$**.** Similarly to (3.37), we find

$$\Big(|R_{\mathbf{v}a}| + |R_{a\mathbf{v}}|\Big)^{2m} \;\leqslant\; \varphi^{-Dm}\Big(\operatorname{Im} S_{\mathbf{v}\mathbf{v}} + \varphi^{C_{\zeta,D}}\Phi\Big)^m \big(1 + N^{m/2}|v_a|^{2m}\big)$$

with 2ζ-high probability, for any constant $D > 0$ and $z \in \mathbf{S}(C_{\zeta,D})$. Therefore (3.39) yields

$$\begin{aligned} |A_{m,3}|(\operatorname{Im} R_{\mathbf{v}\mathbf{v}})^{n-m} \;&\leqslant\; \varphi^{C_\zeta - Dm} N^{-3/2}\Big(\operatorname{Im} S_{\mathbf{v}\mathbf{v}} + \varphi^{C_{\zeta,D}}\Phi\Big)^n \big(1 + N^{m/2}|v_a|^{2m}\big) \\ &\leqslant\; \varphi^{-1}\Big(\operatorname{Im} S_{\mathbf{v}\mathbf{v}} + \varphi^{C_\zeta}\Phi\Big)^n \big(N^{-3/2} + |v_a|^2\big) \end{aligned}$$

with 2ζ-high probability, where we used that $1 \leqslant m \leqslant 3$. Therefore Lemma 3.10 yields

$$\mathbb{E}|A_{m,3}|(\operatorname{Im} R_{\mathbf{v}\mathbf{v}})^{n-m} \;\leqslant\; C\varphi^{-1}\Big(\mathbb{E}\big(\operatorname{Im} S_{\mathbf{v}\mathbf{v}}\big)^n + (\varphi^{C_\zeta}\Phi)^n\Big)\big(N^{-3/2} + |v_a|^2\big)\,, \tag{4.10}$$

which is (4.8). In particular, we have also proved (4.7). Here we omit the details of the estimate on the event of low probability, which are analogous to those following (3.40).

Case (ii): $a \neq b$ **and** $m = 3$**.** By (4.9), we have $s = t = 3$. From (3.37) we get

$$\begin{aligned} \Big(|R_{\mathbf{v}a}| + |R_{a\mathbf{v}}|\Big)^s \Big(|R_{\mathbf{v}b}| + |R_{b\mathbf{v}}|\Big)^t \;\leqslant\; &\bigg(\varphi^{C_\zeta}\frac{\operatorname{Im} S_{\mathbf{v}\mathbf{v}}}{N\eta} + \varphi^{C_\zeta}\Psi^2\bigg)^m + \bigg(\varphi^{C_\zeta}\frac{\operatorname{Im} S_{\mathbf{v}\mathbf{v}}}{N\eta} + \varphi^{C_\zeta}\Psi^2\bigg)^{m-s/2}(C|v_a|)^s \\ &+ \bigg(\varphi^{C_\zeta}\frac{\operatorname{Im} S_{\mathbf{v}\mathbf{v}}}{N\eta} + \varphi^{C_\zeta}\Psi^2\bigg)^{m-t/2}(C|v_b|)^t + (C|v_a|)^s(C|v_b|)^t \end{aligned} \tag{4.11}$$

with 2ζ-high probability. Together with (3.4) and (3.39), this yields

$$\begin{aligned} &\Big(|R_{\mathbf{v}a}| + |R_{a\mathbf{v}}|\Big)^s \Big(|R_{\mathbf{v}b}| + |R_{b\mathbf{v}}|\Big)^t (\operatorname{Im} R_{\mathbf{v}\mathbf{v}})^{n-m} \;\leqslant\; \big(\operatorname{Im} S_{\mathbf{v}\mathbf{v}} + \varphi^{C_{\zeta,D}}\Phi\big)^n \\ &\times\Bigg[\bigg(\frac{\varphi^{C_\zeta}}{N\eta}\bigg)^m + \bigg(\frac{\varphi^{C_\zeta}}{N\eta}\bigg)^{s/2}\big(\varphi^D\Phi\big)^{-t/2}|v_b|^t + \bigg(\frac{\varphi^{C_\zeta}}{N\eta}\bigg)^{t/2}\big(\varphi^D\Phi\big)^{-s/2}|v_a|^s + \big(\varphi^D\Phi\big)^{-s/2-t/2}|v_a|^s|v_b|^t\Bigg] \end{aligned} \tag{4.12}$$

with 2ζ-high probability and for any $D > 0$. Choosing D and C_0 in (2.9) large enough, we get from (2.1), (4.6), Lemma 3.10, and $N^{-1/2} \leqslant \Phi$ that

$$|A_{3,3}|(\operatorname{Im} R_{\mathbf{v}\mathbf{v}})^{n-3} \;\leqslant\; \varphi^{-1}N^{-3/2}\Big((\operatorname{Im} S_{\mathbf{v}\mathbf{v}})^n + (\varphi^{C_\zeta}\Phi)^n\Big)\Big(N^{-1/2} + N^{1/2}|v_b|^2 + N^{1/2}|v_a|^2 + N^{3/2}|v_a|^2|v_b|^2\Big)$$

with 2ζ-high probability. Now (4.8), and hence also (4.7), follows easily (we omit the details of the analysis on the low-probability event).

Case (iii): $a \neq b$ **and** $m = 2$**.** Consider first the case $s = t = 2$. Then $A_{2,3,2,2}$ (see (3.36) and (3.31)) is a finite sum of $O(1)$ terms of the form

$$X_1 \;:=\; R_{\mathbf{v}a}h_{ab}R_{b\mathbf{v}}\,R_{\mathbf{v}a}h_{ab}R_{ba}h_{ab}R_{b\mathbf{v}}\,. \tag{4.13}$$

(The other terms can be obtained from (4.13) by permutation of indices and complex conjugation of factors.) We shall estimate the contribution of X_1; the other terms are dealt with in exactly the same way. Note the presence of an off-diagonal resolvent matrix element R_{ba}, as required by the condition $s = t = 2$. From (3.27) and (4.12) we get, with $m = s = t = 2$, that

$$|X_1| \, (\operatorname{Im} R_{\mathbf{v}\mathbf{v}})^{n-2} \;\leqslant\; \varphi^{C_\zeta} \, \Psi \, N^{-3/2} \bigl(\operatorname{Im} S_{\mathbf{v}\mathbf{v}} + \varphi^{C_{\zeta,D}} \Phi\bigr)^n \\ \times \Biggl[\biggl(\frac{\varphi^{C_\zeta}}{N\eta}\biggr)^2 + \frac{\varphi^{C_\zeta}}{N\eta} \bigl(\varphi^D \Phi\bigr)^{-1} |v_b|^2 + \frac{\varphi^{C_\zeta}}{N\eta} \bigl(\varphi^D \Phi\bigr)^{-1} |v_a|^2 + \bigl(\varphi^D \Phi\bigr)^{-2} |v_a|^2 |v_b|^2 \Biggr]$$

with 2ζ-high probability. Note the factor Ψ arising from the estimate of R_{ba}. Choosing D and C_0 large enough, and recalling (2.9), we find using Lemma 3.10 that

$$|X_1| \, (\operatorname{Im} R_{\mathbf{v}\mathbf{v}})^{n-2} \;\leqslant\; \varphi^{-1} \Bigl((\operatorname{Im} S_{\mathbf{v}\mathbf{v}})^n + (\varphi^{C_\zeta} \Phi)^n \Bigr) \mathcal{E}_{ab}$$

with 2ζ-high probability. This yields (4.8) and hence also (4.7).

Let us therefore consider the case $s = 3$ and $t = 1$. (The case $s = 1$ and $t = 3$ is estimated in the same way.) Using the bounds $\Phi \geqslant (N\eta)^{-1}$ and $\Phi \geqslant N^{-1/2}$, we find

$$\begin{aligned} &|A_{2,3,3,1}| \, (\operatorname{Im} R_{\mathbf{v}\mathbf{v}})^{n-2} \qquad (4.14) \\ &\quad\leqslant\; \varphi^{C_\zeta} \, N^{-3/2} \bigl(\operatorname{Im} S_{\mathbf{v}\mathbf{v}} + \varphi^{C_{\zeta,D}} \Phi\bigr)^n \\ &\qquad \times \Biggl[\biggl(\frac{\varphi^{C_\zeta}}{N\eta}\biggr)^2 + \biggl(\frac{\varphi^{C_\zeta}}{N\eta}\biggr)^{3/2} \bigl(\varphi^D \Phi\bigr)^{-1/2} |v_b| + \biggl(\frac{\varphi^{C_\zeta}}{N\eta}\biggr)^{1/2} \bigl(\varphi^D \Phi\bigr)^{-3/2} |v_a|^2 + \bigl(\varphi^D \Phi\bigr)^{-2} |v_a|^2 |v_b| \Biggr] \\ &\quad\leqslant\; \varphi^{-1} \Bigl((\operatorname{Im} S_{\mathbf{v}\mathbf{v}})^n + (\varphi^{C_\zeta} \Phi)^n \Bigr) \Biggl[N^{-3/2} \frac{\varphi^{C_\zeta}}{N\eta} + N^{-3/2} |v_b| + N^{-1} |v_a|^2 + N^{-1/2} |v_a|^2 |v_b| \Biggr] \qquad (4.15) \end{aligned}$$

with 2ζ-high probability, for D and C_0 large enough. This yields (4.7) in the case $s = 3$ and $t = 1$.

In order to prove the stronger bound (4.8) in the case $s = 3$ and $t = 1$, we note that (3.29), (3.4), (3.5), and the assumption (4.4) yield

$$|R_{\mathbf{v}a}| \;\leqslant\; \varphi^{C_\zeta} \sqrt{\frac{\operatorname{Im} S_{\mathbf{v}\mathbf{v}} + \Phi}{N\eta}} \,. \qquad (4.16)$$

The same bound holds for $R_{a\mathbf{v}}$, $R_{\mathbf{v}b}$, and $R_{b\mathbf{v}}$. Now $A_{2,3,3,1}$ is a finite sum of $O(1)$ terms of the form

$$X_2 \;:=\; R_{\mathbf{v}a} h_{ab} R_{b\mathbf{v}} \, R_{\mathbf{v}a} h_{ab} R_{bb} h_{ba} R_{a\mathbf{v}} \,.$$

(Again, the other terms can be obtained from X_2 by permutation of indices and complex conjugation of factors.) We shall show that

$$\bigl| \mathbb{E} X_2 (\operatorname{Im} R_{\mathbf{v}\mathbf{v}})^{n-2} \bigr| \;\leqslant\; C \varphi^{-1} \mathcal{E}_{ab} \Bigl(\mathbb{E} (\operatorname{Im} S_{\mathbf{v}\mathbf{v}})^n + (\varphi^{C_\zeta} \Phi)^n \Bigr) \,. \qquad (4.17)$$

We split $R_{bb} = (R_{bb} - m) + m$ in the definition of X_2. The first resulting term is estimated, using (3.27), by

$$\varphi^{C_\zeta} \, \Psi \, N^{-3/2} \bigl| R_{\mathbf{v}a} R_{b\mathbf{v}} R_{\mathbf{v}a} R_{a\mathbf{v}} \bigr| (\operatorname{Im} R_{\mathbf{v}\mathbf{v}})^{n-2} \,.$$

The estimate of $|X_1| (\operatorname{Im} S_{\mathbf{v}\mathbf{v}})^{n-2}$ above may now be applied verbatim. What remains is the second term resulting from the above splitting of X_2. Since $|m| \leqslant C$ and h_{ab} is independent of R, we therefore have to show that

$$C N^{-3/2} \Bigl| \mathbb{E} R_{\mathbf{v}a} R_{b\mathbf{v}} R_{\mathbf{v}a} R_{a\mathbf{v}} \, (\operatorname{Im} R_{\mathbf{v}\mathbf{v}})^{n-2} \Bigr| \;\leqslant\; \varphi^{-1} \mathcal{E}_{ab} \Bigl(\mathbb{E} (\operatorname{Im} S_{\mathbf{v}\mathbf{v}})^n + (\varphi^{C_\zeta} \Phi)^n \Bigr) \,. \qquad (4.18)$$

Using (3.7), we expand

$$R_{b\mathbf{v}} \;=\; m\mathcal{R}_{b\mathbf{v}} + \mathcal{R}'_{b\mathbf{v}}\,, \tag{4.19}$$

where we defined (see also (3.9))

$$\mathcal{R}_{b\mathbf{v}} \;:=\; -\sum_k^{(b)} h_{bk} R^{(b)}_{k\mathbf{v}}\,, \qquad \mathcal{R}'_{b\mathbf{v}} \;:=\; v_b R_{bb} + (R_{bb} - m)\mathcal{R}_{b\mathbf{v}}\,. \tag{4.20}$$

Now we observe that, using the bound (3.27), we may repeat the proof of Lemma 3.8 to the letter to find that its statement holds with $(G, \mathcal{G})$ replaced with $(R, \mathcal{R})$. Thus we find

$$|\mathcal{R}_{b\mathbf{v}}| \;\leqslant\; \varphi^{C_\zeta}\sqrt{\frac{\operatorname{Im} R_{\mathbf{v}\mathbf{v}}}{N\eta}} + C|v_b| \;\leqslant\; \varphi^{C_\zeta}\sqrt{\frac{\operatorname{Im} S_{\mathbf{v}\mathbf{v}} + \Phi}{N\eta}} + \varphi^{C_\zeta} N^{-1/4}(N\eta)^{-1/2} \;\leqslant\; \varphi^{C_\zeta}\sqrt{\frac{\operatorname{Im} S_{\mathbf{v}\mathbf{v}} + \Phi}{N\eta}} \tag{4.21}$$

with 2ζ-high probability, where in the second step we used (3.39) and (4.4), and in the last step (3.5). Using (3.27), (4.4), and $\Phi \geqslant (N\eta)^{-1}$, we therefore find

$$|\mathcal{R}'_{b\mathbf{v}}| \;\leqslant\; \biggl(\varphi^{C_\zeta}\frac{\Psi}{\sqrt{N\eta}} + \varphi^{-D} N^{-1/4}\biggr)(\operatorname{Im} S_{\mathbf{v}\mathbf{v}} + \varphi^{C_{\zeta,D}}\Phi)^{1/2} \tag{4.22}$$

with 2ζ-high probability, for any $D \geqslant 0$. Therefore (3.39) and (4.16) yield

$$CN^{-3/2}\Bigl|\mathbb{E} R_{\mathbf{v}a}\mathcal{R}'_{b\mathbf{v}} R_{\mathbf{v}a} R_{a\mathbf{v}}\,(\operatorname{Im} R_{\mathbf{v}\mathbf{v}})^{n-2}\Bigr| \;\leqslant\; N^{-3/2}\varphi^{C_\zeta}(N\eta)^{-3/2}\biggl(\frac{\Psi}{\sqrt{N\eta}} + N^{-1/4}\biggr)\Bigl(\operatorname{Im} S_{\mathbf{v}\mathbf{v}} + \varphi^{C_\zeta}\Phi\Bigr)^n$$

with 2ζ-high probability. Using (2.9), (4.6), and Lemma 3.10, we find that the right-hand side is bounded by

$$\varphi^{-1} N^{-2}\Bigl((\operatorname{Im} S_{\mathbf{v}\mathbf{v}})^n + (\varphi^{C_\zeta}\Phi)^n\Bigr)$$

with 2ζ-high probability. Combined with the usual estimate on the complementary low-probability event, this concludes the estimate of the $\mathcal{R}'_{b\mathbf{v}}$-term. What remains is to prove that

$$CN^{-3/2}\Bigl|\mathbb{E} R_{\mathbf{v}a}\mathcal{R}_{b\mathbf{v}} R_{\mathbf{v}a} R_{a\mathbf{v}}\,(\operatorname{Im} R_{\mathbf{v}\mathbf{v}})^{n-2}\Bigr| \;\leqslant\; \varphi^{-1}\mathcal{E}_{ab}\Bigl(\mathbb{E}(\operatorname{Im} S_{\mathbf{v}\mathbf{v}})^n + (\varphi^{C_\zeta}\Phi)^n\Bigr)\,, \tag{4.23}$$

The key observation behind the estimate of (4.23) is that $\mathbb{E}_b \mathcal{R}_{b\mathbf{v}} = 0$, where $\mathbb{E}_b$ denotes partial expectation with respect to the b-th column of Q. Thus we have

$$\mathbb{E} R_{\mathbf{v}a}\mathcal{R}_{b\mathbf{v}} R_{\mathbf{v}a} R_{a\mathbf{v}}\,(\operatorname{Im} R_{\mathbf{v}\mathbf{v}})^{n-2} \;=\; \mathbb{E}\Bigl[R_{\mathbf{v}a} R_{\mathbf{v}a} R_{a\mathbf{v}}\,(\operatorname{Im} R_{\mathbf{v}\mathbf{v}})^{n-2} - R^{(b)}_{\mathbf{v}a} R^{(b)}_{\mathbf{v}a} R^{(b)}_{a\mathbf{v}}\,(\operatorname{Im} R^{(b)}_{\mathbf{v}\mathbf{v}})^{n-2}\Bigr]\mathcal{R}_{b\mathbf{v}}\,.$$

In order to compare the quantities in the brackets, we use (3.6), (3.27), and (4.16) to get

$$R_{\mathbf{v}a} \;=\; R^{(b)}_{\mathbf{v}a} + \frac{R_{\mathbf{v}b} R_{ba}}{R_{bb}} \;=\; R^{(b)}_{\mathbf{v}a} + O(\varphi^{C_\zeta}\Psi R_{\mathbf{v}b})\,, \tag{4.24}$$

$$R_{\mathbf{v}\mathbf{v}} \;=\; R^{(b)}_{\mathbf{v}\mathbf{v}} + \frac{R_{\mathbf{v}b} R_{b\mathbf{v}}}{R_{bb}} \;=\; R^{(b)}_{\mathbf{v}\mathbf{v}} + O\biggl(\varphi^{C_\zeta}\,\frac{\operatorname{Im} S_{\mathbf{v}\mathbf{v}} + \Phi}{N\eta}\biggr) \tag{4.25}$$

with 2ζ-high probability. In particular, we get from (3.39) and (4.16) that

$$\operatorname{Im} R^{(b)}_{\mathbf{v}\mathbf{v}} \;\leqslant\; (1+\varphi^{-\zeta})\operatorname{Im} S_{\mathbf{v}\mathbf{v}} + \varphi^{C_\zeta}\Phi\,, \qquad |R^{(b)}_{\mathbf{v}a}| \;\leqslant\; \varphi^{C_\zeta}\sqrt{\frac{\operatorname{Im} S_{\mathbf{v}\mathbf{v}}+\Phi}{N\eta}} \tag{4.26}$$

with 2ζ-high probability, for $z \in \mathbf{S}(C'_\zeta)$ with some large enough C'_ζ. A telescopic estimate of the form

$$\prod_{i=1}^k (x_i+y_i) - \prod_{i=1}^k x_i \;=\; \sum_{j=1}^k \Biggl(\prod_{i=1}^{j-1} x_i\Biggr) y_j \Biggl(\prod_{i=j+1}^k (x_i+y_i)\Biggr)$$

therefore gives

$$\begin{aligned}
&CN^{-3/2}\,\Bigl|R_{\mathbf{v}a}R_{\mathbf{v}a}R_{a\mathbf{v}}\,(\operatorname{Im} R_{\mathbf{v}\mathbf{v}})^{n-2} - R^{(b)}_{\mathbf{v}a}R^{(b)}_{\mathbf{v}a}R^{(b)}_{a\mathbf{v}}\,(\operatorname{Im} R^{(b)}_{\mathbf{v}\mathbf{v}})^{n-2}\Bigr|\,|\mathcal{R}_{b\mathbf{v}}| \\
&\quad\leqslant\; \varphi^{C_\zeta}N^{-3/2}|\mathcal{R}_{b\mathbf{v}}|\biggl(\frac{\operatorname{Im} S_{\mathbf{v}\mathbf{v}}+\Phi}{N\eta}\biggr)^{3/2}\Psi\,\bigl(\operatorname{Im} S_{\mathbf{v}\mathbf{v}} + \varphi^{C_\zeta}\Phi\bigr)^{n-2} \\
&\qquad + n\varphi^{C_\zeta}N^{-3/2}|\mathcal{R}_{b\mathbf{v}}|\biggl(\frac{\operatorname{Im} S_{\mathbf{v}\mathbf{v}}+\Phi}{N\eta}\biggr)^{5/2}\bigl(\operatorname{Im} S_{\mathbf{v}\mathbf{v}} + \varphi^{C_\zeta}\Phi\bigr)^{n-3} \\
&\quad\leqslant\; \varphi^{C_\zeta}N^{-3/2}\sqrt{\frac{\operatorname{Im} S_{\mathbf{v}\mathbf{v}}+\Phi}{N\eta}}\,\biggl(\frac{\Psi}{(N\eta)^{3/2}} + \frac{1}{(N\eta)^{5/2}}\biggr)\bigl(\operatorname{Im} S_{\mathbf{v}\mathbf{v}} + \varphi^{C_\zeta}\Phi\bigr)^{n-1/2}
\end{aligned}$$

with 2ζ-high probability, where in the last step we used (4.21) and $n \leqslant \varphi^\zeta$. Now (4.23) follows easily for large enough C_0 in (2.9), using (2.9) and (4.6). This concludes the proof of (4.18) and hence of (4.17).

Case (iv): $a \neq b$ and $m = 1$. Similarly to (4.15), one easily finds the weak bound (4.7). Let us therefore assume (4.4) and prove (4.8). It suffices to prove that

$$N^{-3/2}\bigl|\mathbb{E}\,X_3(\operatorname{Im} R_{\mathbf{v}\mathbf{v}})^{n-1}\bigr| \;\leqslant\; \varphi^{-1}N^{-2}\Bigl(\mathbb{E}\bigl(\operatorname{Im} S_{\mathbf{v}\mathbf{v}}\bigr)^n + (\varphi^{C_\zeta}\Phi)^n\Bigr)\,, \tag{4.27}$$

where X_3 stands for any of the following expressions:

$$R_{\mathbf{v}a}R_{ba}R_{ba}R_{b\mathbf{v}}\,, \qquad R_{\mathbf{v}a}R_{bb}R_{ab}R_{a\mathbf{v}}\,, \qquad R_{\mathbf{v}a}R_{bb}R_{aa}R_{b\mathbf{v}}\,.$$

Here we used that h_{ab} and h_{ba} are independent of R. (Up to an immaterial renaming of indices and complex conjugation, all terms in $A_{1,3}$ are covered by one of these three cases.) Applying the splittings $R_{aa} = m + (R_{aa}-m)$ and $R_{bb} = m + (R_{bb}-m)$, we find that it suffices to prove (4.27) for X_3 being any of

$$\begin{gathered}
R_{\mathbf{v}a}R_{ba}R_{ba}R_{b\mathbf{v}}\,, \qquad R_{\mathbf{v}a}(R_{bb}-m)R_{ab}R_{a\mathbf{v}}\,, \qquad R_{\mathbf{v}a}(R_{bb}-m)(R_{aa}-m)R_{b\mathbf{v}}\,, \\
R_{\mathbf{v}a}R_{ab}R_{a\mathbf{v}}\,, \qquad R_{\mathbf{v}a}(R_{bb}-m)R_{b\mathbf{v}}\,, \qquad R_{\mathbf{v}a}(R_{aa}-m)R_{b\mathbf{v}}\,, \\
R_{\mathbf{v}a}R_{b\mathbf{v}}\,.
\end{gathered}$$

Next, applying the splitting (4.19) to the last line, we find that it suffices to prove (4.27) for X_3 being any of

$$R_{\mathbf{v}a}R_{ba}R_{ba}R_{b\mathbf{v}}\,, \qquad R_{\mathbf{v}a}(R_{bb}-m)R_{ab}R_{a\mathbf{v}}\,, \qquad R_{\mathbf{v}a}(R_{bb}-m)(R_{aa}-m)R_{b\mathbf{v}}\,, \qquad \mathcal{R}'_{\mathbf{v}a}\mathcal{R}'_{b\mathbf{v}}\,, \tag{4.28a}$$

$$R_{\mathbf{v}a}R_{ab}R_{a\mathbf{v}}\,, \qquad R_{\mathbf{v}a}(R_{bb}-m)R_{b\mathbf{v}}\,, \qquad R_{\mathbf{v}a}(R_{aa}-m)R_{b\mathbf{v}}\,, \qquad \mathcal{R}'_{\mathbf{v}a}\mathcal{R}_{b\mathbf{v}}\,, \qquad \mathcal{R}_{\mathbf{v}a}\mathcal{R}'_{b\mathbf{v}}\,, \tag{4.28b}$$

$$\mathcal{R}_{\mathbf{v}a}\mathcal{R}_{b\mathbf{v}}\,. \tag{4.28c}$$

For X_3 in (4.28a), we find from (3.27), (4.16), and (4.22) that

$$|X_3| \leqslant \varphi^{C_\zeta}\biggl(\frac{\Psi^2}{N\eta} + \varphi^{-D}N^{-1/2}\biggr)\bigl(\operatorname{Im} S_{\mathbf{v}\mathbf{v}} + \varphi^{C_{\zeta,D}}\Phi\bigr)$$

with 2ζ-high probability, from which (4.27) easily follows using (2.9), (4.6), (3.39), and Lemma 3.10, having chosen D and C_0 in (2.9) large enough.

Let us now consider $X_3 = R_{\mathbf{v}a}R_{ab}R_{a\mathbf{v}}$. Using (3.7), we split, similarly to (4.19),

$$R_{ab} = m\mathcal{R}_{ab} + (R_{bb}-m)\mathcal{R}_{ab}\,, \qquad \mathcal{R}_{ab} := -\sum_k^{(b)} R_{ak}^{(b)} h_{kb}\,.$$

Using (3.12), (3.4), (3.6), and (3.27), we find

$$|\mathcal{R}_{ab}| \leqslant \varphi^{C_\zeta}\biggl(\frac{1}{N}\sum_k^{(b)}|R_{ak}^{(b)}|^2\biggr)^{1/2} = \varphi^{C_\zeta}\biggl(\frac{1}{N\eta}\operatorname{Im} R_{aa}^{(b)}\biggr)^{1/2} \leqslant \varphi^{C_\zeta}\biggl(\frac{\operatorname{Im} m + \Psi}{N\eta}\biggr)^{1/2} \leqslant \varphi^{C_\zeta}\Psi \tag{4.29}$$

with 2ζ-high probability. For the second part of X_3 resulting from the splitting of R_{ab}, we therefore get the estimate

$$\bigl|R_{\mathbf{v}a}(R_{bb}-m)\mathcal{R}_{ab}R_{a\mathbf{v}}\bigr| \leqslant \varphi^{C_\zeta}\frac{\Psi^2}{N\eta}\bigl(\operatorname{Im} S_{\mathbf{v}\mathbf{v}} + \varphi^{C_\zeta}\Phi\bigr) \leqslant \varphi^{-1}N^{-1/2}\bigl(\operatorname{Im} S_{\mathbf{v}\mathbf{v}} + \varphi^{C_\zeta}\Phi\bigr)$$

with 2ζ-high probability. For the first part, we use $\mathbb{E}_b\mathcal{R}_{ab}$ to write

$$\mathbb{E}R_{\mathbf{v}a}\mathcal{R}_{ab}R_{a\mathbf{v}}(\operatorname{Im} R_{\mathbf{v}\mathbf{v}})^{n-1} = \mathbb{E}\Bigl[R_{\mathbf{v}a}R_{a\mathbf{v}}(\operatorname{Im} R_{\mathbf{v}\mathbf{v}})^{n-1} - R_{\mathbf{v}a}^{(b)}R_{a\mathbf{v}}^{(b)}(\operatorname{Im} R_{\mathbf{v}\mathbf{v}}^{(b)})^{n-1}\Bigr]\mathcal{R}_{ab}\,.$$

This may be estimated using a telescopic sum, exactly as (4.8); we omit the details. This completes the proof of (4.27) in the case $X_3 = R_{\mathbf{v}a}R_{ab}R_{a\mathbf{v}}$. The second and third terms of (4.28b) are estimated similarly.

For the choice $X_3 = \mathcal{R}_{\mathbf{v}a}\mathcal{R}'_{b\mathbf{v}}$, we use $\mathbb{E}_a\mathcal{R}_{\mathbf{v}a} = 0$ to write

$$\mathbb{E}\mathcal{R}_{\mathbf{v}a}\mathcal{R}'_{b\mathbf{v}}(\operatorname{Im} R_{\mathbf{v}\mathbf{v}})^{n-1} = \mathbb{E}\Bigl[\mathcal{R}'_{b\mathbf{v}}(\operatorname{Im} R_{\mathbf{v}\mathbf{v}})^{n-1} - \bigl(\mathcal{R}'_{b\mathbf{v}}\bigr)^{(a)}(\operatorname{Im} R_{\mathbf{v}\mathbf{v}}^{(a)})^{n-1}\Bigr]\mathcal{R}_{\mathbf{v}a}\,, \tag{4.30}$$

where we defined

$$\bigl(\mathcal{R}'_{b\mathbf{v}}\bigr)^{(a)} := v_b R_{bb}^{(a)} + (R_{bb}^{(a)} - m)\mathcal{R}_{b\mathbf{v}}^{(a)}\,, \qquad \mathcal{R}_{b\mathbf{v}}^{(a)} := -\sum_k^{(ab)} h_{bk}R_{k\mathbf{v}}^{(ab)}\,.$$

We find

$$\begin{aligned}\bigl|\mathcal{R}_{b\mathbf{v}}^{(a)} - \mathcal{R}_{b\mathbf{v}}\bigr| &\leqslant |h_{ba}||R_{a\mathbf{v}}^{(b)}| + \biggl|\sum_k^{(ab)} h_{bk}\bigl(R_{k\mathbf{v}}^{(b)} - R_{k\mathbf{v}}^{(ab)}\bigr)\biggr| \\ &\leqslant \varphi^{C_\zeta}N^{-1/2}\sqrt{\frac{\operatorname{Im} S_{\mathbf{v}\mathbf{v}} + \Phi}{N\eta}} + \varphi^{C_\zeta}|R_{a\mathbf{v}}^{(b)}|\biggl(\frac{1}{N}\sum_k |R_{ka}^{(b)}|^2\biggr)^{1/2} \leqslant \varphi^{C_\zeta}\frac{\Psi}{\sqrt{N\eta}}\bigl(\operatorname{Im} S_{\mathbf{v}\mathbf{v}} + \Phi\bigr)^{1/2}\end{aligned} \tag{4.31}$$

with 2ζ-high probability, where we used (4.26), (2.1), (3.12), (3.6), and (4.24). Together with (3.6), (3.27), (4.16), and (4.4), we therefore find

$$\begin{aligned}\big|(\mathcal{R}'_{b\mathbf{v}})^{(a)} - \mathcal{R}'_{b\mathbf{v}}\big| \;&\leqslant\; \big(|v_b| + |\mathcal{R}_{b\mathbf{v}}|\big)|R_{bb}^{(a)} - R_{bb}| + |R_{bb}^{(b)} - m|\varphi^{C_\zeta}\frac{\Psi}{\sqrt{N\eta}}\big(\operatorname{Im} S_{\mathbf{v}\mathbf{v}} + \Phi\big)^{1/2} \\ &\leqslant\; \varphi^{C_\zeta}\frac{\Psi^2}{\sqrt{N\eta}}\big(\operatorname{Im} S_{\mathbf{v}\mathbf{v}} + \Phi\big)^{1/2} \end{aligned} \tag{4.32}$$

with 2ζ-high probability. Recalling (4.21), (4.25), Lemma 3.10, and the usual rough estimate on the complementary low-probability event, a telescopic estimate in (4.30) therefore gives

$$\mathbb{E}\mathcal{R}_{\mathbf{v}a}\mathcal{R}'_{b\mathbf{v}}(\operatorname{Im} R_{\mathbf{v}\mathbf{v}})^{n-1} \;\leqslant\; \varphi^{C_\zeta}\left(\frac{\Psi^2}{N\eta} + \frac{\Psi}{(N\eta)^2}\right)\Big(\mathbb{E}(\operatorname{Im} S_{\mathbf{v}\mathbf{v}})^n + \big(\varphi^{C_\zeta}\Phi\big)^n\Big)\,.$$

Now (4.27) follows.

Now we prove (4.27) for X_3 as in (4.28c). We begin with a graded expansion of $R_{\mathbf{v}\mathbf{v}}$. Using (3.8) we find

$$R_{\mathbf{v}\mathbf{v}} \;=\; R_{\mathbf{v}\mathbf{v}}^{(a)} + \frac{R_{\mathbf{v}a}R_{a\mathbf{v}}}{R_{aa}} \;=\; R_{\mathbf{v}\mathbf{v}}^{(ab)} + \frac{R_{\mathbf{v}b}^{(a)}R_{b\mathbf{v}}^{(a)}}{R_{bb}^{(a)}} + \frac{R_{\mathbf{v}a}R_{a\mathbf{v}}}{R_{aa}}\,.$$

We deal with the last term by applying (3.6) twice, followed by

$$\frac{1}{R_{aa}} \;=\; \frac{1}{R_{aa}^{(b)}} - \frac{R_{ab}R_{ba}}{R_{aa}R_{bb}R_{aa}^{(b)}}\,,$$

itself an immediate consequence of (3.6). This gives the graded expansion

$$R_{\mathbf{v}\mathbf{v}} \;=\; R_{\mathbf{v}\mathbf{v}}^{[ab]} + R_{\mathbf{v}\mathbf{v}}^{[a]} + R_{\mathbf{v}\mathbf{v}}^{[b]} + R_{\mathbf{v}\mathbf{v}}^{[\emptyset]}\,,$$

where

$$R_{\mathbf{v}\mathbf{v}}^{[ab]} \;:=\; R_{\mathbf{v}\mathbf{v}}^{(ab)}\,, \qquad R_{\mathbf{v}\mathbf{v}}^{[a]} \;:=\; \frac{R_{\mathbf{v}b}^{(a)}R_{b\mathbf{v}}^{(a)}}{R_{bb}^{(a)}}\,, \qquad R_{\mathbf{v}\mathbf{v}}^{[b]} \;:=\; \frac{R_{\mathbf{v}a}^{(b)}R_{a\mathbf{v}}^{(b)}}{R_{aa}^{(b)}}$$
$$R_{\mathbf{v}\mathbf{v}}^{[\emptyset]} \;:=\; \frac{R_{\mathbf{v}b}R_{ba}R_{a\mathbf{v}}}{R_{aa}R_{bb}} + \frac{R_{\mathbf{v}a}^{(b)}R_{ab}R_{b\mathbf{v}}}{R_{aa}R_{bb}} - \frac{R_{\mathbf{v}a}^{(b)}R_{a\mathbf{v}}^{(b)}R_{ab}R_{ba}}{R_{aa}R_{bb}R_{aa}^{(b)}}\,.$$

Note that $R_{\mathbf{v}}^{[\mathbb{T}]}$ is independent of the columns of H indexed by $\mathbb{T}$. Moreover, by (3.27), Lemma 3.2, (3.6), (4.16), and (4.26), we have

$$\big|R_{\mathbf{v}\mathbf{v}}^{[a]}\big| + \big|R_{\mathbf{v}\mathbf{v}}^{[b]}\big| \;\leqslant\; \varphi^{C_\zeta}\frac{\operatorname{Im} S_{\mathbf{v}\mathbf{v}} + \Phi}{N\eta}\,, \qquad \big|R_{\mathbf{v}\mathbf{v}}^{[\emptyset]}\big| \;\leqslant\; \varphi^{C_\zeta}\Psi\frac{\operatorname{Im} S_{\mathbf{v}\mathbf{v}} + \Phi}{N\eta} \tag{4.33}$$

with 2ζ-high probability. Thus we write

$$\mathbb{E}\,\mathcal{R}_{\mathbf{v}a}\mathcal{R}_{b\mathbf{v}}(\operatorname{Im} R_{\mathbf{v}\mathbf{v}})^{n-1} \;=\; \sum_{\mathbf{A}}\mathbb{E}\,\mathcal{R}_{\mathbf{v}a}\mathcal{R}_{b\mathbf{v}}\prod_{i=1}^{n-1}\operatorname{Im} R_{\mathbf{v}\mathbf{v}}^{[A_i]} \tag{4.34}$$

where $\mathbf{A} = (A_i)_{i=1}^{n-1}$ and $A_i \in \{\emptyset, a, b, ab\}$ for $i = 1, \dots, n-1$. In order to keep track of the terms in the summation over $\mathbf{A}$, we introduce the counting functions

$$r_1(\mathbf{A}) \;:=\; \sum_{i=1}^{n-1} \big(\mathbf{1}(A_i = a) + \mathbf{1}(A_i = b)\big)\,, \qquad r_2(\mathbf{A}) \;:=\; \sum_{i=1}^{n-1} \mathbf{1}(A_i = \emptyset)\,.$$

We partition the sum in (4.34) as

$$\sum_{\mathbf{A}} \;=\; \sum_{\mathbf{A}} \mathbf{1}\big(r_2(\mathbf{A}) = 0\big)\mathbf{1}\big(r_1(\mathbf{A}) = 0\big) + \sum_{\mathbf{A}} \mathbf{1}\big(r_2(\mathbf{A}) = 0\big)\mathbf{1}\big(r_1(\mathbf{A}) = 1\big) + \sum_{\mathbf{A}} \mathbf{1}\big(r_2(\mathbf{A}) \geqslant 1 \text{ or } r_1(\mathbf{A}) \geqslant 2\big)\,. \tag{4.35}$$

Let us concentrate on the first summand; its condition is equivalent to $A_i = ab$ for all i. Using $\mathbb{E}_a \mathcal{R}_{\mathbf{v}a} = 0$ and $\mathbb{E}_b\big(\mathcal{R}_{b\mathbf{v}} - \mathcal{R}_{b\mathbf{v}}^{(a)}\big) = 0$ we get

$$\mathbb{E}\, \mathcal{R}_{\mathbf{v}a}\mathcal{R}_{b\mathbf{v}} (\operatorname{Im} R_{\mathbf{v}\mathbf{v}}^{[ab]})^{n-1} \;=\; \mathbb{E}\,\big(\mathcal{R}_{\mathbf{v}a} - \mathcal{R}_{\mathbf{v}a}^{(b)}\big)\big(\mathcal{R}_{b\mathbf{v}} - \mathcal{R}_{b\mathbf{v}}^{(a)}\big) (\operatorname{Im} R_{\mathbf{v}\mathbf{v}}^{[ab]})^{n-1}\,.$$

From (4.31), (4.33), (3.39), and Lemma 3.10 we therefore get

$$\big|\mathbb{E}\, \mathcal{R}_{\mathbf{v}a}\mathcal{R}_{b\mathbf{v}} (\operatorname{Im} R_{\mathbf{v}\mathbf{v}}^{[ab]})^{n-1}\big| \;\leqslant\; \varphi^{C_\zeta} \frac{\Psi^2}{N\eta} \big(\mathbb{E}(\operatorname{Im} S_{\mathbf{v}\mathbf{v}})^n + (\varphi^{C_\zeta}\Phi)^n\big) \;\leqslant\; \varphi^{-1} N^{-1/2} \big(\mathbb{E}(\operatorname{Im} S_{\mathbf{v}\mathbf{v}})^n + (\varphi^{C_\zeta}\Phi)^n\big)$$

for large enough C_0.

The second summand of (4.35) consists of n terms of the form

$$\mathbb{E}\, \mathcal{R}_{\mathbf{v}a}\mathcal{R}_{b\mathbf{v}} (\operatorname{Im} R_{\mathbf{v}\mathbf{v}}^{[a]})(\operatorname{Im} R_{\mathbf{v}\mathbf{v}}^{[ab]})^{n-2} \;=\; \mathbb{E}\, \mathcal{R}_{\mathbf{v}a}\big(\mathcal{R}_{b\mathbf{v}} - \mathcal{R}_{b\mathbf{v}}^{(a)}\big) (\operatorname{Im} R_{\mathbf{v}\mathbf{v}}^{[a]})(\operatorname{Im} R_{\mathbf{v}\mathbf{v}}^{[ab]})^{n-2}\,.$$

Recalling (4.33), we estimate this as above by

$$\varphi^{C_\zeta} \frac{\Psi}{(N\eta)^2} \big(\mathbb{E}(\operatorname{Im} S_{\mathbf{v}\mathbf{v}})^n + (\varphi^{C_\zeta}\Phi)^n\big) \;\leqslant\; \varphi^{-1} N^{-1/2} \big(\mathbb{E}(\operatorname{Im} S_{\mathbf{v}\mathbf{v}})^n + (\varphi^{C_\zeta}\Phi)^n\big)$$

for large enough C_0.

What remains is to estimate the third summand in (4.35). From (4.33) and (4.31) we get

$$\begin{aligned} &\sum_{\mathbf{A}} \mathbf{1}\big(r_2(\mathbf{A}) \geqslant 1 \text{ or } r_1(\mathbf{A}) \geqslant 2\big) \big|\mathcal{R}_{\mathbf{v}a}\mathcal{R}_{b\mathbf{v}}\big| \prod_{i=1}^{n-1} \big|\operatorname{Im} R_{\mathbf{v}\mathbf{v}}^{[A_i]}\big| \\ &\qquad\qquad \leqslant\; \varphi^{C_\zeta} \left(\frac{1}{(N\eta)^3} + \frac{\Psi}{(N\eta)^2}\right) \big(\operatorname{Im} S_{\mathbf{v}\mathbf{v}} + \varphi^{C_\zeta}\Phi\big)^n \;\leqslant\; \varphi^{-1} N^{-1/2} \big((\operatorname{Im} S_{\mathbf{v}\mathbf{v}})^n + (\varphi^{C_\zeta}\Phi)^n\big) \end{aligned}$$

with 2ζ-high probability. This completes the proof of (4.27) for $X_3 = \mathcal{R}_{\mathbf{v}a}\mathcal{R}_{b\mathbf{v}}$. □

4.2. Estimate of $G_{\mathbf{vv}} - m$. We now conclude the proof of Proposition 4.1. By polarization and linearity, it is enough to prove the following result.

LEMMA 4.5. *Fix $\zeta > 0$. Then there are constants C_0 and C_ζ, both depending on ζ, such that the following holds. Assume that $z \in \mathbf{S}(C_\zeta)$ satisfies (2.9) with constant C_0. Then we have, for all $n \leqslant \varphi^\zeta$ and all deterministic and normalized $\mathbf{v} \in \mathbb{C}^N$, that*

$$\mathbb{E}\big|G_{\mathbf{vv}}(z) - m(z)\big|^n \;\leqslant\; \big(\varphi^{C_\zeta}\Psi(z)\big)^n\,. \tag{4.36}$$

PROOF. As in the proof of Lemma 3.11, we focus on $\operatorname{Re} G_{\mathbf{vv}} - \operatorname{Re} m$. Assume without loss of generality that n is even. We shall prove that

$$\bigg|\mathbb{E}\Big(B_{m,3}\big(\operatorname{Re} R_{\mathbf{vv}} - \operatorname{Re} m\big)^{n-m}\Big)\bigg| \;\leqslant\; \frac{1}{\log N}\Big(\widetilde{\mathcal{E}}_{ab} + N^{-3/2}\varphi^{C_1}\Psi\Big)\Big[\mathbb{E}\big(\operatorname{Re} S_{\mathbf{vv}} - \operatorname{Re} m\big)^n + (\varphi^{C_\zeta}\Psi)^n\Big] \tag{4.37}$$

for $m = 1, 2, 3$ as well as, assuming

$$|v_a| + |v_b| \;\leqslant\; N^{-1/4}\varphi^{C_1/2}\sqrt{\Psi}\,, \tag{4.38}$$

that

$$\bigg|\mathbb{E}\Big(B_{m,3}\big(\operatorname{Re} R_{\mathbf{vv}} - \operatorname{Re} m\big)^{n-m}\Big)\bigg| \;\leqslant\; \frac{1}{\log N}\widetilde{\mathcal{E}}_{ab}\Big[\mathbb{E}\big(\operatorname{Re} S_{\mathbf{vv}} - \operatorname{Re} m\big)^n + (\varphi^{C_\zeta}\Psi)^n\Big] \tag{4.39}$$

for $m = 1, 2, 3$. Here C_1 is a large enough constant depending on ζ.

Assuming that (4.37) and (4.39) have been proved, we get the claim (4.36) from (3.44) and Lemma 4.3 applied to S; the detains are identical to those of the proof of Lemma 4.2 and the argument following (3.34).

The proof of (4.37) and (4.39) is similar to the proof of (4.7) and (4.8). The key input is the apriori bound

$$\operatorname{Im} S_{\mathbf{vv}} \;\leqslant\; \varphi^{C_\zeta}\Phi \tag{4.40}$$

with 2ζ-high probability, which follows from (4.2) and Markov's inequality. Throughout the proof, we shall consistently (and without further mention) make use of the inequality

$$\Psi^m\big|\operatorname{Re} R_{\mathbf{vv}} - \operatorname{Re} m\big|^{n-m} \;\leqslant\; \varphi^{-D}\Big(\big|\operatorname{Re} S_{\mathbf{vv}} - \operatorname{Re} m\big|^n + (\varphi^{C_{\zeta,D}}\Psi)^n\Big)\,,$$

which follows from the elementary inequality $x^m y^{n-m} \leqslant x^n + y^n$ for $x, y \geqslant 0$, Lemma 3.10, and the estimate

$$|R_{\mathbf{vv}} - S_{\mathbf{vv}}| \;\leqslant\; \varphi^{C_\zeta}\Psi$$

with 2ζ-high probability (as follows from (3.30)). Moreover, as in (4.16), we find that (4.38) implies

$$|R_{\mathbf{v}a}| \;\leqslant\; \varphi^{C_\zeta}\Psi\,. \tag{4.41}$$

The same bound holds for $R_{a\mathbf{v}}$, $R_{\mathbf{v}b}$, and $R_{b\mathbf{v}}$.

As in the proof of Lemma 4.3, we consider four cases.

Case (i): $a = b$ and $m \leqslant 3$. This is easily dealt with using (3.45); we omit further details.

Case (ii): $a \neq b$ **and** $m = 3$**.** Recall that in this case we have $t = s = 3$. From (4.11) we get

$$\Big(|R_{\mathbf{v}a}| + |R_{a\mathbf{v}}|\Big)^3 \Big(|R_{\mathbf{v}b}| + |R_{b\mathbf{v}}|\Big)^3 \leqslant \varphi^{C_\zeta} \bigg(\frac{\operatorname{Im} S_{\mathbf{v}\mathbf{v}}}{N\eta} + \Psi^2\bigg)^{3/2} \Bigg[\bigg(\frac{\operatorname{Im} S_{\mathbf{v}\mathbf{v}}}{N\eta} + \Psi^2\bigg)^{3/2} + |v_a|^2 + |v_b|^2 + \Psi^{-3}|v_a|^2|v_b|^2\Bigg]$$

with 2ζ-high probability. Therefore using (4.40), (3.4), and $\Psi \geqslant cN^{-1/2}$ we get

$$\begin{aligned} |B_{3,3}|\big|\operatorname{Re} R_{\mathbf{v}\mathbf{v}} - \operatorname{Re} m\big|^{n-3} &\leqslant \varphi^{C_\zeta} N^{-3/2}\Psi^3\Big[\Psi^3 + |v_a|^2 + |v_b|^2 + N^{3/2}|v_a|^2|v_b|^2\Big]\,\big|\operatorname{Re} R_{\mathbf{v}\mathbf{v}} - \operatorname{Re} m\big|^{n-3} \\ &\leqslant \varphi^{C_\zeta - D}\Big(\big(\operatorname{Re} R_{\mathbf{v}\mathbf{v}} - \operatorname{Re} m\big)^n + (\varphi^{3D}\Psi)^n\Big)\widetilde{\mathcal{E}}_{ab} \end{aligned}$$

with 2ζ-high probability, where in the last step we used (2.9). Choosing D large enough yields (4.39), and hence also (4.37).

Case (iii): $a \neq b$ **and** $m = 2$**.** In the case $s = t = 2$, the estimate is similar to the estimate of X_1 in (4.13). Using (4.40), (3.4), and $\Psi \geqslant cN^{-1/2}$ we get

$$|X_1| \leqslant \varphi^{C_\zeta}\Psi\,\Psi^2\big(\Psi^2 + |v_a|^2 + |v_b|^2 + N|v_a|^2|v_b|^2\big)$$

with 2ζ-high probability, from which (4.39), and hence also (4.37), easily follows.

Next, consider the case $s = 3$ and $t = 1$. In order to prove (4.37), we estimate using (4.40) and (3.29), similarly to (4.15),

$$\begin{aligned} |B_{2,3,3,1}| &\leqslant \varphi^{C_\zeta} N^{-3/2}\big(\Psi + |v_a|\big)^3\big(\Psi + |v_b|\big) \\ &\leqslant \varphi^{C_\zeta}\Psi^2\bigg[N^{-3/2}\frac{\Phi}{N\eta} + N^{-3/2}|v_b| + N^{-1}|v_a|^2 + N^{-1/2}|v_a|^2|v_b|\bigg] \end{aligned}$$

with 2ζ-high probability from which (4.37) follows. Let us therefore prove (4.39), assuming (4.38). Using (4.41) and (4.40), we find

$$|R_{\mathbf{v}a}| + |R_{a\mathbf{v}}| + |R_{\mathbf{v}b}| + |R_{b\mathbf{v}}| \leqslant \varphi^{C_\zeta}\Psi \tag{4.42}$$

with 2ζ-high probability. We need to prove that

$$N^{-3/2}\Big|\mathbb{E} R_{\mathbf{v}a}R_{bb}R_{a\mathbf{v}}R_{a\mathbf{v}}R_{b\mathbf{v}}\big(\operatorname{Re} R_{\mathbf{v}\mathbf{v}} - \operatorname{Re} m\big)^{n-2}\Big| \leqslant \varphi^{-1}\widetilde{\mathcal{E}}_{ab}\bigg[\mathbb{E}\big(\operatorname{Re} S_{\mathbf{v}\mathbf{v}} - \operatorname{Re} m\big)^n + (\varphi^{C_\zeta}\Psi)^n\bigg]. \tag{4.43}$$

As for (4.18), by splitting $R_{bb} = (R_{bb} - m) + m$ and using (3.27), we find that it is enough to prove

$$N^{-3/2}\Big|\mathbb{E} R_{\mathbf{v}a}R_{a\mathbf{v}}R_{a\mathbf{v}}R_{b\mathbf{v}}\big(\operatorname{Re} R_{\mathbf{v}\mathbf{v}} - \operatorname{Re} m\big)^{n-2}\Big| \leqslant \varphi^{-1}\widetilde{\mathcal{E}}_{ab}\bigg[\mathbb{E}\big(\operatorname{Re} S_{\mathbf{v}\mathbf{v}} - \operatorname{Re} m\big)^n + (\varphi^{C_\zeta}\Psi)^n\bigg]. \tag{4.44}$$

As for (4.18), we use the splitting (4.19). Using (3.27), (4.40), and (4.6), we find that the bounds

$$|\mathcal{R}_{b\mathbf{v}}| \leqslant \varphi^{C_\zeta}\Psi \leqslant \varphi^{C_\zeta}N^{-1/6}, \qquad |\mathcal{R}'_{b\mathbf{v}}| \leqslant \varphi^{C_\zeta}\bigg(\frac{N^{-1/4}}{\sqrt{N\eta}} + \Psi^2\bigg) \leqslant \varphi^{C_\zeta}N^{-1/3} \tag{4.45}$$

hold with 2ζ-high probability. Thus we get (4.44) with $R_{b\mathbf{v}}$ replaced with $\mathcal{R}'_{b\mathbf{v}}$. The remaining term with $\mathcal{R}_{b\mathbf{v}}$ is estimated exactly as (4.23); we omit the details.

Case (iv): $a \neq b$ **and** $m = 1$**.** In order to prove (4.37), we use (4.40) to get

$$|B_{1,3}| \leqslant \varphi^{C_\zeta}\Psi\Big(\Psi + |v_a| + |v_b| + \Psi^{-1}|v_a||v_b| + \Psi^{-1}|v_a|^2 + \Psi^{-1}|v_b|^2\Big)$$

with 2ζ-high probability, from which (4.37) easily follows using $\Psi \geqslant N^{-1/2}$.

As for (4.27), in order to prove (4.37) and (4.39) it suffices to prove the following claim. For X_3 being any expression in (4.28a) – (4.28c), we have

$$N^{-3/2}\big|\mathbb{E}\, X_3(\operatorname{Re} R_{\mathbf{v}\mathbf{v}} - \operatorname{Re} m)^{n-1}\big| \leqslant \varphi^{-1}\Big(\widetilde{\mathcal{E}}_{ab} + N^{-3/2}\varphi^{C_\zeta}\Psi\Big)\Big(\mathbb{E}\big(\operatorname{Re} S_{\mathbf{v}\mathbf{v}} - \operatorname{Re} m\big)^n + (\varphi^{C_\zeta}\Psi)^n\Big)\,, \tag{4.46}$$

as well as, assuming (4.38),

$$N^{-3/2}\big|\mathbb{E}\, X_3(\operatorname{Re} R_{\mathbf{v}\mathbf{v}} - \operatorname{Re} m)^{n-1}\big| \leqslant \varphi^{-1}\widetilde{\mathcal{E}}_{ab}\Big(\mathbb{E}\big(\operatorname{Re} S_{\mathbf{v}\mathbf{v}} - \operatorname{Re} m\big)^n + (\varphi^{C_\zeta}\Psi)^n\Big)\,. \tag{4.47}$$

Note that from (4.20) and (4.40) we get that

$$|\mathcal{R}'_{b\mathbf{v}}| \leqslant C|v_b| + \varphi^{C_\zeta}\Psi^2\,. \tag{4.48}$$

If X_3 is any expression in (4.28a), we get from Lemma 3.8, (3.27), (4.40), and (4.48) that

$$|X_3| \leqslant \varphi^{C_\zeta}\Psi^2\big(\Psi + |v_a|\big)\big(\Psi + |v_b|\big) + \varphi^{C_\zeta}\big(\Psi^2 + |v_a|\big)\big(\Psi^2 + |v_b|\big)$$

with 2ζ-high probability. Now (4.47), and in particular (4.46), follows easily (note that we did not assume (4.38)).

Next, let X_3 be an expression in (4.28b). From Lemma 3.8, (3.27), (4.40), and (4.48) we get

$$|X_3| \leqslant \varphi^{C_\zeta}\Psi\big(\Psi + |v_a|\big)\big(\Psi + |v_a| + |v_b|\big) + \varphi^{C_\zeta}\big(\Psi^2 + |v_a|\big)\big(\Psi + |v_b|\big) + \varphi^{C_\zeta}\big(\Psi + |v_a|\big)\big(\Psi^2 + |v_b|\big)$$

with 2ζ-high probability. Now (4.46) follows easily. Moreover, (4.47) under the assumption (4.38) follows exactly like in paragraphs of (4.29) and (4.30), using the bound $\big|(\mathcal{R}'_{b\mathbf{v}})^{(a)} - \mathcal{R}'_{b\mathbf{v}}\big| \leqslant \varphi^{C_\zeta}\Psi^3$ with 2ζ-high probability, as follows from (4.32) and (4.40).

Finally, we consider the case (4.28c), i.e. $X_3 = \mathcal{R}_{\mathbf{v}a}\mathcal{R}_{b\mathbf{v}}$. Under the assumption (4.38), we find from (4.40), (4.33), and (4.31),

$$\big|\mathcal{R}^{(a)}_{b\mathbf{v}} - \mathcal{R}_{b\mathbf{v}}\big| \leqslant \varphi^{C_\zeta}\Psi^2\,, \qquad \big|R^{[a]}_{\mathbf{v}\mathbf{v}}\big| + \big|R^{[b]}_{\mathbf{v}\mathbf{v}}\big| \leqslant \varphi^{C_\zeta}\Psi^2\,, \qquad \big|R^{[\emptyset]}_{\mathbf{v}\mathbf{v}}\big| \leqslant \varphi^{C_\zeta}\Psi^3$$

with 2ζ-high probability. Then the argument from the proof of Lemma 4.2 can be applied almost unchanged, and we get (4.47) assuming (4.38). □

5. Proof of Theorems 2.3 and 2.5

By Lemma 3.2, if $\eta \leqslant \kappa$ and $|E| > 2$ then the control parameter on the right-hand side of (2.10) can also be expressed as

$$\sqrt{\frac{\operatorname{Im} m(z)}{N\eta}} \asymp N^{-1/2}\kappa^{-1/4}\,, \tag{5.1}$$

where $\kappa \equiv \kappa_E$ was defined in (3.2).

Proof of Theorem 2.3. By polarization and linearity, it is enough to prove that

$$\big|G_{\mathbf{v}\mathbf{v}}(z) - m(z)\big| \;\leqslant\; \varphi^{C_\zeta}\sqrt{\frac{\operatorname{Im} m(z)}{N\eta}} \tag{5.2}$$

with ζ-high probability, for all normalized $\mathbf{v}$. Moreover, by symmetry it suffices to consider the case $2 + \varphi^{C_1} N^{-2/3} \leqslant E \leqslant \Sigma$. In particular, $\kappa \geqslant \varphi^{C_1} N^{-2/3}$. Using Lemma 3.2 we find that Theorem 2.2 implies (5.2) if $\eta \geqslant \eta_0$, where we defined

$$\eta_0 \;:=\; N^{-1/2}\kappa^{1/4}\,.$$

Note that $\eta_0 \leqslant \kappa$.

It remains therefore to establish (5.2) when $0 \leqslant \eta \leqslant \eta_0$. Define

$$z \;:=\; E + \mathrm{i}\eta\,, \qquad z_0 \;:=\; E + \mathrm{i}\eta_0\,.$$

By (5.1) and (5.2) at z_0, it is enough to prove that

$$\big|m(z) - m(z_0)\big| \;\leqslant\; C N^{-1/2}\kappa^{-1/4} \tag{5.3}$$

and

$$\big|G_{\mathbf{v}\mathbf{v}}(z) - G_{\mathbf{v}\mathbf{v}}(z_0)\big| \;\leqslant\; \varphi^{C_\zeta} N^{-1/2}\kappa^{-1/4} \tag{5.4}$$

with ζ-high probability.

Differentiating (2.5), we find

$$m' \;=\; \frac{m^2}{1-m^2}\,, \tag{5.5}$$

which, by Lemma 3.2, implies that $m' \asymp (\kappa+\eta)^{-1/2} = O(\kappa^{-1/2})$. Therefore we get

$$\big|m(z) - m(z_0)\big| \;\leqslant\; C\kappa^{-1/2}\eta_0 \;=\; C N^{-1/2}\kappa^{-1/4}\,,$$

which is (5.3).

Next, by Theorem 3.7 we have $E \geqslant \lambda_N + \eta_0$ with ζ-high probability provided C_1 is large enough. Therefore, since $\eta \leqslant \eta_0 \leqslant E - \lambda_N \leqslant E - \lambda_\alpha$ with ζ-high probability for all $\alpha \leqslant N$, we get

$$\operatorname{Im} G_{\mathbf{v}\mathbf{v}}(z) \;=\; \sum_\alpha \frac{|\langle \mathbf{v}, \mathbf{u}^{(\alpha)}\rangle|^2\eta}{(E-\lambda_\alpha)^2 + \eta^2} \;\leqslant\; 2\sum_\alpha \frac{|\langle \mathbf{v}, \mathbf{u}^{(\alpha)}\rangle|^2\eta_0}{(E-\lambda_\alpha)^2 + \eta_0^2} \;=\; 2\operatorname{Im} G_{\mathbf{v}\mathbf{v}}(z_0) \;\leqslant\; \varphi^{C_\zeta} N^{-1/2}\kappa^{-1/4} \tag{5.6}$$

with ζ-high probability, by (5.2) at z_0 and the estimate $\operatorname{Im} m(z_0) \leqslant C N^{-1/2}\kappa^{-1/4}$. Finally, we estimate the real part from

$$\begin{aligned}\big|\operatorname{Re} G_{\mathbf{v}\mathbf{v}}(z) - \operatorname{Re} G_{\mathbf{v}\mathbf{v}}(z_0)\big| \;&=\; \sum_\alpha \frac{(E-\lambda_\alpha)(\eta_0^2 - \eta^2)|\langle \mathbf{u}^{(\alpha)}, \mathbf{v}\rangle|^2}{\big((E-\lambda_\alpha)^2 + \eta^2\big)\big((E-\lambda_\alpha)^2 + \eta_0^2\big)} \\ &\leqslant\; \frac{\eta_0}{E-\lambda_N}\sum_\alpha \frac{\eta_0|\langle \mathbf{u}^{(\alpha)}, \mathbf{v}\rangle|^2}{(E-\lambda_\alpha)^2 + \eta_0^2} \;\leqslant\; \operatorname{Im} G_{\mathbf{v}\mathbf{v}}(z_0) \end{aligned} \tag{5.7}$$

with ζ-high probability, where in the last step we used that $\eta_0 \leqslant E - \lambda_N$. Combining (5.6) and (5.7) completes the proof of (5.4). □

Proof of Theorem 2.5. We begin with (2.14), whose proof is immediate. Using Theorem 2.2 with Condition **A** and Remark 2.4, we find

$$C \;\geqslant\; \operatorname{Im} G_{\mathbf{v}\mathbf{v}}(\lambda_\alpha + \mathrm{i}\eta) \;=\; \sum_\beta \frac{\eta |\langle \mathbf{u}^{(\beta)}, \mathbf{v}\rangle|^2}{(\lambda_\alpha - \lambda_\beta)^2 + \eta^2} \;\geqslant\; \eta^{-1} |\langle \mathbf{u}^{(\alpha)}, \mathbf{v}\rangle|^2$$

with ζ-high probability, where we used Theorem 3.7 to ensure that $\lambda_\alpha \in [-\Sigma, \Sigma]$ with ζ-high probability. Choosing $\eta = \varphi^\zeta N^{-1}$ yields (2.14).

In order to prove (2.13), we set

$$\eta \;:=\; \gamma_b - \gamma_a\,, \qquad E \;:=\; \gamma_a\,,$$

where γ_α is the classical location of the α-th eigenvalue defined in (3.17). Then we get

$$\sum_{\alpha=a}^b |\langle \mathbf{u}^{(\alpha)}, \mathbf{v}\rangle|^2 \;\leqslant\; \varphi^{C_\zeta} \sum_{\alpha=a}^b \frac{\eta^2 |\langle \mathbf{u}^{(\alpha)}, \mathbf{v}\rangle|^2}{(\lambda_\alpha - E)^2 + \eta^2} \;\leqslant\; \varphi^{C_\zeta} \eta \operatorname{Im} G_{\mathbf{v}\mathbf{v}}(E + \mathrm{i}\eta)\,, \tag{5.8}$$

where in the first step we used Theorem 3.7 to conclude that $(\lambda_\alpha - E)^2 \leqslant \varphi^{C_\zeta} \eta^2$ for $a \leqslant \alpha \leqslant b$. In order to invoke Theorem 2.2 with Condition **B**, we have to satisfy (2.9). Recalling Lemma 3.2, we find that (2.9) holds provided that

$$\eta \;\geqslant\; \varphi^{C_0} N^{-5/6}\,, \qquad \kappa \;\leqslant\; \varphi^{-2C_0} \eta^2 N^{4/3}\,, \tag{5.9}$$

where we abbreviated $\kappa \equiv \kappa_E$. From (3.17) we get

$$\gamma_\alpha + 2 \;\asymp\; \alpha^{2/3} N^{-2/3} \tag{5.10}$$

for $\alpha \leqslant N/2$, from which we deduce, recalling $E = \gamma_\alpha$,

$$\kappa \;\asymp\; a^{2/3} N^{-2/3}\,, \qquad \eta \;\asymp\; (b^{2/3} - a^{2/3}) N^{-2/3}\,.$$

Hence (5.9) is satisfies provided that

$$b^{2/3} - a^{2/3} \;\geqslant\; \varphi^{C_0} N^{-1/6} + \varphi^{C_0} a^{1/3} N^{-1/3}\,.$$

Since $b^{2/3} - a^{2/3} \geqslant b^{-1/3}(b-a)/2$, we find that (5.9), and hence (2.9), holds under the condition (2.12).

Therefore we may apply Theorem 2.2 to the right-hand side of (5.8) to get

$$\sum_{\alpha=a}^b |\langle \mathbf{u}^{(\alpha)}, \mathbf{v}\rangle|^2 \;\leqslant\; \varphi^{C_\zeta} \eta \bigg(\frac{1}{N\eta} + \operatorname{Im} m(E + \mathrm{i}\eta) \bigg) \;\leqslant\; \varphi^{C_\zeta} N^{-1} \Big((b^{2/3} - a^{2/3})^{3/2} + a^{1/3} (b^{2/3} - a^{2/3}) \Big)$$

with ζ-high probability, where we used Lemma 3.2. The claim now follows from the elementary inequalities

$$b^{2/3} - a^{2/3} \;\leqslant\; (b-a)^{2/3}\,, \qquad b^{2/3} - a^{2/3} \;\leqslant\; a^{-1/3}(b-a)\,. \qquad \square$$

For future use, we record the following consequence of Theorem 2.5 which is useful in combination with dyadic decompositions. For any integer $K \leqslant N/4$ we have

$$\sum_{\alpha=K}^{2K} |\langle \mathbf{u}^{(\alpha)}, \mathbf{v}\rangle|^2 \;\leqslant\; \varphi^{C_\zeta} K N^{-1} \tag{5.11}$$

with ζ-high probability.

6. Eigenvalue locations: proof of Theorem 2.7

6.1. Basic facts from linear algebra. We begin by collecting a few well-known tools from linear algebra, on which our analysis of the deformed spectrum relies.

We use the following representation of the eigenvalues of $\widetilde{H}$, which was already used in several papers on finite-rank deformations of random matrices [5–7, 32].

Lemma 6.1. *If $\mu \in \mathbb{R} \setminus \sigma(H)$ and $\det(D) \neq 0$ then $\mu \in \sigma(\widetilde{H})$ if and only if*

$$\det\bigl(V^* G(\mu) V + D^{-1}\bigr) \;=\; 0\,.$$

Proof. For the convenience of the reader, we give the simple proof. The claim follows from the computation

$$\begin{aligned}\det\bigl(\widetilde{H} - \mu\bigr) \;&=\; \det(H-\mu)\det\bigl(\mathbb{1} + (H-\mu)^{-1} V D V^*\bigr)\\ &=\; \det(H-\mu)\det\bigl(\mathbb{1} + V^*(H-\mu)^{-1} V D\bigr)\\ &=\; \det(H-\mu)\det(D)\det\bigl(D^{-1} + V^*(H-\mu)^{-1} V\bigr)\,,\end{aligned}$$

where in the second step we used the identity $\det(\mathbb{1} + AB) = \det(\mathbb{1} + BA)$ which is valid for any $n \times m$ matrix A and $m \times n$ matrix B. □

We shall also make use of the well-known Weyl's interlacing property, summarized in the following lemma.

Lemma 6.2. *If A is an $N \times N$ Hermitian matrix and $B = A + d\,\mathbf{v}\mathbf{v}^*$ with some $d > 0$ and $\mathbf{v} \in \mathbb{C}^N$, then the eigenvalues of A and B are interlaced:*

$$\lambda_1(A) \;\leqslant\; \lambda_1(B) \;\leqslant\; \lambda_2(A) \;\leqslant\; \cdots \;\leqslant\; \lambda_{N-1}(B) \;\leqslant\; \lambda_N(A) \;\leqslant\; \lambda_N(B)\,.$$

We shall occasionally need the eigenvalues of H to be distinct. To that end, we assume without loss of generality that the law of H is absolutely continuous; otherwise consider the matrix $H + \mathrm{e}^{-N} V$ where V is a GOE/GUE matrix independent of H. It is immediate that this perturbation does not change any of H's spectral statistics. Moreover, any Hermitian matrix with an absolutely continuous law has almost surely distinct eigenvalues.

6.2. Warmup: the rank-one case. In order to illustrate our method, we first present a much simplified proof which deals with the case $k = 1$. Let $\mathbf{v} \in \mathbb{C}^N$ be normalized and deterministic, and $d \in \mathbb{R}$ be deterministic (and possibly N-dependent). Define the deformed matrix

$$\widetilde{H} \;:=\; H + d\,\mathbf{v}\mathbf{v}^*\,.$$

For the following we note the elementary estimate

$$\theta(d) - 2 \;\asymp\; (d-1)^2\,, \tag{6.1}$$

as follows from (2.18).

Theorem 6.3. *Fix $\zeta > 0$. Then there is a constant C_ζ such that the following holds. For $0 \leqslant d \leqslant 1$ we have*

$$0 \;\leqslant\; \mu_N - \lambda_N \;\leqslant\; \varphi^{C_\zeta} \frac{d}{N(1 - d + N^{-1/3})}$$

with ζ-high probability. For $1 \leqslant d \leqslant \Sigma - 1$ we have

$$|\mu_N - \theta(d)| \;\leqslant\; \varphi^{C_\zeta} \sqrt{\frac{d - 1 + N^{-1/3}}{N}}$$

with ζ-high probability.

By symmetry, an analogous result holds for $d \leqslant 0$.

Proof. First we note that it is enough to consider $d \in \mathbb{R}_+ \setminus \big[1 - \varphi^D N^{-1/3}, 1 + \varphi^D N^{-1/3}\big]$ for some arbitrary but fixed $D > 0$. This follows from $|\lambda_N - 2| \leqslant \varphi^{C_\zeta} N^{-2/3}$ with ζ-high probability (see Theorem 3.7), the monotonicity of the map $d \mapsto \lambda_N(H + d\,\mathbf{v}\mathbf{v}^*)$ (see Lemma 6.2), and the observation that $\theta(1+\varepsilon) = 1 + \varepsilon^2 + O(\varepsilon^3)$ as $\varepsilon \to 0$ (which implies that $|\theta(d) - 2| \leqslant \varphi^{2D+1} N^{-2/3}$ for $d \in \big[1 - \varphi^D N^{-1/3}, 1 + \varphi^D N^{-1/3}\big]$).

The key identity[3] for the proof is

$$G_{\mathbf{v}\mathbf{v}}(\mu_N) \;=\; -\frac{1}{d}\,,$$

as follows from Lemma 6.1. Let us begin with the case $d \geqslant 1 + \varphi^D N^{-1/3}$. Since $m : \mathbb{R} \setminus (-2,2) \to [-1,1] \setminus \{0\}$ is bijective, we find from (2.5) that $\theta(d)$ is uniquely characterized by

$$m(\theta(d)) \;=\; -\frac{1}{d}\,. \tag{6.2}$$

We therefore have to solve the equation $m(\theta(d)) = G_{\mathbf{v}\mathbf{v}}(x)$ for $x \in [2 + \varphi^{C_1} N^{-2/3}, \infty)$, where C_1 the constant from Theorem 2.3. By Theorem 2.3, we have

$$G_{\mathbf{v}\mathbf{v}}(x) \;=\; m(x) + O\big(\varphi^{C_\zeta} N^{-1/2} \kappa_x^{-1/4}\big) \tag{6.3}$$

with ζ-high probability.

Next, define the interval

$$I_d \;:=\; [x_-(d), x_+(d)]\,, \qquad x_\pm(d) \;:=\; \theta(d) \pm \varphi^D N^{-1/2} (d-1)^{1/2}\,.$$

We claim that

$$\kappa_x \;\asymp\; (d-1)^2\,, \qquad m'(x) \;\asymp\; (d-1)^{-1} \qquad (x \in I_d) \tag{6.4}$$

The first relation of (6.4) follows from

$$|x - \theta(d)| \;\leqslant\; \varphi^D N^{-1/2} (d-1)^{1/2} \qquad \text{and} \qquad \theta(d) - 2 \;\geqslant\; c(d-1)^2 \;\geqslant\; c\varphi^{3D/2} N^{-1/2} (d-1)^{1/2}\,,$$

where in the last step we used $d \geqslant 1 + \varphi^D N^{-1/3}$. In order to prove the second relation of (6.4), we differentiate (5.5) and use Lemma 3.2 to get

$$m'(x) \;\asymp\; \kappa_x^{-1/2}\,, \qquad m''(x) \;\asymp\; \kappa_x^{-3/2}\,. \tag{6.5}$$

Therefore we get from (6.5) and the mean value theorem applied to m' that

$$|m'(x) - m'(\theta(d))| \;\leqslant\; C\varphi^D N^{-1/2} (d-1)^{1/2} (d-1)^{-3} \;\leqslant\; C\varphi^{-D/2} (d-1)^{-1}\,.$$

[3]Here we ignore the possibility that $\mu_N \in \sigma(H)$. Since the law of H is absolutely continuous, it is easy to check that the interlacing inequalities in Lemma 6.2 are strict with probability one; see e.g. the proof of Lemma 6.7.

Therefore (6.4) follows from $m'(\theta(d)) \asymp (d-1)^{-1}$.

Now choose D large enough that $x_-(d) \geqslant 2 + \varphi^{C_1} N^{-2/3}$ for $d \geqslant \varphi^D N^{-2/3}$. Thus (6.3) and (6.4) yield

$$G_{\mathbf{v}\mathbf{v}}(x_-(d)) \;<\; m(\theta(d)) \;<\; G_{\mathbf{v}\mathbf{v}}(x_+(d)) \tag{6.6}$$

with ζ-high probability, provided D is chosen larger than the constant C_ζ in (6.3). Finally we observe that, by Theorem 3.7, with ζ-high probability the function $x \mapsto G_{\mathbf{v}\mathbf{v}}(x)$ is continuous and increasing on $[2+\varphi^{C_1}N^{-2/3}, \infty)$. It follows that with ζ-high probability the equation $G_{\mathbf{v}\mathbf{v}}(x) = m(\theta(d))$ has precisely one solution, $x = \mu_N$, in $[2+\varphi^{C_1}N^{-2/3}, \infty)$. Moreover, this solution lies in I_d, which implies that it satisfies the claim of Theorem 6.3 for $d > 1$.

What remains is the case $d \leqslant 1 - \varphi^D N^{-1/3}$. Choose $x := 2 + \varphi^{C_1} N^{-2/3}$ where C_1 is a large constant to be chosen later. For large enough C_1 we find from Theorem 2.3

$$G_{\mathbf{v}\mathbf{v}}(x) \;=\; m(x) + O\bigl(N^{-1/3}\varphi^{-C_1/4}\bigr) \tag{6.7}$$

with ζ-high probability. From (3.3) we find

$$1 + m(x) \;\asymp\; N^{-1/3}\varphi^{C_1/2}\,, \tag{6.8}$$

which yields

$$1 + G_{\mathbf{v}\mathbf{v}}(x) \;\geqslant\; 0 \;\geqslant\; 1 - \frac{1}{d}$$

with ζ-high probability. Choosing C_1 large enough, we find as above that $y \mapsto G_{\mathbf{v}\mathbf{v}}(y)$ is with ζ-high probability increasing and continuous for $y \geqslant x$, from which we deduce that

$$\lambda_N \;\leqslant\; \mu_N \;\leqslant\; x$$

with ζ-high probability. (The first inequality follows from Lemma 6.2.)

Next, abbreviate $q := \varphi^{C_2}$ for some large constant C_2 to be chosen later. Using Theorem 3.7 we estimate, for $\lambda_N \leqslant \mu_N \leqslant x$ and large enough C_2,

$$\begin{aligned}
\Biggl|\sum_{\alpha \leqslant N-q} \frac{|\langle \mathbf{u}^{(\alpha)}, \mathbf{v}\rangle|^2}{\lambda_\alpha - \mu_N} - \sum_{\alpha \leqslant N-q} \frac{|\langle \mathbf{u}^{(\alpha)}, \mathbf{v}\rangle|^2}{\lambda_\alpha - x}\Biggr| &\leqslant \varphi^{C_\zeta} N^{-2/3} \sum_{\alpha \leqslant N-q} \frac{|\langle \mathbf{u}^{(\alpha)}, \mathbf{v}\rangle|^2}{(\lambda_\alpha - \mu_N)^2} \\
&\leqslant \varphi^{C_\zeta} N^{-2/3} \sum_{k \geqslant 1} \frac{2^k N^{-1}}{(2^{2k/3} N^{-2/3})^2} + \varphi^{C_\zeta} N^{-2/3} \\
&\leqslant \varphi^{C_\zeta} N^{-1/3}
\end{aligned}$$

with ζ-high probability. In the second inequality we estimated the contribution of the eigenvalues $\alpha \geqslant N/2$ using the dyadic decomposition

$$U_k \;:=\; \bigl\{\alpha \in [N/2\,,\, N-q] : N - 2^{k+1} \leqslant \alpha \leqslant N - 2^k\bigr\}$$

combined with Theorem 3.7, the estimate

$$2 - \gamma_\alpha \;\asymp\; (N-\alpha)^{2/3} N^{-2/3} \qquad (\alpha \geqslant N/2)\,,$$

and the delocalization estimate (5.11). A similar (in fact easier) dyadic decomposition works for the remaining eigenvalues $\alpha < N/2$ and yields the last term of the second line. Moreover, we have

$$\sum_{\alpha > N-q} \frac{|\langle \mathbf{u}^{(\alpha)}, \mathbf{v}\rangle|^2}{|\lambda_\alpha - x|} \;\leqslant\; \varphi^{C_\zeta + C_2} N^{-1/3}$$

with ζ-high probability, by Theorems 3.7 and 2.5. Recalling (6.7) and (6.8), we have therefore proved that

$$-\frac{1}{d} \;=\; G_{\mathbf{v}\mathbf{v}}(\mu_N) \;=\; \sum_\alpha \frac{|\langle \mathbf{u}^{(\alpha)}, \mathbf{v}\rangle|^2}{\lambda_\alpha - \mu_N} \;=\; -1 + O\big(\varphi^{C_\zeta + C_1 + C_2} N^{-1/3}\big) + \sum_{\alpha > N-q} \frac{|\langle \mathbf{u}^{(\alpha)}, \mathbf{v}\rangle|^2}{\lambda_\alpha - \mu_N}$$

with ζ-high probability. Therefore

$$\frac{1}{\mu_N - \lambda_N} \sum_{\alpha > N-q} |\langle \mathbf{u}^{(\alpha)}, \mathbf{v}\rangle|^2 \;\geqslant\; \sum_{\alpha > N-q} \frac{|\langle \mathbf{u}^{(\alpha)}, \mathbf{v}\rangle|^2}{\mu_N - \lambda_\alpha} \;=\; \frac{1}{d} - 1 + O\big(\varphi^{C_\zeta + C_1 + C_2} N^{-1/3}\big)$$

with ζ-high probability. Theorem 2.5 implies $|\langle \mathbf{u}^{(\alpha)}, \mathbf{v}\rangle|^2 \leqslant \varphi^{C_\zeta} N^{-1}$, and the claim follows. This concludes the proof of Theorem 6.3. □

6.3. The permissible region. The rest of this section is devoted to the proof of Theorem 2.7.

Definition 6.4. *We choose an event, denoted by Ξ, of ζ-high probability on which the following statements hold.*

(i) The eigenvalues of H are distinct.

(ii) For all $i = 1, \dots, k$ and $\alpha = 1, \dots, N$ we have $\langle \mathbf{v}^{(i)}, \mathbf{u}^{(\alpha)}\rangle \neq 0$.

(iii) All statements of Theorems 2.2, 2.3, 2.5, and 3.7 hold.

We note that such a Ξ exists. As explained in Section 6.1, we assume without loss of generality that the law of H is absolutely continuous. Then conditions (i) and (ii) hold almost surely; we omit the standard proof. That condition (iii) holds with ζ-high probability is a consequence of Theorems 2.2, 2.3, 2.5, and 3.7 (see also Remark 2.4).

For the whole remainder of the proof of Theorem 2.7, we choose and fix an arbitrary realization $H \equiv H^\omega$ with $\omega \in \Xi$. Thus, the randomness of H only comes into play in ensuring that Ξ is of ζ-high probability. The rest of the argument is entirely deterministic.

Fix $k^-, k^+ \in \mathbb{N}$ and define $k^0 := k - k^+ - k^- = \#\{i : |d_i| \leqslant 1\}$. Write

$$\mathbf{d} \;=\; (d_1, \dots, d_k) \;=\; (\mathbf{d}^-, \mathbf{d}^0, \mathbf{d}^+) \qquad \mathbf{d}^\sigma \;=\; (d_1^\sigma, \dots, d_{k^\sigma}^\sigma) \qquad (\sigma \;=\; -, 0, +)\,.$$

We adopt the convention that

$$d_1^- \;\leqslant\; \cdots \;\leqslant\; d_{k^-}^- \;<\; -1 \;\leqslant\; d_1^0 \;\leqslant\; \cdots \;\leqslant\; d_{k^0}^0 \;\leqslant\; 1 < d_1^+ \;\leqslant\; \cdots \;\leqslant\; d_{k^+}^+\,. \tag{6.9}$$

Abbreviate

$$\widetilde{\psi}_N \;\equiv\; \widetilde{\psi} \;:=\; 2k\psi\,. \tag{6.10}$$

For $\widetilde{C}_2 > 0$ define the sets

$$\mathcal{D}^-(\widetilde{C}_2) \;:=\; \Big\{\mathbf{d}^- : -\Sigma + 1 \leqslant d_i^- \leqslant -1 - \varphi^{\widetilde{C}_2}\widetilde{\psi}N^{-1/3}\,,\, i = 1,\dots,k^-\Big\}\,,$$
$$\mathcal{D}^+(\widetilde{C}_2) \;:=\; \Big\{\mathbf{d}^+ : 1 + \varphi^{\widetilde{C}_2}\widetilde{\psi}N^{-1/3} \leqslant d_i^+ \leqslant \Sigma - 1\,,\, i = 1,\dots,k^+\Big\}\,,$$
$$\mathcal{D}^0(\widetilde{C}_2) \;:=\; \Big\{\mathbf{d}^0 : -1 + \varphi^{\widetilde{C}_2}\widetilde{\psi}N^{-1/3} \leqslant d_i^0 \leqslant 1 - \varphi^{\widetilde{C}_2}\widetilde{\psi}N^{-1/3}\,,\, i = 1,\dots,k^0\Big\}\,,$$

the set of allowed $\mathbf{d}$'s,

$$\mathcal{D}(\widetilde{C}_2) \;:=\; \big\{(\mathbf{d}^-,\mathbf{d}^0,\mathbf{d}^+) : \mathbf{d}^\sigma \in \mathcal{D}^\sigma(\widetilde{C}_2)\,,\, \sigma = -,0,+\big\}\,,$$

and the subset

$$\mathcal{D}^*(\widetilde{C}_2) \;:=\; \big\{\mathbf{d} \in \mathcal{D}(\widetilde{C}_2) : d_i \neq 0 \text{ for } i = 1,\dots,k\big\}\,.$$

Let $\widetilde{K} > 0$ denote a constant to be chosen later, and define

$$S(\widetilde{K}) \;:=\; \Big(-\infty\,,\, -2 + \varphi^{\widetilde{K}}N^{-2/3}\Big) \cup \Big(2 - \varphi^{\widetilde{K}}N^{-2/3}\,,\, \infty\Big)\,.$$

We shall only consider eigenvalues of $\widetilde{H}$ in $S(\widetilde{K})$ for some large but fixed $\widetilde{K}$.

Let $\widetilde{C}_3 > 0$ denote some large constant to be chosen later. Define the intervals

$$I_i^-(\mathbf{d}) \;:=\; \Big[\theta(d_i^-) - \varphi^{\widetilde{C}_3}N^{-1/2}(-d_i^- - 1)^{1/2}\,,\, \theta(d_i^-) + \varphi^{\widetilde{C}_3}N^{-1/2}(-d_i^- - 1)^{1/2}\Big] \qquad (i = 1,\dots,k^-)\,,$$
$$I_i^+(\mathbf{d}) \;:=\; \Big[\theta(d_i^+) - \varphi^{\widetilde{C}_3}N^{-1/2}(d_i^+ - 1)^{1/2}\,,\, \theta(d_i^+) + \varphi^{\widetilde{C}_3}N^{-1/2}(d_i^+ - 1)^{1/2}\Big] \qquad (i = 1,\dots,k^+)\,,$$
$$I^0 \;:=\; \Big\{x \in \mathbb{R} : \operatorname{dist}(x,\sigma(H)) \leqslant N^{-2/3}\widetilde{\psi}^{-1}\Big\} \cap S(\widetilde{K})\,.$$

For $\mathbf{d} \in \mathcal{D}(\widetilde{C}_2)$ define

$$\Gamma(\mathbf{d}) \;:=\; I^0 \cup \left(\bigcup_{i=1}^{k^-} I_i^-(\mathbf{d})\right) \cup \left(\bigcup_{i=1}^{k^+} I_i^+(\mathbf{d})\right).$$

The following proposition states that $\Gamma(\mathbf{d})$ is the "permissible region" for the eigenvalues of $\widetilde{H}$. Roughly, the allowed region consists of a small neighbourhood of each $\theta(d_i)$ for $i \in O$, as well as of small neighbourhoods of the eigenvalues of H. The latter regions house the sticking eigenvalues. Proposition (6.5) only establishes where the eigenvalues are allowed to lie; it gives no other information on their locations (such as the number of eigenvalues in each interval). Note that, by definition of $S(\widetilde{K})$, the set $\Gamma(\mathbf{d})$ only keeps track of eigenvalues outside of the interval $\big[-2 + \varphi^{\widetilde{K}}N^{-2/3}, 2 - \varphi^{\widetilde{K}}N^{-2/3}\big]$. This will eventually suffice for the statement (2.21) thanks to the eigenvalue rigidity estimate for H, Theorem 3.7, combined with eigenvalue interlacing; see (6.34) below.

Proposition 6.5. *For $\widetilde{C}_3$ and $\widetilde{C}_2(\widetilde{C}_3)$ large enough (depending on ζ, $\widetilde{K}$, and the constant C_1 from Theorem 2.3) the following holds. For any $\mathbf{d} \in \mathcal{D}(\widetilde{C}_2)$ and $H \equiv H^\omega$ with $\omega \in \Xi$ we have*

$$I_i^\pm(\mathbf{d}) \cap I^0 \;=\; \emptyset \quad \textit{for all} \quad i \;=\; 1,\dots,k^\pm \tag{6.11}$$

as well as

$$\sigma(\widetilde{H}) \cap S(\widetilde{K}) \;\subset\; \Gamma(\mathbf{d})\,. \tag{6.12}$$

Proof. Clearly, it is enough to prove the claim for $\mathbf{d} \in \mathcal{D}^*(\widetilde{C}_2)$. We shall choose the constants $\widetilde{C}_3(\zeta, C_1)$ and $\widetilde{C}_2(\zeta, \widetilde{K}, C_1, \widetilde{C}_3)$ to be large enough during the proof. (Here C_1 is the constant from Theorem 2.3.)

First we prove (6.11). By definition of Ξ (see Theorem 3.7), we find that (6.11) holds if

$$2 + \varphi^{2\widetilde{C}_2} N^{-2/3} - \varphi^{\widetilde{C}_3 + \widetilde{C}_2/2} N^{-2/3} \;>\; 2 + 2\varphi^{C_\zeta} N^{-2/3} \;\geqslant\; \lambda_N + N^{-2/3} \widetilde{\psi}^{-1} ,$$

which is satisfied provided that

$$2\widetilde{C}_2 \;\geqslant\; \widetilde{C}_3 + \widetilde{C}_2/2 + C_\zeta . \tag{6.13}$$

In order to prove (6.12), we define, for each $z \in \mathbb{C} \setminus \sigma(H)$, the $k \times k$ matrix $M(z)$ through

$$M_{ij}(z) \;:=\; G_{\mathbf{v}^{(i)}\mathbf{v}^{(j)}}(z) + \delta_{ij} d_i^{-1} . \tag{6.14}$$

From Lemma 6.1 we find that $x \in \sigma(\widetilde{H}) \setminus \sigma(H)$ if and only if $M(x)$ is singular. The proof therefore consists in locating $x \in \mathbb{R} \setminus \sigma(H)$ for which $M(x)$ is singular.

First we consider the case $x \geqslant 2 + \varphi^{\widetilde{C}_2} N^{-2/3}$. On Ξ we have

$$\lambda_N \;\leqslant\; 2 + \varphi^{\widetilde{C}_2 - 1} N^{-2/3} \qquad \text{and} \qquad \lambda_1 \;\geqslant\; -2 - \varphi^{\widetilde{C}_2 - 1} N^{-2/3} \tag{6.15}$$

provided $\widetilde{C}_2$ is large enough (see Theorem 3.7). In particular, by (6.15) and the definition of Ξ, we have $x \notin \sigma(H)$. By increasing $\widetilde{C}_2$ if necessary we may assume that $\widetilde{C}_2 \geqslant C_1$, where C_1 is the constant from Theorem 2.3. Therefore we get from Theorem 2.3 and Lemma 3.2 that

$$M(x + \mathrm{i}y) \;=\; m(x + \mathrm{i}y) + D^{-1} + O\big(\varphi^{C_\zeta} N^{-1/2} \kappa_x^{-1/4}\big) \tag{6.16}$$

for all $y \in [-\Sigma, \Sigma]$. (We include an imaginary part $y \neq 0$ for later applications of (6.16); for the purposes of this proof we set $y = 0$.)

Let $i \in \{1, \dots, k^+\}$. Then we may repeat to the letter the argument in the proof of Theorem 6.3 leading to (6.4). Provided that $\widetilde{C}_3 \geqslant C_\zeta + 2$, where C_ζ is the constant in (6.16), we therefore get that

$$\left| m(x) + \frac{1}{d_i^+} \right| \;\geqslant\; \varphi^{C_\zeta + 1} N^{-1/2} \kappa_x^{-1/4} \qquad \text{if} \quad x \notin I_i^+(\mathbf{d}) .$$

This takes care of the components $\mathbf{d}^+$ in D^{-1}. In order to deal with the remaining components, $\mathbf{d}^0$ and $\mathbf{d}^-$, we observe that

$$m(x) \;\in\; \big[-1, -c\big]$$

for some $c > 0$ depending on Σ. It is now easy to put all the estimates associated with $i = 1, \dots, k$ together. Recalling (6.16) and choosing $\widetilde{C}_2$ large enough yields, for C_ζ denoting the constant from (6.16),

$$\left| m(x) + \frac{1}{d_i} \right| \;\geqslant\; \varphi^{C_\zeta + 1} N^{-1/2} \kappa_x^{-1/4}$$

for all $i = 1, \dots, k$ provided that

$$x \;\in\; \big[2 + \varphi^{\widetilde{C}_2} N^{-2/3}, \Sigma\big] \setminus \bigcup_{i=1}^{k^+} I_i^+(\mathbf{d}) . \tag{6.17}$$

We conclude[4] from (6.16) that $M(x)$ is regular if (6.17) holds.

An almost identical argument applied to $\mathbf{d}^-$ yields that $M(x)$ is regular if

$$x \in \big[-\Sigma, -2-\varphi^{\widetilde{C}_2}N^{-2/3}\big] \cup \big[2+\varphi^{\widetilde{C}_2}N^{-2/3}, \Sigma\big] \setminus \left(\bigcup_{i=1}^{k^-} I_i^-(\mathbf{d}) \cup \bigcup_{i=1}^{k^+} I_i^+(\mathbf{d})\right). \tag{6.18}$$

Next, we focus on the case

$$x \in \Big[2-\varphi^{\widetilde{K}}N^{-2/3}, 2+\varphi^{\widetilde{C}_2}N^{-2/3}\Big], \qquad \operatorname{dist}(x,\sigma(H)) > N^{-2/3}\widetilde{\psi}^{-1}. \tag{6.19}$$

Our aim is to prove that $M(x)$ is regular for any x satisfying (6.19). Once this is done, the regularity of $M(x)$ for x satisfying (6.18) or (6.19) will imply (6.12). Choose $\eta := N^{-2/3}\widetilde{\psi}^{-1}$ and estimate

$$\begin{aligned}
|G_{\mathbf{v}^{(i)}\mathbf{v}^{(j)}}(x) - G_{\mathbf{v}^{(i)}\mathbf{v}^{(j)}}(x+\mathrm{i}\eta)| &\leqslant \sum_\alpha \frac{|\langle \mathbf{u}^{(\alpha)}, \mathbf{v}^{(i)}\rangle|^2 + |\langle \mathbf{u}^{(\alpha)}, \mathbf{v}^{(j)}\rangle|^2}{2}\left|\frac{1}{\lambda_\alpha - x} - \frac{1}{\lambda_\alpha - x - \mathrm{i}\eta}\right| \\
&\leqslant \sum_\alpha \Big(|\langle \mathbf{u}^{(\alpha)}, \mathbf{v}^{(i)}\rangle|^2 + |\langle \mathbf{u}^{(\alpha)}, \mathbf{v}^{(j)}\rangle|^2\Big)\frac{\eta}{(\lambda_\alpha - x)^2+\eta^2} \\
&= \operatorname{Im} G_{\mathbf{v}^{(i)}\mathbf{v}^{(i)}}(x+\mathrm{i}\eta) + \operatorname{Im} G_{\mathbf{v}^{(j)}\mathbf{v}^{(j)}}(x+\mathrm{i}\eta),
\end{aligned}$$

where in the second step we used (6.19). Therefore, by definition of Ξ (See also Theorem 2.2) and Lemma 3.2, we get (recall that $\widetilde{\psi} \geqslant 1$)

$$G_{\mathbf{v}^{(i)}\mathbf{v}^{(j)}}(x) = \delta_{ij} m(x+\mathrm{i}\eta) + O\left(\varphi^{C_\zeta}\operatorname{Im} m(x+\mathrm{i}\eta) + \frac{\varphi^{C_\zeta}}{N\eta}\right) = -\delta_{ij} + O\left(\varphi^{C_\zeta}N^{-1/3}\Big(\widetilde{\psi} + \varphi^{\widetilde{K}/2} + \varphi^{\widetilde{C}_2/2}\Big)\right).$$

This implies, for any x satisfying (6.19), that

$$M(x) = -\mathbb{1} + D^{-1} + O\left(\varphi^{C_\zeta}N^{-1/3}\Big(\widetilde{\psi} + \varphi^{\widetilde{K}/2} + \varphi^{\widetilde{C}_2/2}\Big)\right). \tag{6.20}$$

Since

$$\left|-1 + \frac{1}{d_i}\right| \geqslant \frac{1}{2}\varphi^{\widetilde{C}_2}\widetilde{\psi}N^{-1/3}$$

for all i, we find that $M(x)$ is regular provided $\widetilde{C}_2$ is chosen large enough that

$$\widetilde{C}_2 - 1 \geqslant C_\zeta + \widetilde{K}/2 + \widetilde{C}_2/2.$$

This completes the analysis of the case (6.19). The case

$$x \in \Big[-2-\varphi^{\widetilde{C}_2}N^{-2/3}, -2+\varphi^{\widetilde{K}}N^{-2/3}\Big], \qquad \operatorname{dist}(x,\sigma(H)) > N^{-2/3}\widetilde{\psi}^{-1}$$

is handled similarly. This completes the proof. $\square$

[4]Here we use the well-known fact that if $\lambda \in \sigma(A+B)$ then $\operatorname{dist}(\lambda, \sigma(A)) \leqslant \|B\|$.

6.4. The initial configuration. In this section we fix a configuration $\mathbf{d}(0) \equiv \mathbf{d}$ that is *independent of* N, and satisfies $k^0 = 0$ as well as

$$-\Sigma + 1 \leqslant d_1^- \;<\; \cdots \;<\; d_{k^-}^- \;<\; -1\,, \qquad 1 \;<\; d_1^+ \;<\; \cdots \;<\; d_{k^+}^+ \;\leqslant\; \Sigma - 1\,. \tag{6.21}$$

Note that $\mathbf{d} \in \mathcal{D}^*(\widetilde{C}_2)$ for large enough N.

First we deal with the outliers.

PROPOSITION 6.6. *For N large enough, each interval $I_i^-(\mathbf{d})$, $i = 1, \dots, k^-$, and $I_i^+(\mathbf{d})$, $i = 1, \dots, k^+$, contains precisely one eigenvalue of $\widetilde{H}$.*

PROOF. Let $i \in \{1, \dots, k^+\}$ and pick a small N-independent positively oriented closed contour $\mathcal{C} \subset \mathbb{C} \backslash [-2, 2]$ that encloses $\theta(d_i^+)$ but no other point of the set $\bigcup_{\sigma = \pm} \bigcup_{i=1}^{k^\sigma} \{\theta(d_i^\sigma)\}$. By Proposition 6.5, it suffices to show that the interior of $\mathcal{C}$ contains precisely one eigenvalue of $\widetilde{H}$. Define

$$f_N(z) \;:=\; \det\bigl(M(z) + D^{-1}\bigr)\,, \qquad g(z) \;:=\; \det\bigl(m(z) + D^{-1}\bigr)\,.$$

The functions g and f_N are holomorphic on and inside $\mathcal{C}$ (for large enough N). Moreover, by construction of $\mathcal{C}$, the function g has precisely one zero inside $\mathcal{C}$, namely at $z = \theta(d_i^+)$. Next, we have

$$\min_{z \in \mathcal{C}} |g(z)| \;\geqslant\; c \;>\; 0\,, \qquad |g(z) - f_N(z)| \;\leqslant\; \varphi^{C_\zeta} N^{-1/2}\,,$$

where the second inequality follows from (6.16). The claim now follows from Rouché's theorem. The eigenvalues near $\theta(d_i^-)$, $i = 1, \dots, k^-$, are handled similarly. ☐

Before moving on, we record the following result on rank-one deformations.

LEMMA 6.7. *Let $\mathbf{v} \in \mathbb{C}^k$ be nonzero. Then for all $i = 1, \dots, k-1$ and all Hermitian $k \times k$ matrices A we have*

$$\lim_{d \to \infty} \lambda_i(A + d\mathbf{v}\mathbf{v}^*) \;=\; \lim_{d \to -\infty} \lambda_{i+1}(A + d\mathbf{v}\mathbf{v}^*)\,.$$

PROOF. By Lemma 6.1, we find that $x \notin \sigma(A)$ is an eigenvalue of $A + d\mathbf{v}\mathbf{v}^*$ if and only if

$$\langle \mathbf{v}\,, (A - x)^{-1}\mathbf{v}\rangle \;=\; -\frac{1}{d}\,.$$

Let

$$E \;:=\; \Bigl\{A : \text{the eigenvalues of } A \text{ are distinct}\,,\ \langle \mathbf{v}\,, \mathbf{u}^{(i)}(A)\rangle \neq 0 \text{ for all } i\Bigr\}\,,$$

where $\mathbf{u}^{(i)}(A)$ denotes the eigenvector of A associated with the eigenvalue $\lambda_i(A)$. (Note that $\mathbf{u}^{(i)}(A)$ is well-defined in E, since the eigenvalues are distinct.) It is not hard to see that E^c is dense in the space of Hermitian matrices.

We write the condition $\langle \mathbf{v}\,, (A - x)^{-1}\mathbf{v}\rangle = -d^{-1}$ as

$$f(x) \;:=\; \sum_i \frac{|\langle \mathbf{v}\,, \mathbf{u}^{(i)}(A)\rangle|^2}{\lambda_i(A) - x} \;=\; -\frac{1}{d}\,,$$

Let $A \in E$. Then f has k singularities at the eigenvalues of H, away from which we have $f' > 0$. Moreover, $f(x) \uparrow 0$ as $x \uparrow \infty$, and $f(x) \downarrow 0$ as $x \downarrow -\infty$. Thus, for any $d \in \mathbb{R} \setminus \{0\}$, the equation $f(x) = -d^{-1}$

has exactly k solutions in $\mathbb{R} \setminus \sigma(A)$. Since $A + d\mathbf{v}\mathbf{v}^*$ has at most k distinct eigenvalues, this proves that $\sigma(A + d\mathbf{v}\mathbf{v}^*) \cap \sigma(A) = \emptyset$ for all $d \in \mathbb{R}$. Moreover, the equation $f(x) = 0$ has exactly $k-1$ solutions, $x_1, \dots, x_{k-1}$. Since $f'(x_i) > 0$ for each $i = 1, \dots, k-1$, it is easy to see that $x_i = \lim_{d \to \infty} \lambda_i(A + d\mathbf{v}\mathbf{v}^*) = \lim_{d \to -\infty} \lambda_{i+1}(A + d\mathbf{v}\mathbf{v}^*)$.

Now the claim follows by approximating an arbitrary matrix A by matrices in E, and by using the Lipschitz continuity of the map $A \mapsto \lambda_i(A)$. □

We now deal with the extremal bulk eigenvalues.

PROPOSITION 6.8. *Fix $0 < \delta < 1/3$ and $\widetilde{K} > 0$. Let $\mathbf{d}$ be N-independent and satisfy* (6.21). *Then for large enough N (depending on δ and $\widetilde{K}$) we have for all α satisfying $\lambda_\alpha \geqslant 2 - \varphi^{\widetilde{K}} N^{-2/3}$ that*

$$|\lambda_\alpha - \mu_{\alpha - k^+}| \;\leqslant\; N^{-1+\delta}\,.$$

Similarly, we have for all α satisfying $\lambda_\alpha \leqslant -2 + \varphi^{\widetilde{K}} N^{-2/3}$ that

$$|\lambda_\alpha - \mu_{\alpha + k^-}| \;\leqslant\; N^{-1+\delta}\,.$$

PROOF. We only prove the first statement; the proof of the second one is almost identical. Abbreviate $\delta' := \delta/2$.

Before embarking on the full proof, we first give a sketch of its main idea, under some simplifying assumptions. Let $A \in \mathbb{N}$ be some fixed constant, and assume that, for each $\alpha \geqslant N - A$, the neighbours of λ_α are further than $N^{-1+\delta'}$ away from λ_α. (This assumption in fact holds with probability $1 - o(1)$, a fact we shall neither use nor prove.) We claim that there is *at least* one eigenvalue of $\widetilde{H}$ in the interval $[x_-^\alpha, x_+^\alpha]$ surrounding λ_α, where

$$x_\pm^\alpha \;:=\; \lambda_\alpha \pm N^{-1+\delta'}/3\,.$$

Before sketching the proof of the above claim, we show how to use it to conclude the argument. By Proposition 6.6, there are at least k^+ eigenvalues in (x_+^N, ∞). Recall that by assumption $k^0 = 0$, i.e. $|d_i| > 1$ for all i. Therefore using interlacing, i.e. a repeated application of Lemma 6.2, we conclude that there are exactly k^+ eigenvalues in (x_+^N, ∞). From the above claim we find that there is at least one eigenvalue in $[x_-^N, x_+^N]$. Using interlacing we find that there are at most $k^+ + 1$ eigenvalues in $[x_-^N, \infty)$. We conclude that there is exactly one eigenvalue in $[x_-^N, x_+^N]$. We may move on to the $(N-1)$-th eigenvalue: we have proved that there are (i) at least $k^+ + 1$ eigenvalues in $[x_-^N, \infty)$ (from the previous step), (ii) at least one eigenvalue in $[x_-^{N-1}, x_+^{N-1}]$ (from the claim), and (iii) at most $k^+ + 2$ eigenvalues in $[x_-^{N-1}, \infty)$ (from interlacing); we conclude that there is exactly one eigenvalue in $[x_-^{N-1}, x_+^{N-1}]$. Continuing in this fashion concludes the proof.

Let us now complete the sketch of the proof of the above claim. Assume for simplicity that H and $\widetilde{H}$ have no common eigenvalues. From Lemma 6.1 we find that x is an eigenvalue of $\widetilde{H}$ if and only if the matrix $M(x)$, defined in (6.14), is singular. Thus, we have to prove that there is an $x \in [x_-^\alpha, x_+^\alpha]$ such that $M(x)$ is singular. The idea of the argument is to do a spectral decomposition of G, and resum all terms not associated with λ_α to get something close to $\operatorname{Re} m(x) \approx -1$. More precisely, we write

$$\begin{aligned} M_{ij}(x) \;&=\; \frac{\langle \mathbf{v}^{(i)}, \mathbf{u}^{(\alpha)} \rangle \langle \mathbf{u}^{(\alpha)}, \mathbf{v}^{(j)} \rangle}{\lambda_\alpha - x} + \sum_{\beta \neq \alpha} \frac{\langle \mathbf{v}^{(i)}, \mathbf{u}^{(\beta)} \rangle \langle \mathbf{u}^{(\beta)}, \mathbf{v}^{(j)} \rangle}{\lambda_\beta - x} + \delta_{ij} d_i^{-1} \\ &\approx\; \frac{\langle \mathbf{v}^{(i)}, \mathbf{u}^{(\alpha)} \rangle \langle \mathbf{u}^{(\alpha)}, \mathbf{v}^{(j)} \rangle}{\lambda_\alpha - x} + \operatorname{Re} m(x) \delta_{ij} + \delta_{ij} d_i^{-1}\,, \end{aligned}$$

where the sum over β was replaced with $\operatorname{Re} m(x)\delta_{ij}$ (up to negligible error terms). This approximation will be justified using Theorems 2.2 and 2.5; it uses that $x \in [x_-^\alpha, x_+^\alpha]$ and consequently all eigenvalues λ_β, $\beta \neq \alpha$, are separated from x by at least $N^{-1+\delta}/3$. Introducing the vector $\mathbf{y} = (y_i) \in \mathbb{C}^k$, defined by $y_i := \langle \mathbf{v}^{(i)}, \mathbf{u}^{(\alpha)} \rangle$, we therefore get

$$M(x) \;\approx\; \frac{\mathbf{y}\mathbf{y}^*}{\lambda_\alpha - x} - \mathbb{1} + D^{-1}\,, \tag{6.22}$$

where we used that $\operatorname{Re} m(x) \approx -1$. By assumption, $|d_i| > 1$ for all i; therefore the matrix $-\mathbb{1} + D^{-1}$ is strictly negative. Also, Theorem 2.5 implies that $|y_i| \leqslant \varphi^{C_\zeta} N^{-1/2}$. Thus it is easy to conclude that all eigenvalues of $M(x_-^\alpha)$ are negative. The first term on the right-hand side of (6.22) is a rank-one matrix. As x approaches λ_α from the left, its nonzero eigenvalue tends to $+\infty$. By continuity, there must therefore exist an $x \in [x_-^\alpha, \lambda_\alpha)$ such that $M(x)$ is singular. This concludes the sketch of the proof of the claim.

Now we turn towards the detailed proof in the general case. Since eigenvalues of H may be separated by less than $N^{-1+\delta'}$, we begin by clumping together eigenvalues of H which are separated by less than $N^{-1+\delta'}$. More precisely, we construct a partition $\mathcal{A} = (A_q)_q$ of $\{1, \dots, N\}$, defined as the finest partition in which α and β belong to the same block if $|\lambda_\alpha - \lambda_\beta| \leqslant N^{-1+\delta'}$. Thus, each block consists of a sequence of consecutive integers. We order the blocks of $\mathcal{A}$ in a "decreasing" fashion, in such a way that if $q < r$ then $\lambda_\alpha > \lambda_\beta$ for all $\alpha \in A_q$ and $\beta \in A_r$.

We now derive a bound on the size of the blocks near the edge. Roughly, we shall show that if $\lambda \in A_q$ and $\lambda \geqslant 2 - \varphi^C N^{-2/3}$ then $|A_q| \leqslant \varphi^{C'}$. Let C_4 be a large constant to be chosen later. Now choose α and β satisfying $0 \leqslant \alpha \leqslant \beta \leqslant \varphi^{C_4}$ such that $N - \alpha$ and $N - \beta$ belong to the same block. Then by definition of Ξ and $\mathcal{A}$ we have

$$c\Big[(\beta/N)^{2/3} - (\alpha/N)^{2/3}\Big] - \varphi^{C_\zeta} N^{-2/3} \;\leqslant\; \lambda_{N-\alpha} - \lambda_{N-\beta} \;\leqslant\; (\beta - \alpha) N^{-1+\delta'}\,,$$

where we used the statement of Theorem 3.7 and the definition (3.17). Thus we get the condition

$$N^{-2/3}\Big[c\beta^{-1/3}(\beta - \alpha) - \varphi^{C_\zeta}\Big] \;\leqslant\; N^{-1+\delta'}(\beta - \alpha)\,.$$

We conclude that if α and β satisfy $0 \leqslant \alpha \leqslant \beta \leqslant \varphi^{C_4}$ and $N - \alpha$ and $N - \beta$ belong to the same block, then

$$\beta - \alpha \;\leqslant\; \varphi^{C_\zeta + C_4/3 + 1}\,. \tag{6.23}$$

Let α_* denote the largest integer such that $\lambda_{N-\alpha_*} \geqslant 2 - \varphi^{\widetilde{K}} N^{-2/3}$. In particular, by definition of Ξ (see Theorem 3.7) we have

$$\alpha_* \;\leqslant\; \varphi^{3\widetilde{K}/2 + C_\zeta}\,. \tag{6.24}$$

Now we choose $C_4 \equiv C_4(\zeta, \widetilde{K})$ large enough that

$$C_4 \;\geqslant\; \max\Big(3\widetilde{K}/2 + C_\zeta\,,\, C_\zeta + C_4/3 + 1\Big) + 2\,.$$

Next, define Q through $N - \alpha_* \in A_Q$. Therefore we get from (6.23) and (6.24) that any $\alpha \leqslant \varphi^{C_4}$ such that $N - \alpha \in A_Q$ satisfies

$$\alpha \;\leqslant\; \alpha^* + \varphi^{C_\zeta + C_4/3 + 1} \;\leqslant\; \varphi^{C_4 - 1}\,.$$

Since blocks are contiguous, we conclude that

$$|A_q| \;\leqslant\; \varphi^{C_4 - 1}\,. \tag{6.25}$$

for each $q = 1, \dots, Q$. Moreover, by definition of Ξ (see Theorem 3.7), we find

$$|\lambda_{N-\alpha} - 2| \;\leqslant\; \varphi^{2C_4/3 + C_\zeta} N^{-2/3}\,.$$

for all $q = 1, \dots, Q$ and all α such that $N - \alpha \in A_q$.

Now we are ready for the main argument. Pick $q \in \{1, \dots, Q\}$ and abbreviate

$$a^q := \min_{\alpha \in A_q} \lambda_\alpha\,, \qquad b^q := \max_{\alpha \in A_q} \lambda_\alpha\,.$$

We introduce the path

$$x_t^q \;:=\; a^q - N^{-1+\delta'}/3 + \big(b^q - a^q + 2N^{-1+\delta'}/3\big)\, t\,, \qquad (t \in [0,1])\,,$$

which will serve to count eigenvalues. (Note that $x_0^q = a^q - N^{-1+\delta'}/3$ and $x_1^q = b^q + N^{-1+\delta'}/3$.) The interval $[x_0^q, x_1^t]$ contains precisely those eigenvalues of H that are in A_q, and its endpoints x_0^q and x_1^q are at a distance greater than $N^{-1+\delta'}/3$ from any eigenvalue of H. Thus, $[x_0^q, x_1^t]$ is the correct generalization of the interval $[x_-^\alpha, x_+^\alpha]$ from the sketch given at the beginning of this proof.

In order to avoid problems with exceptional events, we add some randomness to D. Recall that D satisfies (6.21). Let Δ be a $k \times k$ Hermitian random matrix whose upper triangular entries are independent and have an absolutely continuous law supported in the unit disk. For $\varepsilon > 0$ define

$$\widetilde{H}^\varepsilon \;:=\; H + V(D^{-1} + \varepsilon\Delta)^{-1} V^*\,.$$

From now on we use "almost surely" to mean almost surely with respect to the randomness of Δ. Our main goal is to prove that for each $\varepsilon > 0$, almost surely, there are at least $|A_q|$ eigenvalues of $\widetilde{H}^\varepsilon$ in $[x_0^q, x_1^q] \setminus \sigma(H)$. (Having done this, we shall deduce, by taking $\varepsilon \to 0$, that $\widetilde{H}$ has at least $|A_q|$ eigenvalues in $[x_0^q, x_1^q]$.)

For $x \notin \sigma(H)$ define

$$M_{ij}^\varepsilon(x) \;:=\; G_{\mathbf{v}^{(i)}\mathbf{v}^{(j)}}(x) + \delta_{ij} d_i^{-1} + \varepsilon \Delta_{ij} \qquad (i, j = 1, \dots, k)\,.$$

Then (assuming $x \notin \sigma(H)$) we know that $x \in \sigma(\widetilde{H}^\varepsilon)$ if and only if $M^\varepsilon(x)$ is singular. Split

$$G_{\mathbf{v}^{(i)}\mathbf{v}^{(j)}}(x) \;=\; \sum_{\alpha \in A_q} \frac{\langle \mathbf{v}^{(i)}, \mathbf{u}^{(\alpha)}\rangle \langle \mathbf{u}^{(\alpha)}, \mathbf{v}^{(j)}\rangle}{\lambda_\alpha - x} + \sum_{\alpha \notin A_q} \frac{\langle \mathbf{v}^{(i)}, \mathbf{u}^{(\alpha)}\rangle \langle \mathbf{u}^{(\alpha)}, \mathbf{v}^{(j)}\rangle}{\lambda_\alpha - x}\,.$$

Let $x \in [x_0^q, x_1^q]$. Similarly to the proof of (6.20), we choose $\eta := N^{-1+\delta'}$ and estimate

$$\Bigg| \sum_{\alpha \notin A_q} \frac{\langle \mathbf{v}^{(i)}, \mathbf{u}^{(\alpha)}\rangle \langle \mathbf{u}^{(\alpha)}, \mathbf{v}^{(j)}\rangle}{\lambda_\alpha - x} - \sum_{\alpha \notin A_q} \frac{\langle \mathbf{v}^{(i)}, \mathbf{u}^{(\alpha)}\rangle \langle \mathbf{u}^{(\alpha)}, \mathbf{v}^{(j)}\rangle}{\lambda_\alpha - x - \mathrm{i}\eta} \Bigg| \\ \leqslant\; 2\Big(\operatorname{Im} G_{\mathbf{v}^{(i)}\mathbf{v}^{(i)}}(x + \mathrm{i}\eta) + \operatorname{Im} G_{\mathbf{v}^{(j)}\mathbf{v}^{(j)}}(x + \mathrm{i}\eta) \Big)\,,$$

where we used that $|x - \lambda_\alpha| \geqslant 2N^{-1+\delta'}/3$ for $\alpha \notin A_q$. Moreover,

$$\Bigg| \sum_{\alpha \in A_q} \frac{\langle \mathbf{v}^{(i)}, \mathbf{u}^{(\alpha)}\rangle \langle \mathbf{u}^{(\alpha)}, \mathbf{v}^{(j)}\rangle}{\lambda_\alpha - x - \mathrm{i}\eta} \Bigg| \;\leqslant\; \varphi^{C_\zeta + C_4} N^{-\delta'}\,,$$

where we used (6.23) and the definition of Ξ (see Theorem 2.5). Estimating $G_{\mathbf{v}^{(i)}\mathbf{v}^{(j)}}(x+\mathrm{i}\eta) - m(x+\mathrm{i}\eta)$ therefore yields, similarly to (6.20),

$$M^\varepsilon_{ij}(x) \;=\; \sum_{\alpha \in A_q} \frac{\langle \mathbf{v}^{(i)}\,, \mathbf{u}^{(\alpha)}\rangle \langle \mathbf{u}^{(\alpha)}\,, \mathbf{v}^{(j)}\rangle}{\lambda_\alpha - x} - \delta_{ij} + \delta_{ij} d_i^{-1} + \varepsilon \Delta_{ij} + O\big(\varphi^{C_\zeta + C_4} N^{-\delta'/2}\big)\,.$$

Introducing the vector

$$\mathbf{y}^{(\alpha)} \;=\; (y_i^{(\alpha)})_{i=1}^k\,, \qquad y_i^{(\alpha)} \;:=\; \langle \mathbf{v}^{(i)}\,, \mathbf{u}^{(\alpha)}\rangle\,,$$

we get

$$M^\varepsilon(x) \;=\; \sum_{\alpha \in A_q} \frac{\mathbf{y}^{(\alpha)}(\mathbf{y}^{(\alpha)})^*}{\lambda_\alpha - x} - \mathbb{1} + D^{-1} + \varepsilon \Delta + R(x)\,, \qquad R(x) \;=\; O\big(\varphi^{C_\zeta + C_4} N^{-\delta'/2}\big)\,, \tag{6.26}$$

where $R(x)$ is continuous in x and independent of Δ. Compare this to (6.22) in the sketch given at the beginning of the proof. By Theorem 2.5, for $\alpha \in A_q$ we have

$$|y_i^{(\alpha)}| \;=\; O\big(\varphi^{C_\zeta} N^{-1/2}\big)\,. \tag{6.27}$$

We may now start the counting of the eigenvalues of $\widetilde{H}$ in $[x_0^q, x_1^q]$. We have to prove that there are at least $L := |A_q|$ distinct points x in $[x_0^q, x_1^q]$ at which $M^\varepsilon(x)$ has a zero eigenvalue. As in the simple continuity argument given in the sketch at the beginning of this proof, we shall make use of continuity. However, having to find L such values x instead of just one is a significant complication[5]. Before coming to the full counting argument, we give a sketch of its main idea. See Figure 6.1 for a graphical depiction of this sketch. We extend the real line $\mathbb{R}$, on which the eigenvalues of $M^\varepsilon(x)$ reside, to the real projective line $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\} \cong S^1$. One can think of $\overline{\mathbb{R}}$ as a ring with two distinguished points, 0 at the bottom and ∞ at the top. Thanks to Lemma 6.7, it is possible to label the k eigenvalues of $M^\varepsilon(x_t^q)$ so that they are continuous $\overline{\mathbb{R}}$-valued functions (denoted by $\tilde{e}_1^\varepsilon(t), \dots, \tilde{e}_k^\varepsilon(t)$ below) on $[0,1]$. Thus, we get a family of k beads moving continuously counterclockwise on a ring. At $t = 0$, the eigenvalues are all strictly negative (and finite), i.e. all beads lie in the left half of the ring. As t is continuously increased from 0 to 1, the beads move counterclockwise around the ring. Our goal is to count the number of times 0 is hit by a bead. Thanks to the explicit form of the first term on the right-hand side of (6.26), we know that the point ∞ is hit exactly L times as t ranges from 0 to 1. Since at time $t = 0$ all beads were in the left half of the ring, and since the beads move continuously counterclockwise, we conclude by continuity that 0 is hit at least L times as t ranges from 0 to 1. Below, we denote the times at which ∞ is hit by $s_1, \dots, s_L$, and the times at which 0 is hit by $t_1, \dots, t_L$. One nuisance we have to deal with in the proof is the possibility of several beads crossing one of the two points 0 or ∞ simultaneously. Such events are not admissible for our counting. For instance, if at time t a bead is at 0 while another is at ∞, we cannot conclude that x_t^q is an eigenvalue of $\widetilde{H}$; indeed, because there is a bead at ∞, we know that x_t^q is an eigenvalue of H, and hence Lemma 6.1 is not applicable. However, such pathological events almost surely do not occur. Avoiding them was the reason for introducing Δ. Note that the final result of the counting argument – the number of eigenvalues of $\widetilde{H}^\varepsilon$ in $[x_0^q, x_1^q]$ – is stable under the limit $\varepsilon \to 0$. This will allow us to conclude the proof.

[5]This complication is also visible in the joint arrangement of the eigenvalues of H and $\widetilde{H}$. If all eigenvalues of H are well-separated (by at least $N^{-1+\delta'}$) then, as outlined in the sketch at the beginning of the proof, each eigenvalue λ_α of H has an associated eigenvalue of $\widetilde{H}$, which lies in the interval $[\lambda_\alpha - N^{-1+\delta'}/3, \lambda_\alpha)$. In fact, this eigenvalue typically lies at a distance

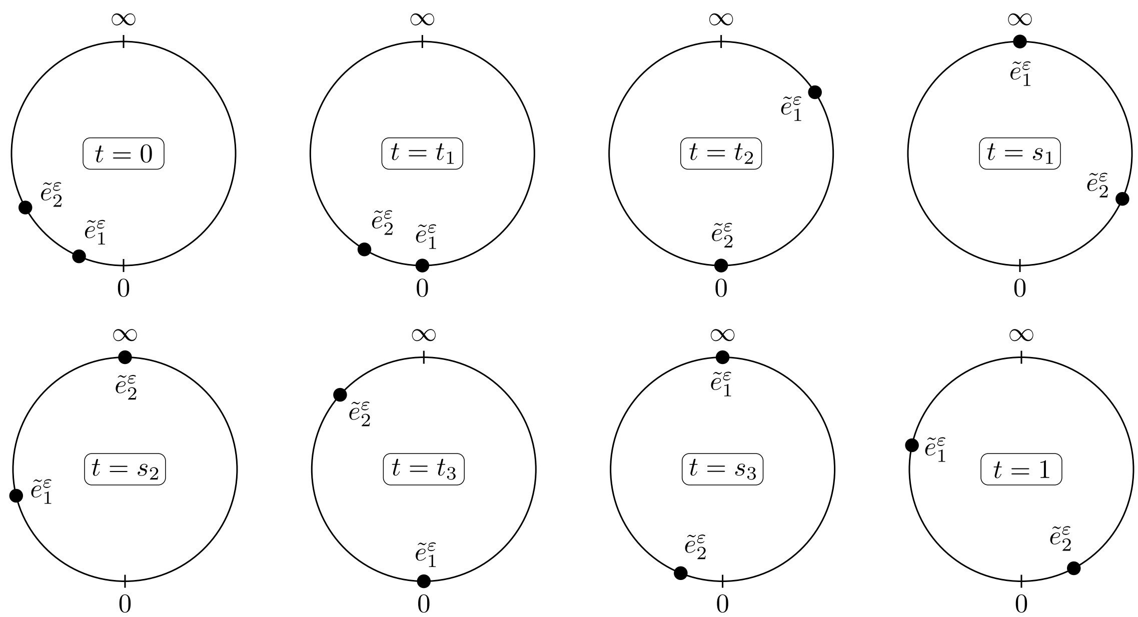

Figure 6.1: A graphical representation of the movement of the eigenvalues (or "beads") $\tilde{e}_1^\varepsilon(t), \tilde{e}_2^\varepsilon(t)$ of $M^\varepsilon(x_t^q)$ as t ranges from 0 to 1. In this example we have $L = 3$, $k = 2$, and $0 < t_1 < t_2 < s_1 < s_2 < t_3 < s_3 < 1$.

Now we give the full proof. Recall that $|d_i| > 1$ is independent of N for all i. Thus we get from (6.26) and (6.27) that, for large enough N and small enough ε, all eigenvalues of $M^\varepsilon(x_0^q)$ are negative. (Here we used that $|\lambda_\alpha - x_0^q| \geqslant N^{-1+\delta'}/3$ for $\alpha \in A_q$.) We shall vary t continuously from 0 to 1 and count the number of eigenvalues crossing the origin. Let $L := |A_q|$ and denote by

$$0 \;<\; s_1 \;<\; s_2 \;<\; \cdots \;<\; s_L \;<\; 1$$

the values of t at which $x_t^q \in \sigma(H)$. (Recall that the eigenvalues of H are distinct.) It is also convenient to write $s_0 = 0$ and $s_{L+1} = 1$. For $t \in [0,1] \setminus \{s_1, \dots s_L\}$, let

$$e_1^\varepsilon(t) \;\leqslant\; e_2^\varepsilon(t) \;\leqslant\; \cdots \;\leqslant\; e_k^\varepsilon(t)$$

denote the ordered eigenvalues of $M^\varepsilon(x_t^q)$. We record the following fundamental properties of $e_1^\varepsilon(t), \dots, e_k^\varepsilon(t)$.

(i) For all $i = 1, \dots, k$, we have $e_i^\varepsilon(0) < 0$ for N large enough and ε small enough (depending on N).

(ii) For every $\ell = 0, \dots, L$ and $i = 1, \dots, k$, the function e_i^ε is continuous on $(s_\ell, s_{\ell+1})$.

(iii) At each singular point s_ℓ, $\ell = 1, \dots, L$, we have

$$e_i^\varepsilon(s_\ell^-) \;=\; e_{i+1}^\varepsilon(s_\ell^+) \qquad (i = 1, \dots k-1)\,.$$

N^{-1} to the left of λ_α, as follows from (6.22) and the fact that the typical size of $\mathbf{y}$ is $N^{-1/2}$. However, if two eigenvalues of H are closer than N^{-1}, this simple ordering breaks down. In general, therefore, all we can say about the eigenvalues of $\widetilde{H}$ associated with the eigenvalues of H in A_q is that they are close to the *group* $\{\lambda_\alpha\}_{\alpha \in A_q}$. Since the diameter of this group is small (see (6.28) below), this will be enough.

(In particular, both one-sided limits exist.)

Property (i) was proved after (6.27). Property (ii) follows from (6.26). Property (iii) follows from Lemma 6.7, using (6.26) and the fact that $R(x)$ is continuous.

Moreover, the two following claims are true almost surely.

(a) For each $\ell = 1, \dots, L$ and $i = 1, \dots, k-1$ we have $e_i^\varepsilon(s_\ell^-) \neq 0$. (The remaining index k satisfies $e_k^\varepsilon(s_\ell^-) = +\infty$.)

(b) If $e_i^\varepsilon(t) = 0$ for some $t \in [0,1] \setminus \{s_1, \dots, s_L\}$ then $e_j^\varepsilon(t) \neq 0$ for all $j \neq i$.

In terms of beads $\tilde{e}_1^\varepsilon(t), \dots, \tilde{e}_k^\varepsilon(t) \in \overline{R}$ (see below), the properties (a) and (b) can be informally summarized as: (a) if a bead is at ∞ then there is no bead at 0, (b) at most one bead is at 0. We omit the standard[6] proofs of (a) and (b), which rely on the fact that the law of Δ is absolutely continuous.

In order to conclude our main argument, it is convenient to regard the eigenvalues $e_1^\varepsilon(t), \dots, e_k^\varepsilon(t)$ as elements of $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\} \cong S^1$, the real projective line. From properties (ii) - (iii), it is apparent that we may rearrange the eigenvalues of $M^\varepsilon(x_t^q)$ as $\tilde{e}_1^\varepsilon(t), \dots, \tilde{e}_k^\varepsilon(t) \in \overline{R}$ and extend them to functions ("beads") on whole interval $[0,1]$ in such a way that, almost surely, each $\tilde{e}_i^\varepsilon$ is a continuous $\overline{\mathbb{R}}$-valued function on $[0,1]$.

We now claim the following.

($*$) Almost surely, there are L distinct times $t_1 < t_2 < \cdots < t_L \in [0,1] \setminus \{s_1, \dots, s_L\}$ such that for each $\ell = 1, \dots, L$ there is an $i = 1, \dots, k$ with $\tilde{e}_i^\varepsilon(t_\ell) = 0$.

Let us prove ($*$). Let $n_i \in \mathbb{N}$ denote the number of times that $\tilde{e}_i^\varepsilon$ hits ∞ as t ranges from 0 to 1. From (6.26) we find that $\sum_{i=1}^k n_i = L$ (recall that the eigenvalues of H are distinct). Moreover, again from (6.26), we find that each such passage of ∞ by $\tilde{e}_i^\varepsilon$ always takes place in the same direction, namely from the positive reals to the negative reals with t increasing. More precisely, if $\tilde{e}_i^\varepsilon(t_*) = \infty$ then there is a neighbourhood $I \ni t_*$ such that for all $t \in I$ we have

$$\tilde{e}_i^\varepsilon(t) \in \mathbb{R}_+ \quad \text{for} \quad t < t_* \qquad \text{and} \qquad \tilde{e}_i^\varepsilon(t) \in \mathbb{R}_- \quad \text{for} \quad t > t_*.$$

Since at time zero we have $\tilde{e}_i^\varepsilon(0) \in \mathbb{R}_-$ (see Property (i) above) we conclude that $\tilde{e}_i^\varepsilon$ has at least n_i distinct zeros. (Recall that n_i was defined as the number of times $\widetilde{e}_i^\varepsilon$ hits ∞.) Moreover, by Property (a), the zeros $\tilde{e}_i^\varepsilon$ are almost surely in $[0,1] \setminus \{s_1, \dots, s_L\}$. By Property (b), the zeros of $e_1^\varepsilon, \dots, e_k^\varepsilon$ are almost surely disjoint. Since $\sum_{i=1}^k n_i = L$, the claim ($*$) follows.

From ($*$) we conclude that, almost surely, $M^\varepsilon(x)$ is singular in at least L points in the set $[x_0^q, x_1^q] \setminus \sigma(H)$. Therefore $\widetilde{H}^\varepsilon$ has almost surely at least L eigenvalues in $[x_0^q, x_1^q]$. Taking $\varepsilon \to 0$, we find that $\widetilde{H}$ has at least $L = |A_q|$ eigenvalues in $[x_0^q, x_1^q]$.

What remains is to prove that $\widetilde{H}$ has at most $|A_q|$ eigenvalues in $[x_0^q, x_1^q]$. We prove this using interlacing, similarly to the corresponding argument given in the sketch at the beginning of the proof. Together with Proposition 6.6, we have proved that there are at least $|A_1| + k^+$ eigenvalues of $\widetilde{H}$ in $[x_0^1, \infty)$. By interlacing (i.e. a repeated application of Lemma 6.2), we find that there are at most $|A_1| + k^+$ eigenvalues of $\widetilde{H}$ in $[x_0^1, \infty)$. We deduce, again using Proposition 6.6, that there are exactly $|A_1|$ eigenvalues of $\widetilde{H}$ in $[x_0^1, x_1^1]$.

[6]The "standard" arguments rely on the fact that the set of singular Hermitian matrices is an algebraic variety of codimension one. In addition, the proof of (a) requires the following fact. Let P be a rank-one orthogonal projector on $\mathbb{C}^k$ and A a Hermitian $k \times k$ matrix; then, as $x \to \pm\infty$, exactly $k-1$ eigenvalues of the matrix $A + xP$ converge, and their limits coincide with the eigenvalues of A restricted to a map from $\ker P$ to $\ker P$. The proof of (b) uses that the set of Hermitian matrices with multiple eigenvalues at zero is an algebraic variety of codimension two.

We have proved that there are at least $|A_1| + |A_2| + k^+$ eigenvalues of $\widetilde{H}$ in $[x_0^2, \infty)$. Using eigenvalue interlacing, we find that there are at most $|A_1| + |A_2| + k^+$ eigenvalues of $\widetilde{H}$ in $[x_0^2, \infty)$. We conclude that there are exactly $|A_2|$ eigenvalues of $\widetilde{H}$ in $[x_0^2, x_1^2]$.

We may now repeat this argument for $q = 3, 4, \ldots, Q$, to get that $\widetilde{H}$ has exactly $|A_q|$ eigenvalues in $[x_0^q, x_1^q]$, for $q = 1, 2, \ldots, Q$. Moreover, by (6.25), we find for any $\alpha \in A_q$ that

$$\sup\Big\{|x - \lambda_\alpha| : \alpha \in A_q\,,\, x \in [x_0^q, x_1^q]\Big\} \;\leqslant\; \varphi^{C_4} N^{-1+\delta'} \;\leqslant\; N^{-1+\delta}\,. \tag{6.28}$$

Therefore the proof is complete. □

6.5. Bootstrapping and conclusion of the proof of Theorem 2.7. We may now complete the proof of Theorem 2.7. In order to extend the statements of Propositions 6.6 and 6.8 to arbitrary N-dependent configurations $\mathbf{d} \in \mathcal{D}(\widetilde{C}_2)$, we continuously deform an N-independent $\mathbf{d}$, for which Propositions 6.6 and 6.8 hold, to the desired N-dependent $\mathbf{d}$. The statements of Propositions 6.6 and 6.8 remain valid for all intermediate $\mathbf{d}$'s; this will follow from the continuity of the eigenvalues of $\widetilde{H}$ as a function of $\mathbf{d}$ and from Proposition 6.5. Roughly, Proposition 6.5 establishes a forbidden region, for arbitrary $\mathbf{d}$, which the eigenvalues of $\widetilde{H}$ cannot cross since they are deformed continuously.

Let $\mathbf{d}(1) \equiv \mathbf{d}_N(1) \in \mathcal{D}^*(\widetilde{C}_2)$ be given (and possibly N-dependent), with associated N-independent indices k^-, k^0, k^+. Choose an N-independent $\mathbf{d}(0) \in \mathcal{D}(\widetilde{C}_2)$ with the same indices k^-, k^0, k^+, such that $\mathbf{d}^0(0) = 0$ and $(\mathbf{d}^-(0), \mathbf{d}^+(0))$ satisfies (6.21). We shall use a bootstrap argument by choosing a continuous (possibly N-dependent) path $(\mathbf{d}(t) : 0 \leqslant t \leqslant 1)$ that connects $\mathbf{d}(0)$ and $\mathbf{d}(1)$. We require the path $\mathbf{d}(t)$ to have the following properties.

(i) For all $t \in [0, 1]$ the point $\mathbf{d}(t)$ satisfies (6.9) and $\mathbf{d}(t) \in \mathcal{D}(\widetilde{C}_2)$.

(ii) If $I_i^+(\mathbf{d}(1)) \cap I_j^+(\mathbf{d}(1)) = \emptyset$ for a pair $1 \leqslant i < j \leqslant k^+$ then $I_i^+(\mathbf{d}(t)) \cap I_j^+(\mathbf{d}(t)) = \emptyset$ for all $t \in [0, 1]$. The same restriction is imposed for $+$ replaced with $-$.

It is easy to see that such a path exists. Informally, condition (ii) states that if the allowed regions for the outliers i and j do not over lap at time $t = 1$ (i.e. the outliers can be distinguished), then they may not overlap at any earlier time.

We continue to work at fixed N and with a fixed realization $H \equiv H^\omega$ with $\omega \in \Xi$. Let $\widetilde{C}_2$ and $\widetilde{C}_3$ be the constants from Proposition 6.5, and choose $\delta > 0$ such that $\widetilde{\psi} \leqslant N^{1/3-\delta}$. Define

$$\widetilde{H}(t) \;:=\; H + V \operatorname{diag}(d_1(t), \ldots, d_k(t)) V^*$$

and abbreviate $\mu_\alpha(t) = \lambda_\alpha(\widetilde{H}(t))$. By Propositions 6.6 and 6.8, we have that

$$\mu_{N-k^++i}(0) \;\in\; I_i^+(\mathbf{d}(0)) \qquad (i = 1, \ldots, k^+)\,, \tag{6.29a}$$

$$\mu_i(0) \;\in\; I_i^-(\mathbf{d}(0)) \qquad (i = 1, \ldots, k^-)\,, \tag{6.29b}$$

as well as

$$\lambda_\alpha \;\geqslant\; 2 - \varphi^{\widetilde{K}} N^{-2/3} \qquad \Longrightarrow \qquad |\lambda_\alpha - \mu_{\alpha-k^+}(0)| \;\leqslant\; N^{-2/3}\widetilde{\psi}^{-1}\,, \tag{6.30a}$$

$$\lambda_\alpha \;\leqslant\; -2 + \varphi^{\widetilde{K}} N^{-2/3} \qquad \Longrightarrow \qquad |\lambda_\alpha - \mu_{\alpha+k^-}(0)| \;\leqslant\; N^{-2/3}\widetilde{\psi}^{-1}\,. \tag{6.30b}$$

In order to invoke a continuity argument, we note that Proposition 6.5 yields

$$\sigma(\widetilde{H}(t)) \cap S(\widetilde{K}) \;\subset\; \Gamma(\mathbf{d}(t)) \tag{6.31}$$

for all $t \in [0,1]$. Moreover, since $t \mapsto \widetilde{H}(t)$ is continuous, we find that $\mu_\alpha(t)$ is continuous in $t \in [0,1]$ for all α.

Let us first analyse the outliers. We focus on the positive outliers associated with $\mathbf{d}^+$; the negative ones are dealt with in the same way. Assume first that the k^+ intervals $I_1^+(\mathbf{d}(t)), \dots, I_{k^+}^+(\mathbf{d}(t))$ are disjoint for $t = 1$. Then, from Property (ii) above, we know that they are disjoint for all $t \in [0,1]$. Thus we find, from (6.29), (6.31), and the continuity of $t \mapsto \mu_\alpha(t)$ that

$$\mu_{N-k^++i}(t) \;\in\; I_i^+(\mathbf{d}(t)) \qquad (i = 1, \dots, k^+) \tag{6.32}$$

for all $t \in [0,1]$, and in particular for $t = 1$.

If $I_1^+(\mathbf{d}(1)), \dots, I_{k^+}^+(\mathbf{d}(1))$ are not disjoint, the situation is only slightly more complicated. Let $\mathcal{B}$ denote the finest partition of $\{1, \dots, k^+\}$ such that i and j belong to the same block of $\mathcal{B}$ if $I_i^+(\mathbf{d}(1)) \cap I_j^+(\mathbf{d}(1)) \neq \emptyset$. Note that the blocks of $\mathcal{B}$ are sequences of consecutive integers. Denote by B_i the block of $\mathcal{B}$ that contains i. Then (6.29) and (6.31) yield, instead of (6.32), that

$$\mu_{N-k^++i}(t) \;\in\; \bigcup_{j \in B_i} I_j^+(\mathbf{d}(t)) \qquad (i = 1, \dots, k^+) \tag{6.33}$$

for all $t \in [0,1]$. At $t = 1$, the right-hand side of (6.33) is an interval that contains $\theta(d_j)$ for all $j \in B_i$. In order to estimate its size, we pick a $j \in B_i$ that is not the largest element of B_i. To streamline notation, abbreviate $d := d_j^+(1)$ and $d' := d_{j+1}^+(1)$. Our first task is to estimate $d' - d$. Since $I_j^+(\mathbf{d}(1)) \cap I_{j+1}^+(\mathbf{d}(1)) \neq \emptyset$, we have

$$\biggl(1 - \frac{1}{(d')^2}\biggr)(d' - d) \;\leqslant\; \theta(d') - \theta(d) \;\leqslant\; 2\varphi^{\widetilde{C}_3} N^{-1/2} (d' - 1)^{1/2}\,.$$

where the second inequality follows from the definition of $I_i^+(\cdot)$. This yields

$$d' - d \;\leqslant\; C\varphi^{\widetilde{C}_3} N^{-1/2} (d' - 1)^{-1/2} \;\leqslant\; C\varphi^{\widetilde{C}_3} N^{-1/2} (d - 1)^{-1/2}\,,$$

where the constant C depends only on Σ. Thus we get

$$(d' - 1)^{1/2} \;\leqslant\; (d - 1)^{1/2}\biggl(1 + \frac{d' - d}{d - 1}\biggr) \;\leqslant\; (d - 1)^{1/2}\Bigl(1 + C\varphi^{\widetilde{C}_3} N^{-1/2}(d - 1)^{-3/2}\Bigr) \;\leqslant\; (d - 1)^{1/2}(1 + o(1))\,,$$

where the last inequality follows from (6.13). Repeating this estimate of $\theta(d_{j+1}^+(1)) - \theta(d_j^+(1))$ for the remaining $j \in B_i$, we find

$$\operatorname{diam}\biggl(\bigcup_{j \in B_i} I_j^+(\mathbf{d}(1))\biggr) \;\leqslant\; (2|B_i| + 2)\,\varphi^{\widetilde{C}_3} N^{-1/2} \min_{j \in B_i} (d_j^+(1) - 1)^{1/2}(1 + o(1))\,.$$

This immediately yields

$$|\mu_{N-k^++i}(1) - \theta(d_i^+)| \;\leqslant\; \varphi^{\widetilde{C}_3+1} N^{-1/2} (d_i^+(1) - 1)^{1/2} \qquad (i = 1, \dots, k^+)\,,$$

and the claim follows.

What remains is the analysis of the extremal bulk eigenvalues. Once again, we make use of a continuity argument. As before, we only consider positive eigenvalues, $\lambda_\alpha \geqslant 2 - \varphi^{\widetilde{K}} N^{-2/3}$ for some $\widetilde{K}$ to be chosen below. Note that by interlacing, Lemma 6.2, we have

$$\lambda_{\alpha-k} \;\leqslant\; \mu_\alpha \;\leqslant\; \lambda_{\alpha+k} \tag{6.34}$$

(using the convention that $\lambda_\alpha = +\infty$ for $\alpha > N$). Recall the role of K from the assumptions of Theorem 2.7. Therefore using the definition of Ξ (see Theorem 3.7), we find that there is a $\widetilde{K} = \widetilde{K}(K)$ such that if $\alpha \geqslant N - \varphi^K$ then

$$\lambda_{\alpha-k} \;\geqslant\; 2 - \varphi^{\widetilde{K}} N^{-2/3} \qquad \text{and} \qquad \mu_\alpha \;\geqslant\; 2 - \varphi^{\widetilde{K}} N^{-2/3} .$$

Let now α satisfy $N - \varphi^K \leqslant \alpha \leqslant N - k^+$. Using (6.30), (6.31), and Proposition 6.5, we find

$$|\lambda_{\alpha+k^+} - \mu_\alpha(0)| \;\leqslant\; N^{-2/3}\widetilde{\psi}^{-1} \qquad \text{and} \qquad \operatorname{dist}(\mu_\alpha(t), \sigma(H)) \;\leqslant\; N^{-2/3}\widetilde{\psi}^{-1} \tag{6.35}$$

for all $t \in [0,1]$. In addition, we know the two following facts about $\mu_\alpha(t)$, for all $t \in [0,1]$.

(i) $\mu_\alpha(t)$ is in the same connected component of $I^0 \subset \mathbb{R}$ as $\mu_\alpha(0)$ (by continuity of $\mu_\alpha(t)$ and Proposition 6.5).

(ii) $\mu_\alpha(t)$ satisfies the interlacing bound (6.34) for all $t \in [0,1]$.

Let B_α be the set of $\beta = 1, \dots, N$ such that λ_β and λ_α are in the same connected component of I^0. Thus we conclude from (i) and (ii) that

$$\mu_\alpha(t) \;\in\; \bigcup_{\substack{\beta \in B_{\alpha+k^+} : \\ |\alpha+k^+ - \beta| \leqslant k}} \bigl[\lambda_\beta - N^{-2/3}\widetilde{\psi}^{-1} \,,\, \lambda_\beta + N^{-2/3}\widetilde{\psi}^{-1}\bigr] .$$

Thus we get

$$|\lambda_{\alpha+k^+} - \mu_\alpha(t)| \;\leqslant\; 2kN^{-2/3}\widetilde{\psi}^{-1} \tag{6.36}$$

for all $t \in [0,1]$. Choosing

$$C_2 \;:=\; \widetilde{C}_2 + 1 \,, \qquad C_3 \;:=\; \widetilde{C}_3 + 1$$

completes the proof of Theorem 2.7 (recall the definition (6.10)).

7. Distribution of the outliers: proof of Theorem 2.14

7.1. Reduction to the law of $G_{\mathbf{v}^{(i)}\mathbf{v}^{(i)}}(\theta(d_i))$. The following proposition reduces the problem to analysing a single explicit random variable.

Proposition 7.1. *There is a constant C_2, depending on ζ, such that the following holds. Suppose that*

$$|d_i| \;\leqslant\; \Sigma - 1 \,, \qquad \bigl||d_i| - 1\bigr| \;\geqslant\; \varphi^{C_2} N^{-1/3}$$

for all $i = 1, \dots, k$. Suppose moreover that for all $i \in O$ (2.24) holds. Recall the definitions (2.16) and (2.17). Then we have for all $i \in O$

$$N^{1/2}(|d_i|-1)^{-1/2}\big(\mu_{\alpha(i)} - \theta(d_i)\big) \;=\; -(1+O(\varphi^{-1}))(|d_i|+1)N^{1/2}(|d_i|-1)^{1/2}\bigg(G_{\mathbf{v}^{(i)}\mathbf{v}^{(i)}}(\theta(d_i)) + \frac{1}{d_i}\bigg)+O(\varphi^{-1})$$

with ζ-high probability.

Before proving Proposition 7.1, we record the following auxiliary result.

Lemma 7.2. *Let C_1 denote the constant from Theorem 2.3. For any*

$$x \;\in\; \big[-\Sigma, -2-\varphi^{C_1}N^{-2/3}\big] \cup \big[2+\varphi^{C_1}N^{-2/3}, \Sigma\big]$$

and any normalized $\mathbf{v} \in \mathbb{C}^N$ we have

$$\big|\partial_x G_{\mathbf{v}\mathbf{v}}(x) - \partial_x m(x)\big| \;\leqslant\; \varphi^{C_\zeta} N^{-1/3}\kappa_x^{-1} \tag{7.1}$$

with ζ-high probability. More generally, we have, for any normalized $\mathbf{v}, \mathbf{w} \in \mathbb{C}^N$,

$$\big|\partial_x G_{\mathbf{v}\mathbf{w}}(x) - \partial_x m(x)\langle \mathbf{v}\,, \mathbf{w}\rangle\big| \;\leqslant\; \varphi^{C_\zeta} N^{-1/3}\kappa_x^{-1} \tag{7.2}$$

with ζ-high probability.

Proof. By symmetry, we may assume that $x \geqslant 0$. Moreover, (7.2) follows from (7.1) and polarization.

We therefore prove (7.1) for $x \geqslant 0$. We have

$$\partial_x G_{\mathbf{v}\mathbf{v}}(x) \;=\; \sum_\alpha \frac{|\langle \mathbf{u}^{(\alpha)}\,, \mathbf{v}\rangle|^2}{(\lambda_\alpha - x)^2}\,.$$

Choose $x \geqslant 2 + N^{-2/3}\varphi^{C_1}$ and abbreviate $\kappa \equiv \kappa_x$. Thus we get, for $\eta \geqslant \varphi^\zeta N^{-1}$,

$$\begin{aligned}
\bigg|\partial_x G_{\mathbf{v}\mathbf{v}}(x) - \frac{1}{\eta}\operatorname{Im} G_{\mathbf{v}\mathbf{v}}(x+\mathrm{i}\eta)\bigg| \;&=\; \bigg|\sum_\alpha \frac{|\langle \mathbf{u}^{(\alpha)}\,, \mathbf{v}\rangle|^2}{(\lambda_\alpha - x)^2} - \sum_\alpha \frac{|\langle \mathbf{u}^{(\alpha)}\,, \mathbf{v}\rangle|^2}{(\lambda_\alpha - x)^2 + \eta^2}\bigg| \\
&\leqslant\; \frac{\eta^2}{(x-\lambda_N)^2}\frac{1}{\eta}\operatorname{Im} G_{\mathbf{v}\mathbf{v}}(x+\mathrm{i}\eta) \\
&\leqslant\; 2\frac{\eta^2}{\kappa^2}\frac{1}{\eta}\operatorname{Im} G_{\mathbf{v}\mathbf{v}}(x+\mathrm{i}\eta)
\end{aligned}$$

with ζ-high probability, where in the last step we used Theorem 3.7. (In the proof of Theorem 2.3, the constant C_1 was chosen large enough for this application of Theorem 3.7; see (5.6).) A similar calculation using the definition (2.4) yields

$$\bigg|\partial_x m(x) - \frac{1}{\eta}\operatorname{Im} m(x+\mathrm{i}\eta)\bigg| \;\leqslant\; \frac{\eta^2}{\kappa^2}\frac{1}{\eta}\operatorname{Im} m(x+\mathrm{i}\eta)\,.$$

Therefore we get, using Theorem 2.3 and Lemma 3.2,

$$\begin{aligned}
\big|\partial_x G_{\mathbf{v}\mathbf{v}}(x) - \partial_x m(x)\big| \;&\leqslant\; \frac{2\eta}{\kappa^2}\Big(\operatorname{Im} G_{\mathbf{v}\mathbf{v}}(x+\mathrm{i}\eta) + \operatorname{Im} m(x+\mathrm{i}\eta)\Big) + \frac{1}{\eta}\varphi^{C_\zeta}\sqrt{\frac{\operatorname{Im} m(x+\mathrm{i}\eta)}{N\eta}} \\
&\leqslant\; C\eta^2\kappa^{-5/2} + \varphi^{C_\zeta}\big(\eta\kappa^{-2} + \eta^{-1}\big)N^{-1/2}\kappa^{-1/4}
\end{aligned}$$

with ζ-high probability. Choosing $\eta := N^{-1/6}\kappa^{3/4}$ yields the claim. □

Proof of Proposition 7.1. We only prove the claim for the case $d_i > 1$; the case $d_i < -1$ is handled similarly.

For $2 + \varphi^{C_1} N^{-2/3} \leqslant x \leqslant \Sigma$, where C_1 is the constant from Theorem 2.3, we define the $k \times k$ Hermitian matrices $A(x)$ and $\widetilde{A}(x)$ through

$$A_{ij}(x) \;:=\; G_{\mathbf{v}^{(i)}\mathbf{v}^{(j)}}(x) - m(x)\delta_{ij} + d_i^{-1}\delta_{ij}\,, \qquad \widetilde{A}_{ij}(x) \;:=\; \delta_{ij}\Big(G_{\mathbf{v}^{(i)}\mathbf{v}^{(i)}}(x) - m(x) + d_i^{-1}\Big)\,.$$

(Here we subtract $m(x)\mathbb{1}$ so as to ensure that $\partial_x A(x)$ is well-behaved; see below.) We denote the ordered eigenvalues of $A(x)$ and $\widetilde{A}(x)$ by $a_1(x) \leqslant \cdots \leqslant a_k(x)$ and $\widetilde{a}_1(x) \leqslant \cdots \leqslant \widetilde{a}_k(x)$ respectively.

For the rest of the proof we fix $i \in O$ satisfying $d_i > 1$. We abbreviate $\theta_i := \theta(d_i)$. We begin by comparing the eigenvalues of $\widetilde{A}(\theta_i)$ and D^{-1}. Define the eigenvalue index $r \equiv r(i) = 1, \ldots, k$ through

$$\widetilde{a}_r(x) \;=\; \frac{1}{d_i} + G_{\mathbf{v}^{(i)}\mathbf{v}^{(i)}}(x) - m(x)\,. \tag{7.3}$$

In particular,

$$\widetilde{a}_r(\theta_i) \;=\; G_{\mathbf{v}^{(i)}\mathbf{v}^{(i)}}(\theta_i) + \frac{2}{d_i}\,.$$

Theorem 2.3 implies that

$$\Big|G_{\mathbf{v}^{(j)}\mathbf{v}^{(j)}}(\theta_i) - m(\theta_i)\Big| \;\leqslant\; \varphi^{C_\zeta} N^{-1/2}(d_i - 1)^{-1/2}\,. \tag{7.4}$$

with ζ-high probability for $j = 1, \ldots, k$. In particular,

$$\bigg|\widetilde{a}_r(\theta_i) - \frac{1}{d_i}\bigg| \;\leqslant\; \varphi^{C_\zeta} N^{-1/2}(d_i - 1)^{-1/2}$$

with ζ-high probability. Moreover, (7.4) and the condition (2.24) yield, for $j \neq i$,

$$\Big|G_{\mathbf{v}^{(j)}\mathbf{v}^{(j)}}(\theta_i) - m(\theta_i)\Big| \;\ll\; |d_i - d_j| \tag{7.5}$$

with ζ-high probability, provided C_2 is chosen large enough. We therefore conclude that

$$\min_{j \neq r}\big|\widetilde{a}_j(\theta_i) - \widetilde{a}_r(\theta_i)\big| \;\geqslant\; \varphi^{C_2 - 1} N^{-1/2}(d_i - 1)^{-1/2} \tag{7.6}$$

with ζ-high probability, provided C_2 is large enough.

Next, we compare the eigenvalues of $A(\theta_i)$ and $\widetilde{A}(\theta_i)$ using second-order perturbation theory (the first-order correction vanishes by definition of $\widetilde{A}$ and A). Theorem 2.3 yields

$$\|A(\theta_i) - \widetilde{A}(\theta_i)\| \;\leqslant\; \varphi^{C_\zeta} N^{-1/2}(d_i - 1)^{-1/2}$$

with ζ-high probability. Therefore (7.6) and nondegenerate second-order perturbation theory yield, for large enough C_2,

$$a_r(\theta_i) \;=\; \widetilde{a}_r(\theta_i) + O\bigg(\frac{\varphi^{C_\zeta} N^{-1}(d_i - 1)^{-1}}{\min_{j \neq r}|\widetilde{a}_j(\theta_i) - \widetilde{a}_r(\theta_i)|}\bigg) \;=\; \widetilde{a}_r(\theta_i) + O\Big(\varphi^{C_\zeta - C_2} N^{-1/2}(d_i - 1)^{-1/2}\Big) \tag{7.7}$$

with ζ-high probability.

Next, we analyse $A(x)$ and make the link to $\mu_{\alpha(i)}$. From Lemma 7.2 we find

$$\big\|\partial_x A(x)\big\| \;\leqslant\; \varphi^{C_\zeta} N^{-1/3} \kappa_x^{-1}$$

with ζ-high probability. In particular, we have for all $j = 1, \dots, k$ that

$$\big|a_j(x) - a_j(y)\big| \;\leqslant\; \varphi^{C_\zeta} N^{-1/3} (\kappa_x^{-1} + \kappa_y^{-1}) |x - y| \tag{7.8}$$

with ζ-high probability, provided that $2 + \varphi^{C_1} N^{-2/3} \leqslant x, y \leqslant \Sigma$.

Recall the definition (2.17) of $\alpha(i)$. From Lemma 6.1 and Theorem 3.7, we know that $\mu_{\alpha(i)}$ is characterized by the property that there is a $q \equiv q(i) \in \{1, \dots, k\}$ such that

$$a_q(\mu_{\alpha(i)}) \;=\; -m(\mu_{\alpha(i)})\,.$$

By Theorem 2.7 we have

$$|\mu_{\alpha(i)} - \theta_i| \;\leqslant\; \varphi^{C_3} N^{-1/2} (d_i - 1)^{1/2} \tag{7.9}$$

with ζ-high probability. Provided C_2 is large enough (depending on C_3), it is easy to see from (7.9) that

$$\mu_{\alpha(i)} - 2 \;\asymp\; \theta_i - 2 \;\asymp\; (d_i - 1)^2 \tag{7.10}$$

with ζ-high probability. Thus we find, using (7.8), (7.9), and (7.10), that for large enough C_2 we have

$$m(\mu_{\alpha(i)}) \;=\; -a_q(\theta_i) + O\Big(\varphi^{C_\zeta} N^{-5/6} (d_i - 1)^{-3/2}\Big) \tag{7.11}$$

with ζ-high probability. (Here we absorbed the constant C_3 into C_ζ.)

We now prove that $q = r$ with ζ-high probability provided C_2 is large enough. Assume by contradiction that $q \neq r$. Then we get, using Theorem 2.3 and the condition (2.24), that

$$\bigg|a_q(\theta_i) - \frac{1}{d_i}\bigg| \;\geqslant\; \varphi^{C_2 - 1} N^{-1/2} (d_i - 1)^{-1/2} \tag{7.12}$$

with ζ-high probability. Moreover, (7.8), (7.9), and (7.10) yield

$$\begin{aligned}
a_q(\theta_i) \;&=\; a_q(\mu_{\alpha(i)}) + O\Big(\varphi^{C_\zeta} N^{-5/6} (d_i - 1)^{-3/2}\Big) \\
&=\; -m(\mu_{\alpha(i)}) + O\Big(\varphi^{C_\zeta} N^{-5/6} (d_i - 1)^{-3/2}\Big) \\
&=\; \frac{1}{d_i} + O\Big(\varphi^{C_\zeta} N^{-1/2} (d_i - 1)^{-1/2} + \varphi^{C_\zeta} N^{-5/6} (d_i - 1)^{-3/2}\Big)
\end{aligned}$$

with ζ-high probability, where in the last step we used (6.5). Together with (7.12), this yields the desired contradiction provided C_2 is large enough. Hence $q = r$.

Putting (7.3), (7.11), and (7.7) together, we get

$$m(\mu_{\alpha(i)}) \;=\; -G_{\mathbf{v}^{(i)}\mathbf{v}^{(i)}}(\theta_i) - \frac{2}{d_i} + O\Big(\varphi^{C_\zeta} N^{-5/6} (d_i - 1)^{-3/2} + \varphi^{C_\zeta - C_2} N^{-1/2} (d_i - 1)^{-1/2}\Big)$$

with ζ-high probability. Thus we find that, for all x between θ_i and $\mu_{\alpha(i)}$, we have

$$m'(x) \;=\; m'(\theta_i) + O(\varphi^{C_3} N^{-1/2}(d_i-1)^{-5/2}) \;=\; m'(\theta_i)(1+O(\varphi^{-1}))$$

with ζ-high probability, where we used (6.5) and (7.9). Using (6.2), (7.10), and (6.5), we conclude that

$$\mu_{\alpha(i)} - \theta_i \;=\; -(1+O(\varphi^{-1}))\frac{G_{\mathbf{v}^{(i)}\mathbf{v}^{(i)}}(\theta_i) + d_i^{-1}}{m'(\theta_i)} + O\Big(\varphi^{C_\zeta} N^{-5/6}(d_i-1)^{-1/2} + \varphi^{C_\zeta - C_2} N^{-1/2}(d_i-1)^{1/2}\Big)$$

with ζ-high probability. The claim now follows for large enough C_2, using the identity (6.2). □

7.2. The GOE/GUE case. By Proposition 7.1, it is enough to analyse the random variable

$$X \;:=\; N^{1/2}(|d|+1)(|d|-1)^{1/2}\bigg(G_{\mathbf{v}\mathbf{v}}(\theta) + \frac{1}{d}\bigg)\,, \tag{7.13}$$

where $\mathbf{v} \in \mathbb{C}^N$ is normalized, d satisfies

$$1 + \varphi^{C_2} N^{-1/3} \;\leqslant\; |d| \;\leqslant\; \Sigma - 1\,, \tag{7.14}$$

and we abbreviated $\theta \;\equiv\; \theta(d)$. For definiteness, we choose $d > 1$ in the following.

The following notion of convergence of random variables is convenient for our needs.

Definition 7.3. *Two sequences of random variables, $\{A_N\}$ and $\{B_N\}$, are* asymptotically equal in distribution*, denoted $A_N \overset{d}{\sim} B_N$, if they are tight and satisfy*

$$\lim_{N\to\infty}\big(\mathbb{E}f(A_N) - \mathbb{E}f(B_N)\big) \;=\; 0 \tag{7.15}$$

for all bounded and continuous f.

Remark 7.4. Definition 7.3 extends the notion of convergence in distribution, in the sense that $\mathbb{E}f(A_N)$ need not have a limit as $N \to \infty$.

Remark 7.5. In order to show that $A_N \overset{d}{\sim} B_N$, it suffices to establish the tightness of either $\{A_N\}$ or $\{B_N\}$ and to verify (7.15) for all $f \in C_c^\infty(\mathbb{R})$. Indeed, if $\{A_N\}$ is tight then so is $\{B_N\}$, by (7.15). By tightness of A_N and B_N, we may replace in (7.15) the bounded and continuous f with a compactly supported continuous function g. Next, we can approximate g uniformly with C_c^∞-functions.

Remark 7.6. Clearly, $A_N \overset{d}{\sim} B_N$ if $A_N \overset{d}{=} B_N$ for all N.

Lemma 7.7. *Let $A_N \overset{d}{\sim} B_N$ and R_N satisfy $\lim_N \mathbb{P}(|R_N| \leqslant \varepsilon_N) = 1$, where $\{\varepsilon_N\}$ is a positive null sequence. Then $A_N \overset{d}{\sim} B_N + R_N$.*

Proof. By Remark 7.5, it suffices to prove (7.15) for $f \in C^1(\mathbb{R})$ such that f and f' are bounded. Then

$$\begin{aligned}
\mathbb{E}f(A_N) - \mathbb{E}f(B_N+R_N) \;&=\; \big(\mathbb{E}f(A_N) - \mathbb{E}f(B_N)\big) + \big(\mathbb{E}f(B_N) - \mathbb{E}f(B_N+R_N)\big) \\
&=\; o(1) + \mathbb{E}\Big[\mathbf{1}(|R_N| \leqslant \varepsilon_N)\big(f(B_N) - f(B_N+R_N)\big)\Big] \\
&=\; o(1)
\end{aligned}$$

where in the last step we used the boundedness of f'. □

LEMMA 7.8. *Let $\{A_N\}$, $\{A'_N\}$, $\{B_N\}$, and $\{B'_N\}$ be sequences of random variables. Suppose that $A_N \overset{d}{\sim} A'_N$, $B_N \overset{d}{\sim} B'_N$, A_N and B_N are independent, and A'_N and B'_N are independent. Then*

$$A_N + B_N \overset{d}{\sim} A'_N + B'_N\,.$$

PROOF. Without loss of generality, we may assume that A_N, B_N, A'_N, B'_N are independent (after replacing A'_N and B'_N with new random variables without changing their laws.) Then for any $\lambda \in \mathbb{R}$ we have

$$\begin{aligned}\mathbb{E}\,\mathrm{e}^{\mathrm{i}\lambda(A_N+B_N)} - \mathbb{E}\,\mathrm{e}^{\mathrm{i}\lambda(A'_N+B'_N)} &= \mathbb{E}\Big[\mathrm{e}^{\mathrm{i}\lambda A_N}(\mathrm{e}^{\mathrm{i}\lambda B_N} - \mathrm{e}^{\mathrm{i}\lambda B'_N}) + (\mathrm{e}^{\mathrm{i}\lambda A_N} - \mathrm{e}^{\mathrm{i}\lambda A'_N})\mathrm{e}^{\mathrm{i}\lambda B'_N}\Big] \\ &= \mathbb{E}\,\mathrm{e}^{\mathrm{i}\lambda A_N}\,\mathbb{E}(\mathrm{e}^{\mathrm{i}\lambda B_N} - \mathrm{e}^{\mathrm{i}\lambda B'_N}) + \mathbb{E}(\mathrm{e}^{\mathrm{i}\lambda A_N} - \mathrm{e}^{\mathrm{i}\lambda A'_N})\,\mathbb{E}\,\mathrm{e}^{\mathrm{i}\lambda B'_N} \\ &\longrightarrow 0\end{aligned}$$

as $N \to \infty$.

Next, we observe that $A_N + B_N$ and $A'_N + B'_N$ are tight. Therefore, recalling Remark 7.5, we find that it suffices to prove

$$\mathbb{E}\,f(A_N + B_N) - \mathbb{E}\,f(A'_N + B'_N) \longrightarrow 0$$

$f \in C_c^\infty$. Denoting by $\hat f$ the Fourier transform of f, we find

$$\mathbb{E}\,f(A_N + B_N) - \mathbb{E}\,f(A'_N + B'_N) = \int \mathrm{d}\lambda\,\hat f(\lambda)\,\Big[\mathbb{E}\,\mathrm{e}^{\mathrm{i}\lambda(A_N+B_N)} - \mathbb{E}\,\mathrm{e}^{\mathrm{i}\lambda(A'_N+B'_N)}\Big] \longrightarrow 0$$

by dominated convergence. □

PROPOSITION 7.9. *Let H be a GOE/GUE matrix. Assume that d satisfies (7.14). Then for large enough C_2 we have*

$$X \overset{d}{\sim} \mathcal{N}\bigg(0, \frac{2(d+1)}{\beta d^2}\bigg)\,.$$

PROOF. By unitary invariance, we have $G_{\mathbf{v}\mathbf{v}} \overset{d}{=} G_{11}$, where $\overset{d}{=}$ denotes equality in distribution. In order to handle the exceptional low-probability events, we add a small imaginary part to the spectral parameter $z := \theta + \mathrm{i}N^{-4}$. Throughout the following we abbreviate $G \equiv G(z)$ and $m \equiv m(z)$. Writing $\mathbf{a}^* := (h_{12}, h_{13}, \dots, h_{1N})$, we get from Schur's formula and (2.5) that

$$\begin{aligned}G_{11} = \frac{1}{h_{11} - z - \mathbf{a}^* G^{(1)}\mathbf{a}} &= \frac{1}{-m - z + h_{11} - \big(\mathbf{a}^* G^{(1)}\mathbf{a} - m\big)} \\ &= m - m^2 h_{11} + m^2\big(\mathbf{a}^* G^{(1)}\mathbf{a} - m\big) + O(|h_{11}|^2) + O\Big(\big|\mathbf{a}^* G^{(1)}\mathbf{a} - m\big|^2\Big)\end{aligned} \tag{7.16}$$

with ζ-high probability. Again by unitary invariance, we have $\mathbf{a}^* G^{(1)}\mathbf{a} \overset{d}{=} \|\mathbf{a}\|^2 G^{(1)}_{22}$. Moreover, both sides are independent of h_{11}, so that

$$-m^2 h_{11} + m^2\big(\mathbf{a}^* G^{(1)}\mathbf{a} - m\big) \overset{d}{=} -m^2 h_{11} + m^2\big(\|\mathbf{a}\|^2 G^{(1)}_{22} - m\big)\,. \tag{7.17}$$

In order to estimate the error term in (7.16), we write

$$\|\mathbf{a}\|^2 G^{(1)}_{22} - m = \big(\|\mathbf{a}\|^2 - 1\big)G^{(1)}_{22} + (G^{(1)}_{22} - m)\,. \tag{7.18}$$

Using (3.6) to estimate $G_{22}^{(1)} - G_{22}$, as well as Theorem 2.3, Lemma 3.5, and Lemma 3.2, we therefore find that

$$\big|\|\mathbf{a}\|^2 G_{22}^{(1)} - m\big| \;\leqslant\; \varphi^{C_\zeta} N^{-1/2}(d-1)^{-1/2} \tag{7.19}$$

with ζ-high probability. Moreover, we have the trivial bound $\mathbb{E}\big|\|\mathbf{a}\|^2 G_{22}^{(1)} - m\big|^k \leqslant (kN)^{Ck}$ for $k \in \mathbb{N}$.

From (7.16), (7.17), (7.18), and (7.19), we conclude that there exist random variables $\widetilde{R}_1$ and $\widetilde{R}_2$ satisfying

$$|\widetilde{R}_1| + |\widetilde{R}_2| \;\leqslant\; \varphi^{C_\zeta} N^{-1}(d-1)^{-1} \tag{7.20}$$

with ζ-high probability, the rough bound

$$\mathbb{E}(|\widetilde{R}_1| + |\widetilde{R}_2|)^k \;\leqslant\; (kN)^{Ck}\,, \tag{7.21}$$

and

$$\begin{aligned}\big(G_{11}^{(2)} - m\big) + \widetilde{R}_1 \;&=\; -m^2 h_{11} + m^2\big(\mathbf{a}^* G^{(1)} \mathbf{a} - m\big)\\ &\overset{d}{=}\; -m^2 h_{11} + m^2\big(\|\mathbf{a}\|^2 G_{22}^{(1)} - m\big)\\ &=\; -m^2 h_{11} + m^3\big(\|\mathbf{a}\|^2 - 1\big) + m^2(G_{22}^{(1)} - m) + \widetilde{R}_2\,.\end{aligned}$$

Defining

$$\begin{aligned} Y_1 \;&:=\; N^{1/2}(d+1)(d-1)^{1/2}\operatorname{Re}\big(G_{11}^{(2)} - m\big)\,, \qquad & Y_2 \;&:=\; N^{1/2}(d+1)(d-1)^{1/2}\operatorname{Re}\big(G_{22}^{(1)} - m\big)\,,\\ W \;&:=\; N^{1/2}\operatorname{Re}\Big(-m^2 h_{11} + m^3\big(\|\mathbf{a}\|^2 - 1\big)\Big)\,, \qquad & R_i \;&:=\; N^{1/2}(d+1)(d-1)^{1/2}\operatorname{Re}\widetilde{R}_i \quad (i = 1,2)\,,\end{aligned}$$

we therefore get

$$Y_1 + R_1 \;\overset{d}{=}\; (d+1)(d-1)^{1/2} W + m^2 Y_2 + R_2\,. \tag{7.22}$$

In order to infer the distribution of Y_1 from (7.22), we observe that the random variables Y_2 and W are independent. Also, $Y_1 \overset{d}{=} Y_2$. Recalling Theorem 2.3 and (3.6), we find the bounds

$$|Y_i| \;\leqslant\; \varphi^{C_\zeta}\,, \qquad |R_i| \;\leqslant\; \varphi^{C_\zeta} N^{-1/2}(d-1)^{-1/2} \qquad (i = 1,2) \tag{7.23}$$

with ζ-high probability, and the rough bounds

$$|Y_i| \;\leqslant\; N^2\,, \qquad \mathbb{E}|R_i|^k \;\leqslant\; (kN)^{Ck} \qquad (i = 1,2)\,. \tag{7.24}$$

Moreover, by the Central Limit Theorem

$$\left(\frac{2(d^2+1)}{\beta d^6}\right)^{-1} W \;\overset{d}{\sim}\; \mathcal{N}(0,1)\,, \tag{7.25}$$

where we used (6.2).

Next, let B and Z_2 be independent random variables whose laws are given by

$$B \;\overset{d}{=}\; \mathcal{N}\bigg(0, \frac{2(d^2+1)}{\beta d^6}\bigg)\,, \qquad Z_2 \;\overset{d}{=}\; \mathcal{N}(0, \xi^2)\,,$$

where we introduced

$$\xi^2 \;\equiv\; \xi_N^2 \;:=\; d^4 \frac{(d^2-1)(d+1)}{d^4-1} \, \frac{2(d^2+1)}{\beta d^6} \;=\; \frac{2(d+1)}{\beta d^2}\,.$$

Defining

$$Z_1 \;:=\; (d+1)(d-1)^{1/2} B + d^{-2} Z_2\,, \tag{7.26}$$

we find that $Z_1 \overset{d}{=} Z_2$. Moreover, a standard moment calculation and the definition of W yield

$$\lim_{N\to\infty} (\mathbb{E}W^k - \mathbb{E}B^k) \;=\; 0\,; \tag{7.27}$$

as usual, only the pairings in the moment expansion of $\mathbb{E}W^k$ survive the limit $N \to \infty$. (See also (7.25), which however cannot be used to deduce (7.27) directly.)

We now compare the distributions of Y_1 and Z_1 by computing moments. Note that the family $\{\mathbb{E}Z_1^k\}_{N\in\mathbb{N}}$ is bounded for each $k \in \mathbb{N}$. We claim that

$$\lim_{N\to\infty} \bigl(\mathbb{E}Y_1^k - \mathbb{E}Z_1^k\bigr) \;=\; 0 \tag{7.28}$$

for all $k \in \mathbb{N}$. (This will imply that $Y_1 \overset{d}{\sim} Z_1$.) We shall prove (7.28) by induction on k. Taking the expectation of (7.22) yields

$$\mathbb{E}Y_1 \;=\; m^2 \mathbb{E}Y_1 + O\Bigl(\varphi^{C_\zeta} N^{-1/2} (d-1)^{-1/2}\Bigr)$$

where we used (7.23), (7.24), and $\mathbb{E}W = O(N^{-1/2})$. Therefore

$$\mathbb{E}Y_1 \;\leqslant\; C\varphi^{C_\zeta} N^{-1/2} (d-1)^{-3/2} \;=\; o(1)$$

provided C_2 in (7.14) is large enough. Here we used that

$$m(z) \;=\; d^{-1} + O(N^{-3})\,, \tag{7.29}$$

as follows from the definition of $z = \theta + \mathrm{i}N^{-4}$, (5.5), Lemma 3.2, and (6.2). Therefore (7.28) for $k = 1$ follows using $\mathbb{E}Z_1 = 0$.

For the induction step, we assume that (7.28) holds for all $k' \leqslant k-1$. From (7.22) we find

$$\begin{aligned} &\mathbb{E}Y_1^k + \sum_{l=1}^k \binom{k}{l} \mathbb{E}\bigl(R_1^l Y_1^{k-l}\bigr) \\ &\qquad = \; \mathbb{E}\bigl((d+1)(d-1)^{1/2} W + m^2 Y_2\bigr)^k + \sum_{l=1}^k \binom{k}{l} \mathbb{E}\Bigl(R_1^l \bigl((d+1)(d-1)^{1/2} W + m^2 Y_2\bigr)^{k-l}\Bigr)\,. \end{aligned} \tag{7.30}$$

We estimate the summands on the left-hand side by

$$\begin{aligned} \bigl|\mathbb{E}\bigl(R_1^l Y_1^{k-l}\bigr)\bigr| \;&\leqslant\; N^C \exp(-\varphi^C) + \Bigl(\varphi^{C_\zeta} N^{-1/2} (d-1)^{-1/2}\Bigr)^l \mathbb{E}|Y_1|^{k-l} \\ &\leqslant\; C\Bigl(\varphi^{C_\zeta} N^{-1/2} (d-1)^{-1/2}\Bigr)^l \varphi^{C_\zeta} \\ &\leqslant\; \varphi^{C_\zeta} N^{-1/2} (d-1)^{-1/2}\,, \end{aligned}$$

where in the first step we used (7.23) and (7.24), in the second step the estimate $\mathbb{E}|Y_1|^{k-l} \leqslant \varphi^{C_\zeta}$ as follows from the induction assumption (7.28) applied to even moments (recall that Y_1 is real) as well as (7.23) and (7.24), and in the third step the fact that $l \geqslant 1$. Note that the constant C_ζ is independent of k. A similar estimate applies to the summands on the right-hand side of (7.30). Thus (7.30) yields

$$\begin{aligned}\mathbb{E}Y_1^k &= \mathbb{E}\big((d+1)(d-1)^{1/2}W + m^2 Y_2\big)^k + O\big(\varphi^{C_\zeta} N^{-1/2}(d-1)^{-1/2}\big) \\ &= m^{2k}\,\mathbb{E}Y_1^k + \sum_{l=2}^{k}\binom{k}{l}\mathbb{E}\big((d+1)(d-1)^{1/2}W\big)^l\,\mathbb{E}\big(m^2 Y_2\big)^{k-l} + O\big(\varphi^{C_\zeta} N^{-1/2}(d-1)^{-1/2}\big)\,,\end{aligned}$$

where in the second step we used the induction assumption and the estimate $\mathbb{E}W = O(N^{-1/2})$. Therefore we get

$$\mathbb{E}Y_1^k = \frac{1}{1-m^{2k}}\sum_{l=2}^{k}\binom{k}{l}\mathbb{E}\big((d+1)(d-1)^{1/2}W\big)^l\,\mathbb{E}\big(m^2Y_2\big)^{k-l} + O\big(\varphi^{C_\zeta}N^{-1/2}(d-1)^{-3/2}\big)\,, \tag{7.31}$$

where we used (7.29).

In order to conclude the proof of (7.28), we deduce from (7.26) that

$$\mathbb{E}Z_1^k = \frac{1}{1-d^{-2k}}\sum_{l=2}^{k}\binom{k}{l}\mathbb{E}\big((d+1)(d-1)^{1/2}B\big)^l\,\mathbb{E}\big(d^{-2}Z_2\big)^{k-l}\,. \tag{7.32}$$

Using the induction assumption (7.28) for $k' = k - l$, (7.29), and the condition $l \geqslant 2$, we get from (7.31), (7.32), and (7.27) that

$$\lim_{N\to\infty}\big(\mathbb{E}Y_1^k - \mathbb{E}Z_1^k\big) = 0$$

for large enough C_2. This concludes the proof of (7.28).

Next, by definition we have $\xi^{-1}Z_1 \overset{d}{=} \mathcal{N}(0,1)$. Moreover, we have that $\xi \in [c, C]$ for some positive constants c and C depending only on Σ. Together with (7.28) for $k = 2$, we infer that the families $\{\xi^{-1}Y_1\}_{N\in\mathbb{N}}$ and $\{\xi^{-1}Z_1\}_{N\in\mathbb{N}}$ are tight. Therefore we get from (7.28) that

$$\lim_{N\to\infty}\big(\mathbb{E}f(\xi^{-1}Y_1) - \mathbb{E}f(\xi^{-1}Z_1)\big) = 0 \tag{7.33}$$

for any continuous bounded function f. Next, we estimate

$$\begin{aligned}\big|G_{11}(\theta) - G_{11}^{(2)}(z)\big| &\leqslant \big|G_{11}(\theta) - G_{11}(z)\big| + \big|G_{11}(z) - G_{11}^{(2)}(z)\big| \\ &\leqslant N^{-4}N^2 + \varphi^{C_\zeta}N^{-1}(d-1)^{-1} \leqslant \varphi^{C_\zeta}N^{-1}(d-1)^{-1}\end{aligned}$$

with ζ-high probability, where in the second step we used Lemma 7.2, (5.5), and Lemma 3.2 to estimate the first term, and Theorem 2.3 and (6.1) to estimate the second term. Therefore

$$X \overset{d}{=} N^{1/2}(d+1)(d-1)^{1/2}\big(G_{11}(\theta) + d^{-1}\big) = Y_1 + O\big(\varphi^{C_\zeta}N^{-1/2}(d-1)^{-1/2}\big) = Y_1 + o(1)$$

with ζ-high probability, where in the second step we used (7.29). Therefore (7.33), the fact that $Z \overset{d}{=} Z_1$, and dominated convergence yield

$$\lim_{N\to\infty}\big(\mathbb{E}f(\xi^{-1}X) - \mathbb{E}f(\xi^{-1}Z)\big) = 0\,. \tag{7.34}$$

The claim now follows from Lemma 7.10 below. □

Lemma 7.10. *Let $\{\xi_N\}$ be a bounded deterministic sequence. Let $A_\infty, A_1, A_2, \dots$ be random variables such that A_N converges weakly to A_∞. Then we have for any bounded continuous function f*

$$\mathbb{E}f(\xi_N A_N) - \mathbb{E}f(\xi_N A_\infty) \longrightarrow 0$$

as $N \to \infty$.

Proof. By Skorokhod's representation theorem, there exist new random variables $\widetilde{A}_\infty, \widetilde{A}_1, \widetilde{A}_2, \dots$ such that $A_\infty \overset{d}{=} \widetilde{A}_\infty$, $A_N \overset{d}{=} \widetilde{A}_N$ for all $N \in \mathbb{N}$, and $\widetilde{A}_N \to \widetilde{A}_\infty$ almost surely. Let ω be such that $\widetilde{A}_N(\omega) \to \widetilde{A}_\infty(\omega)$. By assumption on ξ_N, we find that there exists a $C \equiv C(\omega)$ such that $\xi_N \widetilde{A}_N(\omega) \in [-C, C]$ and $\xi_N \widetilde{A}_\infty(\omega) \in [-C, C]$ for all $N \in \mathbb{N}$. Since f is uniformly continuous on $[-C, C]$, we find that

$$\lim_{N \to \infty} \Big(f(\xi_N \widetilde{A}_N(\omega)) - f(\xi_N \widetilde{A}_\infty(\omega)) \Big) = 0 .$$

The claim now follows by dominated convergence. □

7.3. The almost-GOE/GUE case. As it turns out, replacing the matrix element h_{ij} with a Gaussian in the Green function comparison step below (Section 7.4) is only possible if $|v_i| \leqslant \varphi^{-D}$ and $|v_i| \leqslant \varphi^{-D}$, for some large enough constant $D > 0$. If this assumption is not satisfied, we first have to replace h_{ij} with a Gaussian using a different method, which effectively keeps track of the fluctuations of $G_{\mathbf{v}\mathbf{v}}$ resulting from large components of $\mathbf{v}$. Thus we shall proceed in two steps:

(i) We compare the original Wigner matrix H with $\widehat{H}$, a Wigner matrix obtained from H by replacing the (i, j)-th entry of H with a Gaussian whenever $|v_i| \leqslant \varphi^{-D}$ and $|v_j| \leqslant \varphi^{-D}$.

(ii) We compare the matrix $\widehat{H}$ to a Gaussian matrix.

The step (ii) is performed in this section. To simplify notation, we write H instead of $\widehat{H}$ throughout this section. The step (i) is performed using Green function comparison in Section 7.4 below.

The following shorthand will prove useful.

Definition 7.11. *Let $\{\sigma_N\}$ be a bounded positive sequence. If A_N and B_N are independent random variables with $B_N \overset{d}{\sim} \mathcal{N}(0, \sigma_N^2)$, and if $S_N \overset{d}{\sim} A_N + B_N$, then we write*

$$S_N \overset{d}{\sim} A_N + \mathcal{N}(0, \sigma_N^2) .$$

For the following we write

$$X = \nu N^{1/2} \bigg(G_{\mathbf{v}\mathbf{v}}(\theta) + \frac{1}{d} \bigg) , \qquad \nu \equiv \nu_N := (d+1)(d-1)^{1/2} .$$

Proposition 7.12. *Fix $D > 0$. Let $\mathbf{v} \in \mathbb{C}^N$ be normalized and H be a Wigner matrix such that if $|v_i| \leqslant \varphi^{-D}$ and $|v_j| \leqslant \varphi^{-D}$ then h_{ij} is Gaussian. Then we have*

$$X \overset{d}{\sim} -\nu N^{1/2} d^{-2} \langle \mathbf{v}, H \mathbf{v} \rangle + \mathcal{N}\Bigg(0 , \, \frac{2(d+1)}{\beta d^4} + \frac{4 \nu^2 Q(\mathbf{v})}{d^5} + \frac{\nu^2 R(\mathbf{v})}{d^6} \Bigg) ,$$

where $Q(\mathbf{v})$ and $R(\mathbf{v})$ were defined in (2.22).

PROOF. As before, we consistently drop the spectral parameter $z = \theta$ from our notation.

Let $M \in \mathbb{N}$ denote the number of entries of $\mathbf{v}$ satisfying $|v_i| > \varphi^{-D}$. Since $\mathbf{v}$ is normalized, we have $M \leqslant \varphi^{2D}$. To simplify notation, we assume (after a suitable permutation of the rows and columns of H) that the entries of $\mathbf{v}$ satisfy $|v_i| > \varphi^{-D}$ for $i \leqslant M$ and $|v_i| \leqslant \varphi^{-D}$ for $i > M$. Split $\mathbf{v} = \binom{\mathbf{u}}{\mathbf{w}}$, where $\mathbf{u} \in \mathbb{C}^M$ and $\mathbf{w} \in \mathbb{C}^{N-M}$. (Throughout the following we assume that $\mathbf{w} \neq 0$; the case $\mathbf{w} = 0$ may be easily handled by approximation with nonzero $\mathbf{w}$.) We also split

$$H \;=\; \begin{pmatrix} A & B^* \\ B & H_0 \end{pmatrix},$$

where A is an $M \times M$ matrix and H_0 an $(N-M) \times (N-M)$ matrix with Gaussian entries. Choose a deterministic orthogonal/unitary $(N-M) \times (N-M)$ matrix S such that $S\mathbf{w} = (\|\mathbf{w}\|, 0, \dots, 0)^*$. Thus we get

$$\begin{aligned} G_{\mathbf{v}\mathbf{v}} \;&=\; \mathbf{v}^* \begin{pmatrix} \mathbb{1} & 0 \\ 0 & S^* \end{pmatrix} \begin{pmatrix} \mathbb{1} & 0 \\ 0 & S \end{pmatrix} \begin{pmatrix} A - z & B^* \\ B & H_0 - z \end{pmatrix}^{-1} \begin{pmatrix} \mathbb{1} & 0 \\ 0 & S^* \end{pmatrix} \begin{pmatrix} \mathbb{1} & 0 \\ 0 & S \end{pmatrix} \mathbf{v} \\ &\overset{d}{=}\; \begin{pmatrix} \mathbf{u} \\ \|\mathbf{w}\| \\ 0 \end{pmatrix}^* \begin{pmatrix} A - z & B^* S^* \\ SB & H_0 - z \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{u} \\ \|\mathbf{w}\| \\ 0 \end{pmatrix}, \end{aligned}$$

where we used that $SH_0S^* \overset{d}{=} H_0$ and the fact that A, B, and H_0 are independent.

Next, we split

$$S \;=\; \begin{pmatrix} \mathbf{w}^*/\|\mathbf{w}\| \\ \widetilde{S} \end{pmatrix}, \qquad H_0 \;=\; \begin{pmatrix} g & \mathbf{a}^* \\ \mathbf{a} & H_1 \end{pmatrix},$$

where $\mathbf{a} \in \mathbb{C}^{N-M-1}$ is a vector of i.i.d. Gaussians. Note that $\widetilde{S}^*$ is an isometry, i.e. $\widetilde{S}\widetilde{S}^* = \mathbb{1}$. Thus we may write

$$\begin{aligned} G_{\mathbf{v}\mathbf{v}} \;&\overset{d}{=}\; \begin{pmatrix} \mathbf{u} \\ \|\mathbf{w}\| \\ 0 \end{pmatrix}^* \begin{pmatrix} A - z & B^*\mathbf{w}/\|\mathbf{w}\| & B^*\widetilde{S}^* \\ \mathbf{w}^*B/\|\mathbf{w}\| & g - z & \mathbf{a}^* \\ \widetilde{S}B & \mathbf{a} & H_1 - z \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{u} \\ \|\mathbf{w}\| \\ 0 \end{pmatrix} \\ &=:\; \begin{pmatrix} \mathbf{x} \\ 0 \end{pmatrix}^* \begin{pmatrix} E - z & F^* \\ F & H_1 - z \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{x} \\ 0 \end{pmatrix} \\ &=:\; \Gamma, \end{aligned} \tag{7.35}$$

where the second equality defines the right-hand side using self-explanatory notation. Note that, by definition, $\|\mathbf{x}\| = \|\mathbf{v}\| = 1$.

Next, we claim that

$$(F^*F)_{ij} \;=\; \delta_{ij} + O\big(\varphi^{C_\zeta} N^{-1/2}\big) \tag{7.36}$$

with ζ-high probability. In order to prove (7.36), write

$$F^*F \;=\; \begin{pmatrix} B^*\widetilde{S}^*\widetilde{S}B & B^*\widetilde{S}^*\mathbf{a} \\ \mathbf{a}^*\widetilde{S}B & \mathbf{a}^*\mathbf{a} \end{pmatrix}.$$

We consider four cases. First, if $1 \leqslant i \neq j \leqslant M$ we find using (3.15) that

$$\big|(F^*F)_{ij}\big| \;=\; \Bigg|\sum_{k,l} B^*_{ik}(\widetilde{S}^*\widetilde{S})_{kl}\widetilde{B}_{lj}\Bigg| \;\leqslant\; \frac{\varphi^{C_\zeta}}{N}\Bigg(\sum_{k,l}\big|(\widetilde{S}^*\widetilde{S})_{kl}\big|^2\Bigg)^{1/2} \;=\; \frac{\varphi^{C_\zeta}}{N}\Big(\operatorname{Tr}(\widetilde{S}^*\widetilde{S})^2\Big)^{1/2} \;\leqslant\; \varphi^{C_\zeta}N^{-1/2}$$

with ζ-high probability. Second, if $1 \leqslant i \leqslant M$ we find using (3.13) and (3.14) that

$$\big|(F^*F)_{ii}-1\big| \;=\; \Bigg|\sum_{k,l} B^*_{ik}(\widetilde{S}^*\widetilde{S})_{kl}B_{li}-1\Bigg| \;\leqslant\; \Bigg|\frac{1}{N}\sum_{i}(\widetilde{S}^*\widetilde{S})_{ii}-1\Bigg| + \frac{\varphi^{C_\zeta}}{N}\Bigg(\sum_{k,l}\big|(\widetilde{S}^*\widetilde{S})_{kl}\big|^2\Bigg)^{1/2} \;\leqslant\; \varphi^{C_\zeta}N^{-1/2}$$

with ζ-high probability. Third, for $i = M+1$ we have by (3.13)

$$\big|(F^*F)_{ii}-1\big| \;=\; |\mathbf{a}^*\mathbf{a}-1| \;\leqslant\; \varphi^{C_\zeta}N^{-1/2}$$

with ζ-high probability. Finally, for $1 \leqslant i < j = M+1$ we have by (3.15)

$$\big|(F^*F)_{ij}\big| \;=\; \Bigg|\sum_{k,l} B^*_{ik}\widetilde{S}^*_{kl}a_l\Bigg| \;\leqslant\; \frac{\varphi^{C_\zeta}}{N}\Bigg(\sum_{k,l}|\widetilde{S}^*_{kl}|^2\Bigg)^{1/2} \;=\; \frac{\varphi^{C_\zeta}}{N}\Big(\operatorname{Tr}\widetilde{S}^*\widetilde{S}\Big)^{1/2} \;\leqslant\; \varphi^{C_\zeta}N^{-1/2}$$

with ζ-high probability. This completes the proof of (7.36).

Next, abbreviate $G_1(z) := (H_1 - z)^{-1}$. Since $N^{1/2}(N-M-1)^{-1/2}H_1$ is an $(N-M-1)\times(N-M-1)$ GOE/GUE matrix, we find from (7.36), Theorem 2.3, and Lemma 3.2 that

$$\Big|\big(F^*G_1F\big)_{ij} - \delta_{ij}m\Big| \;\leqslant\; \varphi^{C_\zeta}N^{-1/2}(d-1)^{-1/2} \tag{7.37}$$

with ζ-high probability. Therefore Schur's formula yields

$$\begin{aligned}\Gamma \;&=\; \mathbf{x}^*\Big(-z-m-\big(-E+F^*G_1F-m\big)\Big)^{-1}\mathbf{x}\\ &=\; m\|\mathbf{x}\|^2 - m^2\langle\mathbf{x}, E\mathbf{x}\rangle + m^2\Big(\langle F\mathbf{x}, G_1F\mathbf{x}\rangle - m\|\mathbf{x}\|^2\Big) + O\Big(\varphi^{C_\zeta}N^{-1}(d-1)^{-1}\Big)\,.\end{aligned} \tag{7.38}$$

with ζ-high probability, where in the second step we expanded using (2.5), and estimated the error term using (7.37) as well as the bounds $M \leqslant \varphi^{C_\zeta}$ and $|E_{ij}| \leqslant \varphi^{C_\zeta}N^{-1/2}$. Recalling that $\|\mathbf{x}\| = 1$, we find

$$\begin{aligned}\Gamma - m \;=\; -m^2\begin{pmatrix}\mathbf{u}\\ \|\mathbf{w}\|\end{pmatrix}^*\begin{pmatrix}A & B^*\mathbf{w}/\|\mathbf{w}\|\\ \mathbf{w}^*B/\|\mathbf{w}\| & g\end{pmatrix}\begin{pmatrix}\mathbf{u}\\ \|\mathbf{w}\|\end{pmatrix} + m^2\|F\mathbf{x}\|^2\Bigg(\frac{1}{\|F\mathbf{x}\|^2}\langle F\mathbf{x}, G_1F\mathbf{x}\rangle - m\Bigg)\\ + m^3\big(\|F\mathbf{x}\|^2-1\big) + O\Big(\varphi^{C_\zeta}N^{-1}(d-1)^{-1}\Big)\end{aligned} \tag{7.39}$$

with ζ-high probability.

Next, from $F\mathbf{x} = \widetilde{S}B\mathbf{u} + \|\mathbf{w}\|\mathbf{a}$ we find

$$\begin{aligned}\|F\mathbf{x}\|^2 \;&=\; \big\langle B\mathbf{u}, \widetilde{S}^*\widetilde{S}B\mathbf{u}\big\rangle + 2\|\mathbf{w}\|\operatorname{Re}\big\langle B\mathbf{u}, \widetilde{S}^*\mathbf{a}\big\rangle + \|\mathbf{w}\|^2\|\mathbf{a}\|^2\\ &=\; \big\langle B\mathbf{u}, B\mathbf{u}\big\rangle - \big|\langle\mathbf{w}, B\mathbf{u}\rangle\big|^2 + 2\|\mathbf{w}\|\operatorname{Re}\big\langle B\mathbf{u}, \widetilde{S}^*\mathbf{a}\big\rangle + \|\mathbf{w}\|^2\|\mathbf{a}\|^2\,.\end{aligned}$$

Applying (3.12) to $\langle \mathbf{w}, B\mathbf{u}\rangle = \sum_{i,j} \overline{w}_i u_j B_{ij}$ (with N in (3.12) replaced by $M(N-M)$), we find

$$\bigl|\langle \mathbf{w}, B\mathbf{u}\rangle\bigr|^2 \;\leqslant\; \varphi^{C_\zeta} N^{-1}\,.$$

Similarly, using (3.13) and (3.14) we find that

$$\|B\mathbf{u}\|^2 \;=\; \|\mathbf{u}\|^2 + O\bigl(\varphi^{C_\zeta} N^{-1/2}\bigr)\,, \qquad \|\widetilde{S}B\mathbf{u}\|^2 \;=\; \|\mathbf{u}\|^2 + O\bigl(\varphi^{C_\zeta} N^{-1/2}\bigr)$$

with ζ-high probability, using (3.15) that

$$\bigl|\bigl\langle B\mathbf{u}, \widetilde{S}^*\mathbf{a}\bigr\rangle\bigr| \;\leqslant\; \varphi^{C_\zeta} N^{-1/2}$$

with ζ-high probability, and using (3.13) that

$$\|\mathbf{a}\|^2 \;=\; 1 + O\bigl(\varphi^{C_\zeta} N^{-1/2}\bigr)$$

with ζ-high probability. Using $\|\mathbf{u}\| \geqslant \varphi^{-D}$ (by definition of $\mathbf{u}$), we therefore conclude that

$$\|F\mathbf{x}\|^2 \;=\; \|B\mathbf{u}\|^2 + 2\operatorname{Re}\frac{\|\mathbf{u}\|\|\mathbf{w}\|}{\|\widetilde{S}B\mathbf{u}\|\|\mathbf{a}\|}\bigl\langle \widetilde{S}B\mathbf{u}, \mathbf{a}\bigr\rangle + \|\mathbf{w}\|^2\|\mathbf{a}\|^2 + O\bigl(\varphi^{C_\zeta}N^{-1}\bigr) \;=\; 1 + O\bigl(\varphi^{C_\zeta}N^{-1/2}\bigr) \tag{7.40}$$

with ζ-high probability. Using Theorem 2.3 applied to G_1 (recall that F and H_1 are independent), we therefore get from (7.39) that

$$\begin{aligned}\Gamma - m \;=\; &-m^2\Bigl(\langle \mathbf{u}, A\mathbf{u}\rangle + \|\mathbf{w}\|^2 g + 2\operatorname{Re}\langle \mathbf{w}, B\mathbf{u}\rangle\Bigr) + m^2\left(\frac{1}{\|F\mathbf{x}\|^2}\langle F\mathbf{x}, G_1 F\mathbf{x}\rangle - m\right)\\ &+ m^3\left(\|B\mathbf{u}\|^2 - \|\mathbf{u}\|^2 + 2\operatorname{Re}\frac{\|\mathbf{u}\|\|\mathbf{w}\|}{\|\widetilde{S}B\mathbf{u}\|\|\mathbf{a}\|}\bigl\langle \widetilde{S}B\mathbf{u}, \mathbf{a}\bigr\rangle + \|\mathbf{w}\|^2\bigl(\|\mathbf{a}\|^2 - 1\bigr)\right) + O\Bigl(\varphi^{C_\zeta}N^{-1}(d-1)^{-1}\Bigr)\end{aligned} \tag{7.41}$$

with ζ-high probability. We write this as

$$\Gamma - m \;=\; \Gamma_1 + \cdots + \Gamma_6 + O\Bigl(\varphi^{C_\zeta}N^{-1}(d-1)^{-1}\Bigr) \tag{7.42}$$

with ζ-high probability, where

$$\Gamma_1 \;:=\; -m^2\langle \mathbf{u}, A\mathbf{u}\rangle\,, \qquad \Gamma_2 \;:=\; -m^2\|\mathbf{w}\|^2 g\,, \qquad \Gamma_3 \;:=\; m^2\left(\frac{1}{\|F\mathbf{x}\|^2}\langle F\mathbf{x}, G_1F\mathbf{x}\rangle - m\right),$$

$$\Gamma_4 \;:=\; -2m^2\operatorname{Re}\langle \mathbf{w}, B\mathbf{u}\rangle + m^3\bigl(\|B\mathbf{u}\|^2 - \|\mathbf{u}\|^2\bigr)\,, \qquad \Gamma_5 \;:=\; 2m^3\operatorname{Re}\frac{\|\mathbf{u}\|\|\mathbf{w}\|}{\|\widetilde{S}B\mathbf{u}\|\|\mathbf{a}\|}\bigl\langle \widetilde{S}B\mathbf{u}, \mathbf{a}\bigr\rangle\,,$$

$$\Gamma_6 \;:=\; m^3\|\mathbf{w}\|^2\bigl(\|\mathbf{a}\|^2 - 1\bigr)\,.$$

We now claim that $\Gamma_1, \dots, \Gamma_6$ are independent. In order to prove this, let $f_1, \dots, f_6$ be indicator functions of Borel sets in $\mathbb{R}$. Write $\mathbf{a} = a\boldsymbol{\omega}$ in polar coordinates, where $a > 0$ and $\boldsymbol{\omega} \in S^{N-M-2}$. Since $\mathbf{a}$ is Gaussian,

a and $\boldsymbol{\omega}$ are independent. Denote by $\rho_1, \dots, \rho_6$ the laws of $A, B, g, a, \boldsymbol{\omega}, H_1$ respectively. Then we get

$$
\begin{aligned}
\mathbb{E}\prod_{i=1}^6 f_i(\Gamma_i) &= \int \mathrm{d}\rho_1(A)\,\mathrm{d}\rho_2(B)\,\mathrm{d}\rho_3(d)\,\mathrm{d}\rho_4(a)\,\mathrm{d}\rho_5(\boldsymbol{\omega})\,\mathrm{d}\rho_6(H_1)\prod_{i=1}^6 f_i(\Gamma_i) \\
&= \big(\mathbb{E}f_1(\Gamma_1)\big)\big(\mathbb{E}f_2(\Gamma_2)\big)\big(\mathbb{E}f_6(\Gamma_6)\big)\int \mathrm{d}\rho_2(B)\,\mathrm{d}\rho_5(\boldsymbol{\omega})\,\mathrm{d}\rho_6(H_1)\, f_3(\Gamma_3) f_4(\Gamma_4) f_5(\Gamma_5) \\
&= \big(\mathbb{E}f_1(\Gamma_1)\big)\big(\mathbb{E}f_2(\Gamma_2)\big)\big(\mathbb{E}f_6(\Gamma_6)\big)\big(\mathbb{E}f_3(\Gamma_3)\big)\big(\mathbb{E}f_5(\Gamma_5)\big)\int \mathrm{d}\rho_2(B) f_4(\Gamma_4) \\
&= \prod_{i=1}^6 \mathbb{E}f_i(\Gamma_i)\,,
\end{aligned}
$$

where the second equality follows by definition of the Γ's, and the third from the invariance of the law of $\boldsymbol{\omega}$ under rotations (applied to Γ_5) and from the invariance of the law of H_1 under orthogonal/unitary conjugations (applied to Γ_3). This proves the independence of $\Gamma_1, \dots, \Gamma_6$.

Next, we identify the asymptotic laws of $\Gamma_1, \dots, \Gamma_6$. There is nothing to be done with Γ_1. By definition,

$$
\nu N^{1/2}\Gamma_2 \overset{d}{=} \mathcal{N}\big(0, 2\nu^2\beta^{-1}m^4\|\mathbf{w}\|^4\big)\,. \tag{7.43}
$$

Since $F\mathbf{x}$ is independent of H_1 and $M \leqslant \varphi^{2D}$, we get from Proposition 7.9 that

$$
\nu N^{1/2}\Gamma_3 \overset{d}{\sim} \mathcal{N}\bigg(0, m^4\frac{2(d+1)}{\beta d^2}\bigg)\,. \tag{7.44}
$$

In order to analyse Γ_4, we define $b_i := (B\mathbf{u})_i$ for $i = 1, \dots, N-M$. Then $\{b_i\}_i$ are independent and satisfy

$$
\mathbb{E}b_i = 0\,, \qquad \mathbb{E}|b_i|^2 = \frac{1}{N}\|\mathbf{u}\|^2\,, \qquad \mathbb{E}|b_i|^4 = \frac{4-\beta}{N^2}\|\mathbf{u}\|^4 + \frac{1}{N^2}\sum_j \big(M_{ij}^{(4)} - 4 + \beta\big)|u_j|^4\,.
$$

Thus we find

$$
\Gamma_4 = \sum_i \Big(-2m^2\operatorname{Re}\overline{w}_i b_i + m^3\big(|b_i|^2 - \mathbb{E}|b_i|^2\big)\Big) + O(M/N)\,.
$$

The variance of the term in parentheses is

$$
\begin{aligned}
&\mathbb{E}\Big(-2m^2\operatorname{Re}\overline{w}_i b_i + m^3\big(|b_i|^2 - \mathbb{E}|b_i|^2\big)\Big)^2 \\
&\quad= 4m^4\mathbb{E}(\operatorname{Re}\overline{w}_i b_i)^2 - 4m^5\mathbb{E}\operatorname{Re}\big((\overline{w}_i b_i)|b_i|^2\big) + m^6\mathbb{E}\big(|b_i|^2 - \mathbb{E}|b_i|^2\big)^2 \\
&\quad= 4m^4\beta^{-1}N^{-1}\|\mathbf{u}\|^2|w_i|^2 - 4m^5N^{-3/2}\operatorname{Re}\bigg(\overline{w}_i\sum_j M_{ij}^{(3)}u_j|u_j|^2\bigg) \\
&\qquad\quad + m^6N^{-2}\bigg((3-\beta)\|\mathbf{u}\|^4 + \sum_j\big(M_{ij}^{(4)} - 4 + \beta\big)|u_j|^4\bigg)\,.
\end{aligned}
$$

Since $|w_i| \leqslant \varphi^{-D}$, we get from the Central Limit Theorem and Lemma 7.10 that

$$
\nu N^{1/2}\Gamma_4 \overset{d}{\sim} \mathcal{N}\bigg(0, \nu^2\frac{4m^4}{\beta}\|\mathbf{u}\|^2\|\mathbf{w}\|^2 - 4\nu^2m^5Q(\mathbf{w},\mathbf{u}) + \nu^2m^6\Big(2\beta^{-1}\|\mathbf{u}\|^4 + R(\mathbf{u})\Big)\bigg)\,, \tag{7.45}
$$

where we abbreviated

$$Q(\mathbf{w},\mathbf{u}) \;:=\; N^{-1/2}\operatorname{Re}\sum_{i,j}\overline{w}_i M_{ij}^{(3)} u_j |u_j|^2\,, \qquad R(\mathbf{u}) \;:=\; \frac{1}{N}\sum_{i,j}\big(M_{ij}^{(4)}-4+\beta\big)|u_j|^4\,,$$

and used that $3-\beta = 2\beta^{-1}$ for $\beta = 1,2$. Since $\|\mathbf{u}\| \leqslant 1$ and $\|\mathbf{w}\| \leqslant 1$, we find that $Q(\mathbf{w},\mathbf{u}) \leqslant C$ and $R(\mathbf{u}) \leqslant C$ for some positive constant C. Next, using $\Gamma_5 = 2m^3\|\mathbf{w}\|\operatorname{Re}\langle \widetilde{S}B\mathbf{u},\mathbf{a}\rangle + O(\varphi^{C_\zeta}N^{-1})$ with ζ-high probability and

$$\mathbb{E}\big(2\operatorname{Re}\langle \widetilde{S}B\mathbf{u},\mathbf{a}\rangle\big)^2 \;=\; \frac{4}{\beta N^2}(N-M-1)\|\mathbf{u}\|^2\,,$$

we find from the Central Limit Theorem and Lemma 7.10 that

$$\nu N^{1/2}\Gamma_5 \;\overset{d}{\sim}\; \mathcal{N}\big(0, 4\nu^2\beta^{-1}m^6\|\mathbf{u}\|^2\|\mathbf{w}\|^2\big)\,. \tag{7.46}$$

Finally, we have $\|\mathbf{a}\|^2 - 1 = \|\mathbf{a}\|^2 - \mathbb{E}\|\mathbf{a}\|^2 + O(M/N)$ and

$$\mathbb{E}\big(|a_i|^2 - \mathbb{E}|a_i|^2\big)^2 \;=\; 2\beta^{-1}N^{-2}\,.$$

Thus we conclude from the Central Limit Theorem and Lemma 7.10 that

$$\nu N^{1/2}\Gamma_6 \;\overset{d}{\sim}\; \mathcal{N}\big(0, 2\nu^2\beta^{-1}m^6\|\mathbf{w}\|^4\big)\,. \tag{7.47}$$

Next, (7.43) – (7.47) imply that $\nu N^{1/2}\Gamma_2, \dots, \nu N^{1/2}\Gamma_6$ are tight (as N-dependent random variables). Moreover, an easy variance calculation shows that $\nu N^{1/2}\Gamma_1$ is also tight. Therefore we get from (7.35), (7.42), (7.43) – (7.47), Lemma 7.7, and Lemma 7.8 that (recall the notation from Definition 7.11)

$$X \;\overset{d}{\sim}\; -\nu N^{1/2}m^2\langle \mathbf{u}, A\mathbf{u}\rangle + \mathcal{N}(0, V_1)\,,$$

where

$$V_1 \;:=\; \frac{2(d+1)}{\beta d^6} + \frac{2\nu^2}{\beta d^4}\big(\|\mathbf{w}\|^4 + 2\|\mathbf{u}\|^2\|\mathbf{w}\|^2\big) + \frac{4\nu^2}{d^5}Q(\mathbf{w},\mathbf{u}) + \frac{\nu^2}{d^6}R(\mathbf{u}) + \frac{2\nu^2}{\beta d^6}\Big(\|\mathbf{u}\|^4 + 2\|\mathbf{u}\|^2\|\mathbf{w}\|^2 + \|\mathbf{w}\|^4\Big)\,.$$

Here we used (6.2).

Next, from

$$\langle \mathbf{v}, H\mathbf{v}\rangle \;=\; \langle \mathbf{u}, A\mathbf{u}\rangle + 2\operatorname{Re}\langle \mathbf{w}, B\mathbf{u}\rangle + \langle \mathbf{w}, H_0\mathbf{w}\rangle\,,$$

the Central Limit Theorem, Lemma 7.10, and Lemma 7.8 we find

$$\nu N^{1/2}\langle \mathbf{v}, H\mathbf{v}\rangle \;\overset{d}{\sim}\; \nu N^{1/2}\langle \mathbf{u}, A\mathbf{u}\rangle + \mathcal{N}\bigg(0, \frac{2\nu^2}{\beta}(1-\|\mathbf{u}\|^4)\bigg)\,. \tag{7.48}$$

Moreover, using that the dimension M of $\mathbf{u}$ satisfies $M \leqslant \varphi^{2D}$ and the fact that $\max_i |w_i| \leqslant \varphi^{-D}$, we find

$$Q(\mathbf{w},\mathbf{u}) \;=\; Q(\mathbf{v}) + O(\varphi^{-D})\,, \qquad R(\mathbf{u}) \;=\; R(\mathbf{v}) + O(\varphi^{-2D})\,.$$

Therefore we get, using Lemma 7.8 and recalling that $1 = \|\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{w}\|^2$,

$$X \;\overset{d}{\sim}\; -\nu N^{1/2}d^{-2}\langle \mathbf{v}, H\mathbf{v}\rangle + \mathcal{N}\bigg(0\,,\; \frac{2(d+1)}{\beta d^6} + \frac{4\nu^2}{d^5}Q(\mathbf{v}) + \frac{\nu^2}{d^6}R(\mathbf{v}) + \frac{2\nu^2}{\beta d^6}\bigg)\,.$$

This concludes the proof. □

7.4. Conclusion of the proof of Theorem 2.14. In this section we compute the distribution of $G_{\mathbf{vv}}(\theta)-m(\theta)$ for a general Wigner matrix H, and hence complete the proof of Theorem 2.14. We use the Green function comparison method from the proof of Lemma 3.9.

Let $H=(h_{ij})=(N^{-1/2}W_{ij})$ be an arbitrary real symmetric / Hermitian Wigner matrix, $V=(N^{-1/2}V_{ij})$ a GOE/GUE matrix independent of H, and $\mathbf{v}\in\mathbb{C}^N$ be normalized. For $D>0$ define the subset

$$I_D \;:=\; \bigl\{i=1,\dots,N : |v_i|\leqslant\varphi^{-D}\bigr\}\,.$$

Define a new Wigner matrix $\widehat{H}=(\widehat{h}_{ij})=(N^{-1/2}\widehat{W}_{ij})$ through

$$\widehat{W}_{ij} \;:=\; \begin{cases} V_{ij} & \text{if } i\in I_D \text{ and } j\in I_D \\ W_{ij} & \text{otherwise}\,. \end{cases}$$

Thus, $\widehat{H}$ satisfies the assumptions of Proposition 7.12. Let

$$J_D \;:=\; \bigl\{1\leqslant i\leqslant j\leqslant N : i\in I_D \text{ and } j\in I_D\bigr\}$$

be the set of matrix indices to be replaced. Similarly to (3.21), we choose a bijective map $\phi : J_D\to\{1,\dots,\gamma_{\max}(D)\}$ and denote by $H_\gamma=(h_{ij}^\gamma)$ the matrix defined by

$$h_{ij}^\gamma \;:=\; \begin{cases} N^{-1/2}W_{ij} & \text{if } \phi(i,j)\leqslant\gamma \\ N^{-1/2}\widehat{W}_{ij} & \text{otherwise}\,. \end{cases}$$

In particular, $H_0=\widehat{H}$ and $H_{\gamma_{\max}(D)}=H$. Let now $(a,b)\in J_D$ satisfy $\phi(a,b)=\gamma$. Similarly to (3.22), we write

$$H_{\gamma-1} \;=\; Q+N^{-1/2}V \qquad\text{where}\quad V\;:=\;V_{ab}E^{(ab)}+\mathbf{1}(a\neq b)V_{ba}E^{(ba)}\,,$$

and

$$H_{\gamma} \;=\; Q+N^{-1/2}W \qquad\text{where}\quad W\;:=\;W_{ab}E^{(ab)}+\mathbf{1}(a\neq b)W_{ba}E^{(ba)}\,.$$

In order to avoid singular behaviour on exceptional low-probability events, we add a small imaginary part to the spectral parameter θ, and set $z:=\theta+\mathrm{i}N^{-4}$. Abbreviate

$$x \;:=\; \nu N^{1/2}\,\mathrm{Re}(G_{\mathbf{vv}}(z)-m(z))\,. \tag{7.49}$$

Thus we have the rough bound $|x|\leqslant N^4$ which we shall tacitly use in the following. We use the notation (3.23), which gives rise to the quantities x_R,x_S,x_T defined through (7.49) with G replaced by R,S,T respectively. We may now state the main comparison estimate.

LEMMA 7.13. *Provided D is a large enough constant, the following holds. Let $f\in C^3(\mathbb{R})$ be bounded with bounded derivatives and $q\equiv q_N$ be an arbitrary deterministic real sequence. Then*

$$\mathbb{E}f(x_T+q) \;=\; \mathbb{E}f(x_R+q)+Y_{ab}\mathbb{E}f'(x_R+q)+A_{ab}+O\bigl(\varphi^{-1}\widehat{\mathcal{E}}_{ab}\bigr)\,, \tag{7.50}$$

$$\mathbb{E}f(x_S+q) \;=\; \mathbb{E}f(x_R+q)+A_{ab}+O\bigl(\varphi^{-1}\widehat{\mathcal{E}}_{ab}\bigr)\,, \tag{7.51}$$

where A_{ab} satisfies $|A_{ab}|\leqslant\varphi^{-1}$,

$$Y_{ab} \;:=\; -\nu N^{-1}\,\mathrm{Re}\Bigl(m^4M_{ab}^{(3)}\overline{v}_av_b+m^4M_{ba}^{(3)}\overline{v}_bv_a\Bigr)\,,$$

and

$$\widehat{\mathcal{E}}_{ab} \;:=\; \sum_{\sigma,\tau=0}^{2} N^{-2+\sigma/2+\tau/2}|v_a|^\sigma |v_b|^\tau + \delta_{ab}\sum_{\sigma=0}^{2} N^{-1+\sigma/2}|v_a|^\sigma\,.$$

Before proving Lemma 7.13, we show how it implies Theorem 2.14.

Proof of Theorem 2.14. Fix $D > 0$ large enough that the conclusion of Lemma 7.13 holds. By Remark 7.5, we may assume that $f \in C_c^\infty(\mathbb{R})$. Let $\gamma = \phi(a,b)$. Since $|v_a| \leqslant \varphi^{-D}$ and $|v_b| \leqslant \varphi^{-D}$, we find

$$Y_{ab}^2 \;\leqslant\; \varphi^{-1}\widehat{\mathcal{E}}_{ab}\,. \tag{7.52}$$

Applying (7.50) and (7.51) with f replaced by f' yields

$$Y_{ab}\mathbb{E}f'(x_T+q) \;=\; Y_{ab}\mathbb{E}f'(x_R+q) + Y_{ab}A_{ab} + O\big(\varphi^{-1}\widehat{\mathcal{E}}_{ab}\big)\,.$$

Subtracting this from (7.50) and using $|A_{ab}| \leqslant \varphi^{-1}$ yields

$$\mathbb{E}f(x_T+q) \;=\; \mathbb{E}f(x_R+q) + Y_{ab}\mathbb{E}f'(x_T+q) + A_{ab} + O\big(\varphi^{-1}\widehat{\mathcal{E}}_{ab} + \varphi^{-1}|Y_{ab}|\big)\,.$$

Subtracting (7.51) yields

$$\mathbb{E}f(x_\gamma+q) \;=\; \mathbb{E}f(x_{\gamma-1}+q) + Y_{ab}\mathbb{E}f'(x_\gamma+q) + O\big(\varphi^{-1}\widehat{\mathcal{E}}_{ab} + \varphi^{-1}|Y_{ab}|\big)\,,$$

where we introduced the notation $x_\gamma := \nu N^{1/2}\operatorname{Re}\big((H_\gamma - z)^{-1}_{\mathbf{v}\mathbf{v}} - m(z)\big)$. Using (7.52) we therefore get

$$\mathbb{E}f(x_\gamma + q - Y_{ab}) \;=\; \mathbb{E}f(x_{\gamma-1}+q) + O\big(\varphi^{-1}\widehat{\mathcal{E}}_{ab} + \varphi^{-1}|Y_{ab}|\big)\,. \tag{7.53}$$

We now iterate (7.53), starting at $\gamma = 1$ and $q = 0$. Using that $\sum_{a,b}\widehat{\mathcal{E}}_{ab} \leqslant C$ and $\sum_{a,b}|Y_{ab}| \leqslant C$, we find after $\gamma_{\max}$ iterations of (7.53)

$$\mathbb{E}f\left(x_{\gamma_{\max}(D)} - \sum_{\gamma=1}^{\gamma_{\max}(D)} Y_{\varphi^{-1}(\gamma)}\right) \;=\; \mathbb{E}f(x_0) + O(\varphi^{-1})\,.$$

Moreover, using $|v_a| \leqslant \varphi^{-D}$ and $|v_b| \leqslant \varphi^{-D}$, we find that

$$\begin{aligned}\sum_{\gamma=1}^{\gamma_{\max}(D)} Y_{\varphi^{-1}(\gamma)} \;&=\; -\nu N^{-1}\operatorname{Re}\sum_{a,b\in I_D}\mathbf{1}(a\leqslant b)m(z)^4\Big(M_{ab}^{(3)}\overline{v}_a v_b + M_{ba}^{(3)}\overline{v}_b v_a\Big)\\ &=\; -\nu N^{-1}\operatorname{Re}\sum_{a,b=1}^{N} m(z)^4 M_{ab}^{(3)}\overline{v}_a v_b + O(\varphi^{-2D})\,.\end{aligned}$$

Using Lemma 7.8 we find

$$\nu N^{1/2}\Big((H-z)^{-1}_{\mathbf{v}\mathbf{v}} - m(z)\Big) \;\overset{d}{\sim}\; \nu N^{1/2}\Big((\widehat{H}-z)^{-1}_{\mathbf{v}\mathbf{v}} - m(z)\Big) - \nu N^{-1}\operatorname{Re}\sum_{a,b=1}^{N} m(z)^4 M_{ab}^{(3)}\overline{v}_a v_b\,.$$

Using Lemma 7.2, it is now easy to remove the imaginary part N^{-4} of z to get

$$\nu N^{1/2}\Big((H-\theta)^{-1}_{\mathbf{v}\mathbf{v}}+d^{-1}\Big) \overset{d}{\sim} \nu N^{1/2}\Big((\widehat{H}-\theta)^{-1}_{\mathbf{v}\mathbf{v}}+d^{-1}\Big)-\frac{\nu S(\mathbf{v})}{d^4}\,.$$

Since $\widehat{H}$ satisfies the assumptions of Proposition 7.12, we find

$$\nu N^{1/2}\Big((H-\theta)^{-1}_{\mathbf{v}\mathbf{v}}+d^{-1}\Big) \overset{d}{\sim} -\frac{\nu N^{1/2}\langle \mathbf{v}\,,H\mathbf{v}\rangle}{d^2}-\frac{\nu S(\mathbf{v})}{d^4}+\mathcal{N}\left(0\,,\,\frac{2(d+1)}{\beta d^4}+\frac{4\nu^2 Q(\mathbf{v})}{d^5}+\frac{\nu^2 R(\mathbf{v})}{d^6}\right),$$

using the notation of Definition 7.11. Now Theorem 2.14 follows from Proposition 7.1 and Lemma 7.7. $\square$

Proof of Lemma 7.13. As before, we consistently drop the spectral parameter $z=\theta+\mathrm{i}N^{-4}$ from G and m. We focus on (7.50). From Theorem 2.3, (3.29), and (3.28) (with S replaced by T), we find

$$|T_{\mathbf{v}a}| \leqslant \varphi^{C_\zeta}N^{-1/2}(d-1)^{-1/2}+C|v_a|\,,\qquad |R_{\mathbf{v}a}| \leqslant \varphi^{C_\zeta}N^{-1/2}(d-1)^{-1/2}+C|v_a|+\varphi^{C_\zeta}N^{-1/2}|v_b| \quad (7.54)$$

with ζ-high probability, and similar results hold for $T_{a\mathbf{v}}$, $T_{\mathbf{v}b}$, $T_{b\mathbf{v}}$, $R_{a\mathbf{v}}$, $R_{\mathbf{v}b}$, and $R_{b\mathbf{v}}$. Similarly, from the first inequality of (3.30) (with S replaced by T), we get

$$\big|T_{\mathbf{v}\mathbf{v}}-R_{\mathbf{v}\mathbf{v}}\big| \leqslant \varphi^{C_\zeta}N^{-1/2}\Big(N^{-1}(d-1)^{-1}+N^{-1/2}(d-1)^{-1/2}(|v_a|+|v_b|)+|v_a||v_b|+N^{-1/2}(|v_a|^2+|v_b|^2)\Big)$$

with ζ-high probability. This yields

$$|x_T-x_R| \leqslant \varphi^{\widetilde{C}_\zeta}\Big[N^{-1}(d-1)^{-1/2}+N^{-1/2}(|v_a|+|v_b|)+(d-1)^{1/2}|v_a||v_b|+N^{-1/2}(d-1)^{1/2}(|v_a|^2+|v_b|^2)\Big] \quad (7.55)$$

with ζ-high probability for some constant $\widetilde{C}_\zeta$. Now choose $D\geqslant \widetilde{C}_\zeta+1$. By definition of J_D, we have that $|v_a|\leqslant\varphi^{-D}$ and $|v_b|\leqslant\varphi^{-D}$. Therefore

$$|x_T-x_R|^3 \leqslant \varphi^{-1}\widehat{\mathcal{E}}_{ab}$$

with ζ-high probability. This yields

$$\mathbb{E}f(x_T+q) = \mathbb{E}f(x_R+q)+\mathbb{E}\big(f'(x_R+q)(x_T-x_R)\big)+\frac{1}{2}\mathbb{E}\big(f''(x_R+q)(x_T-x_R)^2\big)+O\big(\varphi^{-1}\widehat{\mathcal{E}}_{ab}\big)\,. \quad (7.56)$$

In order to analyse $x_T-x_R=\nu N^{1/2}\operatorname{Re}(T_{\mathbf{v}\mathbf{v}}-R_{\mathbf{v}\mathbf{v}})$, we write

$$x_T-x_R = y_1+y_2+y_3+y_4\,,$$

where

$$y_k := \begin{cases}\nu N^{1/2-k/2}\operatorname{Re}\big((-RW)^kR\big)_{\mathbf{v}\mathbf{v}} & \text{if } k=1,2,3\\ \nu N^{-3/2}\operatorname{Re}\big((-RW)^4T\big)_{\mathbf{v}\mathbf{v}} & \text{if } k=4\,.\end{cases}$$

Using (7.54), it is easy to check that y_1 is bounded by the right-hand side of (7.55), and that

$$|y_k| \leqslant \varphi^{C_\zeta}N^{1/2-k/2}\Big(N^{-1}(d-1)^{-1/2}+(d-1)^{1/2}\big(|v_a|^2+|v_b|^2\big)\Big) \qquad (k=2,3,4) \quad (7.57)$$

with ζ-high probability. In particular,

$$x_T - x_R \;=\; y_1 + y_2 + y_3 + O\big(\varphi^{-1}\widehat{\mathcal{E}}_{ab}\big)$$

with ζ-high probability. Moreover, using, $|v_a| \leqslant \varphi^{-D}$, $|v_b| \leqslant \varphi^{-D}$, (7.57) for $k = 2$, and the fact that y_1 is bounded by the right-hand side of (7.55), we find that

$$|y_1||y_2| \;\leqslant\; \varphi^{-1}\widehat{\mathcal{E}}_{ab}$$

with ζ-high probability, provided D is chosen large enough. Similarly, using (7.57) we find that $|y_k||y_{k'}| \leqslant \varphi^{-1}\widehat{\mathcal{E}}_{ab}$ for $k, k' \geqslant 2$ for large enough D. Thus we conclude from (7.56) that

$$\mathbb{E} f(x_T + q) \;=\; \mathbb{E} f(x_R + q) + \mathbb{E}\big(f'(x_R + q) y_3\big) + A_{ab} + O\big(\varphi^{-1}\widehat{\mathcal{E}}_{ab}\big)\,,$$

where

$$A_{ab} \;:=\; \mathbb{E}\big(f'(x_R + q)(y_1 + y_2)\big) + \frac{1}{2}\mathbb{E}\big(f''(x_R + q) y_1^2\big)$$

depends on the randomness only through R and the first two moments of W_{ab}. Moreover, from (7.57) and the fact that y_1 is bounded by the right-hand side of (7.55), we conclude that $|A_{ab}| \leqslant \varphi^{-1}$.

What remains is the analysis of the term $\mathbb{E}\big(f'(x_R + q) y_3\big)$. We shall prove that

$$\Big|\mathbb{E}\big(f'(x_R + q) y_3\big) - Y_{ab}\, \mathbb{E} f'(x_R + q)\Big| \;\leqslant\; C\varphi^{-1}\widehat{\mathcal{E}}_{ab}\,. \tag{7.58}$$

If $a = b$, it is easy to see from (7.57) and the definition of Y_{ab} that

$$|y_3| + |Y_{ab}| \;\leqslant\; \varphi^{-1}\widehat{\mathcal{E}}_{ab}\,,$$

from which (7.58) follows.

Let us therefore assume that $a \neq b$. We multiply out the matrix product in $\big((-RW)^3 R\big)_{\mathbf{v}\mathbf{v}}$ and regroup the resulting eight terms according to the number, r, of off-diagonal matrix elements (R_{ab} or R_{ba}) of R. (By convention, the endpoint matrix elements $R_{\mathbf{v}\cdot}$ and $R_{\cdot\mathbf{v}}$ are not counted as off-diagonal.) This gives, in self-explanatory notation, $y_3 = \sum_{r=0}^{2} y_{3,r}$. Using Theorem 2.3 and (7.54), we find

$$|y_{3,1}| + |y_{3,2}| \;\leqslant\; \varphi^{C_\zeta} N^{-3/2}\Big(N^{-1/2}(d-1)^{-1/2} + |v_a| + |v_b|\Big)^2 \;\leqslant\; \varphi^{-1}\widehat{\mathcal{E}}_{ab}$$

with ζ-high probability. Therefore it suffices to prove that

$$\Big|\mathbb{E}\big(f'(x_R + q) y_{3,0}\big) - Y_{ab}\, \mathbb{E} f'(x_R + q)\Big| \;\leqslant\; C\varphi^{-1}\widehat{\mathcal{E}}_{ab} \tag{7.59}$$

for $a \neq b$. By definition,

$$y_{3,0} \;=\; -\nu N^{-1}\operatorname{Re}\Big(R_{\mathbf{v}a} W_{ab} R_{bb} W_{ba} R_{aa} W_{ab} R_{b\mathbf{v}} + R_{\mathbf{v}b} W_{ba} R_{aa} W_{ab} R_{bb} W_{ba} R_{a\mathbf{v}}\Big)\,.$$

Using (7.54) and Theorem 2.3 we find

$$\Big|y_{3,0} + \nu N^{-1}\operatorname{Re}\Big(m^2 |W_{ab}|^2\big(R_{\mathbf{v}a} W_{ab} R_{b\mathbf{v}} + R_{\mathbf{v}b} W_{ba} R_{a\mathbf{v}}\big)\Big)\Big| \;\leqslant\; \varphi^{-1}\widehat{\mathcal{E}}_{ab}$$

with ζ-high probability. We only deal with the first term of $y_{3,0}$; the second one is dealt with analogously. Recalling the definition of Y_{ab}, we conclude that, in order to establish (7.59), it suffices to prove

$$\left|\mathbb{E}\Big(f'(x_R+q)\nu N^{-1}\operatorname{Re}\Big(m^2 M_{ab}^{(3)} R_{\mathbf{v}a}R_{b\mathbf{v}} - m^4 M_{ab}^{(3)}\overline{v}_a v_b\Big)\Big)\right| \;\leqslant\; C\varphi^{-1}\widehat{\mathcal{E}}_{ab} \tag{7.60}$$

with ζ-high probability; here we used that R is independent of W_{ab}.

Setting $\mathbf{u} = (u_i)$ with $u_i := \mathbf{1}(i \notin \{a,b\})v_i$ and recalling (3.9) and (3.10), we get

$$R_{\mathbf{v}a} \;=\; \overline{v}_a R_{aa} + \overline{v}_b R_{ba} + R_{\mathbf{u}a} \;=\; \overline{v}_a m + \overline{v}_a(R_{aa}-m) + \overline{v}_b R_{ba} + m\mathcal{R}_{\mathbf{u}a} + (R_{aa}-m)\mathcal{R}_{\mathbf{u}a}\,, \tag{7.61}$$

where we defined

$$\mathcal{R}_{\mathbf{u}a} \;:=\; -\sum_i^{(a)} R_{\mathbf{u}i}^{(a)} h_{ia}\,, \qquad \mathcal{R}_{b\mathbf{u}} \;:=\; -\sum_i^{(b)} h_{bi} R_{i\mathbf{u}}^{(b)}\,;$$

see (4.20). The second and third terms are estimated using (7.54) and Theorem 2.3:

$$|R_{aa}-m| + |R_{ba}| \;\leqslant\; \varphi^{C_\zeta} N^{-1/2}(d-1)^{-1/2} \tag{7.62}$$

with ζ-high probability. Moreover, since $R^{(a)} = T^{(a)}$, we find from Lemma (3.12), Theorem 2.3, and (3.8) that

$$\begin{aligned}\big|\mathcal{R}_{\mathbf{u}a}\big| \;&\leqslant\; \varphi^{C_\zeta}\Bigg(\frac{1}{N}\sum_i^{(a)}\big|T_{\mathbf{u}i}^{(a)}\big|^2\Bigg)^{1/2} \\ &\leqslant\; \varphi^{C_\zeta}\Bigg(\frac{1}{N}\sum_i\Big(|u_i|^2 + N^{-1}(d-1)^{-1}\Big)\Bigg)^{1/2} \;\leqslant\; \varphi^{C_\zeta} N^{-1/2}(d-1)^{-1/2}\end{aligned} \tag{7.63}$$

with ζ-high probability. A similar estimate holds for $\mathcal{R}_{b\mathbf{u}}$. Using (7.61), (7.62), (7.63), and (7.54) we get

$$\begin{aligned}&\nu N^{-1}\Big|\mathbb{E}\Big(f'(x_R+q)\big(R_{\mathbf{v}a}R_{b\mathbf{v}} - m^2\overline{v}_a v_b\big)\Big)\Big| \\ &\qquad\leqslant\; \nu N^{-1}\Big|\mathbb{E}\Big[f'(x_R+q)\Big(m^2 v_b\mathcal{R}_{\mathbf{u}a} + m^2\overline{v}_a\mathcal{R}_{b\mathbf{u}} + m^2\mathcal{R}_{\mathbf{u}a}\mathcal{R}_{b\mathbf{u}}\Big)\Big]\Big| + C\varphi^{-1}\widehat{\mathcal{E}}_{ab}\end{aligned} \tag{7.64}$$

with ζ-high probability.

What remains is to estimate the right-hand side of (7.64). Defining

$$x_R^{(a)} \;:=\; \nu N^{1/2}\operatorname{Re}(R_{\mathbf{vv}}^{(a)} - m)\,,$$

we find from (3.8) and (7.54) that

$$\big|x_R - x_R^{(a)}\big| \;\leqslant\; \varphi^{C_\zeta}\Big(N^{-1/2}(d-1)^{-1/2} + N^{1/2}(d-1)^{1/2}|v_a|^2 + N^{-1/2}(d-1)^{1/2}|v_b|^2\Big)$$

with ζ-high probability. Using (7.63) and using that the derivative of f is bounded, we may estimate the first term of (7.64) as

$$\nu N^{-1}\Big|\mathbb{E}\Big[f'(x_R+q)v_b\mathcal{R}_{\mathbf{u}a}\Big]\Big| \;\leqslant\; \nu N^{-1}\Big|\mathbb{E}\Big[f'(x_R^{(a)}+q)v_b\mathcal{R}_{\mathbf{u}a}\Big]\Big| + C\varphi^{-1}\widehat{\mathcal{E}}_{ab} \;=\; C\varphi^{-1}\widehat{\mathcal{E}}_{ab}$$

with ζ-high probability. In the second step we used that $x_R^{(a)}$ is independent of the the a-th column of Q and that $\mathbb{E}_a \mathcal{R}_{\mathbf{u}a} = 0$. The second term of (7.64) is similar. In order to estimate the third, we have to make $\mathcal{R}_{b\mathbf{u}}$ independent of the a-th column of Q. (See the definition (3.22).) We estimate, using (3.8), $R^{(b)} = T^{(b)}$, (3.12), and (7.54)

$$\begin{aligned}\left|\mathcal{R}_{b\mathbf{u}} + \sum_i^{(ab)} h_{bi} R_{i\mathbf{u}}^{(ab)}\right| &\leqslant \left|h_{ba} T_{a\mathbf{u}}^{(b)}\right| + \left|\sum_i^{(ab)} h_{bi} \frac{T_{ia}^{(b)} T_{a\mathbf{u}}^{(b)}}{T_{aa}^{(b)}}\right| \\ &\leqslant \varphi^{C\varsigma}\left(N^{-1}(d-1)^{-1/2} + N^{-1/2}|v_a|\right) + \varphi^{C\varsigma} N^{-1/2} \left(\sum_i^{(ab)} \left|T_{ia}^{(b)} T_{a\mathbf{u}}^{(b)} / T_{aa}^{(b)}\right|^2\right)^{1/2} \\ &\leqslant \varphi^{C\varsigma}\left(N^{-1}(d-1)^{-1} + N^{-1/2}(d-1)^{-1/2}|v_a|\right)\end{aligned}$$

with ζ-high probability. Thus we may estimate the third term of (7.64) by

$$\nu N^{-1}\left|\mathbb{E}\Big[f'(x_R+q)\mathcal{R}_{\mathbf{u}a}\mathcal{R}_{b\mathbf{u}}\Big]\right| \;\leqslant\; \nu N^{-1}\left|\mathbb{E}\left[f'(x_R^{(a)}+q)\mathcal{R}_{\mathbf{u}a}\sum_i^{(ab)} h_{bi} R_{i\mathbf{u}}^{(ab)}\right]\right| + C\varphi^{-1}\widehat{\mathcal{E}}_{ab} \;=\; C\varphi^{-1}\widehat{\mathcal{E}}_{ab}\,,$$

where in the second step we again used that $\mathbb{E}_a \mathcal{R}_{\mathbf{u}a} = 0$. This concludes the proof of (7.60), and hence of (7.50).

The proof of (7.51) is almost identical to the proof of (7.50), except that $\mathbb{E}|V_{ab}|^2 V_{ab} = 0$, so that the left-hand side of the analogue of (7.60) vanishes. Note that, by definition, A_{ab} depends only on R and on the first two moments of W_{ab}, which coincide with those of V_{ab}. Hence A_{ab} is the same in (7.50) and (7.51). This concludes the proof. ☐

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