

MEAN CURVATURE FLOW OF HIGHER CODIMENSION IN HYPERBOLIC SPACES

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ABSTRACT. In this paper we investigate the convergence for the mean curvature flow of closed submanifolds with arbitrary codimension in space forms. Particularly, we prove that the mean curvature flow deforms a closed submanifold satisfying a pinching condition in a hyperbolic space form to a round point in finite time.

1. INTRODUCTION

In this paper, we study the convergence of the mean curvature flow of submanifolds in space forms. Let $F : M^n \rightarrow \mathbb{F}^{n+d}(c)$ be a smooth immersion from an n -dimensional closed Riemannian manifold M^n to an $(n+d)$ -dimensional complete simply connected space form $\mathbb{F}^{n+d}(c)$ with constant sectional curvature c . Consider a one-parameter family of smooth immersions $F : M \times [0, T) \rightarrow \mathbb{F}^{n+d}(c)$ satisfying

$$(1.1) \quad \begin{cases} \frac{\partial}{\partial t} F(x, t) &= H(x, t), \\ F(x, 0) &= F(x), \end{cases}$$

where $H(x, t)$ is the mean curvature vector of $F_t(M)$ and $F_t(x) = F(x, t)$. We call $F : M \times [0, T) \rightarrow \mathbb{F}^{n+d}(c)$ the mean curvature flow with initial value F .

The mean curvature flow was proposed by Mullins [17] to describe the formation of grain boundaries in annealing metals. In [3], Brakke introduced the motion of a submanifold by its mean curvature in arbitrary codimension and constructed a generalized varifold solution for all time. For the classical solution of the mean curvature flow, most works have been done on hypersurfaces. Huisken [11, 12] showed that if the initial hypersurface in a Riemannian manifold is uniformly convex, then the mean curvature flow converges to a round point in finite time. Later, Huisken [13] extend this result to hypersurfaces satisfying a pinching condition in a sphere. Many other beautiful results have been obtained, and there are various approaches to study the mean curvature flow of hypersurfaces (see [6, 7], etc.). For the mean curvature flow of submanifolds in higher codimension, some special cases have been studied, see [19, 20, 21, 22, 23, 24] etc. for example. Recently, Andrews-Baker [1] proved a convergence theorem for the mean curvature flow of closed submanifolds satisfying a pinching condition in the Euclidean space. In [2], Baker proved a convergence result for the mean curvature flow of submanifolds in a sphere. In this paper, we study the mean curvature flow of closed submanifolds in hyperbolic

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spaces and extend the convergence result in [1, 2] to the mean curvature flow of arbitrary codimension in space forms.

Theorem 1.1. *Let $F : M^n \rightarrow \mathbb{F}^{n+d}(c)$ be a smooth closed submanifold in a hyperbolic space with constant curvature $c < 0$. Assume F satisfies*

$$(1.2) \quad |A|^2 \leq \begin{cases} \frac{4}{3n}|H|^2 + \frac{n}{2}c, & n = 2, 3, \\ \frac{1}{n-1}|H|^2 + 2c, & n \geq 4. \end{cases}$$

Then the mean curvature flow with F as initial value converges to a round point in finite time.

As an immediate consequence of Theorem 1.1, we obtain the following differentiable sphere theorem.

Corollary 1.2. *Let $F : M^n \rightarrow \mathbb{F}^{n+d}(c)$ be a smooth closed submanifold in a hyperbolic space with constant curvature $c < 0$. Assume F satisfies*

$$|A|^2 \leq \begin{cases} \frac{4}{3n}|H|^2 + \frac{n}{2}c, & n = 2, 3, \\ \frac{1}{n-1}|H|^2 + 2c, & n \geq 4. \end{cases}$$

Then M is diffeomorphic to the unit n -sphere.

Remark 1.3. This differentiable sphere theorem was also obtained by Gu and Xu [10, 25] provided the submanifold is simply connected. In fact, they proved the sphere theorem for submanifolds in a Riemannian manifold by using a different method. For more sphere theorems of submanifolds, we refer the readers to [1, 2, 9, 10, 16, 18, 25, 28], etc.

Combining Theorem 1.1 and the convergence results in [1, 2], we obtain the following theorem.

Theorem 1.4. *Let $F : M^n \rightarrow \mathbb{F}^{n+d}(c)$ be a smooth closed submanifold in a complete simply connected space form with $|H|^2 + n^2c > 0$. Assume F satisfies*

$$(1.3) \quad |A|^2 \leq \begin{cases} \frac{4}{3n}|H|^2 + \frac{1}{12}[7n - 4 + \operatorname{sgn}(c)(n - 4)]c, & n = 2, 3, \\ \frac{1}{n-1}|H|^2 + 2c, & n \geq 4. \end{cases}$$

Then either $F_t(M)$ converges to a round point in finite time, or $c > 0$ and $F_t(M)$ converges to a total geodesic sphere in $\mathbb{F}^{n+d}(c)$ as $t \rightarrow \infty$.

Remark 1.5. For $c > 0$, $|H|^2 + n^2c > 0$ is automatically satisfied. For $c = 0$, $|H|^2 + n^2c > 0$ is equivalent to that the mean curvature is nowhere vanishing. For $c < 0$, $|H|^2 + n^2c > 0$ is implied by condition (1.3).

Remark 1.6. For $c > 0$, the maximal existence time of the mean curvature flow may be finite or infinite. For $c \leq 0$, the mean curvature flow with a closed initial submanifold always has finite maximal existence time.

2. BASIC EQUATIONS

Let $F : M \times [0, T) \rightarrow \mathbb{F}^{n+d}(c)$ be a smooth mean curvature flow with initial closed immersion $F_0 : M \rightarrow \mathbb{F}^{n+d}(c)$. Denote by $g(t)$ and $d\mu_t$ the induced metric and the volume form on M . Let A and H be the second fundamental form and

the mean curvature vector of M in $\mathbb{F}^{n+d}(c)$, respectively. We shall make use of the following convention on the range of indices.

$$1 \leq i, j, k, \dots \leq n, \quad 1 \leq A, B, C, \dots \leq n+d \quad \text{and} \quad n+1 \leq \alpha, \beta, \gamma, \dots \leq n+d.$$

As in [1, 3], we consider the evolution on the spatial tangent bundle. Choose a local orthonormal frame $\{e_i\}$ for the spatial tangent bundle and a local orthonormal frame $\{\nu_\alpha\}$ for the normal bundle. Let $\{\omega_i\}$ be the dual frame of $\{e_i\}$. Then A and H can be written as

$$A = \sum_{i,j,\alpha} h_{ij\alpha} \omega_i \otimes \omega_j \otimes \nu_\alpha = \sum_{i,j} h_{ij} \omega_i \otimes \omega_j, \quad H = \sum_{\alpha} H_\alpha \nu_\alpha.$$

We have the following evolution equations.

$$(2.1) \quad \frac{\partial}{\partial t} |A|^2 = \Delta |A|^2 - 2|\nabla A|^2 + 2R_1 + 4c|H|^2 - 2nc|A|^2,$$

$$(2.2) \quad \frac{\partial}{\partial t} |H|^2 = \Delta |H|^2 - 2|\nabla H|^2 + 2R_2 + 2nc|H|^2,$$

where

$$(2.3) \quad R_1 = \sum_{\alpha,\beta} \left(\sum_{i,j} h_{ij\alpha} h_{ij\beta} \right)^2 + |R^\perp|^2,$$

$$(2.4) \quad |R^\perp|^2 = \sum_{i,j,\alpha,\beta} \left(\sum_p (h_{ip\alpha} h_{jp\beta} - h_{jp\alpha} h_{ip\beta}) \right)^2,$$

$$(2.5) \quad R_2 = \sum_{i,j} \left(\sum_{\alpha} H_\alpha h_{ij\alpha} \right)^2.$$

The contracted form of Simons' identity for traceless second fundamental form $\mathring{A} := A - \frac{1}{n}g \otimes H$ is

$$(2.6) \quad \frac{1}{2} \Delta |\mathring{A}|^2 = \mathring{h}_{ij} \nabla_i \nabla_j H + |\nabla \mathring{A}|^2 + Z + nc|\mathring{A}|^2.$$

Here

$$(2.7) \quad Z = -R_1 + \sum_{i,j,p,\alpha,\beta} H_\alpha h_{ip\alpha} h_{ij\beta} h_{pj\beta}.$$

We also have the following inequality.

$$(2.8) \quad |\nabla A|^2 \geq \frac{3}{n+2} |\nabla H|^2.$$

3. PRESERVED CURVATURE PINCHING CONDITION

Now we prove that the pinching condition (1.2) is preserved under the mean curvature flow with arbitrary codimension in the hyperbolic space.

Lemma 3.1. *For $c < 0$ and $n \geq 2$, if the initial immersion satisfies (1.2), then this condition is preserved along the mean curvature flow.*

Proof. We consider $Q = |A|^2 - \alpha|H|^2 - \beta c$, where the constants

$$\alpha \leq \begin{cases} \frac{4}{3n}, & n = 2, 3, \\ \frac{1}{n-1}, & n \geq 4, \end{cases} \quad \text{and} \quad \beta \geq \begin{cases} \frac{n}{2}, & n = 2, 3, \\ 2, & n \geq 4. \end{cases}$$

By (2.1) and (2.2) we have

$$(3.1) \quad \begin{aligned} \frac{\partial}{\partial t} Q &= \Delta Q - 2(|\nabla A|^2 - \alpha|\nabla H|^2) \\ &\quad + 2R_1 - 2\alpha R_2 - 2nc|\dot{A}|^2 - 2n\left(\alpha - \frac{1}{n}\right)c|H|^2. \end{aligned}$$

We only have to show that if $Q = 0$ at a point $x \in M$, then

$$2R_1 - 2\alpha R_2 - 2nc|\dot{A}|^2 - 2n\left(\alpha - \frac{1}{n}\right)c|H|^2 \leq 0$$

holds at x . We also have $H \neq 0$ at x . Choose $\{\nu_\alpha\}$ such that $\nu_{n+1} = \frac{H}{|H|}$. Let $A_H = \sum_{i,j} h_{ij,n+1} \omega_i \otimes \omega_j$. Set $\dot{A}_H = A_H - \frac{|H|}{n} \text{Id}$ and $|\dot{A}_I|^2 = |\dot{A}|^2 - |\dot{A}_H|^2$.

We replace $|H|^2$ with $\frac{|\dot{A}|^2 - \beta c}{\alpha - \frac{1}{n}}$. Then

$$(3.2) \quad \begin{aligned} &2R_1 - 2\alpha R_2 - 2nc|\dot{A}|^2 - 2n\left(\alpha - \frac{1}{n}\right)c|H|^2 \\ &\leq 2|\dot{A}_H|^2 - 2\left(\alpha - \frac{1}{n}\right)|\dot{A}_H|^2|H|^2 + \frac{2}{n}|\dot{A}_H|^2|H|^2 - \frac{2}{n}\left(\alpha - \frac{1}{n}\right)|H|^4 \\ &\quad + 8|\dot{A}_H|^2|\dot{A}_I|^2 + 3|\dot{A}_I|^2 - 2nc(|\dot{A}_H|^2 + |\dot{A}_I|^2) - 2n\left(\alpha - \frac{1}{n}\right)c|H|^2 \\ &= \left(6 - \frac{2}{n(\alpha - \frac{1}{n})}\right)|\dot{A}_H|^2|\dot{A}_I|^2 + \left(3 - \frac{2}{n(\alpha - \frac{1}{n})}\right)|\dot{A}_I|^4 \\ &\quad + \left(2\beta - 4n + \frac{2\beta}{n(\alpha - \frac{1}{n})}\right)c|\dot{A}_H|^2 + 4\left(\frac{\beta}{n(\alpha - \frac{1}{n})} - n\right)c|\dot{A}_I|^2 \\ &\quad - 2\beta\left(\frac{\beta}{n(\alpha - \frac{1}{n})} - n\right)c^2. \end{aligned}$$

By the definition of α and β , we know that the right hand side of (3.2) is nonpositive for $n \geq 2$. This completes the proof of the lemma. \square

For $\epsilon > 0$, set

$$\alpha_\epsilon = \begin{cases} \frac{4}{3n+n\epsilon}, & n = 2, 3, \\ \frac{1}{n-1+\epsilon}, & n \geq 4, \end{cases} \quad \text{and} \quad \beta_\epsilon = \begin{cases} \frac{n}{2}(1+\epsilon), & n = 2, 3, \\ 2(1+\epsilon), & n \geq 4. \end{cases}$$

If the initial immersion satisfies $|A|^2 < \frac{4}{3n}|H|^2 + \frac{n}{2}c$ for $n = 2, 3$, and $|A|^2 < \frac{1}{n-1}|H|^2 + 2c$ for $n \geq 4$, then there exists an $\epsilon > 0$ such that $|A|^2 \leq \alpha_\epsilon|H|^2 + \beta_\epsilon c$ holds on M_0 . From the proof of Lemma 3.1, this inequality also holds for $t > 0$. On the other hand, if $|A|^2 = \frac{4}{3n}|H|^2 + \frac{n}{2}c$ for $n = 2, 3$, or $|A|^2 = \frac{1}{n-1}|H|^2 + 2c$ for $n \geq 4$ holds somewhere on M_0 , then by the maximum principle, we see that either the equality holds everywhere on M_0 , or the strict inequality holds everywhere for $t > 0$. For the first case, we have $\nabla A = 0$ and $\dot{A}_I = 0$ on M_0 . By [8], M_0 lies in an $(n+1)$ -dimensional total geodesic submanifold of $\mathbb{F}^{n+d}(c)$. Since $\nabla A = 0$, from Theorem 4 of [15], M_0 is either locally isometric to an Euclidean space, or

locally isometric to a product $\mathbb{F}^k(c_1) \times \mathbb{F}^{n-k}(c_2)$ for some $c_1 > 0$, $c_2 < 0$ and $k = 0, \dots, n$. Since M_0 is closed, we see that M_0 is a totally umbilical sphere. Then $|A|^2 \leq \alpha_\epsilon |H|^2 + \beta_\epsilon c$ holds on M_0 for some $\epsilon > 0$. For the second case, we see that after a short time, we also have $|A|^2 \leq \alpha_\epsilon |H|^2 + \beta_\epsilon c$ for some $\epsilon > 0$. Hence, we may assume that $|A|^2 \leq \alpha_\epsilon |H|^2 + \beta_\epsilon c$ for some $\epsilon \in (0, 1)$ and $t \geq t_0 > 0$.

4. PINCHING OF \mathring{A} ALONG THE MEAN CURVATURE FLOW

Assume that $c < 0$. We prove a pinching estimate for the traceless second fundamental form, which guarantees that M_t becomes spherical along the mean curvature flow.

Theorem 4.1. *There are positive constants C_0 and σ_0 independent of t such that*

$$(4.1) \quad |\mathring{A}|^2 \leq C_0 |H|^2 - \sigma_0$$

holds along the mean curvature flow.

Proof. We consider the function $f_\sigma = \frac{|\mathring{A}|^2}{(a|H|^2 + \beta_\epsilon c)^{1-\sigma}}$, where $\sigma \in (0, 1)$ and

$$a = \begin{cases} \frac{1}{3n+n\epsilon}, & n = 2, 3 \\ \frac{1}{n(n-1+\epsilon)}, & n \geq 4. \end{cases}$$

Notice that

$$(4.2) \quad \begin{aligned} & a|H|^2 + \beta_\epsilon c - \left(\left(\alpha_\epsilon - \frac{1}{n} \right) |H|^2 + \beta_\epsilon c \right) \\ & \geq \frac{\epsilon}{3n+n\epsilon} |H|^2 \\ & \geq \frac{\epsilon}{3n+n\epsilon} \cdot \beta(-c) \\ & > 0 \end{aligned}$$

for $n = 2, 3$, and

$$(4.3) \quad \begin{aligned} & a|H|^2 + \beta_\epsilon c - \left(\left(\alpha_\epsilon - \frac{1}{n} \right) |H|^2 + \beta_\epsilon c \right) \\ & \geq \frac{\epsilon}{n(n-1+\epsilon)} |H|^2 \\ & \geq \frac{\epsilon}{n(n-1+\epsilon)} \cdot \beta(-c) \\ & > 0 \end{aligned}$$

for $n \geq 4$. So f_σ is well-defined. From (4.2) and (4.3) we also have

$$(4.4) \quad a|H|^2 + \beta_\epsilon c \geq b|H|^2,$$

where

$$b = \begin{cases} \frac{\epsilon}{3n+n\epsilon}, & n = 2, 3 \\ \frac{\epsilon}{n(n-1+\epsilon)}, & n \geq 4. \end{cases}$$

By a similar computation as in [1], we have

$$\begin{aligned}
(4.5) \quad \frac{\partial}{\partial t} f_\sigma &= \Delta f_\sigma + \frac{2a(1-\sigma)}{a|H|^2 + \beta_\epsilon c} \langle \nabla |H|^2, \nabla f_\sigma \rangle \\
&\quad - \frac{2}{(a|H|^2 + \beta_\epsilon c)^{1-\sigma}} \left(|\nabla A|^2 - \frac{1}{n} |\nabla H|^2 - \frac{a|\mathring{A}|^2}{a|H|^2 + \beta_\epsilon c} |\nabla H|^2 \right) \\
&\quad - \frac{4a^2\sigma(1-\sigma)}{(a|H|^2 + \beta_\epsilon c)^2} f_\sigma |H|^2 \cdot |\nabla |H|^2|^2 - \frac{2a\sigma f_\sigma}{a|H|^2 + \beta_\epsilon c} |\nabla H|^2 \\
&\quad + \frac{2}{(a|H|^2 + \beta_\epsilon c)^{1-\sigma}} \left(R_1 - \frac{1}{n} R_2 - \frac{aR_2|\mathring{A}|^2}{a|H|^2 + \beta_\epsilon c} - nc|\mathring{A}|^2 \right. \\
&\quad \left. - \frac{an(1-\sigma)c|\mathring{A}|^2|H|^2}{a|H|^2 + \beta_\epsilon c} \right) \\
&\quad + \frac{2a\sigma R_2 f_\sigma}{a|H|^2 + \beta_\epsilon c}.
\end{aligned}$$

By (2.8), we have

$$\begin{aligned}
(4.6) \quad &|\nabla A|^2 - \frac{1}{n} |\nabla H|^2 - \frac{a|\mathring{A}|^2}{a|H|^2 + \beta_\epsilon c} |\nabla H|^2 \\
&\geq \left(\frac{3}{n+2} - \frac{1}{n} - \frac{a\left((\alpha_\epsilon - \frac{1}{n})H^2 + \beta_\epsilon c\right)}{a|H|^2 + \beta_\epsilon c} \right) |\nabla H|^2 \\
&\geq \left(\frac{3}{n+2} - \frac{1}{n} - a \right) |\nabla H|^2 \\
&:= \epsilon_\nabla |\nabla H|^2.
\end{aligned}$$

Here ϵ_∇ is a positive constant for $n \geq 2$.

We also have the following estimate.

$$\begin{aligned}
(4.7) \quad &R_1 - \frac{1}{n} R_2 - \frac{aR_2|\mathring{A}|^2}{a|H|^2 + \beta_\epsilon c} \\
&\leq R_1 - \frac{1}{n} R_2 - \frac{R_2|\mathring{A}|^2}{|H|^2} \\
&= R_1 - \frac{R_2|A|^2}{|H|^2} \\
&\leq |\mathring{A}_H|^4 - 2 \left(\frac{|A|^2}{|H|^2} - \frac{2}{n} \right) |\mathring{A}_H|^2 |H|^2 - \frac{2}{n} \left(\frac{|A|^2}{|H|^2} - \frac{1}{n} \right) |H|^4 \\
&\quad - 4|\mathring{A}_H|^2 |\mathring{A}_I|^2 - \frac{3}{2} |\mathring{A}_I|^4 \\
&\leq 0.
\end{aligned}$$

In (4.7) we have used the pinching condition $|A|^2 \leq \alpha_\epsilon |H|^2 + \beta_\epsilon c < \alpha_\epsilon |H|^2$ for $\epsilon \in (0, 1)$.

By (4.4), we have

$$\begin{aligned}
 & -nc|\dot{A}|^2 - \frac{an(1-\sigma)c|\dot{A}|^2|H|^2}{a|H|^2 + \beta_\epsilon c} \\
 (4.8) \quad & \leq -nc|\dot{A}|^2 - \frac{an(1-\sigma)c|\dot{A}|^2(a|H|^2 + \beta_\epsilon c)}{b(a|H|^2 + \beta_\epsilon c)} \\
 & \leq -nc|\dot{A}|^2 - \frac{anc}{b}|\dot{A}|^2 \\
 & := \bar{b}|\dot{A}|^2.
 \end{aligned}$$

For the last term of right hand side of (4.5), we have by (4.4)

$$(4.9) \quad \frac{2a\sigma R_2 f_\sigma}{a|H|^2 + \beta_\epsilon c} \leq \frac{2a\sigma|H|^2|A|^2 f_\sigma}{a|H|^2 + \beta_\epsilon c} \leq \frac{2a\sigma}{b}|A|^2 f_\sigma := \tilde{b}\sigma|A|^2 f_\sigma.$$

Combining (4.5), (4.6), (4.7), (4.8) and (4.9), we have

$$\begin{aligned}
 (4.10) \quad \frac{\partial}{\partial t} f_\sigma & \leq \Delta f_\sigma + \frac{2a(1-\sigma)}{a|H|^2 + \beta_\epsilon c} \langle \nabla|H|^2, \nabla f_\sigma \rangle - \frac{2\epsilon_\nabla}{(a|H|^2 + \beta_\epsilon c)^{1-\sigma}} |\nabla H|^2 \\
 & \quad + 2\bar{b}f_\sigma + \tilde{b}\sigma|A|^2 f_\sigma.
 \end{aligned}$$

To deal with the last term of the right hand side of (4.10), we need the following estimate.

Proposition 4.2. *There exists a positive constant ε independent of t such that*

$$(4.11) \quad Z + nc|\dot{A}|^2 \geq \varepsilon|\dot{A}|^2(a|H|^2 + \beta_\epsilon c)$$

holds for $t \geq t_0$.

Proof. By the argument in the proof of Lemma 5.4 in [2], we only have to show

$$-\frac{\beta_\epsilon}{\alpha_\epsilon - \frac{1}{n}} \left(\frac{1}{n} - \frac{n-2}{2n(n-1)} \right) + n \leq 0.$$

This is true by our choice of α_ϵ and β_ϵ . □

Proposition 4.3. *For any $\eta > 0$, $p \geq 2$ and $t \geq t_0$, we have*

$$\begin{aligned}
 (4.12) \quad \int_{M_t} f_\sigma^p(a|H|^2 + \beta_\epsilon c) d\mu_t & \leq \frac{2p\eta + 5}{b\varepsilon} \int_{M_t} \frac{f_\sigma^{p-1}}{(a|H|^2 + \beta_\epsilon c)^{1-\sigma}} |\nabla H|^2 d\mu_t \\
 & \quad + \frac{2(p-1)}{b\eta\varepsilon} \int_{M_t} f_\sigma^{p-2} |\nabla f_\sigma|^2 d\mu_t.
 \end{aligned}$$

Proof. We have the following estimate.

$$\begin{aligned}
& 2 \int_{M_t} \frac{f_\sigma^{p-1}}{(a|H|^2 + \beta_\epsilon c)^{1-\sigma}} Z d\mu_t + 2nc \int_{M_t} \frac{f_\sigma^{p-1}}{(a|H|^2 + \beta_\epsilon c)^{1-\sigma}} |\mathring{A}|^2 d\mu_t \\
& \leq 2(p-1) \int_{M_t} \frac{f_\sigma^{p-2}}{(a|H|^2 + \beta_\epsilon c)^{1-\sigma}} |\nabla f_\sigma| |\mathring{A}| |\nabla H| d\mu_t \\
& \quad + \frac{2(n-1)}{n} \int_{M_t} \frac{f_\sigma^{p-1}}{(a|H|^2 + \beta_\epsilon c)^{1-\sigma}} |\nabla H|^2 d\mu_t \\
(4.13) \quad & \quad + 4 \int_{M_t} \frac{f_\sigma^{p-1}}{(a|H|^2 + \beta_\epsilon c)^{2-\sigma}} |H| |\mathring{A}| |\nabla H|^2 d\mu_t \\
& \quad + 4(1-\sigma)(p-2) \int_{M_t} \frac{f_\sigma^{p-1}}{(a|H|^2 + \beta_\epsilon c)} |H| |\nabla H| |\nabla f_\sigma| d\mu_t \\
& \quad + 4 \int_{M_t} \frac{f_\sigma^p}{(a|H|^2 + \beta_\epsilon c)^2} |H|^2 |\nabla H|^2 d\mu_t.
\end{aligned}$$

Since $|\mathring{A}|^2 \leq f_\sigma(a|H|^2 + \beta_\epsilon c)^{1-\sigma}$ and $f_\sigma \leq (a|H|^2 + \beta_\epsilon c)^\sigma$, by choosing $\sigma \in (0, 1)$ we have the following estimates.

$$\begin{aligned}
(4.14) \quad & 2(p-1) \int_{M_t} \frac{f_\sigma^{p-2}}{(a|H|^2 + \beta_\epsilon c)^{1-\sigma}} |\nabla f_\sigma| |\mathring{A}| |\nabla H| d\mu_t \\
& \leq \frac{1-\sigma}{\eta} \int_{M_t} f_\sigma^{p-2} |\nabla f_\sigma|^2 d\mu_t + (p-1)\eta \int_{M_t} \frac{f_\sigma^{p-1}}{(a|H|^2 + \beta_\epsilon c)^{1-\sigma}} |\nabla H|^2 d\mu_t,
\end{aligned}$$

$$\begin{aligned}
(4.15) \quad & 4 \int_{M_t} \frac{f_\sigma^{p-1}}{(a|H|^2 + \beta_\epsilon c)^{2-\sigma}} |H| |\mathring{A}| |\nabla H|^2 d\mu_t \\
& \leq \frac{4}{b} \int_{M_t} \frac{f_\sigma^{p-1}}{(a|H|^2 + \beta_\epsilon c)^{1-\sigma}} |\nabla H|^2 d\mu_t,
\end{aligned}$$

$$\begin{aligned}
(4.16) \quad & 4(p-2) \int_{M_t} \frac{f_\sigma^{p-1}}{a|H|^2 + \beta_\epsilon c} |H| |\nabla H| |\nabla f_\sigma| d\mu_t \\
& \leq (p-2) \int_{M_t} \frac{1}{a|H|^2 + \beta_\epsilon c} \left(\frac{2}{\eta} f_\sigma^{p-2} |H|^2 |\nabla f_\sigma|^2 + 2\eta f_\sigma^p |\nabla H|^2 \right) d\mu_t \\
& \leq \frac{2(p-2)}{b\eta} \int_{M_t} f_\sigma^{p-2} |\nabla f_\sigma|^2 d\mu_t \\
& \quad + 2(p-2)\eta \int_{M_t} \frac{f_\sigma^{p-1}}{(a|H|^2 + \beta_\epsilon c)^{1-\sigma}} |\nabla H|^2 d\mu_t,
\end{aligned}$$

$$(4.17) \quad 4 \int_{M_t} \frac{f_\sigma^p}{(a|H|^2 + \beta_\epsilon c)^2} |H|^2 |\nabla H|^2 d\mu_t \leq \frac{4}{b} \int_{M_t} \frac{f_\sigma^{p-1}}{(a|H|^2 + \beta_\epsilon c)^{1-\sigma}} |\nabla H|^2 d\mu_t.$$

In (4.15), (4.16) and (4.17) we have used (4.4).

By (4.11), we have

$$\begin{aligned}
(4.18) \quad & 2 \int_{M_t} \frac{f_\sigma^{p-1}}{(a|H|^2 + \beta_\epsilon c)^{1-\sigma}} (Z + 2nc|\mathring{A}|^2) d\mu_t \\
& \geq 2\varepsilon \int_{M_t} f_\sigma^p (a|H|^2 + \beta_\epsilon c) d\mu_t.
\end{aligned}$$

Combining (4.13)-(4.18), we obtain

$$(4.19) \quad 2\varepsilon \int_{M_t} f_\sigma^p (a|H|^2 + \beta_\varepsilon c) d\mu_t \leq \frac{3p\eta + 10}{b} \int_{M_t} \frac{f_\sigma^{p-1}}{(a|H|^2 + \beta_\varepsilon c)^{1-\sigma}} |\nabla H|^2 d\mu_t \\ + \frac{3(p-1)}{b\eta} \int_{M_t} f_\sigma^{p-2} |\nabla f_\sigma|^2 d\mu_t.$$

Dividing through by 2ε completes the proof. \square

Now we show that the L^p -norm of f_σ is bounded for sufficiently high p .

Lemma 4.4. *For any $p \geq \max\{2, \frac{8}{b\varepsilon\nabla} + 1\}$ and $\sigma \leq \min\left\{\frac{b^2\varepsilon\varepsilon\nabla}{10b\alpha_\varepsilon}, \frac{b^2\varepsilon\sqrt{\varepsilon\nabla}}{4b\alpha_\varepsilon\sqrt{p}}, \frac{1}{2}\right\}$, there exist a constant C independent of t such that for all $t \in [0, T_{\max})$ where $T_{\max} < \infty$, we have*

$$(4.20) \quad \left(\int_{M_t} f_\sigma^p d\mu_t \right)^{\frac{1}{p}} \leq C.$$

Proof. For $t \geq t_0$, from (4.10), we have

$$(4.21) \quad \frac{\partial}{\partial t} \int_{M_t} f_\sigma^p d\mu_t \leq \int_{M_t} p f_\sigma^{p-1} \frac{\partial}{\partial t} f_\sigma d\mu_t \\ \leq -p(p-1) \int_{M_t} f_\sigma^{p-2} |\nabla f_\sigma|^2 d\mu_t \\ + 4(1-\sigma)p \int_{M_t} \frac{f_\sigma^{p-1}}{a|H|^2 + \beta_\varepsilon c} |H| |\nabla|H|| |\nabla f_\sigma| d\mu_t \\ - 2p\varepsilon\nabla \int_{M_t} \frac{f_\sigma^{p-1}}{(a|H|^2 + \beta_\varepsilon c)^{1-\sigma}} |\nabla H|^2 d\mu_t \\ + 2\bar{b}p \int_{M_t} f_\sigma^p d\mu_t + \tilde{b}\sigma p \int_{M_t} |A|^2 f_\sigma^p d\mu_t.$$

As in (4.16), we have

$$(4.22) \quad 4(1-\sigma)p \int_{M_t} \frac{f_\sigma^{p-1}}{a|H|^2 + \beta_\varepsilon c} |H| |\nabla|H|| |\nabla f_\sigma| d\mu_t \\ \leq \frac{2p}{b\mu} \int_{M_t} f_\sigma^{p-2} |\nabla f_\sigma|^2 d\mu_t + 2p\mu \int_{M_t} \frac{f_\sigma^{p-1}}{(a|H|^2 + \beta_\varepsilon c)^{1-\sigma}} |\nabla H|^2 d\mu_t.$$

Substituting (4.22) to (4.21), letting $\mu = \frac{4}{b(p-1)}$ and $p \geq \max\{2, \frac{8}{b\varepsilon\nabla} + 1\}$ we obtain

$$\frac{\partial}{\partial t} \int_{M_t} f_\sigma^p d\mu_t \leq -\frac{p(p-1)}{2} \int_{M_t} f_\sigma^{p-2} |\nabla f_\sigma|^2 d\mu_t \\ - p\varepsilon\nabla \int_{M_t} \frac{f_\sigma^{p-1}}{(a|H|^2 + \beta_\varepsilon c)^{1-\sigma}} |\nabla H|^2 d\mu_t \\ + 2\bar{b}p \int_{M_t} f_\sigma^p d\mu_t + \frac{\tilde{b}\sigma\alpha_\varepsilon p}{b} \int_{M_t} f_\sigma^p (a|H|^2 + \beta_\varepsilon c) d\mu_t.$$

This together with (4.12) implies

$$(4.23) \quad \begin{aligned} \frac{\partial}{\partial t} \int_{M_t} f_\sigma^p d\mu_t &\leq -p(p-1) \left(\frac{1}{2} - \frac{2\tilde{b}\sigma\alpha_\epsilon}{b^2\eta\varepsilon} \right) \int_{M_t} f_\sigma^{p-2} |\nabla f_\sigma|^2 d\mu_t \\ &\quad - \left(p\varepsilon_\nabla - \frac{(2p\eta+5)\tilde{b}\sigma\alpha_\epsilon p}{b^2\varepsilon} \right) \int_{M_t} \frac{f_\sigma^{p-1}}{(a|H|^2 + \beta_\epsilon c)^{1-\sigma}} |\nabla H|^2 d\mu_t \\ &\quad + 2\bar{b}p \int_{M_t} f_\sigma^p d\mu_t. \end{aligned}$$

Now we pick $\eta = \frac{4\tilde{b}\alpha_\epsilon\sigma}{b^2\varepsilon}$ and let $\sigma \leq \min \left\{ \frac{b^2\varepsilon\varepsilon_\nabla}{10b\alpha_\epsilon}, \frac{b^2\varepsilon\sqrt{\varepsilon_\nabla}}{4b\alpha_\epsilon\sqrt{p}}, \frac{1}{2} \right\}$. Then (4.23) reduces to

$$\frac{\partial}{\partial t} \int_{M_t} f_\sigma^p d\mu_t \leq 2\bar{b}p \int_{M_t} f_\sigma^p d\mu_t.$$

This implies

$$(4.24) \quad \int_{M_t} f_\sigma^p d\mu_t \leq e^{2\bar{b}pt} \int_{M_{t_0}} f_\sigma^p d\mu_t.$$

If $t \in [0, t_0]$, by the smoothness of the mean curvature flow we see that $\int_{M_t} f_\sigma^p d\mu_t$ is bounded. For $t \geq t_0$, we only have to show that T_{\max} is finite.

Lemma 4.5. *The maximal existence time T_{\max} of the mean curvature flow is finite.*

Proof. Fixed a point $y \in \mathbb{F}^{n+d}(c)$ and let r be the distance function on $\mathbb{F}^{n+d}(c)$ from y . Denote also by r the composition $r \circ F_t$. We may assume that $r > 0$ on M_t for $t \in [0, T_{\max})$. In fact, if $d = 1$, we may choose y such that it is outside of a geodesic ball in $\mathbb{F}^{n+1}(c)$ that encloses M_0 . By the maximum principle we see that y doesn't lie in any M_t . If $d > 1$, then the Hausdorff dimension of $F(M \times [0, T_{\max}))$ is no more than $n+1$. So we can also pick a point y such that it doesn't lie in any M_t . In both cases, we have $r > 0$ on each M_t .

From [4], we know that

$$(4.25) \quad \Delta r = \langle H, \partial_r \rangle + \text{co}_c(r)(n - |\partial_r^T|^2).$$

Here $\text{co}_c(r) = \frac{\sqrt{-c} \cosh(\sqrt{-c}r)}{\sinh(\sqrt{-c}r)} = \frac{\sqrt{-c}(e^{\sqrt{-c}r} + e^{-\sqrt{-c}r})}{e^{\sqrt{-c}r} - e^{-\sqrt{-c}r}}$, ∂_r is the gradient of r in $\mathbb{F}^{n+d}(c)$, and ∂_r^T is the tangent part of ∂_r to M_t . Clearly we have $\text{co}_c(r) \geq \sqrt{-c}$ and $|\partial_r^T|^2 \leq 1$.

On the other hand, since F_t satisfies (1.1), we have

$$(4.26) \quad \frac{\partial}{\partial t} r = \langle H, \partial_r \rangle.$$

Combining (4.25) and (4.26) we obtain

$$(4.27) \quad \frac{\partial}{\partial t} r = \Delta r - \text{co}_c(r)(n - |\partial_r^T|^2).$$

Suppose that $r(0) < R$. By the maximum principle we see that

$$(4.28) \quad r(t) < R - (n-1)\sqrt{-ct}.$$

Then $T_{\max} < \frac{R}{(n-1)\sqrt{-c}}$, i.e., the maximal existence time of the mean curvature flow is finite. \square

By Proposition 4.5, we finish the proof of Lemma 4.4. \square

Now we can proceed as in [11] or [14] via a Stampacchia iteration procedure to complete the proof of Theorem 4.1. \square

5. A GRADIENT ESTIMATE FOR THE MEAN CURVATURE

We establish a gradient estimate for the mean curvature flow, which will be used to compare the mean curvature at different points of the submanifold. We also assume that $c < 0$.

Theorem 5.1. *For every $\eta > 0$, there exists a constant C_η independent of t such that for all $t \in [0, T_{\max})$, there holds*

$$(5.1) \quad |\nabla H|^2 \leq \eta |H|^4 + C_\eta.$$

Proof. By direct computation, we have

$$(5.2) \quad \frac{\partial}{\partial t} |H|^4 \geq \Delta |H|^4 - 12 |H|^2 |\nabla H|^2 + \frac{4}{n} |H|^6 + 4nc |H|^4,$$

$$(5.3) \quad \frac{\partial}{\partial t} |\nabla H|^2 \leq \Delta |\nabla H|^2 + C_1 |H|^2 |\nabla A|^2 + C_2 |\nabla A|^2,$$

for constants C_1 and C_2 independent of t .

We also have the following estimate for sufficiently large positive constants N_1 and N_2 independent of t .

$$(5.4) \quad \begin{aligned} \frac{\partial}{\partial t} \left((N_1 + N_2 |H|^2) |\dot{A}|^2 \right) &\leq \Delta \left((N_1 + N_2 |H|^2) |\dot{A}|^2 \right) - \frac{4(n-1)}{3n} (N_2 - 1) |H|^2 |\nabla A|^2 \\ &\quad - \frac{4(n-1)}{3n} (N_1 - C(N_2)) |\nabla A|^2 \\ &\quad - C_2(N_1, N_2) |\dot{A}|^2 (|H|^4 + 1) - 2ncN_1 |\dot{A}|^2. \end{aligned}$$

In (5.4), $C(N_2)$ and $C(N_1, N_2)$ are constants depending on N_2 and N_1 , N_2 respectively. Consider the function $f = |\nabla H|^2 + (N_1 + N_2 |H|^2) |\dot{A}|^2 - \eta |H|^4$. From (5.2), (5.3) and (5.4), we have

$$(5.5) \quad \begin{aligned} \frac{\partial}{\partial t} f &\leq \Delta f - \frac{4(n-1)}{3n} (N_2 - 1) |H|^2 |\nabla A|^2 - \frac{4(n-1)}{3n} (N_1 - C_3(N_2)) |\nabla A|^2 \\ &\quad + C_4(N_1, N_2) |\dot{A}|^2 (|H|^4 + 1) + 12\eta |H|^2 |\nabla H|^2 - \frac{4\eta}{n} |H|^6 - 4nc\eta |H|^4. \end{aligned}$$

Here we have consumed $C_1 |H|^2 |\nabla A|^2 + C_2 |\nabla A|^2$ by firstly choosing sufficiently large N_2 and secondly choosing sufficiently large N_1 . Notice that $|\nabla H|^2 \leq n |\nabla A|^2$. We can choose larger N_2 and N_1 depending on η to consume $12\eta |H|^2 |\nabla H|^2$ and make the second, third terms of the right hand side of (5.5) negative. Since $|\dot{A}|^2 \leq C_0 |H|^{2-\sigma_0}$ for $t \geq t_0$ and $|\dot{A}|^2$ is uniformly bounded for $t \in [0, t_0]$, using Young's inequality we get

$$C_4(N_1, N_2) |\dot{A}|^2 (|H|^4 + 1) - 4nc\eta |H|^4 \leq \frac{4\eta}{n} |H|^6 + C_\eta.$$

Here C_η is a constant depending on η and other quantities but independent of t . Then we obtain

$$\frac{\partial}{\partial t} f \leq \Delta f + C_5.$$

Notice that T_{\max} is finite. Then the theorem follows from the maximum principle and the definition of f . \square

6. CONVERGENCE OF MCF IN A HYPERBOLIC SPACE

Theorem 6.1. *Let $F : M^n \rightarrow \mathbb{F}^{n+d}(c)$ be a smooth closed submanifold, where $n \geq 2$ and $c < 0$. Assume F satisfies*

$$(6.1) \quad |A|^2 \leq \begin{cases} \frac{4}{3n}|H|^2 + \frac{n}{2}c, & n = 2, 3, \\ \frac{1}{n-1}|H|^2 + 2c, & n \geq 4. \end{cases}$$

Then $F_t(M)$ converges to a round point in finite time.

Proof. By the curvature estimate in [5], we see that

$$K_{\min}(x) \geq \frac{1}{2} \left(\frac{1}{n-1} - \alpha_\epsilon \right) |H|^2(x) + \frac{1}{2}(2 - \beta_\epsilon)c.$$

By our choices of α_ϵ and β_ϵ , and the preserved pinching condition, we see that there exists a positive constant ε_0 independent of t such that

$$(6.2) \quad K_{\min}(x) \geq \varepsilon_0 |H|^2.$$

Since T_{\max} is finite, $\max_{M_t} |A|^2 \rightarrow \infty$ as $t \rightarrow T_{\max}$. By similar arguments as in [1, 11, 12] we have $\frac{\max_{M_t} |H|}{\min_{M_t} |H|} \rightarrow 1$ as $t \rightarrow T_{\max}$, and M_t 's converge to a single point o as $t \rightarrow T_{\max}$. If we take a rescaling around o (since $\mathbb{F}^{n+d}(c)$ can be consider as a linear space that isomorphic to \mathbb{R}^{n+d}) such that the total area of the expanded submanifolds are fixed, then the rescaled immersions converge to a totally umbilical immersion as $t \rightarrow T_{\max}$. \square

When $p = 1$ and $n = 3$, we have the following proposition.

Proposition 6.2. *Let $F : M^3 \rightarrow \mathbb{F}^4(c)$ be a smooth closed hypersurface in a hyperbolic space with constant curvature $c < 0$. Assume F satisfies*

$$|A|^2 \leq \frac{1}{2}|H|^2 + 2c.$$

Then the mean curvature flow with F as initial value converges to a round point in finite time.

Proof. If $d = 1$, then $\dot{A}_I = 0$. If we take $\alpha \leq \frac{1}{n-1}$ and $\beta \geq 2$ for all $n \geq 2$, the left hand side of (3.2) is nonpositive. Hence $|A|^2 \leq \frac{1}{n-1}|H|^2 + 2c$ is preserved along the mean curvature flow for all $n \geq 2$. When $n = 3$, if we set $a = \frac{1}{3(2+\epsilon)}$, then $\epsilon_\nabla = \frac{3}{4} - \frac{1}{3} - a > 0$, and Theorem 4.1 also holds. Then Theorem 5.1 follows. By a similar argument as in the proof of Theorem 6.1, we get the convergence of the mean curvature flow. \square

Remark 6.3. When $d = 1$ and $n = 3$, the pinching condition $|A|^2 \leq \frac{1}{2}|H|^2 + 2c$ is better than condition (6.1). In fact, we have $\frac{1}{2}|H|^2 + 2c - \frac{4}{9}|H|^2 - \frac{3}{2}c = \frac{1}{18}|H|^2 + \frac{1}{2}c > 0$. Here we have used the fact that $|H|^2 + 9c > 0$, which is implied by $|A|^2 \leq \frac{1}{2}|H|^2 + 2c$.

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