

## A FORMULA OF TWO-PARTITION HODGE INTEGRALS

CHIU-CHU MELISSA LIU, KEFENG LIU, AND JIAN ZHOU

### 1. INTRODUCTION

Let  $\overline{\mathcal{M}}_{g,n}$  denote the Deligne-Mumford moduli stack of stable curves of genus  $g$  with  $n$  marked points. Let  $\pi : \overline{\mathcal{M}}_{g,n+1} \rightarrow \overline{\mathcal{M}}_{g,n}$  be the universal curve, and let  $\omega_\pi$  be the relative dualizing sheaf. The Hodge bundle

$$\mathbb{E} = \pi_* \omega_\pi$$

is a rank  $g$  vector bundle over  $\overline{\mathcal{M}}_{g,n}$  whose fiber over  $[(C, x_1, \dots, x_n)] \in \overline{\mathcal{M}}_{g,n}$  is  $H^0(C, \omega_C)$ . Let  $s_i : \overline{\mathcal{M}}_{g,n} \rightarrow \overline{\mathcal{M}}_{g,n+1}$  denote the section of  $\pi$  which corresponds to the  $i$ -th marked point, and let

$$\mathbb{L}_i = s_i^* \omega_\pi$$

be the line bundle over  $\overline{\mathcal{M}}_{g,n}$  whose fiber over  $[(C, x_1, \dots, x_n)] \in \overline{\mathcal{M}}_{g,n}$  is the cotangent line  $T_{x_i}^* C$  at the  $i$ -th marked point  $x_i$ . A Hodge integral is an integral of the form

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{j_1} \dots \psi_n^{j_n} \lambda_1^{k_1} \dots \lambda_g^{k_g}$$

where  $\psi_i = c_1(\mathbb{L}_i)$  is the first Chern class of  $\mathbb{L}_i$ , and  $\lambda_j = c_j(\mathbb{E})$  is the  $j$ -th Chern class of the Hodge bundle.

The study of Hodge integrals is an important part of intersection theory on  $\overline{\mathcal{M}}_{g,n}$ . Hodge integrals also naturally arise when one computes Gromov-Witten invariants by localization techniques. For example, the following generating series of Hodge integrals arises when one computes local invariants of a toric Fano surface in a Calabi-Yau 3-fold by virtual localization [33]:

$$(1) \quad G_{\mu^+, \mu^-}(\lambda; \tau) = - \frac{(\sqrt{-1}\lambda)^{l(\mu^+) + l(\mu^-)}}{z_{\mu^+} \cdot z_{\mu^-}} [\tau(\tau + 1)]^{l(\mu^+) + l(\mu^-) - 1} \\ \cdot \prod_{i=1}^{l(\mu^+)} \frac{\prod_{a=1}^{\mu_i^+ - 1} (\mu_i^+ \tau + a)}{\mu_i^+!} \cdot \prod_{i=1}^{l(\mu^-)} \frac{\prod_{a=1}^{\mu_i^- - 1} (\mu_i^- \frac{1}{\tau} + a)}{\mu_i^-!} \\ \cdot \sum_{g \geq 0} \lambda^{2g-2} \int_{\overline{\mathcal{M}}_{g, l(\mu^+) + l(\mu^-)}} \frac{\Lambda_g^\vee(1) \Lambda_g^\vee(\tau) \Lambda_g^\vee(-\tau - 1)}{\prod_{i=1}^{l(\mu^+)} \frac{1}{\mu_i^+} \left( \frac{1}{\mu_i^+} - \psi_i \right) \prod_{j=1}^{l(\mu^-)} \frac{\tau}{\mu_j^-} \left( \frac{\tau}{\mu_j^-} - \psi_{l(\mu^+) + j} \right)},$$

where  $\lambda, \tau$  are variables,  $(\mu^+, \mu^-) \in \mathcal{P}_+^2$ , the set of pairs of partitions which are not both empty, and

$$\Lambda_g^\vee(u) = u^g - \lambda_1 u^{g-1} + \dots + (-1)^g \lambda_g.$$

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We will call the Hodge integrals in  $G_{\mu^+, \mu^-}(\lambda; \tau)$  the *two-partition Hodge integrals*.

The purpose of this paper is to prove the following formula conjectured in [34]:

$$(2) \quad G^\bullet(\lambda; p^+, p^-; \tau) = R^\bullet(\lambda; p^+, p^-; \tau),$$

where

$$G^\bullet(\lambda; p^+, p^-; \tau) = \exp\left(\sum_{(\mu^+, \mu^-) \in \mathcal{P}_\pm^2} G_{\mu^+, \mu^-}(\lambda; \tau) p_{\mu^+}^+ p_{\mu^-}^-\right)$$

and

$$\begin{aligned} &R^\bullet(\lambda; p^+, p^-; \tau) \\ &= \sum_{|\mu^\pm|=|\nu^\pm|} \frac{\chi_{\nu^+}(\mu^+)}{z_{\mu^+}} \frac{\chi_{\nu^-}(\mu^-)}{z_{\mu^-}} e^{\sqrt{-1}(\kappa_{\nu^+} + \tau + \kappa_{\nu^-} - \tau^{-1})\lambda/2} \mathcal{W}_{\nu^+, \nu^-}(e^{\sqrt{-1}\lambda}) p_{\mu^+}^+ p_{\mu^-}^-. \end{aligned}$$

$p^\pm = (p_1^\pm, p_2^\pm, \dots)$  are formal variables, and

$$p_\mu^\pm = p_{\mu_1}^\pm \cdots p_{\mu_h}^\pm$$

if  $\mu = (\mu_1 \geq \cdots \geq \mu_h > 0)$ . See Section 2 for the notation in the definition of  $R^\bullet(\lambda; p^+, p^-; \tau)$ .

Formula (2) is motivated by a formula of one-partition Hodge integrals conjectured by M. Mariño and C. Vafa in [25] and proved by us in [23]. See [29] for another approach to the Mariño-Vafa formula. The Mariño-Vafa formula can be obtained by setting  $p^- = 0$  in (2). In a recent paper [4], D.-E. Diaconescu and B. Florea conjectured a relation between three-partition Hodge integrals and the topological vertex [1]. A mathematical theory of the topological vertex will be developed in [22]. Formula (2) of two-partition Hodge integrals is used to derived a formula of three-partition integrals [22, Theorem 8.2].

The generating function  $R^\bullet(\lambda; p^+, p^-; \tau)$  is a combinatorial expression involving the representation theory of Kac-Moody Lie algebras. It is also related to the HOMFLY polynomial of the Hopf link and the Chern-Simon theory [30, 26]. In [35], the third author used (2) and a combinatorial trick called the chemistry of  $\mathbb{Z}_k$ -colored labelled graphs to prove a formula conjectured by A. Iqbal in [12] which expresses the generating function of local Gromov-Witten invariants in all genera of toric surfaces in a Calabi-Yau threefold in terms of  $\mathcal{W}_{\mu, \nu}$ . See [12, 1, 4] for surveys of work on this subject.

Our strategy to prove (2) is based on the following cut-and-join equation of  $R^\bullet$  observed in [34]:

$$(3) \quad \frac{\partial}{\partial \tau} R^\bullet = \frac{\sqrt{-1}\lambda}{2} (C^+ + J^+) R^\bullet - \frac{\sqrt{-1}\lambda}{2\tau^2} (C^- + J^-) R^\bullet,$$

where

$$C^\pm = \sum_{i,j} (i+j) p_i^\pm p_j^\pm \frac{\partial}{\partial p_{i+j}^\pm}, \quad J^\pm = \sum_{i,j} ij p_{i+j}^\pm \frac{\partial^2}{\partial p_i^\pm \partial p_j^\pm}.$$

Equation (3) can be derived by the method in [32, 23]. In [34], the third author proved that

**Theorem 1** (Initial values).

$$(4) \quad G^\bullet(\lambda; p^+, p^-; -1) = R^\bullet(\lambda; p^+, p^-; -1).$$

So (2) follows from the main theorem in this paper:

**Theorem 2** (Cut-and-join equation of  $G^\bullet$ ).

$$(5) \quad \frac{\partial}{\partial \tau} G^\bullet = \frac{\sqrt{-1}\lambda}{2}(C^+ + J^+)G^\bullet - \frac{\sqrt{-1}\lambda}{2\tau^2}(C^- + J^-)G^\bullet.$$

Both [23, Theorem 2] (cut-and-join equation of one-partition Hodge integrals) and Theorem 2 are proved by localization methods. We compute certain relative Gromov-Witten invariants by virtual localization and get an expression in terms of one-partition or two-partition Hodge integrals and certain integrals of target  $\psi$  classes. In [23], we used functorial localization to push forward calculations to projective spaces, where the equivariant cohomology is completely understood, and derived [23, Theorem 2] without using much information about integrals of target  $\psi$  classes. In this paper, we relate integrals of target  $\psi$  classes to double Hurwitz numbers (Proposition 5.4) and use properties of double Hurwitz numbers to prove Theorem 2. More precisely, for each  $(\mu^+, \mu^-) \in \mathcal{P}_+^2$ , we will define a generating function

$$K_{\mu^+, \mu^-}^\bullet(\lambda)$$

of certain relative Gromov-Witten invariants of  $\mathbb{P}^1 \times \mathbb{P}^1$  blown up at a point, and use the localization method to derive the following expression:

$$(6) \quad K_{\mu^+, \mu^-}^\bullet(\lambda) = \sum_{|\nu^\pm|=|\mu^\pm|} \Phi_{\mu^+, \nu^+}^\bullet(-\sqrt{-1}\tau\lambda) z_{\nu^+} G_{\nu^+, \nu^-}^\bullet(\lambda; \tau) z_{\nu^-} \Phi_{\nu^-, \mu^-}^\bullet\left(\frac{-\sqrt{-1}}{\tau}\lambda\right).$$

In (6),  $\Phi_{\mu, \nu}^\bullet(\lambda)$  is a generating function of double Hurwitz numbers, and  $z_\mu$  is defined in Section 2.1. It turns out that (6) is equivalent to the following equation:

$$(7) \quad G_{\mu^+, \mu^-}^\bullet(\lambda; \tau) = \sum_{|\nu^\pm|=|\mu^\pm|} \Phi_{\mu^+, \nu^+}^\bullet(\sqrt{-1}\tau\lambda) z_{\nu^+} K_{\nu^+, \nu^-}^\bullet(\lambda) z_{\nu^-} \Phi_{\nu^-, \mu^-}^\bullet\left(\frac{\sqrt{-1}}{\tau}\lambda\right).$$

So Theorem 2 (cut-and-join equation of  $G^\bullet$ ) follows from the cut-and-join equations of double Hurwitz numbers. As a consequence, one can compute  $K_{\mu^+, \mu^-}^\bullet(\lambda)$  in terms of  $\mathcal{W}_{\nu^+, \nu^-}$  (Corollary 3.5). The cut-and-join equations of double Hurwitz numbers can be derived either from combinatorics (Section 3.3) or from the gluing formula (Section 5.4).

The rest of the paper is arranged as follows. In Section 2, we give the precise statement of (2). In Section 3 we give a combinatorial study of double Hurwitz numbers and derive Theorem 2 (the cut-and-join equation of  $G^\bullet$ ) from (6) and some identities of double Hurwitz numbers. In Section 4, we review J. Li’s works [19, 20] on moduli spaces of relative stable morphisms and virtual localization on such moduli spaces [9, 11]. In Section 5, we give a geometric study of double Hurwitz numbers. In Section 6, we introduce the geometric objects involved in the proof of (6). In Section 7, we prove (6) by arranging the localization contribution in a neat way.

## 2. THE CONJECTURE

The main purpose of this paper is to prove the following formula conjectured by the third author in [34].

**Theorem 3.** *We have the following formula of two-partition Hodge integrals:*

$$(2) \quad G^\bullet(\lambda; p^+, p^-; \tau) = R^\bullet(\lambda; p^+, p^-; \tau).$$

The precise definition of  $G^\bullet(\lambda; p^+, p^-; \tau)$  will be given in Section 2.2 below; the precise definition of  $R^\bullet(\lambda; p^+, p^-; \tau)$  will be given in Section 2.3 below.

**2.1. Partitions.** We recall some notation of partitions. Given a partition

$$\mu = (\mu_1 \geq \mu_2 \geq \dots \geq \mu_h > 0),$$

write  $l(\mu) = h$ , and  $|\mu| = \mu_1 + \dots + \mu_h$ . Define

$$\kappa_\mu = \sum_{i=1}^{l(\mu)} \mu_i(\mu_i - 2i + 1).$$

For each positive integer  $j$ , define

$$m_j(\mu) = |\{i : \mu_i = j\}|.$$

Then

$$|\text{Aut}(\mu)| = \prod_j (m_j(\mu))!.$$

Define

$$z_\mu = \mu_1 \cdots \mu_{l(\mu)} |\text{Aut}(\mu)| = \prod_j (m_j(\mu)! j^{m_j(\mu)}).$$

It is well known that each conjugacy class of  $S_d$ , the permutation group of  $d$  elements, corresponds to a partition  $\mu$  of  $d$ , and the number of elements in this conjugacy class is  $d!/z_\mu$ .

Let  $\mathcal{P}$  denote the set of partitions. We allow the empty partition and take

$$l(\emptyset) = |\emptyset| = \kappa_\emptyset = 0.$$

Let

$$\mathcal{P}_+^2 = \mathcal{P}^2 - \{(\emptyset, \emptyset)\}.$$

**2.2. Generating functions of two-partition Hodge integrals.** For  $(\mu^+, \mu^-) \in \mathcal{P}_+^2$ , define

$$\begin{aligned} & G_{g, \mu^+, \mu^-}(\alpha, \beta) \\ &= \frac{-\sqrt{-1}^{l(\mu^+) + l(\mu^-)}}{|\text{Aut}(\mu^+)| |\text{Aut}(\mu^-)|} \prod_{i=1}^{l(\mu^+)} \frac{\prod_{a=1}^{\mu_i^+ - 1} (\mu_i^+ \beta + a\alpha)}{(\mu_i^+ - 1)! \alpha^{\mu_i^+ - 1}} \prod_{j=1}^{l(\mu^-)} \frac{\prod_{a=1}^{\mu_j^- - 1} (\mu_j^- \alpha + a\beta)}{(\mu_j^- - 1)! \beta^{\mu_j^- - 1}} \\ & \cdot \int_{\overline{\mathcal{M}}_{g, l(\mu^+) + l(\mu^-)}} \frac{\Lambda^\vee(\alpha) \Lambda^\vee(\beta) \Lambda^\vee(-\alpha - \beta) (\alpha\beta(\alpha + \beta))^{l(\mu^+) + l(\mu^-) - 1}}{\prod_{i=1}^{l(\mu^+)} (\alpha(\alpha - \mu_i^+ \psi_i)) \prod_{j=1}^{l(\mu^-)} (\beta(\beta - \mu_j^- \psi_{l(\mu^+) + j})}. \end{aligned}$$

We have the following special cases, which have been studied in [23]:

$$\begin{aligned}
 G_{g,\mu^+,\emptyset}(\alpha, \beta) &= \frac{-\sqrt{-1}^{l(\mu^+)}}{|\text{Aut}(\mu^+)|} \prod_{i=1}^{l(\mu^+)} \frac{\prod_{a=1}^{\mu_i^+-1} (\mu_i^+ \beta + a\alpha)}{(\mu_i^+ - 1)! \alpha^{\mu_i^+-1}} \\
 &\quad \cdot \int_{\mathcal{M}_{g,l(\mu^+)}} \frac{\Lambda^\vee(\alpha) \Lambda^\vee(\beta) \Lambda^\vee(-\alpha - \beta) (\alpha\beta(\alpha + \beta))^{l(\mu^+)-1}}{\prod_{i=1}^{l(\mu^+)} (\alpha(\alpha - \mu_i^+ \psi_i))}, \\
 G_{g,\emptyset,\mu^-}(\alpha, \beta) &= \frac{-\sqrt{-1}^{l(\mu^-)}}{|\text{Aut}(\mu^-)|} \prod_{j=1}^{l(\mu^-)} \frac{\prod_{a=1}^{\mu_j^- - 1} (\mu_j^- \alpha + a\beta)}{(\mu_j^- - 1)! \beta^{\mu_j^- - 1}} \\
 &\quad \cdot \int_{\mathcal{M}_{g,l(\mu^-)}} \frac{\Lambda^\vee(\alpha) \Lambda^\vee(\beta) \Lambda^\vee(-\alpha - \beta) (\alpha\beta(\alpha + \beta))^{l(\mu^-)-1}}{\prod_{j=1}^{l(\mu^-)} (\beta(\beta - \mu_j^- \psi_j))}.
 \end{aligned}$$

By a standard degree argument, one sees that  $G_{g,\mu^+,\mu^-}(\alpha, \beta)$  is homogeneous of degree 0, so

$$G_{g,\mu^+,\mu^-}(\alpha, \beta) = G_{g,\mu^+,\mu^-}(1, \frac{\beta}{\alpha}).$$

Let

$$G_{g,\mu^+,\mu^-}(\tau) = G_{g,\mu^+,\mu^-}(1, \tau).$$

Introduce variables  $\lambda, p^+ = (p_1^+, p_2^+, \dots)$ , and  $p^- = (p_1^-, p_2^-, \dots)$ . Given a partition  $\mu$ , define

$$p_\mu^\pm = p_1^\pm \cdots p_{l(\mu)}^\pm.$$

In particular,  $p_\emptyset^\pm = 1$ . Define

$$\begin{aligned}
 G_{\mu^+,\mu^-}(\lambda; \tau) &= \sum_{g=0}^{\infty} \lambda^{2g-2+l(\mu^+)+l(\mu^-)} G_{g,\mu^+,\mu^-}(\tau), \\
 G(\lambda; p^+, p^-; \tau) &= \sum_{(\mu^+,\mu^-) \in \mathcal{P}_+^2} G_{\mu^+,\mu^-}(\lambda; \tau) p_{\mu^+}^+ p_{\mu^-}^-, \\
 G^\bullet(\lambda; p^+, p^-; \tau) &= \exp(G(\lambda; p^+, p^-; \tau)) \\
 &= \sum_{(\mu^+,\mu^-) \in \mathcal{P}^2} G_{\mu^+,\mu^-}^\bullet(\lambda; \tau) p_{\mu^+}^+ p_{\mu^-}^-, \\
 G_{\mu^+,\mu^-}^\bullet(\lambda; \tau) &= \sum_{\chi \in 2\mathbb{Z}, \chi \leq 2(l(\mu^+)+l(\mu^-))} \lambda^{-\chi+l(\mu^+)+l(\mu^-)} G_{\chi,\mu^+,\mu^-}^\bullet(\tau).
 \end{aligned}$$

**2.3. Generating functions of representations of symmetric groups.** Let

$$q = e^{\sqrt{-1}\lambda}, \quad [m] = q^{m/2} - q^{-m/2}.$$

Define

$$\begin{aligned}
 (8) \quad \mathcal{W}_\mu(q) &= q^{\kappa_\mu/4} \cdot \prod_{1 \leq i < j \leq l(\mu)} \frac{[\mu_i - \mu_j + j - i]}{[j - i]} \prod_{i=1}^{l(\mu)} \prod_{v=1}^{\mu_i} \frac{1}{[v - i + l(\mu)]} \\
 &= \frac{q^{\kappa_\mu/4}}{\prod_{x \in \mu} [h(x)]} = q^{\kappa_\mu/4} \frac{\dim_q R_\mu}{|\mu|!},
 \end{aligned}$$

where  $h(x)$  is the hook length and  $\dim_q R_\mu$  is the quantum dimension of the irreducible representation  $R_\mu$  of  $S_{|\mu|}$  associated with the partition  $\mu$ . Define

$$(9) \quad \mathcal{W}_{\mu,\nu}(q) = q^{|\nu|/2} \mathcal{W}_\mu(q) \cdot s_\nu(\mathcal{E}_\mu(t)),$$

where

$$\mathcal{E}_\mu(t) = \prod_{j=1}^{l(\mu)} \frac{1 + q^{\mu_j - j} t}{1 + q^{-j} t} \cdot \left( 1 + \sum_{n=1}^{\infty} \frac{t^n}{\prod_{i=1}^n (q^i - 1)} \right).$$

Such expressions arise in the study of link invariants [26] and Kac-Moody algebras. In the special case of  $(\mu^+, \mu^-) = (\emptyset, \emptyset)$ , we have

$$\mathcal{W}_{\emptyset,\emptyset} = 1.$$

The right-hand side of (2) is defined by

$$R^\bullet(\lambda; p^+, p^-; \tau) = \sum_{|\nu^\pm|=|\mu^\pm| \geq 0} \frac{\chi_{\nu^+}(C(\mu^+))}{z_{\mu^+}} \frac{\chi_{\nu^-}(C(\mu^-))}{z_{\mu^-}} \cdot e^{\sqrt{-1}(\kappa_{\nu^+} + \tau + \kappa_{\nu^-} - \tau^{-1})\lambda/2} \mathcal{W}_{\nu^+, \nu^-}(e^{\sqrt{-1}\lambda}) p_{\mu^+}^+ p_{\mu^-}^-.$$

3. DOUBLE HURWITZ NUMBERS AND THE CUT-AND-JOIN EQUATIONS OF  $G^\bullet$

In this section, we first derive some identities of double Hurwitz numbers, such as sum formula and cut-and-join equations, which, together with initial values, characterize the double Hurwitz numbers. Then we combine these identities with (6) to obtain Theorem 2 (cut-and-join equation of  $G^\bullet$ ).

**3.1. Double Hurwitz numbers.** Let  $X$  be a Riemann surface of genus  $h$ . Given  $n$  partitions  $\eta^1, \dots, \eta^n$  of  $d$ , denote by  $H_d^X(\eta^1, \dots, \eta^n)^\bullet$  and  $H_d^X(\eta^1, \dots, \eta^n)^\circ$  the weighted counts of possibly disconnected and connected Hurwitz covers of type  $(\eta^1, \dots, \eta^n)$  respectively. We will use the following formula for Hurwitz numbers (see e.g. [5]):

$$(10) \quad H_d^X(\eta^1, \dots, \eta^n)^\bullet = \sum_{|\rho|=d} \left( \frac{\dim R_\rho}{d!} \right)^{2-2h} \prod_{i=1}^n |C_{\eta^i}| \frac{\chi_\rho(C_{\eta^i})}{\dim R_\rho}.$$

It is sometimes referred to as the *Burnside formula*.

Suppose  $C \rightarrow \mathbb{P}^1$  is a genus  $g$  cover which has ramification type  $\mu^+, \mu^-$  at two points  $p_0$  and  $p_1$  respectively, and ramification type (2) at  $r$  other points. By the Riemann-Hurwitz formula,

$$(11) \quad r = 2g - 2 + l(\mu^+) + l(\mu^-).$$

Denote

$$H_g^\circ(\mu^+, \mu^-) = H_d^{\mathbb{P}^1}(\mu^+, \mu^-, \eta^1, \dots, \eta^r)^\circ,$$

$$H_g^\bullet(\mu^+, \mu^-) = H_d^{\mathbb{P}^1}(\mu^+, \mu^-, \eta^1, \dots, \eta^r)^\bullet,$$

for  $\eta^1 = \dots = \eta^r = (2)$ . We have by (10):

$$(12) \quad H_g^\bullet(\mu^+, \mu^-) = \sum_{|\nu|=d} f_\nu(2)^r \frac{\chi_\nu(C_{\mu^+})}{z_{\mu^+}} \frac{\chi_\nu(C_{\mu^-})}{z_{\mu^-}},$$

where  $r$  is given by (11), and

$$f_\nu(2) = |C_{(2)}| \frac{\chi_\nu(C_{(2)})}{\dim R_\nu}.$$

Define

$$\begin{aligned} \Phi_{\mu^+, \mu^-}^\circ(\lambda) &= \sum_{g \geq 0} H_g^\circ(\mu^+, \mu^-) \frac{\lambda^{2g-2+l(\mu^+)+l(\mu^-)}}{(2g-2+l(\mu^+)+l(\mu^-))!}, \\ \Phi_{\mu^+, \mu^-}^\bullet(\lambda) &= \sum_{g \geq 0} H_g^\bullet(\mu^+, \mu^-) \frac{\lambda^{2g-2+l(\mu^+)+l(\mu^-)}}{(2g-2+l(\mu^+)+l(\mu^-))!}, \\ \Phi^\circ(\lambda; p^+, p^-) &= \sum_{\mu^+, \mu^-} \Phi_{\mu^+, \mu^-}^\circ(\lambda) p_{\mu^+}^+ p_{\mu^-}^-, \\ \Phi^\bullet(\lambda; p^+, p^-) &= 1 + \sum_{\mu^+, \mu^-} \Phi_{\mu^+, \mu^-}^\bullet(\lambda) p_{\mu^+}^+ p_{\mu^-}^-. \end{aligned}$$

The usual relationship between connected and disconnected Hurwitz numbers is:

$$(13) \quad \Phi^\circ(\lambda; p^+, p^-) = \log \Phi^\bullet(\lambda; p^+, p^-).$$

By (12) one easily gets

$$(14) \quad \Phi^\bullet(\lambda; p^+, p^-) = 1 + \sum_{d \geq 1} \sum_{|\mu^\pm|=d} \sum_{|\nu|=d} \frac{\chi_\nu(C_{\mu^+})}{z_{\mu^+}} \frac{\chi_\nu(C_{\mu^-})}{z_{\mu^-}} e^{f_\nu(2)\lambda} p_{\mu^+}^+ p_{\mu^-}^-.$$

Equivalently,

$$(15) \quad \Phi_{\mu^+, \mu^-}^\bullet(\lambda) = \sum_{|\nu|=d} \frac{\chi_\nu(C_{\mu^+})}{z_{\mu^+}} \frac{\chi_\nu(C_{\mu^-})}{z_{\mu^-}} e^{f_\nu(2)\lambda}.$$

We also have

$$(16) \quad \Phi_{\mu^+, \mu^-}^\bullet(\lambda) = \sum_{|\nu|=d} (-1)^{l(\mu^+)+l(\mu^-)} \frac{\chi_\nu(C_{\mu^+})}{z_{\mu^+}} \frac{\chi_\nu(C_{\mu^-})}{z_{\mu^-}} e^{-f_\nu(2)\lambda}.$$

**3.2. Sum formula and initial values.** Recall the orthogonality relation for characters of  $S_d$ :

$$(17) \quad \sum_{\mu} \frac{\chi_{\nu^1}(C_\mu) \chi_{\nu^2}(C_\mu)}{z_\mu} = \delta_{\nu^1, \nu^2}.$$

It follows from (16) and (17) that

**Proposition 3.1.**

$$(18) \quad \Phi_{\mu^1, \mu^3}^\bullet(\lambda_1 + \lambda_2) = \sum_{\mu^2} \Phi_{\mu^1, \mu^2}^\bullet(\lambda_1) \cdot z_{\mu^2} \cdot \Phi_{\mu^2, \mu^3}^\bullet(\lambda_2),$$

$$(19) \quad \Phi_{\mu^1, \mu^3}^\bullet(0) = \frac{1}{z_{\mu^1}} \delta_{\mu^1, \mu^3}.$$

Equation (18) is a sum formula for double Hurwitz numbers, and Equation (19) gives the initial values for double Hurwitz numbers.

**Corollary 3.2.** Denote by  $\Phi^\bullet(\lambda)_d$  the matrix  $(\Phi_{\mu,\nu}^\bullet(\lambda))_{|\mu|=|\nu|=d}$ . Then  $\Phi^\bullet(\lambda)_d$  is invertible, and

$$(20) \quad Z_d^{-1} \Phi^\bullet(-\lambda)_d^{-1} = \Phi^\bullet(\lambda)_d Z_d,$$

where  $Z_d = (z_\mu \delta_{\mu,\nu})_{|\mu|=|\nu|=d}$ .

*Proof.* In (18) we take  $\lambda_1 = \lambda$  and  $\lambda_2 = -\lambda$ . Then by (19) we have

$$Z_d^{-1} = \Phi^\bullet(0)_d = \Phi^\bullet(\lambda)_d Z_d \Phi^\bullet(-\lambda)_d.$$

Taking the determinant on both sides one sees that  $\Phi^\bullet(\lambda)_d$  is invertible, and (20) is a straightforward consequence.  $\square$

**3.3. Cut-and-join equation for double Hurwitz numbers.** Recall for any partition  $\nu$  of  $d$ , one has

$$f_\nu(2) \cdot \sum_\mu \frac{\chi_\nu(C_\mu)}{z_\mu} p_\mu = \frac{1}{2} \sum_{i,j} \left( (i+j)p_i p_j \frac{\partial}{\partial p_{i+j}} + ij p_{i+j} \frac{\partial}{\partial p_i} \frac{\partial}{\partial p_j} \right) \sum_\eta \frac{\chi_\nu(C_\eta)}{z_\eta} p_\eta.$$

See e.g. [32, 23]. From this one easily proves the following results.

**Proposition 3.3.** We have the following equations:

$$(21) \quad \frac{\partial \Phi_h^\bullet}{\partial \lambda} = \frac{1}{2} \sum_{i,j \geq 1} \left( ij p_{i+j}^\pm \frac{\partial^2 \Phi_h^\bullet}{\partial p_i^\pm \partial p_j^\pm} + (i+j) p_i^\pm p_j^\pm \frac{\partial \Phi_h^\bullet}{\partial p_{i+j}^\pm} \right),$$

$$(22) \quad \frac{\partial \Phi_h^\circ}{\partial \lambda} = \frac{1}{2} \sum_{i,j \geq 1} \left( ij p_{i+j}^\pm \frac{\partial^2 \Phi_h^\circ}{\partial p_i^\pm \partial p_j^\pm} + ij p_{i+j}^\pm \frac{\partial \Phi_h^\circ}{\partial p_i^\pm} \frac{\partial \Phi_h^\circ}{\partial p_j^\pm} + (i+j) p_i^\pm p_j^\pm \frac{\partial \Phi_h^\circ}{\partial p_{i+j}^\pm} \right).$$

The new feature for the double Hurwitz numbers is that there are two choices to do the cut-and-join, on the  $+$  side or on the  $-$  side. One can rewrite (21) as sequences of systems of ODEs as follows. For each partition  $\mu^-$  of  $d$ , one gets a system of ODEs for  $\{\Phi_{\mu^+,\mu^-}^\bullet(\lambda) : |\mu^+| = d\}$ ; hence they are determined by  $\{\Phi_{\mu^+,\mu^-}^\bullet(0) : |\mu^+| = d\}$ . One can also reverse the roles of  $\mu^+$  and  $\mu^-$ . There are matrices  $CJ_d$  such that the cut-and-join equations in degree  $d$  can be written as

$$(23) \quad \frac{d}{d\lambda} \Phi_d^\bullet = CJ_d \cdot \Phi_d^\bullet = \Phi_d^\bullet \cdot CJ_d^t.$$

**Example 3.4.** When  $d = 2$ , the cut-and-join equation becomes

$$\begin{aligned} \frac{d}{d\lambda} \begin{pmatrix} \Phi_{(2),(2)}^\bullet & \Phi_{(2),(1^2)}^\bullet \\ \Phi_{(1^2),(2)}^\bullet & \Phi_{(1^2),(1^2)}^\bullet \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \Phi_{(2),(2)}^\bullet & \Phi_{(2),(1^2)}^\bullet \\ \Phi_{(1^2),(2)}^\bullet & \Phi_{(1^2),(1^2)}^\bullet \end{pmatrix} \\ &= \begin{pmatrix} \Phi_{(2),(2)}^\bullet & \Phi_{(2),(1^2)}^\bullet \\ \Phi_{(1^2),(2)}^\bullet & \Phi_{(1^2),(1^2)}^\bullet \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \end{aligned}$$

The initial values are

$$\begin{pmatrix} \Phi_{(2),(2)}^\bullet & \Phi_{(2),(1^2)}^\bullet \\ \Phi_{(1^2),(2)}^\bullet & \Phi_{(1^2),(1^2)}^\bullet \end{pmatrix} (0) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}.$$

Hence we have the following solution:

$$\begin{pmatrix} \Phi_{(2),(2)}^\bullet & \Phi_{(2),(1^2)}^\bullet \\ \Phi_{(1^2),(2)}^\bullet & \Phi_{(1^2),(1^2)}^\bullet \end{pmatrix} (\lambda) = \begin{pmatrix} \frac{1}{2} \cosh \lambda & \frac{1}{2} \sinh \lambda \\ \frac{1}{2} \sinh \lambda & \frac{1}{2} \cosh \lambda \end{pmatrix}$$

This is compatible with (16).



**3.4. Cut-and-join equation for two-partition Hodge integrals.** In Section 7, we will define a generating function  $K_{\mu^+, \mu^-}^\bullet(\lambda)$  of relative Gromov-Witten invariants for each  $(\mu^+, \mu^-) \in \mathcal{P}_+^2$ , and derive the following identity by relative virtual localization:

$$(6) \quad K_{\mu^+, \mu^-}^\bullet(\lambda) = \sum_{|\nu^\pm|=|\mu^\pm|} \Phi_{\mu^+, \nu^+}^\bullet(-\sqrt{-1}\tau\lambda) z_{\nu^+} G_{\nu^+, \nu^-}^\bullet(\lambda; \tau) z_{\nu^-} \Phi_{\nu^-, \mu^-}^\bullet\left(\frac{-\sqrt{-1}}{\tau}\lambda\right).$$

Each term on the right-hand side depends on  $\tau$ , which comes from localization weights, but the sum  $K_{\mu^+, \mu^-}^\bullet(\lambda)$  is a generating function of *topological* invariants which do not depend on  $\tau$ .

In matrix form, one has for  $d^+, d^- \geq 0$ ,

$$K^\bullet(\lambda)_{d^+, d^-} = \Phi^\bullet(-\sqrt{-1}\tau\lambda)_{d^+} Z_{d^+} G^\bullet(\lambda; \tau)_{d^+, d^-} Z_{d^-} \Phi^\bullet\left(\frac{-\sqrt{-1}}{\tau}\lambda\right)_{d^-}.$$

Hence by (20) we have

$$G^\bullet(\lambda; \tau)_{d^+, d^-} = \Phi^\bullet(\sqrt{-1}\tau\lambda)_{d^+} Z_{d^+} K^\bullet(\lambda)_{d^+, d^-} Z_{d^-} \Phi^\bullet\left(\frac{\sqrt{-1}}{\tau}\lambda\right)_{d^-}.$$

Taking the derivative in  $\tau$  on both sides, one then gets

$$\frac{\partial}{\partial \tau} G^\bullet(\lambda; \tau)_{d^+, d^-} = \sqrt{-1}\lambda \left( C J_{d^+} \cdot G^\bullet(\lambda; \tau)_{d^+, d^-} - \frac{1}{\tau^2} G^\bullet(\lambda; \tau)_{d^+, d^-} \cdot C J_{d^-}^t \right).$$

This completes the proof of the cut-and-join equation for  $G^\bullet$  and hence the proof of the formula (2) of two-partition Hodge integrals.

**Corollary 3.5.** *We have*

$$K_{\mu^+, \mu^-}^\bullet(\lambda) = \sum_{\eta^\pm} \frac{\chi_{\eta^+}(C_{\mu^+})}{z_{\mu^+}} \cdot \mathcal{W}_{\eta^+, \eta^-}(e^{\sqrt{-1}\lambda}) \cdot \frac{\chi_{\eta^-}(C_{\mu^-})}{z_{\mu^-}}.$$

*Proof.* By (6), (2) and (15) we have

$$\begin{aligned} & K_{\mu^+, \mu^-}^\bullet(\lambda) \\ &= \sum_{|\nu^\pm|=|\mu^\pm|} \Phi_{\mu^+, \nu^+}^\bullet(-\sqrt{-1}\tau\lambda) z_{\nu^+} G_{\nu^+, \nu^-}^\bullet(\lambda; \tau) z_{\nu^-} \Phi_{\nu^-, \mu^-}^\bullet\left(\frac{-\sqrt{-1}}{\tau}\lambda\right) \\ &= \sum_{\nu^\pm, \rho^\pm, \eta^\pm} e^{-\sqrt{-1}f_{\rho^+}(2)\tau\lambda} \frac{\chi_{\rho^+}(C_{\mu^+})}{z_{\mu^+}} \frac{\chi_{\rho^+}(C_{\nu^+})}{z_{\nu^+}} \cdot z_{\nu^+} \\ &\quad \cdot \frac{\chi_{\eta^+}(C_{\nu^+})}{z_{\nu^+}} e^{\sqrt{-1}\kappa_{\eta^+}\tau\lambda/2} \mathcal{W}_{\eta^+, \eta^-}(e^{\sqrt{-1}\lambda}) e^{\sqrt{-1}\kappa_{\eta^-}\tau^{-1}\lambda/2} \frac{\chi_{\eta^-}(C_{\nu^-})}{z_{\nu^-}} \\ &\quad \cdot z_{\nu^-} \cdot \frac{\chi_{\rho^-}(C_{\nu^-})}{z_{\nu^-}} \frac{\chi_{\rho^-}(C_{\mu^-})}{z_{\mu^-}} e^{-\sqrt{-1}f_{\rho^-}(2)\tau^{-1}\lambda} \\ &= \sum_{\eta^\pm} \frac{\chi_{\eta^+}(C_{\mu^+})}{z_{\mu^+}} \cdot \mathcal{W}_{\eta^+, \eta^-}(e^{\sqrt{-1}\lambda}) \cdot \frac{\chi_{\eta^-}(C_{\mu^-})}{z_{\mu^-}}. \end{aligned}$$

In the last equality we have used (17). □

4. RELATIVE STABLE MORPHISMS AND RELATIVE VIRTUAL LOCALIZATION

In this section, we will give a brief review of the moduli spaces of algebraic relative stable morphisms [19, 20] (see [17, 13, 14] for the symplectic approach) and virtual localization on such spaces [9, 11].

**4.1. Relative stable morphisms.** The definitions given in this section are based on J. Li’s work on relative stable morphisms [19, 20], with minor modifications.

Let  $Y$  be a smooth projective variety. Let  $D^1, \dots, D^k$  be disjoint smooth divisors in  $Y$ . For  $\alpha = 1, \dots, k$ , define

$$\Delta(D^\alpha) = \mathbb{P}(\mathcal{O}_{D^\alpha} \oplus \mathcal{N}_{D^\alpha/Y}) \rightarrow D^\alpha,$$

where  $\mathcal{N}_{D/Y}$  denotes the normal sheaf of a subvariety  $D$  in  $Y$ . The projective line bundle  $\Delta(D^\alpha) \rightarrow D^\alpha$  has two distinct sections

$$D_0^\alpha = \mathbb{P}(\mathcal{O}_{D^\alpha} \oplus 0), \quad D_\infty^\alpha = \mathbb{P}(0 \oplus \mathcal{N}_{D^\alpha/Y}).$$

We have

$$\mathcal{N}_{D_0^\alpha/\Delta(D^\alpha)} \cong \mathcal{N}_{D^\alpha/Y}^{-1}, \quad \mathcal{N}_{D_\infty^\alpha/\Delta(D^\alpha)} \cong \mathcal{N}_{D^\alpha/Y}.$$

Let

$$\Delta(D^\alpha)(m) = \Delta(D^\alpha)_1 \cup \Delta(D^\alpha)_2 \cup \dots \cup \Delta(D^\alpha)_m,$$

where  $\Delta(D^\alpha)_i \cong \Delta(D^\alpha)$  for  $i = 1, \dots, m$ . Let  $D_{i,0}^\alpha$  and  $D_{i,\infty}^\alpha$  be the two distinct sections of  $\Delta(D^\alpha)_i$  which correspond to  $D_0^\alpha$  and  $D_\infty^\alpha$ , respectively. Then  $\Delta(D^\alpha)(m)$  is obtained by identifying  $D_{i,\infty}^\alpha$  with  $D_{i+1,0}^\alpha$  for  $i = 1, \dots, m-1$  under the canonical isomorphisms

$$D_{i,\infty}^\alpha \cong D^\alpha \cong D_{i+1,0}^\alpha.$$

Define

$$D_{(0)}^\alpha = D_{1,0}^\alpha, \quad D_{(i)}^\alpha = \Delta(D^\alpha)_i \cap \Delta(D^\alpha)_{i+1}, \quad D_{(m)}^\alpha = D_{m,\infty}^\alpha,$$

where  $i = 1, \dots, m-1$ . The  $\mathbb{C}^*$ -action on  $\mathcal{O}_{D^\alpha}$  induces a  $\mathbb{C}^*$ -action on  $\Delta(D^\alpha)$  such that  $\Delta(D^\alpha) \rightarrow D^\alpha$  is  $\mathbb{C}^*$ -equivariant, where  $\mathbb{C}^*$  acts on  $D^\alpha$  trivially. The two distinct sections  $D_0^\alpha, D_\infty^\alpha$  are fixed under this  $\mathbb{C}^*$ -action. So there is a  $(\mathbb{C}^*)^m$ -action on  $\Delta(D^\alpha)(m)$  fixing  $D_{(0)}^\alpha, \dots, D_{(m)}^\alpha$ , such that  $\Delta(D^\alpha)(m) \rightarrow D^\alpha$  is  $(\mathbb{C}^*)^m$ -equivariant, where  $(\mathbb{C}^*)^m$  acts on  $D^\alpha$  trivially.

The variety

$$Y[m^1, \dots, m^k] = Y \cup \bigcup_{\alpha=1}^k \Delta(D^\alpha)(m^\alpha)$$

with normal crossing singularities is obtained by identifying  $D^\alpha \subset Y$  with  $D_{(0)}^\alpha \subset \Delta(D^\alpha)$  under the canonical isomorphism. There is a morphism

$$\pi[m^1, \dots, m^k] : Y[m^1, \dots, m^k] \rightarrow Y$$

which contracts  $\Delta(D^\alpha)(m^\alpha)$  to  $D^\alpha$ . The  $(\mathbb{C}^*)^{m^\alpha}$ -action on  $\Delta(D^\alpha)(m^\alpha)$  gives a  $(\mathbb{C}^*)^{m^1+\dots+m^k}$  on  $Y[m^1, \dots, m^k]$  such that  $\pi[m^1, \dots, m^k]$  is  $(\mathbb{C}^*)^{m^1+\dots+m^k}$ -equivariant with respect to the trivial action on  $Y$ .

With the above notation, we are now ready to define relative stable morphisms for  $(Y; D^1, \dots, D^k)$ .

**Definition 4.1.** Let  $\beta \in H_2(Y, \mathbb{Z})$  be a nonzero homology class such that

$$d^\alpha = \int_\beta c_1(\mathcal{O}(D^\alpha)) \geq 0.$$

Let  $\mu^\alpha$  be a partition of  $d^\alpha$ . Define

$$\overline{\mathcal{M}}_{g,0}(Y; D^1, \dots, D^k \mid \beta; \mu^1, \dots, \mu^k)$$

to be the moduli space of morphisms

$$f : (C, \{x_i^1\}_{i=1}^{l(\mu^1)}, \dots, \{x_i^k\}_{i=1}^{l(\mu^k)}) \rightarrow Y[m^1, \dots, m^k]$$

such that

- (1)  $(C, \{x_i^1\}_{i=1}^{l(\mu^1)}, \dots, \{x_i^k\}_{i=1}^{l(\mu^k)})$  is a connected prestable curve of arithmetic genus  $g$  with  $\sum_{\alpha=1}^k l(\mu^\alpha)$  marked points.
- (2)  $(\pi[m^1, \dots, m^k] \circ f)_*[C] = \beta \in H_2(Y; \mathbb{Z})$ .
- (3)

$$f^{-1}(D_{(m^\alpha)}^\alpha) = \sum_{i=1}^{l(\mu^\alpha)} \mu_i^\alpha x_i^\alpha$$

as Cartier divisors. In particular, if  $d^\alpha = 0$ , then  $f^{-1}(D_{(m^\alpha)}^\alpha)$  is empty.

- (4) The preimage of  $D_{(l)}^\alpha$  consists of nodes of  $C$ , where  $0 \leq l \leq m^\alpha - 1$ . If  $f(y) \in D_{(l)}^\alpha$  and  $C_1$  and  $C_2$  are two irreducible components of  $C$  which intersect at  $y$ , then  $f|_{C_1}$  and  $f|_{C_2}$  have the same contact order to  $D_{(l)}^\alpha$  at  $y$ .
- (5) The automorphism group of  $f$  is finite.

Two morphisms described above are isomorphic if they differ by an isomorphism of the domain and an element in  $(\mathbb{C}^*)^{m^1 + \dots + m^k}$  acting on the target. In particular, this defines the automorphism group in the stability condition (5) above.

*Remark 4.2.* In [19, 20], the number of divisors  $k = 1$ , but the construction and proofs in [19, 20] show that

$$\overline{\mathcal{M}}_{g,0}(Y; D^1, \dots, D^k \mid \beta; \mu^1, \dots, \mu^k)$$

is a separated, proper Deligne-Mumford stack with a perfect obstruction theory of virtual dimension

$$\int_\beta c_1(TY) + (1 - g)(\dim Y - 3) + \sum_{\alpha=1}^k (l(\mu^\alpha) - |\mu^\alpha|),$$

where  $TY$  is the tangent bundle of  $Y$ .

**Definition 4.3.** We define the moduli space  $\overline{\mathcal{M}}_\chi^\bullet(Y; D^1, \dots, D^k \mid \beta; \mu^1, \dots, \mu^k)$  similarly, with (1) replaced by the following (1) $^\bullet$ , and one additional condition (6):

- (1) $^\bullet$   $(C, \{x_i^1\}_{i=1}^{l(\mu^1)}, \dots, \{x_i^k\}_{i=1}^{l(\mu^k)})$  is a possibly disconnected prestable curve with  $\sum_{\alpha=1}^k l(\mu^\alpha)$  marked points. Let  $C_1, \dots, C_n$  be the connected components of  $C$ , and let  $g_i$  be the arithmetic genus of  $C_i$ . Then

$$\sum_{i=1}^n (2 - 2g_i) = \chi.$$

(6) Let  $\beta_i = \tilde{f}_*[C_i]$ , where  $C_i$  is a connected component of  $C$ . Then  $\beta_i \neq 0$ , and

$$\int_{\beta_i} c_1(\mathcal{O}(D^\alpha)) \geq 0$$

for  $\alpha = 1, \dots, k$ .

By [19, 20],  $\overline{\mathcal{M}}_\chi^\bullet(Y; D^1, \dots, D^k \mid \beta; \mu^1, \dots, \mu^k)$  is a separated, proper Deligne-Mumford stack with a perfect obstruction theory of virtual dimension

$$\int_{\beta} c_1(TY) + \frac{\chi}{2}(\dim Y - 3) + \sum_{\alpha=1}^k (l(\mu^\alpha) - |\mu^\alpha|).$$

**4.2. Tangent and obstruction spaces.** This section is based on [20, Section 5.1]. We first introduce some notation. If  $m^\alpha > 0$ , define line bundles  $L_l^\alpha$  on  $D_{(l)}^\alpha \subset Y[m^1, \dots, m^k]$  by

$$L_l^\alpha = \begin{cases} N_{D_{(0)}^\alpha/Y} \otimes N_{D_{(0)}^\alpha/\Delta(D^\alpha)_1} & l = 0, \\ N_{D_{(l)}^\alpha/\Delta(D^\alpha)_l} \otimes N_{D_{(l)}^\alpha/\Delta(D^\alpha)_{l+1}} & 1 \leq l \leq m^\alpha - 1. \end{cases}$$

Note that  $L_l^\alpha$  is a trivial line bundle on  $D_{(l)}^\alpha$ .

The tangent space  $T^1$  and the obstruction space  $T^2$  of

$$\overline{\mathcal{M}}_{g,0}(Y; D^1, \dots, D^k \mid \beta; \mu^1, \dots, \mu^k)$$

at the moduli point

$$\left[ f : (C, \{x_i^1\}_{i=1}^{l(\mu^1)}, \dots, \{x_i^k\}_{i=1}^{l(\mu^k)}) \rightarrow Y[m^1, \dots, m^k] \right]$$

are given by the following two exact sequences:

$$(24) \quad \begin{array}{ccccccc} 0 & \rightarrow & \text{Ext}^0(\Omega_C(R), \mathcal{O}_C) & \rightarrow & H^0(\mathbf{D}^\bullet) & \rightarrow & T^1 \\ & & \rightarrow & \text{Ext}^1(\Omega_C(R), \mathcal{O}_C) & \rightarrow & H^1(\mathbf{D}^\bullet) & \rightarrow T^2 \rightarrow 0, \end{array}$$

(25)

$$\begin{aligned} 0 & \rightarrow H^0 \left( C, f^* \left( \Omega_{Y[m^1, \dots, m^k]} \left( \sum_{\alpha=1}^k \log D_{(m^\alpha)}^\alpha \right) \right)^\vee \right) \rightarrow H^0(\mathbf{D}^\bullet) \\ & \rightarrow \bigoplus_{\alpha=1}^k \bigoplus_{l=0}^{m^\alpha-1} H_{\text{et}}^0(\mathbf{R}_l^{\alpha\bullet}) \\ & \rightarrow H^1 \left( C, f^* \left( \Omega_{Y[m^1, \dots, m^k]} \left( \sum_{\alpha=1}^k \log D_{(m^\alpha)}^\alpha \right) \right)^\vee \right) \rightarrow H^1(\mathbf{D}^\bullet) \\ & \rightarrow \bigoplus_{\alpha=1}^k \bigoplus_{l=0}^{m^\alpha-1} H_{\text{et}}^1(\mathbf{R}_l^{\alpha\bullet}) \rightarrow 0, \end{aligned}$$

where

$$R = \sum_{\alpha=1}^k \sum_{i=1}^{l(\mu^\alpha)} x_i^\alpha,$$

$$(26) \quad H_{\text{et}}^0(\mathbf{R}_l^{\alpha\bullet}) \cong \bigoplus_{q \in f^{-1}(D_l^\alpha)} T_q(f^{-1}(\Delta(D^\alpha)_l)) \otimes T_q^*(f^{-1}(\Delta(D^\alpha)_l)) \cong \mathbb{C}^{\oplus n_l^\alpha},$$

$$(27) \quad H_{\text{et}}^1(\mathbf{R}_l^{\alpha\bullet}) \cong H^0(D_{(l)}^\alpha, L_l^\alpha)^{\oplus n_l^\alpha} / H^0(D_{(l)}^\alpha, L_l^\alpha),$$

and  $n_l^\alpha$  is the number of nodes over  $D_l^\alpha$ . In (27),

$$H^0(D_{(l)}^\alpha, L_l^\alpha) \rightarrow H^0(D_{(l)}^\alpha, L_l^\alpha)^{\oplus n_l^\alpha}$$

is the diagonal embedding.

We refer the reader to [20] for the definitions of  $H^i(\mathbf{D}^\bullet)$  and the maps between terms in (24), (25). Here we only explain the part relevant to virtual localization calculations. The vector space

$$B_1 = \text{Ext}^0(\Omega_C(R), \mathcal{O}_C)$$

is the space of the infinitesimal automorphisms of the domain curve  $(C, R)$ , and

$$B_4 = \text{Ext}^1(\Omega_C(R), \mathcal{O}_C)$$

is the space of the infinitesimal deformations of  $(C, R)$ . Let  $\hat{C}$  be the normalization of  $C$ ,  $\hat{R} \subset \hat{C}$  be the pull-back of  $R$ , and  $R' \subset \hat{C}$  be the divisor corresponding to nodes in  $C$ . From the local to global spectral sequence, we have an exact sequence

$$0 \rightarrow B_{4,0} \rightarrow B_4 \rightarrow B_{4,1} \rightarrow 0,$$

where

$$B_{4,0} = H^1(C, \mathcal{E}xt_{\mathcal{O}_C}^0(\Omega_C(R), \mathcal{O}_C)) = H^1(C, \Omega_C(R)^\vee)$$

is the space of infinitesimal deformations of the smooth pointed curve  $(\hat{C}, \hat{R} + R')$ , and

$$B_{4,1} = H^0(C, \mathcal{E}xt_{\mathcal{O}_C}^1(\Omega_C(R), \mathcal{O}_C)) \cong \bigoplus_{q \in \text{Sing}(C)} T_{q'}\hat{C} \otimes T_{q''}\hat{C}$$

corresponds to smoothing of nodes of the domain curve. Here  $\text{Sing}(C)$  is the set of nodes of  $C$ , and  $q', q'' \in \hat{C}$  are the two preimages of  $q$  under the normalization map  $\hat{C} \rightarrow C$ . The tangent line of smoothing of the node  $q$  is canonically identified with  $T_{q'}\hat{C} \otimes T_{q''}\hat{C}$ .

The complex vector space

$$B_2 = H^0 \left( C, f^* \left( \Omega_{Y[m^1, \dots, m^k]} \left( \sum_{\alpha=1}^k \log D_{(m^\alpha)}^\alpha \right) \right)^\vee \right)$$

is the space of infinitesimal deformations of the map  $f$  with fixed domain and target, and

$$B_5 = H^1 \left( C, f^* \left( \Omega_{Y[m^1, \dots, m^k]} \left( \sum_{\alpha=1}^k \log D_{(m^\alpha)}^\alpha \right) \right)^\vee \right)$$

is the obstruction space to deforming  $f$  with fixed domain and target.

Finally, let

$$B_3 = \bigoplus_{\alpha=1}^k \bigoplus_{l=0}^{m^\alpha-1} H_{\text{et}}^0(\mathbf{R}_l^{\alpha\bullet}), \quad B_6 = \bigoplus_{\alpha=1}^k \bigoplus_{l=0}^{m^\alpha-1} H_{\text{et}}^1(\mathbf{R}_l^{\alpha\bullet}).$$

The complex vector space  $H_{\text{et}}^1(\mathbf{R}_l^{\alpha\bullet})$  corresponds to obstruction to smoothing the nodes in  $f^{-1}(D_{(l)}^\alpha)$ . More explicitly, let

$$f^{-1}(D_{(l)}^\alpha) = \{q_1, \dots, q_n\},$$

and let  $\nu_i$  be the contact order of  $f$  to  $D_{(l)}^\alpha$  at  $q_i$  (of either of the two branches of  $f$  near  $q_i$ ). Then  $B_4 \rightarrow H^1(\mathbf{D}^\bullet)$  in (24) induces a map

$$\begin{aligned} \bigoplus_{i=1}^n T_{q_{i,1}} \hat{C} \otimes T_{q_{i,2}} \hat{C} &\rightarrow H_{\text{et}}^1(\mathbf{R}_l^{\alpha\bullet}) \cong H^0(D_{(l)}^\alpha, L_l^\alpha)^{\oplus n} / H^0(D_{(l)}^\alpha, L_l^\alpha) \\ (s_1, \dots, s_n) &\mapsto [(s_1^{\nu_1}, \dots, s_n^{\nu_n})], \end{aligned}$$

where we use the isomorphisms

$$H^0(D_{(l)}^\alpha, L_l^\alpha) \cong (L_l^\alpha)_{f(q_i)} \cong (T_{q'_i} \hat{C} \otimes T_{q''_i} \hat{C})^{\otimes \nu_i}.$$

The first isomorphism follows from the triviality of the line bundle  $L_l^\alpha \rightarrow D_{(l)}^\alpha$ . We see that the obstruction vanishes iff the smoothing of the nodes  $q_1, \dots, q_n$  is compatible with the smoothing of the target along the divisor  $D_{(l)}^\alpha$ , which is parametrized by the complex line  $H^0(D_{(l)}^\alpha, L_l^\alpha)$ .

**4.3. Relative virtual localization.** In this section, we assume that a torus  $T = (\mathbb{C}^*)^r$  acts on  $Y$ , and  $D^1, \dots, D^k$  are  $T$ -invariant divisors.

Under our assumption,  $\mathcal{N}_{D^\alpha/Y} \rightarrow D^\alpha$  is  $T$ -equivariant, and the  $T$ -action extends to  $\Delta(D^\alpha)$ . So  $T$  acts on  $Y[m^1, \dots, m^k]$  and acts on  $\overline{\mathcal{M}}_{g,0}(Y; D^1, \dots, D^k \mid \beta; \mu^1, \dots, \mu^k)$  by moving the image.

The  $T$  fixed points set  $\overline{\mathcal{M}}_{g,0}(Y; D^1, \dots, D^k \mid \beta; \mu^1, \dots, \mu^k)^T$  is a disjoint union of

$$\{\mathcal{F}_\Gamma \mid \Gamma \in G_{g,0}(Y; D^1, \dots, D^k \mid \beta; \mu^1, \dots, \mu^k)\},$$

where each  $\Gamma \in G_{g,0}(Y; D^1, \dots, D^k \mid \beta; \mu^1, \dots, \mu^k)$  corresponds to a connected component, or a union of connected components,  $\mathcal{F}_\Gamma$  of  $\overline{\mathcal{M}}_{g,0}(Y; D^1, \dots, D^k \mid \beta; \mu^1, \dots, \mu^k)^T$ . Let

$$[f : (C, \{x_i^1\}_{i=1}^{l(\mu^1)}, \dots, \{x_i^k\}_{i=1}^{l(\mu^k)}) \in \mathcal{F}_\Gamma \subset \overline{\mathcal{M}}_{g,0}(Y; D^1, \dots, D^k \mid \beta; \mu^1, \dots, \mu^k),$$

for some  $\Gamma \in G_{g,0}(\mathbb{P}^1, \mu)$ . The  $T$ -action on  $\overline{\mathcal{M}}_{g,0}(Y; D^1, \dots, D^k \mid \beta; \mu^1, \dots, \mu^k)$  induces  $T$ -actions on the exact sequences (24), (25) which define  $T^1$  and  $T^2$ . Let  $T^{i,f}$  and  $T^{i,m}$  denote the fixing part and the moving part of  $T^i$  under the  $T$ -action, respectively, where  $i = 1, 2$ . Then

$$T^{1,f} - T^{2,f}$$

defines a perfect obstruction theory on  $\mathcal{F}_\Gamma$ , and

$$T^{1,m} - T^{2,m}$$

defines the virtual normal bundle  $N_{\mathcal{F}_\Gamma}^{\text{vir}}$  of  $\mathcal{F}_\Gamma$  in  $\overline{\mathcal{M}}_{g,0}(Y; D^1, \dots, D^k \mid \beta; \mu^1, \dots, \mu^k)$ . More explicitly, let  $B_i^m$  denote the moving part of  $B_i$  under the  $T$ -action, where  $i = 1, \dots, 6$ . Then  $B_3^m = 0$ . Note that there are subtleties due to the  $(\mathbb{C}^*)^{m^1 + \dots + m^k}$ -action on the target  $Y[m^1, \dots, m^k]$ . We have

$$\frac{1}{e_T(N_{\mathcal{F}_\Gamma}^{\text{vir}})} = \frac{e_T(T^{2,m})}{e_T(T^{1,m})} = \frac{e_T(B_1^m)e_T(B_3^m)e_T(B_6^m)}{e_T(B_2^m)e_T(B_4^m)}.$$

In [9], T. Graber and R. Pandharipande proved a localization formula for the virtual fundamental class in the general context of  $\mathbb{C}^*$ -equivariant perfect obstruction theory. In [11], T. Graber and R. Vakil showed that moduli spaces of relative stable morphisms satisfy the technical assumptions required in the general formalism in [9] and derived relative virtual localization under the assumption that the divisor is fixed pointwisely under the  $\mathbb{C}^*$ -action [11, Theorem 3.6]. In our context, the localization formula proved in [9] reads:

$$(28) \quad \begin{aligned} & [\overline{\mathcal{M}}_{g,0}(Y; D^1, \dots, D^k \mid \beta; \mu^1, \dots, \mu^k)]_T^{\text{vir}} \\ &= \sum_{\Gamma \in G_{g,0}(Y; D^1, \dots, D^k \mid \beta; \mu^1, \dots, \mu^k)} (i_{\mathcal{F}_\Gamma})_* \left( \frac{[\mathcal{F}_\Gamma]_T^{\text{vir}}}{e_T(N_{\mathcal{F}_\Gamma}^{\text{vir}})} \right), \end{aligned}$$

where

$$i_{\mathcal{F}_\Gamma} : \mathcal{F}_\Gamma \rightarrow \overline{\mathcal{M}}_{g,0}(Y; D^1, \dots, D^k \mid \beta; \mu^1, \dots, \mu^k)$$

is the inclusion,  $e_T(N_{\mathcal{F}_\Gamma}^{\text{vir}})$  is the  $T$ -equivariant Euler class of the virtual normal bundle  $N_{\mathcal{F}_\Gamma}^{\text{vir}} = T^{1,m} - T^{2,m}$  over  $\mathcal{F}_\Gamma$ ,

$$\begin{aligned} & [\overline{\mathcal{M}}_{g,0}(Y; D^1, \dots, D^k \mid \beta; \mu^1, \dots, \mu^k)]_T^{\text{vir}} \\ & \in A_*^T(\overline{\mathcal{M}}_{g,0}(Y; D^1, \dots, D^k \mid \beta; \mu^1, \dots, \mu^k); \mathbb{Q}) \end{aligned}$$

is the  $T$ -equivariant virtual fundamental class defined by the  $T$ -equivariant perfect obstruction theory  $T^1 - T^2$  on  $\overline{\mathcal{M}}_{g,0}(Y; D^1, \dots, D^k \mid \beta; \mu^1, \dots, \mu^k)$ , and

$$[\mathcal{F}_\Gamma]_T^{\text{vir}} \in A_*^T(\mathcal{F}_\Gamma; \mathbb{Q})$$

is the  $T$ -equivariant virtual fundamental class defined by the perfect obstruction theory  $T^{1,f} - T^{2,f}$  on  $\mathcal{F}_\Gamma$ .

Similarly, we have

$$(29) \quad \begin{aligned} & [\overline{\mathcal{M}}_\chi^\bullet(Y; D^1, \dots, D^k \mid \beta; \mu^1, \dots, \mu^k)]^{\text{vir}} \\ &= \sum_{\Gamma \in G_\chi^\bullet(Y; D^1, \dots, D^k \mid \beta; \mu^1, \dots, \mu^k)} (i_{\mathcal{F}_\Gamma})_* \left( \frac{[\mathcal{F}_\Gamma]^{\text{vir}}}{e_T(N_{\mathcal{F}_\Gamma}^{\text{vir}})} \right). \end{aligned}$$

### 5. DOUBLE HURWITZ NUMBERS AS RELATIVE GROMOV-WITTEN INVARIANTS

In this section, we study double Hurwitz numbers by relative Gromov-Witten theory.

**5.1. Relative morphisms to  $\mathbb{P}^1$ .** Let  $[Z_0, Z_1]$  be the homogeneous coordinates of  $\mathbb{P}^1$ . Let  $\mathbb{C}^*$  act on  $\mathbb{P}^1$  by

$$t \cdot [Z_0, Z_1] = [tZ_0, Z_1]$$

for  $t \in \mathbb{C}^*$ ,  $[Z_0, Z_1] \in \mathbb{P}^1$ . Let

$$s^+ = [0, 1], \quad s^- = [1, 0]$$

be the two fixed points of this  $\mathbb{C}^*$ -action.

Let  $\mu^+$  and  $\mu^-$  be two partitions of  $d > 0$ . Let  $[\mathbb{P}^1] \in H_2(\mathbb{P}^1; \mathbb{Z})$  be the fundamental class. Define

$$\begin{aligned} \overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, \mu^+, \mu^-) &= \overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, s^+, s^-; d[\mathbb{P}^1], \mu^+, \mu^-), \\ \overline{\mathcal{M}}_\chi^\bullet(\mathbb{P}^1, \mu^+, \mu^-) &= \overline{\mathcal{M}}_\chi^\bullet(\mathbb{P}^1, s^+, s^-; d[\mathbb{P}^1], \mu^+, \mu^-). \end{aligned}$$

The virtual dimension of  $\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, \mu^+, \mu^-)$  is

$$2g - 2 + l(\mu^+) + l(\mu^-),$$

and the virtual dimension of  $\overline{\mathcal{M}}_\chi^\bullet(\mathbb{P}^1, \mu^+, \mu^-)$  is

$$-\chi + l(\mu^+) + l(\mu^-).$$

We extend the  $\mathbb{C}^*$ -action on  $\mathbb{P}^1$  to  $\mathbb{P}^1[m^+, m^-]$  by trivial action on  $\Delta^\pm[m^\pm]$ , which is a chain of  $m^\pm$  copies of  $\mathbb{P}^1$ . This induces  $\mathbb{C}^*$ -actions on  $\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, \mu^+, \mu^-)$  and  $\overline{\mathcal{M}}_\chi^\bullet(\mathbb{P}^1, \mu^+, \mu^-)$ . Define the moduli spaces of unparametrized relative stable maps to the triple  $(\mathbb{P}^1, s^+, s^-)$  to be

$$\begin{aligned} \overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, \mu^+, \mu^-) // \mathbb{C}^* &= \left( \overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, \mu^+, \mu^-) \setminus \overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, \mu^+, \mu^-)^{\mathbb{C}^*} \right) / \mathbb{C}^*, \\ \overline{\mathcal{M}}_\chi^\bullet(\mathbb{P}^1, \mu^+, \mu^-) // \mathbb{C}^* &= \left( \overline{\mathcal{M}}_\chi^\bullet(\mathbb{P}^1, \mu^+, \mu^-) \setminus \overline{\mathcal{M}}_\chi^\bullet(\mathbb{P}^1, \mu^+, \mu^-)^{\mathbb{C}^*} \right) / \mathbb{C}^*. \end{aligned}$$

Then  $\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, \mu^+, \mu^-) // \mathbb{C}^*$  is a separated, proper Deligne-Mumford stack with a perfect obstruction theory of virtual dimension

$$2g - 2 + l(\mu^+) + l(\mu^-) - 1,$$

and  $\overline{\mathcal{M}}_\chi^\bullet(\mathbb{P}^1, \mu^+, \mu^-) // \mathbb{C}^*$  is a separated, proper Deligne-Mumford stack with a perfect obstruction theory of virtual dimension

$$-\chi + l(\mu^+) + l(\mu^-) - 1.$$

**5.2. Target  $\psi$  classes.** In the notation in Section 4.1, we have  $\Delta^\pm \cong \mathbb{P}^1$ ,  $\Delta^\pm(m)$  is a chain of  $m$  copies of  $\mathbb{P}^1$ , and  $D_{(l)}^\pm$  is a point, for  $l = 0, \dots, m^\pm$ . Let  $\mathbb{L}^\pm$  be the line bundle on  $\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, \mu^+, \mu^-) // \mathbb{C}^*$  whose fiber at

$$[f : (C, x_1, \dots, x_{l(\mu^+)}, y_1, \dots, y_{l(\mu^-)}) \rightarrow \mathbb{P}^1[m^+, m^-]] \in \overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, \mu^+, \mu^-)$$

is the cotangent line

$$T_{D_{(m^\pm)}^\pm}^* (\mathbb{P}^1[m^+, m^-])$$

of  $\mathbb{P}^1[m^+, m^-]$  at the smooth point  $D_{(m^\pm)}^\pm$ . We then define  $\mathbb{L}^+$  and  $\mathbb{L}^-$  on  $\overline{\mathcal{M}}_\chi^\bullet(\mathbb{P}^1, \mu^+, \mu^-) // \mathbb{C}^*$ , similarly. Define the *target  $\psi$  classes*

$$\psi^0 = c_1(\mathbb{L}^+), \quad \psi^\infty = c_1(\mathbb{L}^-).$$

The following integral of  $\psi^0$  arises in the localization calculations in [23]:

$$\int_{[\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, \mu^+, \mu^-) // \mathbb{C}^*]^{\text{vir}}} (\psi^0)^{2g-2+l(\mu^+)+l(\mu^-)-1}.$$

In Section 5.8, we will relate such integrals of target  $\psi$  classes to double Hurwitz numbers (Proposition 5.4, 5.5).



**5.3. Double Hurwitz numbers.** Let  $\mu^+, \mu^-$  be two partitions of  $d > 0$ . There are branch morphisms

$$\begin{aligned} \text{Br} : \overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, \mu^+, \mu^-) &\rightarrow \text{Sym}^{2g-2+l(\mu^+)+l(\mu^-)} \mathbb{P}^1 \cong \mathbb{P}^{2g-2+l(\mu^+)+l(\mu^-)}, \\ \text{Br} : \overline{\mathcal{M}}_{\chi}^{\bullet}(\mathbb{P}^1, \mu^+, \mu^-) &\rightarrow \text{Sym}^{-\chi+l(\mu^+)+l(\mu^-)} \mathbb{P}^1 \cong \mathbb{P}^{-\chi+l(\mu^+)+l(\mu^-)}. \end{aligned}$$

The double Hurwitz numbers for connected covers of  $\mathbb{P}^1$  can be defined by

$$H_g^{\circ}(\mu^+, \mu^-) = \frac{1}{|\text{Aut}(\mu^+)| |\text{Aut}(\mu^-)|} \int_{[\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, \mu^+, \mu^-)]^{\text{vir}}} \text{Br}^*(H^{2g-2+l(\mu^+)+l(\mu^-)}),$$

where  $H \in H^2(\mathbb{P}^{2g-2+l(\mu^+)+l(\mu^-)}; \mathbb{Z})$  is the hyperplane class. The double Hurwitz numbers for possibly disconnected covers of  $\mathbb{P}^1$  can be defined by

$$H_{\chi, \mu^+, \mu^-}^{\bullet} = \frac{1}{|\text{Aut}(\mu^+)| |\text{Aut}(\mu^-)|} \int_{[\overline{\mathcal{M}}_{\chi}^{\bullet}(\mathbb{P}^1, \mu^+, \mu^-)]^{\text{vir}}} \text{Br}^*(H^{-\chi+l(\mu^+)+l(\mu^-)}).$$

We have

$$H_{2-2g, \mu^+, \mu^-}^{\bullet} = H_g^{\bullet}(\mu^+, \mu^-).$$

Recall that  $H_g^{\circ}(\mu^+, \mu^-), H_g^{\bullet}(\mu^+, \mu^-)$  are defined combinatorially in Section 3.

We define generating functions of double Hurwitz numbers as in Section 3:

$$\begin{aligned} \Phi_{\mu^+, \mu^-}^{\circ}(\lambda) &= \sum_{g=0}^{\infty} \frac{\lambda^{2g-2+l(\mu^+)+l(\mu^-)}}{(2g-2+l(\mu^+)+l(\mu^-))!} H_g^{\circ}(\mu^+, \mu^-), \\ \Phi_{\mu^+, \mu^-}^{\bullet}(\lambda) &= \sum_{\chi \in 2\mathbb{Z}, \frac{\chi}{2} \leq \min\{l(\mu^+), l(\mu^-)\}} \frac{\lambda^{-\chi+l(\mu^+)+l(\mu^-)}}{(-\chi+l(\mu^+)+l(\mu^-))!} H_{\chi, \mu^+, \mu^-}^{\bullet}, \\ \Phi^{\circ}(\lambda; p^+, p^-) &= \sum_{\mu^+, \mu^-} \Phi_{\mu^+, \mu^-}^{\circ}(\lambda) p_{\mu^+}^+ p_{\mu^-}^-, \\ \Phi^{\bullet}(\lambda; p^+, p^-) &= 1 + \sum_{\mu^+, \mu^-} \Phi_{\mu^+, \mu^-}^{\bullet}(\lambda) p_{\mu^+}^+ p_{\mu^-}^-. \end{aligned}$$

Then

$$\Phi^{\bullet}(\lambda; p^+, p^-) = \exp(\Phi^{\circ}(\lambda; p^+, p^-)).$$

Note that

$$(30) \quad \Phi_{\mu^+, \mu^-}^{\bullet}(0) = \frac{\delta_{\mu^+, \mu^-}}{z_{\mu^+}},$$

where  $z_{\nu} = \nu_1 \cdots \nu_{l(\nu)} |\text{Aut}(\nu)|$ , so

$$\Phi^{\bullet}(0, p^+, p^-) = 1 + \sum_{\mu} \frac{p_{\mu}^+ p_{\mu}^-}{z_{\mu}}.$$

**5.4. Gluing formula.** Let  $k^+, k^-$  be positive integers such that

$$k^+ + k^- = -\chi + l(\mu^+) + l(\mu^-).$$

By the gluing formula of algebraic relative Gromov-Witten invariants [20, Corollary 3.16], we have

$$\begin{aligned} & \int_{[\overline{\mathcal{M}}_{\chi}^{\bullet}(\mathbb{P}^1, \mu^+, \mu^-)]^{\text{vir}}} \text{Br}^*(H^{-\chi+l(\mu^+)+l(\mu^-)}) \\ &= \sum_{-\chi^{\pm}+l(\mu^{\pm})+l(\nu)=k^{\pm}} \int_{[\overline{\mathcal{M}}_{\chi^+}^{\bullet}(\mathbb{P}^1, \mu^+, \nu)]^{\text{vir}}} \text{Br}^*(H^{k^+}) \\ & \cdot \frac{a_{\nu}}{|\text{Aut}(\nu)|} \int_{[\overline{\mathcal{M}}_{\chi^-}^{\bullet}(\mathbb{P}^1, \nu, \mu^-)]^{\text{vir}}} \text{Br}^*(H^{k^-}), \end{aligned}$$

where

$$a_{\nu} = \nu_1 \cdots \nu_l(\nu).$$

Therefore, we have the following gluing formula for double Hurwitz numbers:

**Proposition 5.1** (Gluing formula). *Let  $k^+, k^-$  be positive integers such that*

$$k^+ + k^- = -\chi + l(\mu^+) + l(\mu^-).$$

*Then*

$$(31) \quad H_{\chi, \mu^+, \mu^-}^{\bullet} = \sum_{-\chi^{\pm}+l(\mu^{\pm})+l(\nu)=k^{\pm}} H_{\chi^+, \mu^+, \nu}^{\bullet} z_{\nu} H_{\chi^-, \nu, \mu^-}^{\bullet}.$$

Recall that  $z_{\nu} = a_{\nu} |\text{Aut}(\nu)|$ .

Let  $d = |\mu^+| = |\mu^-|$ . It is straightforward to check that Proposition 5.1 implies the *sum formula*

$$(32) \quad \sum_{|\nu|=d} \Phi_{\mu^+, \nu}^{\bullet}(\lambda_1) z_{\nu} \Phi_{\nu, \mu^-}^{\bullet}(\lambda_2) = \Phi_{\mu^+, \mu^-}^{\bullet}(\lambda_1 + \lambda_2),$$

which was derived in Section 3.2 from the combinatoric definition.

The cut-and-join equations (23) for double Hurwitz numbers are special cases  $k_+ = 1, k_- = 1$  of Proposition 5.1. More precisely, differentiate (32) with respect to  $\lambda_1$ , and then set  $\lambda_1 = 0$ . We obtain a cut-and-join equation:

$$(33) \quad \frac{d}{d\lambda} \Phi_{\mu^+, \mu^-}^{\bullet}(\lambda) = \sum_{|\nu|=d} H_{l(\mu^+)+l(\nu)-1, \mu^+, \nu}^{\bullet} z_{\nu} \Phi_{\nu, \mu^-}^{\bullet}(\lambda).$$

Differentiate (32) with respect to  $\lambda_2$ , and then set  $\lambda_2 = 0$ . We obtain another cut-and-join equation:

$$(34) \quad \frac{d}{d\lambda} \Phi_{\mu^+, \mu^-}^{\bullet}(\lambda) = \sum_{|\nu|=d} \Phi_{\mu^+, \nu}^{\bullet}(\lambda) z_{\nu} H_{l(\nu)+l(\mu^-)-1, \nu, \mu^-}^{\bullet}.$$

Define the *cut-and-join coefficients*

$$(CJ)_{\mu\nu} = H_{l(\mu)+l(\nu)-1, \mu, \nu}^{\bullet} z_{\nu}.$$

They are the entries of the matrix  $CJ_d$  in Section 3.3. The cut-and-join equations can be written as

$$(35) \quad \frac{d}{d\lambda} \Phi_{\mu^+, \mu^-}^{\bullet}(\lambda) = \sum_{|\nu|=d} (CJ)_{\mu^+ \nu} \Phi_{\nu, \mu^-}^{\bullet}(\lambda) = \sum_{|\nu|=d} \Phi_{\mu^+, \nu}^{\bullet}(\lambda) (CJ)_{\mu^- \nu},$$

which is equivalent to (23) in Section 3.3:

$$\frac{d}{d\lambda} \Phi_d^{\bullet} = CJ_d \cdot \Phi_d^{\bullet} = \Phi_d^{\bullet} \cdot CJ_d^t.$$

*Remark 5.2.* The cut-and-join equation of the Hurwitz numbers  $H_g^\circ(\mu), H_g^\bullet(\mu)$  was first proved using combinatorics by Goulden, Jackson and Vainshtein [8] and later proved using the gluing formula of symplectic relative Gromov-Witten invariants by Li-Zhao-Zheng [18] and Ionel-Parker [14]. The gluing formula of symplectic relative Gromov-Witten invariants was proved in [17, 14]. The relationship between Hurwitz numbers and relative Gromov-Witten invariants of  $\mathbb{P}^1$  is studied extensively in [28].

**5.5. Localization.** In the rest of this section (Section 5), we will fix  $\chi, \mu^+, \mu^-$  and write

$$r = \chi + l(\mu^+) + l(\mu^-), \quad \alpha_{\mu^+, \mu^-} = |\text{Aut}(\mu^+)| |\text{Aut}(\mu^-)|.$$

In the spirit of [23, Section 7], we lift  $H^r \in H^{2r}(\mathbb{P}^r; \mathbb{Z})$  to

$$\prod_{k=1}^r (H - w_k u) \in H_{\mathbb{C}^*}^{2r}(\mathbb{P}^r; \mathbb{Z}),$$

where  $w_k \in \mathbb{Z}$ , and compute

$$H_{\chi, \mu^+, \mu^-}^\bullet = \frac{1}{\alpha_{\mu^+, \mu^-}} \int_{[\overline{\mathcal{M}}_\chi^\bullet(\mathbb{P}^1, \mu^+, \mu^-)]^{\text{vir}}} \text{Br}^* \left( \prod_{k=1}^r (H - w_k u) \right)$$

by virtual localization.

**5.6. Torus fixed points and admissible triples.** Given a morphism

$$f : (C, \{x_i^1\}_{i=1}^{l(\mu^+)}, \dots, \{x_i^k\}_{i=1}^{l(\mu^k)}) \rightarrow \mathbb{P}^1[m^+, m^-],$$

which represents a point in  $\overline{\mathcal{M}}_\chi^\bullet(\mathbb{P}^1, \mu^+, \mu^-)^{\text{C}^*}$ , let

$$\tilde{f} = \pi[m^+, m^-] \circ f : C \rightarrow \mathbb{P}^1,$$

and let  $C^\pm = \tilde{f}^{-1}(s^\pm)$ . Then

$$C = C^+ \cup L \cup C^-,$$

where  $L$  is a disjoint union of projective lines. Let

$$f^\pm = f|_{C^\pm} : C^\pm \rightarrow \Delta^\pm(m^\pm), \quad f^0 = f|_L : L \rightarrow \mathbb{P}^1.$$

Then  $f^0$  is a morphism of degree

$$d = |\mu^+| = |\mu^-|$$

fully ramified over  $s^+$  and  $s^-$ . The degrees of  $f^0$  restricted to connected components of  $L$  determine a partition  $\nu$  of  $d$ .

Let  $C_1^+, \dots, C_k^+$  be the connected components of  $C^+$ , and let  $g_i$  be the arithmetic genus of  $C_i^+$ . (We define  $g_i = 0$  if  $C_i^+$  is a point.) Define

$$\chi^+ = \sum_{i=1}^k (2 - 2g_i),$$

and define  $\chi^-$  similarly. We have

$$-\chi^+ + 2l(\nu) - \chi^- = -\chi.$$

Note that  $\chi^\pm \leq 2 \min\{l(\mu^\pm), l(\nu)\}$ . So

$$-\chi^+ + l(\mu^+) + l(\nu) \geq 0,$$

and the equality holds if and only if  $m^+ = 0$ . In this case, we have  $\nu = \mu^+$ ,  $\chi^+ = 2l(\mu^+)$ , and  $\chi^- = \chi$ . Similarly,

$$-\chi^- + l(\nu) + l(\mu^-) \geq 0,$$

and the equality holds if and only if  $m^- = 0$ . In this case, we have  $\nu = \mu^-$ ,  $\chi^- = 2l(\mu^-)$ , and  $\chi^+ = \chi$ . There are three cases:

Case 1:  $m^- = 0$ . Then  $f^-$  is a constant map,  $\chi^+ = \chi$ ,  $\nu = \mu^-$ , and  $f^+$  represents a point in

$$\overline{\mathcal{M}}_{\chi}^{\bullet}(\mathbb{P}^1, \mu^+, \mu^-) // \mathbb{C}^*.$$

Case 2:  $m^+ = 0$ . Then  $f^+$  is a constant map,  $\chi^- = \chi$ ,  $\nu = \mu^+$ , and  $f^-$  represents a point in

$$\overline{\mathcal{M}}_{\chi}^{\bullet}(\mathbb{P}^1, \mu^+, \mu^-) // \mathbb{C}^*.$$

Case 3:  $m^+, m^- > 0$ . Up to an element of  $\text{Aut}(\nu)$ ,  $f^+$  represents a point in

$$\overline{\mathcal{M}}_{\chi^+}^{\bullet}(\mathbb{P}^1, \mu^+, \nu) // \mathbb{C}^*,$$

and  $f^-$  represents an element of

$$\overline{\mathcal{M}}_{\chi^-}^{\bullet}(\mathbb{P}^1, \nu, \mu^-) // \mathbb{C}^*.$$

**Definition 5.3.** We say that a triple  $(\chi^+, \nu, \chi^-)$  is *admissible* if

- $\chi^+, \chi^- \in 2\mathbb{Z}$ ;
- $\nu$  is a partition of  $d$ ;
- $\chi^{\pm} \leq 2 \min\{l(\mu^{\pm}), l(\nu)\}$ ;
- $-\chi^+ + 2l(\nu) - \chi^- = -\chi$ .

Let  $G_{\chi}^{\bullet}(\mathbb{P}^1, \mu^+, \mu^-)$  denote the set of all admissible triples.

We define

$$\begin{aligned} \overline{\mathcal{M}}_{\chi, \mu^-, 2l(\mu^-)} &= \overline{\mathcal{M}}_{\chi}^{\bullet}(\mathbb{P}^1, \mu^+, \mu^-) // \mathbb{C}^*, \\ \overline{\mathcal{M}}_{2l(\mu^+), \mu^+, \chi} &= \overline{\mathcal{M}}_{\chi}^{\bullet}(\mathbb{P}^1, \mu^+, \mu^-) // \mathbb{C}^*, \end{aligned}$$

and define

$$\overline{\mathcal{M}}_{\chi^+, \nu, \chi^-} = \left( \overline{\mathcal{M}}_{\chi^+}^{\bullet}(\mathbb{P}^1, \mu^+, \nu) // \mathbb{C}^* \right) \times \left( \overline{\mathcal{M}}_{\chi^-}^{\bullet}(\mathbb{P}^1, \nu, \mu^-) // \mathbb{C}^* \right)$$

if  $(\chi^+, \nu, \chi^-) \in G_{\chi}^{\bullet}(\mathbb{P}^1, \mu^+, \mu^-)$ , and

$$-\chi^+ + l(\mu^+) + l(\nu) > 0, \quad -\chi^- + l(\nu) + l(\mu^-) > 0.$$

For every  $(\chi^+, \nu, \chi^-) \in G_{\chi}^{\bullet}(\mathbb{P}^1, \mu^+, \mu^-)$ , there is a morphism

$$i_{\chi^+, \nu, \chi^-} : \overline{\mathcal{M}}_{\chi^+, \nu, \chi^-} \rightarrow \overline{\mathcal{M}}_{\chi}^{\bullet}(\mathbb{P}^1, \mu^+, \mu^-),$$

whose image  $\mathcal{F}_{\chi^+, \nu, \chi^-}$  is a union of connected components of  $\overline{\mathcal{M}}_{\chi}^{\bullet}(\mathbb{P}^1, \mu^+, \mu^-)^{\mathbb{C}^*}$ .

The morphism  $i_{\chi^+, \nu, \chi^-}$  induces an isomorphism

$$\overline{\mathcal{M}}_{\chi^+, \nu, \chi^-} / A_{\chi^+, \nu, \chi^-} \cong \mathcal{F}_{\chi^+, \nu, \chi^-},$$

where

$$A_{\chi, \mu^-, 2l(\mu^-)} = \prod_{i=1}^{l(\mu^-)} \mathbb{Z}_{\mu_i^-}, \quad A_{2l(\mu^+), \mu^+, \chi} = \prod_{i=1}^{l(\mu^+)} \mathbb{Z}_{\mu_i^+},$$

and for  $-\chi^\pm + l(\mu^\pm) + l(\nu) > 0$ , we have

$$1 \rightarrow \prod_{i=1}^{l(\nu)} \mathbb{Z}_{\nu_i} \rightarrow A_{\chi^+, \nu, \chi^-} \rightarrow \text{Aut}(\nu) \rightarrow 1.$$

Recall that  $a_\nu = \nu_1 \cdots \nu_{l(\nu)}$ , and  $z_\nu = a_\nu |\text{Aut}(\nu)|$ . We have

$$|A_{\chi, \mu^-, 2l(\mu^-)}| = a_{\mu^-}, \quad |A_{2l(\mu^+), \mu^+, \chi}| = a_{\mu^+},$$

and

$$|A_{\chi^+, \nu, \chi^-}| = z_\nu$$

if  $-\chi^\pm + l(\mu^\pm) + l(\nu) > 0$ .

The fixed points set  $\overline{\mathcal{M}}_\chi^\bullet(\mathbb{P}^1, \mu^+, \mu^-)^{\mathbb{C}^*}$  is a disjoint union of

$$\{\mathcal{F}_{\chi^+, \nu, \chi^-} \mid (\chi^+, \nu, \chi^-) \in G_\chi^\bullet(\mathbb{P}^1, \mu^+, \mu^-)\}.$$

**5.7. Contribution from each admissible triple.** Let  $(\chi^+, \nu, \chi^-) \in G_\chi^\bullet(\mathbb{P}^1, \mu^+, \mu^-)$ . We have

$$\text{Br}(\mathcal{F}_{\chi^+, \nu, \chi^-}) = (-\chi^+ + l(\mu^+) + l(\nu))s^+ + (-\chi^- + l(\nu) + l(\mu^-))s^- \in \text{Sym}^r \mathbb{P}^1 = \mathbb{P}^r,$$

where  $r = -\chi + l(\mu^+) + l(\mu^-)$  as before. So

$$i_{\chi^+, \nu, \chi^-}^* \text{Br}^* \left( \prod_{l=1}^r (H - w_l) \right) = \prod_{l=1}^r (-\chi^+ + l(\mu^+) + l(\nu) - w_l) u^r.$$

Let  $N_{\chi^+, \nu, \chi^-}^{\text{vir}}$  on  $\overline{\mathcal{M}}_{\chi^+, \nu, \chi^-}$  be the pull-back of the virtual normal bundle of  $\mathcal{F}_{\chi^+, \nu, \chi^-}$  in  $\overline{\mathcal{M}}_\chi^\bullet(\mathbb{P}^1, \mu^+, \mu^-)$ . Calculations similar to those in [23, Appendix A] show that

$$\begin{aligned} \frac{1}{e_{\mathbb{C}^*}(N_{\chi, \mu^-, 2l(\mu^-)}^{\text{vir}})} &= \frac{a_{\mu^-}}{u - \psi^\infty}, \\ \frac{1}{e_{\mathbb{C}^*}(N_{2l(\mu^+), \mu^+, \chi}^{\text{vir}})} &= \frac{a_{\mu^+}}{-u - \psi^0}, \end{aligned}$$

and for  $-\chi^\pm + l(\mu^\pm) + l(\nu) > 0$ , we have

$$\frac{1}{e_{\mathbb{C}^*}(N_{\chi^+, \nu, \chi^-}^{\text{vir}})} = \frac{a_\nu}{u - \psi_+^\infty} \frac{a_\nu}{-u - \psi_-^0},$$

where  $\psi_+^\infty, \psi_-^0$  are the target  $\psi$  classes on

$$\overline{\mathcal{M}}_{\chi^+}^\bullet(\mathbb{P}^1, \mu^+, \nu), \quad \overline{\mathcal{M}}_{\chi^-}^\bullet(\mathbb{P}^1, \nu, \mu^-),$$

respectively.

Let  $w = (w_1, \dots, w_l)$ . Then

$$\begin{aligned}
 I_{\chi, \mu^-, 2l(\mu^-)}(w) &= \frac{1}{a_{\mu^-}} \int_{[\overline{\mathcal{M}}_{\chi, \mu^-, 2l(\mu^-)}]^{vir}} \frac{i_{\chi, \mu^-, 2l(\mu^-)}^* \text{Br}^* \left( \prod_{l=1}^r (H - w_l u) \right)}{e_{\mathbb{C}^*}(N_{\chi, \mu^-, 2l(\mu^-)}^{vir})} \\
 &= \prod_{l=1}^r (r - w_l) \cdot \int_{[\overline{\mathcal{M}}_{\chi, \mu^-, 2l(\mu^-)}]^{vir}} \frac{u^r}{u - \psi^\infty} \\
 &= \prod_{l=1}^r (r - w_l) \cdot \int_{[\overline{\mathcal{M}}_\chi^\bullet(\mathbb{P}^1, \mu^+, \mu^-) // \mathbb{C}^*]^{vir}} (\psi^\infty)^{r-1}, \\
 I_{2l(\mu^+), \mu^+, \chi}(w) &= \frac{1}{a_{\mu^+}} \int_{[\overline{\mathcal{M}}_{2l(\mu^+), \mu^+, \chi}]^{vir}} \frac{i_{2l(\mu^+), \mu^+, \chi}^* \text{Br}^* \left( \prod_{l=1}^r (H - w_l u) \right)}{e_{\mathbb{C}^*}(N_{2l(\mu^+), \mu^+, \chi}^{vir})} \\
 &= \prod_{l=1}^r (-w_l) \cdot \int_{[\overline{\mathcal{M}}_{2l(\mu^+), \mu^+, \chi}]^{vir}} \frac{u^r}{-u - \psi^0} \\
 &= \prod_{l=1}^r w_l \cdot \int_{[\overline{\mathcal{M}}_\chi^\bullet(\mathbb{P}^1, \mu^+, \mu^-) // \mathbb{C}^*]^{vir}} (\psi^0)^{r-1},
 \end{aligned}$$

and

$$\begin{aligned}
 I_{\chi^+, \nu, \chi^-}(w) &= \frac{1}{z_\nu} \int_{[\overline{\mathcal{M}}_{\chi^+, \nu, \chi^-}]^{vir}} \frac{i_{\chi^+, \nu, \chi^-}^* \text{Br}^* \left( \prod_{l=1}^r (H - w_l u) \right)}{e_{\mathbb{C}^*}(N_{\chi^+, \nu, \chi^-}^{vir})} \\
 &= \frac{a_\nu}{|\text{Aut}(\nu)|} \left( \prod_{l=1}^r (-\chi^+ + l(\mu^+) + l(\nu) - w_l) \right) \\
 &\quad \cdot \int_{[\overline{\mathcal{M}}_{\chi^+, \nu, \chi^-}]^{vir}} \frac{u^r}{(u - \psi_+^\infty)(-u - \psi_-^0)} \\
 &= \frac{a_\nu}{|\text{Aut}(\nu)|} \left( \prod_{l=1}^r (-\chi^+ + l(\mu^+) + l(\nu) - w_l) \right) (-1)^{-\chi^- + l(\nu) + l(\mu^-)} \\
 &\quad \cdot \int_{[\overline{\mathcal{M}}_{\chi^+}^\bullet(\mathbb{P}^1, \mu^+, \nu) // \mathbb{C}^*]^{vir}} (\psi^\infty)^{-\chi^+ + l(\mu^+) + l(\nu) - 1} \\
 &\quad \cdot \int_{[\overline{\mathcal{M}}_{\chi^-}^\bullet(\mathbb{P}^1, \nu, \mu^-) // \mathbb{C}^*]^{vir}} (\psi^0)^{-\chi^- + l(\nu) + l(\mu^-) - 1}.
 \end{aligned}$$

**5.8. Sum over admissible triples.** Recall that  $\alpha_{\mu^+, \mu^-} = |\text{Aut}(\mu^+)| |\text{Aut}(\mu^-)|$ . We have

$$\begin{aligned}
 H_{\chi, \mu^+, \mu^-}^\bullet &= \frac{1}{\alpha_{\mu^+, \mu^-}} \int_{[\overline{\mathcal{M}}_\chi^\bullet(\mathbb{P}^1, \mu^+, \mu^-)]^{vir}} \text{Br}^* \left( \prod_{l=1}^r (H - w_l u) \right) \\
 &= \frac{1}{\alpha_{\mu^+, \mu^-}} \sum_{(\chi^+, \nu, \chi^-) \in G_\chi^\bullet(\mathbb{P}^1, \mu^+, \mu^-)} I_{\chi^+, \nu, \chi^-}(w).
 \end{aligned}$$

Letting  $w = (0, 1, \dots, r - 1)$ , we have

$$H_{\chi, \mu^+, \mu^-}^\bullet = \frac{I_{\chi, \mu^-, 2l(\mu^-)}(w)}{\alpha_{\mu^+, \mu^-}} = \frac{r!}{\alpha_{\mu^+, \mu^-}} \int_{[\overline{\mathcal{M}}_\chi^\bullet(\mathbb{P}^1, \mu^+, \mu^-) // \mathbb{C}^*]^{vir}} (\psi^\infty)^{r-1}.$$

Letting  $w = (1, 2, \dots, r)$ , we have

$$H_{\chi, \mu^+, \mu^-}^\bullet = \frac{I_{2l(\mu^+), \mu^+, \chi}(w)}{\alpha_{\mu^+, \mu^-}} = \frac{r!}{\alpha_{\mu^+, \mu^-}} \int_{[\overline{\mathcal{M}}_\chi^\bullet(\mathbb{P}^1, \mu^+, \mu^-) // \mathbb{C}^*]^{vir}} (\psi^0)^{r-1}.$$

We conclude that

**Proposition 5.4.**

$$\begin{aligned} \frac{H_{\chi, \mu^+, \mu^-}^\bullet}{r!} &= \frac{1}{\alpha_{\mu^+, \mu^-}} \int_{[\overline{\mathcal{M}}_\chi^\bullet(\mathbb{P}^1, \mu^+, \mu^-) // \mathbb{C}^*]^{vir}} (\psi^0)^{r-1} \\ &= \frac{1}{\alpha_{\mu^+, \mu^-}} \int_{[\overline{\mathcal{M}}_\chi^\bullet(\mathbb{P}^1, \mu^+, \mu^-) // \mathbb{C}^*]^{vir}} (\psi^\infty)^{r-1}, \end{aligned}$$

where

$$r = -\chi + l(\mu^+) + l(\mu^-), \quad \alpha_{\mu^+, \mu^-} = |\text{Aut}(\mu^+)| |\text{Aut}(\mu^-)|.$$

If we replace  $\overline{\mathcal{M}}_\chi^\bullet(\mathbb{P}^1, \mu^+, \mu^-)$  by  $\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, \mu^+, \mu^-)$  in Section 5.5, we get

**Proposition 5.5.**

$$\begin{aligned} \frac{H_g^\circ(\mu^+, \mu^-)}{r!} &= \frac{1}{\alpha_{\mu^+, \mu^-}} \int_{[\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, \mu^+, \mu^-) // \mathbb{C}^*]^{vir}} (\psi^0)^{r-1} \\ &= \frac{1}{\alpha_{\mu^+, \mu^-}} \int_{[\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, \mu^+, \mu^-) // \mathbb{C}^*]^{vir}} (\psi^\infty)^{r-1}, \end{aligned}$$

where

$$r = 2g - 2 + l(\mu^+) + l(\mu^-), \quad \alpha_{\mu^+, \mu^-} = |\text{Aut}(\mu^+)| |\text{Aut}(\mu^-)|.$$

### 6. MODULI SPACES AND OBSTRUCTION BUNDLES

In this section, we introduce the geometric objects involved in the proof of (6) and fix notation.

**6.1. The target  $X$ .** Let  $T = (\mathbb{C}^*)^2$  act on  $\mathbb{P}^1 \times \mathbb{P}^1$  by

$$(36) \quad (t_1, t_2) \cdot ([X_1, Y_1], [X_2, Y_2]) = ([t_1 X_1, Y_1], [t_2 X_2, Y_2]),$$

where  $(t_1, t_2) \in T$ ,  $[X_1, Y_1] \in \mathbb{P}^1$ ,  $[X_2, Y_2] \in \mathbb{P}^1$ . Let  $X$  be the toric blowup of  $\mathbb{P}^1 \times \mathbb{P}^1$  at  $([1, 0], [1, 0])$ . Let  $D_5$  be the exceptional divisor, and let  $D_1, D_2, D_3, D_4$  be the proper transforms of

$$\{[0, 1]\} \times \mathbb{P}^1, \quad \{[1, 0]\} \times \mathbb{P}^1, \quad \mathbb{P}^1 \times \{[0, 1]\}, \quad \mathbb{P}^1 \times \{[1, 0]\},$$

respectively. Let  $\beta_i \in H_2(X; \mathbb{Z})$  be the homology class represented by  $D_i$ . We have

$$\begin{aligned} H_2(X; \mathbb{Z}) &= \left( \bigoplus_{i=1}^5 \mathbb{Z}\beta_i \right) / (\mathbb{Z}(\beta_1 - \beta_2 - \beta_5) \oplus \mathbb{Z}(\beta_3 - \beta_4 - \beta_5)) \\ &= \mathbb{Z}\beta_1 \oplus \mathbb{Z}\beta_3 \oplus \mathbb{Z}\beta_5. \end{aligned}$$

Let  $\beta_i^* \in H^2(X; \mathbb{Z})$  be the Poincaré dual of  $\beta_i$ ,  $i = 1, \dots, 5$ . The intersection form on

$$H^2(X; \mathbb{Z}) = \mathbb{Z}\beta_1^* \oplus \mathbb{Z}\beta_3^* \oplus \mathbb{Z}\beta_5^*$$

is given by

$$\begin{matrix} & \beta_1^* & \beta_3^* & \beta_5^* \\ \beta_1^* & 0 & 1 & 0 \\ \beta_3^* & 1 & 0 & 0 \\ \beta_5^* & 0 & 0 & -1 \end{matrix}$$

So

$$\beta_2^* \cdot \beta_2^* = \beta_4^* \cdot \beta_4^* = -1, \quad \beta_2^* \cdot \beta_4^* = 0.$$

The  $T$ -invariant divisor

$$K_X = -D_1 - D_2 - D_3 - D_4 - D_5$$

is a canonical divisor of  $X$ , so

$$c_1(T_X) = 2\beta_1^* + 2\beta_3^* - \beta_5^*.$$

For  $(\mu^+, \mu^-) \in \mathcal{P}_+^2$ , define

$$\overline{\mathcal{M}}_{g,0}(X, \mu^+, \mu^-) = \overline{\mathcal{M}}_{g,0}(X; D_2, D_4 \mid |\mu^+|\beta_3 + |\mu^-|\beta_1; \mu^+, \mu^-),$$

and let  $\overline{\mathcal{M}}_\chi^\bullet(X, \mu^+, \mu^-)$  be the subset of

$$\overline{\mathcal{M}}_\chi^\bullet(X; D_2, D_4 \mid |\mu^+|\beta_3 + |\mu^-|\beta_1; \mu^+, \mu^-)$$

which consists of morphisms

$$f : C \rightarrow X[m^+, m^-]$$

such that for each connected component  $C_i$  of  $C$ ,  $\tilde{f}_*[C_i] \in H_2(X; \mathbb{Z})$  is an element of

$$\{a\beta_3 + b\beta_1 \mid a, b \in \mathbb{Z}_{\geq 0}, (a, b) \neq (0, 0)\}.$$

The virtual dimension of  $\overline{\mathcal{M}}_{g,0}(X, \mu^+, \mu^-)$  is

$$r_{g, \mu^+, \mu^-} = g - 1 + |\mu^+| + l(\mu^+) + |\mu^-| + l(\mu^-),$$

and the virtual dimension of  $\overline{\mathcal{M}}_\chi^\bullet(X, \mu^+, \mu^-)$  is

$$r_{\chi, \mu^+, \mu^-}^\bullet = -\frac{\chi}{2} + |\mu^+| + l(\mu^+) + |\mu^-| + l(\mu^-).$$

The moduli space  $\overline{\mathcal{M}}_{g,0}(X, \mu^+, \mu^-)$  plays the role of  $\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, \mu)$  in the proof of the Mariño-Vafa formula [23].

We have

$$D_2 \cong \mathbb{P}^1 \cong D_4, \quad \mathcal{N}_{D_2/X} \cong \mathcal{O}_{\mathbb{P}^1}(-1) \cong \mathcal{N}_{D_4/X},$$

so

$$\Delta(D_2) \cong \mathbb{F}_1 \cong \Delta(D_4)$$

in the notation of Section 4.1, where  $\mathbb{F}_1$  is the Hirzebruch surface

$$\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)) \rightarrow \mathbb{P}^1.$$



6.2. **The obstruction bundles.** Let

$$\pi : \mathcal{U}_{g,\mu^+,\mu^-} \rightarrow \overline{\mathcal{M}}_{g,0}(X, \mu^+, \mu^-)$$

be the universal domain curve, and let

$$P : \mathcal{T}_{g,\mu^+,\mu^-} \rightarrow \overline{\mathcal{M}}_{g,0}(X, \mu^+, \mu^-)$$

be the universal target. There is an evaluation map

$$F : \mathcal{U}_{g,\mu^+,\mu^-} \rightarrow \mathcal{T}_{g,\mu^+,\mu^-}$$

and a contraction map

$$\tilde{\pi} : \mathcal{T}_{g,\mu^+,\mu^-} \rightarrow X.$$

Let  $\mathcal{D}_{g,\mu^+,\mu^-} \subset \mathcal{U}_{g,\mu^+,\mu^-}$  be the divisor corresponding to the  $l(\mu^+) + l(\mu^-)$  marked points. Define

$$V_{g,\mu^+,\mu^-} = R^1\pi_* \left( \tilde{F}^* \mathcal{O}_X(-D_1 - D_3) \otimes \mathcal{O}_{\mathcal{U}_{g,\mu^+,\mu^-}}(-\mathcal{D}_{g,\mu^+,\mu^-}) \right),$$

where  $\tilde{F} = \tilde{\pi} \circ F : \mathcal{U}_{g,\mu^+,\mu^-} \rightarrow X$ . The fiber of  $V_{g,\mu^+,\mu^-}$  at

$$[f : (C, x_1, \dots, x_{l(\mu^+)}, y_1, \dots, y_{l(\mu^-)}) \rightarrow X[m^+, m^-]] \in \overline{\mathcal{M}}_{g,0}(X, \mu^+, \mu^-)$$

is

$$H^1(C, \tilde{f}^* \mathcal{O}_X(-D_1 - D_3) \otimes \mathcal{O}_C(-R)),$$

where  $\tilde{f} = \pi[m^+, m^-] \circ f$ , and

$$R = x_1 + \dots + x_{l(\mu^+)} + y_1 + \dots + y_{l(\mu^-)}.$$

Note that

$$H^0(C, \tilde{f}^* \mathcal{O}_X(-D_1 - D_3) \otimes \mathcal{O}_C(-R)) = 0$$

and

$$\deg \tilde{f}^* \mathcal{O}_X(-D_1 - D_3) \otimes \mathcal{O}_C(-R) = -|\mu^+| - |\mu^-| - l(\mu^+) - l(\mu^-),$$

so  $V_{g,\mu^+,\mu^-} \rightarrow \overline{\mathcal{M}}_{g,0}(X, \mu^+, \mu^-)$  is a vector bundle of rank

$$r_{g,\mu^+,\mu^-} = g - 1 + |\mu^+| + l(\mu^+) + |\mu^-| + l(\mu^-).$$

The vector bundle  $V_{g,\mu^+,\mu^-} \rightarrow \overline{\mathcal{M}}_{g,0}(X, \mu^+, \mu^-)$  plays the role of the obstruction bundle  $V \rightarrow \overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, \mu)$  in the proof of the Mariño-Vafa formula [23, Section 4.4].

Similarly, we define a vector bundle  $V_{\chi,\mu^+,\mu^-}^\bullet$  of rank

$$r_{\chi,\mu^+,\mu^-}^\bullet = -\frac{\chi}{2} + |\mu^+| + l(\mu^+) + |\mu^-| + l(\mu^-)$$

on  $\overline{\mathcal{M}}_\chi^\bullet(X, \mu^+, \mu^-)$ .

**6.3. Torus action.** Recall that  $T = (\mathbb{C}^*)^2$  acts on  $\mathbb{P}^1 \times \mathbb{P}^1$  by (36). It lifts to a  $T$ -action on  $X$  such that  $X \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$  is  $T$ -equivariant. Let  $T_{\mathbb{R}} = U(1)^2$  be the maximal compact subgroup of  $T$ . The  $T_{\mathbb{R}}$ -action on  $X$  determines a moment map

$$\mu_{T_{\mathbb{R}}} : X \rightarrow \mathfrak{t}_{\mathbb{R}}^*,$$

where  $\mathfrak{t}_{\mathbb{R}}^* \cong \mathbb{R}^2$  is the dual of the Lie algebra  $\mathfrak{t}_{\mathbb{R}}$  of  $T_{\mathbb{R}}$ .

We now lift the  $T$ -action on  $X$  to the line bundle  $\mathcal{O}_X(-D_1 - D_3)$  as follows. We only need to specify the representation of  $T$  on the fiber of one fixed point of the  $T$ -action. The fixed points of the  $T$ -action on  $X$  are

$$z_0 = D_1 \cap D_3, \quad z_+ = D_3 \cap D_2, \quad z_- = D_1 \cap D_4, \quad \tilde{z}_+ = D_2 \cap D_5, \quad \tilde{z}_- = D_4 \cap D_5.$$

Figure 1 shows the image of  $D_1, \dots, D_5$  and the above five fixed points under the moment map  $\mu_{T_{\mathbb{R}}} : X \rightarrow \mathfrak{t}_{\mathbb{R}}^*$ .

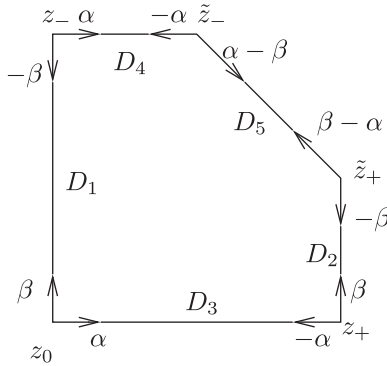


FIGURE 1. The image of  $\mu_{T_{\mathbb{R}}} : X \rightarrow \mathfrak{t}_{\mathbb{R}}^*$

Let  $(w_1, w_2)$  denote the one-dimensional representation given by

$$(t_1, t_2) \cdot z = t_1^{w_1} t_2^{w_2} z$$

for  $(t_1, t_2) \in T, z \in \mathbb{C}$ . The character ring of  $T$  is given by

$$\mathbb{Z}T \cong \mathbb{Z}[\alpha, \beta],$$

where  $\alpha, \beta$  are the characters of the representations  $(1, 0), (0, 1)$ , respectively. The representations of  $T$  on the fibers of  $T_X$  and  $\mathcal{O}_X(-D_1 - D_3)$  at fixed points are given by:

|                             | $z_0$             | $z_+$            | $z_-$            | $\tilde{z}_+$            | $\tilde{z}_-$             |
|-----------------------------|-------------------|------------------|------------------|--------------------------|---------------------------|
| $T_X$                       | $\alpha, \beta$   | $-\alpha, \beta$ | $\alpha, -\beta$ | $\beta - \alpha, -\beta$ | $\alpha - \beta, -\alpha$ |
| $\mathcal{O}_X(-D_1 - D_3)$ | $-\alpha - \beta$ | $-\beta$         | $-\alpha$        | 0                        | 0                         |

Note that

$$\mathfrak{t}_{\mathbb{R}}^* \cong \mathbb{R}\alpha \oplus \mathbb{R}\beta,$$

and the representations of  $T$  on the fibers of  $T_X$  at the fixed points can be read off from the image of the moment map as in Figure 1. The action of  $T$  on  $\Delta(D_2)$  and  $\Delta(D_4)$  can be read off from Figure 2. This extends the action of  $T$  on  $X$  to  $X[m^+, m^-]$ . So  $T$  acts on  $\overline{\mathcal{M}}_{g,0}(X, \mu^+, \mu^-)$  and  $\overline{\mathcal{M}}_{\chi}^{\bullet}(X, \mu^+, \mu^-)$  by moving the image of the morphism.

The  $T$ -action on  $\mathcal{O}_X(-D_1 - D_3)$  induces  $T$ -actions on  $V_{g, \mu^+, \mu^-}$  and on  $V_{\chi, \mu^+, \mu^-}^{\bullet}$ .

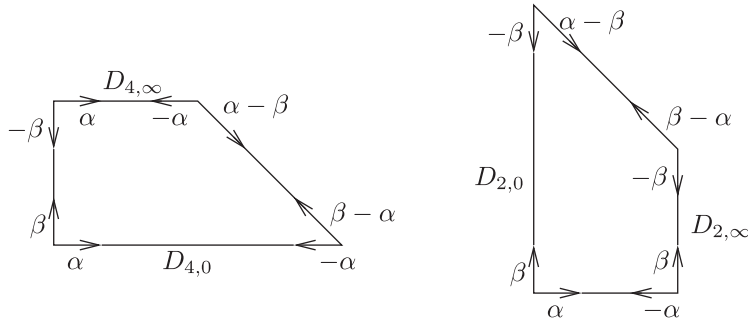


FIGURE 2. The images of  $\mu_{T_{\mathbb{R}}}^- : \Delta(D_4) \rightarrow \mathfrak{t}_{\mathbb{R}}^*$  and  $\mu_{T_{\mathbb{R}}}^+ : \Delta(D_2) \rightarrow \mathfrak{t}_{\mathbb{R}}^*$

7. PROOF OF (6)

Let  $X$  be defined as in Section 6.1. Recall that  $T = (\mathbb{C}^*)^2$  acts on  $X$ . Let  $D_1, \dots, D_5$  be  $T$ -invariant divisors in  $X$  defined in Section 6.1.

Let

$$V_{\chi, \mu^+, \mu^-}^\bullet \rightarrow \overline{\mathcal{M}}_\chi^\bullet(X, \mu^+, \mu^-)$$

be defined as in Section 6.2, with the torus action defined in Section 6.3. Define

$$K_{\chi, \mu^+, \mu^-}^\bullet = \frac{1}{|\text{Aut}(\mu^+)| |\text{Aut}(\mu^-)|} \int_{[\overline{\mathcal{M}}_\chi^\bullet(X, \mu^+, \mu^-)]^{\text{vir}}} e(V_{\chi, \mu^+, \mu^-}^\bullet),$$

$$K_{\mu^+, \mu^-}^\bullet(\lambda) = \sum_{\chi \in 2\mathbb{Z}, \chi \leq 2(l(\mu^+) + l(\mu^-))} \lambda^{-\chi + l(\mu^+) + l(\mu^-)} \frac{(-1)^{|\mu^+| + |\mu^-|}}{\sqrt{-1}^{l(\mu^+) + l(\mu^-)}} K_{\chi, \mu^+, \mu^-}^\bullet.$$

In this section, we will compute

$$K_{\chi, \mu^+, \mu^-}^\bullet = \frac{1}{|\text{Aut}(\mu^+)| |\text{Aut}(\mu^-)|} \int_{[\overline{\mathcal{M}}_\chi^\bullet(X, \mu^+, \mu^-)]^{\text{vir}}} e_T(V_{\chi, \mu^+, \mu^-}^\bullet)$$

by relative virtual localization, and derive the following identity:

**Proposition 7.1.**

$$(6) \quad K_{\mu^+, \mu^-}^\bullet(\lambda) = \sum_{|\nu^\pm| = |\mu^\pm|} \Phi_{\mu^+, \nu^+}^\bullet(-\sqrt{-1}\tau\lambda) z_{\nu^+} G_{\nu^+, \nu^-}^\bullet(\lambda; \tau) z_{\nu^-} \Phi_{\nu^-, \mu^-}^\bullet\left(\frac{-\sqrt{-1}}{\tau}\lambda\right).$$

7.1. **Torus fixed points.** Given a morphism

$$(C, \{x_i\}_{i=1}^{l(\mu^+)}, \{y_j\}_{j=1}^{l(\mu^-)}) \rightarrow X[m^+, m^-]$$

which represents a point in  $\overline{\mathcal{M}}_\chi^\bullet(X, \mu^+, \mu^-)^T$ , let

$$\tilde{f} = \pi[m^+, m^-] \circ f : C \rightarrow X.$$

Then

$$\tilde{f}(C) \subset D_1 \cup D_2 \cup D_3 \cup D_4 \cup D_5,$$

$$\tilde{f}(x_i) \in \{z_+, \tilde{z}_+\}, \quad \tilde{f}(y_j) \in \{z_-, \tilde{z}_-\}.$$

See Figure 1 in Section 6.3 for the configuration of the  $T$ -invariant divisors  $D_1, \dots, D_5$  and the  $T$  fixed points  $z_0, z_+, z_-, \tilde{z}_+, \tilde{z}_-$ .

If

$$\tilde{f}_*(C) = n_1D_1 + n_2D_2 + n_3D_3 + n_4D_4 + n_5D_5$$

as divisors, then

$$\tilde{f}_*[C] = (n_1 + n_2)\beta_1 + (n_3 + n_4)\beta_3 + (n_5 - n_2 - n_4)\beta_5$$

as homology classes.

Let

$$J = \{(n_1, n_2, n_3, n_4, n_5) \in \mathbb{Z}^5 \mid n_i \geq 0, n_1 + n_2 = |\mu^-|, n_3 + n_4 = |\mu^+|, n_5 = n_2 + n_4\}.$$

Given  $\hat{n} = (n_1, n_2, n_3, n_4, n_5) \in J$ , let

$$\mathcal{M}_{\hat{n}} \subset \overline{\mathcal{M}}_{\chi}^{\bullet}(X, \mu^+, \mu^-)^T$$

be the subset which corresponds to

$$\tilde{f}_*(C) = n_1D_1 + n_2D_2 + n_3D_3 + n_4D_4 + n_5D_5.$$

Then  $\overline{\mathcal{M}}_{\chi}^{\bullet}(X, \mu^+, \mu^-)^T$  is a disjoint union of

$$\{\mathcal{M}_{\hat{n}} : \hat{n} \in J\}.$$

We have the following vanishing lemma:

**Lemma 7.2.** *Let  $\hat{n} \in J$ , and let  $i_{\hat{n}} : \mathcal{M}_{\hat{n}} \rightarrow \overline{\mathcal{M}}_{\chi}^{\bullet}(X, \mu^+, \mu^-)^T$  be the inclusion. Then*

$$i_{\hat{n}}^*e_T(V_{\chi, \mu^+, \mu^-}^{\bullet}) = 0$$

unless  $\hat{n} = (|\mu^-|, 0, |\mu^+|, 0, 0)$ .

*Proof.* We use the notation in Section 6.2. Let  $L = \mathcal{O}_X(-D_1 - D_3)$ . We have the following short exact sequence of sheaves on  $\overline{\mathcal{M}}_{\chi}^{\bullet}(X, \mu^+, \mu^-)$ :

$$(37) \quad 0 \rightarrow \tilde{F}^*L(-\mathcal{D}_{\chi, \mu^+, \mu^-}^{\bullet}) \rightarrow \tilde{F}^*L \rightarrow (\tilde{F}^*L)_{\mathcal{D}_{\chi, \mu^+, \mu^-}^{\bullet}} \rightarrow 0.$$

Let

$$s_i : \overline{\mathcal{M}}_{\chi}^{\bullet}(X, \mu^+, \mu^-) \rightarrow \mathcal{U}_{\chi, \mu^+, \mu^-}^{\bullet}$$

be the section corresponding to the  $i$ -th marked point, and let

$$\text{ev}_i = \tilde{F} \circ s_i : \overline{\mathcal{M}}_{\chi}^{\bullet}(X, \mu^+, \mu^-) \rightarrow X$$

be evaluation at the  $i$ -th marked point. Then (37) gives the following long exact sequence:

$$\begin{aligned} 0 &\rightarrow R^0\pi_*\tilde{F}^*L(-\mathcal{D}_{\chi, \mu^+, \mu^-}^{\bullet}) \rightarrow R^0\tilde{F}^*L \rightarrow \bigoplus_{i=1}^{l(\mu^+)+l(\mu^-)} \text{ev}_i^*L \\ &\rightarrow R^1\pi_*\tilde{F}^*L(-\mathcal{D}_{\chi, \mu^+, \mu^-}^{\bullet}) \rightarrow R^1\tilde{F}^*L \rightarrow 0. \end{aligned}$$

We have  $R^0\tilde{F}^*L = 0$ , so

$$\tilde{V}_{\chi, \mu^+, \mu^-}^{\bullet} = R^1\tilde{F}^*L$$

is a vector bundle over  $\overline{\mathcal{M}}_{\chi}^{\bullet}(X, \mu^+, \mu^-)$ . We have the following short exact sequence of vector bundles over  $\overline{\mathcal{M}}_{\chi}^{\bullet}(X, \mu^+, \mu^-)$ :

$$(38) \quad 0 \rightarrow \bigoplus_{i=1}^{l(\mu^+)+l(\mu^-)} \text{ev}_i^*L \rightarrow V_{\chi, \mu^+, \mu^-}^{\bullet} \rightarrow \tilde{V}_{\chi, \mu^+, \mu^-}^{\bullet} \rightarrow 0.$$

The restriction of the above exact sequence to  $\mathcal{M}_{\hat{n}}$  is

$$0 \rightarrow L_{z_-}^{\oplus l(\sigma^1)} \oplus L_{\tilde{z}_-}^{\oplus l(\sigma^2)} \oplus L_{z_+}^{\oplus l(\sigma^3)} \oplus L_{\tilde{z}_-}^{\oplus l(\sigma^4)} \rightarrow i_{\hat{n}}^* V_{\chi, \mu^+, \mu^-}^\bullet \rightarrow i_{\hat{n}}^* \tilde{V}_{\chi, \mu^+, \mu^-}^\bullet \rightarrow 0,$$

where  $\sigma^1, \dots, \sigma^4$  are partitions determined by

$$(39) \quad \{\mu_j^- : \tilde{f}(y_j) \in z_-\}, \{\mu_j^- : \tilde{f}(y_j) \in \tilde{z}_-\}, \{\mu_i^+ : \tilde{f}(x_i) \in z_+\}, \{\mu_i^+ : \tilde{f}(x_i) \in \tilde{z}_+\},$$

respectively. Note that  $\sigma^1, \sigma^2, \sigma^3, \sigma^4$  are constant on each connected component of  $\mathcal{M}_{\hat{n}}$ , and

$$\sigma^1 \cup \sigma^2 = \mu^-, \quad \sigma^3 \cup \sigma^4 = \mu^+.$$

We have seen in Section 6.3 that

$$e_T(L_{z_+}) = -\beta, \quad e_T(L_{z_-}) = -\alpha, \quad e_T(L_{\tilde{z}_+}) = e_T(L_{\tilde{z}_-}) = 0,$$

so

$$i^* e_T(V_{\chi, \mu^+, \mu^-}^\bullet) = 0$$

unless

$$(40) \quad (\sigma^1, \sigma^2, \sigma^3, \sigma^4) = (\mu^-, \emptyset, \mu^+, \emptyset).$$

Let  $\hat{n} = (n_1, n_2, n_3, n_4, n_5) \in J$ ,  $n_5 \neq 0$ . Let  $\mathcal{M}_{\hat{n}}(k)$  be the subset of  $\mathcal{M}_{\hat{n}}$  which consists of the points

$$[f : C \rightarrow X[m^+, m^-]] \in \mathcal{M}_{\hat{n}}$$

such that (40) is true, and

$$f^{-1}(D_5 - \{\tilde{z}^+, \tilde{z}^-\})$$

has  $k$  connected components, where  $1 \leq k \leq n_5$ . Each  $\mathcal{M}_{\hat{n}}(k)$  is a union of connected components of  $\mathcal{M}_{\hat{n}}$ .

We claim that

$$e_T(V_{\chi, \mu^+, \mu^-}^\bullet)|_{\mathcal{M}_{\hat{n}}(k)} = 0$$

for all  $\hat{n} = (n_1, n_2, n_3, n_4, n_5) \in J$ ,  $n_5 \neq 0$ ,  $k = 1, \dots, n_5$ . This will complete the proof.

Let

$$[f : C \rightarrow X[m^+, m^-]] \in \mathcal{M}_{\hat{n}}(k).$$

Then  $C = C_1 \cup C_2$ , where  $C_1$  is the closure of  $f^{-1}(D_5 - \{\tilde{z}^+, \tilde{z}^-\})$ , which is a disjoint union of  $k$  projective lines, and  $C_2$  is the union of other irreducible components of  $C$ . By (40), the ramification divisor  $R \subset C_2$ , and  $C_1$  and  $C_2$  intersect at  $2k$  nodes. We have

$$\begin{aligned} 0 &\rightarrow H^0(C, \tilde{f}^* L(-R)) \rightarrow H^0(C_1, \tilde{f}^* L|_{C_1}) \oplus H^0(C_2, \tilde{f}^* L(-R)|_{C_2}) \rightarrow L_{\tilde{z}^+}^{\oplus k} \oplus L_{\tilde{z}^-}^{\oplus k} \\ &\rightarrow H^1(C, \tilde{f}^* L(-R)) \rightarrow H^1(C_1, \tilde{f}^* L|_{C_1}) \oplus H^1(C_2, \tilde{f}^* L(-R)|_{C_2}) \rightarrow 0, \end{aligned}$$

where

$$H^0(C, \tilde{f}^* L(-R)) = 0 = H^0(C_2, \tilde{f}^* L(-R)|_{C_2}).$$

The restriction of  $L$  to  $D_5$  is (equivariantly) trivial, so

$$H^0(C_1, \tilde{f}^* L|_{C_1}) \cong L_{\tilde{z}^-}^{\oplus k}, \quad H^1(C_1, \tilde{f}^* L|_{C_1}) = 0.$$

We have

$$0 \rightarrow L_{\tilde{z}^+}^{\oplus k} \rightarrow H^1(C, \tilde{f}^* L) \rightarrow H^1(C_2, \tilde{f}^* L) \rightarrow 0,$$

so

$$V_{\chi, \mu^+, \mu^-}^\bullet|_{\mathcal{M}_{\hat{n}}(k)} = L_{\tilde{z}^+}^k \oplus V'$$

and

$$e_T(V_{\chi, \mu^+, \mu^-}^\bullet)|_{\mathcal{M}_{\hat{n}}(k)} = 0.$$

□

Lemma 7.2 tells us that  $\mathcal{M}_{\hat{n}}$  does not contribute to the localization calculation of

$$K_{\chi, \mu^+, \mu^-}^\bullet = \frac{1}{|\text{Aut}(\mu^+)| |\text{Aut}(\mu^-)|} \int_{[\overline{\mathcal{M}}_\chi(X, \mu^+, \mu^-)]^{\text{vir}}} e_T(V_{\chi, \mu^+, \mu^-}^\bullet)$$

if  $\hat{n} \neq (|\mu^-|, 0, |\mu^+|, 0, 0)$ .

**7.2. Admissible labels.** From now on, we only consider

$$\hat{\mathcal{M}} = \mathcal{M}_{(|\mu^-|, 0, |\mu^+|, 0, 0)} \subset \overline{\mathcal{M}}_\chi(X, \mu^+, \mu^-)^T.$$

Given a morphism

$$(C, \{x_i^1\}_{i=1}^{l(\mu^+)}, \dots, \{x_i^k\}_{i=1}^{l(\mu^k)}) \rightarrow X[m^+, m^-]$$

which represents a point in  $\hat{\mathcal{M}}$ , let

$$C^0 = \tilde{f}^{-1}(z_0), \quad C^\pm = \tilde{f}^{-1}(z_\pm),$$

where  $z_0, z_+, z_-$  are defined as in Section 6.3. Then

$$C = C^+ \cup L^+ \cup C^0 \cup L^- \cup C^-,$$

where  $L^+, L^-$  are unions of projective lines,  $f|_{L^+} : L^+ \rightarrow D_3$  is a degree  $d^+ = |\mu^+|$  cover fully ramified over  $z_0$  and  $z_+$ , and  $f|_{L^-} : L^- \rightarrow D_1$  is a degree  $d^- = |\mu^-|$  cover fully ramified over  $z_0$  and  $z_-$ .

Define

$$\mathbb{P}^\pm(m^\pm) = \pi[m^+, m^-]^{-1}(z_\pm).$$

Let

$$f^\pm = f|_{C^\pm} : C^\pm \rightarrow \mathbb{P}^\pm(m^\pm), \quad \tilde{f}^+ = f|_{L^+} : L^+ \rightarrow D_3, \quad \tilde{f}^- = f|_{L^-} : L^- \rightarrow D_1.$$

The degrees of  $\tilde{f}^\pm$  restricted to irreducible components of  $L^\pm$  determine a partition  $\nu^\pm$  of  $d^\pm$ .

Let  $C_1^0, \dots, C_k^0$  be the connected components of  $C^0$ , and let  $g_i$  be the arithmetic genus of  $C_i^0$ . (We define  $g_i = 0$  if  $C_i^0$  is a point.) Define

$$\chi^0 = \sum_{i=1}^k (2 - 2g_i).$$

We define  $\chi^+, \chi^-$  similarly. Then

$$-\chi^+ + 2l(\nu^+) - \chi^0 + 2l(\nu^-) - \chi^- = -\chi.$$

Note that  $\chi^\pm \leq 2 \min\{l(\mu^\pm), l(\nu^\pm)\}$ . So

$$-\chi^+ + l(\nu^+) + l(\mu^+) \geq 0,$$

and the equality holds if and only if  $m^+ = 0$ . In this case, we have  $\nu^+ = \mu^+$ ,  $\chi^+ = 2l(\mu^+)$ . Similarly,

$$-\chi^- + l(\nu^-) + l(\mu^-) \geq 0,$$

and the equality holds if and only if  $m^- = 0$ . In this case, we have  $\nu^- = \mu^-$ ,  $\chi^- = 2l(\mu^-)$ . There are four cases:

Case 1:  $m^+ = m^- = 0$ . Then  $f^+, f^-$  are constant maps, and  $\nu^\pm = \mu^\pm$ .

Case 2:  $m^+ > 0, m^- = 0$ . Then  $f^-$  is a constant map,  $\nu^- = \mu^-$ , and  $f^+$  represents a point in

$$\overline{\mathcal{M}}_{\chi^+}^\bullet(\mathbb{P}^1, \mu^+, \nu^+) // \mathbb{C}^*$$

up to an element in  $\text{Aut}(\nu^+)$ .

Case 3:  $m^+ = 0, m^- > 0$ . Then  $f^+$  is a constant map,  $\nu^+ = \mu^+$ , and  $f^-$  represents a point in

$$\overline{\mathcal{M}}_{\chi^-}^\bullet(\mathbb{P}^1, \nu^-, \mu^-) // \mathbb{C}^*$$

up to an element in  $\text{Aut}(\nu^-)$ .

Case 4:  $m^+, m^- > 0$ . Then  $f^+$  represents a point in

$$\overline{\mathcal{M}}_{\chi^+}^\bullet(\mathbb{P}^1, \mu^+, \nu^+) // \mathbb{C}^*$$

up to an element of  $\text{Aut}(\nu^+)$ , and  $f^-$  represents a point in

$$\overline{\mathcal{M}}_{\chi^-}^\bullet(\mathbb{P}^1, \nu^-, \mu^-) // \mathbb{C}^*$$

up to an element in  $\text{Aut}(\nu^-)$ .

**Definition 7.3.** An *admissible label* is a 5-tuple  $(\chi^+, \nu^+, \chi^0, \nu^-, \chi^-)$  such that

- $\chi^+, \chi^0, \chi^- \in 2\mathbb{Z}$ ;
- $\nu^\pm$  is a partition of  $d^\pm$ ;
- $\chi^0 \leq 2 \min\{l(\nu^+), l(\nu^-)\}$ ,  $\chi^\pm \leq 2 \min\{l(\mu^\pm), l(\nu^\pm)\}$ ;
- $-\chi^+ + 2l(\nu^+) - \chi^0 + 2l(\nu^-) - \chi^- = -\chi$ .

Let  $G_\chi^\bullet(X, \mu^+, \mu^-)$  denote the set of all admissible labels.

For a nonnegative integer  $g$  and a positive integer  $h$ , let  $\overline{\mathcal{M}}_{g,h}$  be the moduli space of stable curves of genus  $g$  with  $h$  marked points.  $\overline{\mathcal{M}}_{g,h}$  is empty for  $(g, h) = (0, 1), (0, 2)$ , but we will assume that  $\overline{\mathcal{M}}_{0,1}$  and  $\overline{\mathcal{M}}_{0,2}$  exist and satisfy

$$\int_{\overline{\mathcal{M}}_{0,1}} \frac{1}{1-d\psi} = \frac{1}{d^2},$$

$$\int_{\overline{\mathcal{M}}_{0,2}} \frac{1}{(1-\mu_1\psi_1)(1-\mu_2\psi_2)} = \frac{1}{\mu_1 + \mu_2}$$

for simplicity of notation. Such an assumption will give the correct final results.

For a nonnegative integer  $g$  and a positive integer  $h$ , let  $\overline{\mathcal{M}}_{\chi,h}^\bullet$  be the moduli of possibly disconnected stable curves  $C$  with  $h$  marked points such that

- If  $C_1, \dots, C_k$  are connected components of  $C$ , and  $g_i$  is the arithmetic genus of  $C_i$ , then

$$\sum_{i=1}^k (2 - 2g_i) = \chi.$$

- Each connected component contains at least one marked point.

The connected components of  $\overline{\mathcal{M}}_{\chi,h}^\bullet$  are of the form

$$\overline{\mathcal{M}}_{g_1, h_1} \times \cdots \times \overline{\mathcal{M}}_{g_k, h_k},$$

where

$$\sum_{i=1}^k (2 - 2g_i) = \chi, \quad \sum_{i=1}^k h_i = h.$$

The restriction of the Hodge bundle  $\mathbb{E} \rightarrow \overline{\mathcal{M}}_{\chi, h}^\bullet$  to the above connected component is the direct sum of the Hodge bundles on each factor, and

$$\Lambda^\vee(u) = \prod_{i=1}^k \Lambda_{g_i}^\vee(u).$$

We define

$$\overline{\mathcal{M}}_{2l(\mu^+), \mu^+, \chi, \mu^-, 2l(\mu^-)} = \overline{\mathcal{M}}_{\chi, l(\mu^+) + l(\mu^-)}^\bullet.$$

For  $-\chi^\pm + l(\mu^\pm) + l(\nu^\pm) > 0$ , we define

$$\begin{aligned} \overline{\mathcal{M}}_{\chi^+, \nu^+, \chi^0, \mu^-, 2l(\mu^-)} &= \left( \overline{\mathcal{M}}_{\chi^+}^\bullet(\mathbb{P}^1, \mu^+, \nu^+) // \mathbb{C}^* \right) \times \overline{\mathcal{M}}_{\chi^0, l(\nu^+) + l(\mu^-)}, \\ \overline{\mathcal{M}}_{2l(\mu^+), \mu^+, \chi^0, \nu^-, \chi^-} &= \overline{\mathcal{M}}_{\chi^0, l(\mu^+) + l(\nu^-)} \times \left( \overline{\mathcal{M}}_{\chi^-}^\bullet(\mathbb{P}^1, \nu^-, \mu^-) // \mathbb{C}^* \right), \\ \overline{\mathcal{M}}_{\chi^+, \nu^+, \chi^0, \nu^-, \chi^-} &= \left( \overline{\mathcal{M}}_{\chi^+}^\bullet(\mathbb{P}^1, \mu^+, \nu^+) // \mathbb{C}^* \right) \times \overline{\mathcal{M}}_{\chi^0, l(\nu^+) + l(\nu^-)} \\ &\quad \times \left( \overline{\mathcal{M}}_{\chi^-}^\bullet(\mathbb{P}^1, \nu^-, \mu^-) // \mathbb{C}^* \right). \end{aligned}$$

For every  $(\chi^+, \nu^+, \chi^0, \nu^-, \chi^-) \in G_\chi^\bullet(X, \mu^+, \mu^-)$ , there is a morphism

$$i_{\chi^+, \nu^+, \chi^0, \nu^-, \chi^-} : \overline{\mathcal{M}}_{\chi^+, \nu^+, \chi^0, \nu^-, \chi^-} \rightarrow \overline{\mathcal{M}}_\chi^\bullet(X, \mu^+, \mu^-),$$

whose image  $\mathcal{F}_{\chi^+, \nu^+, \chi^0, \nu^-, \chi^-}$  is a union of connected components of  $\widehat{\mathcal{M}}$ . The morphism  $i_{\chi^+, \nu^+, \chi^0, \nu^-, \chi^-}$  induces an isomorphism

$$\overline{\mathcal{M}}_{\chi^+, \nu^+, \chi^0, \nu^-, \chi^-} / A_{\chi^+, \nu^+, \chi^0, \nu^-, \chi^-} \cong \mathcal{F}_{\chi^+, \nu^+, \chi^0, \nu^-, \chi^-},$$

where  $A_{2l(\mu^+), \mu^+, \chi, \mu^-, l(\mu^-)}$  is trivial, and for  $\chi^\pm + l(\mu^\pm) + l(\nu^\pm) > 0$ , we have

$$\begin{aligned} 1 &\rightarrow \prod_{i=1}^{l(\nu^+)} \mathbb{Z}_{\nu_i^+} \rightarrow A_{\chi^+, \nu^+, \chi^0, \mu^-, 2l(\mu^-)} \rightarrow \text{Aut}(\nu^+) \rightarrow 1, \\ 1 &\rightarrow \prod_{j=1}^{l(\nu^-)} \mathbb{Z}_{\nu_j^-} \rightarrow A_{2l(\mu^+), \mu^+, \chi^0, \nu^-, \chi^-} \rightarrow \text{Aut}(\nu^-) \rightarrow 1, \\ 1 &\rightarrow \prod_{i=1}^{l(\nu^+)} \mathbb{Z}_{\nu_i^+} \times \prod_{j=1}^{l(\nu^-)} \mathbb{Z}_{\nu_j^-} \rightarrow A_{\chi^+, \nu^+, \chi^0, \nu^-, \chi^-} \rightarrow \text{Aut}(\nu^+) \times \text{Aut}(\nu^-) \rightarrow 1. \end{aligned}$$

So for  $-\chi^\pm + l(\mu^\pm) + l(\nu^\pm) > 0$ , we have

$$\begin{aligned} |A_{\chi^+, \nu^+, \chi^0, \mu^-, 2l(\mu^-)}| &= z_{\nu^+}, & |A_{2l(\mu^+), \mu^+, \chi^0, \nu^-, \chi^-}| &= z_{\nu^-}, \\ |A_{\chi^+, \nu^+, \chi^0, \nu^-, \chi^-}| &= z_{\nu^+} z_{\nu^-}. \end{aligned}$$

The stack  $\widehat{\mathcal{M}}$  is a disjoint union of

$$\{\mathcal{F}_{\chi^+, \nu^+, \chi^0, \nu^-, \chi^-} : (\chi^+, \nu^+, \chi^0, \nu^-, \chi^-) \in G_\chi^\bullet(X, \mu^+, \mu^-)\}.$$



**7.3. Contribution from each admissible label.** Let  $N_{\chi^+, \nu^+, \chi^0, \nu^-, \chi^-}^{\text{vir}}$  on  $\overline{\mathcal{M}}_{\chi^+, \nu^+, \chi^0, \nu^-, \chi^-}$  be the pull-back of the virtual normal bundle of  $\mathcal{F}_{\chi^+, \nu^+, \chi^0, \nu^-, \chi^-}$  in  $\overline{\mathcal{M}}_{\chi}(X, \mu^+, \mu^-)$ . Calculations similar to those in [23, Appendix A] show that

$$\frac{i_{\chi^+, \nu^+, \chi^0, \nu^-, \chi^-}^* e_T(V_{\chi, \mu^+, \mu^-}^\bullet)}{e_T(N_{\chi^+, \nu^+, \chi^0, \nu^-, \chi^-}^{\text{vir}})} = A^+ A^0 A^-,$$

where

$$\begin{aligned} A^0 &= (-1)^{|\nu^+|+|\nu^-|+1} a_{\nu^+} a_{\nu^-} \prod_{i=1}^{l(\nu^+)} \frac{\prod_{a=1}^{\nu_i^+-1} (\nu_i^+ \beta + a\alpha)}{(\nu_i^+ - 1)! \alpha^{\nu_i^+-1}} \prod_{j=1}^{l(\nu^-)} \frac{\prod_{a=1}^{\nu_j^- - 1} (\nu_j^- \alpha + a\beta)}{(\nu_j^- - 1)! \beta^{\nu_j^- - 1}} \\ &\quad \cdot \frac{\Lambda^\vee(\alpha) \Lambda^\vee(\beta) \Lambda^\vee(-\alpha - \beta) (\alpha\beta(\alpha + \beta))^{l(\nu^+)+l(\nu^-)-1}}{\prod_{i=1}^{l(\nu^+)} \alpha(\alpha - \nu_i^+ \psi_i) \prod_{j=1}^{l(\nu^-)} \beta(\beta - \nu_j^- \psi_l(\nu^+)+j)}, \\ A^+ &= \begin{cases} (-1)^{l(\mu^+)}, & \chi^- = 2l(\mu^-), \\ (-1)^{-\frac{\chi^+}{2} + l(\nu^+) + l(\mu^+)} a_{\nu^+} \frac{\beta^{-\chi^+ + l(\mu^+) + l(\nu^+)}}{-\alpha - \psi^+}, & \text{otherwise,} \end{cases} \\ A^- &= \begin{cases} (-1)^{l(\mu^-)}, & \chi^+ = 2l(\mu^+), \\ (-1)^{-\frac{\chi^-}{2} + l(\nu^-) + l(\mu^-)} a_{\nu^-} \frac{\alpha^{-\chi^- + l(\mu^-) + l(\nu^-)}}{-\beta - \psi^-}, & \text{otherwise.} \end{cases} \end{aligned}$$

From the definitions in Section 2.2 and Proposition 5.4, we have

$$\begin{aligned} I_{\chi^+, \nu^+, \chi^0, \nu^-, \chi^-}(\alpha, \beta) &= \frac{1}{|A_{\chi^+, \nu^+, \chi^0, \nu^-, \chi^-}|} \int_{[\overline{\mathcal{M}}_{\chi^+, \nu^+, \chi^0, \nu^-, \chi^-}]^{\text{vir}}} \frac{i_{\chi^+, \nu^+, \chi^0, \nu^-, \chi^-}^* e_T(V_{\chi, \mu^+, \mu^-}^\bullet)}{e_T(N_{\chi^+, \nu^+, \chi^0, \nu^-, \chi^-}^{\text{vir}})} \\ &= \alpha_{\mu^+, \mu^-} \frac{\sqrt{-1}^{l(\mu^+) + l(\mu^-)}}{(-1)^{|\mu^+| + |\mu^-|}} G_{\chi^0, \nu^+, \nu^-}^\bullet(\alpha, \beta) \\ &\quad \cdot z_{\nu^+} \frac{(-\sqrt{-1}\beta/\alpha)^{-\chi^+ + l(\nu^+) + l(\mu^+)}}{(-\chi^+ + l(\nu^+) + l(\mu^+))!} H_{\chi^+, \nu^+, \mu^+}^\bullet \\ &\quad \cdot z_{\nu^-} \frac{(-\sqrt{-1}\alpha/\beta)^{-\chi^- + l(\nu^-) + l(\mu^-)}}{(-\chi^- + l(\nu^-) + l(\mu^-))!} H_{\chi^-, \nu^-, \mu^-}^\bullet, \end{aligned}$$

where  $\alpha_{\mu^+, \mu^-} = |\text{Aut}(\mu^+)| |\text{Aut}(\mu^-)|$ . Let  $\tau = \beta/\alpha$ . Then

$$\begin{aligned} I_{\chi^+, \nu^+, \chi^0, \nu^-, \chi^-}(\alpha, \beta) &= \alpha_{\mu^+, \mu^-} \frac{\sqrt{-1}^{l(\mu^+) + l(\mu^-)}}{(-1)^{|\mu^+| + |\mu^-|}} G_{\chi^0, \nu^+, \nu^-}^\bullet(\tau) \\ &\quad \cdot z_{\nu^+} \frac{(-\sqrt{-1}\tau)^{-\chi^+ + l(\nu^+) + l(\mu^+)}}{(-\chi^+ + l(\nu^+) + l(\mu^+))!} H_{\chi^+, \nu^+, \mu^+}^\bullet \\ &\quad \cdot z_{\nu^-} \frac{(-\sqrt{-1}/\tau)^{-\chi^- + l(\nu^-) + l(\mu^-)}}{(-\chi^- + l(\nu^-) + l(\mu^-))!} H_{\chi^-, \nu^-, \mu^-}^\bullet. \end{aligned}$$

7.4. **Sum over admissible labels.** Let  $\alpha_{\mu^+, \mu^-} = |\text{Aut}(\mu^+)| |\text{Aut}(\mu^-)|$  as above.

$$\begin{aligned}
K_{\chi, \mu^+, \mu^-}^\bullet &= \frac{1}{\alpha_{\mu^+, \mu^-}} \int_{[\overline{\mathcal{M}}_\chi^\bullet(X, \mu^+, \mu^-)]^{\text{vir}}} e_T(V_{\chi, \mu^+, \mu^-}^\bullet) \\
&= \frac{1}{\alpha_{\mu^+, \mu^-}} \sum_{(\chi^+, \nu^+, \chi^0, \nu^-, \chi^-) \in G_\chi^\bullet(X, \mu^+, \mu^-)} \frac{1}{|A_{\chi^+, \nu^+, \chi^0, \nu^-, \chi^-}|} \\
&\quad \cdot \int_{[\overline{\mathcal{M}}]_{\chi^+, \nu^+, \chi^0, \nu^-, \chi^-}^{\text{vir}}} \frac{i_{\chi^+, \nu^+, \chi^0, \nu^-, \chi^-}^* e_T(V_{\chi, \mu^+, \mu^-}^\bullet)}{e_T(N_{\chi^+, \nu^+, \chi^0, \nu^-, \chi^-}^{\text{vir}})} \\
&= \frac{1}{\alpha_{\mu^+, \mu^-}} \sum_{(\chi^+, \nu^+, \chi^0, \nu^-, \chi^-) \in G_\chi^\bullet(X, \mu^+, \mu^-)} I_{\chi^+, \nu^+, \chi^0, \nu^-, \chi^-}(\alpha, \beta) \\
&= \frac{\sqrt{-1}^{l(\mu^+) + l(\mu^-)}}{(-1)^{|\mu^+| + |\mu^-|}} \left( \sum_{(\chi^+, \nu^+, \chi^0, \nu^-, \chi^-) \in G_\chi^\bullet(X, \mu^+, \mu^-)} G_{\chi^0, \nu^+, \nu^-}^\bullet(\tau) \right. \\
&\quad \cdot z_{\nu^+} \frac{(-\sqrt{-1}\tau)^{-\chi^+ + l(\nu^+) + l(\mu^+)}}{(-\chi^+ + l(\nu^+) + l(\mu^+))!} H_{\chi^+, \nu^+, \mu^+}^\bullet \\
&\quad \left. \cdot z_{\nu^-} \frac{(-\sqrt{-1}/\tau)^{-\chi^- + l(\nu^-) + l(\mu^-)}}{(-\chi^- + l(\nu^-) + l(\mu^-))!} H_{\chi^-, \nu^-, \mu^-}^\bullet \right).
\end{aligned}$$

Recall that

$$K_{\mu^+, \mu^-}^\bullet(\lambda) = \sum_{\chi \in 2\mathbb{Z}, \chi \leq 2(l(\mu^+) + l(\mu^-))} \lambda^{-\chi + l(\mu^+) + l(\mu^-)} \frac{(-1)^{|\mu^+| + |\mu^-|}}{\sqrt{-1}^{l(\mu^+) + l(\mu^-)}} K_{\chi, \mu^+, \mu^-}^\bullet.$$

We have

$$K_{\mu^+, \mu^-}^\bullet(\lambda) = \sum_{|\nu^\pm| = |\mu^\pm|} \Phi_{\mu^+, \nu^+}^\bullet(-\sqrt{-1}\tau\lambda) z_{\nu^+} G_{\nu^+, \nu^-}^\bullet(\lambda; \tau) z_{\nu^-} \Phi_{\nu^-, \mu^-}^\bullet\left(\frac{-\sqrt{-1}}{\tau}\lambda\right).$$

This finishes the proof of (6).

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CENTER OF MATHEMATICAL SCIENCES, ZHEJIANG UNIVERSITY, HANGZHOU, ZHEJIANG 310027, PEOPLE'S REPUBLIC OF CHINA AND DEPARTMENT OF MATHEMATICS, HARVARD UNIVERSITY, CAMBRIDGE, MASSACHUSETTS 02138

*Current address:* Department of Mathematics, Columbia University, New York, New York, 10027

*E-mail address:* `ccliu@math.columbia.edu`

CENTER OF MATHEMATICAL SCIENCES, ZHEJIANG UNIVERSITY, HANGZHOU, ZHEJIANG 310027, PEOPLE'S REPUBLIC OF CHINA AND DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA AT LOS ANGELES, LOS ANGELES, CALIFORNIA 90095-1555

*E-mail address:* `liu@cms.zju.edu.cn`, `liu@math.ucla.edu`

DEPARTMENT OF MATHEMATICAL SCIENCES, TSINGHUA UNIVERSITY, BEIJING, 100084, PEOPLE'S REPUBLIC OF CHINA AND CENTER OF MATHEMATICAL SCIENCES, ZHEJIANG UNIVERSITY, HANGZHOU, ZHEJIANG 310027, PEOPLE'S REPUBLIC OF CHINA

*E-mail address:* `jzhou@math.tsinghua.edu.cn`