A SIMPLE PROOF OF MIRZAKHANI'S RECURSION FORMULA OF WEIL-PETERSSON VOLUMES

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ABSTRACT. In this paper, we give a simple proof of Mirzakhani's recursion formula of Weil-Petersson volumes of moduli spaces of curves using the Witten-Kontsevich theorem. We also briefly describe a very general recursive phenomenon in the intersection theory of moduli spaces of curves. In particular, we present several new recursion formulas for higher degree κ classes.

1. Introduction

Following the notations of Mulase and Safnuk [14], let $\mathcal{M}_{g,n}(\mathbf{L})$ denotes the moduli space of bordered Riemann surfaces with n geodesic boundary components of specified lengths $\mathbf{L} = (L_1, \dots, L_n)$ and let $\mathrm{Vol}_{g,n}(\mathbf{L})$ denotes the Weil-Petersson volume $\mathrm{Vol}(\mathcal{M}_{g,n}(\mathbf{L}))$ of moduli spaces of bordered Riemann surfaces. Mirzakhani [12] proved a beautiful recursion formula for these Weil-Petersson volumes.

$$
\begin{aligned}
\mathrm{Vol}_{g,n}(\mathbf{L}) = \frac{1}{2L_1} \sum_{\substack{g_1+g_2=g \\ I\coprod J=\{1,\dots,n\}}} \int_0^{L_1}\int_0^\infty\int_0^\infty xyH(t,x+y)\mathrm{Vol}_{g_1,n_1}(x,\mathbf{L}_I)\mathrm{Vol}_{g_2,n_2}(y,\mathbf{L}_J)dxdydt \\
+ \frac{1}{2L_1}\int_0^{L_1}\int_0^\infty\int_0^\infty xyH(t,x+y)\mathrm{Vol}_{g-1,n+1}(x,y,L_2,\dots,L_n)dxdydt \\
+ \frac{1}{2L_1}\sum_{j=2}^n\int_0^{L_1}\int_0^\infty x\big(H(x,L_1+L_j)+H(x,L_1-L_j)\big) \\
\times \mathrm{Vol}_{g,n-1}(x,L_2,\dots,\hat{L_j},\dots,L_n)dxdt,
\end{aligned}
$$

where the kernel function

$$
H(x,y) = \frac{1}{1+e^{(x+y)/2}} + \frac{1}{1+e^{(x-y)/2}}.
$$

Since

$$
\begin{aligned}
\frac{\mathrm{Vol}_{g,n}(2\pi\mathbf{L})}{(2\pi^2)^{3g+n-3}} &= \frac{1}{(3g+n-3)!}\int_{\mathcal{M}_{g,n}}(\kappa_1+\sum_{i=1}^n L_i^2\psi_i)^{3g+n-3} \\
&= \sum_{\substack{d_0+\cdots+d_n \\ =3g+n-3}} \prod_{i=0}^n \frac{1}{d_i!}\langle\kappa_1^{d_0}\prod\tau_{d_i}\rangle_{g,n}\prod_{i=1}^\infty L_i^{2d_i},
\end{aligned}
$$

by taking derivatives with respect to $\mathbf{L} = (L_1,\dots,L_n)$ in Mirzakhani's recursion, Mulase and Safnuk [14, 16] obtained the following recursion formula of intersection numbers which is clearly equivalent to Mirzakhani's recursion.

$$(2d_1+1)!!\langle\prod_{j=1}^{n}\tau_{d_j}\kappa_1^a\rangle_g \tag{1}$$

$$=\sum_{j=2}^{n}\sum_{b=0}^{a}\frac{a!}{(a-b)!}\frac{(2(b+d_1+d_j)-1)!!}{(2d_j-1)!!}\beta_b\langle\kappa_1^{a-b}\tau_{b+d_1+d_j-1}\prod_{i\neq 1,j}\tau_{d_i}\rangle_g$$

$$+\frac{1}{2}\sum_{b=0}^{a}\sum_{r+s=b+d_1-2}\frac{a!}{(a-b)!}(2r+1)!!(2s+1)!!\beta_b\langle\kappa_1^{a-b}\tau_r\tau_s\prod_{i\neq 1}\tau_{d_i}\rangle_{g-1}$$

$$+\frac{1}{2}\sum_{\substack{c+c'=a-b\\ I\coprod J=\{2,\dots,n\}}}\sum_{b=0}^{a}\sum_{r+s=b+d_1-2}\frac{a!}{c!c'!}(2r+1)!!(2s+1)!!\beta_b$$

$$\times\langle\kappa_1^{c}\tau_r\prod_{i\in I}\tau_{d_i}\rangle_{g'}\langle\kappa_1^{c'}\tau_s\prod_{i\in J}\tau_{d_i}\rangle_{g-g'},$$

where

$$\beta_b=(2^{2b+1}-4)\frac{\zeta(2b)}{(2\pi^2)^b}=(-1)^{b-1}2^b(2^{2b}-2)\frac{B_{2b}}{(2b)!}.$$

This recursion equation is the key formula that Mirzakhani derived and from which she is able to give another proof of the Witten conjecture. Indeed, when $a=0$, identity (1) is just the celebrated Witten-Kontsevich theorem [17, 8, 7, 15, 13, 6, 2].

In Section 2, we give a simple proof of this differential version of Mirzakhani's recursion using Witten-Kontsevich theorem and a simple formula in [5]. In Section 3, we present certain new recursion relations of higher Weil-Petersson volumes in intersection theory of moduli spaces of curves [11].

We remark that, in a previous version of this paper, Theorems 3.2, in Section 3 was stated as a conjecture. However, after we have released that version, we realized that our method can be generalized to prove this conjecture and several other conjectural recursion formulas. In particular, we have obtained explicit formulae for those interesting tautological constants in this theorem. The detailed proofs of the results discussed in this section as well as other results will appear in [11].

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2. Proof of Mirzakhani's Recursion Formula

We first give three simple lemmas that will be used in the proof of Mirzakhani's recursion formula. The following lemma is first used by Mulase and Safnuk [14].

Lemma 2.1. [14] *The constants β_b satisfy the following*

$$\sum_{k=0}^{\infty}(-1)^k\beta_k x^k=\frac{\sqrt{-2x}}{\sin\sqrt{-2x}}.$$

And its inverse

$$(\sum_{k=0}^{\infty}(-1)^k\beta_k x^k)^{-1}=\frac{\sin\sqrt{-2x}}{\sqrt{-2x}}=\sum_{k=0}^{\infty}\frac{2^k}{(2k+1)!}x^k$$

Proof. Since

$$\sum_{n=0}^{\infty} \frac{B_{2n}}{(2n)!} x^{2n} = \frac{x}{2} \frac{e^{x/2} + e^{-x/2}}{e^{x/2} - e^{-x/2}} = \frac{x}{2i} \cot \frac{x}{2i},$$

we have

$$\sum_{k=0}^{\infty} \beta_k x^k = \sqrt{2x}(\cot \sqrt{\frac{x}{2}} - \cot \sqrt{2x}) = \frac{\sqrt{2x}}{\sin \sqrt{2x}}.$$

□

We see that although the expression of β_b is a little complicated, the inverse of its generating function is rather simple.

The following elementary lemma is crucial in our proof.

Lemma 2.2. *Let $F(m,n)$ and $G(m,n)$ be two functions defined on $\mathbb{N} \times \mathbb{N}$, where $\mathbb{N} = \{0, 1, 2, \dots\}$ is the set of nonnegative integers. Let α_k and β_k be real numbers that satisfy*

$$\sum_{k=0}^{\infty} \alpha_k x^k = (\sum_{k=0}^{\infty} \beta_k x^k)^{-1}.$$

Then the following two identities are equivalent.

$$G(m,n) = \sum_{k=0}^{m} \alpha_k F(m-k, n+k), \quad \forall\ (m,n) \in \mathbb{N} \times \mathbb{N}$$

$$F(m,n) = \sum_{k=0}^{m} \beta_k G(m-k, n+k), \quad \forall\ (m,n) \in \mathbb{N} \times \mathbb{N}$$

Proof. Assume the first identity holds, then we have

$$\begin{aligned}
\sum_{i=0}^{m} \beta_i G(m-i, n+i) &= \sum_{i=0}^{m} \beta_i \sum_{j=0}^{m-i} \alpha_j F(m-i-j, n+i+j) \\
&= \sum_{k=0}^{m} \sum_{i+j=k} (\beta_i \alpha_j) F(m-k, n+k) \\
&= \sum_{k=0}^{m} \delta_{k0} F(m-k, n+k) \\
&= F(m,n).
\end{aligned}$$

So we proved the second identity. The proof of the other direction is the same. □

The following combinatorial lemma is a special case of a more general formula proved in [5],

Lemma 2.3. [5] *For $m > 0$,*

$$\langle \prod_{j=1}^{n} \tau_{d_j} \kappa_1^m \rangle_g = \sum_{k=1}^{m} \frac{(-1)^{m-k}}{k!} \sum_{\substack{m_1 + \cdots + m_k = m \\ m_i > 0}} \binom{m}{m_1, \dots, m_k} \langle \prod_{j=1}^{n} \tau_{d_j} \prod_{j=1}^{k} \tau_{m_j+1} \rangle_g.$$

Proof. (sketch) Let $\pi_{n+p,n}:\overline{\mathcal{M}}_{g,n+p}\longrightarrow\overline{\mathcal{M}}_{g,n}$ be the morphism which forgets the last p marked points and denote $\pi_{n+p,n*}(\psi_{n+1}^{a_1+1}\dots\psi_{n+p}^{a_p+1})$ by $R(a_1,\dots,a_p)$, then we have the formula [1]

$$R(a_1,\dots,a_p)=\sum_{\sigma\in\mathbb{S}_p}\prod_{\substack{\text{each cycle } c\\ \text{of } \sigma}}\kappa_{\sum_{j\in c}a_j},$$

where we write any permutation σ in the symmetric group $\mathbb{S}_p$ as a product of disjoint cycles.

By a formal combinatorial argument, we get the following inversion result

$$\kappa_{a_1}\cdots\kappa_{a_p}=\sum_{k=1}^{p}\frac{(-1)^{p-k}}{k!}\sum_{\substack{\{1,\dots,p\}=S_1\coprod\cdots\coprod S_k\\ S_k\neq\emptyset}}R(\sum_{j\in S_1}a_j,\dots,\sum_{j\in S_k}a_j),$$

from which Lemma 2.3 follows easily. ∎

Let LHS and RHS denote the left and right hand side of Mirzakhani's recursion (1) respectively. Apply lemma 2.3 to both sides of identity (1).

$$\begin{aligned}
LHS &= (2d_1+1)!!\langle\prod_{j=1}^{n}\tau_{d_j}\kappa_1^a\rangle_g\\
&= (2d_1+1)!!\sum_{k=0}^{a}\frac{(-1)^{a-k}}{k!}\sum_{\substack{m_1+\cdots+m_k=a\\ m_i>0}}\binom{a}{m_1,\dots,m_k}\langle\prod_{j=1}^{n}\tau_{d_j}\prod_{j=1}^{k}\tau_{m_j+1}\rangle_g\\
&= \sum_{k=0}^{a}\frac{(-1)^{a-k}}{k!}\sum_{\substack{m_1+\cdots+m_k=a\\ m_i>0}}\binom{a}{m_1,\dots,m_k}\Bigg(\sum_{j=2}^{n}\frac{(2(d_1+d_j)-1)!!}{(2d_j-1)!!}\langle\tau_{d_1+d_j-1}\prod_{i\neq1,j}\tau_{d_i}\prod_{i=1}^{k}\tau_{m_i+1}\rangle_g\\
&\quad+\frac{(2(d_1+m_j)+1)!!}{(2m_j+1)!!}\sum_{j=1}^{k}\langle\tau_{d_1+m_j}\prod_{i=2}^{n}\tau_{d_i}\prod_{i\neq j}\tau_{m_i+1}\rangle_g\\
&\quad+\frac{1}{2}\sum_{r+s=d_1-2}(2r+1)!!(2s+1)!!\langle\tau_r\tau_s\prod_{i=2}^{n}\tau_{d_i}\prod_{i=1}^{k}\tau_{m_i+1}\rangle_{g-1}\\
&\quad+\frac{1}{2}\sum_{\substack{I\coprod J=\{2,\dots,n\}\\ I'\coprod J'=\{1,\dots,k\}}}\sum_{r+s=d_1-2}(2r+1)!!(2s+1)!!\langle\tau_r\prod_{i\in I}\tau_{d_i}\prod_{i\in I'}\tau_{m_i+1}\rangle_{g'}\langle\tau_s\prod_{i\in J}\tau_{d_i}\prod_{i\in J'}\tau_{m_i+1}\rangle_{g-g'}\Bigg),
\end{aligned}$$

where we have used the Witten-Kontsevich theorem.

$$RHS = \sum_{b=0}^{a} \beta_b \frac{a!}{(a-b)!} \sum_{j=2}^{n} \frac{(2(b+d_1+d_j)-1)!!}{(2d_j-1)!!} \sum_{k=0}^{a-b} \frac{(-1)^{a-b-k}}{k!} \sum_{\substack{m_1+\cdots+m_k=a-b \\ m_i>0}} \binom{a-b}{m_1,\ldots,m_k}$$
$$\times \langle \tau_{b+d_1+d_j-1} \prod_{i\neq 1,j} \tau_{d_i} \prod_{i=1}^{k} \tau_{m_i+1} \rangle_g$$
$$+ \frac{1}{2} \sum_{b=0}^{a} \sum_{r+s=b+d_1-2} \beta_b \frac{a!}{(a-b)!} (2r+1)!!(2s+1)!! \sum_{k=0}^{a-b} \frac{(-1)^{a-b-k}}{k!} \sum_{\substack{m_1+\cdots+m_k=a-b \\ m_i>0}} \binom{a-b}{m_1,\ldots,m_k}$$
$$\times \langle \tau_r \tau_s \prod_{i=2}^{n} \tau_{d_i} \prod_{i=1}^{k} \tau_{m_i+1} \rangle_{g-1}$$
$$+ \frac{1}{2} \sum_{\substack{I \coprod J=\{2,\ldots,n\} \\ c+c'=a-b}} \sum_{r+s=b+d_1-2} \beta_b \frac{a!}{c!c'!} (2r+1)!!(2s+1)!!$$
$$\times \sum_{k=0}^{c} \frac{(-1)^{c-k}}{k!} \sum_{\substack{m_1+\cdots+m_k=c \\ m_i>0}} \binom{c}{m_1,\ldots,m_k} \times \langle \tau_r \prod_{i\in I} \tau_{d_i} \prod_{i=1}^{k} \tau_{m_i+1} \rangle_{g'}$$
$$\times \sum_{k'=0}^{c'} \frac{(-1)^{c'-k'}}{k'!} \sum_{\substack{m_1+\cdots+m_{k'}=c' \\ m_i>0}} \binom{c'}{m_1,\ldots,m_{k'}} \times \langle \tau_s \prod_{i\in I} \tau_{d_i} \prod_{i=1}^{k'} \tau_{m_i+1} \rangle_{g-g'}.$$

Let

$$F(a,d_1) := \sum_{k=0}^{a} \frac{(-1)^k}{k!} \sum_{\substack{m_1+\cdots+m_k=a \\ m_i>0}} \frac{(2d_1+1)!!}{\prod_{j=1}^{k} m_j!} \langle \prod_{j=1}^{n} \tau_{d_j} \prod_{j=1}^{k} \tau_{m_j+1} \rangle_g$$
$$= \sum_{k=0}^{a} \frac{(-1)^k}{k!} \sum_{\substack{m_1+\cdots+m_k=a \\ m_i>0}} \frac{1}{\prod_{j=1}^{k} m_j!} \Bigg(\sum_{j=2}^{n} \frac{(2(d_1+d_j)-1)!!}{(2d_j-1)!!} \langle \tau_{d_1+d_j-1} \prod_{i\neq 1,j} \tau_{d_i} \prod_{i=1}^{k} \tau_{m_i+1} \rangle_g$$
$$+ \sum_{j=1}^{k} \frac{(2(d_1+m_j)+1)!!}{(2m_j+1)!!} \langle \tau_{d_1+m_j} \prod_{i=2}^{n} \tau_{d_i} \prod_{i\neq j} \tau_{m_i+1} \rangle_g$$
$$+ \frac{1}{2} \sum_{r+s=d_1-2} (2r+1)!!(2s+1)!! \langle \tau_r \tau_s \prod_{i=2}^{n} \tau_{d_i} \prod_{i=1}^{k} \tau_{m_i+1} \rangle_{g-1}$$
$$+ \frac{1}{2} \sum_{\substack{I \coprod J=\{2,\ldots,n\} \\ I' \coprod J'=\{1,\ldots,k\}}} \sum_{r+s=d_1-2} (2r+1)!!(2s+1)!! \langle \tau_r \prod_{i\in I} \tau_{d_i} \prod_{i\in I'} \tau_{m_i+1} \rangle_{g'} \langle \tau_s \prod_{i\in J} \tau_{d_i} \prod_{i\in J'} \tau_{m_i+1} \rangle_{g-g'} \Bigg)$$

and

$$G(a,d_1):=\sum_{j=2}^{n}\frac{(2(d_1+d_j)-1)!!}{(2d_j-1)!!}\sum_{k=0}^{a}\frac{(-1)^k}{k!}\sum_{\substack{m_1+\cdots+m_k=a\\ m_i>0}}\frac{1}{\prod_{i=1}^{k}m_i!}\langle\tau_{d_1+d_j-1}\prod_{i\neq 1,j}\tau_{d_i}\prod_{i=1}^{k}\tau_{m_i+1}\rangle_g$$
$$+\frac{1}{2}(2r+1)!!(2s+1)!!\sum_{k=0}^{a}\frac{(-1)^k}{k!}\sum_{\substack{m_1+\cdots+m_k=a\\ m_i>0}}\frac{1}{\prod_{j=1}^{k}m_j!}\langle\tau_r\tau_s\prod_{i=2}^{n}\tau_{d_i}\prod_{i=1}^{k}\tau_{m_i+1}\rangle_{g-1}$$
$$+\frac{1}{2}\sum_{\substack{I\coprod J=\{2,\dots,n\}\\ c+c'=a}}\sum_{r+s=d_1-2}(2r+1)!!(2s+1)!!\sum_{k=0}^{c}\frac{(-1)^k}{k!}\sum_{\substack{m_1+\cdots+m_k=c\\ m_i>0}}\frac{1}{\prod_{j=1}^{k}m_j!}\langle\tau_r\prod_{i\in I}\tau_{d_i}\prod_{i=1}^{k}\tau_{m_i+1}\rangle_{g'}$$
$$\times\sum_{k'=0}^{c'}\frac{(-1)^{k'}}{k'!}\sum_{\substack{m_1+\cdots+m_{k'}=c'\\ m_i>0}}\frac{1}{\prod_{j=1}^{k}m_j!}\langle\tau_s\prod_{i\in I}\tau_{d_i}\prod_{i=1}^{k}\tau_{m_i+1}\rangle_{g-g'}.$$

Then it is not difficult to see that

$$LHS=(-1)^a a!F(a,d_1)$$

and

$$RHS=(-1)^a a!\sum_{b=0}^{a}(-1)^b\beta_b G(a-b,d_1+b).$$

So by lemmas 2.1 and 2.2, in order to prove Mirzakhani's recursion (1), we need only prove the following identity for $a>0$,

$$\sum_{b=0}^{a}\frac{2^b}{(2b+1)!}F(a-b,d_1+b)=G(a,d_1).$$

It is not difficult to see that,

$$F(a,d_1)-G(a,d_1)$$
$$=\sum_{k=0}^{a}\frac{(-1)^k}{k!}\sum_{\substack{m_1+\cdots+m_k=a\\ m_i>0}}\frac{1}{\prod_{i=1}^{k}m_i!}\sum_{j=1}^{k}\frac{(2(d_1+m_j)+1)!!}{(2m_j+1)!!}\langle\tau_{d_1+m_j}\prod_{i=2}^{n}\tau_{d_i}\prod_{i\neq j}\tau_{m_i+1}\rangle_g$$

and

$$\begin{aligned}
&\sum_{b=1}^{a}\frac{2^b}{(2b+1)!}F(a-b,d_1+b)\\
&\quad=\sum_{b=1}^{a}\frac{2^b}{(2b+1)!}\sum_{k=0}^{a-b}\frac{(-1)^k}{k!}\sum_{\substack{m_1+\cdots+m_k=a-b\\ m_i>0}}\frac{(2(d_1+b)+1)!!}{\prod_{j=1}^k m_j!}\langle\tau_{d_1+b}\prod_{j=2}^n\tau_{d_j}\prod_{j=1}^k\tau_{m_j+1}\rangle_g\\
&\quad=\sum_{k=0}^{a-b}\frac{(-1)^k}{k!}\sum_{\substack{b+m_1+\cdots+m_k=a\\ b>0,m_i>0}}\frac{1}{\prod_{j=1}^k m_j!b!}\frac{(2(d_1+b)+1)!!}{(2b+1)!!}\langle\tau_{d_1+b}\prod_{j=2}^n\tau_{d_j}\prod_{j=1}^k\tau_{m_j+1}\rangle_g\\
&=\frac{1}{k+1}\sum_{k=0}^{a}\frac{(-1)^k}{k!}\sum_{\substack{m_1+\cdots+m_{k+1}=a\\ m_i>0}}\frac{1}{\prod_{i=1}^{k+1} m_i!}\sum_{j=1}^{k+1}\frac{(2(d_1+m_j)+1)!!}{(2m_j+1)!!}\langle\tau_{d_1+m_j}\prod_{i=2}^n\tau_{d_i}\prod_{i\neq j}\tau_{m_i+1}\rangle_g\\
&\quad=-\sum_{k=0}^{a}\frac{(-1)^k}{k!}\sum_{\substack{m_1+\cdots+m_k=a\\ m_i>0}}\frac{1}{\prod_{i=1}^{k} m_i!}\sum_{j=1}^{k}\frac{(2(d_1+m_j)+1)!!}{(2m_j+1)!!}\langle\tau_{d_1+m_j}\prod_{i=2}^n\tau_{d_i}\prod_{i\neq j}\tau_{m_i+1}\rangle_g
\end{aligned}$$

where in the last equation, we substitute $k+1$ by k.

Summing up the above two identities, we proved

$$\sum_{b=0}^{a}\frac{2^b}{(2b+1)!}F(a-b,d_1+b)-G(a,d_1)=0.$$

So we finished the simple proof of Mirzakhani's recursion formula for the Weil-Petersson volumes.

3. Higher Weil-Petersson volumes

Note that Mirzakhani, Mulase and Safnuk's arguments rely heavily on the Wolpert's formula [18]

$$\kappa_1=\frac{1}{2\pi^2}\omega_{WP},$$

where ω_{WP} is the Weil-Petersson Kähler form. We have no similar formulae for higher degree κ classes. So a priori κ_1 may be rather special in the intersection theory. However, as we will see, this is not the case.

First we fix notations as in [5].

Consider the semigroup N^∞ of sequences $\mathbf{m}=(m(1),m(2),\dots)$ where $m(i)$ are nonnegative integers and $m(i)=0$ for sufficiently large i.

Let $\mathbf{m},\mathbf{t},\mathbf{a_1},\dots,\mathbf{a_n}\in N^\infty$, $\mathbf{m}=\sum_{i=1}^n\mathbf{a_i}$, $\mathbf{m}\geq\mathbf{t}$ and $\mathbf{s}:=(s_1,s_2,\dots)$ be a family of independent formal variables.

$$|\mathbf{m}|:=\sum_{i\geq1}im(i),\quad ||\mathbf{m}||:=\sum_{i\geq1}m(i),\quad \mathbf{s^m}:=\prod_{i\geq1}s_i^{m(i)},\quad \mathbf{m}!:=\prod_{i\geq1}m(i)!,$$

$$\binom{\mathbf{m}}{\mathbf{t}}:=\prod_{i\geq1}\binom{m(i)}{t(i)},\quad \binom{\mathbf{m}}{\mathbf{a_1},\dots,\mathbf{a_n}}:=\prod_{i\geq1}\binom{m(i)}{a_1(i),\dots,a_n(i)}.$$

Let $\mathbf{b} \in N^\infty$, we denote a formal monomial of κ classes by

$$\kappa(\mathbf{b}) := \prod_{i\geq 1} \kappa_i^{b(i)}.$$

The following lemma is a generalization of lemma 2.2. The proof can also be generalized.

Lemma 3.1. *Let $F(\mathbf{L},n)$ and $G(\mathbf{L},n)$ be two functions defined on $N^\infty \times \mathbb{N}$, where $\mathbb{N} = \{0,1,2,\dots\}$ is the set of nonnegative integers. Let $\alpha_{\mathbf{L}}$ and $\beta_{\mathbf{L}}$ be real numbers depending only on $\mathbf{L} \in N^\infty$ that satisfy $\alpha_{\mathbf{0}} = \beta_{\mathbf{0}} = 1$ and*

$$\sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} \alpha_{\mathbf{L}}\beta_{\mathbf{L}'} = 0, \qquad \mathbf{b} \neq 0.$$

Then the following two identities are equivalent.

$$G(\mathbf{b},n) = \sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} \alpha_{\mathbf{L}} F(\mathbf{L}', n+|\mathbf{L}|), \quad \forall\ (\mathbf{b},n) \in N^\infty \times \mathbb{N}$$

$$F(\mathbf{b},n) = \sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} \beta_{\mathbf{L}} G(\mathbf{L}', n+|\mathbf{L}|), \quad \forall\ (\mathbf{b},n) \in N^\infty \times \mathbb{N}$$

The following theorem is a vast generalization of Mirzakhani's recursion formula, which further clarifies the combinatorial structure in Mirzakhani's recursion [11].

Theorem 3.2. *There exist (uniquely determined) rational numbers $\alpha_{\mathbf{L}}$ depending only on $\mathbf{L} \in N^\infty$, such that for any $\mathbf{b} \in N^\infty$ and $d_j \geq 0$, the following recursion relation of mixed ψ and κ intersection numbers holds.*

$$\begin{aligned}
&(2d_1+1)!!\langle \kappa(\mathbf{b})\tau_{d_1}\cdots\tau_{d_n}\rangle_g \\
&\quad = \sum_{j=2}^{n} \sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} \alpha_{\mathbf{L}} \binom{\mathbf{b}}{\mathbf{L}} \frac{(2(|\mathbf{L}|+d_1+d_j)-1)!!}{(2d_j-1)!!} \langle \kappa(\mathbf{L}')\tau_{|\mathbf{L}|+d_1+d_j-1} \prod_{i\neq 1,j} \tau_{d_i}\rangle_g \\
&\quad + \frac{1}{2} \sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} \sum_{r+s=|\mathbf{L}|+d_1-2} \alpha_{\mathbf{L}} \binom{\mathbf{b}}{\mathbf{L}} (2r+1)!!(2s+1)!! \langle \kappa(\mathbf{L}')\tau_r\tau_s \prod_{i\neq 1} \tau_{d_i}\rangle_{g-1} \\
&\qquad + \frac{1}{2} \sum_{\substack{\mathbf{L}+\mathbf{e}+\mathbf{f}=\mathbf{b} \\ I\coprod J=\{2,\dots,n\}}} \sum_{r+s=|\mathbf{L}|+d_1-2} \alpha_{\mathbf{L}} \binom{\mathbf{b}}{\mathbf{L},\mathbf{e},\mathbf{f}} (2r+1)!!(2s+1)!! \\
&\qquad\qquad \times \langle \kappa(\mathbf{e})\tau_r \prod_{i\in I}\tau_{d_i}\rangle_{g'} \langle \kappa(\mathbf{f})\tau_s \prod_{i\in J}\tau_{d_i}\rangle_{g-g'}.
\end{aligned}$$

These tautological constants $\alpha_{\mathbf{L}}$ can be determined recursively from the following formula

$$\sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} (-1)^{||\mathbf{L}||} \frac{\alpha_{\mathbf{L}}}{\mathbf{L}!\mathbf{L}'!(2|\mathbf{L}'|+1)!!} = 0, \qquad \mathbf{b}\neq 0,$$

namely

$$\alpha_{\mathbf{b}} = \mathbf{b}! \sum_{\substack{\mathbf{L}+\mathbf{L}'=\mathbf{b} \\ \mathbf{L}'\neq \mathbf{0}}} (-1)^{||\mathbf{L}'||-1} \frac{\alpha_{\mathbf{L}}}{\mathbf{L}!\mathbf{L}'!(2|\mathbf{L}'|+1)!!}, \qquad \mathbf{b}\neq 0,$$

with the initial value $\alpha_{\mathbf{0}} = 1$.

The proofs of Theorem 3.2 uses Lemma 3.1 and the more general version of Lemma 2.3 expressing κ classes in τ classes [5], then follows the steps of our proof of Mirzakhani's recursion formula in Section 2. Details will be given in [11].

From Theorem 3.2, we have generalized almost all results of Mulase and Safnuk [14]. These generalizations are contained in [11].

We have computed a table of $\alpha_{\mathbf{L}}$ for all $|\mathbf{L}| \leq 15$ and have written a Maple program implementing theorem 3.2, which can be downloaded at [19].

We have obtained many identities of higher Weil-Petersson volumes [11] which share similar structures as Theorem 3.2. We present some of them below. Please see also [9, 10].

Theorem 3.3. *Let* $\mathbf{b} \in N^\infty$, $M \geq 2g$ *be an even number and* $d_j \geq 0$.

$$\sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} (-1)^{||\mathbf{L}||} \binom{\mathbf{b}}{\mathbf{L}} \langle \tau_{|\mathbf{L}|+M} \prod_{j=1}^n \tau_{d_j} \kappa(\mathbf{L}')\rangle_g = \sum_{j=1}^n \langle \tau_{d_j+M-1} \prod_{i\neq j} \tau_{d_i} \kappa(\mathbf{b})\rangle_g$$
$$- \frac{1}{2} \sum_{\substack{\mathbf{L}+\mathbf{L}'=\mathbf{b} \\ \underline{n}=I\coprod J}} \sum_{j=0}^{M-2} (-1)^j \binom{\mathbf{b}}{\mathbf{L}} \langle \tau_j \prod_{i\in I} \tau_{d_i} \kappa(\mathbf{L})\rangle_{g'} \langle \tau_{M-2-j} \prod_{i\in J} \tau_{d_i} \kappa(\mathbf{L}')\rangle_{g-g'},$$

where $\kappa(\mathbf{b}) := \prod_{i\geq 1} \kappa_i^{b(i)}$ *is a formal monomial of* κ *classes.*

Theorem 3.4. *For* $\mathbf{b} \in N^\infty$ *and* $d_j \geq 0$,

$$\sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} (-1)^{||\mathbf{L}||} \binom{\mathbf{b}}{\mathbf{L}} \langle \tau_{|\mathbf{L}|} \prod_{j=1}^n \tau_{d_j} \kappa(\mathbf{L}')\rangle_g = \sum_{j=1}^n \langle \tau_{d_j-1} \prod_{i\neq j} \tau_{d_i} \kappa(\mathbf{b})\rangle_g,$$

and

$$\sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} (-1)^{||\mathbf{L}||} \binom{\mathbf{b}}{\mathbf{L}} \langle \tau_{|\mathbf{L}|+1} \prod_{j=1}^n \tau_{d_j} \kappa(\mathbf{L}')\rangle_g = (2g-2+n) \langle \prod_{j=1}^n \tau_{d_j} \kappa(\mathbf{b})\rangle_g.$$

Note that Theorem 3.4 generalizes the results in [3]. Theorems 3.3 and 3.4 both follow easily by applying the formula in [5].

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