The Ground State Energy of Dilute Bose Gas in Potentials with Positive Scattering Length

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Abstract

The leading term of the ground state energy/particle of a dilute gas of bosons with mass m in the thermodynamic limit is $2\pi\hbar^2 a\varrho/m$ when the density of the gas is ϱ, the interaction potential is non-negative and the scattering length a is positive. In this paper, we generalize the upper bound part of this result to any interaction potential with positive scattering length, i.e, $a > 0$ and the lower bound part to some interaction potentials with shallow and/or narrow negative parts.

1 Introduction and main theorems

In Dyson's work [9] and Lieb, Yngvason and Seiringer's work [7, 6], it is rigorously proved that the leading term of the ground state energy/particle of a three dimensional dilute bose gas of mass m in the thermodynamic limit with density ϱ is $2\pi\hbar^2 a\varrho/m$, i.e.,

$$e(\varrho, m) = 2\pi\hbar^2 a\varrho/m(1 + o(1)) \text{ if } a^3\varrho \ll 1 \tag{1.1}$$

where they assumed that the interaction potential is non-negative, the scattering length a is positive. This result is generalized to a two dimensional dilute bose gas in [8]. In this paper, first, in Theorem 1, we generalize the upper bound part of (1.1) to general interaction potentials v with positive scattering length. On the other hand, for the lower bound on the ground energy, it was conjectured in [7] that the lower bound part of (1.1) should hold if the scattering length is positive and v has no N-body bound states for any N. Recently, it is proved in [11] that in some cases with partly shallow negative potential the lower bound part of (1.1) holds. In Theorem 2, we introduce a different method for the lower bound on (1.1) when v can have shallow and/or narrow negative components and provide better(smaller) error term.

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We begin with describing the questions more precisely. We write the Hamiltonian of a system of N interacting bosons which are restricted to a cubic box of volume $\Lambda = L^3$ in the following way (in units where $\hbar = 2m = 1$):

$$H_N \equiv \sum_{i=1}^{N} -\Delta_i + \sum_{1\le i<j\le N} v^a(x_i - x_j) \tag{1.2}$$

Here Δ denotes the Laplacian on Λ with periodic boundary condition and v^a is a scaled interaction potential, i.e.,

$$v^a(x) = a^{-2} \cdot v(x/a), \quad a > 0 \tag{1.3}$$

The pair interaction potential v is spherically symmetric and supported on the set $\{x \in \mathbb{R}^3 : |x| \le R_0\}$ for some $R_0 > 0$.

DEFINITION 1 (Scattering Length)**.** *Assume that w is a pair spherically symmetric interaction potential with compact support. Denote $E[\phi\,]$ as the energy of the complex-valued function ϕ on $\mathbb{R}^3$ as follows,*

$$E[\phi\,] = \int_{\mathbb{R}^3} |\nabla\phi(x)|^2 + \tfrac{1}{2}w(x)|\phi(x)|^2 dx. \tag{1.4}$$

Define the scattering length $SL(w)$ of potential w as the following minimum energy.

$$SL(w) \equiv \min_{\phi} \left\{ \frac{1}{4\pi} E[\phi\,] : \lim_{|x|\to\infty} \phi(x) = 1 \right\} \tag{1.5}$$

Note: If $SL(w) > -\infty$, one can easily prove that the Hamiltonian $-\Delta + \frac{1}{2}w$ has no bound state. In particular, when $w \ge 0$, we have $SL(w) \ge 0$ and w has no bound state. One can see that this definition is equivalent to the definition of scattering length in [5] when $w > 0$.

With the relation between v and v^a in (1.3), we can assume that

$$SL(v) = 1, \quad SL(v^a) = a \tag{1.6}$$

Let $f_1(x)$ be the solution of the zero-energy scattering equation of v, i.e.,

$$-\Delta f_1(x) + \tfrac{1}{2}v(x) f_1(x) = 0, \tag{1.7}$$

then we have that $f_a(x) \equiv f_1(x/a)$ is the solution of the following zero-energy scattering equation of v^a.

$$-\Delta f_a(x) + \tfrac{1}{2}v^a(x) f_a(x) = 0, \tag{1.8}$$

As in [5], one can prove that if f_a is normalized as $\lim_{|x|\to\infty} f_a(x) = 1$, then

$$f_a(x) = 1 - a/x, \;\text{ for }\; |x| > R_0 a \tag{1.9}$$

In this paper, we are interested in the ground energy $E(N, \Lambda)$ of H_N in the thermodynamic limit that $\Lambda \to \infty$, $N \to \infty$ and $N/\Lambda = \varrho$. Low density means that the average inter-particle distance $\varrho^{-1/3}$ is much larger than the scattering length a, i.e. $a^3\varrho \ll 1$.

First, we state that for any fixed v, the upper bound on (1.1) holds for the dilute bose gas.

THEOREM 1. *Fix v with $SL[v]=1$ and v^a satisfying (1.3). Let f_1 be the solution of zero-energy equation of v and normalized as: $f_1(\infty)=1$. In the thermodynamic limit, $\lim_{N\to\infty} N/\Lambda=\varrho$, we have the following upper bound on $E(N,\Lambda)$, which is the ground energy of H_N in (1.2),*

$$\limsup_{N\to\infty} \frac{E(N,\Lambda)}{4\pi a\varrho N} \leq 1+\text{const.}\,(a^3\varrho)^{1/4}, \tag{1.10}$$

for some constant depending on $\|f_1\|_\infty$, provided that $\frac{4\pi}{3}a^3\varrho \leq 1$.

Note: So far, the best proof of the error term on upper bound, when $v\geq 0$, is $O(a^3\varrho)^{1/3}$, as in [5].

On the other hand, for the lower bound in (1.1), we prove that as long as v has a positive core and is bounded from below, (1.1) holds when the negative part is small enough (shallow and/or narrow). In the appendix, we show that if v^a is a continuous function on $\mathbb{R}^3$ and H_N has no bound state for any N, v^a satisfies the above two requirements, i.e.,

$$v^a(0)>0, \quad \min v^a(r) > -\infty \tag{1.11}$$

The above two inequalities (1.11) also hold when v^a is stable [1] (the stability of potential is assumed in [11]).

THEOREM 2. *We assume that $v(x)=v_+(x)+v_-(x)$, $v_+(x)\geq 0$, $v_-(x)\geq -\lambda_-$, $\lambda_->0$ and v_+ has a positive core, i.e. $\exists\, r_1$, such that $v_+(x)\geq \lambda_+>0$ for $|x|\leq r_1$. Here v_- need not be negative.*

There exist $c_1(R_0/r_1)$ and $c_2(R_0/r_1)$, which are greater than one and only depend on R_0/r_1, such that the following holds.

If there exists some positive number t satisfying

$$SL[c_1(R_0/r_1)\cdot(v+tv_-)]\geq 0 \;\text{ and }\; \lambda_+\geq (1+t^{-1})\,c_2(R_0/r_1)\cdot\lambda_-, \tag{1.12}$$

we have the following lower bound on $E(N,\Lambda)$,

$$\liminf_{N\to\infty} \frac{E(N,\Lambda)}{4\pi a\varrho N} \geq 1-\text{const.}\,(a^3\varrho)^{1/17}, \tag{1.13}$$

for some constant depending on v_+ and v_-, provided that $\frac{4\pi}{3}a^3\varrho$ is smaller than some constant depending on v_+, v_- and t.

Note: So far, the best estimation of the error term of the lower bound, when $v>0$, is also $O(a^3\varrho)^{1/17}$, as in [5].

This theorem implies the following two corollaries.

COROLLARY 1. *Assume that*

$$v(x)=v_+(x)+\lambda_-v_-(x), \;\; v_+(x)\geq 0, \;\; v_-(x)\geq -1 \tag{1.14}$$

and v_+ has a positive core, i.e. $\exists r_1$ such that $v_+(x) \geq \lambda_+$ for $|x| \leq r_1$. There exists $\lambda_0(r_1, R_0, \lambda_+, v_-)$ such that, if $0 \leq \lambda_- \leq \lambda_0$, i.e., the potential is shallow enough, we have the following lower bound on $E(N, \Lambda)$,

$$\liminf_{N\to\infty} \frac{E(N,\Lambda)}{4\pi a \varrho N} \geq 1 - \text{const.}\, (a^3 \varrho)^{1/17} \tag{1.15}$$

provided that $\frac{4\pi}{3} a^3 \varrho$ is smaller than some constant depending on v_+ and v_-.

Proof. For fixed R_0, r_1 and λ_+, when λ_- is small enough, we have that

$$SL[c_1(R_0/r_1)(v_+ + 2\lambda_- v_-)] \geq 0 \ \text{ and } \ \lambda_+ \geq 2\, c_2(R_0/r_1)\lambda_-. \tag{1.16}$$

Using Theorem 2, with the choice $t = 1$, we arrive at the desired result. ■

COROLLARY 2. *Assume that*

$$v(x) = v_+(x) + v_-(x), \quad v_+(x) \geq 0 \geq v_-(x) \geq -\lambda_- \tag{1.17}$$

and v_+ has a positive core, i.e. $\exists r_1$ such that $v_+(x) \geq \lambda_+$ for $|x| \leq r_1$. There exist $\lambda_0(R_0/r_1, \lambda_-)$ and $\varepsilon\,(R_0, r_1, \lambda_-)$ such that, if

$$\lambda_+ \geq \lambda_0 \ \text{ and } \int_{x\in\mathbb{R}^3} |v_-(x)| dx \leq \varepsilon\,(R_0, r_1, \lambda_-), \tag{1.18}$$

we have the following lower bound on $E(N, \Lambda)$,

$$\liminf_{N\to\infty} \frac{E(N,\Lambda)}{4\pi a \varrho N} \geq 1 - \text{const.}\, (a^3 \varrho)^{1/17} \tag{1.19}$$

provided that $\frac{4\pi}{3} a^3 \varrho$ is smaller than some constant depending on v_+ and v_-.

Proof. We choose $\lambda_0 = \max\{3, 2\, c_2(R_0/r_1)\}\lambda_-$, then we have that $\lambda_+ \geq \lambda_0 \geq 3\lambda_-$, which implies that

$$[v + v_-](x) \geq \lambda_+/3 \geq 0 \ \text{ for } \ |x| \leq r_1 \tag{1.20}$$

Then we claim that for any $n \geq 1$ and $\lambda_+ \geq 3\lambda_-$, there exists $\xi(n) > 0$,

$$\int_{\mathbb{R}^3} |v_-(x)| dx \leq \xi(n) \Rightarrow SL[n(v + v_-)] \geq 0. \tag{1.21}$$

To prove (1.21), we shall prove that there exists $\xi(n) > 0$, if $\int_{\mathbb{R}^3} |v_-(x)| dx \leq \xi(n)$, for any non-negative radial function f,

$$\int_{|x|\leq R_0} |\nabla f|^2(x) + \frac{n}{2}(v + v_-) f^2(x) dx \geq 0. \tag{1.22}$$

We can see, with (1.20),

$$\begin{aligned} & \int_{|x|\leq R_0} \left[|\nabla f|^2(x) + \frac{n}{2}(v + v_-) f^2(x) \right] dx \qquad (1.23) \\ \geq & \int_{r_1\leq|x|\leq R_0} \left[|\nabla f|^2(x) - \|n v_-\|_\infty f^2(x) \right] dx \\ \geq & \int_{r_1\leq|x|\leq R_0} |\nabla f|^2(x) dx - n\lambda_- \int_{r_1\leq|x|\leq R_0} |f|^2(x) dx \end{aligned}$$

Hence, if (1.22) does not hold, the right side of (1.23) is less than 0. With Sobolev inequality and Schwarz's Inequality, we obtain that there exists $\eta(n)$ such that

$$\left(\int_{r_1\le|x|\le R_0}|f(x)|^4dx\right)^{1/2}\le\eta(n)\int_{r_1\le|x|\le R_0}|f|^2(x)dx \tag{1.24}$$

On the other hand, with (1.20), $v+v_-\ge 2v_-$ and Schwarz's Inequality, we have that

$$\begin{aligned}
&\int_{|x|\le R_0}|\nabla f|^2(x)+\frac{n}{2}(v+v_-)f^2(x)dx \qquad (1.25)\\
\ge\;&\int_{|x|\le R_0}|\nabla f|^2(x)+\int_{|x|\le r_1}\frac{n\lambda_+}{6}f^2(x)-\left|\int_{r_1\le|x|\le R_0}nv_-(x)|f(x)|^2dx\right|\\
\ge\;&\int_{|x|\le R_0}|\nabla f|^2(x)+\int_{|x|\le r_1}\frac{n\lambda_+}{6}f^2(x)\\
&-n\eta(n)\left|\int_{r_1\le|x|\le R_0}v_-^2(x)dx\right|^{1/2}\int_{r_1\le|x|\le R_0}|f|^2(x)dx\\
\ge\;&\int_{|x|\le R_0}|\nabla f|^2(x)+\int_{|x|\le r_1}\frac{n\lambda_+}{6}f^2(x)-n\eta(n)\lambda_-\|v_-\|_1^{1/2}\int_{|x|\le R_0}|f|^2(x)dx
\end{aligned}$$

Thus, for $n\ge 1$, if

$$\|v_-\|_1^{1/2}\le(\xi(n))^{1/2}\equiv\frac{1}{n\eta(n)}\cdot\min_f\frac{\int_{|x|\le R_0}|\nabla f|^2(x)+\int_{|x|\le r_1}\frac{n\lambda_+}{6}f^2(x)dx}{\int_{|r|\le R_0}|f|^2(x)dx},$$

the inequality (1.22) holds. We note that it is easy to see that $\xi(n)>0$. Hence we arrive at the desired result (1.21). At last, choosing

$$\varepsilon\left(R_0,r_1,\lambda_-\right)=\xi\left(c_1(R_0/r_1)\right) \tag{1.26}$$

and using the result of Theorem 2 with $t=1$, we arrive at the desired result (1.19). ■

Remark: Compared with the result of [11], we improve the error term (It was $(a^3\varrho)^{1/31}$ in [11]) and generalize the shapes of potentials, i.e., the negative part of potential can be shallow and/or narrow. In particular, there is no restriction on the depth of the interaction potential v, i.e. for $\forall\lambda_->0$, there $\exists v$ satisfying $\min_{x\in\mathbb{R}^3}v(x)<-\lambda_-$ and Theorem 2 holds.

2 Proofs

2.1 Proof of Theorem one

Proof. As usual, to prove the upper bound on the ground state energy, we only need to construct a sequence of trial states $\Psi_{N,\Lambda}$ satisfying

$$\limsup_{N\to\infty}\frac{\langle\Psi|H_N|\Psi\rangle}{N\langle\Psi|\Psi\rangle}\le 4\pi a\varrho(1+\text{const.}\,Y) \tag{2.1}$$

for some constant that depends only on $\|f_1\|_\infty$. Here we denote Y as

$$Y \equiv \left(\frac{4\pi}{3}a^3\varrho\right)^{1/4} \tag{2.2}$$

Following the ideas in [9, 6], we construct the trial state of the following form,

$$\Psi_N = \prod_{p=1}^{N} F_p \tag{2.3}$$

In [9], F_p depends on the the nearest particle to the x_p among all the x_i with $i < p$, i.e.,

$$F_p = f(t_p), \;\; t_p = \min_{i<p}\{|x_i - x_p|\} \tag{2.4}$$

via the function f which is very close to the zero energy scattering solution and satisfies

$$0 \le f \le 1, \quad f' \ge 0 \tag{2.5}$$

Hence in [9], F_p has the following property

$$F_{p,i} \cdot f(|x_p - x_i|) \le F_p \le F_{p,i} \tag{2.6}$$

Here $F_{p,i}$ is defined in [9] as the value that F_p would take if the point x_i were omitted from consideration.

But in our case where the potential has a negative part, the zero energy scattering solution f_a of v^a may not be an increasing function or bounded by 1 (if it was, the proof would be much simpler). Hence we do *not* have the property (2.6). For this reason, our choice of F_p will be more complicated. Our F_p depends on all *particles* near the x_p, not just the nearest.

We remark that the function F_p should have following properties.

1. F_p is a continuous function of x_i $(1 \le i \le N)$.

2. When $|x_i - x_p|$ is large enough, the position of x_i does not effect F_p, i.e., $\nabla_{x_i} F_p = 0$.

3. F_p has a similar property as (2.6).

First we define $\theta_r(x)$ as the characteristic function of the set $\{x : |x| \le r\}$ and $\bar{\theta}_r \equiv 1 - \theta_r$. Choosing $b = a/Y$, we have

$$a/b = \frac{4\pi}{3}b^3 N/\Lambda = Y. \tag{2.7}$$

Without loss of generality, we assume that $b > \max\{2R_0 a, 4a\}$, as in [9, 5]. We define $f(x)$ as

$$f(x) = \begin{cases} f_a(x)/f_a(b) & b \ge |x| \ge 0 \\ 1 & \text{otherwise}, \end{cases} \tag{2.8}$$

Here f_a is the zero energy scattering solution of v^a, as in (1.8). With the equation (1.9), we note that

$$f(x) = \frac{1 - a/|x|}{1 - a/b}, \text{ for } b \geq |x| \geq R_0 a. \tag{2.9}$$

Let $\widetilde{R} = \max\{R_0 a, 2a\}$, which implies that $f(\widetilde{R}) > \frac{1}{2}$. We define Θ_p^{in}, Θ_p^{out} $(1 < p \leq N)$ as

$$\Theta_p^{in} \equiv \prod_{j<p} \theta_{\widetilde{R}}(x_j - x_p), \quad \Theta_p^{out} \equiv \prod_{j<p} \bar{\theta}_{\widetilde{R}}(x_j - x_p) \tag{2.10}$$

We can see that $\Theta_p^{in} = 1$ when $|x_j - x_p| \leq \widetilde{R}$ for all $j < p$ and $\Theta_p^{out} = 1$ when $|x_j - x_p| > \widetilde{R}$ for all $j < p$. With Θ_p^{in} and Θ_p^{out}, we can define $r_p(x_1, \cdots, x_N)$ and $R_p(x_1, \cdots, x_N)$ as follows, $(x_i \in [0, L]^3, i = 1, \cdots N)$

$$\begin{aligned} r_p \equiv& \\ &(1 - \Theta_p^{out}) \cdot \min_{i<p} \left\{ |x_i - x_p| : f(x_i - x_p) = \min_{j<p} \{ f(x_j - x_p) : |x_j - x_p| \leq \widetilde{R} \} \right\} \\ R_p \equiv& \, \widetilde{R} \cdot \Theta_p^{in} + (1 - \Theta_p^{in}) \times \min_{i<p} \left\{ |x_i - x_p| : |x_i - x_p| > \widetilde{R} \right\} \end{aligned} \tag{2.11}$$

With the definition of R_p and (2.9), we have that

1. $f(R_p) \leq f(x_j - x_p)$ for any $j < p$ satisfying $|x_j - x_p| > \widetilde{R}$,
2. $R_p \leq |x_j - x_p|$ for any $j < p$ satisfying $|x_j - x_p| > \widetilde{R}$.
3. $R_p \geq \widetilde{R}$
4. When $\Theta_p^{in} = 0$, there exists j_p such that $|x_{j_p} - x_p| = R_p$

Similarly, we have

1. $f(r_p) \leq f(x_j - x_p)$ for any $j < p$ satisfying $|x_j - x_p| \leq \widetilde{R}$,
2. $r_p \leq |x_j - x_p|$ for any $j < p$ satisfying $|x_j - x_p| \leq \widetilde{R}$ and $f(x_j - x_p) = f(r_p)$.
3. $r_p \leq \widetilde{R}$
4. When $\Theta_p^{out} = 0$, there exists i_p such that $|x_{i_p} - x_p| = r_p$

Then, we define a *continuous* function T on $\mathbb{R}$ as follows

$$T(|x|) = \begin{cases} 1 & 2\widetilde{R} \geq |x| \\ (|x|^{-1} - b^{-1})(2\widetilde{R}^{-1} - b^{-1})^{-1} & b \geq |x| \geq 2\widetilde{R} \\ 0 & |x| \geq b\,, \end{cases} \tag{2.12}$$

At last we define $F_p(x_1, \cdots, x_N)$ on $[0, L]^{3N}$ as follows ($1 < p \leq N$),

$$F_p \equiv \begin{cases} f(r_p) & \Theta_p^{in} = 1 \\ f(R_p) & \Theta_p^{out} = 1 \\ f(r_p) + T(R_p)\,[f(R_p) - f(r_p)]_- & otherwise\,, \end{cases} \tag{2.13}$$

and $F_1 = 1$. Here $[\cdot]_-$ denotes the negative part, i.e., $[x]_- = x$ when $x < 0$ and $[x]_- = 0$ when $x \geq 0$. We note that for any x

$$[x]_- \leq 0 \tag{2.14}$$

Note: If $v \geq 0$, it is well known that f is an increasing function, which implies the F_p we defined is equal to the F_p in [9].

One can prove that F_p is a continuous function of $(x_1, \cdots, x_N)$ by checking that, for any $j \neq p > 1$ and fixed $x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_N$, F_p is a continuous function of x_j. First we can see that it is trivial for $j > p$, since F_p is independent of x_j when $j > p$. For $j < p$, it only remains to check that F_p is continuous when x_j moves from $|x_j - x_p| = \widetilde{R}$ to $|x_j - x_p| = \widetilde{R} + 0^+$. One can see that when $|x_j - x_p| = \widetilde{R}$, $f(R_p) \geq f(\widetilde{R}) = f(x_j - x_p) \geq f(r_p)$, so $F_p = f(r_p)$, i.e.

$$F_p = \min\left\{ \min_{k: k \neq j, k < p} \{f(x_k - x_p) : |x_k - x_p| \leq \widetilde{R}\}, f(x_j - x_p) \right\} \tag{2.15}$$

On the other hand, when $|x_j - x_p| = \widetilde{R} + 0^+ \leq 2\widetilde{R}$, we can see that $R_p = |x_j - x_p|$, $T(R_p) = 1$ and $f(R_p) = f(\widetilde{R}) + 0^+$. Hence,

$$F_p = \min\left\{ \min_{k: k \neq j, k < p} \{f(x_k - x_p) : |x_k - x_p| \leq \widetilde{R}\}, f(x_j - x_p) \right\} \tag{2.16}$$

Hence we arrive at the desired result that F_p is continuous function.

We can also see that F_p is non-negative and bounded as follows

$$M \equiv \|F_p\|_\infty = \|f\|_\infty \leq (1 - a/b)^{-1} \|f_a\|_\infty = (1 - a/b)^{-1} \|f_1\|_\infty \leq 2\|f_1\|_\infty. \tag{2.17}$$

Here we use the fact $f_a(x) = f_1(x/a)$.

By the definition of F_p, one can see that $F_p = 1$ when $\prod_{q<p} \bar{\theta}_b(x_p - x_q) = 1$ and $F_p \leq 1$ when $\prod_{q<p} \bar{\theta}_{\widetilde{R}}(x_p - x_q) = 1$, so

$$1 - \sum_{q<p} \theta_b(x_p - x_q) \leq F_p \leq 1 + \sum_{q<p} (M - 1)\theta_{\widetilde{R}}(x_p - x_q) \tag{2.18}$$

We now construct the state functions Φ_k as follows ($1 \leq k \leq N$)

$$\Phi_k = \prod_{p=1}^{k} F_p$$

Note: all Φ's are functions on $[0, L]^{3N}$ and Φ_k is independent of x_l for $l > k$. We will choose $\Psi = \Phi_N$ for (2.1).

As in [7], for proving the upper bound on the total energy $\langle\Phi_N|H_N|\Phi_N\rangle\|\Phi_N\|_2^{-2}$, we shall estimate the upper bounds on

$$\|\Phi_N\|_2^{-2}\int\sum_i|\nabla_i\Phi_N|^2\prod_{j=1}^N dx_j \quad\text{and}\quad \|\Phi_N\|_2^{-2}\int\sum_{i<j}v^a(x_i-x_j)|\Phi_N|^2\prod_{k=1}^N dx_k \tag{2.19}$$

Since in our case v^a has negative parts, our strategy is more complicated, i.e., we need to estimate the upper bounds on

$$\|\Phi_N\|_2^{-2}\int\sum_i|\nabla_i\Phi_N|^2\prod_{j=1}^N dx_j \quad\text{and}\quad \|\Phi_N\|_2^{-2}\int\sum_{i<j}[v^a]_+(x_i-x_j)|\Phi_N|^2\prod_{k=1}^N dx_k \tag{2.20}$$

and the lower bound on

$$\|\Phi_N\|_2^{-2}\int\sum_{i<j}\Big|[v^a]_-(x_i-x_j)\Big|\cdot|\Phi_N|^2\prod_{k=1}^N dx_k \tag{2.21}$$

In the remainder of this section we are going to prove the following three inequalities

- $\|\Phi_N\|_2^{-2}\int\sum_i|\nabla_i\Phi_N|^2\prod_{j=1}^N dx_j\le(1+o(1))\frac{N^2}{\Lambda}\int_{\mathbb{R}^3}|\nabla f(x)|^2dx$
- $\|\Phi_N\|_2^{-2}\int\sum_{i<j}[v^a]_+(x_i-x_j)|\Phi_N|^2\prod_{k=1}^N dx_k\le(1+o(1))\frac{N^2}{\Lambda}\int_{\mathbb{R}^3}\frac{1}{2}[v]_+|f(x)|^2dx$
- $\|\Phi_N\|_2^{-2}\int\sum_{i<j}\Big|[v^a]_-(x_i-x_j)\Big|\cdot|\Phi_N|^2\prod_{k=1}^N dx_k\ge(1-o(1))\frac{N^2}{\Lambda}\int_{\mathbb{R}^3}\frac{1}{2}\,|[v]_-|\cdot|f(x)|^2dx$.

To prove these inequalities, we begin with proving the following three inequalities (all Φ's are functions on $[0,L]^{3N}$):

1. For any m-variable function $g_m(x_{i_1}\cdots x_{i_m})$, $m<k\le N$, $i_j\ne i_k$ for $j\ne k$, we have
$$\|\Phi_k F_{i_1}^{-1}\cdots F_{i_m}^{-1}g_m\|_2^2\le(2M)^{2m}\Lambda^{-m}\|\Phi_{k-m}\|_2^2\|g_m\|_2^2 \tag{2.22}$$
2. For any two variable function $g_2(x_i,x_{i'})$ $(i<i')$, we have
$$\|\Phi_N F_{i'}^{-1}g_2\|_2^2\le\|g_2\|_2^2\Lambda^{-2}\|\Phi_N\|_2^2(1+\text{const.}\,Y) \tag{2.23}$$
3. Let $f_{i,i'}=f(x_i-x_{i'})$, for any two variable function $g_2(x_i,x_{i'})$ $(i<i')$, we have
$$\|\Phi_N f_{i,i'}^{-1}g_2\|_2^2\ge\|g_2\|_2^2\Lambda^{-2}\|\Phi_N\|_2^2(1-\text{const.}\,Y) \tag{2.24}$$

Note: if $f_{i,i'}=0$ and $i<i'$, then $F_{i'}=0$, so $\Phi_N f_{i,i'}^{-1}$ is definable.

We will use (2.22) for controlling the error terms. The inequalities (2.23) and (2.24) will be used in estimating the terms (2.20) and (2.21), respectively.

We begin with deriving a lower bound on $\|\Phi_N\|_2^2$. For $i \neq p$, Let $F_{p,i}$ be the value that F_p would take if changing the order of particles as follows,

$$F_{p,i}(x_1 \cdots x_N) \equiv F_{n(p,i)}(x_1 \cdots x_{i-1}, x_{i+1} \cdots x_N, x_i\,) \tag{2.25}$$

Here $n(p,i)$ is defined as follows $(i \neq p)$

$$n(p,i) = \begin{cases} p & i > p \\ p-1 & i < p\,, \end{cases} \tag{2.26}$$

Similarly, we can define $F_{p,i,j}(x_1 \cdots x_N)$ as

$$F_{p,i,j}(x_1 \cdots x_N) \equiv F_{m(p,i,j)}(x_1 \cdots x_{i-1}, x_{i+1} \cdots x_{j-1}, x_{j+1} \cdots x_N, x_i, x_j\,) \;\; for \;\; i < j \tag{2.27}$$

and $F_{p,i,j} = F_{p,j,i}$ for $j < i$. Here $m(p,i,j)$ is defined as the number of the elements of the set $\{1, \cdots, p\} \setminus \{i,j\}$.

Note: As we mentioned F_p we defined is equal to the F_p in [6] in the case when $v \geq 0$. Furthermore, one can see that our definitions of $F_{p,i}$ and $F_{p,i,j}$ are equivalent to those definitions in [6] when $v \geq 0$.

With the definitions of F_p and $F_{p,i}$, we obtain that $F_{p,i}$ is independent of x_i and F_p is bounded from below as follows

$$F_p(x_1 \cdots x_N) \geq \begin{cases} F_{p,i} & i > p \\ F_{p,i}\bar{\theta}_b(x_p - x_i) & i < p\,, \end{cases} \tag{2.28}$$

and

$$F_p \geq \prod_{i<p} \bar{\theta}_b(x_p - x_i).$$

Then Φ_N^2 is bounded from below, for any fixed i, by

$$\begin{aligned} \left| F_1 \cdot F_2 \cdots F_N \right|^2 & \geq \left| F_{1,\,i} \cdots F_{i-1,\,i} F_{i+1,\,i} \cdots F_{N,\,i} \right|^2 \times \prod_{j \neq i} \bar{\theta}_b(x_i - x_j) \\ & \geq \left| F_{1,\,i} \cdots F_{i-1,\,i} F_{i+1,\,i} \cdots F_{N,\,i} \right|^2 \times \Big(1 - \sum_{j \neq i} \theta_b(x_i - x_j) \Big) \end{aligned} \tag{2.29}$$

Integrating both sides with $\int \prod_{j=1}^N dx_j$, we obtain that

$$\|\Phi_N\|_2^2 \geq \|\Phi_{N-1}\|_2^2 \left(1 - \frac{4\pi b^3}{3} N/\Lambda \right) = \|\Phi_{N-1}\|_2^2 (1 - Y) \tag{2.30}$$

Here we used the fact that

$$\|\Phi_{N-1}\|_2^2 = \int \left| F_{1,\,i} \cdots F_{i-1,\,i} F_{i+1,\,i} \cdots F_{N,\,i} \right|^2 \prod_j dx_j. \tag{2.31}$$

Similarly, one can also prove that for $k \leq N$,

$$\|\Phi_k\|_2^2 \geq \|\Phi_{k-1}\|_2^2 (1 - Y) \tag{2.32}$$

Next we are going to prove (2.22) in the case $m = 1, k = N$, i.e.,

$$\|\Phi_N F_i^{-1} g_1\|_2^2 \le 4M^2 \Lambda^{-1} \|\Phi_{N-1}\|_2^2 \, \|g_1\|_2^2 \tag{2.33}$$

One can check that $F_p > F_{p,i}$ only when the following conditions are satisfied:

1. $i < p$
2. $|x_i - x_p| \le \widetilde{R}$,
3. for any other $j < p$, $|x_j - x_p|$ is greater than $\widetilde{R}$,
4. $T(R_p) < 1$, i.e. for any other $j < p$, $|x_j - x_p| > 2\widetilde{R}$,

i.e.,

$$F_p > F_{p,i} \Rightarrow G_{p,i} \equiv \theta_{\widetilde{R}}(x_p - x_i) \prod_{j<p, j\neq i} \bar{\theta}_{2\widetilde{R}}(x_j - x_p) = 1 \tag{2.34}$$

On the other hand, using the fact that $f(\widetilde{R}) > \frac{1}{2}$, one obtains that if $F_p > F_{p,i}$,

$$F_{p,i} \prod_{j<p,j\neq i} \bar{\theta}_{2\widetilde{R}}(x_j - x_p) > f(\widetilde{R}) \prod_{j<p,j\neq i} \bar{\theta}_{2\widetilde{R}}(x_j - x_p) \ge \frac{1}{2} \prod_{j<p,j\neq i} \bar{\theta}_{2\widetilde{R}}(x_j - x_p)$$

Hence when $G_{p,i} = 1$, we have $2MF_{p,i}G_{p,i} \ge M \ge F_p$, i.e.,

$$F_p \le \begin{cases} F_{p,i} & i > p \\ F_{p,i}\Big(1 + (2M-1)G_{p,i}\Big) & i < p\,. \end{cases} \tag{2.35}$$

By the definition of G's, one can see that if $p, q > i$ and $p \neq q$,

$$G_{p,i} G_{q,i} = 0. \tag{2.36}$$

Hence, we have that

$$\prod_{p>i} \Big(1 + (2M-1)G_{p,i}\Big) \le 2M \tag{2.37}$$

Combining (2.35) and (2.37), we have the upper bound on $|\Phi_N F_i^{-1}|$ as follows,

$$\Big|F_1 \cdot F_2 \cdots F_N F_i^{-1}\Big|^2 \le 4M^2 \Big|F_{1,\,i} \cdots F_{i-1,\,i} F_{i+1,\,i} \cdots F_{N,\,i}\Big|^2, \tag{2.38}$$

which implies the desired result (2.33) with (2.31). Furthermore, for any m-variable function $g_m(x_{i_1} \cdots x_{i_m})$

$$\|\Phi_N F_{i_1}^{-1} \cdots F_{i_m}^{-1} g_m\|_2^2 \le (2M)^{2m} \Lambda^{-m} \|\Phi_{N-m}\|_2^2 \|g_m\|_2^2 \tag{2.39}$$

With the inequality (2.30) and the fact $F_i \le M$ for any $i \le N$, we get

$$\begin{aligned} \|\Phi_N g_m\|_2^2 \le & \quad (2M^2)^{2m}\|\Phi_{N-m}\|_2^2\|g_m\|_2^2 \\ \le & \quad (1-Y)^{-m}(2M^2)^{2m}\Lambda^{-m}\|\Phi_N\|_2^2\|g_m\|_2^2 \end{aligned} \tag{2.40}$$

Similarly, we can generalize this result to $m < k \le N$

$$\begin{aligned} \|\Phi_k g_m\|_2^2 \le & \quad (2M^2)^{2m}\|\Phi_{k-m}\|_2^2\|g_m\|_2^2 \\ \le & \quad (1-Y)^{-m}(2M^2)^{2m}\Lambda^{-m}\|\Phi_k\|_2^2\|g_m\|_2^2 \end{aligned} \tag{2.41}$$

Now we shall prove the upper bound on $\|\Phi_N\|_2^2$ with (2.41). Choosing $p = N$, with the bounds of F_p in (2.18), we get that

$$\begin{aligned} \Phi_N^2 \le & \quad F_1^2 \cdot F_2^2 \cdots F_{N-1}^2 \left(1 + \sum_{j<N} M^2\theta_{\widetilde{R}}(x_j - x_N)\right) \\ = & \quad \Phi_{N-1}^2 + \Phi_{N-1}^2 \sum_{j<N} M^2\theta_{\widetilde{R}}(x_j - x_N) \end{aligned} \tag{2.42}$$

Hence, using the inequalities (2.41)($m = 1$) and (2.32), we obtain that

$$\begin{aligned} \|\Phi_N\|_2^2 & \quad \le \|\Phi_{N-1}\|_2^2 + (1-Y)^{-1}(2M^2)^2 Y\|\Phi_{N-2}\|_2^2 \\ & \quad \le \|\Phi_{N-1}\|_2^2(1 + \text{const.}\, Y) \end{aligned} \tag{2.43}$$

Putting (2.43) and (2.30) together, we obtain the relation between $\|\Phi_N\|$ and $\|\Phi_{N-1}\|$

$$\|\Phi_N\|_2^2 = \|\Phi_{N-1}\|_2^2(1 + O(Y)) \tag{2.44}$$

Similarly, for $k \le N$

$$\|\Phi_k\|_2^2 = \|\Phi_{k-1}\|_2^2(1 + O(Y)) \tag{2.45}$$

Next, we shall prove (2.23), i.e.,

$$\|\Phi_N F_{i'}^{-1} g_2\|_2^2 \le \|g_2\|_2^2\Lambda^{-2}\|\Phi_N\|_2^2(1 + \text{const.}\, Y). \tag{2.46}$$

Using the inequalities $F_i \le \left(1 + \sum_{l<i}(M-1)\theta_{\widetilde{R}}(x_l - x_i)\right)$ and (2.35), with the property of the G's in (2.36), we get

$$\prod_{k \ne i'} F_k \quad \le \left(1 + \sum_{l<i}(M-1)\theta_{\widetilde{R}}(x_l - x_i)\right) \prod_{k \ne i', i} F_{k\,,i} \cdot \left(1 + \sum_{j>i} 2MG_{j,i}\right) \tag{2.47}$$

Similarly, replacing $F_{p,i}$'s with $F_{p,i,i'}$'s and using the fact that $G_{k,l} \le \theta_{\widetilde{R}}(x_k - x_l)$, we get

$$\begin{aligned} \Phi_N F_{i'}^{-1} \le & \quad \prod_{k \ne i', i} F_{k,i,i'} \left(1 + \sum_{l<i} M\theta_{\widetilde{R}}(x_l - x_i)\right) \\ & \times \left(1 + \sum_{j>i} 2M\theta_{\widetilde{R}}(x_j - x_i)\right) \times \left(1 + \sum_{j'>i'} 2M\theta_{\widetilde{R}}(x'_j - x'_i)\right) \end{aligned} \tag{2.48}$$

Expanding (2.48), multiplying $g_2(x_i, x_{i'})$ to each side and integrating them with $\prod_{k=1}^N dx_k$, with the result of (2.41, 2.45), we obtain that

$$\|\Phi_N F_{i'}^{-1} g_2\|_2^2 \le \|g_2\|_2^2 \Lambda^{-2} \|\Phi_N\|_2^2 (1 + \text{const.}\, Y) \tag{2.49}$$

So far we proved some upper bounds of the expectation value of Φ_N. Next we are going to prove the following lower bound on $\|\Phi_N f_{i,i'}^{-1} g_2\|_2^2$:

$$\|\Phi_N f_{i,i'}^{-1} g_2\|_2^2 \ge \|g_2\|_2^2 \Lambda^{-2} \|\Phi_N\|_2^2 (1 - \text{const.}\, Y) \tag{2.50}$$

Here we denote $f_{i,i'} = f(x_i - x_{i'})$ $(i < i')$. First, by the definition of $F_{i'}$, one can see that $F_{i'} = f(x_i - x_{i'})$ when $\prod_{k<i', k\neq i} \bar{\theta}_b(x_k - x_i') = 1$, i.e.,

$$F_{i'}^2 \ge f(x_i - x_{i'})^2 \left(1 - \sum_{k<i', k\neq i} \theta_b(x_k - x_{i'})\right) \tag{2.51}$$

Using this inequality and (2.39) with $m = 3$, $i_1 = i$, $i_2 = i'$ and $i_3 = k$, we obtain that

$$\left\|\Phi_N f_{i,i'}^{-1} g_2\right\|_2^2 \ge \left\|\Phi_N F_{i'}^{-1} g_2\right\|_2^2 - \text{const.}\, Y \, \|g_2\|_2^2 \, \Lambda^{-2} \, \|\Phi_N\|_2^2 \tag{2.52}$$

Then with the lower bound on F_i^2 in (2.18), i.e., $F_i^2 \ge 1 - \sum_{k<i} \theta_b(x_k - x_i)$, we obtain that

$$\|\Phi_N f_{i,i'}^{-1} g_2\|_2^2 \ge \|\Phi_N F_i^{-1} F_{i'}^{-1} g_2\|_2^2 - \text{const.}\, Y \|g_2\|_2^2 \Lambda^{-2} \|\Phi_N\|_2^2 \tag{2.53}$$

Again, using the bound on F_p in (2.28), we see that

$$\Phi_N F_i^{-1} F_{i'}^{-1} \ge \prod_{k\neq i,i'} F_{k,i,i'} \left(1 - \sum_{l<i} \theta_b(x_l - x_i)\right) \times \left(1 - \sum_{l'<i',\, l'\neq i} \theta_b(x_{l'} - x_{i'})\right)$$

Then using (2.41) and (2.45), we arrive at the desired result (2.50).

So far, we have proved the inequalities we need for calculating the value of $\langle \Phi_N | \sum_{i,j} v^a(x_i - x_j) | \Phi_N \rangle$. Then we need to calculate $\nabla_i \Phi_N$. We denote i_p as the particle satisfying $i_p < p$ and $|x_{i_p} - x_p| = r_p$ and n_p^r as the unit vector in the direction of $x_p - x_{i_p}$. Similarly, denote j_p as the particle satisfying $j_p < p$ and $|x_{j_p} - x_p| = R_p$ and n_p^R as the unit vector in the direction of $x_p - x_{j_p}$. We remark that such i_p or j_p may not exist in some cases, but we do define them as 0. We denote $\nabla_0 F_p = 0$. Recall the definition of F_p in (2.13). We have

$$\begin{aligned} -\nabla_p F_p &= \nabla_{i_p} F_p + \nabla_{j_p} F_p \\ \nabla_{i_p} F_p &= -n_p^r f'(r_p) \Big(\Theta_p^{in} + \Theta_p^- (1 - T(R_p)) + \Theta_p^+ \Big) \\ \nabla_{j_p} F_p &= -n_p^R \Big(\Theta_p^{out} f'(R_p) + \Theta_p^- T(R_p) f'(R_p) + \Theta_p^- T'(R_p) \big(f(R_p) - f(r_p)\big) \Big) \end{aligned} \tag{2.54}$$

Here Θ_p^+ is the function of $(x_1 \cdots x_N)$ which is defined as

$$\Theta_p^+ \equiv \big[1 - \Theta^{in} - \Theta^{out}\big] \cdot h\big[f(R_p) - f(r_p)\big] \tag{2.55}$$

and Θ_p^- is defined as

$$\Theta_p^- \equiv \big[1 - \Theta^{in} - \Theta^{out}\big] \cdot h\big[f(r_p) - f(R_p)\big] \tag{2.56}$$

Here h is the Heaviside step function. By the definition of Φ_N, we obtain that

$$\frac{|\nabla_p \Phi_N|^2}{|\Phi_N|^2} = \left| -F_p^{-1}\nabla_{i_p}F_p - F_p^{-1}\nabla_{j_p}F_p + \sum_{q,i_q=p} F_q^{-1}\nabla_p F_q + \sum_{q,j_q=p} F_q^{-1}\nabla_p F_q \right|^2$$

Then with (2.54), we have that

$$\begin{aligned}
\sum_p |\nabla_p \Phi_N|^2 \leq \quad & 2|\Phi_N|^2 \sum_p F_p^{-2}\Big(|f'(r_p)|^2 \big(\Theta_p^{in} + \Theta_p^- |1 - T(R_p)|^2 + \Theta_p^+\big) \\
& + |T(R_p) f'(R_p)|^2 \Theta_p^- + |T'(R_p)|^2 |f(R_p) - f(r_p)|^2 \Theta_p^- \\
& + |T(R_p)| \cdot |1 - T(R_p)| \cdot |f'(r_p)| \cdot |f'(R_p)| \Theta_p^- + |f'(R_p)|^2 \Theta_p^{out} \Big) \\
& + 2|\Phi_N|^2 \sum_{k<p<q} F_p^{-1} F_q^{-1} \Big(|\nabla_k F_p| \cdot |\nabla_p F_q| + |\nabla_k F_p| \cdot |\nabla_k F_q| \Big)
\end{aligned}$$

Because $0 \leq T \leq 1$ and $i_p \neq j_p$, one can easily prove that for any fixed p,

$$\begin{aligned}
& \Big(|f'(r_p)|^2 \big(\Theta_p^{in} + \Theta_p^- |1 - T(R_p)|^2 + \Theta_p^+\big) + |f'(R_p)|^2 \Theta_p^{out} \\
+ \quad & |T(R_p) f'(R_p)|^2 \Theta_p^- + |T(R_p)| \cdot |1 - T(R_p)| \cdot |f'(r_p)| \cdot |f'(R_p)| \Theta_p^- \Big) \\
\leq \quad & \sum_{k:k<p} f'(|x_p - x_k|)^2,
\end{aligned}$$

and

$$|T'(R_p)|^2 |f(R_p) - f(r_p)|^2 \Theta_p^- \leq M^2 \sum_{k:k<p} \left(T'(|x_p - x_k|)^2 \sum_{j:j\neq k,p} \theta_{\widetilde{R}}(x_j - x_p) \right) \tag{2.57}$$

Hence, we obtain that

$$
\begin{aligned}
&\langle\Phi_N|H_N|\Phi_N\rangle \leq 2\sum_{i<j}\int |\Phi_N|^2 F_j^{-2}\left(\tfrac{1}{2}f(x_i-x_j)^2[v(x_i-x_j)]_+ + f'(x_i-x_j)^2\right)\\
&-2\sum_{i<j}\int |\Phi_N|^2 f^{-2}(x_i-x_j)\left(\tfrac{1}{2}f^2(x_i-x_j)\Big|[v(x_i-x_j)]_-\Big|\right)\\
&+2\sum_{i<j}\int |\Phi_N|^2 M^2\sum_{k<p}\left(T'(|x_p-x_k|)^2\sum_{j:j\neq k,p}\theta_{\widetilde{R}}(x_j-x_p)\right) \qquad (2.58)\\
&+2\sum_{k<p<q}\int |\Phi_N|^2 F_p^{-1}F_q^{-1}\Big(|\nabla_k F_p|\cdot|\nabla_p F_q| + |\nabla_k F_p|\cdot|\nabla_k F_q|\Big)
\end{aligned}
$$

Here $[\cdot]_+$ and $[\cdot]_-$ denote the positive and negative part, respectively and we used the fact that $F_j \leq f(x_i-x_j)$ when $i<j$ and $|x_i-x_j|\leq \widetilde{R}$, which implies that

$$
[v(x_i-x_j)]_+ \leq F_j^{-2} f(x_i-x_j)^2[v(x_i-x_j)]_+ \qquad (2.59)
$$

With the results in (2.49) and (2.50), we can obtain the upper bound on the main part of $\langle\Phi_N|H_N|\Phi_N\rangle$, i.e.,

$$
\begin{aligned}
&2\sum_{i<j}\int |\Phi_N|^2 \times\\
&\frac{1}{2}\left(F_j^{-2}\left[f^2(x_i-x_j)[v(x_i-x_j)]_+ + f'(x_i-x_j)^2\right] - \frac{f^2(x_i-x_j)}{f^2(x_i-x_j)}\Big|[v(x_i-x_j)]_-\Big|\right)\\
&\leq 4\pi a N^2/\Lambda(1+\text{const.}\,Y)\|\Phi_N\|_2^2 \qquad (2.60)
\end{aligned}
$$

With the definition of T in (2.12) and (2.40), we obtain that the third line of (2.58) is bounded as const. $aN^2Y\|\Phi_N\|_2^2/\Lambda$. For the other terms, we have

$$
|\nabla_{i_p}F_p| \leq |f'(|x_{i_p}-x_p|)|, \quad |\nabla_{j_p}F_p| \leq |f'(|x_{j_p}-x_p|)| + MT'(|x_{j_p}-x_p|) \qquad (2.61)
$$

Hence, with the inequality (2.39), we can prove that the last line in (2.58) are bounded as

$$
\text{const.}\, N^3\Lambda^{-2}(K+L)^2\|\Phi_N\|_2^2 \qquad (2.62)
$$

Here K and L are defined as follows

$$
K \equiv \int_{\mathbb{R}^3} |f'(|x-y|)|\,dy \quad L \equiv \int_{\mathbb{R}^3} T'(|x-y|)\,dy. \qquad (2.63)
$$

Note that K and L are independent of x. By the definitions of f in (2.8) and T in (2.12), we get that

$$
K = O(ab), \quad L = O(\widetilde{R}b) = O(ab) \qquad (2.64)
$$

Hence we obtain that the last line in (2.58) are bounded by const. aN^2Y^2. Combining this result with (2.60), we get the following result,

$$
\frac{\langle\Phi_N|H_N|\Phi_N\rangle}{\|\Phi_N\|_2^2} \leq 4\pi a N^2/\Lambda(1+\text{const.}\,Y) \qquad (2.65)
$$

At last, by choosing $\Psi = \Phi_N$, we arrive at the desired result (2.1), which implies Theorem 1. ■

2.2 Proof of Theorem Two

Proof. Following the ideas in [7], we need to replace the hard potential by a *soft* potential at the expense of local kinetic energy. This method has been used in many papers on dilute bose or fermi gases [7, 8, 6, 2, 3]. But in this method the kinetic energy of particle i only can be used for the hard-soft potential replacement between the particle i and *one* other j (the nearest particle [7]). In our case that v^a is partly negative, we can not ignore the potential between i and other k's for the lower bound on the energy. To solve this problem, we begin with separating the whole Hamiltonian into two parts, (1) The Hamiltonian of the energy when two particles are close to each other and they are far away from the others. (2) The Hamiltonian of the remaining energy. In the remainder of this section, we prove that the first part is greater than $4\pi a N^2 \Lambda^{-1}(1 - O(a^3\varrho)^{1/17})$ and the second part is non-negative.

Another important property Lieb and Yngvason used in [7] is the superadditivity of the ground energy $E(n, \ell)$ of n particles in $[0, \ell]^3$ with Neumann boundary condition, i.e.,

$$E(n + n', \ell) \geq E(n, \ell) + E(n', \ell) \tag{2.66}$$

This property is trivial in the case $v^a \geq 0$. In our proof, we are not going to prove any similar property, actually we only need the property (2.134) that for fixed ℓ, when n is larger than $4\varrho\ell^3$, the energy/particle is greater than $8\pi a\varrho$, as in (2.62) of [5], i.e.,

$$E(n, \ell)/n \geq 8\pi a\varrho(1 - \text{const.}\,(a^3\varrho)^{1/17}) \tag{2.67}$$

which will be proved in Lemma 1.

Choosing

$$R = a(a^3\varrho)^{-5/17} \geq 2R_0 a, \tag{2.68}$$

we define $F_{i,j}$ for $i \neq j$ as follows:

$$F_{i,j} = \theta_R(x_i - x_j) \prod_{k \neq i,j} \bar{\theta}_{2R}(x_i - x_k) \tag{2.69}$$

Here θ_R is the characteristic function of the open set $|x| < R$, and $\bar{\theta}_R = 1 - \theta_R$. We note that $F_{i,j} \neq F_{j,i}$ and $F_{i,j}$ is equal to 1 only when x_j is close to x_i, but the other x_k's are not. It is easy to check that $\sum_{i:i\neq j} F_{i,j} \leq 1$, so

$$-\nabla_j \sum_{i:i\neq j} F_{i,j} \nabla_j \leq -\Delta_j \tag{2.70}$$

for any fixed $x_1, \cdots x_{j-1}, x_{j+1}, \cdots, x_N$.

Then we denote v_+^a and v_-^a as scaled potentials as follows,

$$v_+^a(r) = a^{-2} v_+(r/a), \;\; v_-^a(r) = a^{-2} v_-(r/a) \tag{2.71}$$

Choosing

$$Y = (a^3 \varrho)^{1/17} \tag{2.72}$$

and ε satisfying

$$3 \cdot \left(\min \left\{ 1, SL[v_+] \right\} \right)^{-1} \cdot Y = \varepsilon < \frac{t}{2(1+t)}, \tag{2.73}$$

with the definition

$$v^a_\varepsilon \equiv v^a - \varepsilon v^a_+, \tag{2.74}$$

we separate the Hamiltonian H_N as follows

$$\begin{aligned} &H_N = \\ &(1-\varepsilon)\sum_j -\nabla_j \sum_i F_{i,j} \nabla_j + \sum_{i\neq j} F_{ij} \frac{v^a_\varepsilon}{2}(x_i - x_j) - \varepsilon \sum_j \Delta_j + \varepsilon \sum_{i \neq j} \frac{v^a_+}{2}(x_i - x_j) \\ &+(1-\varepsilon)\sum_j -\nabla_j (1 - \sum_i F_{i,j}) \nabla_j + \sum_{i\neq j} (1 - F_{ij}) \frac{v^a_\varepsilon}{2}(x_i - x_j) \end{aligned} \tag{2.75}$$

First, we claim the following Lemma 1, which will be proved in next section.

Lemma 1. *Define Y, $F_{i,j}$, ε, v^a_ε and R as in (2.72), (2.69), (2.73), (2.74) and (2.68) respectively. There exists C depending only on v such that*

$$\begin{aligned} &H'' \equiv \\ &(1-\varepsilon)\sum_j -\nabla_j \sum_i F_{i,j} \nabla_j + \sum_{i\neq j} F_{ij} \frac{v^a_\varepsilon}{2}(x_i - x_j) - \varepsilon \sum_j \Delta_j + \varepsilon \sum_{i \neq j} \frac{v^a_+}{2}(x_i - x_j) \\ &\geq 4\pi a N^2/\Lambda(1 - CY) \end{aligned} \tag{2.76}$$

Hence, to obtain Theorem 2, it only remains to prove that the last line of (2.75), as an operator, is bounded from below by zero, i.e.,

$$(1-\varepsilon)\sum_j -\nabla_j (1 - \sum_i F_{i,j}) \nabla_j + \sum_{i\neq j} (1 - F_{ij}) \tfrac{1}{2} v^a_\varepsilon (x_i - x_j) \geq 0$$

By the assumptions $\varepsilon < t\,(2+2t)^{-1}$, we have

$$v^a_\varepsilon \geq \frac{2+t}{2+2\,t} v^a_+ + v^a_-.$$

Hence, it remains to prove that

$$0 \leq H'_N \equiv \frac{2+t}{2+2\,t} \sum_j -\nabla_j (1 - \sum_i F_{i,j}) \nabla_j + \frac{1}{2} \sum_{i \neq j} (1 - F_{ij}) \left(\frac{2+t}{2+2\,t} v^a_+ + v^a_- \right) (x_i - x_j)$$

Because $\lim_{N\to\infty} E(N,\Lambda)/N$ exists, for proving Theorem 2, we can assume that N is even, i.e., $N = 2N_1$. Consider any partition $P = (\pi_1, \pi_2)$ of $1, ..., N$ into two disjoint sets with N_1 integers in π_1 and π_2 respectively. For each P, we define that

$$\begin{aligned} H_P = H_{(\pi_1,\pi_2)} \equiv \quad & \frac{2+t}{1+t}\sum_{j\in\pi_1} -\nabla_j(1-\sum_{i\neq j}F_{i,j})\nabla_j + \sum_{i,j\in\pi_1}(1-F_{i,j})\tfrac{1}{2}v^a_{1,1}(x_i-x_j) \\ + \quad & \sum_{i\in\pi_2, j\in\pi_1}(1-F_{i,j})\tfrac{1}{2}v^a_{2,1}(x_i-x_j) + \sum_{i,j\in\pi_2}(1-F_{i,j})\tfrac{1}{2}v^a_{2,2}(x_i-x_j) \end{aligned}$$

Here we denote $v^a_{\alpha,\beta}$ as the interaction potential between particles in π_α and π_β, which are chosen as

$$v^a_{1,1} = v^a_{2,2} = \frac{t}{1+t}v^a_+ \geq 0, \quad v^a_{2,1} = \frac{4}{1+t}v^a_+ + 4v^a_-, \tag{2.77}$$

so

$$\frac{1}{4}\left(v^a_{1,1,} + v^a_{2,1} + v^a_{2,2}\right) = \frac{2+t}{2+2\,t}v^a_+ + v^a_- \leq v^a_\varepsilon.$$

It is easily to check that

$$H'_N = \sum_P H_P / \sum_P 1 \tag{2.78}$$

Hence, to obtain $H'_N \geq 0$, it remains to prove that for $\forall P$, $H_P \geq 0$. Because there is no kinetic energy of particles in π_2, we can fix the configuration of x_i's with $i \in \pi_2$. Since permutation of the labels in π_1 and π_2 is irrelevant, we assume that $\pi_1 = \{1, \cdots, N_1\}$, $\pi_2 = \{N_1+1, \cdots, N\}$.

As we can see $v^a_{2,1}$ is the only partly negative component in H_P. For fixed π_2 particles, we can write $v^a_{2,1}(x_j - x_i)$ as

$$v^a_{2,1}(x_j-x_i) = v^a_{2,1}(x_j-x_i)(1-\chi_A(x_i)) + v^a_{2,1}(x_j-x_i)\chi_A(x_i) \tag{2.79}$$

Here χ_A is the characteristic function of A, which is a subset of $[0,L]^3$ (2.96). We shall show A is the area where the density of π_2 particles is less than some fixed number. To obtain $H_P \geq 0$, our strategy is to prove that

1. The total energy of the interaction potential $v^a_{1,1}$ and $v^a_{2,2}$ cancels out the negative part of $v^a_{2,1}(1-\chi_A)$.

2. The total kinetic energy and the positive part of $v^a_{2,1}$ cancels out the negative part of $v^a_{2,1}\chi_A$.

To make the strategy more clear, we shall define A where the density of π_2 particles is less than some fixed number. First we divide the cubic box $[0,L]^3$ into small cubes B_n $(n \in \mathbb{N})$ of side length ℓ, with

$$\ell = \tfrac{1}{2}r_1 a.$$

Then, with fixed x_i's, $i \in \pi_2$, for any $x \in [0,L]^3$, we define the $G(x)$ as the set of i's which satisfy $i \in \pi_2$ and $|x_i - x| \leq R_0 a$, i.e.,

$$G(x) \equiv \{i \in \pi_2 : |x_i - x| \leq R_0 a\} \tag{2.80}$$

We denote $|G(x)|$ as the number of the elements of $G(x)$.

We denote $d(x, B_n)$ as the distance between the cube $B_n \subset \mathbb{R}^3$ and $x \in \mathbb{R}^3$. Since $|G(y)|$ is uniformly bounded ($|G(y)| \leq N_1$), there must exist a point $X(B_n) \in \mathbb{R}^3$ satisfying $d(X(B_n), B_n) \leq 2R_0 a$ and

$$|G(X(B_n))| = \max\{|G(y)| : d(y, B_n) \leq 2R_0 a\} \tag{2.81}$$

We define $G(B_n) \equiv G(X(B_n))$. We are going to prove that there exists $n_1 \in \mathbb{N}$ depending on R_0/r_1 such that

1. The total energy of the interaction potential $v^a_{1,1}$ and $v^a_{2,2}$ cancels out the negative parts of $v^a_{2,1}(x_j, x_i)$'s when x_i is in a cube B_n such that $|G(B_n)| > n_1$.
2. The total kinetic energy and the positive part of $v^a_{2,1}$ cancel out the negative part of the remaining $v^a_{2,1}$'s.

First, we derive the lower bound on the total energy of $v^a_{2,2}$, i.e. (2.86, 2.88). With the definition of $G(B_n) = G(X(B_n))$, we know that the set $\{x_k : k \in G(B_n)\}$ can be covered by a sphere of radius $R_0 a$. So the number of the cubes which one need to cover this set is less than const. $(R_0/r_1)^3$. We denote these cubes as $B_{n_1} \cdots B_{n_m}$ ($m \leq \text{const.}\,(R_0/r_1)^3$) and assume the number of i's satisfying $i \in G(B_n)$ and $x_i \in B_{n_k}$ is a_{n_k}. Because the side length of B_{n_k} is equal to $r_1 a/2$, the distance between the two particles in the same cube is no more than $\frac{\sqrt{3}}{2} r_1 a < r_1 a$. Hence we have

$$\begin{aligned}
\sum_{i,j \in G(B_n)} \theta_{r_1 a}(x_i - x_j) \geq & \sum_{k=1}^{m} \left[(a_{n_k})^2 - (a_{n_k}) \right] \\
\geq & \frac{(\sum_{k=1}^m a_{n_k})^2}{m} - (\sum_{k=1}^m a_{n_k}) \\
\geq & \text{const.}\,(R_0/r_1)^{-3} |G(B_n)|^2 - |G(B_n)|
\end{aligned} \tag{2.82}$$

Hence, we obtain that there exist $n_1 \geq 3$ and $n_1, n_2 = \text{const.}\,(R_0/r_1)^3$ such that when $|G(B_n)| \geq n_1$,

$$\sum_{i,j \in G(B_n)} \theta_{r_1 a}(x_i - x_j) \geq \frac{1}{n_2} |G(B_n)|^2, \tag{2.83}$$

which implies

$$\sum_{i,j \in G(B_n)} v^a_{2,2}(x_i - x_j) \geq \frac{t\lambda_+ a^{-2}}{(1+t) n_2} |G(B_n)|^2 \tag{2.84}$$

Here, we used (2.77) and (2.71), i.e.,

$$v^a_{2,2}(r) = \frac{t}{(1+t)} v^a_+(r) = \frac{t}{(1+t)} a^{-2} v_+(r/a) \tag{2.85}$$

Again, with the fact that the set $\{x_k : k \in G(B_n)\}$ can be covered with a sphere of diameter $2R_0 a \le R$, one can see that if $i \in G(B_n)$ and $|G(B_n)| \ge 3$, we have $F_{i,j} = 0$ for any $j \neq i$. Hence we obtain that, for any fixed B_n satisfying $|G(B_n)| \ge n_1$,

$$\sum_{i,j \in G(B_n)} (1 - F_{i,j}) v^a_{2,2}(x_i - x_j) = \sum_{i,j \in G(B_n)} v^a_{2,2}(x_i - x_j) \ge \frac{t\lambda_+ a^{-2}}{(1+t)n_2} |G(B_n)|^2 \quad (2.86)$$

Then, we are going to sum up all the cubes satisfying $|G(B_n)| \ge n_1$. It is easy to see that

$$d(x_i, B_n) \le 3R_0 a, \quad \text{for} \quad i \in G(B_n), \quad (2.87)$$

which implies that for any fixed $i \in \pi_2$, the number of cubes B_n's satisfying $i \in G(B_n)$ is less than some constant n_3, which is less than const. $(R_0/r_1)^3$. Hence, summing up all the blocks satisfying $|G(B_n)| \ge n_1$, with the inequality (2.86), we get that

$$\begin{aligned} \sum_{i,j \in \pi_2} (1 - F_{i,j}) v^a_{2,2}(x_i - x_j) &\ge \sum_{n:|G(B_n)| \ge n_1} \sum_{i,j \in G(B_n)} (1 - F_{i,j}) v^a_{2,2}(x_i - x_j) \\ &\ge \sum_{n:|G(B_n)| \ge n_1} \frac{t\lambda_+ a^{-2}}{(1+t)n_2 n_3} |G(B_n)|^2 \end{aligned} \quad (2.88)$$

Second, we derive the lower bound on the interaction potential between particles in π_1. Because the distance between any two points in the same cube is less than $r_1 a$, we have $v^a_{1,1}(x_i - x_j) \ge a^{-2}\lambda_+ t(1+t)^{-1}$ when $i, j \in \pi_1$ and $x_i, x_j \in B_n$, i.e.,

$$\sum_{i,j \in \Pi_1(B_n)} v^a_{1,1}(x_i - x_j) \ge \frac{a^{-2} t\lambda_+}{1+t} \Big(|\Pi_1(B_n)|^2 - |\Pi_1(B_n)| \Big) \quad (2.89)$$

Here $\Pi_1(B_n)$ is defined as the set of i's such that $i \in \pi_1$ and $x_i \in B_n$ and $|\Pi_1(B_n)|$ is the number of the elements of $\Pi_1(B_n)$. Furthermore, if $x_i \in B_n$ and $|G(B_n)| \ge 1$, there must be a $k \in \pi_2$ satisfying $|x_i - x_k| \le 4R_0 a \le 2R$, hence $F_{i,j} = 0$ for any other $j \in \pi_1$. Using this result, for any B_n satisfying $|G(B_n)| \ge 1$, we have that

$$\sum_{i,j \in \Pi_1(B_n)} (1 - F_{i,j}) v^a_{1,1}(x_i - x_j) \ge \frac{t\lambda_+ a^{-2}}{1+t} \Big(|\Pi_1(B_n)|^2 - |\Pi_1(B_n)| \Big) \quad (2.90)$$

At last, we derive the lower bound on $v^a_{2,1}$. By the definitions of $|G(B_n)|$ and $v^a_{2,1}$, we have that $\forall x \in B_n$,

$$\sum_{i \in \pi_2} [v^a_{2,1}]_-(x - x_i) \ge -4\lambda_- a^{-2} |G(B_n)|.$$

Here we denote $[v^a_{2,1}]_-$ as the negative part of $v^a_{2,1}$ which is equal to $4[v^a]_-$. With the facts $0 \ge 4[v^a]_- \ge -4\lambda_- a^{-2}$ and $0 \le F_{i,j} \le 1$, we have the following inequality

$$\sum_{j \in \Pi_1(B_n),\ i \in \pi_2} (1 - F_{i,j}) [v^a_{2,1}]_-(x_i - x_j) \ge -4\lambda_- a^{-2} \cdot |\Pi_1(B_n)| \cdot |G(B_n)| \quad (2.91)$$

One can check that if $|G(B_n)| \geq n_1$ and

$$\lambda_+ \geq (1+t^{-1})\lambda_- \cdot \max\{2\sqrt{n_2 n_3}, \frac{n_2 n_3}{4n_1}\} \sim \text{const.}\,(1+t^{-1})\lambda_-(R_0/r_1)^3, \tag{2.92}$$

the sum of the right sides of (2.90) and (2.91) is bounded from below as follows,

$$\begin{aligned} & -4\lambda_- \cdot |\Pi_1(B_n)| \cdot |G(B_n)| + \frac{t\lambda_+}{1+t}\Big(|\Pi_1(B_n)|^2 - |\Pi_1(B_n)|\Big) \qquad (2.93)\\ \geq \quad & -\frac{t\lambda_+}{(1+t)n_2 n_3}|G(B_n)|^2 \end{aligned}$$

Hence, with (2.90) and (2.91), we obtain that if (2.92) holds and $|G(B_n)| \geq n_1$,

$$\begin{aligned} 0 \leq \quad & \frac{t}{1+t}\frac{\lambda_+ a^{-2}}{n_2 n_3}|G(B_n)|^2 + \sum_{i,j\in\Pi_1(B_n)}(1-F_{i,j})v^a_{1,1}(x_i - x_j) \qquad (2.94)\\ + \quad & \sum_{j\in\Pi_1(B_n),\ i\in\pi_2}(1-F_{i,j})[v^a_{2,1}]_-(x_i - x_j), \end{aligned}$$

Then summing up all the B_n's satisfying $|G(B_n)| > n_1$, with (2.88) and $v_{11} \geq 0$, we obtain that as long as (2.92) holds,

$$\begin{aligned} 0 \leq \quad & \sum_{i,j\in\pi_2}(1-F_{i,j})\tfrac{1}{2}v^a_{2,2}(x_i - x_j) + \sum_{i,j\in\pi_1}(1-F_{i,j})\tfrac{1}{2}v^a_{1,1}(x_i - x_j) \qquad (2.95)\\ & + \sum_{j\in\pi_1, i\in\pi_2}(1-F_{i,j})\tfrac{1}{2}[v^a_{2,1}]_-(x_i - x_j)\big(1-\chi_A(x_j)\big) \end{aligned}$$

Here A is defined as the set $\cup_{|G(B_n)|\leq n_1} B_n$.

$$A = \cup_{|G(B_n)|\leq n_1} B_n \tag{2.96}$$

So far, we proved the interaction potential between particles of the same groups cancels out the negative part of the $v^a_{2,1}(1-\chi_A)$ term in (2.79). We shall show that the kinetic energy and the positive part of $v^a_{2,1}$ cancel out the remaining negative part of $v^a_{2,1}$.

For the other terms in the Hamiltonian H_P, we claim that as long as

$$SL[4\,n_1(v + tv_-)] \geq 0 \tag{2.97}$$

we have

$$\begin{aligned} 0 \leq \quad & \frac{1}{2}\sum_{j\in\pi_1, i\in\pi_2}(1-F_{i,j})\Big([v^a_{2,1}]_+(x_i - x_j) + [v^a_{2,1}]_-(x_i - x_j)\chi_A(x_j)\Big) \qquad (2.98)\\ & + \frac{2+t}{1+t}\sum_{j\in\pi_1} -\nabla_j\left(1 - \sum_i F_{i,j}\right)\nabla_j \end{aligned}$$

As we can see that (2.95) and (2.98) implies that $H_P \geq 0$ when $SL[4n_1(v+tv_-)] \geq 0$ and (2.92) holds, i.e., $\lambda_+ \geq \text{const.}\,(1+t^{-1})\lambda_-(R_0/r_1)^3$, which completes the proof of Theorem 2.

To prove (2.98), we only need to prove the following operator inequality, for any fixed $x_2, \cdots, x_N$,

$$0 \leq \quad -\frac{2+t}{1+t}\nabla_1(1-\sum_{i=2}^{N}F_{i,1})\nabla_1 \qquad (2.99)$$
$$+\tfrac{1}{2}\sum_{j\in\pi_2}(1-F_{j,1})\Big([v^a_{2,1}]_+(x_1-x_j)+[v^a_{2,1}]_-(x_1-x_j)\chi_A(x_1)\Big)$$

First, if $[v^a_{2,1}]_-(x_1-x_j)\chi_A(x_1)\neq 0$, then $d(B_n^{x_1},x_j)\leq R_0 a$, here the $B_n^{x_1}$ is the cube where x_1 is. We obtain that $j\in\pi_2'\subset\pi_2$, here π_2' is defined as

$$\pi_2'\equiv\{j\,'\in\pi_2:\exists B_n, D(x_{j\,'},B_n)\leq R_0 a, |G(B_n)|\leq n_1\} \qquad (2.100)$$

Hence, it only remains to prove that

$$0\leq \quad -\frac{2+t}{1+t}\nabla_1(1-\sum_{i=2}^{N}F_{i,1})\nabla_1+\tfrac{1}{2}\sum_{j\in\pi_2'}(1-F_{j,1})v^a_{2,1}(x_1-x_j) \qquad (2.101)$$

Second, we claim the following inequality which will be proved later.

$$n_1\left(1-\sum_{i=2}^{N}F_{i,1}\right)\geq\sum_{j\in\pi_2'}(1-F_{j,1})\theta_{(R_0a)}(x_1-x_j) \qquad (2.102)$$

which implies that

$$-\nabla_1\left(1-\sum_{i=2}^{N}F_{i,1}\right)\nabla_1\geq-\frac{1}{n_1}\nabla_1\sum_{j\in\pi_2'}(1-F_{j,1})\theta_{(R_0a)}(x_1-x_j)\nabla_1 \qquad (2.103)$$

With (2.103), we obtain that the right side of (2.101) is not less than

$$\sum_{j\in\pi_2'}\left(-\frac{2+t}{n_1(1+t)}\nabla_1(1-F_{j,1})\theta_{(R_0a)}(x_1-x_j)\nabla_1+\frac{1}{2}(1-F_{j,1})v^a_{2,1}(x_j-x_1)\right)$$
$$\geq \quad \sum_{j\in\pi_2'}\left(1-\prod_{k\neq 1\,or\,j}\bar\theta_{2R}(x_k-x_j)\right)\times\frac{2}{n_1(1+t)}$$
$$\times\Big(-\nabla_1\theta_{(R_0a)}(x_1-x_j)\nabla_1+2n_1(v^a+tv^a_-)(x_j-x_1)\Big) \qquad (2.104)$$

Here we used the definition of $F_{j,1}$ and (2.77), i.e., $v^a_{2,1}=\frac{4}{1+t}[v^a+tv^a_-]$. With the assumption $SL[4\,n_1(v+v_-)]\geq 0$, we obtain that (2.104)≥ 0, which implies inequality (2.101).

Hence, it only remains to prove (2.102). For $x_2, \cdots, x_N$ fixed, we define π_3 as following,

$$\pi_3 = \left\{ 2 \le j \le N : \prod_{2\le k\le N, k\ne j} \bar{\theta}_{2R}(x_j - x_k) = 1 \right\} \tag{2.105}$$

With the definition of π_3, we obtain that

$$F_{j,1} = \begin{cases} \theta_R(x_j - x_1) & j \in \pi_3 \\ 0 & j \notin \pi_3 \, , \end{cases} \tag{2.106}$$

Hence, it only remains to prove that

$$n_1 \left(1 - \sum_{i\in\pi_3} \theta_R(x_1 - x_i) \right) \ge \sum_{j\in\pi_2', j\notin\pi_3} \theta_{(R_0 a)}(x_1 - x_j) \tag{2.107}$$

or

$$\max_{x\in\mathbb{R}^3} \left(n_1 \sum_{i\in\pi_3} \theta_R(x - x_i) + \sum_{j\in\pi_2', j\notin\pi_3} \theta_{(R_0 a)}(x - x_j) \right) \le n_1 \tag{2.108}$$

Because the distance between x_i $(i \in \pi_3)$ and x_j $(2 \le j \le N, j \ne i)$ are not less than $2R$, we have that if $i \in \pi_3$

$$\theta_R(x - x_i) = 1 \Rightarrow \sum_{j\ne 1, j\ne i} \theta_R(x - x_j) = 0 \tag{2.109}$$

So, it only remains to prove that

$$\max_x \left(\sum_{j\in\pi_2', j\notin\pi_3} \theta_{(R_0 a)}(x - x_j) \right) \le n_1 \tag{2.110}$$

By the definition of π_2' in (2.100), if $j \in \pi_2'$ and $\theta_{(R_0 a)}(x - x_j) = 1$, there exist B_n satisfying $|G(B_n)| \le n_1$ and $d(x, B_n) \le 2R_0 a$. Hence by the definition of $G(B_n)$ in (2.81) and (2.80), we obtain that, for $\forall x \in \mathbb{R}^3$

$$\sum_{i\in\pi_2'} \theta_{(R_0 a)}(x - x_i) = 1 \Rightarrow \sum_{i\in\pi_2} \theta_{(R_0 a)}(x - x_i) \le n_1 \tag{2.111}$$

With the fact that $\pi_2' \subset \pi_2$, we arrive at the desired result (2.110) and complete the proof of Theorem 2. ∎

2.3 Proof of Lemma 1

Proof. Let $\delta\Omega$ be any infinitismal solid angle. With the definition of scattering length, we have that if ϕ is a complex-valued function such that

$$\phi(x) = 1 \text{ for } x \in \delta\Omega \otimes \mathbb{R} \text{ and } |x| = R' \ge R_0 a \tag{2.112}$$

then

$$\delta\Omega \cdot a \leq \int_{\delta\Omega\otimes[0,R']} |\nabla\phi(x)|^2 + \tfrac{1}{2}v^a(x)|\phi(x)|^2 dx \tag{2.113}$$

Hence we obtain that

$$a\int_{\mathbb{R}^3} \delta(|x|-R')|\phi(x)|^2 dx \leq \int_{|x|\leq R'} |\nabla\phi(x)|^2 + \tfrac{1}{2}v^a(x)|\phi(x)|^2 dx \tag{2.114}$$

which says that for any non-negative radial function $U_0(x)$, supported in the annulus $R_0 a \leq |x| \leq R$, with $\int_{\mathbb{R}^3} U_0(x)\,dx = 4\pi$, we have

$$-\nabla\theta_R(x)\nabla + \tfrac{1}{2}v^a \geq aU_0 \tag{2.115}$$

Note: The result of lemma 2.5 of [5] shows the $\theta_R(x)$ in above inequality can be replaced with the characteristic function of any star-shaped set when $v^a \geq 0$.

Furthermore, one can easily prove that for fixed v ($SL[v]=1$), v_+ and small enough ε,

$$SL[v-\varepsilon v_+] > 0, \quad |f_\varepsilon|_\infty \leq \text{const.} \tag{2.116}$$

Here we denote f_ε as the normalized solution ($\lim_{|x|\to\infty} f_\varepsilon(x) = 1$) of the zero-energy scattering equation of $v-\varepsilon v_+$. Hence, by the definition of scattering length, using f_ε as the trial function for v, we obtain that

$$SL[v] \leq SL[v-\varepsilon v_+] + \varepsilon\|v_+\|_1 \cdot |f_\varepsilon|_\infty \leq SL[v-\varepsilon v_+] + \text{const.}\,\varepsilon \tag{2.117}$$

Combining this result with (2.115) and the definition of v^a_ε, we have,

$$-(1-\varepsilon)\nabla\theta_R(x)\nabla + \tfrac{1}{2}v^a_\varepsilon \geq (1-\text{const.}\,\varepsilon)aU_0 \tag{2.118}$$

and

$$-(1-\varepsilon)\nabla_j F_{i,j}\nabla_j + \tfrac{1}{2}F_{i,j}v^a_\varepsilon(x_i-x_j) \geq (1-\text{const.}\,\varepsilon)aF_{i,j}U_0(x_i-x_j) \tag{2.119}$$

Hence, we obtain the following lower bound on H'', which is defined in (2.76)

$$H'' \geq \varepsilon\sum_j -\Delta_j + \varepsilon\sum_{i\neq j}\frac{v^a_+}{2}(x_i-x_j) + (1-\text{const.}\,\varepsilon)a\sum_{i\neq j}W_{i,j} \tag{2.120}$$

Here $W_{i,j}$ is defined as

$$W_{i,j} = F_{ij}U_0(x_i-x_j) \geq 0 \tag{2.121}$$

As in [4], we choose

$$\ell = aY^{-6} \tag{2.122}$$

and divide Λ into small cubes with side length ℓ. Then we have

$$H'' \geq H^{(3)} \equiv \varepsilon\sum_j -\Delta_j + \varepsilon\sum_{i\neq j}\frac{v^a_+}{2}(x_i-x_j) + (1-\text{const.}\,\varepsilon)a\sum_{i\neq j}W'_{i,j} \tag{2.123}$$

Here $W'_{i,j}$ is defined as

$$W'_{i,j} = G_{ij} U_0(x_i - x_j), \quad G_{ij} \equiv F_{ij}\chi(x_i)\chi(x_j) \geq 0, \tag{2.124}$$

and $\chi(x)$ is equal to 1 when the distance between x and the edges of the small cubes is greater than $2R$; otherwise it is equal to 0. As we can see the particles in different cubes don't affect each other in $H^{(3)}$.

We are going to estimate the ground energy $E^{(3)}(n,\ell)$ of $H^{(3)}$ for n particles in $[0,\ell]^3$ with *Neumann* boundary condition.

First, in the case that $n \leq \frac{8}{3}\varrho\ell^3 Y^{-1}$, with the definition of ε in (2.73), we have that

$$H^{(3)} \quad \geq 3Y\sum_j -\Delta_j + (1 - \text{const.}\, Y)a\sum_{i\neq j} W'_{i,j} \tag{2.125}$$

Then with the Temple inequality in [10], as in [5] (Ineq. 2.60, 2.66), we have that

$$\begin{aligned}\frac{E^{(3)}(n,\ell)}{n} \geq \quad & 4\pi\frac{an}{\ell^3}\left(1-\frac{1}{n}\right)(1-\text{const.}\, Y)\left(1-\frac{2R}{\ell}\right)^3\left(1+\frac{4\pi n}{3}\left(\frac{2R}{\ell}\right)^3\right)^{-1} \\ & \left(1-\frac{3}{\pi}\frac{an}{\left(R^3-(aR_0)^3\right)(3\pi Y\ell^{-2}-4a\ell^{-3}n^2)}\right) \\ \geq \quad & 4\pi\frac{an}{\ell^3}(1-\frac{1}{n})(1-\text{const.}\, Y)(1-\frac{n}{6\ell^3\varrho}Y)\end{aligned} \tag{2.126}$$

Second, when $n \geq \frac{8}{3}\varrho\ell^3 Y^{-1}$, using the fact $W' \geq 0$, we obtain that

$$H^{(3)} \geq \varepsilon H^{(4)} \equiv \varepsilon\left(\sum_j -\Delta_j + \sum_{i\neq j}\frac{v_+^a}{2}(x_i - x_j)\right) \tag{2.127}$$

Using superadditivity of the ground state energy of $H^{(4)}$, we obtain that the ground energy $E^{(4)}(n,\ell)$ of $H^{(4)}$ is bounded from below as follows, $(n \geq p)$

$$E^{(4)}(n,\ell) \geq \left[\frac{n}{p}\right]E^{(4)}(p,\ell) \geq \frac{n}{2p}E^{(4)}(p,\ell) \tag{2.128}$$

Here $[n/p]$ is the largest integer not greater than n/p. Actually, $H^{(4)}$ is just the Hamiltonian for the pure non-negative interaction potential, as in [7]. Denote a_+ as follows:

$$a_+ = \min\{SL(v^a), SL(v_+^a)\} \leq a \tag{2.129}$$

Replacing v_+^a with soft potential, we obtain that,

$$H^{(4)} \geq 3Y\sum_j -\Delta_j + (1-\text{const.}\, Y)a_+\sum_{i\neq j} W'_{i,j} \tag{2.130}$$

As (2.126), we can prove that when $p = \frac{8}{3}\varrho\ell^3 Y^{-1}$,

$$E^{(4)}(p,\ell)/p \geq \frac{32}{3}\pi a_+ \varrho Y^{-1}\left(1-\frac{1}{2}\right)(1-\text{const.}\,Y) \geq \frac{16}{3}\pi a_+\varrho Y^{-1} \tag{2.131}$$

Hence when $n \geq \frac{8}{3}\varrho\ell^3 Y^{-1}$, we have the following lower bound on the ground energy $E^{(3)}(n,\ell)$ of $H^{(3)}$

$$E^{(3)}(n,\ell)/n \geq \varepsilon E^{(4)}(n,\ell)/n \geq \frac{8}{3}\pi\varepsilon a_+\varrho Y^{-1} \geq 8\pi a\varrho \tag{2.132}$$

For the last inequality, we used the definition of ε in (2.73). So far we proved that

$$\frac{E^{(3)}(n,\ell)}{n} \geq \begin{cases} 4\pi\frac{an}{\ell^3}(1-\frac{1}{n})(1-\text{const.}\,Y)(1-\frac{n}{6\ell^3\varrho}Y) & 1\leq n\leq \frac{8}{3}\ell^3\varrho Y^{-1} \\ 8\pi a\varrho & n\geq \frac{8}{3}\ell^3\varrho Y^{-1}\,, \end{cases} \tag{2.133}$$

which implies that when Y is small enough,

$$\frac{E^{(3)}(n,\ell)}{n(1-\text{const.}\,Y)} \geq \begin{cases} 4\pi\frac{an}{\ell^3}(1-\frac{1}{n}) & 1\leq n\leq 4\ell^3\varrho \\ 8\pi a\varrho & n\geq 4\ell^3\varrho\,, \end{cases} \tag{2.134}$$

Recall the following two facts,

1. the interaction potential $W'_{i,j}$ only depends on the particles in the same cubes as i and j,
2. the particles in different cubes have no interaction.

Using the inequality above (2.134), with the method in [7] (Ineq. 2.55, 2.56), one can prove that the ground state $E^{(3)}(N,\Lambda)$ of $H^{(3)}$ of N particles in big cubic Λ is greater than

$$E^{(3)}(N,\Lambda)/N \geq 4\pi a\varrho(1-\text{const.}\,Y). \tag{2.135}$$

Here Y is defined in (2.72), which implies the desired result (2.76). ■

3 Appendix

In this appendix, we show that if v^a is a continuous function and H_N has no bound state for any N, v^a has a positive core and bounded from below, i.e.,

$$v^a(0) > 0, \quad \min v^a(r) \neq -\infty \tag{3.1}$$

And these inequalities also hold when v^a is stable [1] in the sense of (3.2). One can see that $\min v^a(r)\neq-\infty$ is trivial when v^a is continuous. So it only remains to prove that $v^a(0) > 0$.

First, we prove the statement in the case when v^a is stable, which is defined as follows: there exists constant C, for any N, $x_1,\cdots,x_N$,

$$\sum_{1\leq i\neq j\leq N} v^a(x_i-x_j) \geq -CN, \tag{3.2}$$

Inserting

$$x_1 = x_2 = \cdots = x_{[N/2]} = 0, \quad x_{[N/2]+1} = x_{[N/2]+2} = \cdots = x_N = x_0 \tag{3.3}$$

into the left side of (3.2), for some $x_0 \in \mathbb{R}^3$ satisfying $v^a(x_0) < 0$, we obtain that

$$\text{const.}\, v^a(0)N^2 - \text{const.}\, v^a(x_0)N^2 \geq -CN, \tag{3.4}$$

which implies the desired result that $v^a(0) > 0$.

Next, we prove the statement in the case that H_N has no bounded state for any N. Because v^a is not pure non-negative, there exist $x_0 \in \mathbb{R}^3, r_1, C \in R$ satisfying that

$$v^a(x) < -C, \text{ for } x \in B(x_0, r_1) \subset \mathbb{R}^3 \tag{3.5}$$

Here $B(x_0, r_1)$ is the sphere of radius r_0 centered at x_0. If $v^a(0) \leq 0$, there exists $r_2 < r_1/2$ satisfying that

$$v^a(x) < C/2, \text{ for } x \in B(0, r_2) \tag{3.6}$$

We construct the trial state such that $x_1, x_2, \cdots, x_{[N/2]}$ are localized in $B(0, r_2)$ with the Dirichelet boundary condition and $x_{[N/2]+1}, x_{[N/2]+2}, \cdots, x_N$ are localized in $B(x_0, r_2)$ with the same boundary condition. The energy of this state is less than

$$-\frac{C}{8}N^2 + \frac{\text{const.}}{r_2^2}N \tag{3.7}$$

Here the first term is potential energy and the second term is kinetic energy. When N goes to infinity, the energy of this trial state is negative and hence there are bound states, which is a contradiction with our assumptions. So we arrive at the desired result that $v^a(0) > 0$.

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