

CIRCLE ACTION AND SOME VANISHING RESULTS ON MANIFOLDS

PING LI AND KEFENG LIU

ABSTRACT. Kawakubo and Uchida showed that, if a closed oriented $4k$ -dimensional manifold M admits a semi-free circle action such that the dimension of the fixed point set is less than $2k$, then the signature of M vanishes. In this note, by using G -signature theorem and the rigidity of the signature operator, we generalize this result to more general circle actions. Combining the same idea with the remarkable Witten-Taubes-Bott rigidity theorem, we explore more vanishing results on spin manifolds admitting such circle actions. Our results are closely related to some earlier results of Conner-Floyd, Landweber-Stong and Hirzebruch-Slodowy.

1. INTRODUCTION AND RESULTS

Unless otherwise stated, all the manifolds discussed in this paper are closed smooth manifolds and all involutions and circle actions on the manifolds are smooth. We denote by superscripts the corresponding dimensions of the manifolds.

The following is a classical result of Conner and Floyd ([5], §27.2).

Theorem 1.1 (Conner-Floyd). *Suppose $g : M^{2n} \rightarrow M^{2n}$ is an involution on a manifold and M^g is the fixed point set of g . If $\dim(M^g) < n$, then the Euler characteristic of M is even.*

Here by $\dim(M^g)$ we mean the dimension of highest dimensional connected component of M^g .

Using their famous G -signature theorem, Atiyah and Singer reproved ([3], p.582-p.583) Theorem 1.1 when n is even, M is oriented and g is orientation preserving.

We recall that a circle action (S^1 -action) is called *semi-free* if it is free on the complement of the fixed point set or equivalently, the isotropy subgroup of any non fixed point on the manifold is trivial. Using bordism techniques developed by Conner and Floyd in [5], Kawakubo and Uchida showed the following result ([11], Theorem 1.2), which could be taken as a counterpart in the circle case to Theorem 1.1 in some sense.

Theorem 1.2 (Kawakubo-Uchida). *Suppose M^{4k} admits a semi-free S^1 -action and M^{S^1} is the fixed point set of this action. If $\dim(M^{S^1}) < 2k$, then the signature of M , $\text{sign}(M^{4k})$, is zero.*

Our first purpose in this note is, by closer looking at the G -signature theorem in the circle case, to generalize Theorem 1.2 to more general cases. Before stating our first main result, we will introduce some notations, which will be used throughout this paper without further explanation.

2000 *Mathematics Subject Classification.* 19J35, 57R20, 58J20.

Key words and phrases. circle action, signature, twisted signature operator, rigidity.

The first author is supported by Program for Young Excellent Talents in Tongji University.

Suppose M^{2n} is a oriented manifold admitting a S^1 -action. Let F^{2m} be a connected component of this action. With respect to this S^1 -action, the tangent bundle of M^{2n} restricted to F^{2m} , $TM^{2n}|_{F^{2m}}$, has the following equivariant decomposition:

$$TM^{2n}|_{F^{2m}} = L_1 \oplus \cdots \oplus L_{n-m} \oplus TF^{2m},$$

where each L_i is a real 2-plane bundle of F^{2m} . We can identify L_i with a complex line bundle relative to which the representation of S^1 on each fiber of L_i is given by $e^{\sqrt{-1}\theta} \rightarrow e^{\sqrt{-1}k_i\theta}$ with $k_i \in \mathbb{Z} - \{0\}$. These k_1, \cdots, k_{n-m} are called *weights* of this S^1 -action on the connected component F^{2m} and uniquely determined up to signs.

Definition 1.3. Let the notations be as above. We call a S^1 -action *prime* if there exists a number $\xi \in S^1$ such that, for any $k \in \bigcup_{F^{2m}} \{k_1, \cdots, k_{n-m}\}$, we have $\xi^k = -1$.

Remark 1.4. Note that the weights of a semi-free circle action are ± 1 . Hence semi-free circle actions are prime.

Now we can state our first result, which generalizes Theorem 1.2.

Theorem 1.5. *Suppose M^{4k} admits a prime S^1 -action and M^{S^1} is the fixed point set of this action. If $\dim(M^{S^1}) < 2k$, then $\text{sign}(M^{4k}) = 0$.*

We will prove this result in Section 2. The signature of a oriented manifold can be realized as an index of some elliptic operator ([3], §6), now called signature operator. Besides the G -signature theorem, the key ingredient of the proof of Theorem 1.5 is the rigidity of the signature operator (see Section 2 for more details). The rigidity of signature operator is only the beginning of a remarkable rigidity theorem: Witten-Taubes-Bott rigidity theorem. Our second purpose in this note is, by using this rigidity theorem, to replace the conclusion of $\text{sign}(M) = 0$ in Theorem 1.5 by those of vanishing indices of some twisted signature operators.

In order to state our second result, let us begin with the rigidity of elliptic operators.

Let $D : \Gamma(E) \rightarrow \Gamma(F)$ be an elliptic operator acting on sections of complex vector bundles E and F over a manifold M . Ellipticity guarantees that both $\ker(D)$ and $\text{coker}(D)$ are finite-dimensional. Then the index of D is defined as

$$\text{ind}(D) = \dim_{\mathbb{C}} \ker(D) - \dim_{\mathbb{C}} \text{coker}(D).$$

If M admits an S^1 -action preserving D , i.e., acting on E and F and commuting with D , then both $\ker(D)$ and $\text{coker}(D)$ admit an S^1 -action and hence are S^1 -modules. Therefore the virtual complex vector space $\ker(D) - \text{coker}(D)$ has a Fourier decomposition into a finite sum of complex one-dimensional irreducible representations of S^1 :

$$\ker(D) - \text{coker}(D) = \sum a_i \cdot L^i,$$

where $a_i \in \mathbb{Z}$ is the representation of S^1 on \mathbb{C} given by $\lambda \mapsto \lambda^i$. The equivariant index of D at $g \in S^1$, $\text{ind}(g, D)$, is defined to be

$$\text{ind}(g, D) = \sum a_i \cdot g^i.$$

The elliptic operator D is called *rigid* with respect to this S^1 -action if $a_i = 0$ for all $i \neq 0$, i.e., $\ker(D) - \text{coker}(D)$ consists of the trivial representation with multiplicity a_0 . Consequently, $\text{ind}(g, D) \equiv \text{ind}(D)$ for any $g \in S^1$. An elliptic operator is called *universally rigid* if it is rigid with respect to *any* S^1 -action. The fundamental examples of universally rigid operators are

signature operator and Dirac operator (on spin manifolds). The reason for the former is that both of its kernel and cokernel can be identified with subspaces of the deRham cohomology group ([3], §6) on which S^1 always induces a trivial action. The latter is a classical result of Atiyah and Hirzebruch [2].

Let $\Omega_{\mathbb{C}}^+$ and $\Omega_{\mathbb{C}}^-$ be the even and odd complex differential forms on a oriented Riemann manifold M^{2n} under the Hodge $*$ -operator. Then the signature operator

$$d_s : \Omega_{\mathbb{C}}^+ \rightarrow \Omega_{\mathbb{C}}^-$$

is elliptic and the index of d_s equals to $\text{sign}(M)$ ([3], §6).

Let W be a complex vector bundle over M . By means of a connection on W , the signature operator can be extended to a twisted operator ([15], IV, §9)

$$d_s \otimes W : \Omega_{\mathbb{C}}^+(W) \rightarrow \Omega_{\mathbb{C}}^-(W).$$

This operator is also elliptic and the index of $d_s \otimes W$ is denoted by $\text{sign}(M, W)$.

Let $T_{\mathbb{C}}$ be the complexified tangent bundle of M . For an indeterminate t , set

$$\Lambda_t T_{\mathbb{C}} = \sum_{k=0}^{\infty} t^k \Lambda^k T_{\mathbb{C}}, \quad S_t T_{\mathbb{C}} = \sum_{k=0}^{\infty} t^k S^k T_{\mathbb{C}},$$

where $\Lambda^k T_{\mathbb{C}}$ and $S^k T_{\mathbb{C}}$ are the k -th exterior power and symmetry power of $T_{\mathbb{C}}$ respectively ([1], §3.1).

Let R_i be the sequence of bundles defined by the formal series

$$\sum_{i=0}^{+\infty} q^i R_i = \bigotimes_{i=1}^{+\infty} \Lambda_{q^i} T_{\mathbb{C}} \otimes \bigotimes_{j=1}^{+\infty} S_{q^j} T_{\mathbb{C}}.$$

The first few terms of this sequence are

$$R_0 = 1, \quad R_1 = 2T_{\mathbb{C}}, \quad R_2 = 2(T_{\mathbb{C}} \otimes T_{\mathbb{C}} + T_{\mathbb{C}}), \quad \dots$$

With all this understood we have the following rigidity theorem.

Theorem 1.6 (Witten-Taubes-Bott). *For a spin manifold M^{2n} , each of the elliptic operators $d_s \otimes R_i$ is universally rigid.*

This rigidity theorem was conjectured and given a string-theoretic interpretation by Witten [18]. It was first proved by Taubes [16]. A simpler proof was then presented by Bott and Taubes [4]. Using modular invariance of Jacobi functions, the second author gave a more simpler and unified new proof in [13] and further generalized them in [14].

Using this remarkable rigidity theorem, Hirzebruch and Slodowy showed that ([10], p.317), among other things, if g is an involution contained in a circle acting on a spin manifold M^{4k} and $\dim(M^g) < 2k$, then $\text{sign}(M, R_i) = 0$ for all i .

We are now ready to state our second main result, which could be taken as the counterpart to Hirzebruch-Slodowy's above mentioned result in the circle case.

Theorem 1.7. *Suppose M^{2n} is a spin manifold admitting a prime S^1 -action. If $\dim(M^{S^1}) < n$, then $\text{sign}(M^{2n}, R_i) = 0$ for all i .*

Corollary 1.8. *Suppose M^{2n} is a spin manifold admitting a semi-free S^1 -action. If $\dim(M^{S^1}) < n$, then $\text{sign}(M^{2n}, R_i) = 0$ for all i .*

Remark 1.9. In [12], Landweber and Stong proved two results concerning the signature and the indices of three twisted Dirac operators, which also have the same feature as our results in some sense. More precisely, they showed that ([12], Theorem 1), if a closed spin manifold M^{2n} admits a circle action of *odd type*, then $\text{sign}(M) = 0$. Moreover, if this action is semi-free, then ([12], Theorem 2) $\hat{A}(M, T_{\mathbb{C}}) = \hat{A}(M, \Lambda^2 T_{\mathbb{C}}) = \hat{A}(M, \Lambda^3 T_{\mathbb{C}} + T_{\mathbb{C}}^2) = 0$, where $\hat{A}(M, E)$ is the index of the Dirac operator on M twisted by a complex vector bundle E .

2. PROOF OF RESULTS

Let M^{2n} be a oriented manifold admitting an S^1 -action. Let F^{2m} be a connected component of the fixed point set M^{S^1} . As pointed out in Introduction, $\text{TM}^{2n}|_{F^{2m}}$ can be decomposed into

$$\text{TM}^{2n}|_{F^{2m}} = L_1 \oplus \cdots \oplus L_{n-m} \oplus \text{TF}^{2m}.$$

Here L_i could be taken as a complex line bundle over F^{2m} with weight k_i , $1 \leq i \leq n - m$.

F^{2m} can be oriented so that all orientations of L_1, \dots, L_{n-m} and F^{2m} taken together yield the orientation of M^{2n} . Let $c_1(L_i) \in H^2(F^{2m}; \mathbb{Z})$ be the first Chern class of L_i . Suppose the total Pontrjagin class of F^{2m} has the following formal decomposition

$$p(F^{2m}) = 1 + p_1(F^{2m}) + \cdots = \prod_{i=1}^m (1 + x_i^2),$$

i.e., $p_i(F^{2m})$ is the i -th elementary symmetry polynomial of x_1^2, \dots, x_m^2 .

With these notations set up, we have the following important lemma, which should be well-known for experts (cf. [9], §5.8), although, according to the authors acknowledge, nobody state it explicitly as follows.

Lemma 2.1. *Let g be an indeterminate. Then the rational function of g*

$$\sum_{F^{2m}} \left\{ \left[\left(\prod_{i=1}^m x_i \frac{1 + e^{-x_i}}{1 - e^{-x_i}} \right) \left(\prod_{j=1}^{n-m} \frac{1 + g^{k_j} e^{-c_1(L_j)}}{1 - g^{k_j} e^{-c_1(L_j)}} \right) \right] \cdot [F^{2m}] \right\}$$

identically equals to $\text{sign}(M)$. Here $[F^{2m}]$ is the fundamental class of F^{2m} determined by the orientation and the sum is over all the connected components of M^{S^1} .

Proof. Let $g \in S^1$ be a topological generator. Then the fixed point set of the action of g on M are exactly M^{S^1} . So the G -signature theorem ([3], p.582) tells us that

$$(2.1) \quad \text{sign}(g, M^{2n}) = \sum_{F^{2m}} \left\{ \left[\left(\prod_{i=1}^m x_i \frac{1 + e^{-x_i}}{1 - e^{-x_i}} \right) \left(\prod_{j=1}^{n-m} \frac{1 + g^{k_j} e^{-c_1(L_j)}}{1 - g^{k_j} e^{-c_1(L_j)}} \right) \right] \cdot [F^{2m}] \right\}$$

Here $\text{sign}(g, M^{2n})$ is the equivariant index of the signature operator at $g \in S^1$. According to the rigidity of the signature operator, we have $\text{sign}(g, M^{2n}) \equiv \text{sign}(M^{2n})$. Therefore (2.1) holds for a dense subset of S^1 (the topological generators are dense in S^1), which means (2.1) is in fact an identity for an indeterminate g . \square

Remark 2.2. (1) Lemma 2.1 was used in ([9], §5.8), by putting $g = 0$, to obtain the famous formula

$$\text{sign}(M^{2n}) = \sum_{F^{2m}} \text{sign}(F^{2m}),$$

of which several proofs are given by Atiyah-Hirzebruch ([2], §3), Hattori-Taniguchi ([8], §4), and Witten ([17], §3) respectively.

- (2) When M^{S^1} consists of isolated points, Lemma 2.1 was used by Ding ([7], p.3947) to obtain some interesting results concerning the representations on the isolated fixed points.

Proof of Theorem 1.5.

Proof. Now suppose M^{4k} has a prime S^1 -action. Let $\xi \in S^1$ be the desired element as in Definition 1.3. Then we have

$$\begin{aligned}
(2.2) \quad \left[\prod_{j=1}^{2k-m} \frac{1 + g^{k_j} e^{-c_1(L_j)}}{1 - g^{k_j} e^{-c_1(L_j)}} \right] \Big|_{g=\xi} &= \prod_{j=1}^{2k-m} \frac{1 - e^{-c_1(L_j)}}{1 + e^{-c_1(L_j)}} \\
&= \prod_{j=1}^{2k-m} \frac{c_1(L_j) - \frac{1}{2}c_1^2(L_j) + \cdots}{2 - c_1(L_j) + \cdots} \\
&= \left(\prod_{j=1}^{2k-m} c_1(L_j) \right) \cdot \prod_{j=1}^{2k-m} \frac{1 - \frac{1}{2}c_1(L_j) + \cdots}{2 - c_1(L_j) + \cdots} \\
&= e(\nu F^{2m}) \cdot \prod_{j=1}^{2k-m} \frac{1 - \frac{1}{2}c_1(L_j) + \cdots}{2 - c_1(L_j) + \cdots},
\end{aligned}$$

where $e(\nu F^{2m}) \in H^{4k-2m}(F^{2m}; \mathbb{Z})$ is the Euler class of the normal bundle of F^{2m} in M^{4k} .

If $\dim(M^{S^1}) < 2k$, then $4k - 2m > 2m$, which means $e(\nu F^{2m}) = 0$ and so by Lemma 2.1 $\text{sign}(M^{4k}) = 0$. This completes the proof. \square

Proof of Theorem 1.7.

Proof. Let R_i be the complex vector bundles defined in Introduction. Then for each topological generator $g \in S^1$, the equivariant index of the elliptic operator $d_s \otimes R_i$, $\text{sign}(g, M^{2n}, R_i)$, like G -signature theorem, could be computed in terms of the local invariants of the fixed point set M^{S^1} . This is given by a general Lefschetz fixed point formula of Atiyah-Bott-Segal-Singer ([3], p.254-p.258). Instead of writing down the general form of this formula, we only indicate that, for $\text{sign}(g, M^{2n}, R_i)$, this formula is of the following form.

$$\sum_{i=0}^{+\infty} q^i \cdot \text{sign}(g, M^{2n}, R_i) = \sum_{F^{2m}} \left\{ \prod_{i=1}^m \left[\left(x_i \frac{1 + e^{-x_i}}{1 - e^{-x_i}} \right) \cdot u_i \right] \prod_{j=1}^{n-m} \left[\left(\frac{1 + g^{k_j} e^{-c_1(L_j)}}{1 - g^{k_j} e^{-c_1(L_j)}} \right) \cdot v_j \right] \right\} \cdot [F^{2m}],$$

where

$$u_i = \prod_{r=1}^{+\infty} \frac{(1 + q^r e^{-x_i})(1 + q^r e^{x_i})}{(1 - q^r e^{-x_i})(1 - q^r e^{x_i})},$$

and

$$v_j = \prod_{r=1}^{+\infty} \frac{(1 + q^r g^{k_j} e^{-c_1(L_j)})(1 + q^r g^{-k_j} e^{c_1(L_j)})}{(1 - q^r g^{k_j} e^{-c_1(L_j)})(1 - q^r g^{-k_j} e^{c_1(L_j)})}.$$

We recommend the readers the references (([13], §1 and §5) or ([6], §2.4)) for a detailed description of $\text{sign}(g, M^{2n}, R_i)$ in terms of the local data of M^{S^1} .

Rigidity theorem 1.6 says that, for an indeterminate g , the following identity holds

$$\sum_{i=0}^{+\infty} q^i \cdot \text{sign}(M^{2n}, R_i) \equiv \sum_{F^{2m}} \left\{ \prod_{i=1}^m \left[\left(x_i \frac{1 + e^{-x_i}}{1 - e^{-x_i}} \right) \cdot u_i \right] \prod_{j=1}^{n-m} \left[\left(\frac{1 + g^{k_j} e^{-c_1(L_j)}}{1 - g^{k_j} e^{-c_1(L_j)}} \right) \cdot v_j \right] \right\} \cdot [F^{2m}],$$

By using the same idea as in the proof of Theorem 1.5, we can get the conclusion of Theorem 1.7. \square

3. CONCLUSION REMARKS

As we have seen, the key idea in the proofs is to extract a cohomology class $e(\nu F^{2m})$ from the right-hand side of (2.1), by giving a special value on g . In fact, for a general compact Lie group G acting on M^{2n} and $g \in G$, the G -signature theorem is of the following form ([3], p.582)

$$(3.1) \quad \text{sign}(g, M) = \sum_{F \in M^g} [e(N^g(-1)) \cdot u] \cdot [F],$$

where F is a connected component of the fixed point set of g , M^g (rather than the fixed point set of the whole G), $e(N^g(-1))$ is the Euler class of a subbundle of the normal bundle on M^g , corresponding to the eigenvalue -1 of the representation of g on the normal bundle of F , and $u \in H^*(F)$.

Consequently, the right-hand side of (3.1) vanishes if, for every F , the fiber dimension of $N^g(-1)$ is greater than the dimension of F . This is the corollary 6.13 in [3] on page 582. It is this corollary that makes Atiyah and Singer to reprove Theorem 1.1 in some cases, because for an involution, -1 is the *only* eigenvalue on the normal bundle of the fixed point set. While in the circle case, for a topological generator $g = e^{2\pi\sqrt{-1}\theta} \in S^1$, -1 is *not* the eigenvalue (g is a topological generator if and only if θ is irrational, then $g^k \neq -1$ for all weights k). So we have to construct a cohomology class ($e(\nu F^{2m})$ in (2.2)) similar to $e(N^g(-1))$ in (3.1) artificially. This is the origin of our Definition 1.3.

REFERENCES

- [1] M.F. Atiyah: *K-theory*, New-York-Amsterdam, 1967.
- [2] M.F. Atiyah, F. Hirzebruch: *Spin-Manifolds and Group Actions*, 1970 Essays on Topology and Related Topics (Mémoires dédiés à Georges de Rham) pp. 18-28 Springer, New York.
- [3] M.F. Atiyah, I.M. Singer: *The index theory of elliptic operators: III*, Ann. Math. **87** (1968), 546-604.
- [4] R. Bott, C. Taubes: *On the rigidity theorems of Witten*, J. Amer. Math. Soc. **2** (1989), 137-186.
- [5] P.E. Conner, E.E. Floyd: *Differential periodic maps*, Springer, Berlin, 1964.
- [6] A. Dessai, R. Jung: *On the rigidity theorem for elliptic genera*, Trans. Amer. Math. Soc. **350** (1998), 4195-4220.
- [7] P. Ding: *Topological obstructions to certain Lie group actions on manifolds*, Trans. Amer. Math. Soc. **358** (2006), 3937-3967.
- [8] A. Hattori, H. Taniguchi: *Smooth S^1 -action and bordism*, J. Math. Soc. Japan. **24** (1972), 701-731.
- [9] F. Hirzebruch, T. Berger, R. Jung: *Manifolds and modular forms*, Aspects of Mathematics, E20, Friedr. Vieweg and Sohn, Braunschweig, 1992.
- [10] F. Hirzebruch, P. Slodowy: *Elliptic genera, involutions, and homogeneous spin manifolds*, Geom. Dedicata. **35** (1990), 309-343.
- [11] K. Kawakubo, F. Uchida: *On the index of a semi-free S^1 -action*, J. Math. Soc. Japan. **23** (1971), 351-355.
- [12] P. Landweber, R.E. Stong: *Circle actions on spin manifolds and characteristic numbers*, Topology **27** (1988), 145-161.
- [13] K. Liu: *On elliptic genera and theta-functions*, Topology. **35** (1996), 617-640.
- [14] K. Liu: *On modular invariance and rigidity theorems*, J. Differential Geom. **41** (1995), 343-396.

- [15] R. Palais: *Seminar on the Atiyah-Singer index theorem*, Ann. of Math. Stud., vol. 57, Princeton Univ. Press, Princeton, 1965.
- [16] C. Taubes: *S^1 actions and elliptic genera*, Comm. Math. Phys. 122 (1989), 455–526.
- [17] E. Witten: *Supersymmetry and Morse theory*, J. Differential Geom. **17** (1982), 661–692.
- [18] E. Witten: *The index of the Dirac operator in loop space*, Elliptic curves and modular forms in algebraic topology (Princeton, NJ, 1986), 161–181, Lecture Notes in Math., 1326, Springer, Berlin, 1988.

DEPARTMENT OF MATHEMATICS, TONGJI UNIVERSITY, SHANGHAI 200092, CHINA

E-mail address: pingli@tongji.edu.cn

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA AT LOS ANGELES, LOS ANGELES, CA 90095,
USA AND CENTER OF MATHEMATICAL SCIENCE, ZHEJIANG UNIVERSITY, 310027, CHINA

E-mail address: liu@math.ucla.edu